16.1 Double Integrals over Rectangular Regions

Arnav Patil

University of Toronto

1 Volumes of Solids

We assume $z = f(x, y)$ is a non-negative continuous function on a rectangular region $R = \{(x, y) : a \leq x \leq b, c \leq y \leq d\}$. A partition of R is formed by dividing R into n rectangular regions using lines running parallel to the x and y axes. Essentially, we have $\Delta A_k = \Delta x_k \Delta y_k$.

Therefore, the volume of the k-th box:

$$f(x_k^*, y_k^*)\Delta A_k = f(x_k^*, y_k^*)\Delta x_k \Delta y_k$$

The sum of the volumes of the n boxes gives an approximation to the volume of the solid:

$$V \approx \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k$$

1.0.1 Definition – Double Integrals

A function f defined on a rectangular region R in the xy-place is integrable if $\lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k$ exists for all partitions of R. The limit is the double integral of f over R, denoted as:

$$\int\int f(x, y)dA = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k$$

2 Iterated Integrals

$$V = \int\int f(x, y)dA = \int\int (6 - 2x - y)dA$$
$$V = \int_0^1 A(x)dx$$
$$A(x) = \int_0^2 (6 - 2x - y)dy$$
$$V = \int_0^1 A(x)dx = \int_0^1 (\int_0^2 (6 - 2x - y)dy)dx$$

The expression that appears on the right side of this equation is called an **iterated integral**.

2.0.1 Theorem 16.1 – (Fubini) Double Integrals over Rectangular Regions

Let f be continuous on a rectangular region. The double integral of f over R may be evaluated by either of two integrated integrals:

$$\int\int f(x,y)dA = \int_c^d \int_a^b f(x,y)dxdy = \int_a^b \int_c^d f(x,y)dydx$$

The importance of Fubini's Theorem is that it says double integrals may be evaluated using iterated integrals, and also that the order of integration doesn't matter.

3 Average Value

Recall the average value of the integrable function f over the interval $[a,b]$ is:

$$\bar{g} = \frac{1}{b-a}\int_a^b f(x)dx$$

To find the average value of an integrable function f over a region R, we integrate f over R and divide the result by the "size" of R, which is the area of R in the two-variable case.

3.0.1 Definition – Average Value of a Function Over a Plane Region

The average value of an integrable function over f over a region R is:

$$\bar{f} = \frac{1}{\text{area of } R}\int\int f(x,y)dA$$