18.2 Cauchy-Goursat Theorem

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In this chapter we concentrate on domains where the contour C is a simple closed curve with a positive (CCW) curve. We shall see that when f is analytic in a special kind of domain D, the value of the contour integral $\oint_C f(z)dz$ is the same for ANY simple closed curve that lies entirely within D.

1 Simply and Multiply Connected Domains

A domain D is said to be simply connected if every simple closed contour C lying entirely within D can be shrunk to a point without leaving D.

A domain that is not simply connected is called a multiply connected domain, that is, a multiply connected domain has 'holes' in it. A domain with two holes is called triply connected, and the pattern continues.

2 Cauchy's Theorem

Cauchy's Theorem says: "Suppose a function f is analytic in a simply connected domain D and that f' is continuous in D. Then, for every simple closed contour C in D, we have $\oint_C f(z)dz = 0$."

$$\oint_C f(z)dz = \int\int_D \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) + i \int\int_D \left(\frac{\partial v}{\partial x} - \frac{\partial v}{\partial y} \right) dA \quad (1)$$

In 1883, French mathematician proved Cauchy's Theorem without the need for assuming f' is continuous. Thus, we have a modified version of the theorem called the Cauchy-Goursat Theorem.

2.0.1 Theorem 18.2.1 – Cauchy-Goursat Theorem

Suppose a function of f is analytic in a simply connected domain D. Then for every simple closed contour C in D, we have $\oint_C f(z)dz = 0$.

3 Cauchy-Gourmat Theorem for Multiply Connected Domains

3.0.1 Theorem 18.2.2 – Cauchy-Gourmat Theorem for Multiply Connected Domains

If f is analytic on each contour and at each point interior to C but exterior to all the C_k, $k = 1, 2, 3, ..., n$, then:

$$\oint_C f(z)dz = \sum_{k=1}^{n} \oint_{C_k} f(z)dz \quad (2)$$