4️⃣

The Fourier Transform

4 The Continuous-Time Fourier Transform

4.1 Representation of Aperiodic Signals: The Continuous-Time Fourier Transform

4.1.1 Development of the Fourier Transform Representation of an Aperiodic Signal

Let’s start by revisiting the Fourier series representation for the continuous-time periodic square wave. Over one period,

$$x(t) = \begin{cases} 1, |t| < T_1 \\ 0, T_1 < |t| < T/2 \end{cases}$$

and periodically repeats with period T. The Fourier series coefficients a_k are given by:

$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T}$$

where $\omega_0 = 2\pi/T$.

We consider an aperiodic signal as the limit of a periodic signal as the period becomes large, and we examine the Fourier representation of that signal. For some number T_1, $x(t) = 0$ is $|t| > T_1$. From this aperiodic signal, we construct a periodic signal $\tilde{x}(t)$ for which $x(t)$ is one period. As we select a longer period, $\tilde{x}(t)$ becomes identical to $x(t)$.

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}\,dt$$

By defining the envelope $X(j\omega)$ of Ta_k as

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-jk\omega t}\,dt$$

we have the coefficients a_k

$$a_k = \frac{1}{T} X(jk\omega_0)$$

By combining the above summation and the second formula for the coefficients, we get

$$\tilde{x}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} X(jk\omega_0) e^{jk\omega_0 t} \omega_0$$

Ultimately, we have the equations that give us the Fourier transform pair, with $X(j\omega)$ referred to as the **Fourier Transform** or **Fourier integral** of $x(t)$ and $x(t)$ gives us the **inverse Fourier transform**, or **synthesis** equation.

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{jwt} \, d\omega$$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-jwt} \, dt$$

- For periodic signals, these complex exponentials have amplitudes $\{a_k\}$ which occur at a discrete set of harmonically related frequencies $k\omega_0$.
- For aperiodic signals, the complex exponentials occur at a continuum of frequencies.
  - These frequencies have an ‘amplitude’ given by $X(j\omega(d\omega/2\pi)$
  - The transform $X(j\omega)$ of an aperiodic signal is referred to as the spectrum
    - This is because the transform gives us the information we require to reconstruct the signal as a linear combination of sinusoidal signals at different frequencies

$$a_k = \frac{1}{T} X(j\omega) \bigg|_{\omega = k\omega_0}$$

4.1.2 Convergence of Fourier Transforms

- To derive the Fourier equations above, we assume that $x(t)$ is of a finite duration, however, the equations are valid for many infinite-duration signals too.
  - Our derivation of the Fourier transform suggests we can apply the same definition and criteria for convergence.
- Just as with periodic signals, there is a set of conditions (the **Dirichlet conditions**) that guarantee that $\tilde{x}(t)$ is equal to $x(t)$ for any t except for at a discontinuity.

- Where there is a discontinuity, the value at that t is the is average value on either side of the discontinuity.

The Dirichlet conditions require that:

1. $x(t)$ is absolutely integrable
2. $x(t)$ have a finite number of maxima and minima within any finite interval
3. $x(t)$ have a finite number of discontinuities within any finite interval. Each discontinuity must be finite.

4.3 Properties of the Continuous-Time Fourier Transform

Sometimes we will refer to $X(j\omega)$ with the notation $\mathcal{F}\{x(t)\}$ and similarly $x(t)$ with the notation $\mathcal{F}^{-1}\{X(j\omega)\}$. We will use this notation to refer to a Fourier transform pair:

$$x(t) \xleftrightarrow{\mathcal{F}} X(j\omega)$$

4.3.1 Linearity

$$ax(t) + by(t) \xleftrightarrow{\mathcal{F}} aX(j\omega) + bX(j\omega)$$

4.3.2 Time Shifting

$$x(t - t_0) \xleftrightarrow{\mathcal{F}} e^{-j\omega t_0}X(j\omega)$$

4.3.3 Conjugation and Conjugate Symmetry

$$x^*(t) \xleftrightarrow{\mathcal{F}} X(-j\omega)$$

4.3.4 Differentiation and Integration

$$\frac{dx(t)}{dt} \xleftrightarrow{\mathcal{F}} j\omega X(j\omega)$$

$$\int_{-\infty}^{t} x(\tau)\, dt \xleftrightarrow{\mathcal{F}} \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$$

4.3.5 Time and Frequency Scaling

$$x(at) \xleftrightarrow{\mathcal{F}} \frac{1}{|a|}X(j\omega/a)$$

$$x(-t) \xleftrightarrow{\mathcal{F}} X(-j\omega)$$

4.3.6 Duality

$$-jt\ x(t) \xleftrightarrow{\mathcal{F}} \frac{dX(j\omega)}{d\omega}$$

$$e^{j\omega_0 t}x(t) \xleftrightarrow{\mathcal{F}} X(j(\omega - \omega_0))$$

$$-\frac{1}{jt}x(t) + \pi x(0)\delta(t) \xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\omega} x(\tau)\ d\tau$$

4.3.7 Parseval’s Relation

$$\int_{-\infty}^{+\infty} |x(t)|^2\ dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2\ d\omega$$

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5 The Discrete-Time Fourier Transform

5.1 Representation of Aperiodic Signals: The Discrete-Time Fourier Transform

5.1.1 Development and the Discrete-Time Fourier Transform

- We know that Fourier series coefficients for a continuous-time periodic square wave can be viewed as samples of an envelope function.
  - As the period of the square wave increases, these samples become more finely spaced.
  - Suggestion in chapter 4: represent an aperiodic signal $x(t)$ by constructing a periodic signal $\tilde{x}(t)$ that equals $x(t)$ over one period.
    - As this period approaches infinity $\tilde{x}(t)$ was equal to $x(t)$ over larger and larger intervals of time.

The discrete-time Fourier transform is given by the following equations:

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\ d\omega$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$

We see that the discrete-time Fourier transform shares many similarities with the continuous-time case. The major differences between the two are the periodicity of the DT transform and the finite interval of interrogation in the synthesis equation.

5.1.3 Convergence Issues Associated with the Discrete-Time Fourier Transform

Conditions on $x[n]$ that guarantee the convergence of this sum are direct counterparts of the convergence conditions for the CT Fourier transform. The first DT Fourier transform equation will converge if $x[n]$ is absolutely summable.

5.3 Properties of the Discrete-Time Fourier Transform

We will use the following to represent the DT Fourier transform

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega})$$

5.3.1 Periodicity of the DT Fourier Transform

$$X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$$

5.3.2 Linearity of the Fourier Transform

$$ax_1[n] + bx_2[n] \overset{\mathcal{F}}{\longleftrightarrow} aX_1(e^{j\omega}) + bX_2(e^{j\omega})$$

5.3.3 Time Shifting and Frequency Shifting

$$x[n-n_0] \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega n_0} X(e^{j\omega})$$

5.3.4 Conjugation and Conjugate Symmetry

$$x^*[n] \overset{\mathcal{F}}{\longleftrightarrow} X^*(e^{-j\omega})$$

5.3.5 Differencing and Accumulation

$$x[n] - x[n-1] \overset{\mathcal{F}}{\longleftrightarrow} (1 - e^{-j\omega})X(e^{j\omega})$$

5.3.6 Time Reveral

$$x[-n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{-j\omega})$$

5.3.7 Time Expansion

$$x_{(k)}[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{jk\omega})$$

5.3.8 Differentiation in Frequency

$$nx[n] \overset{\mathcal{F}}{\longleftrightarrow} j\frac{dX(e^{j\omega})}{d\omega}$$
