Companion to Mathematics

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Introduction

1 Number Theory

Pythagorean Triples: Three numbers are a Pythagorean triple if and only if they can be written in the form $2mn, m^2 - n^2, m^2 + n^2$, where $m, n \in \mathbb{Z}^+$.

Fermat's Last Theorem: There does not exist positive integers a, b, c, n with n greater than 2 such that $a^n + b^n = c^n$

2 Algebra

Exponents and Roots: An exponent is defined with repeated multiplication. $a^b = c$ implies $a * a * a * ... * a$ a total of b times is equal to c. When $b = \frac{1}{2}$, it is expressed as $\sqrt{a}$, and when $b = \frac{1}{n}$, it is expressed as $\sqrt[n]{a}$.

Exponent Laws:

- $x^a x^b = x^{a+b}$
- $\frac{x^a}{x^b} = x^{a-b}$
- $(x^a)^b = x^{ab}$ if $x, a, b \in \mathbb{R}$

Rationalizing Denominators: Given an expression of the form $\frac{n}{a\sqrt{b}+c\sqrt{d}}$, multiply the numerator and denominator by the conjugate $a\sqrt{b} - c\sqrt{d}$, and simplify to rationalize the denominator.

Logarithm: Informally, if $a^b = c$ and $a, b, c \in \mathbb{R}$, then a logarithm is defined such that $\log_a c = b$.

Logarithm Laws:

- $\log_a b + \log_a c = \log_a bc$
- $\log_a b - \log_a c = \log_a \frac{b}{c}$
- $\log_a b^c = c \log_a b$
- $\log_a c = \frac{\log_b c}{\log_b a}$

Quadratic Equation: A quadratic equation is of the form $y = ax^2 + bx + c$. The roots are the values of x for which $f(x) = 0$.

Quadratic Formula: For all quadratics $f(x) = ax^2 + bx + c$, the roots of f are: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Imaginary and Complex Numbers: The imaginary unit i is defined such that $i^2 = -1$. A complex number is a number of the form $a + bi$, where a is the real part, and b is the imaginary part.

Euler's Identity: $(x_1^2+x_2^2+x_3^2+x_4^2)(y_1^2+y_2^2+y_3^2+y_4^2) = (x_1y_1+x_2y_2+x_3y_3+x_4y_4)^2 + (x_1y_2-x_2y_1+x_3y_4-x_4y_3)^2 + (x_1y_3-x_3y_1+x_4y_2-x_2y_4)^2 + (x_1y_4-x_4y_1+x_2y_3-x_3y_2)^2$

Hyperbolic Sine Functions: Hyperbolic sin, noted as sinh is defined as $\sinh x = \frac{e^x-e^{-x}}{2}$, and cosh as $\cosh x = \frac{e^x+e^{-x}}{2}$. Other hyperbolic functions like tanh, and sech are defined as expected in terms of sinh and cosh. The inverses of the functions are arsinh and arcosh.

Hyperbolic Sine Function Properties:

- $\sinh x = -i \sin ix$
- $\cosh x = \cos ix$
- arsinh $x = \ln x + \sqrt{x^2+1}$
- arcosh $x = \ln x + \sqrt{x^2-1}$

3 Trigonometry

Trig Identities in a Triangle: If α, β, γ are the angles of a triangle, then:

- $\cos\alpha + \cos\beta + \cos\gamma = 1 + 4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$
- $\sin\alpha + \sin\beta + \sin\gamma = 4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$
- $\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 4\sin\alpha\sin\beta\sin\gamma$
- $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2 + 2\cos\alpha\cos\beta\cos\gamma$
- $\sin^2\frac{\alpha}{2} + \sin^2\frac{\beta}{2} + \sin^2\frac{\gamma}{2} + 2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2} = 1$
- $\tan\frac{\alpha}{2}\tan\frac{\beta}{2} + \tan\frac{\beta}{2}\tan\frac{\gamma}{2} + \sin\frac{\gamma}{2}\sin\frac{\alpha}{2} = 1$
- $\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma$
- $\cot\frac{\alpha}{2} + \cot\frac{\beta}{2} + \cot\frac{\gamma}{2} = \cot\frac{\alpha}{2}\cot\frac{\beta}{2}\cot\frac{\gamma}{2}$

Three Variable Trig Identities:

