# Math 145: Algebra (Advanced Level) Notes

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These are takeaway notes for the fall 2022 offering of Math 145, instructed by Blake Madill at the University of Waterloo.

| Note | Example | Definition | Results |
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| Rings | A ring, $(R,+,\times)$, is a set, $R$, along with two functions, $+{:}R^2\to R$ and $\times{:}R^2\to R$, which we call addition and multiplication. We must have the following ring axioms:<br>1. There exists an additive identity.<br>2. Every element has an additive inverse.<br>3. Addition is commutative.<br>4. Addition is associative.<br>5. Multiplication is associative.<br>6. Multiplication distributes over addition.<br>We often simply denote this ring by it set of elements, $R$. |
| Unital Rings, Units, and the Group of Units | A unital ring is a ring with a multiplicative identity. In a unital ring, $R$, an element with a multiplicative inverse is called a unit. The group of units of $R$ is $R^\times := \{a\in R : a^{-1}\,\text{exists}\}$. |
| Commutative Rings | A ring, $R$, is commutative if multiplication commutes, which is to say that $(\forall a,b\in R)(a\,b=b\,a)$. |
| Properties of Rings | From the ring axioms, we get the following results for a ring, $R$:<br>1. The additive inverse of any $a\in R$ is unique and denoted $-a$.<br>2. The additive identity is unique, and denoted $0_R$.<br>3. $(\forall a\in R)(a\cdot 0_R=0_R\cdot a=0_R)$.<br>4. $(\forall a,b\in R)((-a)\,b=a\,(-b)=-(a\,b))$<br>And for a unital ring, $R$, we see:<br>1. The multiplicative identity is unique, and denoted $1_R$.<br>2. If $a\in R^\times$, then the multiplicative inverse of $a$ is unique and denoted $a^{-1}$. |
| Ring Notation | In a ring, $R$, and for some $n\in\mathbb{N}$, we generally say that<br>1. $a-b\equiv a+(-b)$<br>2. $n\,a\equiv\underbrace{a+a+\cdots+a}_{n\,\text{times}}$.<br>3. If $R$ is unital, $n\equiv n\,1\equiv\underbrace{1+1+\cdots+1}_{n\,\text{times}}$<br>4. $a^n\equiv\underbrace{a\times a\times\cdots\times a}_{n\,\text{times}}$ |
| Checking Commutativity | Let $R$ be a ring.<br>$$(R\,\text{is commutative})\Leftrightarrow(\forall a,b\in R)((a+b)^2=a^2+2\,a\,b+b^2)$$ |

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| Binomial Theorem | Let $R$ be a commutative ring. $(\forall a, b \in R)(\forall n \in \mathbb{N})$ $$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$ |
| Subring | A subring, $S$, of $R$ is a subset of $R$ which maintains the same ring structure of $R$. This means it is a ring under the same addition and multiplication definitions for $R$. |
| Subring Test | Let $R$ be a ring, and let $\varnothing \neq S \subseteq R$. $$(S \,\text{subring of}\, R) \Leftrightarrow (\forall a, b \in S)(a \times b \in S \wedge a - b \in S)$$ |
| Center of a Ring | Let $R$ be a ring. The center of the ring is defined by: $$Z(R) := \{a \in R : (\forall b \in R)(a\, b = b\, a)\}$$ We have that $Z(R)$ is always a subring of $R$, and is commutative. |