Part I Summary of Answers

1.
    a. $P(A) = \frac{1}{5}$, $P(B) = \frac{4}{5}$
    b. $P(A|X) = 11\%$
    c. $P(X|A) = 1.4\%$, $P(X|B) = 2.8\%$
2.
    a. The MAP hypothesis is that the person DOES NOT have the disease: $P(ND|X) = 0.018\%$, $P(D|X) = 15\%$
    b. The ML hypothesis is that the person has the disease: $P(X_{Method\ 1} = D|ND) = 15\%$, $P(X_{Method\ 1} = D|D) = 90\%$
    c. $P(D|X) = 2.32\%$
3.
    a. $(1-\theta)^4\theta^2$
    b. $4log(1-\theta) + 2log\theta$
    c. $\widehat{\theta} = \frac{1}{3}$
4.
    a. $\widehat{\theta_1} = \frac{2}{5}$
    b. $\widehat{\theta_1} = \frac{3}{8}$
    c. $\widehat{\theta_1} = \frac{t+1}{N+2}$
5.
    a. $P(X|"-") = 13.6\%$
    b. The ML Estimate is Class $"-"$, in fact: $P(X|"+") = 2.58\%$, $P(X|"-") = 13.6\%$.
    c. The MAP Estimate is Class $"-"$, in fact: $P("+")P(X|"+") = 1.03\%$, $P("-")P(X|"-") = 8.16\%$.

Problem 1 (a)

$$X_A \sim N(9.2, 1.6)$$

$$X_B \sim N(9.6, 1.2)$$

(b) $\mathbb{P}(\text{"ROCKS} \in A\text{"}) = p_A$

$\mathbb{P}(\text{"ROCKS} \in B\text{"}) = p_B$

$$\begin{cases} p_B = 4 p_A \\ p_A + p_B = 1 \end{cases} \longrightarrow \quad p_A + 4p_A = 1$$

$$5p_A = 1$$

$$p_A = \frac{1}{5} = 20\%$$

$$p_B = \frac{4}{5} = 80\%$$

Problem 1 (b) -p1

(b)

$X_1 = 9.3$; $X_2 = 8.8$; $X_3 = 9.8$

WE NEED :

$$P(A|X) = \frac{P(A) \cdot p(X|A)}{p(X)}$$

WHERE

BECAUSE X_i's ARE INDEPENDENT

$$p(X|A) = \underbrace{p(X_1|\mu_1,\sigma_1) \cdot p(X_2|\mu_1,\sigma_1) \cdot p(X_3|\mu_1,\sigma_1)}$$

AND $$p(x|\mu,\sigma) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$$

AND $$p(X) = p(X|A) \cdot p(A) + p(X|B) \cdot p(B)$$

NOW,

$$p(X_1|\mu_1,\sigma_1) = p(9.3\,|\,9.2, 1.6) = 0.24885$$

$$p(X_2|\mu_1,\sigma_1) = p(8.8\,|\,9.2, 1.6) = 0.24167$$

$$p(X_3|\mu_1,\sigma_1) = p(9.8\,|\,9.2, 1.6) = 0.23241$$

$$\Rightarrow p(X|A) = 0.01398$$

FOR LOCATION B, WE USE $\mu_2 = 9.6$ AND $\sigma_2 = 1.2$

Problem 1 (b) -p2 and (c)

(b) $p(9.3 \mid 9.6, 1.2) = 0.32222$

$p(8.8 \mid 9.6, 1.2) = 0.26621$

$p(9.8 \mid 9.6, 1.2) = 0.32787$

$\Rightarrow p(X \mid B) = 0.02812$

FROM (a), WE HAVE $\mathbb{P}(A) = p_A = \frac{1}{5}$
AND $\mathbb{P}(B) = p_B = \frac{4}{5}$

SO, $p(X) = 0.01398 \cdot \frac{1}{5} + 0.02812 \cdot \frac{4}{5}$

$= 0.02529$

FINALLY, $\mathbb{P}(A \mid X) = 0.1105 \approx 11\%$

(c) $p(X \mid A) \approx 1.4\%$ FROM (b)

$p(X \mid B) = 2.8\%$ FROM (b)

SO THE ML HYPOTHESIS IS THAT THE ROCKS COME FROM LOCATION B

Problem 2 (a)

PROBLEM # 2

SOME DEFINITIONS:

D = "HAS DISEASE" ; ND = "DOES NOT HAVE DISEASE"

