

Cyclic Quadrilaterals and More Triangles

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1 Cyclic Quadrilaterals

Definition 1. We call a quadrilateral which can be inscribed in a circle a **cyclic quadrilateral** (or a **concylic quadrilateral**) and we say that the circle circumscribes the quadrilateral.

From the previous lesson about points in a triangle, we learnt about the properties of triangles inscribed in a circle, and because we know that there are $\binom{4}{3} = 4$ distinct triangles in a cyclic quadrilateral, we can make some statements with our knowledge of angles while dealing with circumcircles.

Theorem 1. Any 4 points A, B, C , and D are concyclic iff $\angle ABC = \angle ADC$.

2 Related Theorems

Theorem 2. (Power of a Point) If $ABCD$ is a convex quadrilateral with AB and CD intersecting at P and AC and BD intersecting at Q , then $ABCD$ is cyclic iff either:

1. $AQ \cdot QC = BQ \cdot QD$ (or equivalently $QAD \sim QBC$)
2. $PA \cdot PB = PC \cdot PD$ (or equivalently $PAD \sim PCB$)

Theorem 3. (Ptolemy's Theorem) Quadrilateral $ABCD$ is cyclic iff

$$AB \cdot CD + AD \cdot BC = AC \cdot BD$$

Theorem 4. (Brahmagupta's Formula) If $ABCD$ is a cyclic quadrilateral with sides of length a, b, c , and d , then let the semiperimeter be $s := \frac{1}{2}(a + b + c + d)$. We have that:

$$[ABCD] = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

Corollary 1. (Heron's Formula) For a triangle ABC , if s is the semiperimeter, $\frac{1}{2}(a + b + c)$, we have:

$$[ABC] = \sqrt{s(s-a)(s-b)(s-c)}$$

3 Exercises

Exercise 1. Let ABC be a triangle with heights AD, BE , and CF . Find the measures of the internal angles of $\triangle DEF$.

Exercise 2. This is a well known fact. Prove that if l is a line tangent to the circumcircle of ABC at A , and D lies on l such that $\angle DAC > \angle DAB$, then $\angle DAB = \angle ACB$.

Exercise 3. (2022 KJMO Q1) The inscribed circle of an acute triangle ABC meets the segments AB and BC at D and E respectively. Let I be the incenter of the triangle ABC . Prove that the intersection of the line AI and DE is on the circle whose diameter is AC .

KJMO refers to the Korean Junior Math Olympiad.