- $\sin\alpha + \sin\beta + \sin\gamma - \sin(\alpha+\beta+\gamma) = 4\sin\frac{\alpha+\beta}{2}\sin\frac{\beta+\gamma}{2}\sin\frac{\gamma+\alpha}{2}$
- $\cos\alpha + \cos\beta + \cos\gamma + \cos(\alpha+\beta+\gamma) = 4\cos\frac{\alpha+\beta}{2}\cos\frac{\beta+\gamma}{2}\cos\frac{\gamma+\alpha}{2}$

4 Combinatorics

Squares of Binomial Coefficients: For $n \in \mathbb{W}$, we have $\sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n}$.

Vandermonde's identity: For $m, n, r \in \mathbb{W}$, we have $\binom{m+n}{r} = \sum_{k=0}^{r}\binom{m}{k}\binom{n}{r-k}$

Generaliaztion of Vandermonde's Identity: For $n_1, n_2, \ldots, n_p, m \in \mathbb{W}$, we have $\binom{n_1+n_2+\cdots+n_p}{m} = \sum_{k_1+k_2+\cdots+k_p=m}\binom{n_1}{k_1}\binom{n_2}{k_2}\cdots\binom{n_p}{k_p}$

5 Inequalities

Trivial Inequalities:

- If $x \leq y$ and $y \leq z$, then $x \leq z$ for any $x, y, z \in \mathbb{R}$
- If $x \leq y$ and $a \leq b$, then $x + a \leq y + b$ for any $x, y, a, b \in \mathbb{R}$
- If $x \leq y$, then $x + z \leq y + z$ for any $x, y, a, b \in \mathbb{R}$
- If $x \leq y$ and $a \leq b$, then $xa \leq yb$ for any $x, y \in \mathbb{R}^+$ or $a, b \in \mathbb{R}^+$
- For $A_i \in \mathbb{R}^+$ and $x_i \in \mathbb{R}$ for all $i = 1, 2, \ldots, n$, we have $\sum_{i=0}^{n} A_i x_i^2 \geq 0$ with equality if and only if $x_1 = x_2 = x_3 = \cdots = x_n$

Nesbitt's Inequality: For $a, b, c \in \mathbb{R}^+$, we have $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$

Root Mean Square: For any $a_1, a_2, a_3, \ldots, a_n \in \mathbb{R}^+$, we define the RMS(root mean square) to be $\sqrt{\frac{a_1^2+a_2^2+\cdots+a_n^2}{n}}$.

Arithmetic Mean: For any $a_1, a_2, a_3, \ldots, a_n \in \mathbb{R}^+$, we define the AM(arithmetic mean) to be $\frac{a_1+a_2+\cdots+a_n}{n}$.

Geometric Mean: For any $a_1, a_2, a_3, \ldots, a_n \in \mathbb{R}^+$, we define the GM(geometric mean) to be $\sqrt[n]{a_1 a_2 \cdots a_n}$.

Harmonic Mean: For any $a_1, a_2, a_3, \ldots, a_n \in \mathbb{R}^+$, we define the HM(harmonic mean) to be $\frac{n}{a_1^{-1}+a_2^{-1}+\cdots+a_n^{-1}}$.

Inequalities Between Means: For any $a_1, a_2, a_3, \ldots, a_n \in \mathbb{R}^+$, let RMS, AM, GM, and HM be the root mean square, arithmetic mean, geometric mean, and harmonic mean respectively. Now we have $RMS \geq AM \geq GM \geq HM$ with equality if and only if $a_1 = a_2 = a_3 = \cdots = a_n$.

Jensen's Inequality: Let $f : (a, b) \to \mathbb{R}$ be a concave up function on (a, b). For $n \in \mathbb{N}, \alpha_1, \alpha_2, \ldots, \alpha_n \in (0, 1)$ such that $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$ and any $x_1, x_2, \ldots, x_n \in (a, b)$, we have $f(\sum_{i=1}^{n} \alpha_i x_i) \leq \sum_{i=1}^{n} \alpha_i f(x_i)$.

Young's Inequality: For $a, b > 0$, and $p, q > 1$ with $a, b, p, q \in \mathbb{R}$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$. Equality exists if and only if $a^p = b^q$.

