5️⃣

Magnetostatics

5.1 Magnetic Forces and Torques

- We defined electric field $\vec{E}$ at point in space as an electric force per unit charge acting on a test charge
- Now, we define the magnetic flux density $\vec{B}$ at a point in space in terms of magnetic force that acts on a moving test charge

$$\vec{F}_m = q\vec{u} \times \vec{B}$$

- The strength of $\vec{B}$ is measured in teslas. For a positively charged test particle, the direction of $\vec{F}$ is that of the cross product containing $\vec{u}$ and $\vec{B}$ governed by the right hand rule.
- The strength of $\vec{F}_m$ is given by

$$F_m = quB\sin\theta$$

5.1.1 Magnetic Force on a Current-Carrying Conductor

- For a closed circuit of contour C carrying a current I, the magnetic force is

$$\vec{F}_m = I\oint_C d\vec{l} \times \vec{B}$$

5.2 The Biot-Savart Law

- Magnetic flux and magnetic field are related by:

$$\vec{B} = \mu\vec{H}$$

- The Biot-Savart law states that the differential magnetic field $d\vec{H}$ generated by a steady current I flowing through a differential length $d\vec{l}$ is:

$$d\vec{H} = \frac{I}{4\pi}\frac{dl \times \hat{R}}{R^2}$$

- The magnetic field is orthogonal to the plane containing the direction of the current element and the distance vector

5.2. Magnetic Field Due to Surface and Volume Current Distributions

$$I\,d\vec{l} = \vec{J}_S\,ds = \vec{J}\,dV$$

- For an infinitely long wire:

$$\vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\phi}$$

5.3 Maxwell's Magnetostatic Equations

5.3.1 Gauss' Law for Magnetism

- Just as we had Gauss' Law for Electricity, we can find a magnetic counterpart, the Gauss' Law for Magnetism

$$\nabla \cdot \vec{B} = 0 \Longleftrightarrow \oint_S \vec{B} \cdot d\vec{s} = 0$$

- Magnetic field lines, in contrast to electric field lines, always form continuous closed loops from North to South
- Gauss' Law is constrained to a choice of a Gaussian surface enclosing the charges, similarly, Ampere's Law is constrained to a choice of an Amperian loop encircling the current

5.4 Vector Magnetic Potential

- We introduce a quantity called the vector magnetic potential A:

$$\vec{B} = \nabla \times \vec{A}$$

5.4.1 Vector Poisson's Equation

Given the equations:

$$\nabla(\nabla \cdot A) - \nabla^2 A = \mu J$$

and

$$\nabla \cdot A = 0$$

gives us the vector Poisson's equation:

$$\nabla^2 A = -\mu J$$

Since we can express this equation for each of the coordinate components of A and J, we can write the vector equation:

$$\vec{A} = \frac{\mu}{4\pi} \int_V \frac{\vec{J}}{R} \, dV$$

5.4.2 Magnetic Flux

- The magnetic flux Φ linking a surface S is defined as the total magnetic flux density passing through it:

$$\Phi = \int_S \vec{B} \cdot d\vec{s}$$

- In free space, we modify $\vec{B} = \mu_0 \vec{H}$ to:

$$\vec{B} = \mu_0 \vec{H} + \mu_0 \vec{M} = \mu_0 (\vec{H} + \vec{M})$$

5.5 Magnetic Properties of Materials

- we can classify materials as diamagnetic, paramagnetic, or ferromagnetic

5.5.2 Magnetic Permeability

- In free space, $\vec{B} = \mu_o \vec{H}$ is modified to:

$$\vec{B} = \mu_0 \vec{H} + \mu_0 \vec{M}$$

where the magnetization vector $\vec{M}$ is defined as the vector sum of the magnetic dipole moments of toms contained in a unit volume of the material

- In mose magnetic materials, we have $\vec{M} = \chi_m \vec{H}$ where χ is the magnetic susceptibility of the material

$$\mu = \mu_0 (1 + \chi_m)$$

5.5.3 Magnetic Hysteresis of Ferromagnetic Materials

- Discusses magnetic domain theory
- Hysteresis means to “lab behind”
  - the existence of a hysteresis loop implies that the magnetization process depends not only on the magnetic field $\vec{H}$ but also on the magnetic history of the material

