Complex Analysis

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Complex Analysis - Comprehensive Note

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Part I

Functions of a Complex Variable

Chapter 1

Sets in the Complex Plane

We already have scratched the basics of working with the **imaginary unit** $j = \sqrt{-1}$ in other coursework, and it will not be covered again for the 845th time! If you need a refresher, try this out! In this chapter we will be talking about types of sets in the complex plane, some formal definitions and what they geometrically are, we do this so that we can eventually work our way up to understanding functions of a complex variable.

1.1 Terminology

Lets start by introducing some essential terminology about sets in the complex plane.

Suppose we have some $z_0 = x_0 + jy_0$. Since the expression $|z - z_0|$ represents the distance between points z and z_0 on the complex plane, we can denote **a circle**, in other words set of circular points lying on some radial distance ρ from a center point z_0 as Equation 1.1

$$|z - z_0| = \rho \tag{1.1}$$

This means that $|z| = 1$ is the unit circle centered at the origin, and that $|z - 1 - 2j| = 5$ is a circle of radius 5, centered at the point $1 + 2j$. What happens if we modify the equality in Equation 1.1 to the inequality seen in Equation 1.2?

$$|z - z_0| < \rho \tag{1.2}$$

The inequality in Equation 1.2 is called the **neighborhood of z_0** or an **open disk**. It defines the set of points that lie within, but not on the circle defined by Equation 1.1.

1.2 *Interior, Exterior & Boundary Points*

Using this definition of the neighborhood in Equation 1.2 we can formally define an **interior point**. A point z_0 is said to be an interior point of some set S of the complex plane if there exists some (possibly infintely small!) neighborhood of z_0 that lies entirely within S.

Through the same logic, a point z_0 is said to be an **exterior point** of some set S if there exists some (possibly infinitely small!) neighborhood of z_0 that lies entirely outside of S.

We can also say that a point z_0 is said to be a **boundary point** of the set S if *every* neighborhood of $z_0$contains at least one point within S, and one point outside of S.

1.3 *Open & Closed Sets*

Now that we have formally defined the three types of points, we can talk about **open-sets** and **closed-sets**.

If every point within a set S is an interior-point, the set S is said to be an **open-set**. If this condition is not met, this means that the set posesses boundary points, and it is called a **closed-set**

This is basically a strange way of saying that open-sets have no boundaries. The set defined in Equation 1.2 is an example of an open-set because for every point within the set we can get infinitely closer and closer to the boundary without ever touching the boundary. On the other hand a closed-set does have a boundary and if we take a point on the boundary, some of this points neighborhood will lie outside the set S, rendering it a non-open (closed) set.

Open sets manifest when there is no semblance of an 'equals' in the inequality, and closed sets manifest when there are semblances of the 'equals'. For example, the symbols $\geq, \leq$ define a closed set, while the symbols $<, >$ define an open set. Here are some examples of open-sets

- $\text{Im}(z) > 0$ - The upper-half of the complex plane
- $1 < \text{Re}(z) < 2$ - An infinitely tall strip
- $|z| > 1$ - The exterior of a unit circle
- $1 < |z| < 2$ - The open annulus (donut) centered at the origin with inner radius 1 and outer radius 2.

Through these examples we can see that the inequality in Equation 1.3 represents the general **open-annulus** centered at z_0, with inner and outer radii of ρ_1 and ρ_2 respectively. There is a lot of other stuff that we can cover related to sets, but it is not required to understand the preceeding chapters.

$$\rho_1 < |z - z_0| < \rho_2 \tag{1.3}$$

Chapter 2

Functions of a Complex Variable

Now that we have defined what sets are in the complex plane, we will have noticeably easier time thinking about what **functions of a complex variable** are. Recall that a function is essentially a sort of **correspondance** between two sets: the input, and the output. Each element in the input set is mapped to only one element in the output set. While the same element in the output set can be reached through multiple elements in the input set. Every x produces only one y, but every y may have more than one x that produces it.

