4.3 Translation Theorems

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1 Translation on the s-axis

In general, if we know $\mathcal{L}\{f(t)\} = F(s)$, it is possible to compute the Laplace transform of an exponential multiple of the function f with no additional effort other than translating or shifting $F(s)$ to $F(s-a)$.

1.0.1 Theorem 4.3.1 – First Translation Theorem

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

1.0.2 Inverse Form of Theorem 4.3.1

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t)$$

2 Translation on the t-axis

2.1 Unit Step Function

It is convenient to define a special function that is 0 until a specified time a, then 1 after that time. This is also called the **Heaviside function**.

2.1.1 Definition 4.3.1 – Unit Step Function

$$\mathcal{U}(t-a) = \begin{cases} 0, 0 \leq t \leq a \\ 1, t \geq a \end{cases}$$

This definition implies that the function

$$f(t) = \begin{cases} g(t), 0 \leq t < a \\ h(t), t \geq a \end{cases}$$

is the same as the function

$$f(t) = g(t) - g(t)\mathcal{U}(t-a) + h(t)\mathcal{U}(t-a)$$

2.1.2 Theorem 4.3.2 Second Translation Theorem

$$\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}F(s)$$

2.1.3 Inverse Form of Theorem 4.3.2

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathcal{U}(t-a)$$

Using Definition 4.1.1, we can derive an alternative version of the above theorem.

$$\mathcal{L}\{g(t)\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{g(t+a)\}$$