4️⃣

Electrostatics

4.1 Maxwell's Equations

- Modern electromagnetic theory is based on four fundamental relations known as **Maxwell's equations**

$$\nabla \cdot \mathbf{D} = \rho_v$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

- $\mathbf{E}$ and $\mathbf{D}$ are the electric field intensity and flux density
    - Correlated by $\mathbf{D} = \varepsilon \mathbf{E}$ where ε is the **electrical permittivity**
- $\mathbf{H}$ and $\mathbf{B}$ are the magnetic field intensity and flux density
    - Correlated by $\mathbf{B} = \mu \mathbf{H}$ where μ is the **magnetic permeability**
- James Clerk Maxwell published these equations in 1873 and established the **first unified theory of electricity and magnetism**

> 💡 Under **static** conditions, all functions of time go to zero

Electrostatics

$$\nabla \cdot \mathbf{D} = \rho_v$$
$$\nabla \times \mathbf{E} = 0$$

Magnetostatics

$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{H} = \mathbf{J}$$

- We can study electricity and magnetism as separate phenomena so long as distributions of charge and current flow stay constant

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4.2 Charge and Current Distributions

Charge Densities

- Volume charge density ρ_v

$$\rho_v = \lim_{\Delta v \to 0} \frac{\Delta q}{\Delta V} = \frac{dq}{dV} \quad (C/m^3)$$

- The total charge contained in a volume V is

$$Q = \int_v \rho_v \, dV \quad (C)$$

- The surface charge density ρ_s is given by:

$$\rho_s = \lim_{\Delta s \to 0} \frac{\Delta q}{\Delta s} = \frac{dq}{ds} \quad (C/m^2)$$

- Line charge density ρ_l is given by:

$$\rho_l = \lim_{\Delta l \to 0} \frac{\Delta q}{\Delta l} = \frac{dq}{dl} \quad (C/m)$$

Current Density

- Let $\mathbf{u}$ be the velocity at which charges move in a tube. Then, the current density is given by:

$$\mathbf{J} = \rho_v \mathbf{u} \quad (A/m^2)$$

- Then, the total current flowing through a surface is

$$I = \int_S \mathbf{J} \cdot d\mathbf{s} \quad (A)$$

💡 When a current is generated by actual movement of charged matter, it is called **convection current**, and $\mathbf{J}$ is called a **convectional current density**.

Otherwise, if the current is generated by movement of charged particles relative to the host material, we call it **conduction current**.

4.3 Coulomb's Law

- Coulomb's Law was first introduced for electrical charges in air, and was later generalized to other media
- Coulomb's Law implies that:
    - An isolated charge q induces an electric field $\mathbf{E}$ at every point in space, where $\mathbf{E}$ is given by

$$\mathbf{E} = \frac{q}{4\pi\varepsilon R^2}\hat{R} \quad (V/m)$$

    - In the presence of an electric field $\mathbf{E}$ at any given point in space, the force acting on a small, positive test charge is

$$\mathbf{F} = q\mathbf{E}$$

Electric Field Due to Multiple Point Charges

- The electric field at any given point is the vector sum of the field caused by all point charges

Electric Field Due to Charge Distribution

- Volume distrubtion

$$\mathbf{E} = \frac{1}{4\pi\varepsilon}\int_V \frac{\rho_v}{R^2}\, dV$$

- Surface distribution

$$\mathbf{E} = \frac{1}{4\pi\varepsilon}\int_S \frac{\rho_s}{R^2}\, dS$$

- Line distribution

$$\mathbf{E} = \frac{1}{4\pi\varepsilon}\int_l \frac{\rho_l}{R^2}\, dl$$

- For an infinite sheet of charge we have

$$\mathbf{E} = \frac{\rho_s}{2\varepsilon_0}$$

4.4 Gauss' Law

- We begin by restating the differential form of Gauss' Law: $\nabla \cdot \mathbf{D} = \rho_v$

$$\int_v \nabla \cdot \mathbf{D}\, dV = \oint_S \mathbf{D} \cdot d\mathbf{s} = Q$$

