Complex Analysis

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Complex Analysis - Comprehensive Note

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Part I

Functions of a Complex Variable

Chapter 1

Sets in the Complex Plane

We already have scratched the basics of working with the **imaginary unit** $j = \sqrt{-1}$ in other coursework, and it will not be covered again for the 845th time! If you need a refresher, try this out! In this chapter we will be talking about types of sets in the complex plane, some formal definitions and what they geometrically are, we do this so that we can eventually work our way up to understanding functions of a complex variable.

1.1 *Terminology*

Lets start by introducing some essential terminology about sets in the complex plane.

Suppose we have some $z_0 = x_0 + jy_0$. Since the expression $|z - z_0|$ represents the distance between points z and z_0 on the complex plane, we can denote **a circle**, in other words set of circular points lying on some radial distance ρ from a center point z_0 as Equation 1.1

$$|z - z_0| = \rho \tag{1.1}$$

This means that $|z| = 1$ is the unit circle centered at the origin, and that $|z - 1 - 2j| = 5$ is a circle of radius 5, centered at the point $1 + 2j$. What happens if we modify the equality in Equation 1.1 to the inequality seen in Equation 1.2?

$$|z - z_0| < \rho \tag{1.2}$$

The inequality in Equation 1.2 is called the **neighborhood of z_0** or an **open disk**. It defines the set of points that lie within, but not on the circle defined by Equation 1.1.

1.2 *Interior, Exterior & Boundary Points*

Using this definition of the neighborhood in Equation 1.2 we can formally define an **interior point**. A point z_0 is said to be an interior point of some set S of the complex plane if there exists some (possibly infintely small!) neighborhood of z_0 that lies entirely within S.

Through the same logic, a point z_0 is said to be an **exterior point** of some set S if there exists some (possibly infinitely small!) neighborhood of z_0 that lies entirely outside of S.

We can also say that a point z_0 is said to be a **boundary point** of the set S if *every* neighborhood of $z_0$contains at least one point within S, and one point outside of S.

1.3 *Open & Closed Sets*

Now that we have formally defined the three types of points, we can talk about **open-sets** and **closed-sets**.

If every point within a set S is an interior-point, the set S is said to be an **open-set**. If this condition is not met, this means that the set posesses boundary points, and it is called a **closed-set**

This is basically a strange way of saying that open-sets have no boundaries. The set defined in Equation 1.2 is an example of an open-set because for every point within the set we can get infinitely closer and closer to the boundary without ever touching the boundary. On the other hand a closed-set does have a boundary and if we take a point on the boundary, some of this points neighborhood will lie outside the set S, rendering it a non-open (closed) set.

Open sets manifest when there is no semblance of an 'equals' in the inequality, and closed sets manifest when there are semblances of the 'equals'. For example, the symbols $\geq, \leq$ define a closed set, while the symbols $<, >$ define an open set. Here are some examples of open-sets

- $\mathrm{Im}\,(z) > 0$ - The upper-half of the complex plane
- $1 < \mathrm{Re}\,(z) < 2$ - An infinitely tall strip
- $|z| > 1$ - The exterior of a unit circle
- $1 < |z| < 2$ - The open annulus (donut) centered at the origin with inner radius 1 and outer radius 2.

Through these examples we can see that the inequality in Equation 1.3 represents the general **open-annulus** centered at z_0, with inner and outer radii of ρ_1 and ρ_2 respectively. There is a lot of other stuff that we can cover related to sets, but it is not required to understand the preceeding chapters.

$$\rho_1 < |z - z_0| < \rho_2 \tag{1.3}$$

Chapter 2

Functions of a Complex Variable

Now that we have defined what sets are in the complex plane, we will have noticeably easier time thinking about what **functions of a complex variable** are. Recall that a function is essentially a sort of **correspondance** between two sets: the input, and the output. Each element in the input set is mapped to only one element in the output set. While the same element in the output set can be reached through multiple elements in the input set. Every x produces only one y, but every y may have more than one x that produces it.

