An Interesting Integral

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1 Introduction

This is on the following integral where $a, b \in \mathbb{W}$.

$$I(a,b) = \int_0^1 x^a (1-x)^b \,\mathrm{d}x$$

If my work resembles that of someone else, it is by pure coincidence, and I have independently found the following result.

2 Analysis

Let's analyze this integral by considering different parameters for I. First let's consider $b = 0$.

$$\begin{aligned} I(a,0) &= \int_0^1 x^a \,\mathrm{d}x \\ &= \left[\frac{x^{a+1}}{a+1}\right]_0^1 \\ &= \frac{1}{a+1} \end{aligned}$$

Now let's observe a certain property of $I(a,b)$.

$$\begin{aligned} I(a,b) &= \int_0^1 x^a (1-x)^b \,\mathrm{d}x \\ &= \int_0^1 x^a (1-x)^{b-1}(1-x) \,\mathrm{d}x \\ &= \int_0^1 x^a (1-x)^{b-1} \,\mathrm{d}x - \int_0^1 x^{a+1} (1-x)^{b-1} \,\mathrm{d}x \\ &= I(a, b-1) - I(a+1, b-1) \end{aligned}$$

Now using this property, it is easy to find the value of $I(a,1)$ and $I(a,2)$.

$$\begin{aligned} I(a,1) &= I(a,0) - I(a+1,0) \\ &= \frac{1}{a+1} - \frac{1}{a+2} \\ &= \frac{1}{(a+1)(a+2)} \end{aligned}$$

$$
\begin{aligned}
I(a,2) &= I(a,1) - I(a+1,1) \\
&= \frac{1}{(a+1)(a+2)} - \frac{1}{(a+2)(a+3)} \\
&= \frac{2}{(a+1)(a+2)(a+3)}
\end{aligned}
$$

$$
\begin{aligned}
I(a,3) &= I(a,2) - I(a+1,2) \\
&= \frac{2}{(a+1)(a+2)(a+3)} - \frac{2}{(a+2)(a+3)(a+4)} \\
&= \frac{6}{(a+1)(a+2)(a+3)(a+4)}
\end{aligned}
$$

Now with this information, I hypothesize that the following which agrees with the examples I have found.

$$I(a,b) = \frac{a!b!}{(a+b+1)!}$$

I now prove this hypothesis with induction.

Base Case: In the base case, $b=0$, and this has already been verified that $I(a,0) = \frac{a!0!}{(a+0+1)!}$ since $I(a,0) = \frac{1}{a+1}$

Inductive Step: The inductive hypothesis is that $I(a,k-1) = \frac{a!(k-1)!}{(a+(k-1)+1)!}$. Now I prove that $I(a,k) = \frac{a!k!}{(a+k+1)!}$. Note the following.

$$
\begin{aligned}
I(a,k) &= I(a,k-1) - I(a+1,k-1) \\
&= \frac{a!(k-1)!}{(a+(k-1)+1)!} - \frac{(a+1)!(k-1)!}{((a+1)+(k-1)+1)!} \\
&= \frac{a!(k-1)!}{(a+k)!}\left(1 - \frac{a+1}{a+k+1}\right) \\
&= \frac{a!(k-1)!}{(a+k)!}\left(\frac{k}{a+k+1}\right) \\
&= \frac{a!k!}{(a+k+1)!}
\end{aligned}
$$

And since this is what we wanted, the induction is complete. Q.E.D.