- Maxwell’s equations incorporate Gauss’ law in themselves
  - For a simple case such as an isolated point charge, we can use Coulomb’s law
  - For more complex systems, we can still use Coulomb’s law, but Gauss’ law is much easier to apply
    - A shortcoming is that it can only be applied to symmetrical charge distributions

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4.5 Electric Scalar Potential

- Operation of an electric circuit usually described in terms of currents flowing through branches and voltage at nodes.
- Voltage difference V b/w two nodes represents the amount of work or **potential energy** required to move a unit charge from one terminal to the other.
- Subject of this section is relationship between $\vec{E}$ and V.

Electric Potential as a Function of Electric Field

- When a charged particle is in an electric field there is

$$\vec{F}_{ext} = -q\vec{E}$$

- Work done in moving any object a vector differential distance $d\vec{l}$ while exerting a force $\vec{F}_{ext}$ is:

$$dW = \vec{F}_{ext} \cdot d\vec{l} = -q\vec{E} \cdot d\vec{l}$$

- Differential electric potential energy dW per unit charge is called the differential electric potential dV. That is,

$$dV = \frac{dW}{q} = -\vec{E} \cdot d\vec{l}$$

- Units are (J/C) or (V)

$$V_{21} = V_2 - V_1 = -\int_{P_1}^{P_2} \vec{E} \cdot d\vec{l}$$

The voltage difference between two nodes in an electric circuit has the same value regardless of which path in the circuit we follow in between the nodes.

Moreover, Kirchoff’s voltage law states the net voltage drop around a closed loop is zero.

💡 The line integral of the electrostatic field $\vec{E}$ around any closed contour C is 0.

- Conservative property of the electrostatic field can be deduced from Maxwell's second equation. If $\partial/\partial t = 0$, then

$$\nabla \times \vec{E} = 0$$

- If we integrate this over an open surface S and apply Stokes' Theorem to convert the surface integral into a line integral, we obtain

$$\int_S (\nabla \times \vec{E}) \cdot d\vec{s} = \oint_C \vec{E} \cdot d\vec{l} = 0$$

Electric Potential Due to Point Charges

Electric field due to a point charge q is given by:

$$\vec{E} = \hat{R}\frac{q}{4\pi\varepsilon R^2}$$

Electric Potential Due to Continuous Distributions

Volume distribution

$$V = \frac{1}{4\pi\varepsilon}\int_{v'} \frac{\rho_v}{R'}\, dv'$$

Charge distribution

$$V = \frac{1}{4\pi\varepsilon}\int_{S'} \frac{\rho_s}{R'}\, ds'$$

Line distribution

$$V = \frac{1}{4\pi\varepsilon}\int_{l'} \frac{\rho_l}{R'}\, dl'$$

Electric Field as a Function of Electric Potential

$$\vec{E} = -\nabla V$$

> 💡 This differential relationship between V and $\vec{E}$ allows us to determine $\vec{E}$ for any charge distribution by first calculating V and then taking the negative gradient of V.

- An electric dipole consists of two point charges, equal magnitude but opposite polarity separated by a distance d.
- The dipole moment is given by $\vec{p} = q\vec{d}$. Then, we have:

$$V = \frac{\vec{p} \cdot \hat{R}}{4\pi\varepsilon_0 R^2}$$

Poisson's Equation

$$\nabla \cdot \vec{E} = \frac{\rho_v}{\varepsilon}$$

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon} \text{ (Poisson's equation)}$$

4.6 Conductors

- A material medium has electromagnetic constitutive parameters:
  - Electrical permittivity ε
  - Magnetic permeability μ
  - Conductivity σ
- **Homogeneous** means the constitutive parameters do not vary by position
- **Isotropic** means the constitutive parameters do not vary from point to point
- Conduction current density is given by:

$$\vec{J} = \sigma \vec{E}$$

- A **perfect dielectric** has $\sigma = 0$ and a **perfect conductor** has $\sigma = \infty$