5.6 Magnetic Boundary Conditions

- By analogy of Gauss' Law, we find that

$$\oint_S \vec{B} \cdot d\vec{s} \Longrightarrow B_{1_n} = B_{2_n}$$

- we can further represent that as

$$\mu_1 H_{1_n} = \mu_2 H_{1_2}$$

$$\hat{n} \times (\vec{H}_1 - \vec{H}_2) = \vec{J}_s$$

- surface currents can exist only on the surfaces of perfect conductors and superconductors. hence, at the interface between media with finite conductivities, we have $\vec{J}_s = 0$ and

$$H_{1t} = H_{2t}$$

- To summarize, we may say that boundary conditions require:
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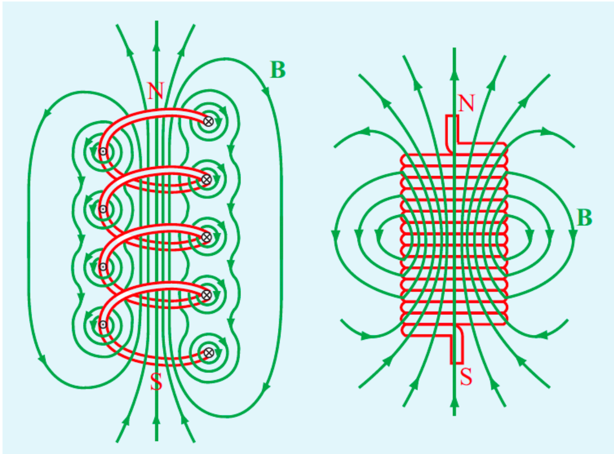
$$\vec{B}_{1n} = \vec{B}_{2n} \text{ and } \frac{\vec{B}_{1t}}{\mu_1} = \frac{\vec{B}_{2t}}{\mu_2}$$

5.7 Inductance

- a typical inductor consists of multiple turns of wire helically coiled around a cylindrical core, such a structure is called a solenoid

5.7.1 Magnetic Field in a Solenoid

$$\vec{H} = \hat{z} \frac{Ia^2}{2(a^2 + z^2)^{3/2}}$$



(a) Loosely wound solenoid
(b) Tightly wound solenoid

$$\vec{B} = \frac{\mu N I}{l}\hat{z}$$

- self-inductance is the magnetic flux linkage of a coil or circuit with itself
- mutual inductance involves the magnetic flux linkage in a circuit due to the magnetic field generated by a current in another one

5.7.2 Self-Inductance of a Solenoid

$$\Phi = \int_S \vec{B} \cdot d\vec{s}$$

- Magnetic flux linkage Λ is defined as the total magnetic flux linking a given circuit or conducting structure

$$\Lambda = N\Phi = \mu \frac{N^2}{l} I S$$

- The self-inductance of any conducting structure is defined as the ratio of the magnetic flux linkage Λ to the current I flowing through the structure

$$L = \frac{\Lambda}{I}$$

5.7.3 Self-Inductance of Other Conductors

- for a two-conductor configuration for either two parallel wires or a coaxial wire, the inductance is given by:

$$L = \frac{\Lambda}{I} = \frac{\Phi}{I} = \frac{1}{I}\int_S \vec{B} \cdot d\vec{s}$$

5.7.4 Mutual Inductance

$$L_{12} = \frac{\Lambda_{12}}{I_1} = \frac{N_{12}}{I_1} \int_{S_2} \vec{B}_1 \cdot d\vec{s}$$

5.8 Magnetic Energy

$$W_m = \frac{1}{2} L i^2$$

- this is the magnetic energy stored in the inductor
- we can also define the magnetic energy density

$$w_m = \frac{W_m}{V} = \frac{1}{2} \mu H^2$$

- for any volume V containing a material with permeability μ the total magnetic energy stored in a magnetic field is

$$W_m = \frac{1}{2} \int_V \mu H^2 \, dV$$