15.4 The Chain Rule

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1 The Chain Rule with One Independent Variable

If we have y is a function of u and u is a function of t, then we have:

$$\frac{dy}{dt} = \frac{dy}{du}\frac{du}{dt}$$

Now we consider composite functions of the form $z = f(x, y)$ where x and y are functions of t. Then, what is dz/dt. We can illustrate the relationships among these variables using a tree diagram.

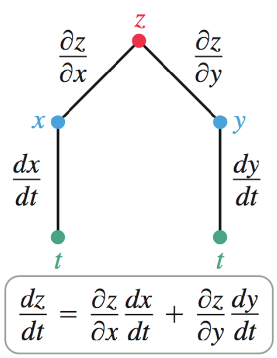

Figure 1: Tree Diagram

Theorem 15.7 – Chain Rule (One-Independent Variable)

Let z be a differentiable function of x and y on its domain, where x and y are differentiable functions of t on an interval I. Then,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Several comments are in order:

- With $z = f(x(t), y(t))$, the dependent variable is z and the sole independent variable is t. x and y are intermediate variables.
- The above theorem generalizes directly to functions of more than two variables.

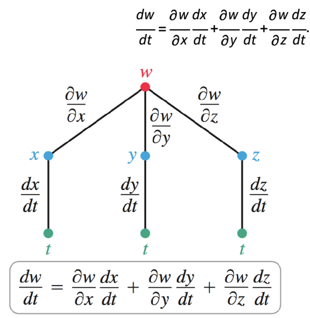

Figure 2: Diagram Tree for a Function of Three Variables

2 The Chain Rule with Several Independent Variables

Suppose z depends o two independent variables s and t. Once again, we can use a diagram tree to organize the relationships among variables.

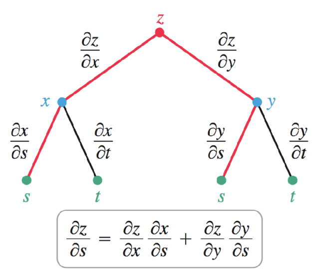

Figure 3: Chain Rule with Multiple Variables

Theorem 15.8 – Chain Rule (Two Independent Variable)

Let z be a differentiable function of x and y, who are in turn functions of s and t. Then:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

3 Implicit Differentiation

Theorem 15.9 – Implicit Differentiation

Let F be differentiable on its domain and suppose $F(x, y) = 0$ defined y as a differentiable function of x. Provided $F_y \neq 0$

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$