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Digital Systems & Signal Processing

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Preface

The purpose of this document is to act as a comprehensive note for my understanding on the subject matter. I may also use references aside from the lecture material to further develop my understanding, and these references will be listed here.

This document should eventually serve as a standalone reference for learning or review of the subject matter. There is also a lot of organization within these documents, please refer to the table of contents within your PDF viewer for ease of navigation.

References

- Course Material & Provided Lecture Notes
- Applied Digital Signal Processing - G. Manoalkis, K. Ingle

Sampling & Reconstruction of Signals

Frequency

Consider the definitions¹ of a CT-signal.

$$x_a(t) = A \cos(\omega t + \phi)$$

$$x_a(t) = A \cos(2\pi f t + \phi)$$

The value f is called the signals **fundamental frequency** measured in Hertz².

Given some value of f , we can define its inverse, $T = 1/f$ as the signals **fundamental period**³.

If some signal x_a has fundamental frequency f_s and fundamental period T_s , we can say that:⁴

$$x(t + T_s) = x(t)$$

Extending this to DT-signals isn't as straightforward as one would think ...

A DT-signal is periodic with period N if :

$$x(n + N) = x(n)$$

¹ One with radial and one with temporal frequency

² Hz [cycles/sec]

³ Measured in Hz^{-1} [sec/cycle]

⁴ This is the formal definition of fundamental period

The fundamental period of a DT signal, is the smallest value of N such that the above holds true.

Properties of Discrete-Time Signals

DT signals have some strange properties.

- DT-signals are only periodic if their frequency f is a rational number
- DT-signals whose frequencies ω are separated by some integer multiple of 2π , are identical
- DT-sinusoids are distinct only within a certain frequency range, sinusoids with frequencies outside this range are an alias of a sinusoid with frequency within this range
- The highest frequency possible for a DT-signal is at $\omega = \pm\pi$ or $f = \pm 1/2$

Lets consider the first property. We require the following to hold.⁵

⁵ Consider that $x(n) = \cos(2\pi fn)$

$$x(n + N) = x(n)$$

$$\cos(2\pi f(n + N)) = \cos(2\pi fn)$$

$$\cos(2\pi fn + 2\pi fN) = \cos(2\pi fn)$$

This⁶ implies a rational frequency:

$$f = \frac{k}{N}$$

⁶ $\cos(x) = \cos(x + 2\pi k)$ only holds true for some integer value of k , this implies that $2\pi fN = 2\pi k$

The second property can easily be proven as shown below, suppose $k \in \mathbb{Z}$:

$$x_k(n) = \cos(\omega_0 + 2\pi k)n + \phi$$

$$x_k(n) = \cos(\omega_0 n + 2\pi kn + \phi)$$

Then, the following two are indistinguishable.

$$x_k(n) = \cos(\omega_0 n + 2\pi kn + \phi) = \cos(\omega_0 n + \phi)$$

The third and fourth property are basically to be treated as fact, try creating a sequence of this on a graphing calculator to verify that this is true for yourself.

Analog to Digital Conversion

To modify analog signals with a digital signal processor, we must first convert them into digital form⁷. This process is called analog to digital conversion (abbreviated A/D conversion). In many cases we also convert the signal back into analog form (D/A conversion).

⁷ a sequence of binary numbers

A/D Conversion is done in three main steps:

- Sampling
- Quantization
- Encoding

Sampling concerns turning an analog signal into a discrete time signal. **Quantization** involves turning it from continuous valued into a discrete valued signal. Lastly, **Encoding** is about turning it into a binary sequence for transmission.

Sampling

Given some analog input $x_a(t)$ we can perform **periodic** (or **uniform sampling**) by storing its value at certain time intervals described by a **sampling frequency** f_s or **sampling period** T_s .

- Mathematically this would be the equivalent of taking a CT-sinusoid and performing a substitution $t = n/f_s = nT_s$ for some $n \in \mathbb{Z}$
- Essentially we turn a signal into a sequence, $x[n] := x_a(nT_s)$

We know that one of the properties of DT-signals is that if the frequency of two are separated by some integer value, they are indistinguishable from one another. Consider some analog frequency f_1 and f_2 .

$$\frac{f_1}{f_s} - \frac{f_2}{f_s} = k$$

If the above holds true, the DT-signals will look the same after being sampled at frequency f_s . This effect is called **aliasing**.

Consider two CT sinusoids⁸ being sampled at $f_s = 1$ Hz. It can be shown that the frequencies of both signals satisfy the equation above, and are identical after sampling (conversion into a sequence).

$$^8 x_1(t) = \cos(2\pi \cdot 1/8) \text{ and } x_2(t) = \cos(-2\pi \cdot 7/8)$$