Hölder's Inequality: For $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n \in \mathbb{R}^+$, and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then $\sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}}$ with equality if and only if $\frac{a_1^p}{b_1^q} = \frac{a_2^p}{b_2^q} = \cdots = \frac{a_n^p}{b_n^q}$.

Weighted Hölder's Inequality: For $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n, m_1, m_2, \ldots, m_n \in \mathbb{R}^+$, and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then $\sum_{i=1}^{n} a_i b_i m_i \leq \left(\sum_{i=1}^{n} a_i^p m_i\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} b_i^q m_i\right)^{\frac{1}{q}}$ with equality if and only if $\frac{a_1^p}{b_1^q} = \frac{a_2^p}{b_2^q} = \cdots = \frac{a_n^p}{b_n^q}$.

Generalized Hölder's Inequality: Let $a_{i,j}, i \leq m, j \leq n$ and $i, j \in \mathbb{R}^+$, and $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}^+$ such that $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$. Now $\sum_{i=1}^{m} \left(\prod_{j=1}^{m} a_{i,j}^{\alpha_j} \right) \leq \prod_{j=1}^{m} \left(\sum_{i=1}^{m} a_{i,j} \right)^{\alpha_j}$

Minkowski's First Inequality: For $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n \in \mathbb{R}^+$ and $p > 1$, we have $\left(\sum_{i=1}^{n} (a_i + b_i)^p \right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} a_i^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} b_i^p \right)^{\frac{1}{p}}$ with equality if and only if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}$

Minkowski's Second Inequality: For $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n \in \mathbb{R}^+$ and $p > 1$, we have $\left(\left(\sum_{i=1}^{n} a_i \right)^p + \left(\sum_{i=1}^{n} b_i \right)^p \right)^{\frac{1}{p}} \leq \sum_{i=1}^{n} \left(a_i^p + b_i^p \right)^{\frac{1}{p}}$ with equality if and only if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}$

Minkowski's Third Inequality: For $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n \in \mathbb{R}^+$, we have $\sqrt[n]{a_1 a_2 \dots a_n} + \sqrt[n]{b_1 b_2 \dots b_n} \leq \sqrt[n]{(a_1 + b_1)(a_2 + b_2) \dots (a_n + b_n)}$ with equality if and only if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}$

Weighted Minkowski's Inequality: For $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n, m_1, m_2, \dots, m_n \in \mathbb{R}^+$ and $p > 1$, we have $\left(\sum_{i=1}^{n} (a_i + b_i)^p m_i \right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} a_i^p m_i \right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} b_i^p m_i \right)^{\frac{1}{p}}$ with equality if and only if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}$

Weighted Cauchy-Schwarz Inequality: For $a_i, b_i \in \mathbb{R}, m_i \in \mathbb{R}^+$, for $i = 1, 2, dots, n$, now we have $\left(\sum_{i=1}^{n} a_i b_i m_i \right)^2 \leq \left(\sum_{i=1}^{n} a_i^2 m_i \right) \left(\sum_{i=1}^{n} b_i^2 m_i \right)$ with equality if and only if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}$

On Sequences with the same Orientation: Let there be non-negative real numbers $a_1, a_2, \dots a_n, b_1, b_2, \dots, b_n$ and positive real numbers $c_1, c_2, \dots c_n$, with the property that $\frac{a_1}{c_1} \geq \frac{a_2}{c_2} \geq \cdots \geq \frac{a_n}{c_n}$ and $\frac{b_1}{c_1} \geq \frac{b_2}{c_2} \geq \cdots \geq \frac{b_n}{c_n}$. Now we have $\sum_{i=1}^{n} \frac{a_i b_i}{c_i} \geq \frac{\left(\sum_{i=1}^{n} a_i \right) \left(\sum_{i=1}^{n} b_i \right)}{\sum_{i=1}^{n} c_i}$.

6 Geometry

Cyclic Quadrilaterals: A convex quadrilateral $ABCD$ is cyclic if and only if $\angle ADB = \angle ACB$ or if and only if $\angle DAB + \angle BCD = 180°$

Power of a Point: In a convex quadrilateral $ABCD$, let AB and CD intersect at P, and let AC and BD intersect at Q. $ABCD$ is cyclic if and only if either $AQ \cdot CQ = BQ \cdot DQ$ or if and only if $AP \cdot BP = CP \cdot DP$.

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