19.1 Sequences and Series

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1 Sequences

A sequence is a function whose domain is the set of positive integers. If $\lim_{n\to\infty} z_n = L$, we say the sequence is convergent.

1.0.1 Theorem 19.1.1 – Criterion for Convergence

A sequence $\{z_n\}$ converges to a complex number L iff $Re(z_n)$ converges to $Re(L)$ and $Im(z_n)$ converges to $Im(L)$.

2 Series

An infinite series of complex numbers

$$\sum_{k=1}^{\infty} z_k = z_1 + z_2 + z_3 + ... + z_n + ...$$

is convergent if the sequence of partial sums $\{S_n\}$ where

$$S_n = z_1 + z_2 + ... + z_n$$

converges. If $S_n \to L$ as $n \to \infty$, we say the sum of the series is L.

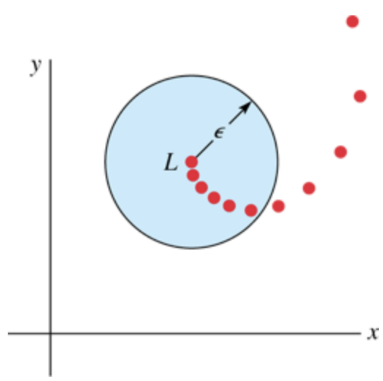

Figure 1: If $\{z_n\}$ converges to L, all but a finite number of terms are in any ϵ-neighbourhood of L

3 Geometric Series

For the geometric series

$$\sum_{k=1}^{\infty} ak^{k-1} = a + az + az^2 + ... + az^{n-1} + ...$$

the nth term of the sequence of partial sums is

$$S_n = a + az + az^2 + ... + az^{n-1}$$

Solving for S_n gives

$$S_n = \frac{a(1-z^n)}{1-z} \tag{1}$$

Since $z^n \to 0$ as $n \to \infty$, whenever $|z| < 1$, we conclude that the series converges to

$$\frac{a}{1-z} \tag{2}$$

3.0.1 Theorem 19.1.2 – Necessary Conditions for Convergence

If $\sum_{k=1}^{\infty} z_k$ converges, then $lim_{n\to\infty} z_n = 0$.

3.0.2 Theorem 19.1.3 – The n-th Term Test for Divergence

If $\lim_{n\to\infty} z_n = \neq 0$, then the series diverges.

3.0.3 Theorem 19.1.1 – Absolute Convergence

An infinite series is said to be absolutely convergent if the equivalent series with the sum term with absolute value brackets also converges.

3.0.4 Theorem 19.1.4 – Ratio Test

Suppose we have an infinite series such that

$$\lim_{n\to\infty} \left| \frac{z_{n+1}}{z_n} \right| = L \tag{3}$$

Then:

1. If $L < 1$ the series converges absolutely.
2. If $L > 1$ or $L = \infty$, then the series diverges.
3. If $L = 1$ the test is inconclusive

3.0.5 Theorem 19.1.5 – Root Test

Suppose we have an infinite series such that

$$\lim_{n\to\infty} = \sqrt[n]{|z_n|} = L \tag{4}$$

Then:

1. If $L < 1$ the series converges absolutely.
2. If $L > 1$ or $L = \infty$, then the series diverges.
3. If $L = 1$ the test is inconclusive

4 Power Series

An infinite series of the form

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + ... \tag{5}$$

where coefficients a_k are complex coefficients is called a power series. The power series is said to be centered at z_0.

5 Circle of Convergence

Every complex power series has a radius of convergence R. It also has a circle of convergence defined by $|z - z_0| = R$. The power series converges absolutely when $|z - z_0| < R$ and diverges for $|z - z_0| > R$. The radius R can be:

1. 0,
2. a finite number, or
3. infinity.