Lecture 17 – Line Integrals of Vector Dot Products

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1 Geometrical Properties

We will describe in pictorial form the types of geometries to consider when performing integrals over curved lines and curved surfaces. We will consider 2D problems as well as 3D problems.

1. $\mathbf{T}$ – unit vector tangent to the contour
2. $\mathbf{n}_s$ – unit vector normal to the contour
3. ds – differential arc length
4. $\mathbf{T}ds = \mathbf{ds}$ – differential arc length

1.0.1 Open Contour Geometries

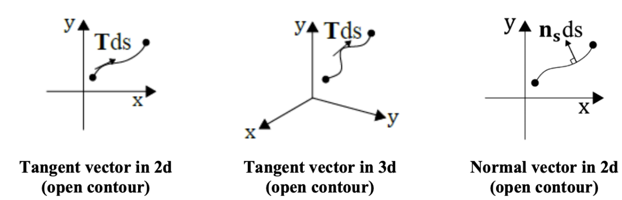

Figure 1: Open Contour Geometries

1.0.2 Closed Contour Geometries

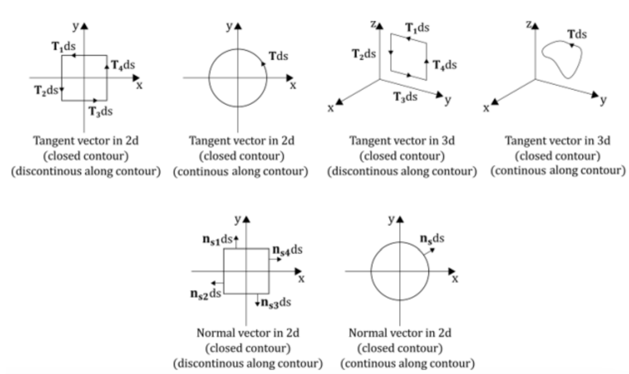

Figure 2: Closed Contour Geometries

1.0.3 Closed and Open Surfaces

It is obvious to see that any closed surface can be decomposed into a group of open surfaces.

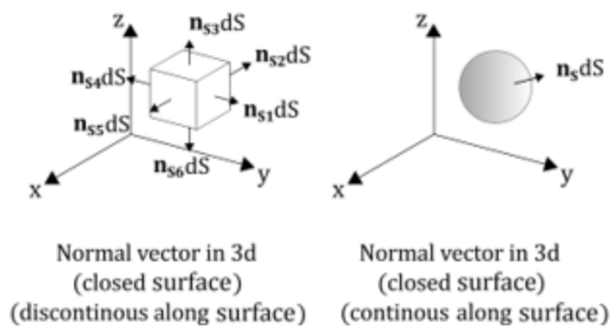

Figure 3: Orientation of a Differential Surface Area Vector

1. $\mathbf{n}_S$ – unit vector normal to a differential surface area
2. dS – magnitude of differential surface area
3. $\mathbf{n}_S dS = \mathbf{dS}$ – differential surface area

1.0.4 Flux and Circulation

Flux means something that leaves or enters a 2D contour in a plane or a surface in a 3D domain.

Circulation is a measure of how much rotation exists in a vector field. Both of these values are scalars.

2 Integral Definitions

2.1 Contour Integrals

1. Integrating the scalar density along the length of an open contour

$$\int_C \lambda(s) ds \tag{1}$$

2. Integrating the scalar density along the length of a closed contour

$$\oint_C \lambda(s) ds \tag{2}$$

3. Average of f over an open contour

$$f_{avg} = \frac{\int_C f ds}{\int_C ds} \tag{3}$$

4. Average of f over a closed contour

$$f_{avg} = \frac{\oint_C f ds}{\oint_c ds} \tag{4}$$

5. Flux from an open contour

$$\int_C \mathbf{F} \cdot \mathbf{n}_s ds \tag{5}$$

6. Flux from a closed contour

$$\oint_C \mathbf{F} \cdot \mathbf{n}_s ds \tag{6}$$

7. Circulation of an open contour

$$\int_C \mathbf{F} \cdot \mathbf{T} ds \tag{7}$$

8. Circulation of a closed contour

$$\oint_C \mathbf{F} \cdot \mathbf{T} ds \tag{8}$$

2.2 Surface Area Integrals

1. Integrating the scalar density over an open surface

$$\iint_S \sigma(S) dS \tag{9}$$

2. Integrating the scalar density over a closed surface

$$\oiint_S \sigma(S) dS \tag{10}$$

3. Average of f over an open surface

$$f_{avg} = \frac{\iint_S f dS}{\iint_S dS} \tag{11}$$

4. Average of f over a closed surface

$$f_{avg} = \frac{\oiint_S f dS}{\oiint_S dS} \tag{12}$$

5. Flux from an open surface

$$\iint_S \mathbf{F} \cdot \mathbf{n}_s dS \tag{13}$$

6. Flux from a closed surface

$$\oiint_S \mathbf{F} \cdot \mathbf{n}_s dS \tag{14}$$