A Visualized Smoothed Particle Hydrodynamics Model for Simulating Liquid and its Interaction with Boundaries

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Abstract

This paper shows a simplified model to do simulations of liquid and its interaction with boundaries. Smoothed Particle Hydrodynamics (SPH) model is a model to simulate the liquid with particle basis. Currently, it was used in many fields of study including biophysics and astrophysics. This method solves Navier–Stokes equations and use the solution to guide the particles moving, generating a final liquid visual effect. Three force states, including pressure, viscosity and gravity are calculated. Boundary conditions are particle-based, following Newton's law, with some loss of kinetic energy. Liquid and interaction simulations are implemented in CPU codes; the final visualization is mostly done in GPU codes, with apparent speed-up than in CPU codes.

1 Navier–Stokes equations

Navier–Stokes equation describes the motion of fluids with several forces considered. In this model, only $F_{gravity}$, $F_{viscosity}$ and $F_{pressure}$ are considered for the sake of simplification. The Navier–Stokes equation is given by the differential equation:

$$\frac{d\boldsymbol{u}}{dt} = g - \frac{\nabla p}{\rho} + \frac{\mu \nabla^2 \boldsymbol{u}}{\rho} \tag{1}$$

where

- $\boldsymbol{u}$ is the liquid relative velocity,
- g is the acceleration of gravity,
- $\nabla\cdot$ is the divergence,
- p is the pressure,
- μ is the viscosity constant,
- t is time,
- ρ is the density,

In this form of Navier–Stokes equation, other more complex forces, like surface tension force, are not taken into account for convenience.

2 Initial Conditions

SPH (Smoothed Particle Hydrodynamics) model is particle-based, and initial conditions involving x (initial positions), v (velocities), m (mass), p (pressure), and N (particle numbers, based on liquid volume) are needed. Noted that there is no need for defining initial densities, which can be solved with initial positions and mass in SPH model.

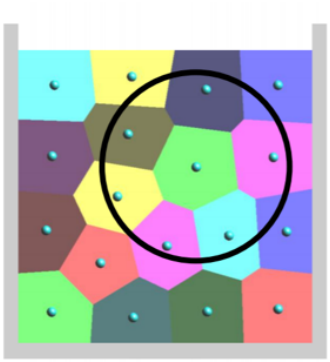

Figure 1: Diagram of Kernel range (black circle) of a green particle (top right) with its distinguishable neighbors, of which the radius is h.

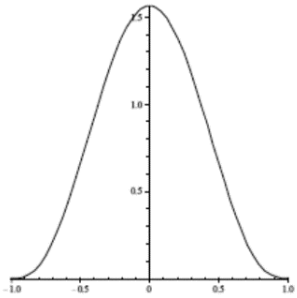

Figure 2: Graph of kernal function, which is Gaussian-like.

3 Kernel Function

In the idealist liquid simulation model, density and pressure of a certain particle need to be calculated by considering all particles in the environment. However, the usual case is that we will not consider the entire neighboring particles, but only a fraction of them, demonstrated by the circle in Figure 1. The radius of the circle, or the numbers of neighboring particles are subject to the capacity of CPU and is defined as h.

We use a Gaussian-like function, shown in Figure 2., named Kernel function, to represent the neighborhoods. Let $\sum_j$ be all the neighbors of particle in position $\boldsymbol{r}$. Use m_j and ρ_j to represent the mass and density of particle j. $W()$ is the Kernel function. Then the $A(\boldsymbol{r})$, the value we want to sum up using Kernel function, can be given by the equation:

$$\boldsymbol{A}(\boldsymbol{r_i}) = \sum_j \boldsymbol{A_j} \frac{m_j}{\rho_j} W(\boldsymbol{r_i} - \boldsymbol{r_j}, h) \tag{2}$$

3.1 Property of Kernel function

Like Gaussian function Kernel function is even function:

$$W(+\boldsymbol{r}) = W(-\boldsymbol{r}) \tag{3}$$

Also, its area size sums up as one:

$$\int W(\boldsymbol{r}) dr = 1 \tag{4}$$

3.2 Density

Noted that in Eq.1 ρ_j is an unknown value, we need to first solve ρ_j by plugging ρ_j into Eq.1 and write our density function:

$$\rho(\boldsymbol{r_i}) = \sum_j m_j W(\boldsymbol{r_i} - \boldsymbol{r_j}, h) \tag{5}$$

$$u_i = \sum_j u_j \frac{m_j}{\rho_j} W(\boldsymbol{r_i} - \boldsymbol{r_j}, h) \tag{6}$$

here $W(\boldsymbol{r} - \boldsymbol{r_j}, h)$ we use W_{poly6} function:

$$W_{poly6}(\boldsymbol{r_i} - \boldsymbol{r_j}, h) = \begin{cases} K_{poly6}(h^2 - \|\boldsymbol{r_i} - \boldsymbol{r_j}\|^2)^3, & 0 \leq \|\boldsymbol{r_i} - \boldsymbol{r_j}\| \leq h \\ 0, & otherwise \end{cases} \tag{7}$$

then according to Eq.4, in 2D:

$${K_{poly6}}^{-1} = \int_0^{2\pi} \int_0^h r(h^2 - r^2)^3 dr d\theta \tag{8}$$

which states:

$$K_{poly6} = \frac{4}{\pi h^8} \tag{9}$$

and:

$$\rho(\boldsymbol{r_i}) = \frac{4}{\pi h^8} \sum_j m_j r(h^2 - r^2)^3 \tag{10}$$