2.1 *Complex Functions*

Suppose we have some real valued function $y = f(x)$. If the domain of this function is extended to include the complex numbers z, we call it a **function of a complex variable** - or a **complex function** for short. We also write it in a slightly differently as Equation 2.1 to imply that it is a complex function.

The image (output) w, of a complex number (input) z, will be some complex number $u + jv$ Where u and v are the real and imaginary parts of w, and are entirely-real valued functions. In other words, we can define some complex function in terms of purely u and v. This fact is also shown in Equation 2.1

$$w = f(z) = u(x,y) + jv(x,y) \tag{2.1}$$

Although we cannot draw a 4-dimensional graph to represent a complex-function. We can think of it as a **mapping between planes**. more specifically, from the input-plane (z-plane) to the output-plane (w-plane) This is shown in Figure 2.1

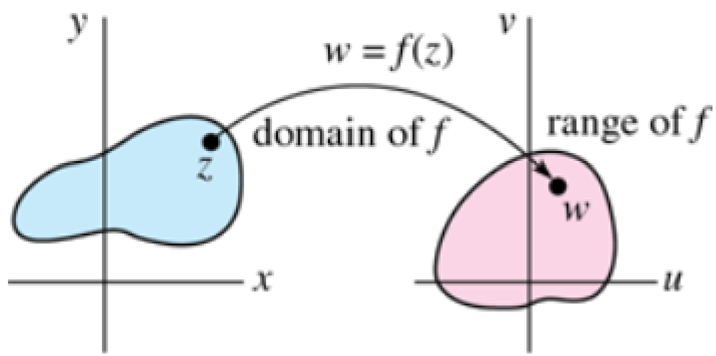

Figure 2.1: The mapping from z-plane to w-plane

2.2 *Limits & Continuity*

The **limit-definition of a complex function** looks the same as the epsilon-delta limit-definition for real valued functions. All it means is that the points $f(z)$ can be made arbitrarily close to the point L if we choose a point z close to, but never equal to the point z_0.

$$\lim_{z \to z_0} f(z) = L \tag{2.2}$$

The fundamental difference between the limits of real-valued functions and complex functions lies in the geometric meaning of the limit. For real-valued functions, we can approach from either the left or the right, but for complex functions we should approach the same thing no matter what path we follow. In other words: when we say that the limit in Equation 2.2 exists, we mean that $f(z)$ approaches L as the point z approaches z_0 from *any* direction. The properties of limits are the exact-same as real valued limits.

A complex function is said to be **continuous** at a point z_0 if the equality in Equation 2.3 holds true, which makes sense.

$$\lim_{z \to z_0} f(z) = f(z_0) \tag{2.3}$$

2.3 *Complex Derivatives*

Now that we have discussed the basics of limits and continuity for complex functions, the next question to ask is "what is a complex derivative?". As we will see, this is very different from asking that its real and imaginary parts have partial derivatives with respect to x and y.

We can say that a complex function $f(z)$ is **complex differentiable** at z_0 if the limit in equation 2.4 or 2.5 exists. These are the regular definitions of a derivative, thus the **real-valued rules of differentiation are the same** as the complex differentiation.

If a complex function is differentiable at point z_0 and at every point in some neighborhood of z_0 we call that function an **analytic function**.

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \tag{2.4}$$

$$f'(c) = \lim_{z \to c} \frac{f(z) - f(z_0)}{z - z_0} \tag{2.5}$$

Another useful thing to know about analyticity of functions is that **if we can express a complex-function as a rational function** (numerator and denominator are polynomials) we can say that **the function is analytic** wherever the denominator is not equal to zero. This is shown below. The only requrement here is that N and D are polynomials.

