16.5 Triple Integrals in Cylindrical and Spherical Coordinates

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1 Cylindrical Coordinates

1.0.1 Transformations Between Cylindrical and Rectangular Coordinates

- Rectangular $\rightarrow$ Cylindrical
  - $r^2 = x^2 + y^2$
  - $\tan\theta = y/x$
  - $z = z$
- Cylindrical $\rightarrow$ Rectangular
  - $x = r\cos\theta$
  - $y = r\sin\theta$
  - $z = z$

2 Integration in Cylindrical Coordinates

2.0.1 Theorem 16.6 – Change of Variables for Triple Integrals in Cylindrical Coordinates

Let f be continuous over the region D, expressed in cylindrical coordinates as:

$$D = \{(r, \theta, z) : 0 \leq g(\theta) \leq r \leq h(\theta), \alpha \leq \theta\beta, G(x, y) \leq z \leq H(x, y)\}$$

Then, f is integrable over D, and the triple integral of f over D:

$$\int\int\int f(x, y, z)dV = \int_\alpha^\beta \int_g^h \int_G^H f(r\cos\theta, r\sin\theta, z)dzrdrd\theta$$

3 Spherical Coordinates

- ρ is the distance from the origin to P.
- ϕ is the angle between the positive z-axis and the line OP.
- θ is the same angle as in cylindrical coordinates; it measures rotation about the z-axis relative to the positive x-axis.

4 Integration in Spherical Coordinates

4.0.1 Theorem 16.7 – Change of Variables for Triple Integrals in Spherical Coordinates

Let f be continuous over the region D, expressed in spherical coordinates as:

$$D = \{(\rho, \phi, \theta) : 0 \leq g(\phi, \theta) \leq \rho \leq h(\phi, \theta), a \leq \phi \leq b, \alpha\theta\beta\}$$

Then, f if integrable over D, and the triple integral of f over D is:

$$\int\int\int f(x, y, z)dV = \int_{\alpha}^{\beta}\int_{a}^{b}\int_{g}^{h} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi d\rho d\phi d\theta$$