

Tricks with Finite Series and Products

BY ARNAV KUMAR

We often come across series and products that seem quite difficult to compute, but with some tricks, we can simplify the problem significantly. This involves being able to correctly identify when a series or product telescopes and being able to pair or group terms in a way which simplifies the sum or product.

This handout is aimed to only deal with finite series and products, since extra caution and completely different rules apply when dealing with infinite series and products.

1 Telescoping Series and Products

There are multiple instances where a given series (consisting of several terms) can be simplified to a value that does not require the full computation of the sum. The nature of the sum means that terms in the middle cancel out with each other.

Example 1. Suppose we want to evaluate $\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \cdots + \frac{1}{19 \times 20}$. Notice that

$$\begin{aligned} \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \cdots + \frac{1}{19 \times 20} &= \sum_{n=1}^{19} \frac{1}{n(n+1)} \\ &= \sum_{n=1}^{19} \left(\frac{1}{n} - \frac{1}{n+1} \right) \\ &= 1 - \frac{1}{20} \end{aligned}$$

Thus our answer is $\boxed{\frac{19}{20}}$.

As we saw in the previous example, we were able to express each term in a way that meant that it canceled out with its neighbours. A similar process is the case for telescoping products.

This next example uses the Fibonacci numbers which can be defined quite nicely with recursion.

Definition 1. We define the **Fibonacci sequence** recursively, where for $n \in \mathbb{Z}^{\geq 0}$, the n^{th} Fibonacci number is given by

$$F_n := \begin{cases} 1 & \text{if } n \leq 1 \\ F_{n-1} + F_{n-2} & \text{otherwise} \end{cases}$$

Problem 1. Express the following sum as a function of F_{99} , F_{100} , and F_{101} :

$$\frac{1}{F_0 F_2} + \frac{1}{F_1 F_3} + \cdots + \frac{1}{F_{n-1} F_{n+1}}$$

So far, what we've been doing in the last two examples is splitting up some sort of expression in the denominator in a way that leaves us with two terms that cancel out with neighbours. This method of breaking a term down into two terms is an informal introduction to **partial fraction decomposition**. Not all telescoping series use this method, as you will see in the exercises.

2 Term Grouping

Choosing to group terms differently is a very effective strategy in making a difficult sum become approachable.

As a testimony to this, the derivation of the formula for an arithmetic series can be done with term grouping.

Theorem 1. (Finite Arithmetic Series) For $a, d \in \mathbb{R}$ and $n \in \mathbb{Z}^+$, we have the following:

$$a + (a + d) + (a + 2d) + \cdots + (a + (n - 1)d) = \sum_{i=0}^{n-1} (a + id) = \frac{1}{2}n(2a + (n - 1)d)$$

Furthermore, term grouping can help out with reducing unfamiliar sums if we are well acquainted with some standard sums.

3 Common Series

This is a good place to introduce the equivalent formula for a finite geometric series and some other sums that are very common. It is advised to have these memorized.

Theorem 2. (Finite Geometric Series) For $a, r \in \mathbb{R}$, $r \neq 1$, and $n \in \mathbb{Z}^+$, we have that

$$a + ar + ar^2 + \cdots + ar^{n-1} = \sum_{i=0}^{n-1} ar^i = a \frac{r^n - 1}{r - 1}$$

Theorem 3. (Sum of Powers of Natural Numbers) Let $n \in \mathbb{N}$. Now, we have the following results:

1. $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$
2. $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$
3. $1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$

Note 1. Proofs to all of these common series can be done with mathematical induction.

4 Exercises

Exercise 1. (1999 IMC A5) Calculate $1999^2 - 1998^2 + 1997^2 - 1996^2 + \cdots + 3^2 - 2^2 + 1^2$

Exercise 2. Evaluate the following if $n = 10$:

$$\sum_{i=0}^{n-1} (i+2)(i+1)$$

Exercise 3. (2009 IMC A6) Simplify as a fraction in lowest terms

$$\frac{(2^4 + 2^2 + 1)(4^4 + 4^2 + 1) \cdots (10^4 + 10^2 + 1)}{(3^4 + 3^2 + 1)(5^4 + 5^2 + 1) \cdots (11^4 + 11^2 + 1)}$$

Exercise 4. (Brilliant) Evaluate

$$\frac{1}{\sqrt{1} + \sqrt{2}} + \frac{1}{\sqrt{2} + \sqrt{3}} + \cdots + \frac{1}{\sqrt{99} + \sqrt{100}}$$

IMC refers to the Junior High School Division International Mathematics Competition.