4.2 Inverse Transforms and Transforms of Derivatives

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1 Inverse Transforms

1.1 Linearity of the Inverse Laplace Transformation

$$\mathcal{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha\mathcal{L}^{-1}\{F(s)\} + \beta\mathcal{L}^{-1}\{G(s)\} \tag{1}$$

1.2 Transform of a Derivative

$$\begin{aligned}
\mathcal{L}\{f'(t)\} &= \int_0^{t_0} e^{-st} f'(t)dt + \int_{t_0}^{\infty} e^{-st} f'(t)dt \\
&= \left[e^{-st} f(t)\Big|_0^{t_0} + s\int_0^{t_0} e^{-st} f(t)dt \right] + \left[e^{-st} f(t)\Big|_{t_0}^{\infty} + s\int_{t_0}^{\infty} e^{-st} f(t)dt \right] \\
&= -f(0) + \lim_{t\to\infty} e^{-st} f(t) + s\left[\int_0^{t_0} e^{-st} f(t)dt + \int_{t_0}^{\infty} e^{-st} f(t)dt \right] \\
&= -f(0) + \lim_{t\to\infty} e^{-st} f(t) + s\int_0^{\infty} e^{-st} f(t)dt
\end{aligned}$$

We may condense this into:

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0) \tag{2}$$

$$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0) \tag{3}$$

$$\mathcal{L}\{f'''(t)\} = s^3F(s) - s^2f(0) - sf'(0) - f''(0) \tag{4}$$

2 Solving Linear ODEs

$$a_n\mathcal{L}\{\frac{d^ny}{dt^n}\} + a_{n-1}\mathcal{L}\{\frac{d^{n-1}y}{dt^{n-1}}\} + ... + a_0\mathcal{L}\{y\} = \mathcal{L}\{g(t)\} \tag{5}$$

From (5) we get:

$$a_n[s^nY(s) - s^{n-1}y(0) - ... - y^{(n-1)}(0)] + ... + a_0Y(s) = G(s) \tag{6}$$

Essentially this states that the Laplace transform of a linear differential equation with constant coefficients becomes an algebraic equation in $Y(s)$. If we solve the transformed function (6) for $Y(s)$, we obtain $P(s)Y(s) = Q(s) + G(s)$, then:

$$Y(s) = \frac{Q(s) + G(s)}{P(s)} \tag{7}$$

2.0.1 Theorem 4.2.3 – Behaviour of $F(s)$ as $s \to \infty$

If a function f is piecewise continuous on $[0, \infty)$ and of exponential order with c as specified in Definition 4.1.2 and $\mathcal{L}\{f(t)\} = F(s)$, then $\lim_{s\to\infty} F(s) = 0$