# 19.6 Evaluation of Real Integrals

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In this section we will see how residue theory can help us real integrals of the forms:

$$\int_0^{2\pi} F(\cos\theta, \sin\theta)d\theta \tag{1}$$

$$\int_{-\infty}^{\infty} f(x)dx \tag{2}$$

$$\int_{-\infty}^{\infty} f(x)\cos\alpha x dx \text{ and } \int_{-\infty}^{\infty} f(x)\sin\alpha x dx \tag{3}$$

## 1 Integrals of the Form $\int_0^{2\pi} F(\cos\theta, \sin\theta)d\theta$

Convert an integral form into a complex integral where the contour $C$ is the unit circle centered at the origin.

$$dz = ie^{i\theta}d\theta, \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}, \cos\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} \tag{4}$$

we replace, $d\theta, \cos\theta, \sin\theta$ with

$$d\theta = \frac{dz}{iz}, \cos\theta = \frac{1}{2}(z+z^{-1}), \sin\theta = \frac{1}{2i}(z-z^{-1}) \tag{5}$$

The integral in question then becomes

$$\oint_C F\left(\frac{1}{2}(z+z^{-1}), \frac{1}{2i}(z-z^{-1})\right)\frac{dz}{iz} \tag{6}$$

## 2 Integrals of the Form $\int_{-\infty}^{\infty} f(x)dx$

When $f$ is continuous throughout the domain, recall from calculus that the improper integral is defined in terms of two limits.

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{r\to\infty}\int_{-r}^{0} f(x)dx + \lim_{R\to\infty}\int_0^R f(x)dx \tag{7}$$

If both limits exist, the integral is said to be convergent, otherwise, the integral is divergent. If the limit is convergent, we can use a single integral:

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)dx \tag{8}$$

The limit is called the Cauchy Principal Value of the integral and is written as

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)dx \tag{9}$$

To evaluate an integral $\int_{-\infty}^{\infty} f(x)dx$, where $f(x) = P(x)/Q(x)$ is continuous on $(-\infty, \infty)$, by residue theory we replace $x$ by the complex value $z$ and integrate the complex function $f$ over a closed contour $C$, that consists of the interval $[-R, R]$ on the real axis and a semicircle $C_R$ of radius large enough to enclose all the poles of $f(x) = P(z)/Q(z)$ in the upper half-plane $\Re(z) > 0$.

$$\oint_C f(z)dz = \int_{C_R} f(z)dz + \int_{-R}^{R} f(x)dx = 2\pi i \sum_{k=1}^{n} \text{Res}(f(z), z_k) \tag{10}$$

#### 2.0.1 Theorem 19.6.1 – Behaviour of Integral as $R \to \infty$

Suppose $f(z) = P(z)/Q(z)$ where the degree of $P(z)$ is $n$ and the degree of $Q(z)$ is $m \geq n + 2$. If $C_R$ is a semicircular contour $z = \Re e^{ie}$, $0 \leq \theta \leq \pi$ then $\int_{C_R} f(z)dz \to 0$ as $R \to \infty$.

## 3 Integrals of the Forms $\int_{-\infty}^{\infty} f(x) \cos \alpha x dx$ and $\int_{-\infty}^{\infty} f(x) \sin \alpha x dx$

When $\alpha > 0$, these integrals are referred to as **Fourier integrals**. Using Euler's formula $e^{i\alpha x} = \cos \alpha x + i \sin \alpha x$, we get:

$$\int_{-\infty}^{\infty} f(x)e^{i\alpha x}dx = \int_{-\infty}^{\infty} f(x) \cos \alpha x dx + i \int_{-\infty}^{\infty} f(x) \sin \alpha x dx \tag{11}$$

Before continuing, we give without proof sufficient conditions under which the contour integral along $C_R$ approaches zero as $R \to \infty$.

## 4 Indented Contours

The improper integrals that we have considered are continuous throughout their domain. If $f$ has poles on the real axis, we use an indented contour. The symbol $C_r$ denotes a semicircle contour centered at $z = c$ oriented in the positive direction. The next theorem is important to this discussion.

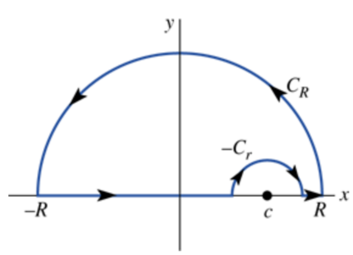


Figure 1: Indented Contour

#### 4.0.1 Theorem 19.6.3 – Behaviour of Integral as $r \to 0$

Suppose $f$ has a simple pole $z = c$ on the real axis. If $C_r$ is the contour defined $z = c + re^{i\theta}$, then

$$\lim_{r \to 0} \int_{C_r} f(z)dz = \pi i \, \text{Res}(f(z), c) \tag{12}$$