Welcome to Hilbert's Grand Hotel

MATH 231

Hilbert's Grand Hotel has a countable infinite number of rooms where for each natural number n it has a room numbered n. Consider one day the hotel is completely full with every room occupied. The ghost of Keith Moon arrives looking for a hotel room to trash. How can this new guest to accommodated? Hilbert simply instructs the guest in room n to move to room $n+1$ for each $n \in \mathbb{N}$. Now room 0 is unoccupied and can get rented to the ghost of Keith Moon.

Here we find what feels like a paradox in that the hotel is completely full yet it can still accommodate a further guest. Such a situation cannot be true of a hotel with a finite number of rooms. We will see below that when there is a countably infinite number of rooms the hotel can accommodate many new guests even when completely full.

Problem 1. Hilbert's Grand Hotel is completely full with all rooms occupied, then 4 new guests arrive each needing a room. How can these 4 guests be fit into the hotel?

Problem 2. Hilbert's Grand Hotel is completely full with all rooms occupied, then a bus arrives with a countable infinite number of new guests. How can all the new guests be accommodated at the hotel?

Problem 3. This time Hilbert's Grand Hotel is completely empty, but a countably infinite number of busses arrive each with a countably infinite number of guests. How can all the guests be accommodated at the hotel?