13.6 Cylinders and Quadric Surfaces

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In Section 13.5, we discussed how lines and planes are described by parametric linear equations. In this section, we will look at the geometry of three-dimensional objects described by quadratic equations in three variables. These are *quadric surfaces*.

1 Cylinders and Traces

In the vernacular, we use cylinder to describe the curved wall of a paint can, for example. In the context of this textbook, we will use **cylinder** to refer to a a surface that is parallel to a line, specifically in this text, we focus on cylinders parallel to one of the coordinate axes.

An example in $\mathbb{R}^3$ is $y = x^2$, which you will notice does not contain z. This means z is arbitrary and can take on any value.

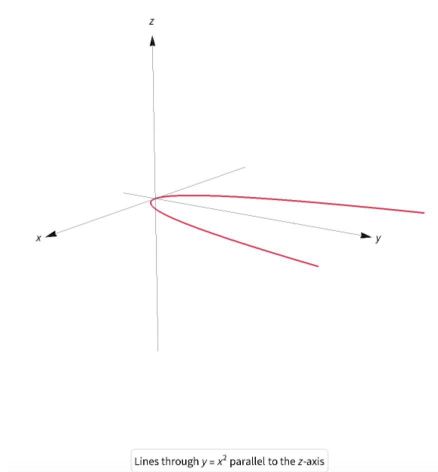


Figure 1: Cylinder parallel to the z-axis

A surface's **trace** is the set of points at which the surface intersects a plane that is parallel to one of the coordinate planes.

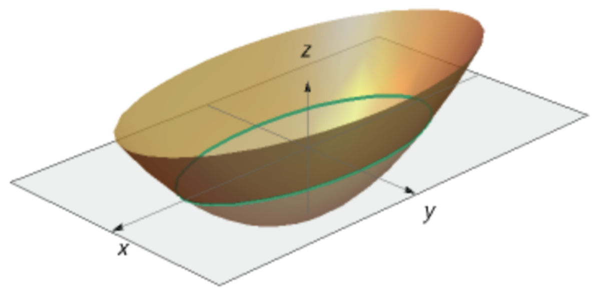


Figure 2: xy-trace given by the green line

2 Quadric Surfaces

Quadric surfaces are given by the general quadratic equation in three variables:

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

where not all of A, B, C, D, E, F are zero.

In this text, we will not conduct a detailed study of quadric surfaces, but there are some of particular interest, which have various practical uses. For examples, paraboloids have the same reflective properties as their 2D equivalents, and are used to design satellite dishes and mirrors in telescopes. Hyperboloids inspire the shape of nuclear plant cooling towers. Lastly, ellipsoids help design water tanks and gears.

Here are some tips to keep in mind when drawing quadric surfaces:

1. **Intercepts** – Determine the points where the surface intersects the coordinate axes. Do this by setting x, y, and z to zero two-at-a-time and solving for the third.
2. **Traces** – Finding traces helps visualizes the surface. By setting $z = 0$, we can find all points parallel to the xy-plane.
3. **Completing the figure** – Sketch at least two more traces, say $z = 1$, then draw curves that pass through all traces to complete the surface.

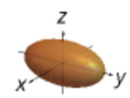
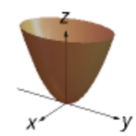
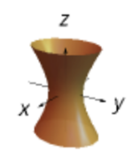
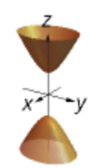
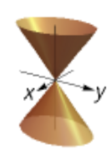
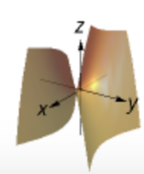
Name	Standard Equation	Features	Graph
Ellipsoid	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	All traces are ellipses.	
Elliptic paraboloid	$z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$	Traces with $z=z_0>0$ are ellipses. Traces with $x=x_0$ or $y=y_0$ are parabolas.	
Hyperboloid of one sheet	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$	Traces with $z=z_0$ are ellipses for all z_0. Traces with $x=x_0$ or $y=y_0$ are hyperbolas.	
Hyperboloid of two sheets	$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	Traces with $z=z_0$ with $\lvert z_0 \rvert > \lvert c \rvert$ are ellipses. Traces with $x=x_0$ or $y=y_0$ are hyperbolas.	
Elliptic cone	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}$	Traces with $z=z_0\neq 0$ are ellipses. Traces with $x=x_0$ or $y=y_0$ are hyperbolas or intersecting lines.	
Hyperbolic paraboloid	$z=\frac{x^2}{a^2}-\frac{y^2}{b^2}$	Traces with $z=z_0\neq 0$ are hyperbolas. Traces with $x=x_0$ or $y=y_0$ are parabolas.	

Figure 3: Some standard quadric surfaces (taken directly from the textbook)