Lecture 21 – Definitions of Circulation and Flux, Geometric Interpretation of Divergence and Curl Operators, and Vector Identities

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Any scalar function $f(x)$ can be decomposed into a function with odd and even characteristics.

$$f_{odd}(x) = \frac{1}{2}(f(x) - f(-x))$$
$$f_{even}(x) = \frac{1}{2}(f(x) + f(-x))$$
$$f(x) = f_{odd}(x) + f_{even}(x)$$

1 Vector Decomposition Description

Imagine we have a 2d vector field $\mathbf{F} = \langle x - y, y + x \rangle$. We can write this vector out as a sum of two vector fields which we claim have different properties.

$$\mathbf{F}_{flux} = \langle x, y \rangle \tag{1}$$
$$\mathbf{F}_{circ} = \langle -y, x \rangle \tag{2}$$
$$\mathbf{F} = \mathbf{F}_{flux} + \mathbf{F}_{circ} \tag{3}$$

1.1 Description of $\mathbf{F}_{flux}$

The field $\mathbf{F}_{flux}$ is a radial flux density that originates from a single point, so the point can be viewed as a source or sink. The flux integral can be linked to the strength of the source.

We propose the divergence of a vector field as the strength of the source, which is the flux integral divided by the area enclosed by the contour.

1.2 Description of $\mathbf{F}_{circ}$

The field $\mathbf{F}_{circ}$ is a circulation density whose field lines form closed contours that are centered at the origin. We can characterize the field pattern using the following minimum set of properties:

- The field lines encircle an axis which we call an axis of rotation.

- The circulation integral divided by the area enclosed by the contour can be linked to the strength of rotation.

We also need to define a pseudovector as the axis of rotation does not fir the description of a standard vector.

We define the curl of the vector field $\mathbf{F}$ by the product of the strength of rotation, and the unit pseudovector representing the direction of the axis of rotation.

2 Intuitive Understanding of the Divergence and Curl of a Vector Field

- Compute the flux and circulation over the closed contour for the vectors $\mathbf{F}$, $\mathbf{F}_{flux}$, and $\mathbf{F}_{circ}$.
- Normalize the flux and circulation integrals. In this case, we divide by the area enclosed by the contour so that the final result does not depend on the radius of the contour.
- Show the results in the table form and analyze them.

We will consider an example: the contour for a circle of radius r centered at the origin is parameterized as:

$$C : \langle r\cos(t), r\sin(t)\rangle \quad 0 \le t \le 2\pi$$

	Flux	Flux/Area	Circulation	|Circulation/Area|
	$\oint_C \mathbf{F}\cdot\mathbf{n}_s ds$	$\frac{\oint_C \mathbf{F}\cdot\mathbf{n}_s ds}{\pi r^2}$	$\oint_C \mathbf{F}\cdot\mathbf{T} ds$	$\frac{\oint_C \mathbf{F}\cdot\mathbf{T} ds}{\pi r^2}$
$\mathbf{F}_{circ}$	0	0	$2\pi r^2$	2
$\mathbf{F}_{flux}$	$2\pi r^2$	2	0	0
$\mathbf{F}$	$2\pi r^2$	2	$2\pi r^2$	2

Table 1: Flux, Circulation, and Normalized Vector Values for the Vectors $\mathbf{F}$, $\mathbf{F}_{flux}$, and $\mathbf{F}_{circ}$

Some observations of note:

1. Normalizing the circulation and flux integrals for a simple linear problem allowed us to remove the dependency on the shape and size of the contour.
2. The magnitude of the curl of a vector field is attributed to a rotational field $\mathbf{F}_{circ}$ and the divergence of a vector field is attributed to a source field $\mathbf{F}_{flux}$.
3. The flux integral acts as a filter that only selects $\mathbf{F}_{flux}$ and likewise for $\mathbf{F}_{circ}$.
4. These concepts can easily be generalized to 3d cases.

3 Formal Definitions of the Divergence and Curl of a Vector Field

3.0.1 Definition – Divergence of a Vector Field $\mathbf{F}$

$$\lim_{A\to 0} \frac{\oint_C \mathbf{F}\cdot\mathbf{n}_s ds}{V} \tag{4}$$

$$\mathbf{F} = \langle f(x,y)g(x,y), h(x,y)\rangle = \langle F_x, F_y, F_z\rangle \tag{5}$$

3.0.2 Definition – Curl of a Vector Field $\mathbf{F}$ in 2d

$$\lim_{A_{\hat{z}}\to 0} \frac{\oint_C \mathbf{F}\cdot\mathbf{T} ds}{A_{\hat{z}}}\hat{z} \tag{6}$$

$$\mathbf{F} = \langle f(x,y), g(x,y), 0)\rangle = \langle F_x, F_y, 0\rangle = \langle F_x, F_y\rangle \tag{7}$$

4 Concluding Observations

4.1 Physical Meaning of $\nabla \cdot \mathbf{F}$ (a scalar)

We mentioned earlier that the divergence of a vector field indicates the strength of a source at a particular point. Now assume that we give the source a physical meaning, like mass or charge density.
If we know the field and sweep over a region computing at each point $\nabla \cdot \mathbf{F}$, we are actually filtering out information telling us where charge exists and the magnitude of the charge density at any given location.

4.2 Physical Meaning of $\nabla \times \mathbf{F}$ (a pseudovector)

Hence, $\nabla \times \mathbf{F} = \mathbf{J}$.If we give $\mathbf{J}$ a physical meaning such as a current flux density, then it implies that the field $\mathbf{F}$ represents a circulation density. If we know the field $\mathbf{F}$ and we sweep over the the region while computing $\nabla \times \mathbf{F}$ at all points, then we are filtering out information telling us where current flux density exists and what its magnitude and direction is at any given location.

5 Vector Operators

Vector Operation	Expression
Gradient $\nabla(\phi(x,y,z))$	$\nabla(\phi) = \left\langle \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right\rangle$
Curl $\nabla \times \mathbf{F}$	$\nabla \times \mathbf{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \hat{x}, \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \hat{y}, \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \hat{z}$
Divergence $\nabla \cdot \mathbf{F}$	$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$

Table 2: A Summary of the Primitive Vector Operators