

# 4

## The Fourier Transform

### 4 The Continuous-Time Fourier Transform

#### 4.1 Representation of Aperiodic Signals: The Continuous-Time Fourier Transform

##### 4.1.1 Development of the Fourier Transform Representation of an Aperiodic Signal

Let's start by revisiting the Fourier series representation for the continuous-time periodic square wave. Over one period,

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$

and periodically repeats with period  $T$ . The Fourier series coefficients  $a_k$  are given by:

$$a_k = \frac{2 \sin(k\omega_0 T_1)}{k\omega_0 T}$$

where  $\omega_0 = 2\pi/T$ .

We consider an aperiodic signal as the limit of a periodic signal as the period becomes large, and we examine the Fourier representation of that signal. For some number  $T_1$ ,  $x(t) = 0$  is  $|t| > T_1$ . From this aperiodic signal, we construct a periodic signal  $\tilde{x}(t)$  for which  $x(t)$  is one period. As we select a longer period,  $\tilde{x}(t)$  becomes identical to  $x(t)$ .

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} dt$$

By defining the envelope  $X(j\omega)$  of  $Ta_k$  as

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-jk\omega t} dt$$

we have the coefficients  $a_k$

$$a_k = \frac{1}{T} X(jk\omega_0)$$

By combining the above summation and the second formula for the coefficients, we get

$$\tilde{x}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} X(jk\omega_0) e^{jk\omega_0 t}$$

Ultimately, we have the equations that give us the Fourier transform pair, with  $X(j\omega)$  referred to as the **Fourier Transform** or **Fourier integral** of  $x(t)$  and  $x(t)$  gives us the **inverse Fourier transform**, or **synthesis** equation.

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega$$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

- For periodic signals, these complex exponentials have amplitudes  $\{a_k\}$  which occur at a discrete set of harmonically related frequencies  $k\omega_0$ .
- For aperiodic signals, the complex exponentials occur at a continuum of frequencies.
  - These frequencies have an 'amplitude' given by  $X(j\omega)(d\omega/2\pi)$
  - The transform  $X(j\omega)$  of an aperiodic signal is referred to as the spectrum
    - This is because the transform gives us the information we require to reconstruct the signal as a linear combination of sinusoidal signals at different frequencies

$$a_k = \frac{1}{T} X(j\omega) \Big|_{\omega=k\omega_0}$$

#### 4.1.2 Convergence of Fourier Transforms

- To derive the Fourier equations above, we assume that  $x(t)$  is of a finite duration, however, the equations are valid for many infinite-duration signals too.
  - Our derivation of the Fourier transform suggests we can apply the same definition and criteria for convergence.
- Just as with periodic signals, there is a set of conditions (the **Dirichlet conditions**) that guarantee that  $\tilde{x}(t)$  is equal to  $x(t)$  for any  $t$  except for at a discontinuity.

- Where there is a discontinuity, the value at that  $t$  is the average value on either side of the discontinuity.

The Dirichlet conditions require that:

1.  $x(t)$  is absolutely integrable
2.  $x(t)$  have a finite number of maxima and minima within any finite interval
3.  $x(t)$  have a finite number of discontinuities within any finite interval. Each discontinuity must be finite.

## 4.3 Properties of the Continuous-Time Fourier Transform

Sometimes we will refer to  $X(j\omega)$  with the notation  $\mathcal{F}\{x(t)\}$  and similarly  $x(t)$  with the notation  $\mathcal{F}^{-1}\{X(j\omega)\}$ . We will use this notation to refer to a Fourier transform pair:

$$x(t) \xleftrightarrow{\mathcal{F}} X(j\omega)$$

### 4.3.1 Linearity

$$ax(t) + by(t) \xleftrightarrow{\mathcal{F}} aX(j\omega) + bX(j\omega)$$

### 4.3.2 Time Shifting

$$x(t - t_0) \xleftrightarrow{\mathcal{F}} e^{-j\omega t_0} X(j\omega)$$

### 4.3.3 Conjugation and Conjugate Symmetry

$$x^*(t) \xleftrightarrow{\mathcal{F}} X(-j\omega)$$

### 4.3.4 Differentiation and Integration

$$\frac{dx(t)}{dt} \xleftrightarrow{\mathcal{F}} j\omega X(j\omega)$$

$$\int_{-\infty}^t x(\tau) dt \xleftrightarrow{\mathcal{F}} \frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(\omega)$$

### 4.3.5 Time and Frequency Scaling

$$x(at) \xleftrightarrow{\mathcal{F}} \frac{1}{|a|} X(j\omega/a)$$

$$x(-t) \xleftrightarrow{\mathcal{F}} X(-j\omega)$$

### 4.3.6 Duality

$$\begin{aligned}
 -jt x(t) &\xleftrightarrow{\mathcal{F}} \frac{dX(j\omega)}{d\omega} \\
 e^{j\omega_0 t} x(t) &\xleftrightarrow{\mathcal{F}} X(j(\omega - \omega_0)) \\
 -\frac{1}{jt} x(t) + \pi x(0)\delta(t) &\xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\omega} x(\tau) d\tau
 \end{aligned}$$

#### 4.3.7 Parseval's Relation

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

## 5 The Discrete-Time Fourier Transform

### 5.1 Representation of Aperiodic Signals: The Discrete-Time Fourier Transform

#### 5.1.1 Development and the Discrete-Time Fourier Transform

- We know that Fourier series coefficients for a continuous-time periodic square wave can be viewed as samples of an envelope function.
  - As the period of the square wave increases, these samples become more finely spaced.
  - Suggestion in chapter 4: represent an aperiodic signal  $x(t)$  by constructing a periodic signal  $\tilde{x}(t)$  that equals  $x(t)$  over one period.
    - As this period approaches infinity  $\tilde{x}(t)$  was equal to  $x(t)$  over larger and larger intervals of time.

The discrete-time Fourier transform is given by the following equations:

$$\begin{aligned}
 x[n] &= \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega \\
 X(e^{j\omega}) &= \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}
 \end{aligned}$$

We see that the discrete-time Fourier transform shares many similarities with the continuous-time case. The major differences between the two are the periodicity of the DT transform and the finite interval of interrogation in the synthesis equation.

#### 5.1.3 Convergence Issues Associated with the Discrete-Time Fourier Transform

Conditions on  $x[n]$  that guarantee the convergence of this sum are direct counterparts of the convergence conditions for the CT Fourier transform. The first DT Fourier transform equation will converge if  $x[n]$  is absolutely summable.

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