ECS 020 Midterm1 Solution

Question 1 (20 points)

Expand the following notation:

a) $\sum_{i=1}^{i=3}((2y)(i) - 1)$

b) $\frac{(n+y)!}{(n+y-2)!}$

Solution:

Assuming y = 4

a) $((2y)(1) - 1) + ((2y)(2) - 1) + ((2y)(3) - 1) = ((2 * 4)(1) - 1) + ((2 * 4)(2) - 1) + ((2 * 4)(3) - 1) = ((12 * 4) - 3)$

b) $\frac{(n+4)!}{(n+2)!} = \frac{(n+4)(n+3)(n+2)!}{(n+2)!} = (n + 4)(n + 3)$

Question 2 (30 points)

Determine if the following statements are propositions if yes, determine the truth value, if no explain why

a) $(2 + 7 = 9) \rightarrow (1 + 1 = 3)$
b) There exist a number y that is less than x, where $(x, y) \in R$
c) $x + 3 = 1$ is a proposition
d) $\sqrt{2}$ is a rational number
e) $x + 3 = 2$ is solvable if x is in the domain of real numbers
f) $2 + 3 = 2$ is and only if $1 + 1 = 3$

Solution:

Question	**Proposition**	**Truth Value**	**Why**
a	yes	F	$(2+7=9)$ is *true* and $(1+1=3)$ is *false*, from the truth table for the conditional statement, we know that the truth value would be *false*
b	yes	T	there exists a number y in R which is less than x in R therefore, this is declarative statement that is *true*.
c	yes	F	"$x+3=1$ is a proposition" is a declarative statement that can be either *true* or *false* but not both, which in this case is *true*
d	yes	F	"$\sqrt{2}$ is a proposition" as it is a declarative statement which can be either *true* or *false* but not both. In this case, it is *false* as $\sqrt{2}$ is a irrational number
e	yes	T	Given "x+3=2 is solvable" is q and "x is in the domain of real number" is p. This is a conditional statement $p \rightarrow q$, and the truth value of q is *true*, therefore no matter what the truth value of p is, from the truth table of the conditional statement, we know that the truth value would be *true*
f	yes	T	$2+3=2$ is *false* and $1+1=3$ is *false*, from the truth table for the biconditional statement, we know that the truth value would be *true*

Question 3 (15 points)

Let a, b, and c be three propositions. Show that this is a tautology, using a truth table. Decide if it is satisfiable- explain your answer.

$$(a \oplus b) \wedge (b \vee c) \longrightarrow (\neg a \vee c)$$

Solution:

Let $A = (a \oplus b) \wedge (b \vee c)$
The truth table is as follows:

a	b	c	$a \oplus b$	$b \vee c$	A	$\neg a \vee c$	$A \longrightarrow (\neg a \vee c)$
T	T	T	F	T	F	T	T
T	T	F	F	T	F	F	T
T	F	T	T	T	T	T	T
T	F	F	T	F	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T
F	F	T	F	T	F	T	T
F	F	F	F	F	F	T	T

Since the last column of the above truth table has all True values, the expression is a tautology. Yes, it is satisfiable because we were able to prove that the expression is a tautology.

Question 4 (15 points)

Restate the propositions in English:
$p =$ "I love cats and dogs"
$q =$ "I love all animals"

Solutions

1. $(\neg p) \rightarrow (\neg q)$

   If I don't love cats or I don't love dogs, then I don't love all animals.

2. $(\neg p) \iff q$

   I love all animals if and only if I don't love cats or I don't love dogs.

3. $p \wedge (\neg q)$

   I love cats and dogs, and I don't love all animals

Question 5 (10 points)

This exercise relates to the inhabitants of the island of knights and knaves created by Smullyan, where knights always tell the truth and knaves always lie. You encounter two people, A and B. Determine, if possible, what A and B are if they address you in the way described. If you cannot determine what these two people are, can you draw any conclusions?

A says "We are both the same"

B says "A is a knight."

