3️⃣

The Fourier Series

3 Fourier Series Representation of Periodic Signals

3.3 Fourier Series Representation of Continuous-Time Periodic Signals

3.3.1 Linear Combinations of Harmonically Related Complex Exponentials

- For a signal $\phi_k(t) = e^{j\omega_0 t}$ with a fundamental frequency ω_0, there is a set of **harmonically related** complex exponentials $\phi_k(t) = e^{jk\omega_0 t}$ for all integers k
    - The terms for $k = 1$ and $k = -1$ are called the **fundamental components** or **first harmonic components**
    - Components after $k = |1|$ are known as the Nth components

3.3.2 Determination of the Fourier Series Representation of a Continuous-Time Periodic Signal

Given

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

if we multiply both sides by $e^{-jn\omega_0 t}$ then integrate from 0 to T

$$x(t)e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t}$$

$$\int_0^T x(t)e^{-jn\omega_0 t}\,dt = \sum_{k=-\infty}^{\infty} a_k \left[\int_0^T e^{jk\omega_0 t} e^{-jn\omega_0 t}\,dt\right]$$

From MAT290 we can recall how to evaluate the integral in the brackets. Rewriting using Euler’s formula:

$$\int_0^T e^{j(k-n)\omega_0 t}\,dt = \int_0^T \cos(k-n)\omega_0 t\,dt + j\int_0^T \sin(k-n)\omega_0 t\,dt$$

Thus,

$$\int_0^T e^{j(k-n)\omega_0 t}\,dt = \begin{cases} T, k = n \\ 0, k \neq n \end{cases}$$

and

$$a_n = \frac{1}{T}\int_T x(t)e^{-jn\omega_0 t}\,dt$$

To summarize, if $x(t)$ can be expressed as a linear combination of harmonically related complex exponentials (or, if it has a Fourier series representation), then the coefficients are given by the equation above. This pair of equations defines the Fourier series of a periodic continuous-time signal.

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t} \tag{1}$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}\,dt \tag{2}$$

3.4 Convergence of the Fourier Series

While Fourier maintained that any periodic signal can be represented by a Fourier series, this is not actually true. However, a Fourier series exists for all functions that we are concerned with in this course.

Let $x_N(t)$ be a finite series of the form

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$

Let $e_N(t)$ denote the approximation error, represented by

$$e_N(t) = x(t) - x_N(t)$$

We need to specify a quantitative measure of how good any particular approximation is. We can use the criterion of the energy in the error over one period

$$E_N = \int_T |e_N(t)|^2\,dt$$

A set of conditions known as the **Dirichlet conditions** guarantees that $x(t)$ equals its Fourier series representation, except where $x(t)$ is discontinuous. These conditions will apply to all of the functions studied in this course.

1. Over any period, $x(t)$ must be absolutely integrable, that is

$$\int_T |x(t)|\, dt < \infty$$

2. In any finite interval of time, $x(t)$ is of bounded variation. There are no more than a finite number of maxima and minima during any single period of the signal.
3. In any finite interval of time, there are only a finite number of discontinuities. Furthermore, each of these discontinuities is finite.

3.5 Properties of the Continuous-Time Fourier Series

Suppose that the function $x(t)$ is a periodic signal with period T and fundamental frequency ω_0 and the Fourier series coefficients of $x(t)$ are denoted by a_k.

3.5.1 Linearity

$$z(t) = Ax(t) + By(t) \text{ becomes } c_k = Aa_k + Bb_k$$

3.5.2 Time Shifting

The Fourier series coefficients b_k of the resulting signal $y(t) = x(t - t_0)$ may be expressed as

$$b_k = \frac{1}{T} \int_T x(t - t_0) e^{-jk\omega_0 t}\, dt$$

If we say that

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k$$

then we have

$$x(t - t_0) \xleftrightarrow{\mathcal{FS}} e^{-jk\omega_0 t}\, a_k$$

The implication of this property is that when a periodic signal is shifted in time, the magnitude of its Fourier series coefficients does not change.

3.5.3 Time Reversal

If we say that

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k$$

then we have

$$x(-t) \xleftrightarrow{\mathcal{FS}} a_{-k}$$

3.5.4 Time Scaling

$$x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(\alpha\omega_0)t}$$

3.5.5 Multiplication

$$x(t)y(t) \xleftrightarrow{\mathcal{FS}} h_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

3.5.6 Conjugation and Conjugate Symmetry

If we take the complex conjugate of a periodic signal $x(t)$, then we apply both complex conjugation and time reversal on the Fourier coefficients.

$$x^*(t) \xleftrightarrow{\mathcal{FS}} a^*_{-k}$$

3.5.7 Parseval’s Relation for Continuous-Time Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 \, dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

3.6 Fourier Series Representation of Discrete-Time Periodic Signals

3.6.1 Linear Combinations of Harmonically Related Complex Exponentials

A discrete-time signal is periodic if

$$x[n] = x[n+N]$$

The set of all discrete-time complex exponential signals periodic with N is given by

$$\phi_k[n] = e^{jk\omega_0 n} = e^{jk(2\pi/N)n}, k = 0, \pm 1, \pm 2, \ldots$$

The summation as k varies of a range of successive N integers is

$$x[n] = \sum_{k=\langle N\rangle} a_k \phi_k[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$$

Determination of the Fourier Series Representation of a Periodic Signal

$$\sum_{n=\langle N\rangle} a_k e^{jk(2\pi/N)n} = \begin{cases} N, k = 0, \pm N, \pm 2N, \dots \\ 0, \text{ otherwise} \end{cases}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N\rangle} x[n] e^{-jk(2\pi/N)n}$$

3.7 Properties of Discrete-Time Fourier Series

$$x[n] \xleftrightarrow{\mathcal{FS}} a_k$$

3.7.1 Multiplication

$$x[n]y[n] \xleftrightarrow{\mathcal{FS}} d_k = \sum_{l=\langle N\rangle} a_l b_{k-l}$$

3.7.2 First Difference

$$x[n] - x[n-1] \xleftrightarrow{\mathcal{FS}} (1 - e^{-jk(2\pi/N)}) a_k$$

3.7.3 Parseval’s Relation for Discrete-Time Periodic Signals

$$\frac{1}{N} \sum_{n=\langle N\rangle} |x[n]|^2$$