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Term Test I Theorems

Theorem 2.3

Let $N_0 \in \mathbf{Z}_{\geq 1}$ be a desired period. Then there are exactly N_0 distinct DT complex exponential signals of period N_0, given by

$$\phi_k[n] = e^{jk\omega_0 n}, k \in \{0, 1, \ldots, N_0 - 1\}$$

where $\omega_0 = 2\pi N_0$.

Theorem 3.1

If x is a real-valued, even, and periodic CTS with a fundamental period T_0, then we can represent x using the **cosine Fourier series**

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \alpha_k \cos(k\omega_0 t), \qquad \omega_0 = \frac{2\pi}{T_0}$$

$$\alpha_k = \frac{2}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t) \cos(k\omega_0 t)\, dt, \qquad k \in \{0, 1, 2, \ldots\}$$

Theorem 3.2

If x is a real, odd, and periodic CTS with a fundamental period T_0, then we represent x using the **sine Fourier series**

$$x(t) = \sum_{k=1}^{\infty} b_k \sin(k\omega_0 t), \qquad b_k = \frac{2}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t) \sin(k\omega_0 t)\, dt$$

Theorem 3.3

Let x be a periodic CTS with a fundamental period T_0 and angular frequency $\omega_0 = 2\pi / T_0$. Then,

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{jk\omega_o t}$$

is called the continuous-time Fourier series (CTFS) of the signal x, where the Fourier series coefficients α_k are given by

$$\alpha_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-jk\omega_0 t}\, dt$$

Theorem 3.5

If x has finite action, $x \in L_1^{per}$, then the Fourier coefficients α_k are well-defined and satisfy

$$\lim_{k \to \pm\infty} |\alpha_k| = 0$$

Definition 3.1

Let $e_K = \hat{x}_K - x$ denote the energy between the approximation and signal x. We say $\hat{x}_K$ converges to x

1. **Pointwise at time** t_0 if $\lim_{K \to \infty} e_K(t_0) = 0$
2. **Uniformly** if $\lim_{K \to \infty} ||e_K||_\infty = 0$
3. **In energy** if $\lim_{K \to \infty} ||e_K||_2 = 0$

Theorem 3.6

Suppose that x has finite action.

1. If x has a continuous derivative at time t_0, then $\hat{x}_K$ converges to x pointwise at t_0.
2. If the left side limits $x(t_0^-)$, $\frac{dx}{dt}(t_0^-)$ and the right side limits $x(t_0^+)$, $\frac{dx}{dt}(t_0^+)$ all exist at t_0, then

$$\lim_{K \to \infty} \hat{x}_K(t_0) = \frac{1}{2}(x(t_0^-) + x(t_0^+))$$

Theorem 3.7

Suppose that x has finite action. If x has a derivative which is continuous everywhere, then $\hat{x}_K$ converges to x uniformly.

Theorem 3.8

If x has finite energy, then

1. $\hat{x}_K$ converges to x in energy.
2. The CTFS coefficients α_k have finite energy.
3. The signal and the coefficients both satisfy **Parseval's relation**.

$$\frac{1}{T_0} ||x||_2^2 = ||\alpha||_2^2$$

Theorem 3.9

If x is a periodic DTS with a fundamental period N_0, then the discrete-time Fourier series of x is the sum

$$x[n] \sum_{k=0}^{N_0 - 1} \alpha_k e^{jk\omega_0 n}$$

where the Fourier coefficients are given by

$$\alpha_k = \frac{1}{N_0} \sum_{k=0}^{N_0 - 1} x[l] e^{-jk\omega_0 l}$$