3.2 Reduction of Order

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In Chapter 3.1 we saw that the general solution of a homogeneous linear second-order differential equation:

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \tag{1}$$

was a linear combination $y = c_1y_1 + c_2y_2$, where y_1 and y_2 are solutions that constitute a linearly independent set on some interval I. We will learn a method to find solutions in the next section. It turns out we can construct a second solution y_2 od a homogeneous equation (1) provided we know one nontrivial solution y_1 of the DE.

1 Reduction of Order

Suppose $y(x)$ denotes a known solution of equation (1). We seek a second solution $y_2(x)$ of (1) that is linearly independent to $y_1(x)$. The idea is to find $u(x)$ where $y_2(x) = u(x)y_1(x)$, this method is called reduction of order, since we can reduce a second-order ODE into a first-order ODE.

Let us explore this using an example:

1.0.1 Example: Finding a Second Solution

Given that $y_1 = e^x$ is a solution of $y'' - y = 0$ on the interval $(-\infty, \infty)$ use reduction of order to find a second solution y_2.

If $y = u(x)y_1(x) = u(x)e^x$, then the first two derivatives of y are obtained from the product rule:

$$y' = ue^x + e^xu', y'' = ue^x + 2e^xu' + e^xu''$$

By substituting y and y'' into the original DE, we can simplify to:

$$y'' - y = e^x(u'' + 2u') = 0$$

Since $e^x \neq 0$, the last equation requires $u'' + 2u' = 0$. If we make the substitution $w = u'$, this linear second-order equation in u becomes $w' + 2w = 0$, which is a linear first-order equation in w. Using the integrating factor e^{2x}, we can write $d/dx[e^{2x}w] = 0$. After integrating we have $w = c_1e^{-2x}$, which becomes $u' = c_1e^{-2x}$. Integrating again gives us: $u = -\frac{1}{2}c_1e^{-2x} + c_2$. Thus, we have:

$$y = u(x)e^x = -\frac{c_1}{2}e^{-x} + c_2e^x \tag{2}$$

By choosing $c_2 = 0$ and $c_1 = -2$, we obtain the desired $y_2 = e^{-x}$.

2 General Case

Suppose we divide by $a_2(x)$ in order to put equation (1) in the standard form:

$$y'' + P9x)y' + Q(x)y = 0 \tag{3}$$

where $P(x)$ and $Q(x)$ are continuous on some interval I. Suppose further that $y_1(x)$ is a solution of (3) on I and that $y_1(x) \neq 0$ for every x in the interval. If we define $y = u(x)y_1(x)$ it follows that:

$$\begin{aligned}
&y' = uy_1' + y_1u', \\
&y'' = uy_1'' + 2y_1'u' + y_1u'' \\
&y'' + Py' + Qy = u[y_1'' + Py_1' + Qy_1] + y_1u'' + (2y_1' + Py_1)u' = 0
\end{aligned}$$

Implying that we have:

$$y_1u'' + (2y_1' + Py_1)u' = 0 \text{ or } y_1w' + (2y_1' + Py_1)w = 0 \tag{4}$$

if have $w = u'$. Observe that the last equation in (4) is both linear and separable. We can then separate variables and integrate:

$$\frac{dw}{w} + 2\frac{y_1'}{y_1}dx + Pdx = 0$$

$$\ln|wy_1^2| = -\int Pdx + c \text{ or } wy_1^2 = c_1e^{-\int Pdx}$$

We can solve the last equation for w using $w = u'$ and integrating again:

$$u = c_1 \int \frac{e^{-\int Pdx}}{y_1^2}dx + c_2$$

Theorem 3.2.1 Reduction of Order

Let $y_1(x)$ be a solution of the homogeneous differential equation (3) on an interval I and that $y_1(x) \neq 0$ for all x in I. Then:

$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1^2} \tag{5}$$

is a second solution of (3) on the interval I.

To prove linear independence of the functions, we can use **Theorem 3.1.3 The Wronskian** of the function is:

$$W(y_1, y_2) = det \begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix} = e^{-\int P(x)dx \neq 0}$$