13.5 Lines and Planes in Space

Arnav Patil

University of Toronto

1 Lines in Space

We can define a unique line in $\mathbb{R}^3$ using either two distinct points or one point and a direction. We can use these properties to derive two different descriptions of lines: parametric equations, and vector equations.

Component Form

$$\langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle$$

Vector Form

$$\vec{r} = \vec{r}_0 + t\vec{v}$$

This gives us $x = x_0 + at$, $y = y_0 + bt$, and $z = z_0 + ct$.

2 Distance from a Point to a Line

Consider a line l given by $\vec{r} = \vec{r}_0 + t\vec{v}$. Our goal is to find the distance d between Q and l. We form another perpendicular line from Q to Q' on l to make a right-angle triangle PQQ', thus, the shortest distance from Q to l is the distance from Q to Q'.

We know that $d = |\vec{PQ}|\sin\theta$, where θ is the angle between $\vec{v}$ and $\vec{PQ}$. We have:

$$|\vec{v} \times \vec{PQ}| = |\vec{v}||\vec{PQ}|\sin\theta = |\vec{v}|d$$

From which we can derive:

$$d = \frac{|\vec{v} \times \vec{PQ}|}{|\vec{v}|}$$

3 Equations of Planes

A plane is a flat surface that infinitely extends in all directions. Three points, where not all points are on the same line, determine a unique plane in $\mathbb{R}^3$. A plane can also be determined by one point on the plane and a nonzero vector orthogonal to it. This vector is called a **normal vector** and specifies the orientation of the plane.

We can formally define a plane as: Given a fixed point P_0 and a nonzero vector $\vec{n}$, the set of points P for which $\vec{P_0P}$ is orthogonal to $\vec{n}$ is called a plane. Just as a slope defines the orientation of a line in $\mathbb{R}^2$, a normal vector determines the orientation of a plane in $\mathbb{R}^3$.

The equation of a plane in $\mathbb{R}^3$ can be given by:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \text{ OR } ax + by + cz = d$$

Note: a vector $\vec{n} = \langle a, b, c \rangle$ is used to describe a plane by specifying a direction orthogonal to the plane. On the other hand, a vector $\vec{v} = \langle a, b, c \rangle$ is used to describe a direction parallel to the line.

4 Parallel and Orthogonal Planes

Normal vectors tell us about the orientation of planes. In particular, there are two cases of interest: one where the planes are parallel, and one where the planes are orthogonal relative to each other. If $\vec{n}_1$ and $\vec{n}_2$ are parallel, then the planes are parallel. If $\vec{n}_1 \cdot \vec{n}_2 = 0$, then the planes are orthogonal.

We will take a look at an example to solidify our understanding. Find an equation of the line of intersection of the planes $Q : x + 2y + z = 5$ and $R : 2x + y - z = 7$. First note $\vec{n}_1 = \langle 1, 2, 1 \rangle$ and $\vec{n}_2 \langle 2, 1, -1 \rangle$ are not multiples of each other. Thus, the planes are not parallel and they must intersect in a line l. To find l, we need a point on l and a vector pointing in the direction of l. Setting $z = 0$ in the equations of the planes gives equations of the lines in which the planes intersect. By setting $z = 0$, we find a point that lies on both planes and on the xy plane ($z = 0$).

$$\begin{aligned} x + 2y &= 5 \\ 2x + y &= 7 \end{aligned}$$

After solving this system, we see that $x = 3$ and $y = 1$. We see that $(3, 1, 0)$ is a point on l. Next, we need to find a vector parallel to l. Because l lies in Q and R, it is orthogonal to $\vec{n}_Q$ and $\vec{n}_R$. The cross product is a vector parallel to l, which in this case is $\langle -3, 3, -3 \rangle$.

Therefore, any point on the line $\vec{r} = \langle 3, 1, 0 \rangle + t\langle -3, 3, -3 \rangle$ satisfies the equations of both planes. In other words, any point (x, y, z) satisfying $x = 3 - 3t$, $y = 1 + 3t$, $z = -3t$ also satisfies both plane equations.