15.3 Partial Derivatives

Arnav Patil

University of Toronto

1 Limit of a Function of Two Variables

Suppose we are standing on the surface below, at the point $P(0, 0, f(0, 0))$. If we walk east-west, we will walk uphill, and if we walk north-south, we will walk downhill. Essentially, the function value changes at a different rate in every direction we walk from P. So, how do we define the slope (or rate of change) at a given point on this surface? The answer is partial derivatives.

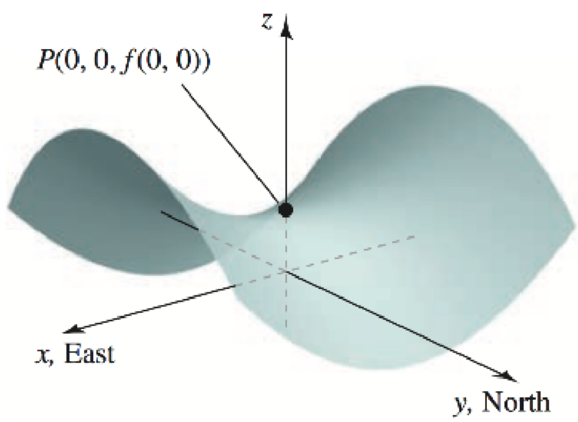

Figure 1: Saddle Graph

Partial derivatives are when we hold all but one independent variable fixed, then compute an ordinary derivative with respect to it.

Let's say we are moving on the surface $z = f(x, y)$ starting at the point $(a, b, f(a, b))$ in a way such that $y = b$ is fixed and only x is allowed to vary. The resulting path is a trace on the surface with vertical plane $y = b$. This path is described by $z = f(x, b)$ which is a function of the single variable x. We can calculate the slope of this curve as the ordinary derivative of $f(x, b)$ with respect to x. This derivative is called the partial derivative of f with respect to x, and is denoted as $\frac{\partial f}{\partial x}$ or f_x. We can define the limit:

$$f_x(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h}$$

Similarly, we can move along $z = f(x, y)$ in such a way that $x = a$ is fixed and y varies. Now, the result is a trace described by $z = f(a, y)$, the intersection of the surface and the plane $x = a$. Now, we can take the ordinary derivative of this with respect to y.

Definition: The partial derivative of f with respect to x at the point (a, b) is:

$$f_x(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h}$$

The partial derivative of f with respect to y at the point (a, b) is:

$$f_y(a,b) = \lim_{h\to 0} \frac{f(a, b+h) - f(a,b)}{h}$$

provided these limits exist.

Partial derivatives may be denoted in any of the following ways:

$$\frac{\partial f}{\partial x}(a,b) = \left.\frac{\partial f}{\partial x}\right|_{(a,b)} = f_x(a,b)$$

Partial derivatives from the definition

EXAMPLE: Suppose $f(x, y) = x^2y$. Use the limit definition of partial derivatives to compute $f_x(x, y)$ and $f_y(x, y)$.

SOLUTION: We compute the partial derivatives at an arbitrary point (x, y) in the domain. The partial derivative with respect to x is:

$$\begin{aligned}
f_x(x,y) &= \lim_{h\to 0} \frac{f(x+h, y) - f(x,y)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^2y - x^2y}{h} \\
&= \lim_{h\to 0} \frac{(x^2 + 2xh + h^2 - x^2)y}{h} \\
&= \lim_{h\to 0} (2x+h)y \\
&= 2xy
\end{aligned}$$

The partial derivative with respect to y is:

$$\begin{aligned}
f_y(x,y) &= \lim_{h\to 0} \frac{f(x, y+h) - f(x,y)}{h} \\
&= \lim_{h\to 0} \frac{x^2(y+h) - x^2y}{h} \\
&= \lim_{h\to 0} \frac{x^2(y+h-y)}{h} \\
&= x^2
\end{aligned}$$

There is a shortcut revealed to evaluate partial derivatives: to compute the partial derivative of f with respect to x, we treat y as a constant and differentiate:

$$\frac{\partial}{\partial x}(x^2y) = y\frac{\partial}{\partial x}(x^2) = 2xy$$

We can apply the same shortcut to differentiate with respect to y:

$$\frac{\partial}{\partial y}(x^2y) = x^2\frac{\partial}{\partial y}(y) = 2x$$

2 Higher-Order Partial Derivatives

Just as we can take second, third (and so on) derivatives of functions of one variable, we can have higher-order partial derivatives as well. Say we take the partial derivative f_x of a function f, then we can further differentiate f_x with respect to x or y. This means there are four possible second-order derivatives to f.

2.1 Equality of Mixed Partial Derivatives

A mixed partial derivative is taken with respect to x the first time and y the next, or vice versa. Essentially, do not differentiate with respect to the same variable both times.

Theorem: Clairaut Equality of Mixed Partial Derivatives – Assume f is defined on an open set D of $\mathbb{R}^2$, and f_{xy} and F_{yx} are continuous throughout D. Then $f_{xy} = f_{yx}$ at all points of D.

3 Functions of Three Variables

Everything covered thus far about partial of derivatives of functions with two variables carries over to functions of three or more variables.

4 Differentiability

Although we have learned to differentiate partial derivatives of a function of several variables, we have not yet covered what it means for a function to be *differentiable* at a point. Recall that a function f of one variable is differentiable at $x = a$ provided:

$$f'(a) = \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x}$$

If f can be differentiated at a, that means there is a smooth curve at a (no jumps, corners, or cusps). The function also has a unique tangent line at that point with value $f'(a)$.

For a function of several variables, the surface should be smooth at that point, and there should also exist something analogous to a tangent line. We define the quantity:

$$\varepsilon = \frac{f(a + \Delta x) - f(a)}{\Delta x} - f'(a)$$

where ε is a function of Δx. We can multiply both sides by Δx to give:

$$\varepsilon \Delta x = f(a + \Delta x) - f(a)\Delta x$$

which when further rearranged gives the change in function $y = f(x)$:

$$\Delta y = f(a + \Delta x) - f(a) = f'(a)\Delta x + \varepsilon \Delta x$$

DEFINITION: Differentiability – The function $z = f(x, y)$ is differentiable at (a, b) provided the change $\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$ equals $\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$. A function is differentiable on an open region R if it is differentiable at every point on R.

THEOREM: Conditions for Differentiability – Suppose a function f has partial derivatives f_x and f_y defined on an open set containing (a, b) with f_x and f_y being continuous at (a, b). Then, f is differentiable at (a, b).