17.6 Exponential and Logarithmic Functions

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We will now examine exponential, logarithmic, trigonometric, and hyperbolic functions of complex numbers.

1 Exponential Functions

From Calculus II we already have defined a complex exponential function.

$$e^z = e^{x+yi} = e^x(\cos y + i \sin y)$$

Note that when $y = 0$, the function reduces to e^x.

2 Properties

We will begin by finding the derivative of e^z:

$$f'(z) = e^x \cos y + i(e^x \sin y) = f(z)$$

Therefore, we have that $\frac{d}{dz}e^z = e^z$.

3 Periodicity

The complex function $f(z) = e^z$ is periodic with period $2\pi i$. The strip $-\pi < y \leq \pi$ is called the **fundamental region** of the exponential function.

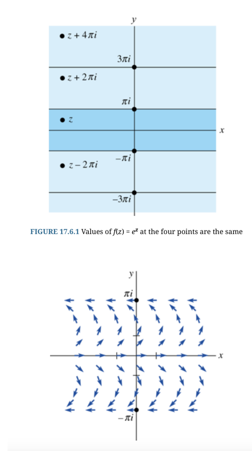

Figure 1: Flow over the fundamental region

4 Polar Form of a Complex Number

As we saw in section 17.2, we can write the polar form of a complex number as:

$$z = re^{i\theta}$$

5 Logarithmic Functions

The logarithm of a nonzero complex number is defined as the inverse of the exponential function, so $w = \ln z$.

Logarithm of a Complex Number

$$\ln z = \log_e |z| + i(\theta + 2k\pi)$$

6 Principal Value

In real calculus, $\log_e 5 = 1.6094$ has only one value, but in complex calculus, $\log_e 5 = 1.6094 + 2k\pi i$. The value of $\ln 5$ corresponding to $k = 0$ is called the **principal value** of $\ln 5$.

$$\ln(z_1 z_2) = \ln z_1 + \ln z_2 \text{ and } \ln\left(\frac{z_1}{z_2}\right) = \ln z_1 - \ln z_2$$

7 Analyticity

$$\frac{d}{dz} \ln z = \frac{1}{z}$$