# First & Second Order Systems

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*Control Systems - Note 2*

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# Chapter 1

# Time Domain Response

Now that we have learned how to obtain the characteristics of a system, and model it in the $s$-domain. We should begin our study on important characteristics of a system: transient response and steady-state errors. In this note we look at the time domain response (transient response) of first and second order systems.

## 1.1 *Characteristics of Systems*

Systems (in the context of control systems), refer to the transfer function. Systems in general can be written in terms of their numerator and denominator polynomials. The order of a system refers to the number of poles it posesses (degree of the $D$ function)

$$G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0} = \frac{N(s)}{D(s)}$$

We can also formally define the zeroes and poles of a system. Zeroes are values that make the function equal to 0, where poles are values where the function goes to plus or minus infinity. Formally, we can write the following definitions.
The system $f$ has a **zero** at $x_0$ if:

$$f(x)\,|_{x=x_0} = 0$$

The system $f$ has a pole at $x_p$ if :

$$f(x)\,|_{x=x_p} = \pm\infty$$

For our system $G(s)$, This means that the roots of $N$ determines its zeroes, and that the roots of $D$ determines its poles.

## 1.2 Responses of a System

The **step response** of a system can be determined by applying a step input to the transfer function, and determining the output in the time-domain. For first order systems, this will generate two terms called the **forced response**, and the **natural response**. We can also find various other response of a system, for example the impulse response, and the ramp response.

# Chapter 2

# First Order Systems

## 2.1 Characteristic Form

The general form of a **first order system** is given by:

$$T(s) = \frac{A}{s+a}$$

## 2.2 Step Response

We can easily find the step response of a general first order system. Consider an output $R$, and some input $C$. To find the step response, we set $C(s) = s^{-1}$

$$R(s) = \frac{A}{s+a} C(s)$$

$$R(s) = \frac{A}{s(s+a)} = \frac{k_1}{s} + \frac{k_2}{s+a}$$

$$r(t) = k_1 + k_2 e^{-at}$$

We can easily solve for coefficients $k_1$ and $k_2$to obtain the following step response

$$r(t) = \frac{A}{a} - \frac{A}{a}e^{-at} = \frac{A}{a}\left(1 - e^{-at}\right)$$

## 2.3 The Time Constant, $\tau$

The **time constant** $(\tau)$ is a value of $t$ that will provide a 1 into the argument of the exponential terms. The only exponential term present here is $\exp(-at)$. Thus setting values of $t$ equal to the reciprocal of $a$ will always generate 1 for the argument. The time constant is helpful because it helps us evaluate the step response of a first order system, as the output will rise to 63% of the final value within one time-constant, and 95% of the final value at 3 time-constants.

$$\tau = \frac{1}{a}$$

## 2.4 Initial Speed

The **initial speed** of a first order system's step response can be derived by differentiating the expression, and evaluating it at $t = 0$. We can see that the intial slope (speed) is equal to the number $A$ in the numerator of the transfer function. A system is said to be faster than another system if it has an intial speed greater than that of the other.

$$\frac{d}{dt}\left[\frac{A}{a}\left(1 - e^{-at}\right)\right]|_{t=0} = \frac{A}{a}[a] = A$$

## 2.5 Rise Time, $T_r$

The **rise time** $(T_r)$ of a system is defined as the time it takes for the system to go from 10% to 90% of its final value. We can easily find this for the general first order system.

$$T_r = t_{90} - t_{10}$$

It is trivial to determine $t_{90}$ and $t_{10}$ analytically.

$$t_{90} = -\frac{\ln(0.1)}{a}$$

$$t_{10} = -\frac{\ln(0.9)}{a}$$

Thus we can write it in terms of the value $a$, or the time constant $\tau$.

$$T_r = \frac{1}{a}\left(\ln(0.9) - \ln(0.1)\right) = \frac{2.2}{a} = 2.2\tau$$

## 2.6 Settling Time, $T_s$

The settling time $(T_s)$ of a system is defined as the time it takes for the system to reach, and stay within 2% of its final value. In general this is also trivial to determine analytically. We can write it in terms of the value $a$, or the time-constant $\tau$.

$$aT_s = -\ln(0.02)$$

$$T_s = \frac{3.91}{a} = 3.91\tau$$

## 2.7 Graph Methods

We can determine (approximate) the transfer function from a plot of the step response for a first-order system. Here are the steps to do so.

