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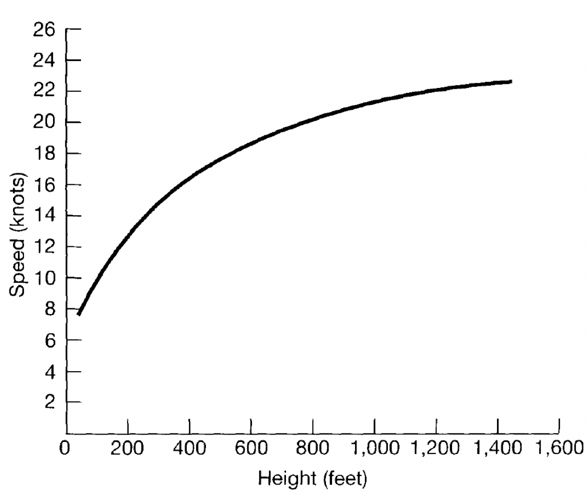
Fundamentals of Continuous- and Discrete-Time Signals

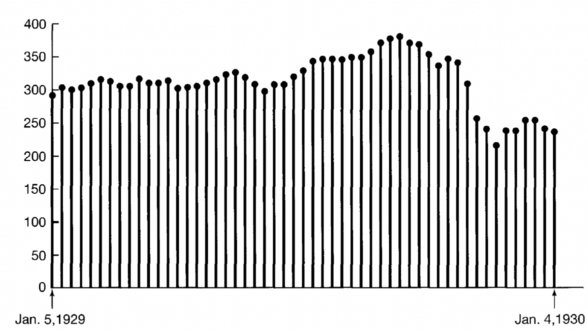
1 Signals and Systems

1.1 Continuous-Time and Discrete-Time Signals

1.1.1 Examples and Mathematical Representation

- Signals can be represented in many ways, but the information in a signal is contained as a pattern of variations.
- Signals are represented mathematically as equations in one or more variables
- We will consider two basic types of signals
  - One where the independent variable is continuous — continuous-time signals, and
  - One where the independent variable is discrete — discrete-time signals





- Conventionally, we use the variable t to represent continuous independent variables, and n to represent discrete variables
  - $x(t)$
  - $x[n]$

- To emphasize the fact that discrete-time signals are only defined for integer values, we sometimes call them discrete-time *sequences*
- One can derive a discrete-time signal by sampling a continuous-time signal at regular intervals

1.1.2 Signal Energy and Power

If $v(t)$ and $i(t)$ are the voltage and current across a resistor, then the instantaneous power is given by:

$$p(t) = v(t)i(t) = \frac{1}{R}v^2(t)$$

The total energy dissipated by over the interval $t_1 < t < t_2$ is

$$\frac{1}{R}\int_{t_1}^{t_2} v^2(t)\,dt$$

and the average power over this interval is

$$\frac{1}{t_2 - t_1}\frac{1}{R}\int_{t_1}^{t_2} v^2(t)\,dt$$

We can define the time-averaged power over an infinite interval as

$$P_\infty = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|x(t)|^2\,dt = \lim_{T\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{+N}|x[n]|^2$$

1.2 Transformations of the Independent Variable

1.2.1 Examples of Transformations of the Independent Variable

- In this section we will focus on a number of elementary signal transformations that modify the independent variable (time axis)
- This will allow us to discuss important basic properties of signals and systems
- The simplest transformation is the **time shift**, represented mathematically as $x(t - t_0)$ or $x[n - n_0]$. It represents a translation along the independent axis
- Another transformation is **time reversal**, which is represented by $x(-t)$ or $x[-n]$ and can be obtained by reflecting the signal along the dependent axis
- Then there is **time scaling**, which is represented by $x[\alpha n]$ or $x(\alpha t)$.

1.2.2 Periodic Signals

- A signal is **periodic** if there is a positive value T such that $x(t) = x(t + T)$ for all t.
    - We call x periodic with period T
    - The smallest T_0 for which the above identity holds is called the **fundamental period** T_0.

1.2.3 Even and Odd Signals

A signal is called **odd** if

$$x(-t) = x(t)$$
$$x[-n] = x[n]$$

and **even** if

$$x(-t) = -x(t)$$
$$x[-n] = -x[n]$$

- Every signal can be decomposed into even and odd component signals

1.3 Exponential and Sinusoidal Signals

1.3.1 Continuous-Time Complex Exponential and Sinusoidal Signals

The continuous-time complex exponential signal is of the form

$$x(t) = Ce^{at}$$

If C and a are real, then the signal is a real exponential. If a is purely imaginary, then consider

$$x(t) = e^{j\omega_0 t}$$

For $x(t)$ to be periodic, we must have a period T such that

$$e^{j\omega_0(t+T)} = e^{j\omega_0 t}e^{j\omega_0 T}$$

where $e^{j\omega_0 T} = 1$. Then, the smallest possible value for which $x(t)$ is still periodic is

$$T_0 = \frac{2\pi}{|\omega_0|}$$

Furthermore, by using Euler's formula, we can write

$$e^{j\omega_0 t} = \cos(\omega_0 t) + j\sin(\omega_0 t)$$

The inverse of the fundamental period T_0 is called the **fundamental frequency** ω_0.

