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Maxwell’s Equations for Time-Varying Fields

6.1 Faraday’s Law

- Faraday hypothesized that if a current produces a magnetic field, then the converse should also be true:
    - A magnetic field should produce a current in a wire

$$\Phi = \int_S \vec{B} \cdot d\vec{s}$$

- current is only induced when the magnetic flux changes, and the direction of the current is dependent on whether the flux is increasing or decreasing
- when a galvanometer detects the flow of current through the coil, it means that a voltage has been induced across the galvanometer terminals
    - called the electromotive force V_{emf} (it’s a voltage not a force)

$$V_{emf} = -N\frac{d\Phi}{dt} = -N\frac{d}{dt}\int_S \vec{B} \cdot d\vec{s}$$

- an emf can be generated in a closed conducting loop under any of the following three conditions:
    - a time-varying magnetic field linking a stationary loop, called a **transformer emf**
    - a moving loop with a time-varying area in a static field, called the **motional emf**
    - a moving loop in a time-varying field
- total emf is given by:

$$V_{emf} = V_{emf}^{tr} + V_{emf}^{m}$$

6.2 Stationary Loop in a Time-Varying Magnetic Field

$$V_{emf}^{tr} = -N \int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s}$$

- the transformer emf is the voltage difference that would appear across the small opening in terminals

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

6.3 The Ideal Transformer

$$\frac{V_1}{V_2} = \frac{N_1}{N_2} \text{ and } \frac{I_1}{I_2} = \frac{N_2}{N_1}$$

6.4 Moving Conductor in a Static Magnetic Field

- the field generated by the motion of the charged particle is called a motional electric field

$$\vec{E}_m = \frac{\vec{F}_m}{q} = \vec{u} \times \vec{B}$$

- in general, if any segment of a closed circuit with contour C moves with a velocity $\vec{u}$ across a static magnetic field $\vec{B}$ then the induced motional emf is given by:

$$V_{emf}^{m} = \oint_C (\vec{u} \times \vec{B}) \cdot d\vec{l}$$

6.5 The Electromagnetic Generator

- the magnetic field is

$$\vec{B} = \hat{z} B_0$$

- as the loop rotates with an angular velocity ω about its own axis, the segments move with:

$$\vec{u} = \hat{n} \omega \frac{w}{2}$$

$$\hat{n} \times \hat{z} = \hat{x} \sin \alpha$$

- we then obtain the result

$$V_{emf}^{m} = wl\omega B_0 \sin\alpha = A\omega B_0 \sin\alpha$$

- the angle α is realted to ω by

$$\alpha = \omega t + C_0$$

6.6 Moving Conductor in a Time-Varying Magnetic Field

$$V_{emf} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int_S \vec{B} \cdot d\vec{s}$$

6.7 Displacement Current

- Ampere's Law in differential form is given by

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

- we integrate both sides over an arbitrary open surface S and contour C.
  - the surface integral of $\vec{J}$ is the conduction current I_c flowing through S, and
  - the surface integral of $\nabla \times \vec{H}$ becomes a line interal of $\vec{H}$ over C

$$\oint_C \vec{H} \cdot d\vec{l} = I_c + \int_S \frac{\partial \vec{D}}{\partial t} \cdot d\vec{l}$$

- the second term on the right has units of amperes obviously, and is proportional to the time derivative of the electric flux density $\vec{D}$
  - this is called the **displacement current** I_d

$$I_d = \int_S = \vec{J}_d \cdot d\vec{s} = \int_S \frac{\partial \vec{D}}{\partial t} \cdot d\vec{s}$$

- from the above two equations, we can say that

$$\oint_C \vec{H} \cdot d\vec{l} = I_c + I_d = I$$

6.9 Charge-Current Continuity Relation

- we define I as the net current flowing across S out of V
  - thus, I is the negative rate of change of Q

$$I = -\frac{dQ}{dt} = -ddt \int_V \rho_V \; dV$$

- the current I is also defined as the outward flux of the current density $\vec{J}$ through the surface S

$$\oint_S \vec{J} \cdot d\vec{s} = -\frac{d}{dt} \int_V \rho_V \; dV$$

- apply the divergence theorem and convert the surface integral of $\vec{J}$ into a volume integral of its divergence

$$\oint_S \vec{J} \cdot d\vec{s} = \int_V \nabla \cdot \vec{J} \, dV = -\frac{d}{dt} \int_V \rho_V \; dV$$

- we can move the time derivative inside the integral and express as a partial derivative of ρ_V, then drop both volume integrals

$$\nabla \cdot \vec{J} = -\frac{\partial \rho_V}{\partial t}$$

- this is known as the charge-current continuity relation, or the **charge continuity equation**.
- another expression for Kirchhoff’s current law:

$$\oint_S \vec{J} \cdot d\vec{s} = 0$$