15.1 Graphs and Level Curves

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1 Functions of Two Variables

Concepts related to functions of several variables is explained intuitively using the case of two independent variables, and this understanding can then be generalized for functions of more than two variables. In general, functions of two variables can be written **explicitly** as:

$$z = f(x, y)$$

or **implicitly** as:

$$F(x, y, z) = 0$$

The idea of domain and range in these functions is the same as in functions of one variable.

A function $z = f(x, y)$ assigns a point (x, y) in $\mathbb{R}^2$ to unique number z in $\mathbb{R}$.

2 Graphs of Functions of Two Variables

A graph of a function $f(x, y)$ is the set of points (x, y, z) that satisfies $z = f(x, y)$. We may also write this set of points as $(x, y, f(x, y))$ or $F(x, y, z) = 0$.

Functions of two variables must pass the vertical line test, i.e., a relation $F(x, y, z) = 0$ is a function if and only if every line parallel to the z-axis intersects the graph at only one point.

2.1 Level Curves

Functions of two variables are represented in $\mathbb{R}^3$ as surfaces, which can be represented in a manner similar to a contour map.

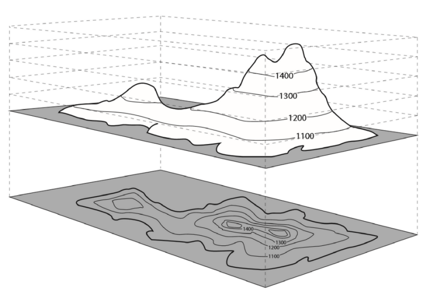

Figure 1: How to read a contour map

Consider a surface defined by $z = f(x, y)$, and imagine walking on this surface such that your elevation is always a constant value $z = z_0$. This path is called a **contour curve**. This curve in the xy-plane is called a

level curve. A contour curve is indeed a **trace** in the plane $z = z_0$. A level curve is not necessarily a trace, it may consist of a single point, a group of points, or a group of curves.

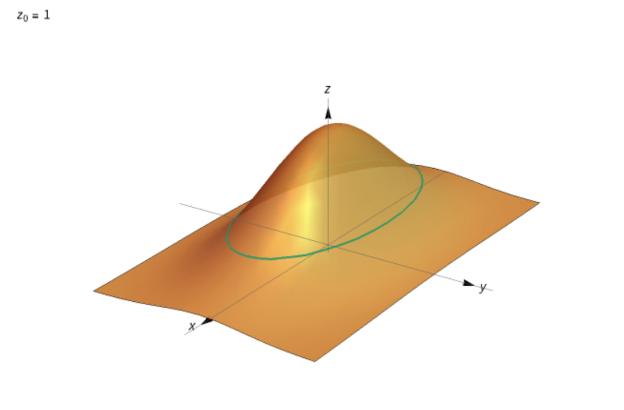

Figure 2: Exmaple of a level curve

3 Applications of Functions of Two Variables

3.1 A Probability Function of Two Variables

On a particular day, the fraction of students with the flu is r such that $0 \leq r \leq 1$. If you have n encounters with students, the probability of meeting at least one infected student is $p(n, r) = 1 - (1 - r)^n$.

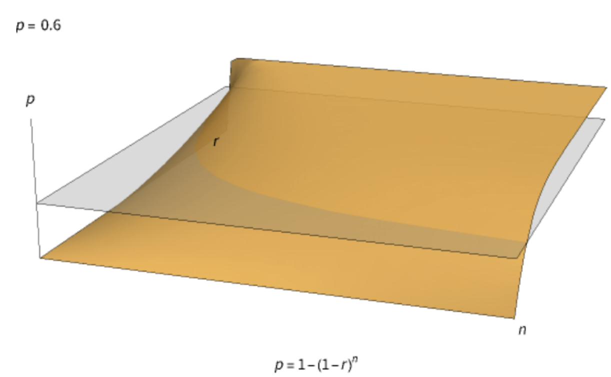

Figure 3: Probability expressed as a graph

r is restricted to $[0, 1]$ as it's a fraction of the population, and n is any non-negative integer. Since $0 \leq r \leq 1$, we have $0 \leq 1 - r \leq 1$. And since n cannot be negative, we have $0 \leq (1 - r)^n \leq 1$, which makes the range of the function $[0, 1]$, consistent with the fact that it is a probability.

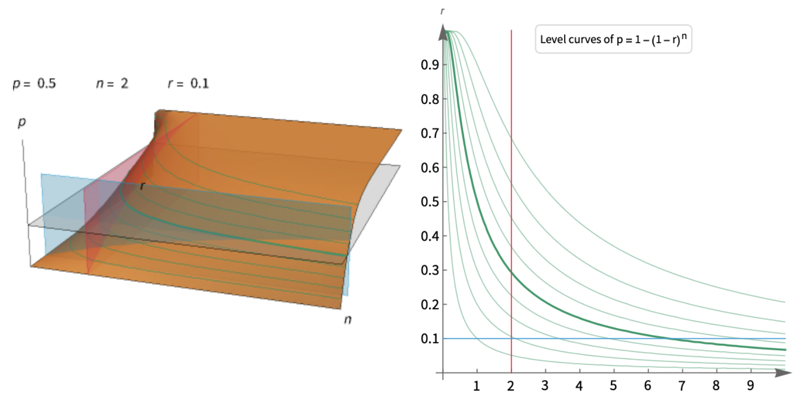

Figure 4: Level curve graph of the problem

4 Functions of More Than Two Variables

The properties of functions of two variables extend to functions of more than two variables. For example, a function in three variables is defined explicitly as $w = f(x, y, z)$ and implicitly as $F(x, y, z, w) = 0$.

We can more formally define a function with n independent variables as follows: the function $x_{n+1} = f(x_1, x_2, ..., x_n)$ assigns a unique real number x_{n+1} to each point $(x_1, x_2, ..., x_n)$ in $\mathbb{R}^n$.

5 Graphs of Functions of More Than Two Variables

Graphing functions of two independent variables required a three-dimensional coordinate system, which is the limit of ordinary graphing functions. There are two approaches:

The idea of level curves can be extended. Given function $w = f(x, y, z)$, we can set $w = 0$ as a constant. Then, the surface formed by all points for which $f(x, y, z) = 0$ holds becomes a **level surface**.

Another approach is to use colour to portray the fourth dimension. See the image below:

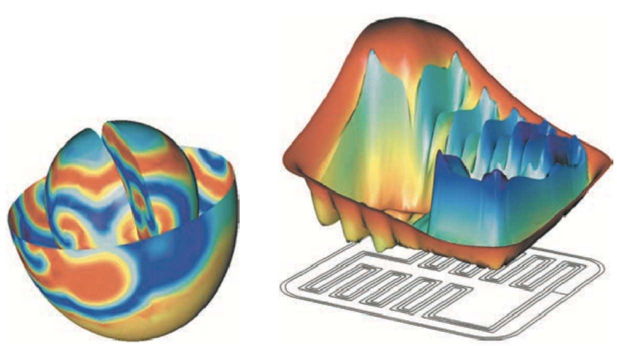

Figure 5: Using a colour gradient as a representation of the fourth dimension