3.1 Theory of Linear Equations

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1 Initial-Value and Boundary-Value Problems

1.1 Initial-Value Problem

Review from Section 1.2

1.2 Existence and Uniqueness

Review from Section 1.2

1.3 Boundary-Value Problem

Another type of problem which entails solving a linear differential equation or order 2 or more, where the dependent variable y or its derivatives are specified at different points.

$$\textbf{Solve: } a_2(x)\frac{d^y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$
$$\textbf{Subject to: } y(a) = y_0, y(b) = y_1$$

This sort of problem is called a boundary-value problem because the prescribed values $y(a) = y_0$ and $y(b) = y_1$ are called boundary conditions.

We may define general boundary conditions as:

$$A_1y(a) + B_1y'(a) = C_1$$
$$A_2y(a) + B_2y'(b) = C_2$$

2 Homogeneous Equations

$$a_n(x)\frac{d^n}{dx^n} + a_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + ... + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

The above DE is homogeneous.

$$a_n(x)\frac{d^n}{dx^n} + a_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + ... + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

The above DE is non-homogeneous. In order to solve the above non-homogeneous DE, we will first need to solve the associated heterogeneous equation.

2.1 Differential Operators

Differentiation is denoted by the capital letter D, such that $dy/dx = Dy$. Higher-order derivatives can be expressed in a similar manner. Remember from Calculus II that differentiation matches the criteria for linear transformations.

2.2 Superposition Principle

Theorem 3.1.2 – Superposition Principle for Homogeneous Equations

The sum, or **superposition** of two or more solutions to a homogeneous DE is also a solution.

2.3 Linear Dependence and Linear Independence

We carry over the same definitions of linear dependence and Independence from linear algebra.

2.4 Solutions of Differential Equations

Suppose each of the functions $f_1(x), f_2(x), ..., f_n(x)$ possesses at least $n-1$ derivatives. The determinant of the matrix

$$\begin{bmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{bmatrix}$$

is called the Wronskian of the function.

Theorem 3.1.3 – Criterion for Linearly Independent Solutions

Let $y_1, y_2, .., y_n$ be n solutions of the homogeneous linear nth-order DE on an interval I. Then the set of solutions is linearly independent on I iff $W(y_1, y_2, ..., y_n) \neq 0$ for every x in the interval.

Definition 3.1.3 – Fundamental Set of Solutions

Any set of $y_1, y_2, ..., y_n$ of n linearly independent solutions of the homogeneous linear nth-order DE on an interval I is said to be a fundamental set of solutions on that interval.

Theorem 3.1.4 – Existence of a Fundamental Set

There exists a fundamental set of solutions for the homogeneous linear nth-order DE on an interval I.

Theorem 3.1.5 – General Solutions for Homogeneous Equations

The general solution of the equation on the interval I is given by:

$$y = c_1 y_1(x) + c_2 y_2(x) + ... + c_n y_n(x)$$

3 Nonhomogeneous Equations

General Solution for Nonhomogeneous Equations

The general solution of the equation on the interval I is:

$$y = c_1 y_1(x) + c_2 y_2(x) + ... + c_n y_n(x) + y_p(x)$$

The linear combination part of the solution is called the complementary function. This gives us:

$$y = y_c + y_p$$