3.3 Pressure

Based on Eq.1 and Eq.2, we can obtain pressure p and pressure force $\boldsymbol{F}_{pressure}$ with the equation:

$$\boldsymbol{F}_{pressure}(\boldsymbol{r_i}) = -\nabla p_i = -\sum_j p_j \frac{m_j}{\rho_j} \nabla W(\boldsymbol{r_i} - \boldsymbol{r_j}, h) \tag{11}$$

however, notice that we normally take the average value of p_i and p_j, since particles in different pressure region share different pressure force, which means:

$$\boldsymbol{F}_{pressure}(\boldsymbol{r_i}) = -\sum_j \frac{m_j(p_j + p_i)}{2\rho_j} \nabla W(\boldsymbol{r_i} - \boldsymbol{r_j}, h) \tag{12}$$

where for Kernel function $W(\boldsymbol{r_i} - \boldsymbol{r_j}, h)$, we use:

$$W_{spiky}(\boldsymbol{r_i} - \boldsymbol{r_j}, h) = \begin{cases} K_{spiky}(h - \|\boldsymbol{r_i} - \boldsymbol{r_j}\|)^3, & 0 \leq \|\boldsymbol{r_i} - \boldsymbol{r_j}\| \leq h \\ 0, & otherwise \end{cases} \tag{13}$$

and for p_i or p_j, based on ideal gas law, given fluid constant K and static density ρ_0 is:

$$p_i = K(\rho_i - \rho_0) \tag{14}$$

similar to Eq.7, based on Eq.4, we can get:

$$K_{spiky} = (\int_0^{2\pi} \int_0^h r(h - r)^3 dr d\theta)^{-1} = \frac{10}{\pi h^5} \tag{15}$$

$$\nabla W(\boldsymbol{r_i} - \boldsymbol{r_j}, h) = \nabla \frac{10}{\pi h^5} (h - \|\boldsymbol{r_i} - \boldsymbol{r_j}\|)^3 = -\hat{\boldsymbol{r}} \frac{30}{\pi h^5} (h - r)^2 \tag{16}$$

finally, we can get:

$$\boldsymbol{F}_{pressure}(\boldsymbol{r_i}) = \boldsymbol{a}_{pressure}(\boldsymbol{r_i})\rho_i = \hat{\boldsymbol{r}} \sum_j \frac{m_j K(\rho_i + \rho_j - 2\rho_0)}{2\rho_j} \frac{30}{\pi h^5} (h - r)^2 \tag{17}$$

3.4 Viscosity

In agreement with Eq.1, the viscosity force can be calculated as:

$$\boldsymbol{F}_{viscosity}(\boldsymbol{r_i}) = \mu\nabla^2\boldsymbol{u}(\boldsymbol{r_i}) = \mu\sum_j \boldsymbol{u_j}\frac{m_j}{\rho_j}\nabla^2 W(\boldsymbol{r_i}-\boldsymbol{r_j},h) \tag{18}$$

and for Kernel function here:

$$W_v(\boldsymbol{r_i}-\boldsymbol{r_j},h) = \begin{cases} K_v(-\frac{r^3}{2h^3}+\frac{r^2}{h^2}+\frac{h}{2r}-1), & 0\le r\le h \\ 0, & otherwise \end{cases} \tag{19}$$

then based on Eq.4:

$$K_v = (\int_0^{2\pi}\int_0^h r(-\frac{r^3}{2h^3}+\frac{r^2}{h^2}+\frac{h}{2r}-1)drd\theta)^{-1} = \frac{10}{3\pi h^2} \tag{20}$$

also, notice that $\boldsymbol{u_j}$ here is relative velocity, which means:

$$\boldsymbol{F}_{viscosity}(\boldsymbol{r_i}) = -\mu\sum_j \frac{(\boldsymbol{u_j}-\boldsymbol{u_i})m_j}{\rho_j}\nabla^2\frac{30r^4-20hr^3-10h^4}{3\pi h^5 r^3} \tag{21}$$

and according to reference[1], to increase stability of simulation, we get:

$$\nabla^2\frac{30r^4-20hr^3-10h^4}{3\pi h^5 r^3} \approx \frac{45}{\pi h^6}(h-r) \tag{22}$$

3.5 Summary

Based on Eq.1, we can rewrite Navier–Stokes equation as [1]:

$$\rho\frac{d\boldsymbol{u}}{dt} = \rho g + \hat{\boldsymbol{r}}\sum_j \frac{m_j K(\rho_i+\rho_j-2\rho_0)}{2\rho_j}\frac{30}{\pi h^5}(h-r)^2 - \mu\sum_j\frac{(\boldsymbol{u_j}-\boldsymbol{u_i})m_j}{\rho_j}\frac{45}{\pi h^6}(h-r) \tag{23}$$

4 Boundary Condition

5 Visulization GPU speed-up

6 Application in Blood Circulation in Heart

6.1 FitzHugh-Nagumo (FHN) equation

References

[1] D. Charypar M. Muller and M. Gross ¨. “Particle-based fluid simulation for interactive applications”. In: *Eurographics/SIGGRAPH Symposium on Computer Animation* (2003).