Introduction to Control Theory

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Control Systems - Note 1

Contents

Chapter 1

Control Engineering

Control Engineering is a very vast subfield of engineering and spans practically any system-level topic. In this introductory course (note series) we will talk about the basics of control theory, control systems, and how to design them.

1.1 Control Systems

A **Control System** is a system consisting of subsystems and processes assembled for the purposes of obtaining some *desired output* (with desired *specifications*), all given some *specified input*. This is a very confusing and high level definition, but it will make sense as we do more and more work on this topic. For now, a quick example of a control system would be a rotary potentiometer controlling the height of a platform. How far we turn our potentiometer is our *specified input*, for which we want to encode the height of our platform (our *desired output*) But upon turning the knob, how will the platform respond? slowly? quickly? accelerate fast? accelerate slow? These are the *specifcations* for our control systems, and we would design it to our liking.

1.2 *Open-loop & Closed-loop*

A control system can fall under two categories, **open-loop** and **closed-loop**. The only difference between these types of system is the presence of something called **feedback**, which is a pathway that connects a node further down the line, to one earlier in the line - called a feedback loop. Open-loop systems have no feedback loops, while closed-loop systems contain at least one feedback loop.

An example of each is given below:

- Toaster oven (open-loop) - how much you turn the time knob determines how long it toasts for.
- Cruise control (closed-loop) - checks current speed and compares it to desired speed adjusting in the correct direction.

Chapter 2

The Frequency Domain

The first step in every control design case is to use the laws of physics and engineering to generate a **differential equation** that describes the relationship between inputs and outputs of a system. We often find working with differential equations unwieldy and unintuitive, thus we convert them into algebraic equations via the **laplace transform**. This chapter will be primarly creating a model for the following rotational mechanical system, modelling how the output angle $\theta(t)$ changes with respect to a changing torque $\tau(t)$.

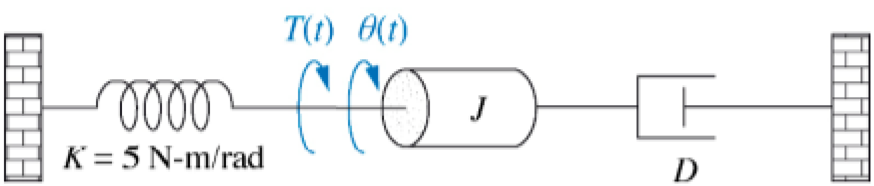

Figure 2.1: A Rotational-Mechanical System

2.1 *Basic Kinematics*

Our system is shown above in Figure 2.1, it consists of a torsional (rotational) spring with constant K, a mass with inertia J, and a rotational viscous damping of some constant D. We can start to generate the differential equation by writing newton's second law for rotational kinematics.

$$\sum \tau = I\ddot{\theta}$$

The complete list of torques on the system are:

- Input Torque - $+\tau$
- Torsional spring - $-K\theta$
- Viscous Damping - $-D\dot{\theta}$

This means we can write newton's second law for this system as follows, which generates a (generally non-homogenous) second order linear ordinary differential equation in equation 2.1.

$$\tau\left(t\right) - K\theta - D\dot{\theta} = J\ddot{\theta}$$

$$\tau\left(t\right) = J\ddot{\theta} + D\dot{\theta} + K\theta \tag{2.1}$$

2.2 *The Transfer Function*

In order to better understand and interpret what is actually happening in equation 2.1, we take the laplace transform of it to convert it into an algebraic equation in the frequency domain. This lets us understand how the inputs are mapped to outputs via something called the **transfer function**, which is defined as the ratio between output and input, $T\left(s\right)$below:

$$\frac{\text{output}}{\text{input}} = T\left(s\right)$$

Upon taking the laplace transform of equation 2.1 (neglecting intial conditions), we arrive at the following transfer function for our general rotational-mechanical system in the complex-frequency domain, shown in equation 2.2.

$$\frac{\theta(s)}{\tau(s)} = \frac{1}{Js^2 + Ds + K} \tag{2.2}$$

Suppose we were interested in a different output though, consider obtaining the output for $\omega(s)$ the angular velocity of the motor. We can easily find this from Equation 2.2 by considering the relationship between omega and theta in the time domain.

$$\dot{\theta}(t) = \omega(t)$$

We know that differentiating in the time-domain is the same as multiplying by s in the frequency domain, thus we can write this relationship in the s-domain and generate an expression for omega in terms of theta.

