Lecture 19 – Scalar Function and Flux with Parametric Representation

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1 Surface Parameterization

First step is to map the surface to a plane, there are two ways of doing so:

1.0.1 Parameterized Representation

We can make the mapping $(x, y, z) = (x(u, v), y(u, v), f(x(u, v), y(u, v)))$ on the surface S.

1.0.2 Explicit Representation

Points of the xy plane have coordinates (x, y), so we can map the point (x, y) to $(x, y, z) = (x, y, f(x, y))$.

2 Surface Integrals of Scalar-Values Functions

We now develop the surface integral of a function f on a smooth parameterized surface S described by:

$$\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle \tag{1}$$

We then need to compute dS by finding the two special vectors tangent to the surface at P.

- $\vec{t}_u$ is a vector tangent to the surface corresponding to a change in u with v held constant, and
- $\vec{t})_v$ is a vector tangent to the surface corresponding to a change in v with u held constant.

$$\vec{t}_u = \frac{\partial \vec{r}}{\partial u} = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle \tag{2}$$

$$\vec{t}_v = \frac{\partial \vec{r}}{\partial v} = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle \tag{3}$$

We then apply the cross product to find the area of the parallelogram:

$$|\vec{t}_u \Delta u \times \vec{t}_v \Delta v| = |\vec{t}_u \times \vec{t}_v| \Delta u \Delta v = \Delta S_k = dS \tag{4}$$

3 Computing Flux

3.0.1 Definition – Flux Integral (Parameterization)

$$\iint_S \mathbf{F} \cdot \mathbf{n}_S dS = \iint_R \langle g(u, v), h(u, v), p(u, v) \rangle \cdot \mathbf{t}_u \times \mathbf{t}_v du dv \tag{5}$$