2.1 *Complex Functions*

Suppose we have some real valued function $y = f(x)$. If the domain of this function is extended to include the complex numbers z, we call it a **function of a complex variable** - or a **complex function** for short. We also write it in a slightly differently as Equation 2.1 to imply that it is a complex function.

The image (output) w, of a complex number (input) z, will be some complex number $u + jv$ Where u and v are the real and imaginary parts of w, and are entirely-real valued functions. In other words, we can define some complex function in terms of purely u and v. This fact is also shown in Equation 2.1

$$w = f(z) = u(x, y) + jv(x, y) \tag{2.1}$$

Although we cannot draw a 4-dimensional graph to represent a complex-function. We can think of it as a **mapping between planes**. more specifically, from the input-plane (z-plane) to the output-plane (w-plane) This is shown in Figure 2.1

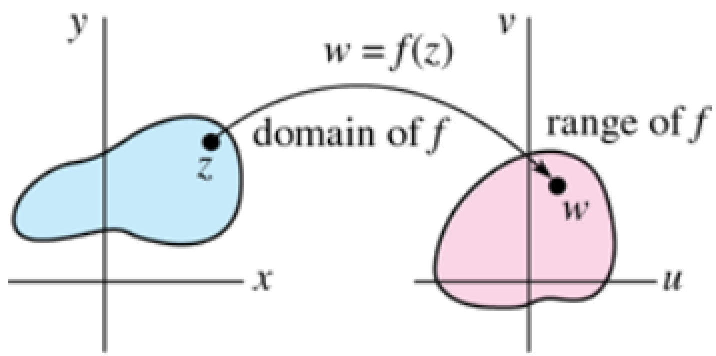

Figure 2.1: The mapping from z-plane to w-plane

2.2 *Limits & Continuity*

The **limit-definition of a complex function** looks the same as the epsilon-delta limit-definition for real valued functions. All it means is that the points $f(z)$ can be made arbitrarily close to the point L if we choose a point z close to, but never equal to the point z_0.

$$\lim_{z \to z_0} f(z) = L \tag{2.2}$$

The fundamental difference between the limits of real-valued functions and complex functions lies in the geometric meaning of the limit. For real-valued functions, we can approach from either the left or the right, but for complex functions we should approach the same thing no matter what path we follow. In other words: when we say that the limit in Equation 2.2 exists, we mean that $f(z)$ approaches L as the point z approaches z_0 from *any* direction. The properties of limits are the exact-same as real valued limits.

A complex function is said to be **continuous** at a point z_0 if the equality in Equation 2.3 holds true, which makes sense.

$$\lim_{z \to z_0} f(z) = f(z_0) \tag{2.3}$$

2.3 *Complex Derivatives*

Now that we have discussed the basics of limits and continuity for complex functions, the next question to ask is "what is a complex derivative?". As we will see, this is very different from asking that its real and imaginary parts have partial derivatives with respect to x and y.

We can say that a complex function $f(z)$ is **complex differentiable** at z_0 if the limit in equation 2.4 or 2.5 exists. These are the regular definitions of a derivative, thus the **real-valued rules of differentiation are the same** as the complex differentiation.

If a complex function is differentiable at point z_0 and at every point in some neighborhood of z_0 we call that function an **analytic function**.

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \tag{2.4}$$

$$f'(c) = \lim_{z \to c} \frac{f(z) - f(z_0)}{z - z_0} \tag{2.5}$$

Another useful thing to know about analyticity of functions is that **if we can express a complex-function as a rational function** (numerator and denominator are polynomials) we can say that **the function is analytic** wherever the denominator is not equal to zero. This is shown below. The only requrement here is that N and D are polynomials.

