Companion to Mathematics

Arnav Kumar

Contents

Introduction

1 Number Theory

Fermat's Last Theorem: There does not exist positive integers a, b, c, n with n greater than 2 such that $a^n + b^n = c^n$

2 Algebra

Exponents and Roots: An exponent is defined with repeated multiplication. $a^b = c$ implies $a * a * a * ... * a$ a total of b times is equal to c. When $b = \frac{1}{2}$, it is expressed as $\sqrt{a}$, and when $b = \frac{1}{n}$, it is expressed as $\sqrt[n]{a}$.

Exponent Laws:

- $x^a x^b = x^{a+b}$
- $\frac{x^a}{x^b} = x^{a-b}$
- $(x^a)^b = x^{ab}$ if $x, a, b \in \mathbb{R}$

Rationalizing Denominators: Given an expression of the form $\frac{n}{a\sqrt{b}+c\sqrt{d}}$, multiply the numerator and denominator by the conjugate $a\sqrt{b} - c\sqrt{d}$, and simplify to rationalize the denominator.

Logarithm: Informally, if $a^b = c$ and $a, b, c \in \mathbb{R}$, then a logarithm is defined such that $\log_a c = b$.

Logarithm Laws:

- $\log_a b + \log_a c = \log_a bc$
- $\log_a b - \log_a c = \log_a \frac{b}{c}$
- $\log_a b^c = c \log_a b$
- $\log_a c = \frac{\log_b c}{\log_b a}$

Quadratic Equation: A quadratic equation is of the form $y = ax^2 + bx + c$. The roots are the values of x for which $f(x) = 0$.

Quadratic Formula: For all quadratics $f(x) = ax^2 + bx + c$, the roots of f are: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Imaginary and Complex Numbers: The imaginary unit i is defined such that $i^2 = -1$. A complex number is a number of the form $a + bi$, where a is the real part, and b is the imaginary part.

Euler's Identity: $(x_1^2 + x_2^2 + x_3^2 + x_4^2)(y_1^2 + y_2^2 + y_3^2 + y_4^2) = (x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4)^2 + (x_1y_2 - x_2y_1 + x_3y_4 - x_4y_3)^2 + (x_1y_3 - x_3y_1 + x_4y_2 - x_2y_4)^2 + (x_1y_4 - x_4y_1 + x_2y_3 - x_3y_2)^2$

Hyperbolic Sine Functions: Hyperbolic sin, noted as sinh is defined as $\sinh x = \frac{e^x - e^{-x}}{2}$, and cosh as $\cosh x = \frac{e^x + e^{-x}}{2}$. Other hyperbolic functions like tanh, and sech are defined as expected in terms of sinh and cosh. The inverses of the functions are arsinh and arcosh.

Hyperbolic Sine Function Properties:

- $\sinh x = -i \sin ix$
- $\cosh x = \cos ix$
- $\text{arsinh } x = \ln x + \sqrt{x^2 + 1}$
- $\text{arcosh } x = \ln x + \sqrt{x^2 - 1}$

3 Trigonometry

Trig Identities in a Triangle: If α, β, γ are the angles of a triangle, then:

- $\cos\alpha + \cos\beta + \cos\gamma = 1 + 4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$
- $\sin\alpha + \sin\beta + \sin\gamma = 4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$
- $\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 4\sin\alpha\sin\beta\sin\gamma$
- $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2 + 2\cos\alpha\cos\beta\cos\gamma$
- $\sin^2\frac{\alpha}{2} + \sin^2\frac{\beta}{2} + \sin^2\frac{\gamma}{2} + 2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2} = 1$
- $\tan\frac{\alpha}{2}\tan\frac{\beta}{2} + \tan\frac{\beta}{2}\tan\frac{\gamma}{2} + \sin\frac{\gamma}{2}\sin\frac{\alpha}{2} = 1$
- $\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma$
- $\cot\frac{\alpha}{2} + \cot\frac{\beta}{2} + \cot\frac{\gamma}{2} = \cot\frac{\alpha}{2}\cot\frac{\beta}{2}\cot\frac{\gamma}{2}$

Three Variable Trig Identities:

- $\sin\alpha + \sin\beta + \sin\gamma - \sin(\alpha + \beta + \gamma) = 4\sin\frac{\alpha+\beta}{2}\sin\frac{\beta+\gamma}{2}\sin\frac{\gamma+\alpha}{2}$
- $\cos\alpha + \cos\beta + \cos\gamma + \cos(\alpha + \beta + \gamma) = 4\cos\frac{\alpha+\beta}{2}\cos\frac{\beta+\gamma}{2}\cos\frac{\gamma+\alpha}{2}$

4 Combinatorics

5 Inequalities

Trivial Inequalities:

- If $x \leq y$ and $y \leq z$, then $x \leq z$ for any $x, y, z \in \mathbb{R}$
- If $x \leq y$ and $a \leq b$, then $x + a \leq y + b$ for any $x, y, a, b \in \mathbb{R}$
- If $x \leq y$, then $x + z \leq y + z$ for any $x, y, a, b \in \mathbb{R}$
- If $x \leq y$ and $a \leq b$, then $xa \leq yb$ for any $x, y \in \mathbb{R}^+$ or $a, b \in \mathbb{R}^+$

- For $A_i \in \mathbb{R}^+$ and $x_i \in \mathbb{R}$ for all $i = 1, 2, \ldots, n$, we have $\sum_{i=0}^{n} A_i x_i^2 \geq 0$ with equality if and only if $x_1 = x_2 = x_3 = \cdots = x_n$

Nesbitt's Inequality: For $a, b, c \in \mathbb{R}^+$, we have $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$

Root Mean Square: For any $a_1, a_2, a_3, \ldots, a_n \in \mathbb{R}^+$, we define the RMS(root mean square) to be $\sqrt{\frac{a_1^2+a_2^2+\cdots+a_n^2}{n}}$.

Arithmetic Mean: For any $a_1, a_2, a_3, \ldots, a_n \in \mathbb{R}^+$, we define the AM(arithmetic mean) to be $\frac{a_1+a_2+\cdots+a_n}{n}$.

Geometric Mean: For any $a_1, a_2, a_3, \ldots, a_n \in \mathbb{R}^+$, we define the GM(geometric mean) to be $\sqrt[n]{a_1 a_2 \cdots a_n}$.

Harmonic Mean: For any $a_1, a_2, a_3, \ldots, a_n \in \mathbb{R}^+$, we define the HM(harmonic mean) to be $\frac{n}{a_1^{-1}+a_2^{-1}+\cdots+a_n^{-1}}$.

Inequalities Between Means: For any $a_1, a_2, a_3, \ldots, a_n \in \mathbb{R}^+$, let RMS, AM, GM, and HM be the root mean square, arithmetic mean, geometric mean, and harmonic mean respectively. Now we have $RMS \geq AM \geq GM \geq HM$ with equality if and only if $a_1 = a_2 = a_3 = \cdots = a_n$.

Index