ECO 311: Game Theory Assignment
(Take Home/Open Book)

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Definition: An action $a_i \in A_i$ is rationalizable in the strategic game $(N, \{A_i\}_{i \in N}, \{u_i\}_{i \in N})$ if for each $j \in N$, there is a set $Z_j \subseteq A_j$ such that:-
• $a_i \in Z_i$
• every action $a_j \in Z_j$ is a best response to a belief μ_j of player j that assigns positive probabilities to actions in some subset of Z_{-j}.

(A Course in Game Theory by Martin J. Osborne and Ariel Rubinstein)

1.

(a) In the following two-player simultaneous move game, find the set of rationalizable actions (pure actions, not probabilistic) for each player.

Player 2

Player 1	a	b	c	d
A	$(7,0)$	$(2,5)$	$(0,7)$	$(0,1)$
B	$(5,2)$	$(3,3)$	$(5,2)$	$(0,1)$
C	$(0,7)$	$(2,5)$	$(7,0)$	$(0,1)$
D	$(0,0)$	$(0,-2)$	$(0,0)$	$(9,-1)$

Essentially, We have to eliminate never-best responses. After performing said elimination, the actions that are left can be called rationalizable actions.

Let's perform said Elimination of Never Best Responses:-

For Player-2

We'll check if any of Player 2's actions can be classified as never best responses a is Player-2's best response to Player 1 when they play C, i.e., **7> 5 , 7>0 and 7>1 respectively when compared to b,c, and d**

b is Player-2's best response to Player 1 when they play B, i.e. **3>2, 3>2, and 3>1, respectively when compared to a,c, and d.**

c is Player-2's best response to Player 1 when they play A, i.e. **7>0, 7>5, and 7>1, respectively when compared to a,b, and d.**

d is never the best response for Player 2 to use against Player 1. c is the best response against A. b is the best response against B. a is the best response against C. Both a and c are the best responses against D.

Since d is a 'never-best response,' d is removed from Player-2's action profile.

Rationalizable actions for Player-2=> {a, b, c}

Player 2

Player 1		a	b	c
	A	$(7,0)$	$(2,5)$	$(0,7)$
	B	$(5,2)$	$(3,3)$	$(5,2)$
	C	$(0,7)$	$(2,5)$	$(7,0)$
	D	$(0,0)$	$(0,-2)$	$(0,0)$

For Player-1:

We'll check if any of Player 1's actions can be classified as never best responses

A is Player-1's best response to Player 2 when they play a, i.e., **7> 5, 7>0, and 7>0, respectively**

B is Player-1's best response to Player 2 when they play b, i.e. **3>2, 3>2, and 3>0, respectively**

C is Player-1's best response to Player 2 when they play c, i.e. **7>0, 7>5, and 7>0, respectively**

D is never Player-1's best response to Player-2 for any strategy pursued by Player-2, as could be inferred from the last three lines and the revised matrix

Since D is a 'never-best response', D is removed from Player-1's action profile.

Rationalizable actions for Player-1=> {A, B, C}

This is the final result. {A,B,C} and {a,b,c} are the rationalizable actions for Player-1 and Player-2 respectively

Player 2

Player 1		a	b	c
	A	$(7,0)$	$(2,5)$	$(0,7)$
	B	$(5,2)$	$(3,3)$	$(5,2)$
	C	$(0,7)$	$(2,5)$	$(7,0)$

(b) Do you think a weakly dominated action can be rationalizable? Why/Why not? What about strictly dominated actions?

Lemma 60.1 from Osborne and Rubenstein: "An action of a player in a finite strategic game is a never best response if and only if it is strictly dominated."