Recall: This is the case because the limits of complex-polynomials exist everywhere, so they are differentiable and analytic everywhere, thus the rational function is only non-analytic wherever the fraction is undefined (denominator is 0).

$$\text{if } f(z) = \frac{N(z)}{D(z)}, \text{ then } f \text{ is analytic wherever } D(z) \neq 0$$

2.4 Cauchy-Riemann Equations

We now turn to the question of systematically deciding when some complex function is differentiable. If the complex derivative exists at some point z then we should be able to compute it via any path we choose. Let's choose to approach them via purely horizontal and vertical paths. We can use Equation 2.4 to carry this out.
Note: In each of the two limits in the equation below corresponds to approaching the limit horizontally (along a purely real t) or vertically (along a purely imaginary t), all for some $t \in \mathbb{R}$.

$$f'(z) = \lim_{t\to 0} \frac{f(z+t) - f(z)}{t} = \lim_{t\to 0} \frac{f(z+jt) - f(z)}{jt}$$

We can rewrite this in terms of u and v. First lets approach the derivative along a horizontal line. We can see that this simplifies into the definitions of the partial derivatives of u and v in Equation 2.6

$$f'(z) = \lim_{t\to 0} \frac{u(x+t,y) + jv(x+t,y) - u(x,y) - jv(x,y)}{t}$$

$$f'(z) = \lim_{t\to 0} \frac{u(x+t,y) - u(x,y)}{t} + j \lim_{t\to 0} \frac{v(x+t,y) - v(x,y)}{t}$$

$$f'(z) = \frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x} \tag{2.6}$$

Now if we approach the derivative along a vertical line, we can see similar partial derivatives appear in Equation 2.7, but this time with respect to y instead of x

$$f'(z) = \lim_{t\to 0} \frac{u(x,y+t) + jv(x,y+t) - u(x,y) - jv(x,y)}{jt}$$

$$f'(z) = -j \lim_{t \to 0} \frac{u(x, y+t) - u(x, y)}{t} + \lim_{t \to 0} \frac{v(x, y+t) - v(x, y)}{t}$$

$$f'(z) = \frac{\partial v}{\partial y} - j \frac{\partial u}{\partial y} \tag{2.7}$$

We can now equation Equations 2.6 and 2.7 by equating real and imaginary components. In doing so we obtain the **Cauchy-Riemann Equations**. Which tell us the complex values (z values) where a function is differentiable.

We can use these equations to test analyticity of a function on some domain D by checking if they are differentiable everywhere on this domain D (are the cauchy-riemann equations satisfied everywhere in D?).

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

2.5 *Using the Cauchy-Riemann Equations*

Lets use the equations we just derived to show that the function $f(z) = |z|^2$ is differentiable only at $z = 0$, and analytic nowhere.

$$f(z) = |z|^2 = \left(x^2 + y^2\right) + j(0)$$

This gives $u = x^2 + y^2$ and $v = 0$. Thus the Cauchy-Riemann Equations reduce to the equations below, which are only simultaneously satisfied when $x = y = 0$. This function is only differentiable at a single point, $z = 0$ thus it is analytic nowhere (*Recall:* analyticity is a neighborhood property)

$$2x = 0$$

$$0 = -2y$$

2.6 *Relation to Laplace's Equation*

Suppose we have some analytic function (aka function that is differentiable everywhere) $f(z) = u + jv$ we call its constiuent functions u and v **harmonic functions** because they can be related to **Laplace's Equation** (Steady State Heat) through the **Cauchy-Riemann Equations**.
Let's write the Cauchy-Riemann Equatons, and partially differentiate one of these equations w.r.t x, the other to y.

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial v}{\partial x \partial y}$$

$$-\frac{\partial v}{\partial x \partial y} = \frac{\partial^2 u}{\partial y^2}$$

We can combine these to get the following relation to laplace's equation. We can follow a similar process (switching the equation differentiation variables) to prove the second line.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$
$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Chapter 3

Complex Function Families

There are a couple families of functions that behave differently when their domain is extended to include the complex plane. In this chapter we will learn about them.

