Goal: The goal of this assignment is to work on recursion, set theory, modular arithmetic and equivalence relations.

Instructions: Solve the problems below. Any of these problems may appear in the exams. The numbered propositions are problems in "The Art of the Proof" by Beck and Geoghegan. You can work with other students but you must write the homework solutions on your own, using your own words and thought process.

Writing: Write solutions and proofs in full sentences. Your goal should be that any classmates or any first year undergraduate can understand your argument.

1. A triangulation of a polygon in the plane is a subdivision of the polygon into triangular pieces by adding edges that go between vertices. Let C_n be the number of triangulations of a polygon in the plane with $n+2$ sides. Find a recursion for C_{n+1} in terms of the previous elements in the sequence. Example: C_1 are triangulations of the triangle, and there is only one, so $C_1 = 1$. The next term C_2 counts the number of triangulations of a square, we can add either of the two diagonals to divide into triangles, so there are two ways and $C_2 = 2$. One can check that $C_3 = 5$ and $C_4 = 14$. The sequence thus starts as

$$(C_1, C_2, C_3, C_4, \ldots) = (1, 2, 5, 14, \ldots)$$

Hint: if we let $C_0 = 1$, $C_5 = C_0C_4 + C_1C_3 + C_2C_2 + C_3C_1 + C_4C_0$. Consider also the parenthesis problem you saw in discussion.

2. Prove the following propositions:

   (a) Given sets A, B and C. Show that $A \times (B \cup C) = (A \times B) \cup (A \times C)$

   (b) Let $B - C = \{x \mid x \in B, x \notin C\}$. For sets A, B, C Show that $A \times (B - C) = (A \times B) - (A \times C)$.

   Suppose that $x \in A \times (B - C)$. Then, $x = (a, y)$ where $a \in A$ and $y \in B - C$. Thus, y is not contained in B and $y \in C$. Therefore, $(a, y) \in (A \times B)$. Moreover, $(a, y) \notin (A \times C)$ since $y \notin C$. Thus, $A \times (B - C) \subset (A \times B) - (A \times C)$. Suppose now that $(x, y) \in (A \times B) - (A \times C)$. Then, $(x, y) \in (A \times B)$ and $(x, y) \notin (A \times C)$. Therefore $x \in A$, and $y \in B$. Since $(x, y) \notin (A \times C)$ as well, then $y \notin C$. Thus, $(x, y) \in A \times (B - C)$, and $(A \times B) - (A \times C) \subset A \times (B - C)$. We can then conclude that $(A \times B) - (A \times C) = A \times (B - C)$

3. Prove or disprove the following statement. Let X and Y be sets. Let A, B be subsets of X and C, D subsets of Y. Then

$$(A \times C) \cup (B \times D) = (A \cup B) \times (C \cup D).$$

Let $X = \{1, 2\}$ and let $A = \{1\}$, $B = \{2\}$. Let $Y = \{3, 4\}$ and let $C = \{3\}, D = \{3\}$. Then, $(A \times C) \cup (B \times D) = \{(1, 3), (2, 4)\}$, while $(A \cup B) \times (C \cup D) = \{(1, 3), (2, 3), (1, 4), (2, 4)\}$. Note that $(2, 3) \in (A \cup B) \times (C \cup D)$, but $(2, 3) \notin (A \times C) \cup (B \times D)$, so these sets are not equal.

4. Show that $\{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x = y \pmod{10}\} \subset \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x = y \pmod{5}\}$

   Let $(x, y) \in \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x = y \pmod{10}\}$ be arbitrary. Then, there exists an integer k such that $x - y = 10k$. We know that $10 = 2 \cdot 5$, so $x - y = 5(2k)$. Therefore $x = y$ modulo 5. Therefore $(x, y) \in \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x = y \pmod{5}\}$. Thus, $\{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x = y \pmod{10}\} \subseteq \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x = y \pmod{5}\}$. Finally, since $(0, 5) \in \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x = y \pmod{5}\}$, and $(0, 5) \notin \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x = y \pmod{10}\}$, we have that it is a proper subset.

5. Consider the relation $\mathbb{R} = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x - y \in \mathbb{Z}\}$ on $\mathbb{R}$. Prove that this relation is reflexive, symmetric and transitive

6. Prove the following divisibility statements

   (a) 5^{600} is not divisible by 3.

   (b) $7 | 8^{n+2} + 7n^2 + 3 \cdot 8^n - 4$

   For (b): Note that

$$\begin{aligned} 8^{n+2} + 7n^2 + 3 \cdot 8^n - 4 &=_7 1^{n+2} + 0 \cdot n^2 + 3 \cdot 1^n - 4 \\ &=_7 1 + 0 + 3 - 4 = 0. \end{aligned}$$

   , So, $8^{n+2} + 7n^2 + 3 \cdot 8^n - 4$ is divisible by 7.