3.3 Linear Equations with Constant Coefficients

Arnav Patil

University of Toronto

We will explore whether exponential solutions exist for homogeneous linear higher-order ODEs such as:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_1 y' + a_0 y = 0 \tag{1}$$

Where all coefficients a_i are real and $a_n \neq 0$.

1 Auxiliary Equation

We can construct an auxiliary equation, which takes us from:

$$ay'' + by' + cy = 0 \tag{2}$$

to this:

$$am^2 + bm = c = 0 \tag{3}$$

1.1 Case I: Distinct Real Roots

If we have two unequal real roots m_1 and m_2, then we have two solutions: $y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$. We see that these functions must be linearly independent over x, and form a fundamental set.

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} \tag{4}$$

1.2 Case II: Repeated Real Roots

When $m_1 = m_2$ we obtain only one exponential solution. The general solution is:

$$y = c_1 e^{m_1 x} + c_2 x e^{m_1 x} \tag{5}$$

1.3 Case III: Complex Conjugate Roots

If we have a negative determinant then the roots m_1 and m_2 are complex. We use Euler's formula, and with a little manipulation we end up with:

$$y = e^{ax}(c_1 \cos \beta x + c_2 \sin \beta x \tag{6}$$

2 Two Equations Worth Knowing

The two linear DEs

$$y'' + k^2 y = 0 \text{ and } y'' - k^2 y = 0$$

are important in applied mathematics. $y'' + k^2 y = 0$ has imaginary roots $m_1 = ki$ and $m_2 = -ki$ for its auxiliary equation, and the general solution of the DE is:

$$y = c_1 \cos kx + c_2 \sin kx \tag{7}$$

Likewise, the auxiliary equation for $y'' - k^2 y = 0$ has distinct real roots $m_1 = k$ and $m_2 = -k$, and the general solution of the DE is:

$$y = c_1 e^{kx} + c_2 e^{-kx} \tag{8}$$

Notice if we choose $c_1 = \frac{1}{2}$ and $c = \frac{1}{2}$ respectively, we end up with $y = \frac{1}{2} \cosh kx$ and $y = \frac{1}{2} \sinh kx$. An alternative form of this general solution would be:

$$y = c_1 \cosh kx + c_2 \sinh kx \tag{9}$$

3 Higher-Order Derivatives

Generally, to solve an n-th order DE

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_1 y' + a_0 y = 0 \tag{10}$$

we must solve the n-th degree polynomial equation

$$a_n m^n + a_{n-1} m^{n-1} + ... + a_1 m + a_0 = 0 \tag{11}$$