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Digital Systems & Signal Processing

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Preface

The purpose of this document is to act as a comprehensive note for my understanding on the subject matter. I may also use references aside from the lecture material to further develop my understanding, and these references will be listed here.

This document should eventually serve as a standalone reference for learning or review of the subject matter. There is also a lot of organization within these documents, please refer to the table of contents within your PDF viewer for ease of navigation.

References

- Course Material & Provided Lecture Notes
- Applied Digital Signal Processing - G. Manoalkis, K. Ingle

Sampling & Reconstruction of Signals

Frequency

Consider the definitions[1] of a CT-signal.

$$x_a(t) = A\cos(\omega t + \phi)$$

$$x_a(t) = A\cos(2\pi f t + \phi)$$

The value f is called the signals **fundamental frequency** measured in Hertz[2]. Given some value of f, we can define its inverse, $T = 1/f$ as the signals **fundamental period**[3].

If some signal x_a has fundamental frequency f_s and fundamental period T_s, we can say that:[4]

$$x(t + T_s) = x(t)$$

Extending this to DT-signals isn't as straightforward as one would think ...

A DT-signal is periodic with period N if :

$$x(n + N) = x(n)$$

[1] One with radial and one with temporal frequency

[2] Hz [cycles/sec]

[3] Measured in Hz^{-1} [sec/cycle]

[4] This is the formal definition of fundamental period

The fundamental period of a DT signal, is the smallest value of N such that the above holds true.

Properties of Discrete-Time Signals

DT signals have some strange properties.

- DT-signals are only periodic if their frequency f is a rational number
- DT-signals whose frequencies ω are seperated by some integer multiple of 2π, are identical
- DT-sinusoids are distinct only within a certain frequency range, sinusoids with frequencies outside this range are an alias of a sinusoid with frequency within this range
- The highest frequency possible for a DT-signal is at $\omega = \pm\pi$ or $f = \pm 1/2$

Lets consider the first property. We require the following to hold.[5]

$$x(n+N) = x(n)$$

$$\cos(2\pi f(n+N)) = \cos(2\pi f n)$$

$$\cos(2\pi f n + 2\pi f N)) = \cos(2\pi f n)$$

This[6] implies a rational frequency:

$$f = \frac{k}{N}$$

[5] Consider that $x(n) = \cos(2\pi f n)$

[6] $\cos(x) = \cos(x + 2\pi k)$ only holds true for some integer value of k, this implies that $2\pi f N = 2\pi k$

The second property can easily be proven as shown below, suppose $k \in \mathbb{Z}$:

$$x_k(n) = \cos{(\omega_0 + 2\pi k)n + \phi}$$

$$x_k(n) = \cos{(\omega_0 n + 2\pi k n + \phi)}$$

Then, the following two are indistinguishable.

$$x_k(n) = \cos{(\omega_0 n + 2\pi k n + \phi)} = \cos{(\omega_0 n + \phi)}$$

The third and fourth property are basically to be treated as fact, try creating a sequence of this on a graphing calculator to verify that this is true for yourself.

Analog to Digital Conversion

To modify analog signals with a digital signal processor, we must first convert them into digital form[7]. This process is called analog to digital conversion (abbreviated A/D conversion). In many cases we also convert the signal back into analog form (D/A conversion).

[7] a sequence of binary numbers

A/D Conversion is done in three main steps:

- Sampling
- Quantization
- Encoding

Sampling concerns turning an analog signal into a discrete time signal. **Quantization** involves turning it from continuous valued into a discrete valued signal. Lastly, **Encoding** is about turning it into a binary sequence for transmission.

Sampling

Given some analog input $x_a(t)$ we can perform **periodic** (or **uniform sampling**) by storing its value at certain time intervals described by a **sampling frequency** f_s or **sampling period** T_s.

- Mathematically this would be the equivalent of taking a CT-sinusoid and performing a substitution $t = {}^{n}/_{f_s} = nT_s$ for some $n \in \mathbb{Z}$
- Essentially we turn a signal into a sequence, $x[n] := x_a(nT_s)$

We know that one of the properties of DT-signals is that if the frequency of two are seperated by some integer value, they are indistinguishable from one another. Consider some analog frequency f_1 and f_2.

$$\frac{f_1}{f_s} - \frac{f_2}{f_s} = k$$

If the above holds true, the DT-signals will look the same after being sampled at frequency f_s. This effect is called **aliasing**.

Consider two CT sinusoids[8] being sampled at $f_s = 1$ Hz. It can be shown that the frequencies of both signals satisfy the equation above, and are identical after sampling (conversion into a sequence).

[8] $x_1(t) = \cos(2\pi \cdot {}^{1}/_{8})$ and $x_2(t) = \cos(-2\pi \cdot {}^{7}/_{8})$

Consider a real-life signal expressed as a summation:

$$x_a(t) = \sum_{i=1}^{N} A\cos\left(2\pi f_i t + \phi_i\right)$$

Lets also say that we have filtered this signal and that there is some max frequency useful to us, denoted $f_{\max}$.

In order to avoid aliasing we need to ensure that the below holds true for all values of i.

$$-\frac{1}{2} \leq \frac{f_i}{f_s} \leq \frac{1}{2}$$

$$-\frac{f_s}{2} \leq f_i \leq \frac{f_s}{2}$$

In other words, we need to ensure that each frequency f_i is at least two times lower in magnitude than the chosen sampling frequency f_s.

This can easily be done by choosing to sample at a rate **twice as high as $f_{\max}$.** This special frequency is called the **Nyquist Rate**

$$\text{Nyquist Rate } = 2 \cdot f_{\max}$$