Reading Assignment 3 (Short): Triple Integrals in Rectangular Coordinates

Arnav Patil

University of Toronto

1 Triple Integrals in Rectangular Coordinates

Consider a function $w = f(x, y, z)$ that is defined on a closed and bounded region D of $\mathbb{R}^3$; despite the difficulty in representing it, we can still define the integral of f over D. We partition D into boxes that are wholly contained within D, with the kth box having $\Delta V_k = \Delta x_k, \Delta y_k, \Delta z_k$.

$$\int\int\int_D f(x,y,z) = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta V_k \tag{1}$$

Notice the analogy between double and triple integrals:

- Area of $R = \int\int_R dA$, and
- Volume of $D = \int\int\int_D dV$

2 Finding Limits of Integration

We can write the triple integral as an iterated integral.

$$\int\int\int_D f(x,y,z)dV = \int\int_R \left(\int_G^H f(x,y,z)dz\right)dA \tag{2}$$

2.0.1 Theorem – Triple Integrals

To integrate over all points of D, we carry out the following:

1. Integrate with respect to z from $z = G(x, y)$ to $z = H(x, y)$; the result is generally a function of x and y.
2. Integrate with respect to y from $y = g(x)$ to $y = h(x)$; the result is generally a function of x.
3. Integrate with respect to x from $x = a$ to $x = b$; the result is always a number that doesn't depend on $x, y,$ or z.

Note: this Theorem is a version of Fubini's Theorem.

3 Changing the Order of Integration

We can simplify the solving of a triple integral by choosing an appropriate order of integration. Often we don't know whether a particular order will work, some trial and error is required.

4 Average Value of a Function of Three Variables

4.0.1 Definition – Average Value of a Function of Three Variables

If f is continuous on a region D of $\mathbb{R}^3$, then the average value of f over D is:

$$\bar{f} = \frac{1}{\text{volume of} D} \int\int\int_D f(x, y, z) dV \tag{3}$$