Lecture 8 – Small Signal Analysis

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$$z = f(x, y) = f(a, b) + \left.\frac{\partial f}{\partial x}\right|_{\substack{x=a \\ y=b}} (x - a) + \left.\frac{\partial f}{\partial y}\right|_{\substack{x=a \\ y=b}} (y - b)$$

We need to introduce a definition for the Jacobian matrix before we can describe the small signal model procedure.

1 Jacobian Matrix

Express $F(x, y, z)$ in new variables as follows:

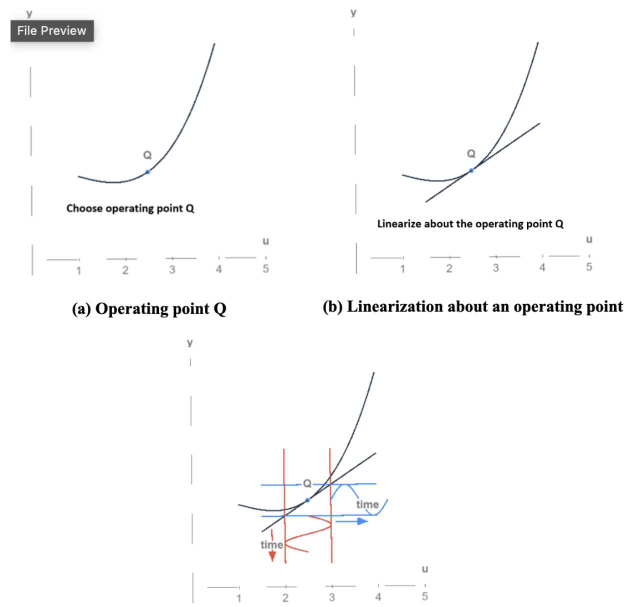
$$x = f(u, v, w) \tag{1}$$
$$y = g(u, v, w) \tag{2}$$
$$z = h(u, v, w) \tag{3}$$

1.0.1 Definition – Jacobian Matrices

A 3×3 Jacobian matrix **J** has the form:

$$J = \begin{bmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} & \frac{\partial f}{\partial w} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} & \frac{\partial g}{\partial w} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} & \frac{\partial h}{\partial w} \end{bmatrix} = \frac{\partial(x, y, z)}{\partial(u, v, w)} \tag{4}$$

2 Small-Signal Model for a Single Input and Single Output



(a) Operating point Q

(b) Linearization about an operating point

(c) Apply small signal time varying input to find small signal output

Figure 1: Small-signal analysis assuming a time varying input variable

2.0.1 Definition

The nomenclature Δu and $\hat{u}(t)$ are interchangeable. Usually we use the latter to denote small variations associated with time.

$$Y + Output(t) = g(U + Input(t))$$

3 Small-Signal Model for the Multivariable Case Given a System Model

We start with some definitions:

3.0.1 Definitions – state variable x, input variable u, and output variable y

State Variable x – variable in a first order derivative $\frac{dx}{dt}$
Input Variable u – variable representing an excitation signal $\frac{dx}{dt} = f(x, u)$
Output Variable y – Function of the State Variable x and input variable u : $y = g(x, u)$

3.0.2 Definition – State Space Equations

$$\frac{dx_1}{dt} = f_1(x_1(t), ..., x_n(t), u_1(t), ..., u_m(t))$$
$$\vdots$$
$$\frac{dx_n}{dt} = f_2(x_1(t), ..., x_n(t), u_1(t), ..., u_m(t))$$

3.0.3 Definition – Output Equations

$$
\begin{aligned}
y_1 &= g_1(x_1(t), ..., x_n(t), u_1(t), ..., u_m(t)) \\
&\vdots \\
y_k &= g_k(x_1(t), ..., x_n(t), u_1(t), ..., u_m(t))
\end{aligned}
$$

Solving DEs for the general case is beyond the scope of this course as it requires knowledge of discrete mathematics and numerical methods.

3.1 Observations on Linear and Non-Linear Systems of Equations

1. If the differential equation set is nonlinear in any of the state variables or input variables, then the differential equation set is nonlinear.
2. If an output equation is nonlinear in any of the state variables or input variables, then the output equation is nonlinear.
3. In the nonlinear case, the right-hand side of the differential equation set, or the right-hand side of the output equation cannot be expressed in a matrix form since one or more of the expressions is non-linear.

4 Small-Signal Modelling Procedure

1. Expand the differential state space equations and algebraic output equations and neglect all terms with derivatives greater than first order. The result is a set of coupled linear DEs and linear output equations.
2. Set all time derivatives and small-signal perturbation terms to zero, then:
   (a) Determine the equilibrium solution for the state variables given in the equilibrium values for the input variables.
   (b) Determine the equilibrium solution for the output variables.
3. Determine the elements of the Jacobian matrices for the small-signal model. The elements of the Jacobian matrices are a function of the equilibrium values for the input variables and state variables.