Functions as told by MATH 231

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1 Functions are triples

A *function* $f : A \to B$ is really a triple: (1) a set A called the *domain*, (2) a set B called the *codomain*, and (3) a rule which assigns every element $x \in A$ to some unique element $f(x) \in B$. Often times only part (3) the rule for assignment in focused on and often given as a formula. However, it is important to look at (1), (2), and (3).

Remark. In programming you encounter a statement like `int my_function(int x, float y)` which declares a function. The `(int x, float y)` says the domain of the function is order pairs of integers and floating point numbers (i.e. a Cartesian product). The `int` outside is saying the function will output on integer (i.e. the specifying the codomain). The rule for computing the function would be given after this declaration.

Problem 1. Consider $f : \mathbb{Z} \to \mathbb{Z}$ with $f(x) = x$. Also, consider $g : \mathbb{N} \to \mathbb{Z}$ with $g(x) = x$. Lastly, consider $h : \mathbb{Z} \to \mathbb{N}$ with $h(x) = x$. Are these all the same function? Are they all actually functions?

Problem 2. We want a function $f : A \to \mathbb{N}$ given by $f(a) = \frac{a}{10}$. What is a possible choice for the domain A?

Problem 3. We want a function $g : \mathbb{Z} \to B$ given by $g(a) = a\pi$. What is a possible choice for the codomain B?

2 Injective, surjective, and bijective

Let $f : A \to B$ be a function, then

- f is *injective* if $\forall x \in A, \forall y \in A, x \neq y \to f(x) \neq f(y)$.
- f is *surjective* if $\forall y \in B, \exists x \in A, f(x) = y$.
- f is *bijective* if it is both injective and surjective.

Remark. You may be more familiar with the term *one-to-one* in place of injective and the term *onto* in place of surjective. These mean the same thing and can also be used.

Problem 4. Find a quantified expression for both not injective and not surjective by negating the expressions in the definitions above.

Problem 5. For each of the following function determine if they are injective, surjective, both (i.e. bijective), or neither:

- $f : \mathbb{Z} \to \mathbb{Z}$, $f(n) = n + 3$
- $g : \mathbb{N} \to \mathbb{N}$, $g(n) = n + 3$
- $\alpha : \mathbb{Q} \to \mathbb{Q}, \alpha(x) = 2x$
- $\beta : \mathbb{Z} \to \mathbb{Z}, \beta(x) = 2x$
- $\phi : \mathbb{N} \times \mathbb{Z} \to \mathbb{Z}, \phi(a, b) = ab$
- $\psi : \mathbb{N} \times \mathbb{Z} \to \mathbb{N}, \psi(a, b) = a^3b^2$
- $h : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}, h(x) = (x, x + 1)$

3 Infinity is weird

The natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$ and the integers $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ have infinitely many elements. Also, $\mathbb{N} \subseteq \mathbb{Z}$ so the natural numbers are a subset of the integers. In this sense there are "more" integers than natural numbers. In another sense there is the same amount of each. We now explore this. Let us define two functions $f : \mathbb{Z} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{Z}$ by

$$f(n) = \begin{cases} 2n & \text{if } x \geq 0 \\ -2n - 1 & \text{if } n < 0 \end{cases} \qquad g(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{-n-1}{2} & \text{if } n \text{ is odd} \end{cases}$$

Problem 6. Show that both $f : \mathbb{Z} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{Z}$ are bijections.

Problem 7. Show that for $f : \mathbb{Z} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{Z}$ we have $f(g(n)) = n$ for all $n \in \mathbb{N}$ and $g(f(n)) = n$ for all $n \in \mathbb{Z}$.

The above problem says that the functions $f : \mathbb{Z} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{Z}$ are *inverse* to each other. Having an inverse function is equivalent to being a bijection.

That is, for any function $f : A \to B$ there is an inverse function $g : B \to A$ with $f(g(n)) = n$ for all $n \in B$ and $g(f(n))$ for all $n \in A$ if and only if f is a bijection.

The *cardinality* of a set A is denoted $|A|$ and is roughly "the number of elements in the set." Two sets have the same cardinality if and only if there is bijection between them. For finite sets it is simply the number of elements in the set. So, $|\{0, 5, 11\}| = 3$. For infinite sets it can be a bit weird. We saw $|\mathbb{N}| = |\mathbb{Z}|$ since we exhibited a bijection. Perhaps surprisingly, it is also the case that $|\mathbb{N}| = |\mathbb{Q}|$. The next problem is a slightly easier version of showing that the sets of natural numbers and rational numbers have the same cardinality.

Problem 8. Find a bijection between $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$. Remember you need a rule for assigning each element of $\mathbb{N}$ to an element of $\mathbb{N} \times \mathbb{N}$ it doesn't have to be what we usually think of as a "formula."

Remark. It is good to know that $|\mathbb{N}| \neq |\mathbb{R}|$. This means there is no bijection and "there are strictly more real numbers than natural numbers." However, we won't worry about proving this fact. The natural numbers are called *countably infinite* while the real numbers are called *uncountable*.

Problem 9. Let A and B be two finite sets with $|A| = m$ and $|B| = n$. Find $|\mathcal{P}(A)|$ and $|A \times B|$. Try some examples for small sets A and B.