The Catalan Numbers

BY ARNAV KUMAR

This handout provides a somewhat more formal (but for brevity, still incomplete) take on the lecture content.

Proposition 1. *We have that given two (finite) sets A and B, the (size) cardinality of A and B is equal if there is a bijection between the sets. In other words, if there is a map $f: A \to B$ such that for any $b \in B$ there is a unique $a \in A$ such that $f(a) = b$, then $|A| = |B|$.*

Proposition 2. *We have that $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$ is the number of ways to choose k objects from a collection of n where $0! = 0$ and $n! = 1 \times 2 \times \cdots \times n$ for $n > 0, n \in \mathbb{Z}$.*

Problem 1. We are playing a game on the Cartesian plane. Every move, if we were at the position (x, y) we may move to either $(x+1, y)$ or $(x, y+1)$. For $n \in \mathbb{N}$ arbitrary, how many distinct paths are there from $(0,0)$ to (n, n) given that we never move above the line $y = x$?

I choose to share the following solution because it is perhaps the easiest elementary solution which still provides a closed form solution, rather than a recurrence relation.

Solution. Lets represent a path as a finite sequence of moves. A move is one of U for up or R for right. Additionally, if S is a path, we define $S(x, y)$ to be the point we end up at as a result of following the path S starting at (x, y). By convention, if $k = 0$, we'll take $S(x, y) = (x, y)$. Let $\#_U(S)$ and $\#_R(S)$ be the number of Us and Rs in our path respectively.

Let T be the set of all paths S such that $S(0,0) = (n, n)$. Additionally, define $A \subseteq T$ to be all of those paths which have a point on the path above $y = x$.

Note that the value we are searching for is $|T \setminus A| = |T| - |A|$ since T and $A \subseteq T$ are finite. This is because we want the number of paths from $(0,0)$ to (n, n) which are never above $y = x$.

Further, we know that for any $S \in T$, we have $\#_R(S) = \#_U(S) = n$. Thus we see that $|T| = \binom{2\,n}{n}$ which is the number of ways to choose which of the $2\,n$ moves is U (the rest have to be Rs).

For any $S \in A$, let's suppose that move l was the first move that brought us above $y = x$. Since it is the first, we know that after move l, we are on the line $y = x + 1$. We split up S into S_0 and S_1 which are the parts of S before reaching $y = x + 1$ for the first time, and after reaching $y = x + 1$ for the first time. We see that $\#_R(S_0) = 1 + \#_U(S_0)$. Additionally, $\#_R(S_0) + \#_R(S_1) = \#_R(S) = \#_U(S) = \#_U(S_0) + \#_U(S_1)$ which means that $\#_R(S_1) + 1 = \#_U(S_1)$.

Now take $\widetilde{S}_1$ to be the path S_1 but with every R now a U and vice versa. We see that $\#_R(\widetilde{S}_1) = \#_U(S_1) = 1 + \#_R(S_1) = \#_U(\widetilde{S}_1)$. Thus, we see that the path S' (which is the combination of the paths S_0 and $\widetilde{S}_1$) satisfies $\#_R(S') = 2 + \#_U(S')$. This means that $S'(0,0) = (n-1, n+1)$ since path S' has the same length as S. Furthermore, every path from $(0,0)$ to $(n-1, n+1)$ must pass through $y = x + 1$, so can demonstrate a bijection between paths from $(0,0)$ to $(n-1, n+1)$ and those in A (if it's not obvious to you, see if you can figure out what the bijection is).

This means $|A| = \binom{2\,n}{n+1}$ which is the number of ways to choose which of the $2\,n$ terms in the sequence are going to be Us.

Overall, $|T \setminus A| = \binom{2\,n}{n} - \binom{2n}{n+1} = \binom{2\,n}{n}\left(1 - \frac{n}{n+1}\right) = \frac{1}{n+1}\binom{2\,n}{n}$. Q.E.F.

Definition 1. *For $n \geq 0$, we define the n^{th}* ***Catalan Number*** *to be*

$$C_n := \frac{1}{n+1}\binom{2\,n}{n} = \frac{(2n)!}{n!\,(n+1)!}$$

Remark 1. Note that C_n is the solution to the previous problem.

By finding bijections between the set in the first problem and sets in others, we know that we can relate them to the Catalan numbers as well. Here are some examples (note that there are many more):

1. The number of well formed bracket expressions with $2\,n$ brackets.
2. The number of ways to draw n non-overlapping chords between the $2\,n$ verticies of a convex $2\,n$-gon such that every vertex is only part of 1 chord.