2.1 Solution Curves Without a Solution

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We start with two ways of analyzing a differential equation to get an approximate sense of that the solution curve looks like, without having to solve the equation.

1 Direction Fields

1.1 Slope

The slope of a tangent line at $(x, y(x))$ is the value of the first derivative evaluated at that point. If we take (x, y) to be any point in a region o the xy-plane where f is defined, then the value assigned to it by the function represents the slope of a line called the **lineal element**.

For example, consider the equation $y' = 0.2xy$, where $f(x, y) = 0.2xy$. At the point $(2, 3)$ the slope of a lineal element is $f(2, 3) = 0.2(2)(3) = 1.2$. Therefore, if a solution curve passes through the point, it does so tangent to the line segment – in other words, we can say the lineal element is a miniature tangent line.

1.2 Direction Field

If we draw a lineal element for f at each point (x, y) over a rectangular region, then we call this collection of lineal points a **direction or slope field**. The direction field suggests the shape of a family of curves, and it may be possible to notice certain points where the solutions act weird.

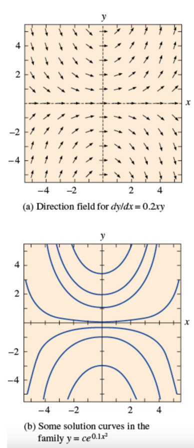

Figure 1: Examples of Direction Fields

1.3 Increasing/Decreasing

If $y' > 0$ for all x in an interval I, then we know that the differentiable function $y = y(x)$ is increasing on I, and vice versa for $y' < 0$.

2 Autonomous First-Order DEs

2.1 DEs Free of the Independent Variable

An ordinary differential equation in which the independent variable does not appear explicitly is said to be autonomous. It is expressed normally as:

$$\frac{dy}{dx} = f(y)$$

2.2 Critical Points

The zeroes of the function $f(x)$ above are of special importance. We say a real number c is a critical point if it's a zero of f. If we plug in $y(x) = c$ into the above equation, then both sides equal zero.

If c is a critical point of $f(x)$, then $y(x) = c$ is a constant solution of the autonomous differential equation.

2.3 Solution Curves

Suppose a function has two critical points at c_1 and c_2, then the graphs of $y = c_1$ and $y = c_2$ are horizontal lines. These lines partition the region R into three subregions.

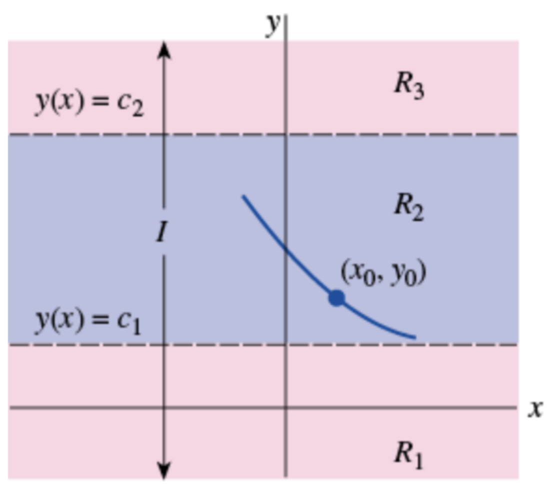

Figure 2: Subregions given by equilibrium solutions

Here are some observations we can make without proof:

- If (x_0, y_0) is in any subregion R_1, R_2, R_3 and $y(x)$ is a solution whose graph passes through this point, then $y(x)$ stays in that subregion for all x. This is because the function may not cross equilibrium solutions.
- By continuity of f we have either $f(y) > 0$ or $f(y) < 0$ for all x. In other words, $f(y)$ cannot change signs in a subregion.
- Since $dy/dx = f(y(x))$ is either positive or negative in a subregion R_i, a solution $y(x)$ is strictly monotonic. This means that $y(x)$ can neither be oscillatory nor have local extrema.

2.4 Attractors and Repellers

Suppose $y(x)$ is a nonconstant solution of an autonomous DE and that c is a critical point of the DE; there are three types of behaviour $y(x)$ can exhibit near c.

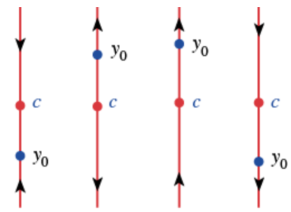

Figure 3: Stable, unstable, and two semi-stable critical points

In the diagram above, when both arrowheads on either side of c point towards c, then all solutions $y(x)$ that start from an initial point (x_0, y_0) near c will exhibit the behaviour $\lim_{x\to\infty} y(x) = c$. This critical point is said to be stable, or an attractor.

When both arrowheads face away from c, then all solutions will move away from c as x grows; this is known as an unstable critical point, or a repeller.

When both arrows point in the same direction, a solution $y(x)$ starting sufficiently near c will be attracted to c from one side and repelled from the other. These critical points are called semi-stable.

2.5 Autonomous DEs and Direction Fields

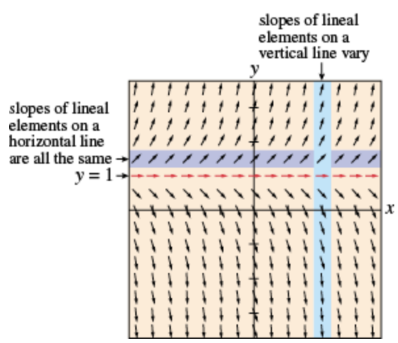

Figure 4: Direction Field for an Autonomous DE

2.6 Translation Property

If $y(x)$ is a solution of an autonomous differential equation $dy/dx = f(y)$, then $y_1(x) = y(x - k)$, where k is a constant, is also a solution. Essentially, applying a horizontal translation to $y(x)$ will have no effect on the long-term behaviour of the function.