3.1 *The Exponential Function*

Recall that real-valued exponential function has the following properties

$$\frac{d}{dx}\left[e^x\right] = e^x$$

$$e^{x+y} = e^x e^y$$

If we define the **complex-valued exponential** in terms of the complex variable $z = x + jy$.

$$w = e^z = e^{x+jy}$$

Then by properties of the real-valued exponential we can write the following. Giving the formal definition of the complex-argument exponential function in Equation 3.1
Note: the complex-valued exponential reduces to the real-valued one when $y = 0$

$$e^z = e^x e^{jy}$$
$$e^z = e^x \left(\cos y + j \sin y\right) = e^x \cos y + j e^x \sin y \tag{3.1}$$

Through Equation 2.6 or 2.7 you can prove that **the complex exponential satisfies both properties** shown at the start of this section. The complex exponential is also said to be periodic with a complex period $2\pi j$.

3.2 *Logarithmic Function*

The **logarithm of a complex number** $z = x + jy$ is defined as the **inverse of the complex-exponential** function. That is $w = \ln z$ if $z = e^w$. Just like the real-valued logarithm, the complex-logarithm is not defined at $z = 0$.

to find the the real and imaginary parts of the logarithm, we write $w = u + jv$.

$$z = e^w$$
$$x + jy = e^u \cos v + j e^u \sin v$$

This implies that $x = e^u \cos v$, and that $y = e^u \sin v$. We can solve these for u and v (the real and complex parts of the ln) by squaring both of these results and adding them together. This gives a solution for u.
Note: here the u value is the nautral logartihm of the modulus (a real-number)

$$e^{2u}\left(1\right) = x^2 + y^2 = |z|^2$$
$$u = \ln|z|$$

To solve for v, we have to divide both equations, to obtain the following. The below equation means that v is θ, the argument of z

$$v = \arctan \frac{y}{x}$$

This gives a final definition of the complex-logarithm, which depends on the mod and (non-unique) argument of z

$$\ln z = \ln |z| + j \arg (z)$$

Part II

Integration in the Complex Plane

Chapter 4

Complex Contour Integration

Integration in the complex plane is defined in a somewhat similar manner to that of a line integral in the plane. In other words, we will be dealing with an integral of a complex function $f(z)$ that is defined along a curve C in the complex plane.

These curves are parametrized in $t \in \mathbb{R}$ with equations $x = x(t)$ and $y = y(t)$. By using x and y as the real and imaginary parts, we can parametrize a curve C in the complex plane by means of a real-variable.

4.1 Defining the Complex Integral

Since the complex plane is a 2-dimensional plane, every complex integral is at least a complex contour integral, leading to the nomenclature of calling them **complex integrals**. We can start to define the complex integral similarly to how we would a real integral. This leads to the general definition of a complex integral in Equation 4.1

$$\int_C f(z)\ dz = \lim_{\Delta z \to 0} \sum f(z)\, \Delta z$$

$$\int_C f(z)\ dz = \lim_{\Delta x,\ \Delta y \to 0} \sum (u + jv)\,(\Delta x + j\Delta y)$$

$$\int_C f(z)\ dz = \lim_{\Delta x,\ \Delta y \to 0} \sum (u\Delta x - v\Delta y) + j\,(v\Delta x + u\Delta y)$$

$$\int_C f(z)\ dz = \int_C u\,dx - v\,dy + j\int_C v\,dx + u\,dy \tag{4.1}$$

4.2 *Evaluation of a Complex Integral*

Evaluate the integral below, given that the curve C can be parametrized as $x = 3t$, $y = t^2$, for $t \in [-1, 4]$.

$$\int_C \bar{z}\,dz$$

To begin lets write $z(t) = x(t) + jy(t)$. Lets also find $z'(t)$. This means that the integrand (complex conjugate) has its middle sign flipped. Note that we are also finding dz and not $d\bar{z}$.

$$z(t) = 3t + jt^2$$

$$\frac{dz}{dt} = 3 + j2t$$

$$\bar{z}(t) = 3t - jt^2$$

We can now entirely parametrize the integral. We get the following computable line integral upon doing so. At this point the question is basically done, we evalute the integral and simplify the result into its real and complex parts.

$$\int_{-1}^{4} \left(3t - jt^2\right)(3 + j2t)\ dt = 196 + 65j$$