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Linear Dynamic Circuits

6 Capacitance and Inductance

6.1 The Capacitor

- Capacitors are dynamic elements involving time variation of an electric field produced by a voltage.
- A uniform electric field $\vec{E}(t)$ exists between the metal plates when a voltage exists across the capacitor.
- When the separation d is small compared to the dimension of the plates, the electric field between the plates is

$$\vec{E}(t) = \frac{q(t)}{\varepsilon A}$$

- where ε is the permittivity of the dielectric, A is the area of the plates, and $q(t)$ is the magnitude of the electric charge on each plate.
- The relationship between the electric field and voltage across the capacitor is given by:

$$\vec{E}(t) = \frac{v_C(t)}{d}$$

- The proportionality constant inside the brackets in this equation is the capacitance C, which is by definition:

$$C = \frac{\varepsilon A}{d}$$

$i - v$ Relationship

$$\frac{dq(t)}{dt} = \frac{d[Cv_C(t)]}{dt}$$

In practice, the time t_0 is established by a physical event such as closing a switch or the start of a particular clock pulse.

$$v_C(t) = v_C(0) + \frac{1}{C}\int_0^t i_C(x)\,dx$$

By the passive sign convention, the power in dissipated by a capacitor is given by:

$$p_C(t) = i_C(t)\, v_C(t)$$

To determine the stored energy, we take the first derivative. Since power is the time rate of change of energy, the quantity inside the brackets must be the energy stored in the capacitor.

$$w_C(t) = \frac{1}{2} C v_C^2(t)$$

- Stored energy is never negative and is proportional to the square of the voltage.
- The relationship also implies voltage is a continuous function of time.
- Since power is the time derivative of energy, a discontinuous change in energy implies infinite power, which is physically impossible.
- The capacitor voltage is a **state variable** because it determines the energy state of the element.

To summarize, the capacitor is a dynamic circuit element with the following properties:

1. The current through the capacitor is zero unless the voltage is changing. The capacitor acts like an open circuit to DC excitations.
2. The voltage across the capacitor is a **continuous function** of time. A discontinuous change in capacitor voltage would require infinite current and power, which is physically impossible.
3. The capacitor absorbs power from the circuit when storing energy and returns previously stored energy when delivering power. The net energy transfer is nonnegative, indicating that the capacitor is a passive element.

6.2 The Inductor

- The inductor is a dynamic circuit element involving the time variation of a magnetic field produced by a current.
- In a linear magnetic medium, the flux is proportional to both the current and the number of turns in the coil.
- The total flux is given by:

$$\phi(t) = k_1 N i_L(t)$$

- k_1 is a constant of proportionality dependent on the coil material, it's geometry, and its permeability, μ

- Flux linkage in a coil is given by the symbol λ with units weber-turns. It is proportional to the number of turns in the coil and to the total magnetic flux

$$\lambda(t) = N\phi(t)$$

- The proportionality constant inside the brackets in the inductance L, given by:

$$L = k_1 N^2$$

- The unit of inductance is the henry (H)

$$\lambda(t) = L i_L(t)$$

$i - v$ Relationship

$$\frac{d[\lambda(t)]}{dt} = \frac{d[L i_L(t)]}{dt}$$

$$v_L(t) = L \frac{d i_L(t)}{dt}$$

The reference time t_0 is established by some physical event such as closing or opening a switch. Without losing any generality, we assume $t_0 = 0$ and solving for $i_L(t)$ in the form

$$i_L(t) = i_L(0) + \frac{1}{L} \int_0^t v_L(x)\, dx$$

Power and Energy

Just as for the capacitor, we have power given by:

$$p_L(t) = i_L(t)\, v_L(t)$$

As is with capacitor energy, the constant in this expression is zero since it is energy stored in an instant t. Thus we have:

$$w_L(t) = \frac{1}{2} L i_L^2(t)$$

In this case, **current** is the state variable of the inductor as it determined the energy state of the element.

To summarize, we have the following points for the inductor:

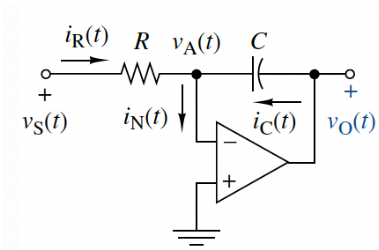
1. The voltage across the inductor is zero unless the current through it is changing. For DC excitations, the inductor is a short circuit.

2. The current through the inductor is a continuous function of time. A discontinuous change would require infinite voltage and power, which is physically impossible.
3. The inductor absorbs power from the circuit when storing energy and delivers power to the circuit. The net energy is nonnegative, indicating that the inductor is a passive element.

More About Duality

KVL	$\leftrightarrow$	KCL
Loop	$\leftrightarrow$	Node
Resistance	$\leftrightarrow$	Conductance
Voltage source	$\leftrightarrow$	Current source
Thévenin	$\leftrightarrow$	Norton
Short circuit	$\leftrightarrow$	Open circuit
Series	$\leftrightarrow$	Parallel
Capacitance	$\leftrightarrow$	Inductance
Flux linkage	$\leftrightarrow$	Charge

6.3 Dynamic OP AMP Circuits



Let’s determine the signal processing function of the circuit. Start by writing the KCL equation at node A.

$$i_R(t) + i_C(t) = i_N(t)$$

Write the resistor and capacitor device equations using the fundamental property of node voltages.

$$\begin{cases} i_C(t) = C\frac{d[v_O(t) - v_A(t)]}{dt} \\ i_R(t) = \frac{1}{R}[v_S(t) - v_A(t)] \end{cases}$$

