Term Test II Theorems

Theorem 3.9

If x is a periodic DTS with a fundamental period N_0, then the discrete-time Fourier series of x is the sum

$$x[n] \sum_{k=0}^{N_0-1} \alpha_k e^{jk\omega_0 n}$$

where the Fourier coefficients are given by

$$\alpha_k = \frac{1}{N_0} \sum_{k=0}^{N_0-1} x[l] e^{-jk\omega_0 l}$$

Comparison of CTFS and DTFS

CTFS	DTFS
represents periodic $x(t)$ as an infinite discrete sum of CT complex exponentials	represents periodic $x[n]$ as a finite discrete sum of DT complex exponentials
α_k captures the amount of harmonic $k\omega_0$ contained in x	α_k captures the amount of harmonic $k\omega_0$ contained in x
In general, sequence of coefficients is aperiodic	Sequence of coefficients is always N_0 periodic

Definition 4.1

The continuous-time Fourier transform (CTFT) of a CT signal is a complex-valued signal $X : \mathbf{R} \to \mathbf{C}$ defined pointwise by

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$$

Definition 4.2

The inverse continuous-time Fourier transform is defined pointwise by

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t}\, d\omega$$

General principle: if x is very concentrated in the time domain, the spectrum will be spread out in the frequency domain.

Theorem 4.1

If x has finite action then the CTFT is well-defined and satisfies $\lim_{\omega \to \infty} |X(j\omega) = 0$

Theorem 4.2

If x has finite energy then:

1. X exists and has finite energy
2. The signal and its spectrum satisfy Parseval's Relation

Definition 4.3

The discrete-time Fourier transform is defined pointwise as:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Definition 4.4

The inverse discrete-time Fourier transform is defined pointwise as:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\, d\omega$$

Proposition 4.1

If x is a CTS with CTFT X, then the iCTFT of X recovers the original signal x.

Definition 5.1

A CTS x is bandlimited with $B > 0$ if its CTFT spectrum satisfies $X(j\omega) = 0$ for all $|\frac{\omega}{2\pi}| \geq B$

Theorem 5.1 (Nyquist-Shannon Theorem)

Let x be a bandlimited CTS with bandwidth B. If x is sampled with sampling period T_s with $T_s \leq \frac{1}{2B}$, then x can be exactly recovered from the samples.

The minimum sampling frequency ω_s which satisfies this is $4\pi B$ and is called the Nyquist frequency.

Definition 6.1

A CT system is linear for any two input signals and two constants it holds that:

$$T\{\alpha x + \hat{\alpha}\hat{x}\} = \alpha T\{x\} + \hat{\alpha}T\{\hat{x}\}$$

Definition 6.2

A CT system is time-invariant if for any input x with output $y = T\{x\}$ it holds that $y_\tau = T\{x_\tau\}$ for all time shifts $\tau \in \mathbf{R}$

Defintion 6.3

A CT system is causal for all $t \in \mathbf{R}$, the output value $y(t)$ depends only on past and present input values

Definition 6.5

A CT system is memoryless if for all times the output value depends only on the input value at that same time.

Definition 6.6

A CT system T is invertible if there exists another CT system T_{inv} such that for all inputs

$$T_{inv}\{T\{x\}\} = x$$

Definition 6.7

A CT linear system is bounded-input-bounded-output (BIBO) stable if there is a constant $K \geq 0$ such that for all bounded inputs

$$||y||_\infty \leq K||x||_\infty$$

Definition 6.8

The impulse response h of a CT LTI system T is the response to a unit CT impulse applied at $t = 0$

Proposition 6.1

1. If v and w are right-sided then $v * w$ exists and is also right-sided
2. If v or w have finite duration, then $v * w$ exists. If both have finite duration, then $v * w$ also has finite duration of $len(v) + len(w)$
3. If v, w have finite action then $v * w$ also has finite action. Moreover, $||v * w||_1 \leq ||v||_1 \cdot ||w||_1$
4. If v has finite action and w has finite energy, then $v * w$ exists and has finite energy. Moreover, $||v * w||_2 \leq ||v||_1 \cdot ||w||_2$
5. If v has finite action and w and finite amplitude, then $v * w$ exists and has finite amplitude. Moreover, $||v * w||_\infty \leq ||v||_1 \cdot ||w||_\infty$

Proposition 6.1

A CT LTI system T with impulse response $h = T\{\delta\}$ is:

1. Causal iff $h(t) = 0$ for all $t < 0$
2. Memoryless iff $h(t) = \alpha\delta(t)$ for some complex α
3. BIBO stable if h has finite action, in which case $||y||_\infty = ||h||_1 \cdot ||x||_\infty$

Theorem 6.2

If x is right-sided from time 0, then $y = T\{x\}$ is right-sided from 0 and

$$y(t) = \int_{-\infty}^{\infty} h(t - \tau)x(\tau)\ d\tau$$

Lemma 6.1

Suppose that T is an LTI system with impulse response h. Then T is invertible iff there exists another impulse response h_{inv} such that $h * h_{inv} = \delta$

Series Response of LTI Systems

$$y = (h_2 * h_1) * x$$

Parallel Response of LTI Systems

$$y = (h_1 + h_2) * x$$

Negative Feedback of LTI Systems

$$h = (\delta + h_1 * h_2)^{-1} * h_1$$

Theorem 6.3

For each right-sided input the LICC-ODE $Q(D)y(t) = P(D)x(t)$ possesses exactly one right-sided solution $y(t)$ and therefore defines a system . Moreover:

- the system T is linear, time-invariant, and causal
- the system T is BIBO stable iff $m \leq n$ and all roots of

$$Q(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$$