		PREDICTED CLASS D	PREDICTED CLASS ND
ACTUAL CLASS	D	TP	FN
	ND	FP	TN

$TP = \mathbb{P}(X = D \mid D)$

$FN = \mathbb{P}(X = ND \mid D)$

$FP = \mathbb{P}(X = D \mid ND)$

$TN = \mathbb{P}(X = ND \mid ND)$

WE KNOW: $\mathbb{P}(X_{Method\ 1} = D \mid ND) = 15\%$

$\mathbb{P}(X_{Method\ 1} = ND \mid D) = 10\%$

$\mathbb{P}(X_{Method\ 2} = D \mid ND) = 5\%$

$\mathbb{P}(X_{Method\ 2} = ND \mid D) = 3\%$

THE PRIOR IS $\mathbb{P}(D) = 0.02\%$

(a) THE POSTERIOR THAT THE PERSON HAS THE DISEASE IS:

$$\mathbb{P}(D \mid X_{M1}) \propto \mathbb{P}(D) \cdot \mathbb{P}(X_{M1} = D \mid D)$$

$$= \mathbb{P}(D) \cdot [1 - \mathbb{P}(X_{M1} = ND \mid D)]$$

$$= 0.0002 \cdot (1 - 0.1) = 0.018\%$$

THE POSTERIOR THAT THE PERSON DOES NOT HAVE THE DISEASE IS:

$$\mathbb{P}(ND \mid X) \propto \mathbb{P}(ND) \cdot \mathbb{P}(X_{M1} = D \mid ND)$$

$$= [1 - \mathbb{P}(D)] \cdot \mathbb{P}(X_{M1} = D \mid ND)$$

$$= (1 - 0.0002) \cdot 0.15 = 14.997\%$$

THE MAP HYPOTHESIS IS THAT THE PERSON DOES NOT HAVE THE DISEASE (ND).

Problem 2 - (b), (c)

(b) THE ML HYPOTHESIS IS THAT THE PERSON HAS THE DISEASE;

$$\mathbb{P}(X_{M1} = D \mid D) =$$

$$= [1 - \mathbb{P}(X_{M1} = ND \mid D)]$$

$$= (1 - 0.1) = 90\%$$

$$\mathbb{P}(X_{M1} = D \mid ND) = 15\%$$

(c) WE CAN UPDATE OUR PRIOR BELIEF AFTER TEST 1 AND USE IT AS PRIOR OF TEST 2.

SO,

$$\mathbb{P}_2(D) = \mathbb{P}(D \mid X_{M1}) = \frac{\mathbb{P}(D)\,\mathbb{P}(X_{M1} = D \mid D)}{\mathbb{P}(X_{M1})}$$

$$\mathbb{P}(X_{M1}) = \mathbb{P}(X_{M1} = D \mid D)\,\mathbb{P}(D) + \mathbb{P}(X_{M1} = D \mid ND)\,\mathbb{P}(ND)$$

$$= 0.9 \cdot 0.0002 + 0.15 \cdot 0.9998$$

$$= 0.1505 \Rightarrow \mathbb{P}_2(D) = \frac{0.0002 \cdot 0.9}{0.1505} = 0.0012$$

NOW,

$$\mathbb{P}(D \mid X_{M2}) = \frac{\mathbb{P}_2(D)\,\mathbb{P}(X_{M2} = D \mid D)}{\mathbb{P}(X_{M2})}$$

$$\mathbb{P}(X_{M2}) = \mathbb{P}(X_{M2} = D \mid D)\,\mathbb{P}(D) + \mathbb{P}(X_{M2} = D \mid ND)\,\mathbb{P}(ND)$$

$$= (1 - 0.03) \cdot 0.0002 + 0.05 \cdot 0.9998$$

$$= 0.050184$$

$$= \frac{0.0012 \cdot 0.97}{0.050184} = 0.02317$$

$$\approx 2.32\%$$

Problem 3

(a) $\mathbb{P}(H|\theta) = \theta$

$\mathbb{P}(T|\theta) = 1-\theta$

IT IS REASONABLE TO ASSUME EVERY TOSS OF THE COIN RUNS INDEPENDENTLY FROM THE OTHERS, SO:

$$\mathbb{P}(TTHTTH|\theta) = [\mathbb{P}(T|\theta)]^4 \cdot [\mathbb{P}(H|\theta)]^2$$

$$= (1-\theta)^4 \theta^2$$

(b) $\log[\mathbb{P}(TTHTTH|\theta)] = \log[(1-\theta)^4 \theta^2]$

$$= \log(1-\theta)^4 + \log \theta^2$$

$$= 4 \log(1-\theta) + 2 \log \theta$$

(c) $\underset{\theta}{\operatorname{argmax}} [4 \log(1-\theta) + 2 \log \theta]$ ← WE NEED TO TAKE THE DERIVATIVE WRT θ AND EQUAL IT TO 0.