Drift Velocity

$$\vec{u}_e = -\mu_e \vec{E} \text{ (m/s)}$$

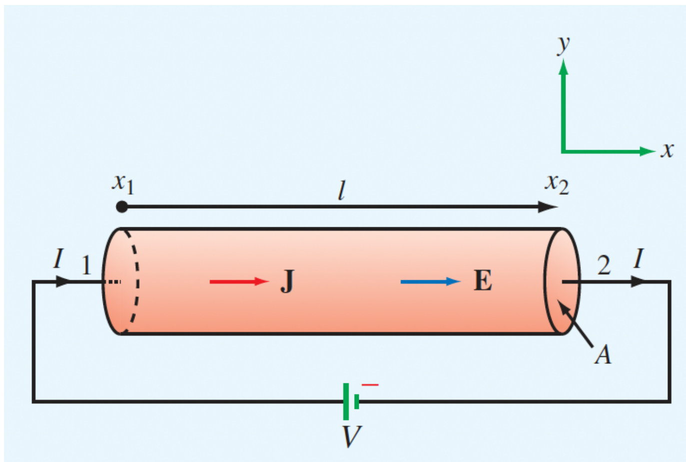
- μ_e is a property called the electron mobility. Similarly, we have hole drift velocity and hole mobility
- The **total conduction current density** is given by:

$$\vec{J} = \vec{J}_e + \vec{J}_h = \rho_{ve}\vec{u}_e + \rho_{vh}\vec{u}_h \text{ (A/m\^{}2)}$$

$$\vec{J} = (-\rho_{ve}\mu_e + \rho_{vh}\mu_h)\vec{E} = \sigma\vec{E}$$

- For a perfect dielectric we would have $\vec{J} = 0$ and $\vec{E} = 0$

Resistance



$$R = \frac{l}{\sigma A}$$

- Reciprocal of R is called G which is conductance, with a unit of Ω^{-1}. For a linear resistor:

$$G = \frac{1}{R} = \frac{\sigma A}{l}$$

- If $\vec{J}$ is in the $\hat{r}$ direction, the inner conductor must be at a higher potential than the outer conductor. The voltage difference is given by:

$$V_{ab} = -\int_b^a \vec{E} \cdot d\vec{l} = -\int_b^a \frac{I}{2\pi\sigma l} \frac{\hat{r} \cdot \hat{r}\, dr}{r} = \frac{I}{2\pi\sigma l} \ln\left(\frac{b}{a}\right)$$

Joule’s Law

- The work expended by the electric field in moving q_e a differential distance Δl_e and moving a q_h a distance Δl_h is:

$$\Delta W = \vec{F}_e \cdot \Delta\vec{l}_e + \vec{F}_h \cdot \Delta\vec{l}_h$$

- Power P is measured in units of watts (W) and is defined as the time rate of change in energy. For a volume V, the total dissipated power is:

$$P = \int_v \vec{E} \cdot \vec{J}\, dV \text{ (Joule's Law)}$$

4.7 Dielectrics

- In a conductor electrons are loosely bound to their atom, whereas in dielectrics the atoms are tightly bound

Polarization Field

In a free space with $\vec{D} = \varepsilon_0 \vec{E}$, the presence of microscopic dipoles in a dielectric material alters that relationship to:

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$

where $\vec{P}$ is the electric polarization field. $\vec{P}$ is directly proportional to $\vec{E}$ and is expressed as

$$\vec{P} = \varepsilon_0 \chi_e \vec{E}$$

where χ_e is the electric susceptibility of the material.

Permittivity of a material ε is given by:

$$\varepsilon = \varepsilon_e (1 + \chi_e)$$

Dielectric Breakdown

The preceding model assumes that the magnitude $\vec{E}$ will not exceed a certain critical value, the dielectric strength $\vec{E}_{ds}$. Beyond this, electrons will detach and accelerate though the material as a conduction current. This is known as dielectric behaviour.

$$V_{br} = E_{ds} d$$

4.8 Electric Boundary Conditions

> 💡 A vector field is spatially continuous if it does not exhibit abrupt changes in either magnitude of direction when expressed as a function of position.