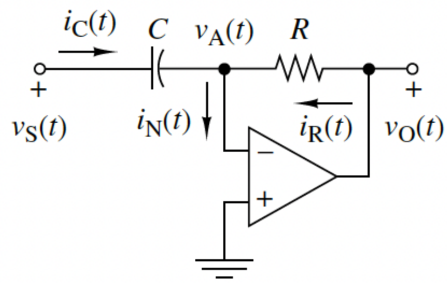
Then we substitute the element constraints into the KCL connection constraint produces

$$\frac{v_S(t)}{R} + C\frac{dv_O(t)}{dt} = 0$$

By integrating and rearranging a bit, we can find that:

$$v_O(t) = v_O(0) - \frac{1}{RC} \int_0^t v_S(x)\, dx$$

Similarly, we have an inverting differentiator circuit



6.4 Equivalent Capacitance and Inductance

Capacitors

$$C_{EQ} = C_1 + C_2 + \cdots + C_N \text{ (parallel connection)}$$

$$\frac{1}{C_{EQ}} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_N} \text{ (series connection)}$$

Inductors

$$\frac{1}{L_{EQ}} = \frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_N} \text{ (parallel connection)}$$

$$L_{EQ} = L_1 + L_2 + \cdots + L_N \text{ (series connection)}$$

DC Equivalent Circuits

- Under DC conditions, a capacitor acts as a open circuit and an inductor acts as a short circuit.
- DC capacitor voltage and inductor current becomes initial conditions for a transient response that begins at $t = 0$ when something in the circuit changes.

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7 First- and Second-Order Circuits

7.1 RC and RL Circuits

- RC and RL circuits are called **first-order circuits** because their behaviour is governed by a first-order DE.

Zero-Input Response of First-Order Circuits

The response depends on three factors:

1. The inputs driving the circuit $v_T(t)$
2. The values of the circuit parameters R_T and C
3. The value of $v(t)$ at $t = 0$ (initial condition)

The classical approach is to try an exponential solution

$$v(t) = Ke^{st}$$

7.2 First-Order Circuit Step Response

- Designing a circuit to meet transient response specifications requires making compromises with respect to the circuit's steady-state performance.
- Because the circuit is linear, we choose a method that uses superposition to divide the solution for $v(t)$ into two components:
  - $v_N(t)$ is the natural response and is the general solution when the input is set to zero
  - $v_F(t)$ is the forced responde, which is a particular solution when the input is a step function

$$v(t) = v_N(t) + v_F(t)$$

The step response of a first-order circuit depends on three quantities:

1. The amplitude of the step input (V_A or I_A)
2. The circuit time constant ($R_T C$ or L/R_N)
3. The value of the state variable at $t = 0$ (V_0 or I_0)

Zero-State Response

- The zero-state response is proportional to the amplitude of the input step function.
- The total response is not directly proportional to the input amplitude.
- The zero-state response occurs when the input is zero ($V_A = 0$ or $I_A = 0$)

7.3 Initial and Final Conditions

$$x(t) = [x(0) - x(\infty)]e^{-t/T_C} + x(\infty)$$

The state variable response in switched dynamic circuits is found using the following steps:

1. Find the initial value by applying DC analysis to the circuit configuration for $t < 0$with the element replaced with an open circuit.
2. Find the final value by applying DC analysis to the circuit configuration for $t > 0$ with the element replaced with an open circuit.
3. Find the time constant T_C of the circuit in the configuration for $t > 0$.
4. Write the step response directly without formulating and solving for the circuit differential equation.

7.5 The Series RLC Circuit

- Second order circuits contain two energy storage elements that cannot be replaced by a single equivalent element.
    - They are called second-order circuits
- There are two types, series and parallel RLC circuits

Solve for the roots of the characteristic equation

$$LCs^2 + R_T Cs + 1 = 0$$

- The solutions then define the natural frequencies of the circuit
- We can have:
    - s as the complex frequency
    - α as the neper frequency
    - β as the radian frequency or the damped natural frequency

7.6 The Parallel RLC Circuit

Characteristic equation

$$LCs^2 + \frac{L}{R_N}s + 1 = 0$$

7.7 Second-Order Circuit Step Response

General second-order differential equation with a step function has the form

$$a_2 \frac{d^y(t)}{dt^2} + a_1 \frac{dy(t)}{dt} + a_0 y(t) = Au(t)$$

The step response is found by the partitioning $y(t)$ into forced and natural components:

$$y(t) = y_N(t) + y_F(t)$$

In a second order circuit, the zero-state and natural responses studied take one of the three possible forms:

- overdamped,
- critically damped,
- underdamped

Now we need to introduce two new parameters

$$\omega_0^2 = \frac{a_0}{a_2} \text{ and } 2\zeta\omega_0 = \frac{a_1}{a_2}$$

- ω_0 is the undamped natural frequency
- ζ is the damping ratio

Now we can write the general homogeneous equation in the form:

$$\frac{d^2 y_N(t)}{dt^2} + 2\zeta\omega_0 \frac{dy_N(t)}{dt} + \omega_0^2 y_N(t) = 0$$

From this we can get

$$\omega_0 = \sqrt{\frac{1}{LC}}$$

and

$$\zeta = \frac{R_T}{2L\omega_0}$$

7.8 Summary

- Circuits containing one storage element are described by first-order differential equations
- The zero-input response in a first-order circuit is an exponential whose time constant depends on circuit parameters.
- The total response is the sum of the forced and natural responses.
- The zero-state response results from the input driving forces, and the zero-input response is caused by the initial energy stored in the storage element.
- For a sinusoidal input, the forced response is called the sinusoidal steady-state response, or the AC response.
- Circuits containing two energy storage elements are called second-order circuits
- Circuit damping ratio ζ and undamped natural frequency ω_0 determine the form of the zero-input and natural responses of any second-order circuit.