Recall: This is the case because the limits of complex-polynomials exist everywhere, so they are differentiable and analytic everywhere, thus the rational function is only non-analytic wherever the fraction is undefined (denominator is 0).

$$\text{if } f(z) = \frac{N(z)}{D(z)}, \text{ then } f \text{ is analytic wherever } D(z) \neq 0$$

2.4 Cauchy-Riemann Equations

We now turn to the question of systematically deciding when some complex function is differentiable. If the complex derivative exists at some point z then we should be able to compute it via any path we choose. Let's choose to approach them via purely horizontal and vertical paths. We can use Equation 2.4 to carry this out.
Note: In each of the two limits in the equation below corresponds to approaching the limit horizontally (along a purely real t) or vertically (along a purely imaginary t), all for some $t \in \mathbb{R}$.

$$f'(z) = \lim_{t \to 0} \frac{f(z+t) - f(z)}{t} = \lim_{t \to 0} \frac{f(z+jt) - f(z)}{jt}$$

We can rewrite this in terms of u and v. First lets approach the derivative along a horizontal line. We can see that this simplifies into the definitions of the partial derivatives of u and v in Equation 2.6

$$f'(z) = \lim_{t \to 0} \frac{u(x+t, y) + jv(x+t, y) - u(x, y) - jv(x, y)}{t}$$

$$f'(z) = \lim_{t \to 0} \frac{u(x+t, y) - u(x, y)}{t} + j \lim_{t \to 0} \frac{v(x+t, y) - v(x, y)}{t}$$

$$f'(z) = \frac{\partial u}{\partial x} + j \frac{\partial v}{\partial x} \tag{2.6}$$

Now if we approach the derivative along a vertical line, we can see similar partial derivatives appear in Equation 2.7, but this time with respect to y instead of x

$$f'(z) = \lim_{t \to 0} \frac{u(x, y+t) + jv(x, y+t) - u(x, y) - jv(x, y)}{jt}$$

$$f'(z) = -j \lim_{t \to 0} \frac{u(x, y+t) - u(x, y)}{t} + \lim_{t \to 0} \frac{v(x, y+t) - v(x, y)}{t}$$
$$f'(z) = \frac{\partial v}{\partial y} - j \frac{\partial u}{\partial y} \tag{2.7}$$

We can now equation Equations 2.6 and 2.7 by equating real and imaginary components. In doing so we obtain the **Cauchy-Riemann Equations**. Which tell us the complex values (z values) where a function is differentiable.

We can use these equations to test analyticity of a function on some domain D by checking if they are differentiable everywhere on this domain D (are the cauchy-riemann equations satisfied everywhere in D?).

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

2.5 *Using the Cauchy-Riemann Equations*

Lets use the equations we just derived to show that the function $f(z) = |z|^2$ is differentiable only at $z = 0$, and analytic nowhere.

$$f(z) = |z|^2 = \left(x^2 + y^2\right) + j(0)$$

This gives $u = x^2 + y^2$ and $v = 0$. Thus the Cauchy-Riemann Equations reduce to the equations below, which are only simultaneously satisfied when $x = y = 0$. This function is only differentiable at a single point, $z = 0$ thus it is analytic nowhere (*Recall:* analyticity is a neighborhood property)

$$2x = 0$$
$$0 = -2y$$

2.6 *Relation to Laplace's Equation*

Suppose we have some analytic function (aka function that is differentiable everywhere) $f(z) = u + jv$ we call its constiuent functions u and v **harmonic functions** because they can be related to **Laplace's Equation** (Steady State Heat) through the **Cauchy-Riemann Equations**.
Let's write the Cauchy-Riemann Equatons, and partially differentiate one of these equations w.r.t x, the other to y.

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial v}{\partial x \partial y}$$

$$-\frac{\partial v}{\partial x \partial y} = \frac{\partial^2 u}{\partial y^2}$$

We can combine these to get the following relation to laplace's equation. We can follow a similar process (switching the equation differentiation variables) to prove the second line.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Chapter 3

Complex Function Families

There are a couple families of functions that behave differently when their domain is extended to include the complex plane. In this chapter we will learn about them.

