3.8 Linear Models: Initial-Value Problems

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We will consider some linear dynamical systems where each model is a linear second-rder differential equation with constant coefficients with ICs specified at t_0:

$$a_2\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_0y = g(t), y(t_0) = y_0, y'(t_0) = y_1$$

The function g is variously called the input, driving, or forcing function of the system. The output or response of the system is a function $y(t)$ defined on an interval I containing t_0.

1 Spring/Mass Systems: Free Undamped Motion

1.1 Hooke's Law

Hooke's Law states that the spring exerts a restoring force F opposite to the direction of elongation and proportional to the amount of elongation. We generally simply state it as $F = -kx$.

1.2 Newton's Second Law

We have a condition of equilibrium $mg = ks$ or $mg - ks = 0$. Assuming no other forces are acting on the system and assuming the mass is free of other external forces (called **free motion**), we can equate Newton's Second Law with the net force of the restoring force and weight:

$$m = \frac{d^2x}{dt^2} = -k(s + x) + mg = -kx + mh - ks = -kx \tag{1}$$

1.3 DE of Free Undamped Motion

By dividing (1) by mass m we can obtain the second-order DE:

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \tag{2}$$

where $\omega^2 = k/m$. Equation (2) is said to describe **simple harmonic motion** or **free undamped motion**.

1.4 Solution and Equation of Motion

The solutions of the auxiliary equation $m^2 + \omega^2 = 0$ are complex numbers $m_1 = \omega i$, $m_2 = -\omega i$. Thus we get the solution:

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t \tag{3}$$

The period of free vibration is $T = 2\pi/\omega$ and the frequency is $f = 1/T = \omega/2\pi$. The period is the time interval between two consecutive maxima or two consecutive minima. We refer to either as an **extreme displacement** of the mass. When we use initial conditions to solve for constants c_1 and c_2, we say the resulting particular solution is the **equation of motion**.

1.5 Alternative Form of $x(t)$

Note that in the following equations $A = \sqrt{c_1^2 + c_2^2}$ is the amplitude of free vibrations and ϕ is a phase angle.

$$y = A\sin(\omega t + \phi) \text{ and } y = A\cos(\omega t - \phi) \tag{4}$$

where

$$\tan\phi = \frac{c_1}{c_2} \text{ and } \tan\phi = \frac{c_2}{c_1} \tag{5}$$

To verify this, we can expand $\sin(\omega t + \phi)$ by the double angle sine formula:

$$A\sin(\omega t + \phi) = A\sin(\omega t)\cos\phi = (A\sin\phi)\cos\omega t + (A\cos\phi)\sin\omega t, \tag{6}$$

1.6 Double Spring Systems

For two parallel springs, we define the effective spring constant:

$$k_{eff} = k_1 + k_2$$

1.7 Systems with Variable Spring Constants

In a model for an **aging spring**, the spring constant k is replaced with $K(t) = ke^{-\alpha t}, k > 0, a > 0$. The linear differential equation:

$$mx'' + ke^{-\alpha t}x = 0$$

cannot be solved by the methods considered in this chapter.

2 Spring/Mass Systems: Free Damped Motion

2.1 DE of Free Damped Motion

When no other external forces are impressed on the system, we have:

$$m\frac{d^2x}{dt^2} = -kx - \beta\frac{dx}{dt} \tag{7}$$

By putting (10) in standard form, we find that the DE of free damped motion is:

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0 \tag{8}$$

where

$$2\lambda = \frac{\beta}{m}, \omega^2 = \frac{k}{m} \tag{9}$$

2.1.1 Case I: $\lambda^2 - \omega^2 > 0$

This system is said to be overdamped because β is large compared to k.

$$x(t) = e^{-\lambda t}(c_1 e^{\sqrt{\lambda^2-\omega^2}t} + c_2 e^{-\sqrt{\lambda^2-\omega^2}t} \tag{10}$$

2.1.2 Case II: $\lambda^2 - \omega^2 = 0$

The system is said to be critically damped because any slight decrease in damping force would result in oscillatory motion.

$$x(t) = e^{-\lambda t}(c_1 + c_2 t) \tag{11}$$

2.1.3 Case III: $\lambda^2 - \omega^2 < 0$

In this case the damping coefficient is small compared to the spring constant, thus we call it underdamped.

$$x(t) = e^{-\lambda t}(c_1 \cos \sqrt{\omega^2 - \lambda^2} t) + c_2 \sin \sqrt{\omega^2 - \lambda^2} t) \tag{12}$$

2.2 Alternative Form of $x(t)$

$$x(t) = Ae^{-\lambda t} \sin(\sqrt{\omega^2 - \lambda^2} t + \phi) \tag{13}$$

where $A = \sqrt{c_1^2 + c_2^2}$ and $\tan \phi = \frac{c_1}{c_2}$.

The coefficient $Ae^{-\lambda t}$ is called the **damped amplitude**, and $2\pi / \sqrt{\omega^2 - \lambda^2}$ is called the **quasi period**.

3 Spring/Mass Systems: Driven Motion

3.1 DE of Driven Motion with Damping

The inclusion of $f(t)$ in the formulation of Newton's Second Law gives the differential equation of **driven** or **forced motion**:

$$m \frac{d^2 x}{dt^2} = -kx - \beta \frac{dx}{dt} + f(t) \tag{14}$$

which when put into standard form:

$$\frac{d^2 x}{dt^2} + 2\lambda \frac{dx}{dt} + \omega^2 x = F(t) \tag{15}$$

3.2 Transient and Steady-State Terms

The general solution is the sum of a nonperiodic function $x_c(t)$ and a periodic function $x_p(t)$. Moreover, $x_c(t)$ goes to zero as time goes on, thus, the mass displacement is approximated by $x_p(t)$. The complementary function is called a transient solution and the particular function is called a steady-state solution.

3.3 Pure Resonance

$$\begin{aligned} x(t) &= \lim_{\gamma \to \omega} F_0 \frac{-\gamma \sin \omega t + \omega \sin \gamma t}{\omega(\omega^2 - \gamma^2)} \\ &= F_0 \lim_{\gamma \to \omega} \frac{-\sin \omega t + \omega t \cos \gamma t}{-2\omega\gamma} \\ &= \frac{F_0}{2\omega^2} \sin \omega t - \frac{F_0}{2\omega} t \cos \omega t \end{aligned}$$

4 Series Circuit Analogue

$$m \frac{d^2 x}{dt^2} + \beta \frac{dx}{dt} + kx = f(t) \tag{16}$$

$$L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{1}{C} q = E(t) \tag{17}$$