$$\frac{d}{d\theta}\left(4 \log(1-\theta) + 2 \log \theta\right) = \frac{-4}{1-\theta} + \frac{2}{\theta} = 0$$

$$\frac{2}{\theta-1} + \frac{1}{\theta} = 0$$

$$\frac{2\theta + \theta - 1}{(\theta-1)\theta} = 0$$

$\hat{\theta} = \frac{1}{3}$ IS THE ML ESTIMATE OF θ

Problem 4

(a) $S = 3 \Rightarrow V = \{1, 2, 3\}$

$\mathcal{X} = \{3, \underbrace{1, 1}, 2, 3\}$ $N = \#\text{EXAMPLES} = 5$

$t = \#$ OCCURRENCES OF "1" $= 2$

$$\Rightarrow \hat{\theta}_1 = \frac{t}{N} = \frac{2}{5} = 0.40$$

(b) USING ADD-1 SMOOTHING,

$$\hat{\theta}_1 = \frac{t+1}{N+S} = \frac{2+1}{5+3} = \frac{3}{8} = 0.375$$

(c) $S = 2 \Rightarrow V = \{1, 2\}$

PRIOR :

$$P(\theta) = 6\,\theta(1-\theta)$$

$\mathcal{X} = \{\underbrace{x_1, x_2, \dots, x_N}_{\#(x_i\text{'s THAT ARE } 1) = t}\}$ $N = \#(\text{EXAMPLES})$

FOR THE MAP ESTIMATE $\hat{\theta}_1$, WE HAVE TO SOLVE

$$\underset{\theta}{\arg\max}\ \mathbb{P}(\theta)\ \mathbb{P}(x = 1 \mid \theta) =$$

$$= \underset{\theta}{\arg\max}\ \underbrace{6\theta(1-\theta)\ \theta^t (1-\theta)^{N-t}}_{\text{TAKING THE LOGS,}}$$

$$\log 6 + (t+1)\log\theta + (N-t+1)\log(1-\theta)$$

NOW WE DERIVE WRT θ AND TAKE IT TO 0:

$$\frac{t+1}{\theta} + \frac{t-N-1}{1-\theta} = 0$$

$$(t+1)(1-\theta) + \theta(t-N-1) = 0$$

$$t+1 \cancel{- t\theta} - \theta \cancel{+ t\theta} - N\theta - \theta = 0$$

$$\theta(N+2) = t+1$$

$$\hat{\theta}_1 = \frac{t+1}{N+2}$$

Problem 5 - (a), (b)

(a) USING $\theta_c = \dfrac{t+m}{N+sm}$, WHERE

t = # CLASSES
N = # EXAMPLES
s = # POSSIBLE VALUES OF x_i
$m = 0.3$

$$\mathbb{P}(x_1 = \text{Low} \mid +) = \frac{1+0.3}{2+3\cdot 0.3} = \frac{1.3}{2.9}$$

$$\mathbb{P}(x_2 = \text{Yes} \mid +) = \frac{0+0.3}{2+2\cdot 0.3} = \frac{0.3}{2.6}$$

$$\mathbb{P}(x_3 = \text{Green} \mid +) = \frac{1+0.3}{2+2\cdot 0.3} = \frac{1.3}{2.6}$$

$$\mathbb{P}(x_1 = \text{Low} \mid -) = \frac{1+0.3}{3+3.03} = \frac{1.3}{3.9}$$

$$\mathbb{P}(x_2 = \text{Yes} \mid -) = \frac{2+0.3}{3+2\cdot 0.3} = \frac{2.3}{3.6}$$

$$\mathbb{P}(x_3 = \text{Green} \mid -) = \frac{2+0.3}{3+2\cdot 0.3} = \frac{2.3}{3.6}$$

(b) $\mathbb{P}(x \mid +) = \mathbb{P}(\text{Low} \mid +)\,\mathbb{P}(\text{Yes} \mid +)\,\mathbb{P}(\text{Green} \mid +) =$

$$= \frac{1.3}{2.9} \cdot \frac{0.3}{2.6} \cdot \frac{1.3}{2.6} = 0.0258$$

$\mathbb{P}(x \mid -) = \mathbb{P}(\text{Low} \mid -)\,\mathbb{P}(\text{Yes} \mid -)\,\mathbb{P}(\text{Green} \mid -) =$

$$= \frac{1.3}{3.9} \cdot \frac{2.3}{3.6} \cdot \frac{2.3}{3.6} = 0.136$$

Problem 5 (c)

(c) THE ML LABEL FOR $x = [\text{Low}, \text{Yes}, \text{Green}]$ IS "$-$" AS $\mathbb{P}(x|-) > \mathbb{P}(x|+)$.

(d) $\mathbb{P}(+) \cdot \mathbb{P}(x|+) = \frac{2}{5} \cdot 0.0258 = 0.0103$

$\mathbb{P}(-) \cdot \mathbb{P}(x|-) = \frac{3}{5} \cdot 0.136 = 0.0816$

HENCE, THE MAP LABEL FOR $x = [\text{Low}, \text{Yes}, \text{Green}]$ IS "$-$".