15.5 Directional Derivatives and the Gradient

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1 Directional Derivatives

Let $(a, b, f(a, b))$ be a point on the surface of $z = f(a, b)$ and let **u** be a unit vector in the xy-plane. If we want to find the rate of change of f in the direction **u**at $P_0(a, b)$, we can't simply use $f_x(a, b)$ or $f_y(a, b)$ unless **u** $= \langle 1, 0 \rangle$ or **u** $= \langle 0, 1 \rangle$, but it is a combination of the above.

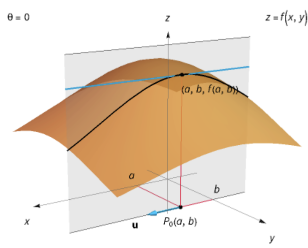

Figure 1: Sample **u** on a surface given by z

The derivative must be computed along a line l in the xy-plane that faces the same direction as **u**. Now imagine Q, the plane perpendicular to the xy-plane containing l. This plane cuts into a surface $z = f(x, y)$ in a curve C. If we consider two points P_0 and P, then we can find the slope of the secant line between these two points:

$$\frac{f(a + hu_1, b + hu_2) - f(a, b)}{h}$$

The derivative of f in the direction of **u** is obtained by letting $h \to 0$; when this limit exists, it's called the directional derivative of f at (a, b) in the direction of **u**.

Definition – Directional Derivative

Let f be differentiable at (a, b) and let $u = \langle u_1, u_2 \rangle$ be a unit vector in the xy-plane. The directional derivative of f at (a, b) in the direction of **u** is:

$$D_u f(a, b) = \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h}$$

Theorem 15.10 – Directional Derivative

Let f be differentiable at (a, b) and let **u** $= \langle u_1, u_2 \rangle$ be a unit vector in the xy-plane. The directional derivative of f at (a, b) in the direction of **u** is:

$$D_u f(a, b) = \langle f_x(a, b), f_y(a, b) \rangle \cdot \langle u_1, u_2 \rangle$$

2 The Gradient Vector

The vector $\langle f_x(a,b), f_y(a,b)\rangle$ that appears in the above dot product is important in it's own right; it is called the gradient of f.

Definition – Gradient (Two Dimensions)

Let f be differentiable at the point (x,y). The gradient of f at (x,y) is the vector-valued function:

$$\nabla f(x,y) = \langle f_x(a,b), f_y(a,b)\rangle = f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j}$$

With the definition of the gradient, we can write the directional derivative of f at (a,b) in the direction of **u** as:

$$D_u f(a,b) = \nabla f(a,b) \cdot \mathbf{u}$$

3 Interpretations of the Gradient

Using the properties of the dot product, we can see that:

$$\begin{aligned} D_u f(a,b) &= \nabla)(a,b) \cdot \mathbf{u} \\ &= |\nabla f(a,b)||\mathbf{u}| \cos\theta \\ &= |\nabla f(a,b)| \cos\theta \end{aligned}$$

Theorem 15.11 – Directions of Change

Let f be a differentiable function at (a,b) with $\nabla f(a,b) \neq 0$:

1. f has its maximum rate of increase at (a,b) in the direction of the gradient $\nabla f(a,b)$. The rate of increase in this direction is $|\nabla f(a,b)|$.
2. f has its maximum rate of decrease at (a,b) in the direction $-\nabla f(a,b)$. The rate of increase in this direction is $-|\nabla f(a,b)|$.
3. The directional derivative is zero in any direction orthogonal to $\nabla f(a,b)$

4 The Gradient and Level Curves

Theorem 15.12 – The Gradient and Level Curves

Given a function f differentiable at (a,b), the tangent line to the level curve of f at (a,b) is orthogonal to the gradient $\nabla f(a,b)$ provided by $\nabla f(a,b) \neq 0$

5 The Gradient in Three Dimensions

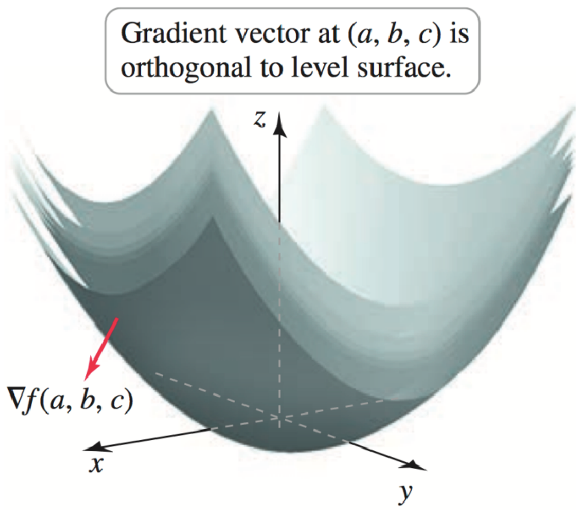

Figure 2: Visualized Gradient in Three Dimensions

Definition – Directional Derivative and Gradient in Three Dimensions

Let f be differentiable at (a, b, c) and let **u** $= \langle u_1, u_2, u_3 \rangle$ be a unit vector. The directional derivative of f at (a,b,c) in the direction of **u** is:

$$D_u f(a, b, c) = \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2, c + hu_3) - f(a, b, c)}{h}$$

The gradient of f at the point (x, y, z) is the vector valued function:

$$\begin{aligned} \nabla f(x, y, z) &= \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle \\ &= f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k} \end{aligned}$$