Proof. "*Let the strategic game be G = hN,(Ai),(ui)i and let a ∗ i ∈ Ai . Consider the auxiliary strictly competitive game G0 (see Definition 21.1) in which the set of actions of player 1 is Ai \{a ∗ i }, that of player 2 is A−i , and the preferences of player 1 are represented by the payoff function v1 given by v1(ai , a−i) = ui(a−i , ai)−ui(a−i , a∗ i). (Note that the argument (ai , a−i) of v1 is a pair of actions in G0 while the arguments (a−i , ai) and (a−i , a∗ i) are action profiles in G.) For any given mixed strategy profile (m1, m2) in G0 we denote by v1(m1, m2) the expected payoff of player 1. The action a ∗ i is a never-best response in G if and only if for any mixed strategy of player 2 in G0 there is an action of player 1 that yields a positive payoff; that is, if and only if minm2 maxai v1(ai , m2) > 0. This is so if and only if minm2 maxm1 v1(m1, m2) > 0 (by the linearity of v1 in m1). Now, by Proposition 33.1 the game G0 has a mixed strategy Nash equilibrium, so from part (b) of Proposition 22.2 applied to the mixed extension of G0 we have minm2 maxm1 v1(m1, m2) > 0 if and only if maxm1 minm2 v1(m1, m2) > 0; that is, if and only if there exists a mixed strategy m∗ 1 of player i in G0 for which v1(m∗ 1 , m2) > 0 for all m2 (that is, for all beliefs on A−i). Since m∗ 1 is a probability measure on Ai \ {a ∗ i } it is a mixed strategy of player 1 in G; the condition v1(m∗ 1 , m2) > 0 for all m2 is equivalent to Ui(a−i , m∗ 1) − Ui(a−i , a∗ i) > 0 for all a−i ∈ A−i , which is equivalent to a ∗ i being strictly dominated.*"

Weakly Dominated Actions: Yes

Although, It’s enough to cite Lemma 60.1 to prove why weakly dominated actions can be rationalizable. Let me offer a more concise, informal proof. Weakly Dominated actions can be rationalisable since, by definition they are either less than or equal to another action’s expected utilities. Now, if they are equal to another action’s expected utility, and that action in itself is a best response to some action by the other player, the weakly dominated action will be the best response to some action, and consequently, It will not be a never-best response. So, It won’t be removed while eliminating never-best responses, and will count as a rationalisable action.

Let me use an example to clarify this further:-

Player-1 \ Player-2	X	Y
A	(8,2)	(12,9)
B	(8,5)	(4,3)

In this example, A Weakly Dominates B, i.e. B is a weakly dominated strategy. However, Player-1’s best response against Player-2, playing X, is both A and B.

So, **B is rationalizable here despite being weakly dominated.**

Strictly Dominated Actions: No

Though Lemma 60.1 does the trick, I’ll offer an easier explanation. Strictly Dominated Actions, by definition, provide a strictly worse response (less utility) than another action from the player’s action profile for all actions in their rival’s action profile. Consequently, they can never be the best responses for any action performed by the rival player [The strategy that dominates them is always a better alternative]. Since they are never-best responses, they can never be rationalizable.
(Rationalizable actions are the actions that are left after eliminating never-best actions. That is to say, Strictly Dominated actions will be eliminated and thus will not be rationalizable)

(c) Every action profile (ai, a−i) where ai ∈ Zi, a−i ∈ Z−i is a Nash equilibrium. True or False? Why?

False

Proof by Contradiction: Take the following example (from part (a))

		Player 2		
		a	b	c
Player 1	A	$(7,0)$	$(2,5)$	$(0,7)$
	B	$(5,2)$	$(3,3)$	$(5,2)$
	C	$(0,7)$	$(2,5)$	$(7,0)$

In this example, all action profiles ∈ Zi i.e. they are all rationalizable. However, only the action profile (B,b) is a Nash Equilibrium. (3>2 for both 3s)

Let's expand upon our claim by elaborating on how (A,a) is rationalizable, but not a Nash Equilibrium. Given that Player 1 plays A, Player 2 would like to play c, not A. Since (A, a) is not the best response for Player-2 , ergo, (A, a) is not a Nash equilibrium.