3.1 The Exponential Function

Recall that real-valued exponential function has the following properties

$$\frac{d}{dx}\left[e^{x}\right] = e^{x}$$

$$e^{x+y} = e^{x}e^{y}$$

If we define the **complex-valued exponential** in terms of the complex variable $z = x + jy$.

$$w = e^{z} = e^{x+jy}$$

Then by properties of the real-valued exponential we can write the following. Giving the formal definition of the complex-argument exponential function in Equation 3.1
Note: the complex-valued exponential reduces to the real-valued one when $y = 0$

$$e^z = e^x e^{jy}$$
$$e^z = e^x (\cos y + j \sin y) = e^x \cos y + j e^x \sin y \tag{3.1}$$

Through Equation 2.6 or 2.7 you can prove that **the complex exponential satisfies both properties** shown at the start of this section. The complex exponential is also said to be periodic with a complex period $2\pi j$.

3.2 Logarithmic Function

The **logarithm of a complex number** $z = x + jy$ is defined as the **inverse of the complex-exponential** function. That is $w = \ln z$ if $z = e^w$. Just like the real-valued logarithm, the complex-logarithm is not defined at $z = 0$.

to find the the real and imaginary parts of the logarithm, we write $w = u + jv$.

$$z = e^w$$
$$x + jy = e^u \cos v + j e^u \sin v$$

This implies that $x = e^u \cos v$, and that $y = e^u \sin v$. We can solve these for u and v (the real and complex parts of the ln) by squaring both of these results and adding them together. This gives a solution for u.
Note: here the u value is the nautral logartihm of the modulus (a real-number)

$$e^{2u}(1) = x^2 + y^2 = |z|^2$$
$$u = \ln |z|$$

To solve for v, we have to divide both equations, to obtain the following. The below equation means that v is θ, the argument of z

$$v = \arctan \frac{y}{x}$$

This gives a final definition of the complex-logarithm, which depends on the mod and (non-unique) argument of z

$$\ln z = \ln |z| + j \arg (z)$$

Part II

Complex Integration Theorems

Chapter 4

Complex Contour Integration

Integration in the complex plane is defined in a somewhat similar manner to that of a line integral in the plane. In other words, we will be dealing with an integral of a complex function $f(z)$ that is defined along a curve C in the complex plane.

These curves are parametrized in $t \in \mathbb{R}$ with equations $x = x(t)$ and $y = y(t)$. By using x and y as the real and imaginary parts, we can parametrize a curve C in the complex plane by means of a real-variable.

4.1 *Defining the Complex Integral*

Since the complex plane is a 2-dimensional plane, every complex integral is at least a complex contour integral, leading to the nomenclature of calling them **complex integrals**. We can start to define the complex integral similarly to how we would a real integral. This leads to the general definition of a complex integral in Equation 4.1

$$\int_C f(z)\ dz = \lim_{\Delta z \to 0} \sum f(z)\, \Delta z$$

$$\int_C f(z)\ dz = \lim_{\Delta x,\ \Delta y \to 0} \sum (u + jv)\,(\Delta x + j\Delta y)$$

$$\int_C f(z)\ dz = \lim_{\Delta x,\ \Delta y \to 0} \sum (u\Delta x - v\Delta y) + j\,(v\Delta x + u\Delta y)$$

$$\int_C f(z)\ dz = \int_C u\,dx - v\,dy + j\int_C v\,dx + u\,dy \tag{4.1}$$

4.2 Evaluation of a Complex Integral

Evaluate the integral below, given that the curve C can be parametrized as $x = 3t$, $y = t^2$, for $t \in [-1, 4]$.

$$\int_C \bar{z}\,dz$$

To begin lets write $z(t) = x(t) + jy(t)$. Lets also find $z'(t)$. This means that the integrand (complex conjugate) has its middle sign flipped. Note that we are also finding dz and not $d\bar{z}$.

$$z(t) = 3t + jt^2$$

$$\frac{dz}{dt} = 3 + j2t$$

$$\bar{z}(t) = 3t - jt^2$$

We can now entirely parametrize the integral. We get the following computable line integral upon doing so. At this point the question is basically done, we evalute the integral and simplify the result into its real and complex parts.

$$\int_{-1}^{4} \left(3t - jt^2\right)(3 + j2t)\ dt = 196 + 65j$$

4.3 Fundamental Theorem of Complex Contour Integrals

Just like in multivariate calculus, we have a formulation called the **independence of path** for a complex line integral.