1. Determine the final value. *Note*: it is equal to $A/a$
2. Determine the time constant by finding the time where the response reaches 63% of the final value. *Note:* it is equal to $1/a$
3. Now we can determine $A$, because we have determined $a$.

# Chapter 3

# Second Order Systems

In this chapter we are going to study the time response of a **second order system**. Unlike first order systems, a second ordersystem can exhibit a wide range of responses. Developing an understanding of first and second order systems are quite important within control engineering, as we can often represent higher order systems as consituents containing first order and second order terms.

## 3.1 Characteristic Form

A general second order system is represented as the following, and note that it represents a second order linear differential equation. Some of the variables in this equation are a bit strange at first, but lets define them and what they mean.

- **The Damping Factor** $\zeta$ - Determines the **damping regime** of a system - if it is underdamped, critically damped, or overdamped.
- **The Natural Frequency** $\omega_0$ - Determines the **undamped oscillation frequency** of the system

$$T(s) = \frac{K}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

## 3.2 Damping Regimes

In order to get a better understanding of the system we will have to solve the denominator to find the system poles. Via the quadratic formula we can write the following solutions for the poles

$$s_{1,2} = -\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2 - 1} \tag{3.1}$$

Now it is quite obvious as to why we consider the variable $\zeta$, as the one that determines the behaviours of the system.

1. Repeated, real roots - $\zeta = 1$
2. Distinct, real roots - $\zeta > 1$
3. Distinct, complex roots - $\zeta < 1$

In case 1, we call the system **crtitically damped**. In case 2 we call the system **overdamped**. In case 3, we call the system **underdamped**. This is because the value of $\zeta$ determines the number of poles, and the type of each pole that we have. There is also the case where $\zeta = 0$ which is referred to as **undamped**, however this still produces distinct complex roots, thus we deem it a *special case* and stick it under case 3.

## 3.3 Undamped Systems

An **undamped system arises when $\zeta = 0$,** and this results in a marginally unstable system that oscillates indefintely. Lets take a look at the step response for a general undamped system below.

$$T(s) = \frac{\omega_0^2}{s^2 + \omega_0^2}$$

The output $R$ for some applied step response is shown below. We can also write partial fraction decomposition for this expression.

$$R(s) = \frac{\omega_0^2}{s^2 + \omega_0^2} \cdot \frac{1}{s}$$

$$R(s) = \frac{k_1}{s} + \frac{k_2}{s - j\omega_0} + \frac{k_3}{s + j\omega_0}$$

We can then take the inverse laplace of this expression and get the response in the time domain. Note that the **natural frequency** $\omega_0$ appears in the argument of the sinuosoidal terms, and that these are everpresent for all $t$. This means that the output oscillates forever, even after reaching the final constant value.

$$r(t) = k_1 + k_2 \exp(j\omega_0 t) + k_3 \exp(-j\omega_0 t)$$

$$r(t) = k_1 + (k_2 + k_3)\cos(\omega_0 t) + (k_2 - k_3)\sin(\omega_0 t)$$

## 3.4 *Underdamped Systems*

An **underdamped system arises when $\zeta < 1$,** this results in a time-domain response with damped oscillations. Lets take a look at the general underdamped system below.