(An action Profile is a Nash Equlibrium if both actions are the best responses for either player should their rival choose to play the action assigned to them in the action profile)

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2. Consider a sequential game with $N = \{1, 2, \ldots, \infty\}$ players. Suppose each player $i \in N$ moves only in time period i: he chooses an action $a_i \in \{Yes, No\}$. For example, player $i = 1$ chooses $a_1 \in \{Yes, No\}$ in time period 1. In time period 2, player 2 chooses $a_2 \in \{Yes, No\}$, and so on for each $i \in N$. Each player can observe the actions of all the preceding players in the game. In every time period i, if the new player i chooses Yes, 1 dollar is deposited in the account of each player $j \leq i$. If i chooses No, then 2 dollars is deposited in i's account and the accounts of all $j < i$ players is reduced to 0.

Rationale: Maximising players' account balances

For this question, I will rely on Chapter 12 [Repeated Games] of the MIT Course on the 'Economic Applications of Game Theory' and the Wikipedia article on Repeated Games.

(a) How will you find equilibrium in this game? Briefly explain the key technique used and find the equilibria.

The discount factor δ represents each player's patience to wait for the next iteration of the game. Now, proceeding further, we make two cases. (i) where there exists no phenomenon of a discount factor (the players are patient), and (ii) where there is a discount factor

(i) If there is no discount factor: The folk theorem for this case suggests that "every individually rational and feasible payoff profile in the basic game is a Nash equilibrium payoff profile in the repeated game." Now let us take two models of rationality:-

(a) The player's rationale is to ensure the best utility for all players:-

In this case, the Subgame Perfect Equilibrium and the Nash Equilibrium would appear to be cooperation, where every player plays 'Yes,' (except player1) ensuring infinitely large utilities for every player(mean utility slightly greater than 1).

Using the one-shot dev principle, It is clear that deviating and picking 'No' would be counterintuitive for every player other than Player-1,
who can Choose 'No' without harming other players' limit inferior to mean utility. (others will add 1 to their scores, but reduce k (where k>=1) from others' scores, bringing down the group's limit inferior to the mean utility.

(b) The player's rationale is to maximize their own utility

'Yes' cooperation appears to be a Nash Equilibrium if Tit for Tat trigger is used, i.e. players are punished for choosing 'No,' and breaking the 'Yes' cooperation. After player k plays 'No' and gets +2 for themselves, And 0 for players x where 1<=x<k, Tit for Tat suggests that the player k+1 play 'No' to neutralize player k's ill gotten Gains.However Consequently, Player-k+2 will have to settle for an inefficient option if they wish to encourage cooperation and prevent a volunteer's dilemma. Since these triggers do not ostensibly harm the punishers, it is an SPE. It is interesting how a player would just about hope for the player before them to violate the cartel's rules, so as to increase their average by another point.

	Time Periods											
Players	0	1	2	.	.	.	i	i+1	.	.	.	.
Player i (Coop)	.	.	.	.	.	0	1	2	3	4	5	6
Player i (Defect)	.	.	.	.	.	0	2	0	1	2	3	4

Player i might see a temporary gain, but in the long run, his mean is worse off.

(ii) If the discount factor matters

The folk theorem for this case requires that the payoff profile dominates the minmax Profile. Consequently, It suggests that the punishers must be rewarded for punishing the defector for it to have . However, It isn't possible in this case since only i+1 can gain by punishing i, the rest, are indifferent to the punishment. Moreover, It isn't possible to minmax the defector forever, which prevents the existence of Nash Equilibrium too. Moreover, for a high δ, this case should closely resemble (i)

(b) Suppose each player i > 1 forms a belief about the "types" of players who will join the game after him by observing the actions of the previous players. In this setting, can the action profile (Yes, Yes, . . .) be sustained in equilibrium? Explain your answer.

The action profile (Yes, Yes, …) can be preserved in equilibrium as showcased in the (i)(b) part of Q2(a). This is so because no player would wish to unilaterally deviate from the Yes chain since their mean would be detrimentally affected. If player i chooses to play 'No', the only beneficiary would be i+1, which has no bearing on the decisions of player i. Consequently, If we choose to adhere to the One-Shot Principle, and the concept of triggers, it wouldn’t be beneficial for any existing player for player i to deviate from the (Yes, Yes…..) action profile.

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