Midterm Information

The second midterm will take place during class on Monday, May 20^{th}. The exam takes 50 minutes to complete. You are allowed an index card with your own notes on it. The exam will consist of three parts: I) short answer, where you state an important definition/theorem, count something, do some quick computation, or decide on the truth/falsity of some statement. II) Prove or disprove some statement by either providing some counterexample or writing a short proof. III) Prove a statement (often longer proofs than in part II). The exam will cover everything we learned on graph theory including

- The language and definitions of graphs
- The two handshaking lemmas
- Special types of graphs (e.g. $K_n, K_{n,m}, P_n, C_n, W_n, Q_n, \ldots$)
- (Induced) Subgraphs
- Graph Isomorphisms
- Walks, Paths, Cycles, Trails, and Circuits (We follow the definitions in Siniora's notes NOT in the textbook).
- Connectivity
- Bipartite graphs
- Eulerian circuits and trails
- Hierholzer's algorithms
- Hamiltonian paths and cycles
- Planarity of graphs
- Euler's formula for planar graphs
- Non-planar graphs

To study for the exam, make sure to

- Know your definitions and statements of theorems well! I will ask you to state some of them, which should be more or less free points for you.
- Read the notes carefully, understanding all theorems AND their proofs. Reading math is not like reading literature... make sure to have a paper and pencil and work through the proofs yourself as if you are doing them on your own!
- Review homework problems and solve the optional ones too
- Solve the problems below and come to office hours to ask me for help and/or extra practice problems.

Practice Problems

In what follows, the word "graph" means finite simple graph.

1. If G is a graph of order n, what is the maximum number of edges in G?

2. Prove that in any graph, there exists two vertices u, v with $\deg(u) = \deg(v)$.

3. Let G be a graph that is not connected, what is the maximum size of G?

4. Consider the complete graph K_5 with vertices labelled $\{a, b, c, d, e\}$. How many path have a as an end vertex?

5. Is it true that any graph having exactly two vertices of odd degree must contain a path from one to the other?

6. Let G be a graph such that $\deg(v) \geq 2$ for all vertices v in G. Prove that G contains a cycle.

7. Let P and Q be two path of maximum length in a connected graph. Prove that P and Q have a vertex in common.

8. Prove that an edge e is bridge if and only if e lies on no cycle.

9. Prove or disprove each of the following:

    - If G has no bridges, then G has exactly one cycle.
    - If G has no cut vertices, then G has no bridges.
    - If G has no bridges, the G has no cut vertices.

10. If $K_{m,n}$ is regular, prove that $m = n$.

11. Prove or disprove: K_4 is a subgraph of $K_{4,4}$.

12. List all connected subgraphs of C_{17}, up to isomorphism.

13. The concept of a complete bipartite graphs can be generalized to define the complete multipartite graph $K_{n_1,n_2,\ldots,n_k}$, where we have k sets of vertices $A_1, \ldots, A_k$ (containing $n_1, \ldots, n_k$ vertices, respectively), where all possible edges between any two different sets are present, and no edges are present between two vertices in the same set. Find an expression for the order and size of $K_{n_1,\ldots,n_k}$.

14. For each of our special graphs $K_n, K_{n,m}, P_n, C_n, W_n, Q_n, \ldots$ decide whether they contain an Eulerian trail or an Eulerian circuit. If one exists, find it in a step-by-step application of Hierholzer's algorithm. If none exist, prove it using the theorems we learned.

15. Think of necessary and sufficient conditions for a graph G and its complement $\overline{G}$ to both be Eulerian. Prove your result.

16. For which m, n is the complete bipartite graph $K_{m,n}$ hamiltonian?

17. Show that there is a knight tour on a 3×4 chessboard.

18. Show that there is no knight tour in a 4×4 chessboard.

19. Draw a 5-regular *planar* graph
20. Explain why embedding a graph in the plane is essentially the same as embedding the graph on a sphere.
21. Let G be a planar graph. Prove that G is bipartite if and only if $b(R)$ is even for every region R.
22. If G is planar 3-regular graph of order 24, how many regions are in a planar representation of G?
23. Let G be a connected planar graph of order less than 12. Prove that $\delta(G) \leq 4$.
24. Prove that Euler's formula fails for disconnected graphs.
25. Let G be a planar graph which contains NO copy of C_3 as a subgraph. Prove that G has no more than $2n - 4$ edges.
26. Prove that there is no bipartite planar graph with minimum degree at least 4.
27. Let G be a planar graph with k connected components. Prove that $V - E + F = 1 + k$.
28. Let G be of order $n \geq 11$. Show that at least one of G and $\overline{G}$ is nonplanar.
29. Show that the average degree of a planar graph is less than 6.
30. Find a 4-regular planar graph. Prove that it is unique (up to isomorphism).
31. A planar graph G is called maximal planar if the addition of any edge to G creates a nonplanar graph.
    - Show that every region of maximal planar graph is a triangle.
    - If a maximal planar graph has order n, how many edges and regions does it have?