3.5 Variation of Parameters

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This method is named after the Italian astronomer and mathematician Joseph-Louis Lagrange.

1 Some Assumptions

We start with the linear second-order DE:

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x) \tag{1}$$

and put it in standard form:

$$y'' + P(x)y' + Q(x)y = f(x) \tag{2}$$

2 Method of Variation of Parameters

We seek a solution of the form:

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) \tag{3}$$

where y_1 and y_2 form a fundamental set of solutions on I. We then differentiate y_p twice to get:

$$\begin{aligned} y_p' &= u_1y_1' + y_1u_1' + u_2y_2' + y_2u_2' \\ y_p'' &= u_1y_1'' + y_1'u_1' + y_1u_1'' + u_1'y_1' + u_2y_2'' + y_2'u_2' + y_2u_2'' + u_2'y_2' \end{aligned}$$

Substituting (3) and its derivatives into (2) and grouping terms together yields:

$$y_p'' + P(x)y_p' + Q(x)y_p = \frac{d}{dx}[y_1u_1' + y_2u_2'] + P[y_1u_1' + y_2u_2'] + y_1'u_1' + y_2'u_2' = f(x) \tag{4}$$

By Cramer's Rue, we can solve the system:

$$\begin{aligned} y_1u_1' + y_2u_2' &= 0 \\ y_1'u_1' + y_2'u_2' &= f(x) \end{aligned}$$

can be expressed in terms of determinants

$$u_1' = \frac{W_1}{W} = -\frac{y_2f(x)}{W} \text{ and } u_2' = \frac{W_2}{W} = \frac{y_1f(x)}{W} \tag{5}$$

where:

$$W = \det\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}, W_1 = \det\begin{bmatrix} 0 & y_2 \\ f(x) & y_2' \end{bmatrix}, W_2 = \det\begin{bmatrix} y_1 & 0 \\ y_1' & f(x) \end{bmatrix} \tag{6}$$

3 Constants of Integration

When computing the infinite integrals of u_1' and u_2', we don't need to introduce any constants. This is because:

$$\begin{aligned} y = y_c + y_p &= c_1y_1 + c_2y_2 + (u_1 + a_1)y_1 + (u_2 + b_1)y_2 \\ &= (c_1 + a_1)y_1 + (c_2 + b_1)y_2 + u_1y_1 + u_2y_2 \\ &= C_1y_1 + C_2y_2 + u_1y_1 + u_2y_2 \end{aligned}$$

4 Integral-Defined Functions

Going back to (3), we can use the following to solve linear second-order DE:

$$u_1(x) = -\int_{x_0}^{x} \frac{y_2(t)f(t)}{W(t)}dt \text{ and } u_2(x) = \int_{x_0}^{x} \frac{y_1(t)f(t)}{W(t)}dt$$

5 Higher-Order Equations

Non-homogeneous second order equations can be put into the form:

$$y^{(n)} + P_{n-1}(x)y^{(n-1)} + ... + P_1(x)y' + P_0(x)y = f(x) \tag{7}$$

If $y_c = c_1y_1 + ... + c_ny_n$ is the complementary function, then a particular solution is:

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) + ... + u_n(x)y_n(x)$$

Cramer's Rule gives us:

$$u_k' = \frac{W_k}{W}, k = 1, 2, ...n$$

$$u_1' = \frac{W_1}{W}, \quad u_2' = \frac{W_2}{W}, \quad u_3' = \frac{W_3}{W}, \tag{10}$$

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}, \quad W_1 = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ f(x) & y_2'' & y_3'' \end{vmatrix}, \quad W_2 = \begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & f(x) & y_3'' \end{vmatrix}, \text{ and } W_3 = \begin{vmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & f(x) \end{vmatrix}.$$

Figure 1: How to Determine u_1, u_2, and u_3