16.3 Double Integrals in Polar Coordinates

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1 Moving from Rectangular to Polar Coordinates

In polar coordinates, a polar rectangle has the form $R = \{(r, \theta) : 0 \leq a \leq r \leq b, \alpha \leq \theta \leq \beta\}$. The volume of the solid region beneath the surface $z = f(x, y)$ with a base R is approximately:

$$V = \sum_{k=1}^{n} f(x_k^*, y_k^*) \Delta A_k$$

This approximation to the volume is a Riemann sum. We let Δ be the maximum value of Δr and $\Delta\theta$. If f is continuous on R, then as $n \to \infty$ and $\Delta \to 0$, the sum approaches a double integral:

$$\int\int f(x, y) dA = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*) \Delta A_k \tag{1}$$

1.0.1 Theorem 16.3 – Change of Variables for Double Integrals over Polar Rectangle Regions

Let f be continuous on the region R in the xy plane expressed in polar coordinates $R = \{(r, \theta) : 0 \leq a \leq r \leq b, \alpha \leq \theta \leq \beta\}$, where $\beta - \alpha \leq= 2\pi$. Then f is integrable over R, and the double integral of f over R:

$$\int\int f(x, y) dA = \int_{\alpha}^{\beta} \int_{a}^{b} f(r\cos\theta, r\sin\theta) r dr d\theta$$

2 More General Polar Regions

2.0.1 Theorem 16.4 – Change of Variables for Double Integrals over More General Polar Regions

Let f be continuous on the region R in the xy-plane expressed in polar coordinate as:

$$R = \{(r, \theta) : 0 \leq g(\theta) \leq r \leq h(\theta), \alpha \leq \theta \leq \beta\}$$

where $0 < \beta - \alpha \leq 2\pi$. Then:

$$\int\int f(x, y) dA = \int_{\alpha}^{\beta} \int_{g(\theta)}^{h(\theta)} f(r\cos\theta, r\sin\theta) r dr d\theta$$

3 Areas of Regions

3.0.1 Areas of Polar Regions

The are of the polar region $R = \{(r, \theta) : 0 \leq g(\theta) \leq r \leq h(\theta), \alpha \leq \theta \leq \beta\}$, is:

$$A = \int\int dA = \int_{\alpha}^{\beta} \int_{g(\theta)}^{h(\theta)} r dr d\theta$$