Suppose f is continuous (analytic) in D. If there exists a function, F such that it is antiderivative of f in D. Then for any contour C in D with an initial point z_0 and terminating point z_1 we can write:

$$\int_C f(z)\, dz = F(z_1) - F(z_0)$$

Chapter 5

Cauchy-Goursat Theorem

In this chapter we will learn the **cauchy-goursat** theorem, which is just a complex sounding term for something very simple!

5.1 The Cauchy-Goursat Theorem

The **Cauchy-Goursat Theorem** tells us that if we are taking the closed contour integral of some complex function $f(z)$ on some closed contour C, the result will be zero is f is analytic in the region enclosed by C.

$$\text{if } f \text{ is analytic within } C \ldots \oint_C f(z)\ dz = 0$$

Note: When evaluating complex-integrals, another theorem for analyticity that is very useful was briefly covered in Section 2.3. The theorem tells us that if a complex-function can be expressed as a ratio between polynomials in z. Then the function is differentiable and analytic everywhere except where the denominator is equal to 0.

5.2 Deforming Contours

Suppose the function we are trying to take the contour integral of is analytic everywhere, except for a point within the closed contour. The simple process of **deformation of contours** helps us avoid having to actually take the line integral here. Consider the orange region in Figure 5.1 which represents the regions of analyticity. What would we do if we were trying to find the contour integral of C_1?

Consider the fact that we can connect the two contours in (a) by going through points A and B, and that by doing this, we only encircle a region where the function is fully analytic, thus the entire thing would equal to 0. We would then arrive at the following formulation. Given that the two line integrals from A to B and B to A cancel out, we are left with Equation 5.1.

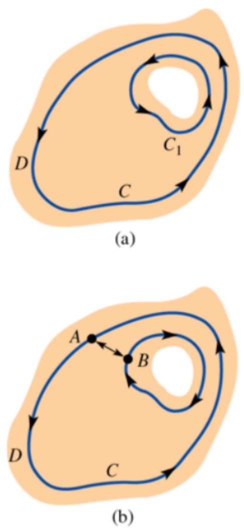

Figure 5.1: Deforming a contour for a Doubly-Connected Domain

$$\oint_C + \int_{AB} - \oint_{C_1} + \int_{BA} = 0$$

$$\oint_C f(z)\, dz = \oint_{C_1} f(z)\, dz \tag{5.1}$$

We can do this for any number of smaller closed contours C_n within a larger (encompassing) contour C. We can repeat the process above for **multiply connected domains** and come to Equation 5.2. Figure 5.2 shows the process for a triply-connected-domain where $n = 2$

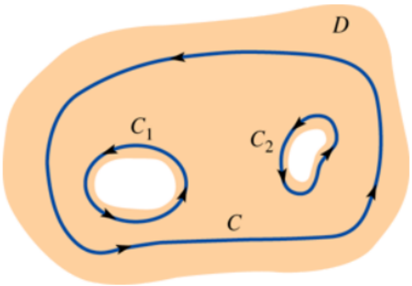

Figure 5.2: Deforming a contour for a Triply-Connected Domain

$$\oint_C f(z)\, dz = \sum_n \oint_{C_n} f(z)\, dz \tag{5.2}$$

5.3 Applying Deformation of Contours

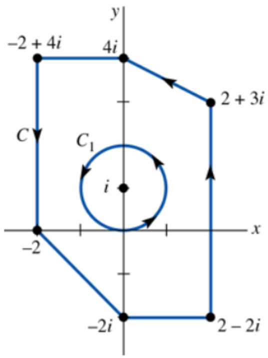

Figure 5.3: Evaluate $\oint_C (z-j)^{-1}\ dz$ where C is the outer contour

Using deformation of contours (our function z^2 is a polynomial and is analytic everywhere) we can conclude that the complicated contour integral around C is equal to the simpler contour integral around the circle $|z-j|=1$, which we can parametrize as $x=\cos t$, $y=1+\sin t$, for $t\in[0,2\pi]$. Now we can write our defintion of z

