18.1 Contour Integrals

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1 A Definition

A piecewise smooth curve C is called a contour or path. An integral of $f(z)$ on C is denoted by $\int_C f(z)$ or $\oint_C f(z)$ if the contour is closed.

1.0.1 Definition 18.1.1 – Contour Integral

The contour integral of f along C is:

$$\int_C f(z)dz = \lim_{||P||\to 0} \sum_{k=1}^{n} f(z)_k^* \Delta z_k \tag{1}$$

2 Method of Evaluation

$$\int_C f(z)dz = \int_C udx - vdy + i\int_C vdx + udx \tag{2}$$

In other words, we see that the contour integrals is a combination of two real line integrals. Thus, we arrive at the following theorem:

2.0.1 Theorem 18.1.1 – Evaluation of a Contour Integral

If f is continuous on a smooth curve C given by $z = x + it$, then

$$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt \tag{3}$$

3 Circulation and Net Flux

When we interpret the complex function $f(z) = u(x, y) + iv(x, y)$ as a vector, we get the following:

$$\oint_C f \cdot \mathbf{T}ds = \oint_C udx + vdy \tag{4}$$

$$\oint_C f \cdot \mathbf{n}ds = \oint_C udy - idx \tag{5}$$

The integral in (4) is called the **circulation** around C, which measures the tendency of the flow to rotate the curve C. The integral in (5) is called the **net flux** across C, which measures the presence of sources or sinks for the fluid inside C. We see that:

$$\left(\oint_C f \cdot \mathbf{T}ds\right) + i\left(\oint_C f \cdot \mathbf{n}ds\right) = \oint_C (u - iv)(dx + idy) = \oint f(\bar{z})dz \tag{6}$$