Steady State Error

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Control Systems - Note 5

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Chapter 1

Error

Designing control systems depends primarily on three specifications:

1. Transient Response (Note 3)
2. Stability (Note 4)
3. Steady-State Errors (This Note)

Which we will be finishing by the end of this note.

1.1 Formulating Error

Consider a simple **unity-feedback** control system with an open-loop transfer function of $G(s)$, and closed-loop transfer function of $T(s)$. This system is shown in Figure 1.1.

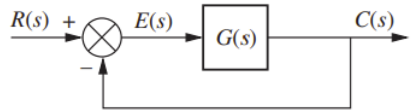

Figure 1.1: Unity Feedback Control System

$$T(s) = \frac{G(s)}{1 + G(s)}$$

The error $E(s)$ in this control system would be the difference between the input $R(s)$ and output $C(s)$. This gives us our first error formulation shown in Equation 1.1.

$$E(s) = R(s) - C(s) \tag{1.1}$$

To generate our next error formulation, consider that $C(s) = R(s)T(s)$. The output is the input multiplied by the closed-loop transfer function. We can substitute this into Equation 1.1 to get our second error formulation in Equation 1.2

$$E(s) = R(s)[1 - T(s)] \tag{1.2}$$

To arrive at our third formulation for error, consider the fact that the output is equal to the error times the open-loop transfer function, $C(s) = E(s)G(s)$. We can substitute this into Equation 1.1 to get our third formulation in Equation 1.3

$$E(s) = \frac{R(s)}{1 + G(s)} \tag{1.3}$$

So far we have found three different formulations for error shown in Equations 1.1, 1.2 and 1.3. All of these formulations depend on the input $R(s)$, but one depends on the output, another depends on the closed-loop transfer function, and the last depends on the open-loop transfer function. The error $E(s)$ is equal to each of the formulations below.

$$R(s) - C(s) \qquad R(s)[1 - T(s)] \qquad \tfrac{R(s)}{1+G(s)}$$

1.2 Steady-State-Error

Although we can calculate the error at any time $e(t)$ through the inverse laplace transform of the $E(s)$ function, we are more interested in the final value of the error, $e(\infty)$. We can easily find this out through final-value theorem. Which is given in Equation 1.4.

$$f(\infty) = \lim_{s\to 0} sF(s) \tag{1.4}$$

1.3 Static Error-Constants & Steady-State Error

Consider a step input applied to a unity-gain feedback system. If we apply this step input into Equation 1.3 we can get the following simplification.

$$\begin{aligned} e_{\text{step}}(\infty) &= \lim_{s\to 0} \frac{s(1/s)}{1+G(s)} \\ e_{\text{step}}(\infty) &= \frac{1}{1+\lim_{s\to 0} G(s)} \end{aligned} \tag{1.5}$$

In order for the step response error at infinity to be equal to 0, we require our denominator to go to zero. This means that we require our limit (The DC-Gain of the open-loop transfer function) to go to infinity. If this condition is not met, we do not achieve a zero error at $t \to \infty$.

$$\text{DC Gain} = \lim_{s\to 0} G(s) = \infty$$

Although this limit is the DC-Gain, we define it as a value called the **position-error constant** K_p. In other words, we require our $K_p = \infty$ for a 0 steady state error from a step response.

$$K_p = \lim_{s\to 0} G(s) \tag{1.6}$$

We can do the exact same process for a ramp input $\left(1/s^2\right)$, and a parabolic input $\left(1/s^3\right)$. In order to define the **velocity-error constant** K_v and the **acceleration-error constant** K_a respectively.

$$K_v = \lim_{s\to 0} sG(s) \tag{1.7}$$

$$K_a = \lim_{s \to 0} s^2 G(s) \tag{1.8}$$

1.4 Satisfying the Error-Constant Conditions

We can think of this as differentiating the open-loop transfer function (as differentiating is equivalent to multiplying by s). Which helps us remember the order of the s term in front of the open-loop transfer function. For their respective inputs each error constant needs to be equal to infinty to give a zero steady state error. If they are not equal to infinity, it means that the steady state error approaches some nonzero ratio (see Equation 1.5)

To satisfy this constraint, our open-loop transfer function needs to be of the form below. Our denominator needs to go to infinity, thus **we require an n'th order pole at the origin**, such that the limit approaches infinity at $s = 0$. The system is also said to be a **type-n system**.

$$G(s) = \frac{(s - z_1)(s - z_2) \ldots (s - z_n)}{s^n (s - p_1)(s - p_2) \ldots (s - p_m)}$$

For the position-error constant, we require that $n \geq 1$. This means that we require at least a 1st order pole at the origin, and consequently at least one integration must be present in the forward path. For the velocity-error constant, we require that $n \geq 2$, as one of these will effectively be "cancelled out" by the s in Equation 1.7, this means at least a 2nd order pole at the origin, and at least 2 integrations in the forward path. Similarly, for the acceleration-error constant, we require that n≥ 3, at least a 3rd order pole at the origin, and at least 3 integrations along the forward path.

Lets consider what happens if we do not satisfy these conditions. Consider a parabolic input with transform $(1/s^3)$. We require at least a 3rd order pole at the origin. Lets examine what happens with a 1st order and 2nd order pole at the origin instead. Let's start by considering a 1st order pole at the origin.

$$K_a = \lim_{s \to 0} s^2 \frac{(s - z_1)(s - z_2) \ldots (s - z_n)}{s (s - p_1)(s - p_2) \ldots (s - p_m)} = 0$$

When we have the error-constant go to 0, this implies that the steady-state error is infinite.

$$e(\infty) = \lim_{s \to 0} \frac{1}{s^2 + s^2 G(s)} = \frac{1}{0} = \infty$$

~	Type-0	Type-1	Type-2
Step	$\frac{1}{1+K_p}$	0	0
Ramp	∞	$\frac{1}{K_v}$	0
Parabolic	∞	∞	$\frac{1}{K_a}$

Figure 1.2: Table Summarizing Steady-State Errors

Lets consider a 2nd order pole now.

$$K_a = \lim_{s\to 0} s^2 \frac{(s-z_1)(s-z_2)\dots(s-z_n)}{s^2(s-p_1)(s-p_2)\dots(s-p_m)} = \frac{z_1 z_2 \dots z_n}{p_1 p_2 \dots p_m}$$

When we have the error-constant equal some rational number, this implies the stead-state error is some finite value.

$$e(\infty) = \lim_{s\to 0} \frac{1}{s^2 + s^2 G(s)} = \frac{1}{\frac{z_1 z_2 \dots z_n}{p_1 p_2 \dots p_m}} = \frac{p_1 p_2 \dots p_m}{z_1 z_2 \dots z_n}$$

We can conclude that when the error-constants are equal to 0, the steady-state error goes to infinity. When the error-constants are equal to some finite value, the steady-state error is also some finite value. It is only when the error-constants go to infinity that the stead-state error goes to 0. The following is summarized in the table below. Each cell represents the steady-state error. The rows are for each type of input (s, $1/s^2$, and $1/s^3$) and the columns are for differing system types.

1.5 Non-Unity Feedback

To use all of this for non-unity feedback, you must derive the error function $E(s)$. Then use these methods to determine the steady-state errors for different types of inputs.