17.7 Trigonometric and Hyperbolic Functions

Arnav Patil

University of Toronto

1 Trigonometric Functions

Using Euler's formula, we see that the real functions of sine of cosine can be represented as a combination of exponential functions.

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

2 Derivatives

Given:

$$\frac{d}{dz} e^z = e^z$$

we can use the Chain Rule to see that:

$$\frac{d}{dz} \sin z = \frac{d}{dz} \frac{e^{iz} - e^{-iz}}{2i} = \frac{e^{iz} + e^{-iz}}{2} = \cos z$$

We can extend this to the other trigonometric functions and see that all of their derivatives are the same as their real counterparts. Trigonometric identities are also the same in complex calculus.

3 Zeroes

To find zeroes of $\sin z$ and $\cos z$, we need to express both functions in the form $u + vi$. But first, recall that if y is real, then the hyperbolic sine and cosine functions are defined as:

$$\sinh y = \frac{e^y - e^{-y}}{2} \text{ and } \cosh y = \frac{e^y + e^{-y}}{2}$$

Fast forward to the definitions of $\sin z$ and $\cos z$:

$$\sin z = \sin x \cosh y + i \cos x + \sinh y$$

$$\cos z = \cos x \cosh y - i \sin x \sinh y$$