$$z = x + jy = \cos t + j\,(1+\sin t)$$

$$z = \cos t + j\sin t + j = j + e^{jt}$$

$$z - j = e^{jt}$$

$$\frac{dz}{dt} = je^{jt}$$

Now we can do our integral, which gives us the following result

$$\oint_C \frac{dz}{z-j} = \int_0^{2\pi} \frac{je^{jt}}{e^{jt}}\, dt = 2\pi j$$

Chapter 6

Cauchy-Integrals

6.1 A Useful Theorem

The integral we come across at the end of section 5.3 can be **generalized using deformation of contours**. Suppose we had to evalute the following integral instead...

$$\oint_C \frac{1}{(z-z_0)^n}\,dz$$

Lets **pick a simpler contour** C_1 with radius 1 such that the point z_0 is an interior point to this integral, for simplicity lets simply center the contour C_1 at z_0 such that it is guaranteed to be an interior point. Via deformation of contours, we can say that the below integral is equal to our original one.

$$\oint_{C_1} \frac{1}{(z-z_0)^n}\,dz$$

Now lets parametrize our complex z with $x = \text{Re}\{z_0\} + \cos t$, $y = \text{Im}\{z_0\} + \sin t$, for $t \in [0, 2\pi]$.

$$z = x + jy = (\text{Re}\{z_0\} + j\text{Im}\{z_0\}) + (\cos t + j \sin t)$$

$$z = z_0 + e^{jt}$$
$$z - z_0 = e^{jt}$$
$$(z - z_0)^n = e^{njt}$$
$$\frac{dz}{dt} = je^{jt}$$

Our integral now becomes

$$j \int_0^{2\pi} \frac{e^{jt}}{e^{njt}}\, dt$$
$$j \int_0^{2\pi} \exp\left((1-n)\, jt\right)\, dt = \left.\frac{\exp\left((1-n)\, jt\right)}{1-n}\right|_0^{2\pi}$$

Which evaluates to the following. The numerator is a complex exponential with an argument that is always a multiple of $2\pi j$, which means it is a phasor of magnitude 1, and angle always on the x-axis (always just a 1). This is the case for all values except for $n = 1$, where the output is undefined.

We also must evaluate the expression as $n = 1$, which we can do with the limit below as it gives an indeterminate form.

$$\lim_{n \to 1} \frac{\exp\left((1-n)\, 2\pi j\right) - 1}{1-n} = 2\pi j$$

We did all this work to show that the following is true, considering the earlier expression is **the result of our very first integral**, we can say that for some $n \in \mathbb{Z}$:

$$\oint_C \frac{1}{(z - z_0)^n}\, dz = \begin{cases} 2\pi j, & n = 1 \\ 0, & n \neq 1 \end{cases} \tag{6.1}$$

6.2 *Cauchy's Integral Formula*

Suppose we have some analytic f over some simply-connected domain D. Let C be a simple-closed-contour lying entirely within D. Then if z_0 is any point within C, we can say that equation 6.2 (called **Cauchy's Integral Formula**) is true. We will also see that this formula is just a special case of the formula in section 6.3.

$$f(z_0) = \frac{1}{2\pi j} \oint_c \frac{f(z)}{z - z_0} \, dz \tag{6.2}$$

This can be easily shown by adding and subtracting $f(z_0)$ in the numerator.

