Cyclic Quadrilaterals

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1 Cyclic Quadrilaterals

Definition 1. *We call a quadrilateral which can be incribed in a circle a* ***cyclic quadrilateral*** *(or a* ***concyclic quadrilateral****) and we say that the circle circumscribes the quadrilateral.*

From the previous lesson about points in a triangle, we learnt about the properties of triangles inscribed in a circle, and because we know that there are $\binom{4}{3} = 4$ distinct triangles in a cyclic quadrilateral, we can make some statements with our knowledge of angles while dealing with circumcircles.

Theorem 1. *Any 4 points A, B, C, and D are concyclic iff $\measuredangle ABC = \measuredangle ADC$.*

2 Related Theorems

Theorem 2. (Power of a Point) *If $ABCD$ is a convex quadrilateral with AB and CD intersecting at P and AC and BD intersecting at Q, then $ABCD$ is cyclic iff either:*

1. $AQ \cdot QC = BQ \cdot QD$ *(or equivalently $QAD \sim QBC$)*
2. $PA \cdot PB = PC \cdot PD$ *(or equivalently $PAD \sim PCB$)*

Theorem 3. (Ptolemy's Theorem) *Quadrilateral $ABCD$ is cyclic iff*

$$AB \cdot CD + AD \cdot BC = AC \cdot BD$$

Theorem 4. (Brahmagupta's Formula) *If $ABCD$ is a cyclic quadrilateral with sides of length a, b, c, and d, then let $s := \frac{1}{2}(a + b + c + d)$. We have that:*

$$[ABCD] = \sqrt{(s-a)\,(s-b)\,(s-c)\,(s-d)}$$

3 Exercises

Exercise 1. Let ABC be a triangle with heights AD, BE, and CF. Find the measures of the internal angles of $\triangle DEF$.

Exercise 2. Although not related to cyclic quadrilaterals, this is a well known theorem. Prove that if l is a line tangent to the circumcircle of ABC at A, and D lies on l such that $\angle DAC > \angle DAB$, then prove $\angle DAB = \angle ACB$.

Exercise 3. (2022 KJMO Q1) The inscribed circle of an acute triangle ABC meets the segments AB and BC at D and E respectively. Let I be the incenter of the triangle ABC. Prove that the intersection of the line AI and DE is on the circle whose diameter is AC.

KJMO refers to the Korean Junior Math Olympiad.