18.4 Cauchy's Integral Formulae

Arnav Patil

University of Toronto

The most significant result of the Cauchy-Goursat Theorem is that the value of an analytic function f at any point z_0 in a simply connected domain can be represented by a contour integral. We will further show that an analytic function f in a simply connected domain possesses derivatives of all orders.

1 First Formula

1.0.1 Theorem 18.4.1 – Cauchy's Integral Formula

Let f be analytic in a simply connected domain D, and let C be a simple closed contour lying entirely within D. If z_0 is any point within C, then

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}dz \tag{1}$$

2 Second Formula

2.0.1 Theorem 18.4.2 – Cauchy's Integral Formula for Derivatives

Let f be analytic in a simply connected domain D, and let C be a simple closed contour lying entirely within D. If z_0 is any point interior to C, then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz \tag{2}$$

3 Liouville's Theorem

If we take the contour $C$to be the circle $|z-z+0|=r$, it follows from (2) that:

$$|f^{(n)}(z_0)| = \frac{n!}{2\pi}\left|\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz\right| \leq \frac{n!}{2\pi}M\frac{1}{r^{n+1}}2\pi r = \frac{n!M}{r^n} \tag{3}$$

This result is called Cauchy's inequality, and we can use it to prove the following theorem.

3.0.1 Theorem 18.4.3 – Liouville's Theorem

The only bounded entire functions are constants.

4 Fundamental Theorem of Algebra

If $P(z)$ is a nonconstant polynomial, then the equation $P(z)=0$ has at least one root.