Solution

Let K represent knight and N represent knave.
Also let,
P_1: *A says "We are both the same"*
P_2: *B says "A is a knight."*

The truth table is as follows:

A	**B**	$P_1 \wedge P_2$
K	K	True
K	N	False
N	K	False
N	N	False

From the table, we can say that **both A and B are knights.**

Question 6 (40 points)

a) Prove $\exists n \in \mathbb{N}$, $4 + n = 13$

b) Prove $\neg(\forall n \in N 4 + n = 8)$

c) Prove that if n is a positive integer, then n is even if and only if $5n + 8$ is even

d) Let n be a natural number and let $a_1, a_2, a_3, ..., a_n$ be a set of n real numbers. Prove that at least one of these numbers is more than, or equal to the average of these numbers.

e) let n be an integer. Show that if $n^2 + 1$ is even, then n is odd, using: a proof by contradiction.

Solutions:

a)

If $n = 9$, then $4 + 9 = 13$. Since 9 is in the set of $\mathbb{N}$, then $\exists n \in \mathbb{N}$, $4 + n = 13$

b)

We want to prove that the statement $\neg(\forall n \in \mathbb{N}\, 4+n = 8)$ is true. In other words, we want to show that there exists at least one natural number n for which $4 + n \neq 8$.

Let $n = 2$, which is in N. Then, we have:

$$4 + n = 4 + 2 = 6 \tag{1}$$

Since $6 \neq 8$, we have shown that there exists at least one natural number n for which $4 + n \neq 8$. Therefore, the statement $\neg(\forall n \in \mathbb{N}\, 4 + n = 8)$ is true.

c)
We have to prove two implications.

- If n is even, then $5n + 8$ is even
- If $5n + 8$ is even, then n is even

First, we know that n is even. Therefore, we can express n as $n = 2a$, where a is any positive integer. We need to show that $5n+8$ is even. If $n = 2a$, then $5n+8 = 5(2a)+8$. After further simplification, $5n+8 = 10a+8$. We can say that $5n+8 = 2(5a+4)$. Since $2(5a+4)$ is an even integer, $5n+8$ is even.

Second, we will prove using the contrapositive.
Contrapositive- "If n is odd, then $5n + 8$ is odd."
We know that n is odd. Therefore, we can express n as $n = 2b + 1$, where b is any positive integer. We need to show that $5n + 8$ is odd. If $n = 2b + 1$, then $5n + 8 = 5(2b + 1) + 8$. After further simplification, $5n+8 = 10b+5+8$. We can say that $5n+8 = 10b+12+1$. Next, $5n+8 = 2(5b+6)+1$ Since $2(5b + 6) + 1$ is an odd integer, $5n + 8$ is odd.

d)
We use a proof by contradiction. Suppose none of the real numbers $a_1, a_2, \ldots, a_n$ is greater than or equal to the average of these numbers, denoted by a. By definition, we have:

$$a = \frac{a_1 + a_2 + \cdots + a_n}{n} \tag{2}$$

Our hypothesis is that:

$$\begin{gathered} a_1 < a \\ a_2 < a \\ \cdots \\ a_n < a \end{gathered}$$

We sum up all these equations and get the following:

$$a_1 + a_2 + \cdots + a_n < n \cdot a \tag{3}$$

Replacing a in equation (3) by its value given in equation (2), we get:

$$a_1 + a_2 + \cdots + a_n < a_1 + a_2 + \cdots + a_n \tag{4}$$

This is not possible: a number cannot be strictly smaller than itself: we have reached a contradiction. Therefore, our hypothesis was wrong, and the original statement was correct.

e)
Suppose $P(n) \to Q(n)$ is false, i.e., that $\neg(P(n) \to Q(n))$ is true, i.e., that $P(n) \wedge \neg Q(n)$ is true. This is only the case if $P(n)$ is true and $\neg Q(n)$ is true.
If $\neg Q(n)$ is true, then n is even. By definition of even numbers, there exists an integer k such that $n = 2k$. Then,

$$n^2 + 1 = (2k)^2 + 1 = 4k^2 + 1 \tag{5}$$

$4k^2 + 1 = 2(2k^2) + 1$ can be written in the form of $2k' + 1$, where k' is an integer; therefore, by the definition of odd, $n^2 + 1$ is odd, i.e., $\neg P(n)$ is true. However, we have supposed that $P(n)$ is true: we have reached a contradiction. Hence, the original statement is true.