$$s\theta(s) = \omega(s)$$

$$\theta(s) = \frac{1}{s}\omega(s)$$

We can plug this into equation 2.2 to get our desired transfer function for omega in terms of tau in Equation 2.3.

$$\frac{\omega(s)}{\tau(s)} = \frac{s}{Js^2 + Ds + K} \tag{2.3}$$

Chapter 3

The Electric Motor

After modelling a general rotiation-mechanical system in chapter 2, we will be modelling an electromechanical system - the electric (induction) motor. We will being by modeling the electrical circuit, and then combining that with our model from chapter 2 to get the model of an electric motor.

3.1 *The Electric Motor*

Consider the circuit in Figure 3.1, It is an extremely basic model electric motor circuit model. consisting of a circuit, and a rotational mechanical system from the motor's rotating head. Before we start modeling, we will need some background information about motors. It is common to model off some background information in the real world as well. Here are some interesting relations describing the angular velocity and torque of a motor, given a machine constant and the motor voltage and current.

$$\omega_m = K_m v_b$$

$$\tau_m = K_m i_a$$

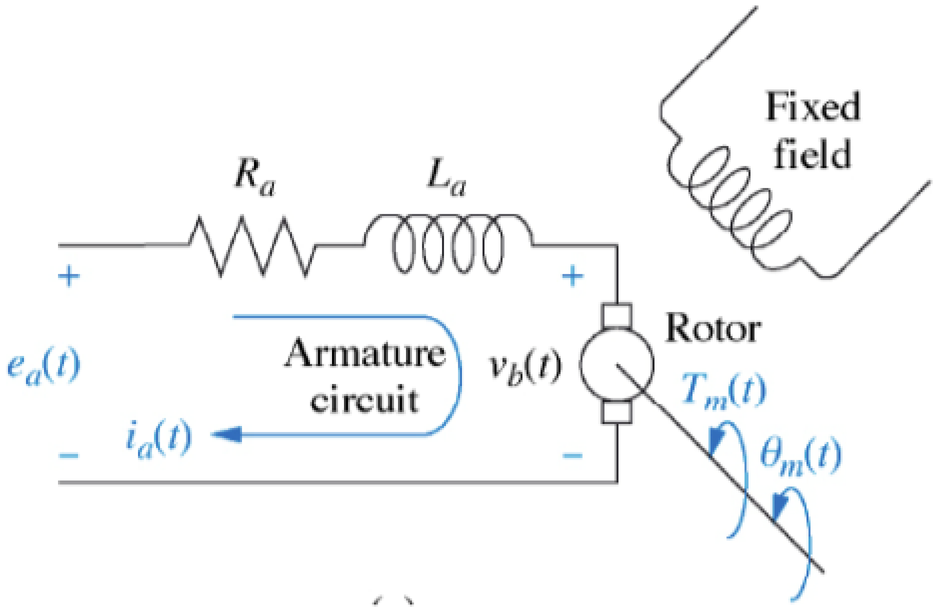

Figure 3.1: The Electric Circuit of an Induction Motor

3.2 *The Model*

Let's start by writing KVL for the circuit.

$$v_a - v_b = R_a i_a + L_a \frac{di_a}{dt}$$

Taking laplace of both sides and generating the transfer function gives

$$\frac{i_a}{v_a - v_b} = \frac{1}{R + sL}$$

From the provided formulae in section 3.1, we can write

$$i_a = \frac{1}{K_m}\tau_m$$

And substitue this into our circuit's transfer function to obtain a transfer function for motor torque, with respect to voltage difference.

$$\frac{\tau_m}{v_a - v_b} = \frac{K_m}{R + sL}$$

We can then use the transfer function for a rotational-mechanical system we derived earlier in chapter 2, to relate torque to an output velocity. *Note:* In this system there is no spring, thus $K = 0$.

$$\frac{\omega_m(s)}{v_a - v_b} = \frac{K_m}{R + sL} \cdot \frac{1}{Js + D}$$

This is the complete model of an electric motor. We can introduce some simplification like ignoring the sL term as inductance values (L) are generally extremely small. Doing some algebra and ultimately we get something of the form, where K and a are calculable through the systems unique parameters (resistance, inertia, damping, etc.).

$$\frac{\omega_m(s)}{v_a - v_b} = \frac{K}{s(s + a)}$$