At the boundary of two distinct media, one notices that:

$$(\vec{E}_1 - \vec{E}_2) \cdot \hat{l}_1 = 0$$

I other words, the tangential component of the electric field is continuous across the boundary between any two media.

$$\vec{E}_{1t} = \vec{E}_{2t}$$

Dielectric-Conductor Boundary

If medium 1 is a dielectric and medium 2 is a perfect conductor, then because the electric fields and fluxes vanish in a conductor, it follows that $\vec{E}_2 = \vec{D}_2 = 0$. This implies that the tangential and normal components to the interface are both zero.

Conductor-Conductor Boundary

If medium 1 has permittivity ε_1 and conductivity σ_1, and medium 2 has permittivity ε_2 and conductovity σ_2, then the interface between them holds a surface charge density ρ_s.

$$\vec{E}_{1t} = \vec{E}_{2t} \text{ and } \varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = \rho_s$$

The normal component of $\vec{J}$ has to be continuous across the boundary between two different media under electrostatic conditions.

$$J_{1n}\left(\frac{\varepsilon_1}{\sigma_1} - \frac{\varepsilon_2}{\sigma_2}\right) = \rho_s$$

4.9 Capacitance

- When separated by a dielectric, any two conducting bodies form a capacitor.
- If a DC voltage is connected across the surfaces, the positive and negative source terminals accumulate charges of $+Q$ and $-Q$ respectively.

> 💡 When a conductor has excess charge, it distributes the charge on its surface to maintain a zero electric field everywhere within the conductor.

$$C = \frac{Q}{V} \text{ (C/V or F)}$$

- The tangential component of $\vec{E}$ always vanishes at a conductor's surface, so $\vec{E}$ is always perpendicular. The normal component is then given by:

$$E_n = \hat{n} \cdot \vec{E} = \frac{\rho_s}{\varepsilon}$$

- The charge Q is equal to the integral of ρ_s over the surface S.

$$Q = \int_S \varepsilon \vec{E} \cdot d\vec{S}$$

$$C = \frac{\int_S \varepsilon \vec{E} \cdot d\vec{s}}{-\int_l \vec{E} \cdot d\vec{l}}$$

The value of C obtained for any specific capacitor configuration is always independent of the magnitude of $\vec{E}$.

If the material between the conductors is not a perfect dielectric but has a small conductivity σ, then the general expression for the R resistance is:

$$R = \frac{-\int_l \vec{E} \cdot d\vec{l}}{\int_S \sigma \vec{E} \cdot d\vec{s}}$$

For a uniform conductivity and permittivity, we then obtain

$$RC = \frac{\varepsilon}{\sigma}$$

The voltage difference between the plates is:

$$V = -\int_0^d \vec{E} \cdot d\vec{l} = i \int_0^d (-\hat{z}E) \cdot \hat{z}\, dz = Ed$$

And the capacitance would then be:

$$C = \frac{Q}{V} = \frac{Q}{Ed} = \frac{\varepsilon A}{d}$$

4.10 Electrostatic Potential Energy

- The energy spent in charging a capacitor using a power supply us stored in the dielectric medium in the form of electrostatic potential energy.
- The volage v across a capacitor is related to the charge stored q by

$$v = \frac{q}{C}$$

- The amount of work W required to charge the capacitor can be given by

$$W_e = \frac{1}{2} C V^2$$

where V is the final voltage.

- The **electrostatic energy density** w_e is defined as the electrostatic potential energy W_e per unit volume:

$$w_e = \frac{W_e}{\nu} = \frac{1}{2}\varepsilon E^2$$

- The opposing charged plates are also attracted to each other by an electrical force

$$\vec{F}_e = -\hat{z}F_e$$

where F_e is given by:

$$F_e = \frac{1}{2}\varepsilon\frac{AV^2}{d^2}$$

- We generalize this result for any $d\vec{l}$ along any direction as:

$$\vec{F}_e = -\nabla W_e$$
