16.7 Change of Variables in Multiple Integrals

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1 Recap of Change of Variables

We use the change of variables strategy to help simplify a single-variable integral. For example, to simplify the integral $\int_0^1 s\sqrt{2x+1}dx$, we choose a $u = 2x + 1$ and $du = 2dx$. Therefore, we have:

$$\int_0^1 2\sqrt{2x+1}dx = \int_1^3 \sqrt{u}du$$

2 Transformations in the Plane

A change of variables in a double integral is a transformation that relates a pair of variables to another. $(x, y) = T(u, v)$ is compactly written as:

$$T : x = g(u, v) \text{ and } y = h(u, v)$$

2.0.1 One-to-One Transformation

A transformation T from a region S to a region R is one-to-one on S if $T(P) = T(Q)$ only when $P = Q$.

2.0.2 Jacobian Determinant of a Transformation of Two Variables

Given a transformation $T : x = g(u, v), y = h(u, v)$ where g and h are differentiable on a region of the uv-plane, the Jacobian determinant of T is:

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \tag{1}$$

2.0.3 Theorem 16.8 – Change of Variables for Double Integrals

Let T be a transformation that maps a closed bounded region S in the uv-plane to a region R in the xy-plane. Assume T is injective on the interior of S and g, h have continuous first partial derivatives there. If f is continuous on R, then:

$$\int\int_R f(x, y)dA = \int\int_S f(g(u, v)h(u, v))|J(u, v)|dA \tag{2}$$

3 Change of Variables in Triple Integrals

3.0.1 Definition – Jacobian Determinant of a Transformation of Three Variables

Given a transformation T where g, h, p are differentiable on a region of uvw-space, the Jacobian is given by:

$$J(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{bmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{bmatrix} \tag{3}$$

3.0.2 Theorem 16.9 – Change of Variables for Triple Integrals

Let $T(u,v,w)$ be a transformation that maps a closed bounded region S to a region $D = T(S)$. Assume T is one-to-one on the interior of S and g, h, p have continuous first partial derivatives there. If f is continuous on D, then:

$$\int\int\int_D f(x,y,z)dV = \int\int\int_S f(g(u,v,w), h(u,v,w), p(u,v,w))|J(u,v,w)|dV \tag{4}$$

4 Strategies for Choosing New Variables

Here are some suggestions for finding new variables of integration. These apply to both double and triple integrals.

1. **Aim for simple regions of integration in the uv plane.** The new region should be as simple as possible. For e.g., double integrals are easiest to evaluate over rectangular regions with sides parallel to the coordinate axes.

2. **Is $(x,y) \to (u,v)$ or $(u,v) \to (x,y)$ better?** Depending on the problem, inverting the transformation may be easy, difficult, or impossible.

3. **Let the integrand suggest new variables.**

4. **Let the region suggest new variables.**