2.3 Linear Equations

Arnav Patil

University of Toronto

1 A Definition

A first-order DE of the form:

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

is said to be a linear equation in the variable y. If $g(x) = 0$, then the DE is also said to be homogeneous.

2 Standard Form

We can divide both sides of the above equation by $a_1(x)$ to obtain a more useful form of the DE:

$$\frac{dy}{dx} + P(x)y = f(x)$$

3 The Property

The DE given above has the property that its solution is the sum of the two solutions $y = y_c + y_p$, where y_c is a solution of the associated homogeneous equation:

$$\frac{dy}{dx} + P(x)y = 0$$

$$\frac{d}{dx}[y_c + y_p] + P(x)[y_c + y_p] = \left[\frac{dy_c}{dx} + P(x)y_c\right] + \left[\frac{dy_p}{dx} + P(x)y_p\right] = f(x)$$

4 Method of the Integrating Factor

Check Lecture 2-2 notes to see this method of solving Linear First-Order ODEs.

1. Put a linear first-order equation into standard form then determine $P(x)$ and the integrating factor $e^{\int P(x)dx}$.

2. Multiply the standard form by the integrating factor. The left side is the derivative of the product of the integrating factor and y. Write:

$$\frac{d}{dx}[e^{\int P(x)dx}y] = e^{\int P(x)dx}f(x)$$

   then integrate both sides of the equation.

4.1 Singular Points

When we divide by $a_1(x)$ to get the standard form, we need to be careful about what happens for values of x where $a_1(x) = 0$. The discontinuity may carry over to functions in the general solution of the differential equation.

5 Error Function

$$\frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \int_0^{x} e^{-t^2} dt + \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt = 1$$

where the first part is $erf(x)$ and the second part is $erfc(x)$. We also see that:

$$erf(x) + erfc(x) = 1$$