Lecture 25A – Computing a Vector Field $\vec{F}$ when $\nabla \times \vec{F}$ and $\nabla \cdot \vec{F}$ are Given

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Many problems in nature (electrostatics, magnetostatics, heat transfer, fluid flow, etc.) are described by either of the following equations:

$$\nabla \cdot \vec{F} = \rho(x,y,z) \qquad \nabla \times \vec{F} = 0 \text{ irrotational field} \tag{1}$$

$$\nabla \cdot \vec{F} = 0 \qquad \nabla \times \vec{F} = \vec{J}(x,y,z) \text{ source-free field} \tag{2}$$

1 Symmetries

There are various types of symmetries. The main criterion is that the vector $\vec{F}$ is aligned with one of the basis vectors in the appropriate coordinate system.

1.1 Planar Symmetry

Consider $\vec{F} = f(x)\hat{x}$ in 3d Cartesian coordinates. The expression indicates that the vector points in the $\hat{x}$ direction and the magnitude depends only on the x position.

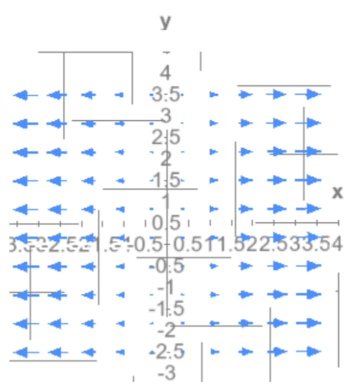

Figure 1: Planar Symmetry

1.2 Cylindrical Symmetry

Let $\vec{F} = f(\sqrt{x^2+y^2})(-\frac{y}{\sqrt{x^2+y^2}}\hat{x} + \frac{x}{\sqrt{x^2+y^2}}\hat{y}) = f(r)\hat{\theta}$. The vector points in the $\hat{\theta}$ direction and its magnitude depends only on position r where $r = \sqrt{x^2+y^2}$.

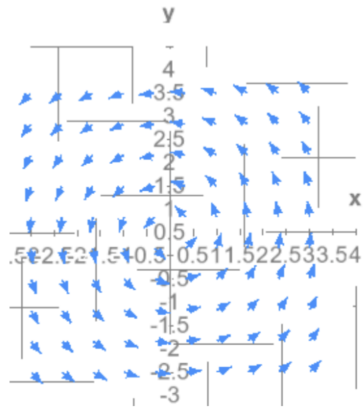

Figure 2: Cylindrical Symmetry

1.3 Cylindrical Symmetry

Let $\vec{F}$ be the vector such that $\vec{F} = f(\sqrt{x^2+y^2})(\frac{x}{\sqrt{x^2+y^2}}\hat{x} + \frac{x}{\sqrt{x^2+y^2}}\hat{y}) = f(r)\hat{r}$. The vector points in the $\hat{r}$ direction and its magnitude depends only on position r where $r = \sqrt{x^2+y^2}$.

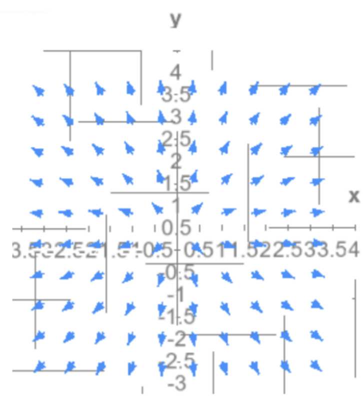

Figure 3: Cylindrical Summary

1.4 Spherical Symmetry

Let $\vec{F}$ be the vector $\vec{F} = f(\sqrt{x^2+y^2+z^2})(\frac{y}{\sqrt{x^2+y^2+z^2}}\hat{x} + \frac{x}{\sqrt{x^2+y^2+z^2}}\hat{y} + \frac{z}{\sqrt{x^2+y^2+z^2}}\hat{z}) = f(r)\hat{r}$. The vector points in the direction of $\hat{r}$ and its magnitude depends only on the position r where $r = \sqrt{x^2+y^2+z^2}$.

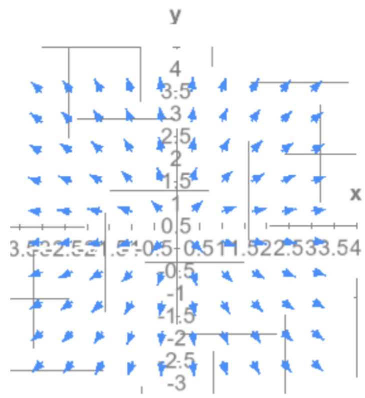

Figure 4: Spherical Symmetry

2 Cases That Can be Solved Without Specialized Tools or Knowledge

1. An irrotational field $\vec{F}$ and symmetry $\rho(x, y, z)$:

$$\nabla \cdot \vec{F} = \rho(x, y, z) \tag{3}$$
$$\nabla \times \vec{F} = 0 \tag{4}$$

   We compute $\vec{F}$ using the Divergence Theorem:

$$\oiint_S d\vec{S} = \iiint_V \nabla \cdot \vec{F} dV = \iint_V \rho dV \tag{5}$$

2. A source free field $\vec{F}$, and symmetry for $\vec{J}(x, y, z)$:

$$\nabla \cdot \vec{F} = 0 \tag{6}$$
$$\nabla \times \vec{F} = \vec{J}(x, y, z) \tag{7}$$

   $\vec{F}$ is computed using Stokes' Theorem:

$$\oint_C \vec{F} \cdot \vec{ds} = \iint_S \nabla \times \vec{F} \cdot \vec{ds} = \iint_S \vec{J} \cdot \vec{ds} \tag{8}$$

3 Summary of Conditions for Symmetry

We invoke symmetry on equations (3) and (4) if ρ is a function of only variable in an appropriately chosen coordinate system. Likewise, we can only invoke symmetry on (6) and (7) if $\vec{J}$ is directed along one of the basis vectors in an appropriate coordinate system and has a magnitude that is a function of only one coordinate system.