System Stability

Arnav Goyal

Control Systems - Note 4

Contents

Chapter 1

Stability

As previously discussed, we can easily find the time-domain response of a system through laplace transform methods. In this note we will be learning about stability and the criteria for a system to be stable.

1.1 *Types of Stability*

Within control systems, there exist three kinds of stability.

1. **Stable** - A system is said to be stable if the natural response approaches zero as time approaches infinity.

2. **Unstable** - A system is said to be unstable if the natural response grows without bound as time approaches infinity

3. **Marginally Stable** - A system is said to be marginally stable if the natural response neither decays nor grows without bound, but remains constant or oscillates as time approaches infinity.

1.2 *Stability & Pole Location*

The first step to taking inverse laplace transforms is partial fraction decomposition. Recall that decomposition (s-domain) terms of the following form, always result in exponential time-domain terms. In order for a system to be stable, we require these exponential terms to be decaying terms (i.e their argument must be negative).

$$\frac{K}{s+z} \leftrightarrow K \exp\left(-zt\right)$$

In the case that a is some complex value $z = a+bj$, we get a decaying sinusoid term, we require that these decaying exponentials also have negative arguments.

$$\frac{K}{s+z} \leftrightarrow K \exp\left(-at\right) \cos\left(-bjt\right)$$

Both of these cases imply that **stability demands that ALL poles in the transfer function lie on the LHS of the complex plane**. If all but one are on the LHS, the response will grow without bounds and become unstable. In all cases, this requires solving the denominator polynomial which isnt always very simple to do. Often times we wont even have to do this, the most basic criteria for stability is that all terms in the denominator's polynomial standard frm have the same sign, and that there are no missing terms. Which, isnt very helpful. For this reason we introduce something called the Routh-Hurwitz Criterion in the next chapter.

Chapter 2

Routh-Hurwitz Criterion

In this chapter we will learn a method that yields stability information without needing to solve the denominator polynomial. Using this method we can tell *how many* poles lie on the LHS and RHS of the complex plane. (Note: that we say how many, not where. We cannot find their coordinates with this method). This method involves constructing a **Routh Table**, and then interpreting it.

2.1 Routh Tables

Consider the following equivalent (closed-loop) transfer function, we will learn how to construct a routh-table for it. This routh table is used for the Routh-Hurwitz criterion. Figure 2.1 holds the initial layout for a Routh-Table, where we start in the row for the highest degree term and write every other term in the neighbouring columns.

$$G(s) = \frac{N(s)}{a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2			
s^1			
s^0			

Figure 2.1: Initial Layout for a Routh-Table

Completing the Routh-Table

The Routh-Table is completed through the repeated application of determinants. Recall the formula for the determinant of a 2x2 matrix

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

The Routh-Table uses the negative determinant, divided by the term in the row above, and the leftmost column. The completed routh-table looks like this.

TABLE 6.2 Completed Routh table

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$\dfrac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$\dfrac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$\dfrac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
s^1	$\dfrac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$\dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$\dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
s^0	$\dfrac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$\dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

Figure 2.2: Completed Routh-Table

Interpreting the Routh-Table

Now that we know how to make basic Routh-tables, we can look to learn how to interpret them. The way to interpret these tables is to look at the number of sign changes in the first-column. The number of sign changes in the first-column tell us the number of poles on the RHS of the s-plane. Knowing how many are on the RHS, we can use symmetry and the fundamental theorem of algebra to deduce the number on the LHS (having a sign change means the system is unstable). We might sometimes run into special cases when generating this table though.

2.2 Routh-Table: Zero in the First Column

When we encounter a zero in only the first column, we replace it with a variable ϵ, which represents a very small positive or negative number. After we will get a table dependent on epsilon. We see what happens as epsilon goes to 0+ and 0- and count the sign changes for both cases. In both cases they should be the same.

2.3 Routh-Table: Entire Row of Zeroes

When we encounter this the following entries need to be handles differently. In this case we go directly to the **row right above the row of zeros**. Using it to construct an **auxilliary polynomial**, where the highest power of s in the polynomial is the same as the power of s in that row. We then d/ds this polynomial, and use the new coefficients as the replacements for the row of zeroes, continuining the table in the standard way. The interpretation changes as well.

The row (and every row below) from which the auxilliary polynomial was constructed now corresponds with the auxilliary polynomial. This auxilliary polynomial is always an even polynomial and its roots will always have rotational symmetry about the origin. The rows above the ones for the aux polynomial are for the original function. Overall we add up every sign change once again to get the number of poles on the RHS of the complex plane.