$$\oint_c \frac{f(z)}{z - z_0} \, dz = \oint_c \frac{f(z) + f(z_0) - f(z_0)}{z - z_0} \, dz$$

$$\oint_c \frac{f(z)}{z - z_0} \, dz = \oint_c \frac{f(z_0)}{z - z_0} \, dz + \oint_c \frac{f(z) - f(z_0)}{z - z_0} \, dz$$

$$\oint_c \frac{f(z)}{z - z_0} \, dz = f(z_0) \oint_c \frac{1}{z - z_0} \, dz + \oint_c \frac{f(z) - f(z_0)}{z - z_0} \, dz$$

We have shown in equation 6.1 that the first term is equal to $2\pi j$, and through some fancy pure math stuff the second term cancels out. Leaving the following as an equivalent expression to equation 6.2

$$\oint_c \frac{f(z)}{z - z_0} \, dz = 2\pi j f(z_0)$$

This formula is useful, because we can avoid doing the integral completely if we only have a 1st order pole within our contour C, and we recognize an $f(z)$ from an integral in the form of equation 6.2. We evaluate the integral by finding the value of $f(z_0)$, and maybe multiplying it by $2\pi j$.

Note: sometimes we may have another pole outside the contour, in which case we can still use this formula, we simple bring the exterior pole up into our $f(z)$ and only consider the pole within the contour as our denominator.

6.3 *Cauchy's Integral Formula for Derivatives*

Following up on the previous section, if we have a higher order pole within our contour? We can now use **Cauchy's Integral Formula for Derivatives**. Which is shown by equation 6.3

$$f^{(n)}(z_0) = \frac{n!}{2\pi j} \oint_c \frac{f(z)}{(z - z_0)^{n+1}} \, dz \tag{6.3}$$

Just like at the end of the previous section, we can use this formula if a higher order pole is present within our contour (regardless of however many poles are outside the contour, we bring these poles up into the numerator and include them in our f). Now we have a pretty diverse toolbox to deal with these complex integrals (by rarely actually doing integrals too!)

6.4 *Applying Cauchy's Integral Formulations*

Evaluate the following, where $C : |z| = 1$

$$\oint_c \frac{z+1}{z^4 + 4z^3} \, dz$$

Immediately, it looks good to factor out the z^3 from the denominator. giving the following polynomial in the denominator. Here we have a 3rd order pole at the origin, and a first order pole at -4. Only the 3rd order pole is within our contour, the other one is outside, so we can use one of cauchy's formulations but which one? Here we have a higher order pole (not a first order pole) inside the contour, which means we must use cauchy's integral formula for derivatives (equation 6.3).

$$z^3 (z+4)$$

Lets reformat our integral to make it look like the form of equation 6.3. in our integral, $f(z)$ is everything but the pole, thus we include the other factor in our defintion of $f(z)$, for clarity's sake this factor has been rewritten in the numertor

$$\oint_c \frac{(z+1)(z+4)^{-1}}{z^3} \, dz$$

Then by equation 6.3

$$\oint_c \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi j}{n!} f^{(n)}(z_0)$$

$$n+1 = 3 \rightarrow n = 2$$

$$f(z) = (z+1)(z+4)^{-1}$$

$$z_0 = 0$$

$$f(0) = \frac{1}{4}$$

Now we must find the second derivative of f, and its value evaluated at 0, Which is not shown here (is trivial compared to the rest of the problem). Regardless we get a final answer as below.

$$f''(z_0) = -\frac{3}{32}$$

$$\oint_c \frac{z+1}{z^4+4z^3}\,dz = -\frac{2\pi j}{2!}\left(\frac{3}{32}\right)$$

$$\oint_c \frac{z+1}{z^4+4z^3}\,dz = -\frac{3\pi j}{32}$$

Chapter 7

Series Expansions

7.1 Taylor Series Expansion

The **Taylor Series** for complex functions is defined in exactly the same way as real-valued functions. Suppose we have some f that is analytic on some domain D, and that we are trying to fnd an expansion at a point a within D. We can write the taylor series as equation 7.1

$$f(a) = \sum_{k} \frac{f^{(n)}(a)}{n!} (x-a)^n \tag{7.1}$$

Equation 7.1 is formally called the "Taylor series for $f(z)$ centered at a", and it is valid for the largest circle centered around a that still lies entirely within D. This series is sometimes called the **MacLaurin Series** when the series is centered at 0 i.e $(a = 0)$.

