15.6 Tangent Planes and Linear Approximation

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1 Tangent Planes

Recall that a surface in $\mathbb{R}^3$ may be defined in at least two different ways:

1. **Explicitly** in the form $z = f(x, y)$, or
2. **Implicitly** in the form $F(x, y, z) = 0$.

$$\frac{d}{dt}[F(x(t), y(t), z(t))] = \nabla F(x, y, z) \cdot \mathbf{r}'(t)$$

Definition – Equation of the Tangent Plane for $F(x, y, z) = 0$

Let F be differentiable at the point $P_0(a, b, c)$ with $\nabla F(a, b, c) \neq 0$. The plane tangent to surface at P_0 is called the tangent plane, and it is the plane that passes through P_0 orthogonal to $\nabla F(a, b, c)$.

Important – Tangent Plane for $z = f(x, y)$

Let f be differentiable at that point (a, b). An equation of the plane tangent to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$ is:

$$z = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b)$$

2 Linear Approximation

A tangent line at a point gives a good approximations to the function's behaviour near that point. This method is called linear approximation.

Definition – Linear Approximation

Let f be differentiable at (a, b). The linear approximation to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$ is the tangent plane at that point, given by the equation:

$$L(x, y) = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b)$$

We can construct a formula for linear approximations in three variables the same way.

3 Differentials and Change

The exact change in the function between the points (a, b) and (x, y) is:

$$\Delta z = f(x, y) - f(a, b)$$

Replacing $f(x, y)$ by its linear approximation, the change Δz is approximated by:

$$\Delta z \approx L(x, y) - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

Definition – The Differential dz

Let f be differentiable at the point (x, y). The change in $z = f(x, y)$ as the independent variables change from (x, y) to $(x + dx, y + dy)$ is denoted Δz and is approximated by the differential dz:

$$\Delta z \approx dz = f_x(x, y)dx + f_y(x, y)dy$$