Number Bases and Induction

by Arnav Kumar

1 Number Bases

We often write numbers in base 10, which is to say that the position of any digit in the number represents a power of 10 that is multiplied by the number. Consider $342 = 3 \cdot 10^2 + 4 \cdot 10 + 2$. We can use another number to achieve a different number base.

Definition 1. *Given $n \in \mathbb{Z}$, and a number base b, the* ***base b representation of n****, $\overline{d_k d_{k-1} \ldots d_1 d_0}_{(b)}$, satisfies*

$$n = \sum_{i=0}^{k} d_k \cdot b^k$$

where $0 \leq d_i < b$ and $d_i \in \mathbb{Z}$ for all integers i.

2 Principle of Mathematical Induction

Theorem 1. (Principle of Mathematical Induction) *Let $P(n)$ be a statement that depends on a $n \in \mathbb{Z}^+$. If $P(1)$ is true (called the* ***base case****), and if $(\forall k \in \mathbb{Z}^+)(P(k) \Rightarrow P(k+1))$ (called the* ***inductive step****), then $P(n)$ is true for all positive integers n.*

Exercise 1. Let $n \in \mathbb{Z}^+$. Show that

1. $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$
2. $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

Theorem 2. (Strong Induction) *Let $P(n)$ be a statement that depends on a $n \in \mathbb{Z}^+$. If $P(1)$ is true, and if $(\forall k \in \mathbb{Z}^+)((P(1) \wedge P(2) \wedge \cdots \wedge P(K)) \Rightarrow P(k+1))$, then $P(n)$ is true for all positive integers n.*

When using induction, it may not be true that what you are trying to prove CAN be proven by induction because it may be a weak result that doesn't give you enough information for the inductive step. In that case, you have to prove something even stronger. For example, consider the following:

Example 1. (AoPS Induction Handout) It is true that for all $n \geq 1$, we have

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} < 2$$

but proving this is not possible with induction. We have to instead prove a stronger result with induction, that for all $n \geq 1$, we have

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} < 2 - \frac{1}{n}$$

3 2022 Korean Junior Math Olympiad Q3

Problem. For a given odd prime number p, define $f(n)$ the remainder of d divided by p, where d is the biggest divisor of n which is not a multiple of p. For example when $p=5$, $f(6)=1$, $f(35)=2$, $f(75)=3$. Define the sequence $a_1, a_2, \ldots, a_n, \ldots$ of integers as the followings:

- $a_1 = 1$
- $a_{n+1} = a_n + (-1)^{f(n)+1}$

Determine all integers m, such that there exist infinitely many positive integers k such that $m = a_k$.

Lemma 1. *If we write n in base p as $n = \overline{c_k c_{k-1} \ldots c_1 c_0 0 \ldots 0}_{(p)}$, where $c_0 \neq 0$, then we see that $f(n) = c_0$.*

Proof. From class □

Lemma 2. *If $n = \overline{c_k c_{k-1} \ldots c_1 c_0}_{(p)}$, then $a_{n+1} = 1 + \sum_{i=1}^{k} (c_i \mod 2)$. Here, $(x \mod 2)$ is an operation with value 0 or 1.*

Proof. You can observe this by doing examples, and it can be proven with induction.

Base case: $n=1$

In this base case, we indeed have that $a_2 = 1 + (1 \mod 2) = 2$

Inductive step: Assuming that if $l = \overline{c_k c_{k-1} \ldots c_1 c_0}_{(p)}$ satisfies $a_{l+1} = 1 + \sum_{i=1}^{l} (c_i \mod 2)$, we will show that $l+1 = \overline{d_k d_{k-1} \ldots d_1 d_0}_{(p)}$ satisfies $a_{l+2} = 1 + \sum_{i=1}^{l+1} (d_i \mod 2)$.

Let j be the smallest integer such that $c_j \neq 0$ and $c_j \neq p-1$. Now, we see that $l+1 = \overline{c_k c_{k-1} \ldots c_{j+1} (c_j+1) 0 \ldots 0}_{(p)}$, thus $d_i = c_i$ if $i > j$, $d_i = c_{i+1}$ if $i = j$, and $d_i = 0$ otherwise. Then,

$$\begin{aligned}
a_{l+2} &= a_{l+1} + (-1)^{f(l+1)+1} \\
&= 1 + \left(\sum_{i=1}^{l} (c_i \mod 2) \right) + (-1)^{f(l+1)+1} \\
&= 1 + \left(\sum_{i=j+1}^{l} (c_i \mod 2) \right) + (c_j \mod 2) + (-1)^{f(l+1)+1} \\
&= 1 + \left(\sum_{i=j+1}^{l} (c_i \mod 2) \right) + (c_j \mod 2) + (-1)^{c_j+1+1} \\
&= 1 + \left(\sum_{i=j+1}^{l} (c_i \mod 2) \right) + (c_j + 1 \mod 2) \\
&= 1 + \left(\sum_{i=j+1}^{l} (d_i \mod 2) \right) + (d_j \mod 2) \\
&= 1 + \left(\sum_{i=1}^{l} (d_i \mod 2) \right)
\end{aligned}$$

Which is what we wanted to show.

Thus, the induction is complete, and the lemma is proven □

Now, for any $m \in \mathbb{Z}^+$, let $s := \frac{(p^m - 1)}{(p-1)}$. From Lemma 2, we see that for any non-negative i, we have $a_{s \cdot p^i + 1} = m$. Thus, $(\forall m \in \mathbb{Z}^+)(\exists \text{ infinitely many } k \in \mathbb{Z}^+)(m = a_k)$.