15.2 Limits and Continuity

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1 Limit of a Function of Two Variables

A function f of two variables has a limit L as $P(x, y)$ approaches a fixed point $P_0(a, b)$ if $|f(x, y) - L|$ can become arbitrarily small for all P. If this limit exists, we can write:

$$\lim_{(x,y)\to(a,b)} f(x, y) = \lim_{P\to P_0} f(x, y) = L$$

We can construct a more formal definition of a limit of a function of two variables: The function f has the limit L as $P(x, y)$ approaches $P_0(a, b)$, written

$$\lim_{(x,y)\to(a,b)} f(x, y) = \lim_{P\to P_0} f(x, y) = L$$

if, given any $\varepsilon > 0$, there exists a $\sigma > 0$ such that $|f(x, y) - L| < \varepsilon$ whenever:

$$0 < |PP_0| = \sqrt{(x-a)^2 + (y-b)^2} < \sigma$$

Therefore, the limit only exists if $f(x, y)$ approaches L as P approaches P_0 along all possible paths.

All the limit laws apply as they do for functions in a single variable.

2 Limits at Boundary Points

Let R be a region in $\mathbb{R}^2$. An **interior point** of R lies entirely in R, meaning we can construct a disk centred at that point containing only points within R also. A **boundary point** of R lies on the edge of R such that every disk centred at that point contains at least one point outside and inside R. We may translate these definitions to $\mathbb{R}^3$ by replaces *disk* with *ball*.

With these definitions, we may also define **open sets** as regions consisting entirely of interior points, and **closed sets** as regions containing all of their boundary points.

Let us consider a few examples:

Example: limits at boundary points

Evaluate $\displaystyle\lim_{(x,y)\to(4,1)} \frac{xy - 4y^2}{\sqrt{x} - 2\sqrt{y}}$.

Solution: Points in the domain must satisfy $x \geq 0$ and $y \geq 0$ and $x \neq 4y$. Due to the last condition, we see that the point $(4, 1)$ lies on the boundary of the domain.

$$\begin{aligned}\lim_{(x,y)\to(4,1)} \frac{xy-4y^2}{\sqrt{x}-2\sqrt{y}} &= \lim_{(x,y)\to(4,1)} \frac{(xy-4y^2)(\sqrt{x}+2\sqrt{y})}{(\sqrt{x}-2\sqrt{y})(\sqrt{x}+2\sqrt{y})} \\ &= \lim_{(x,y)\to(4,1)} \frac{y(x-4y)(\sqrt{x}+2\sqrt{y})}{x-4y} \\ &= \lim_{(x,y)\to(4,1)} y(\sqrt{x}+2\sqrt{y}) \\ &= 4.\end{aligned}$$

Example: nonexistence of a limit

Investigate the limit $\lim\limits_{(x,y)\to(0,0)} \dfrac{(x+y)^2}{x^2+y^2}$.
Solution: The domain of this function is $D : \{(x,y)|(x,y) \neq (0,0)\}$, meaning the limit is at a boundary point *outside* of the domain. Suppose we let (x,y) approach $(0,0)$ along the line $y = mx$; we can substitute $y = mx$. Thus:

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{(x+y)^2}{x^2+y^2} &= \lim_{x\to 0} \frac{(x+mx)^2}{(x^2+m^2x^2)} \\ &= \lim_{x\to 0} \frac{x^2(1+m)^2}{x^2(1+m^2)} \\ &= \frac{(1+m)^2}{1+m^2}.\end{aligned}$$

where the constant m determines the direction of approach to $(0,0)$. In other words, the function approaches different values as $(x,y)\to(0,0)$, depending on the value of m.

The Two-Path Test for Nonexistence of Limits – If (x,y) approaches two different values as $(x,y)\to(a,b)$, then the limit does not exist.

3 Continuity of Functions of Two Variables

The definition of continuity for functions of two variables is the same as the definition for functions of one variable.

The function f is continuous at the point (a,b) given:

1. f is defined at (a,b),
2. $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ exists, and
3. $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

3.1 Composite Functions

Recall that a composition of a continuous function (in a single variable) is also continuous. If $u = g(x,y)$ is continuous at (a,b) and $z = f(u)$ is continuous at $g(a,b)$, then the composite function $z = f(g(x,y))$ is also continuous at (a,b).

4 Functions of Three Variables

Work done with limits and continuity of functions of two variables extends of three+ variables. Limits of rational and polynomial functions may be evaluated by directly substituting at all points within their domains. Compositions of continuous functions $f(g(x, y, z))$ are continuous at points at which $g(x, y, z)$ is within the domain of f.