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Analysis of Continuous-Time Signals Using the Laplace Transform

9 The Laplace Transform

9.1 The Laplace Transform

- When we replace the $j\omega$ complex exponential variable with a general variable s, we have the Laplace transform

$$X(s) = \int_{-\infty}^{+\infty} x(t)e^{-st}\,dt$$

- THe range of values of s for which the Laplace integral converges is called the region of convergence
- If we express $X(s)$ as a rational function with numerator $N(s)$ and denominator $D(s)$, then we say:
    - the roots of the numerator are called the zeroes of the function
    - the roots of the denominator are called the poles of the function

9.2 The Region of Convergence for Laplace Transforms

- There are some specific constraints on the ROC for various classes of signals
- Property 1: the ROC of $X(s)$ consists of strips parallel to the $j\omega$-axis of the s-plane.
- Property 2: for rational Laplace transforms, the ROC does not contain any poles.
- Property 3: if $x(t)$ is of finite duration and is absolutely integrable, then the ROC is the entire s-plane
- Property 4: if $x(t)$ is right-sided, and if the line $Re\{s\} = \sigma_0$ is in the ROC, then all values of s for which $Re\{s\} > \sigma_0$ will also be in the ROC.
- Property 5: if $x(t)$ is left-sided, and if the line $Re\{s\} = \sigma_0$ is in the ROC, then all values of s for which $Re\{s\} < \sigma_0$ will also be in the ROC.

- Property 6: If $x(t)$ is two-sided, and if the line $Re\{s\} = \sigma_0$ is in the ROC, then the ROC will consist of the strip in the s-place that includes the line $Re\{s\} = \sigma_0$
- Property 7: If the Laplace transform $X(s)$ of $x(t)$ is rational, then its ROC is bounded by poles or extends to infinity. In addition, no poles of $X(s)$ are contained in the ROC.

9.3 The Inverse Laplace Transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\sigma + j\omega) e^{(\sigma + j\omega)t} \, d\omega = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(s) e^{st} \, d\omega$$

9.5 Properties of the Laplace Transform

9.5.1 Linearity of the Laplace Transform

$$a x_1(t) + b x_2(t) \longleftrightarrow a X_1(s) + b X_2(s)$$

9.5.2 Time Shifting

$$x(t - t_0) \longleftrightarrow e^{-s t_0} X(s)$$

9.5.3 Shifting in the s-Domain

$$e^{s_0 t} x(t) \longleftrightarrow X(s - s_0)$$

9.5.4 Time Scaling

$$x(at) \longleftrightarrow \frac{1}{|a|} X(s/a)$$

9.5.5 Conjugation

$$x^*(t) \longleftrightarrow X^*(s^*)$$

9.5.6 Convolution Property

$$x_1(t) * x_s(t) \longleftrightarrow X_1(s) X_2(s)$$

9.5.7 Differentiation in the Time Domain

$$\frac{dx(t)}{dt} \longleftrightarrow s X(s)$$

9.5.8 Differentiation in the s-Domain

$$-tx(t) \longleftrightarrow \frac{dX(s)}{ds}$$

9.9 The Unilateral Laplace Transform

- We now introduce the unilateral Laplace transform

$$X(s) = \int_0^{\infty} x(t)e^{-st}\,dt$$