Sometimes we hear about a term called the **radius of convergence** R. This term describes how large around the center point that the series converges to the original function. In terms of complex functions, this R is just the distance between the center point and the nearest **isolated singularity**.

An **isolated singularity** is a point at which f fails to be analytic, but where f remains analytic in some neighbourhood around that point (think point discontinuity). An example of an isolated singularity is shown below. Furthermore, if f is an entire function, then $R = \infty$.

$$f(z) = \frac{1}{z - s_i} \text{ implies that } z = s_i \text{ is an isolated singularity}$$

7.2 Laurent Series

So far we've seen some things coming up: **singular points**, **singularities**, and **poles**. Which are all examples of when $f(z)$ is not analytic. These points are found as the zeroes of the denominator in f. Contrast this with **entire functions** which are analytic everywhere.

Here we can define Laurent's Theorem (leading to a Laurent Series) which lets us find a series expansion for f on an annular domain, centered at an **isolated singularity**.

Let f be analytic within an annular domain D defined by $r < |z - z_0| < R$. Then f has a series expansion, valid on D, that is given by equation 7.2 and 7.3

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k \tag{7.2}$$

$$a_k = \frac{1}{2\pi j} \oint_c \frac{f(s)}{(s - z_0)^{k+1}} ds \tag{7.3}$$

We can seperate this series into the positive and negative indices k & $-k$. This gives two terms the first term (negative exponent) is called the **principal part** of the series, and the second one (postive exponent) is called the **analytic part** of the series.

$$f(z) = \sum_{k=1}^{\infty} a_{-k} (z - z_0)^{-k} + \sum_{k=1}^{\infty} a_k (z - z_0)^k$$

Chapter 8

Residue Theorem

8.1 Residues

The **residue** of a function f at a point $z = z_0$ is simplify the coefficient a_{-1} of the Laurent Series expansion of f at $z = z_0$. We can write this shorthand as Equation 8.1

$$\text{Res}\ \{f(z), z_0\} = a_{-1} \tag{8.1}$$

Lets also try to find the value of this a_{-1} value. Lets begin by writing the laurent-series-expansion for f at some point z_0 (which is a simple pole, aka a pole of order 1). This simple procedure gives equation 8.2

$$\begin{aligned}
f(z) &= \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots \\
(z-z_0) f(z) &= a_{-1} + a_0(z-z_0) + a_1(z-z_0)^2 + a_2(z-z_0)^3 \\
\lim_{z\to z_0} (z-z_0) f(z) &= a_{-1} + \lim_{z\to z_0} a_0(z-z_0) + a_1(z-z_0)^2 + a_2(z-z_0)^3 \\
\lim_{z\to z_0} (z-z_0) f(z) &= a_{-1}
\end{aligned} \tag{8.2}$$

This means that we can write Equation 8.3 as the much easier way to find **the residue of a first order pole**...

$$\text{Res}\ \{f(z), z_0\} = \lim_{z \to z_0} (z - z_0) f(z) \tag{8.3}$$

We can also do a similar process, giving Equation 8.4 which shows us how to find **the residue of a higher order pole with order n**...

$$\text{Res}\,\{f(z), z_0\} = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z) \tag{8.4}$$

8.2 Cauchy's Residue Theorem

Let D be a simply connected domain, and C a simple closed-contour lying entirely within D. If a function f is analytic on/within C, except at a finite number of singular points $z_0, z_1, ...z_k$ within C then:

$$\oint_c f(z)\ dz = 2\pi j \cdot \sum_k \text{Res}\ \{f(z), z_k\}$$

Essentially, the contour integral of a function is equal to $2\pi j$ multipled by the sum of the residues of the function. Which are extremely easy to find through equation 8.3 and 8.4. This is an **extremely important theorem** which we will keep revisiting.