17.2 Powers and Roots

Arnav Patil

University of Toronto

In Calculus II, we discussed how points (x, y) may be expressed in a polar form (r, θ). We will now discuss how being able to do the same to a complex number helps us easily find its powers and roots.

1 Polar Form

A nonzero complex number $z = x + yi$ can be written as $z = (r\cos\theta) + i(r\sin\theta)$. We call this the polar form of the complex number z. We call r the modulus, or length of z, denoted as $|z|$. The angle θ is the argument of z and is denoted as $\theta = \arg z$. Remember that $\theta \pm 2k\pi$ where $k \in \mathbb{Z}$ are all arguments of z.

The **principal argument** of z is in the interval $(-\pi, \pi]$ and is denoted by Arg z.

2 Multiplication and Division

Multiplying two complex numbers is given by:

$$z_1 z_2 = r_1 r_2 e^{\theta_1 + \theta_2}$$

Division of two complex numbers is given by:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{\theta_1 - \theta_2}$$

Proofs of the above are elementary in that they were covered in kindergarten.

3 Integer Powers of z

The n^{th} power of a given complex number z is given by:

$$z^n = r^n e^{n\theta i}$$

4 De Moivre's Formula

When $z = \cos\theta + i\sin\theta$, we have $|z| = 1$, which yields:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

5 Roots

See class lecture notes below because that explanation is better than the textbook.

Roots

We will prove this much later in the course

- Exercise → Compute all cube roots of $z = -8$
- Fact → For any complex $z \neq 0$, the equation $w^n = z$ has n complex solutions
- First write the given z in polar form

  ↳ $z = -8 = 8e^{i(\pi + 2k\pi)}$

  $x: -8 = 8\cos\theta$, $y: 0 = 8\sin\theta$ $\Big\}$ $\cos\theta = -1$, $\sin\theta = 0$ $\therefore \theta = \pi + 2k\pi$

- Now write w in polar form $w = \rho e^{i\alpha}$

  ↳ $w^3 = \rho^3 e^{3\alpha i} = 8e^{i(\pi+2k\pi)} \rightarrow \rho^3 = 8 \rightarrow \rho = 2$

  $3\alpha = \pi + 2k\pi \rightarrow \alpha = \frac{\pi}{3} + \frac{2}{3}k\pi$

  ↳ $k=0$, $w_0 = 2e^{i\pi/3} = 2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}) = 1 + \sqrt{3}\,i$ → check

  $k=1$, $w_1 = 2e^{i(\frac{\pi}{3} + \frac{2\pi}{3})} = 2e^{i\pi} = -2$ → check

  $k=2$, $w_2 = 2e^{i(\frac{\pi}{3} + \frac{4\pi}{3})} = 2e^{i\frac{5\pi}{3}} = 1 - \sqrt{3}\,i$ → check

☆ If finding n roots, must consider $w=0$ to $w=n-1$

Figure 1: How to Find Roots of a Complex Number