18.3 Independence of Path

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In real calculus we have the Fundamental Theorem of Calculus:

$$\int_a^b f(x)dx = F(b) - F(a) \tag{1}$$

In this case we can say that the line integral is independent of the path. So now we ask the question: is there a complex version of the Fundamental Theorem of Calculus.

1 Path Independence

1.0.1 Definition 18.3.1 – Independence of Path

Let z_0 and z_1 be points in a domain D. A contour integral $\int_C f(z)dz$ is said to be independent of the path if its value is the same for all contours C in D. We see from the Cauchy-Goursat Theorem that

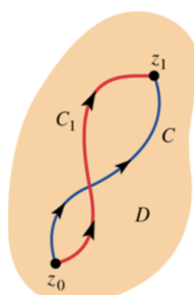

Figure 1: If f is analytic in D, integrals over C and C_1 are the same

$$\int_C f(z)dz + \int_{-C_1} f(z)dz = 0 \tag{2}$$

which is equivalent to

$$\int_C f(z)dz = \int_{C_1} f(z)dz \tag{3}$$

1.0.2 Theorem 18.3.1 – Analyticity Implies Path Independence

If f is an analytic function in a simply connected domain D, then $\int_C f(z)dz$ is independent of the chosen path C.

1.0.3 Definition 18.3.2 – Antiderivative

Suppose f is continuous in a domain D. If there exists a function F such that $F'(z) = f(z)$ for each z in D, then F is called an **antiderivative** of F.

1.0.4 Theorem 18.3.2 – Fundamental Theorem for Contour Integrals

Suppose f is continuous in a domain D and F is an antiderivative of f in D. Then for any contour C in D with initial point z_0 and terminal point z_1,

$$\int_C f(z)dz = F(z_1) - F(z_0) \tag{4}$$

In proving the above theorem, we can make the following two statements:

1. If a continuous function f has an antiderivative F in D, then $\int_C f(z)dz$ is independent of the path.
2. If f is continuous and $\int_C f(z)dz$ is independent of the path in a domain D, then f has an antiderivative everywhere in D.

1.0.5 Theorem 18.3.3 – Existence of an Antiderivative

If f is analytic in a simply connected domain D, then f has an antiderivative in D; that is, there exists a function F such that $F'(z) = f(z)$ for all z in D.