Pigeonhole Principle

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The pigeonhole principle is used in proofs usually to show certain boundary cases satisfy a property.

1 What is the Pigeonhole Principle?

The Pigeonhole Principle, also known as Dirichlet's Box Principle, is presented as a statement about with placing pigeons into pigeonholes, but applies to much more.

Theorem 1. (Pigeonhole Principle) *Suppose there are at least $n \times k + 1$ pigeons and k pigeonholes. If every pigeon is in a pigeonhole, then the there exists at least one pigeonhole with $n + 1$ pigeons in it.*

The pigeonhole principle is quite intuitive, and more generalized versions of the theorem do exist. If you have to use a generalized version of the pigeonhole principle, it is generally not required to cite a theorem or give a proof for it in a contest. Here is one of the main generalizations of the pigeonhoole principle (though others do exist).

Theorem 2. (Generalized Pigeonhole Principle) *If $q_1, q_2, \ldots, q_n \in \mathbb{Z}^+$, then if $\left(\sum_{i=1}^{n} q_i\right) - n + 1$ pigeons are placed into n holes, then we have that at least one $1 \leq k \leq n$ satisfies that the k^{th} pigeonhole has at least q_k pigeons in it.*

This theorem can be used quite trivially to justify statements like the following:

Problem 1. Prove that if Jean has 10 socks in his drawer, and he takes out 6 of them, that he will have at least one matching pair of socks.

Solution. By the Pigeonhole Principle, since there are 5 different types of socks, and since he has taken out 6, he has taken out at least $1 + 1 = 2$ socks of the same type, meaning that he has taken out a matching pair of socks.

2 Exercises

Exercise 1. (1961 SUMO 8.4) Prove that it is possible to place 7 markers in a 4×4 grid but impossible to place 6 markers in a 4×4 board such that regardless of which two rows and two columns are chosen in the grid, there is a marker that is in none of the chosen rows or columns.

Exercise 2. There are 5 points chosen in a 2×2 square. Prove that there are always two points which are at a distance of at most $\sqrt{2}$ from each other.

Exercise 3. (1947 HMC) Prove that among any 6 people, you can always find 3 people who all know each other, or 3 people who all do not know each other. Assume that if person A knows person B, then person B knows person A. This problem is a specific case of the more general Ramsey's Problem.

Exercise 4. Given a set, S, of n integers, prove that there exists a nonempty subset, T, of S such that the sum of all the elements of T is divisible by n.

SUMO refers to the Soviet Union Mathematical Olympiad and HMO refers to the Hungarian Mathematics Competition.