$$T(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

We can find poles of the denominator through the quadratic formula (equation 3.1). Lets also remember that we are treating it as $\zeta < 1$ thus lets factor out a $j$ as we expect to get complex roots. Let's also rewrite this complex thing into its real and complex parts $\sigma_d$ and $\omega_d$ respectively. Here $\sigma_d$ represents the exponential damping of the oscillations and $\omega_d$ represents the frequency of the damped oscillations.

$$s_{1,2} = -\zeta\omega_0 \pm j\sqrt{1 - \zeta^2}$$

$$s_{1,2} = \sigma_d \pm j\omega_d$$

The output $R$ for some applied step response is shown below. We can also write partial fraction decomposition for this expression.

$$R(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2} \cdot \frac{1}{s}$$

$$R(s) = \frac{k_1}{s} + \frac{k_2}{s - \sigma_d - j\omega_d} + \frac{k_3}{s - \sigma_d + j\omega_d}$$

We can now take inverse laplace to get the following (disgusting expression) which we will have to simplify further. *Note:* $\sigma_d$ is a negative number, hence these oscillations will eventually die out as $t \to \infty$. We can also rewrite the final line in terms of a single sinusoid, which is omitted from this note, but can easily be done through basic phasor addition.

$$r(t) = k_1 + k_2 \exp((\sigma_d + j\omega_d)t) + k_3 k_2 \exp((\sigma_d - j\omega_d)t)$$

$$r(t) = k_1 + \exp(\sigma_d t)(k_2 \exp(j\omega_d t) + k_3 \exp(-j\omega_d t))$$

$$r(t) = k_1 + \exp(\sigma_d t)[(k_2 + k_3)\cos(\omega_d t) + (k_3 - k_2)\sin(\omega_d t)]$$

## 3.5 *Critically Damped*

A **critically damped system arises when $\zeta = 1$**, This results in the **fastest possible response (without any overshoot)** that a second order system can possibly give. Let's take a look at the critically damped system below

$$T(s) = \frac{\omega_0^2}{s^2 + 2\omega_0 s + \omega_0^2}$$

The output $R$ for some applied step response is shown below, we can also easily factor this as follows to form the partial fraction decomposition.

$$R(s) = \frac{\omega_0^2}{s^2 + 2\omega_0 s + \omega_0^2} \cdot \frac{1}{s}$$

$$R(s) = \frac{k_1}{s} + \frac{k_2}{s + \omega_0} + \frac{k_3}{(s + \omega_0)^2}$$

Taking inverse laplace give the following time domain response

$$r(t) = k_1 + k_2 \exp(-\omega_0 t) + k_3 t \exp(-\omega_0 t)$$

## 3.6 Overdamped Systems

An **overdamped system arises when $\zeta < 1$,** this results in a time-domain response with two time constants. Lets take a look at the general overdamped system below.

$$T(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

The output $R$ for some applied step response is shown below. We can find poles of the denominator through the quadratic formula (equation 3.1) which will both be distinct real numbers here, which I will denote $s_1$ and $s_2$.

$$R(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2} \cdot \frac{1}{s}$$

$$R(s) = \frac{k_1}{s} + \frac{k_2}{s - s_1} + \frac{k_3}{s - s_2}$$

We can then take inverse laplace to get the following time domain expression with two time constants (double exponential)

$$r(t) = k_1 + k_2 \exp(-s_1 t) + k_3 \exp(-s_2 t)$$

## 3.7 Performance Specifications

We need some way to measure how good our second order systems are doing, thus we can introduce some new concepts in addition to the **rise time $T_r$** and the **settling time $T_s$** that we defined earlier.

- Peak Time $T_p$ - The time required for the response to reach its peak, can be obtained as the time where the first derivative becomes zero.
- Percent Overshoot - Defined as the amount the system overshoots at the peak time, when compared to the final value.
- Settling Time $T_s$ - Once again, defined as the amount of time it takes for the response to reach and stay within 2% of the final value.
- Rise Time $T_r$- The time it takes for the response to go from 10% to 90% of the final value. It is not possible to determine a general expression analytically for second order systems.