1.3.3 Periodicity Properties of Discrete-Time Complex Exponentials

$e^{j\omega_0 t}$	$e^{j\omega_0 n}$
Distinct signals for distinct values of ω_0	Identical signals for values of ω_0 separated by multiples of 2π
Periodic for any choice of ω_0	Periodic only if $\omega_0 = 2\pi m/N$ for some integers $N > 0$ and m
Fundamental frequency ω_0	Fundamental frequency ω_0/m
Fundamental period $\omega_0 = 0:$ undefined $\omega_0 \neq 0: \frac{2\pi}{\omega_0}$	Fundamental period $\omega_0 = 0:$ undefined $\omega_0 \neq 0: \frac{2\pi m}{\omega_0}$

1.4 The Unit Impulse and Unit Step Functions

1.4.1 The Discrete-Time Unit Impulse and Unit Step Functions

The **unit impulse** or **unit sample** is defined as

$$\delta[m] = \begin{cases} 0, n \neq 0 \\ 1, n = 0 \end{cases}$$

The **unit step** function, another basic discrete-time signal, is defined as

$$u[n] = \begin{cases} 0, n < 0 \\ 1, n \geq 0 \end{cases}$$

We can see that the discrete-time unit impulse is the first difference of the discrete-time unit step functions

$$\delta[n] = u[n] - u[n-1]$$

Conversely, the discrete-time unit step is the running sum of the unit sample

$$u[n] = \sum_{m=-\infty}^{n} \delta[m]$$

1.4.2 The Continuous-Time Unit Step and Unit Impulse Functions

The unit step in continuous-time is defined similarly to its discrete-time counterpart. One difference is that the unit step function in continuous-time is undefined at 0

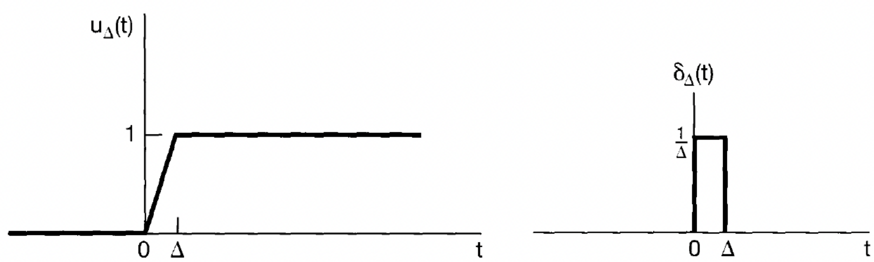
$$u(t) = \begin{cases} 0, t < 0 \\ 1, t > 0 \end{cases}$$

The unit step is the **running integral** of the unit impulse

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau$$

Conversely, the unit impulse function is considered the first derivative of the unit step

$$\delta(t) = \frac{du(t)}{dt}$$



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2 Linear Time-Invariant Systems

2.5 Singularity Functions

- We introduce a set of related signals called the **singularity functions** to learn more about the idealized unit impulse function in continuous-time
- These signals are defined in terms of how they behave under convolution with other signals

2.5.1 The Unit Impulse as an Idealized Short Impulse

By the sifting property, the unit impulse can be seen as the impulse response of the identity system. We have for any $x(t)$,

$$x(t) = x(t) * \delta(t)$$

2.5.2 Defining the Unit Impulse Through Convolution

All properties of the unit impulse can be obtained through the operational definition given above. If we let $x(t) = 1$ for all t, then

$$1 = x(t) = \delta(t) * x(t) = \int_{-\infty}^{+\infty} \delta(\tau) x(t-\tau)\, d\tau = \int_{-\infty}^{+\infty} \delta(\tau)\, d\tau$$

With some more manipulation, we can see that

$$f(t)\delta(t) = f(0)\delta(t)$$

2.5.3 Unit Doublets and Other Singularity Functions

Consider the LTI system for which the output is the derivative of the input $y(t) = \frac{dx(t)}{dt}$

The unit impulse response of this system is the derivative of the unit impulse, called the unit doublet $u_1(t)$

$$\frac{dx(t)}{dt} = x(t) * u_1(t)$$

Taking the derivative again,

$$\begin{aligned}\frac{d^2x(t)}{dt^2} &= \frac{d}{dt}\frac{dx(t)}{dt} \\ &= x(t) * u_1(t) * u_1(t) \\ &= x(t) * u_2(t)\end{aligned}$$

We see that in general, we have

$$u_k(t) = u_1(t) * \cdots * u_1(t) \text{ (k times)}$$

Sometimes we use alternative notations for $\delta(t)$ and $u(t)$

$$\begin{aligned}\delta(t) &= u_0(t) \\ u(t) &= u_{-1}(t)\end{aligned}$$
