Pricing Problem Statement

This challenge centers on designing and implementing a pricing system for DeliveryContracts, a financial instrument that offers optionality in the delivery of underlying assets. The system you develop should handle both the underlying asset ValueNotes and the derivative instrument itself.

Your solution will be evaluated on four aspects: implementation accuracy (20 marks), system design (40 marks), documentation (20 marks) and bonus (10 marks), totaling 90 possible marks. While accurate pricing is essential, more emphasis is placed on creating maintainable, well-structured code that could adapt to evolving market requirements. We expect solutions to demonstrate strong object-oriented principles, appropriate use of design patterns, and modern C++ features, supported by clear architecture and comprehensive documentation. Your code should be well-commented and demonstrate consideration for maintainability, scalability and performance. Robust error handling is essential - your implementation should gracefully handle edge cases, invalid inputs, and potential numerical instabilities with appropriate error messages and exception handling mechanisms. You are encouraged to make reasonable assumptions where specifications may feel incomplete or unclear - this is a natural part of system design. However, all assumptions must be clearly documented with sound justification explaining your reasoning.

The challenge consists of two main components. First, you'll implement ValueNotes - the underlying instruments that form the foundation of our pricing system. Then, you'll develop the DeliveryContract pricing framework, which builds upon your ValueNote implementation to handle more complex financial derivative calculations.

Question 1

A ValueNote is a financial instrument representing a loan between an investor and an issuer. When purchasing a ValueNote, an investor lends money to the issuer for a predetermined period. In return, the issuer commits to making periodic interest payments and repaying the principal amount at maturity. ValueNotes are tradable securities that can be bought or sold in the open market at prevailing market prices.

Some key aspects of a ValueNote are:

1) **Notional (N):** The principal amount of the loan (this is usually 100).

2) **Maturity (M):** The date when the ValueNote expires and the issuer must repay the notional amount, along with any remaining interest payments.

3) **Value Rate (VR):** The annualized interest rate expressed as a percentage (e.g., VR = 5 represents a 5% annual rate) that determines periodic interest payments based on the payment frequency.

4) **Payment Frequency (PF):** The number of interest payments per year. For instance, a ValueNote with a 5% semi-annual rate makes two payments of 2.5% each year until maturity.

5) **Price (VP):** This is the current market price of the ValueNote.

6) **Effective Rate (ER):** This is the total return an investor can expect to receive from a ValueNote depending on its current market price and the remaining cashflows.

The relationship between a ValueNote's today price and its today's Effective Rate can be expressed through the following three different conventions.

A) **Linear Rate:**

This simple linear relationship between today price VP_o and effective rate ER_0 is given by:

$$VP_o = N\left(1 - ER_0 * \frac{M}{100}\right)$$

Where M represents the maturity in years from today.

B) **Cumulative Rate:**

This convention accounts for the time value of money through discounting:

$$VP_o = \left(\sum_{i=1}^{n-1} \frac{VF_i}{\left(1+\frac{ER_0}{100*PF}\right)^{PF*t_i}}\right) + \frac{VF_n+N}{\left(1+\frac{ER_0}{100*PF}\right)^{PF*t_n}}$$

Where VF_i is ith future interest payment $\left(VF_i = \frac{VR*N}{100*PF}\right)$, PF is the interest payment frequency of the ValueNote (e.g. PF = 2 for semi-annual payments, PF = 4 for quarterly payments, PF = 12 for monthly payments), t_i is the time to each remaining payment expressed in years from today and n is the number of remaining interest payments until maturity.

C**) Recursive Rate:**

This convention uses a recursive relationship to determine ValueNote's today price:

$$VP_o = \frac{N + FV_n}{1+ER_0*M/100}$$

Where:

$$FV_0 = 0$$

$$FV_i = (FV_{i-1} + VF_i)(1 + ER_0 * \frac{m_i}{100})$$

Here n is the number of remaining payments until maturity, $ER_0$is the today's Effective rate of the ValueNote, M is the maturity in years from today, VF_i is ValueNote's ith interest flow, m_i is the time between i and i+1 th flow (m_n is assumed to be 0) in years.

Tasks for question 1

Q1.1) Define a ValueNote and implement calculations to a) Determine the Price given the Effective Rate (Price to Rate) b) Determine the Effective Rate given the Price (Rate to Price) for all three rate conventions.

Q1.2) For all three rate conventions: a) Derive and implement the analytical expressions for $\frac{\partial VP_0}{\partial ER_0}$ (Price sensitivity to Effective Rate also represented as $VP_0'\left(ER_0\right)$) b) Derive and implement the analytical expressions for $\frac{\partial ER_0}{\partial VP_0}$ (Effective Rate sensitivity to Price). *Note: The solution must show complete analytical derivation. Any numerical approximation methods (such as using price or rate shocks) will result in zero marks for the entire pricing problem.*

Q1.3) For all three rate conventions: a) Derive and implement the analytical expressions for $\frac{\partial^2 VP_0}{\partial ER_0^{\ 2}}$ (Price sensitivity to Effective Rate also represented as $VP_0''\left(ER_0\right)$)) b) Derive and implement the analytical expressions for $\frac{\partial^2 ER_0}{\partial VP_0^{\ 2}}$ (Effective Rate sensitivity to Price). *Note: The solution must show complete analytical derivation. Any numerical approximation methods (such as using price or rate shocks) will result in zero marks for the entire pricing problem.*

Question 2

A DeliveryContract is a financial derivative instrument that obligates the buyer to purchase, and the seller to sell, the most economical ValueNote to deliver from a basket of ValueNotes specified in the contract at a predetermined price on a future date.

Some key aspects of a DeliveryContract are:

1) **Expiration Date (T_{ED})**: This is the date when the seller of the DeliveryContract must deliver the most economical ValueNote, and the buyer of the DeliveryContract must pay the fixed delivery price (agreed at contract initiation) multiplied by the RelativeFactor of the delivered ValueNote. For this question, expiration periods are set at either 3, 6, or 12 months from today.
2) **Basket of ValueNotes (BoVN)**: This is a pre-specified set of ValueNotes that are eligible for delivery under the DeliveryContract. The seller of the DeliveryContract has the right to choose which ValueNote from this basket to deliver at expiration.
3) **Most Economical ValueNote (MEV)**: This is the ValueNote in the delivery basket that is most economical to deliver at the expiration of DeliveryContract.

4) **RelativeFactor (RF)**: This is the multiplier assigned to each deliverable ValueNote to equalize the value of different ValueNotes for delivery purposes. This is essentially done to normalize the differences in value rates and maturities of multiple deliverables for the DeliveryContract. The calculation uses a **Standardized Value Rate (SVR)**, which is a fixed interest rate assigned to a hypothetical ValueNote as a standardized basis to differentiate between different ValueNotes from the basket. Some RelativeFactor Calculation Methods:
    A) **UnityFactor**: The simplest method where the RelativeFactor is set to 1 for all ValueNotes in the delivery basket.
    B) **CumulativeFactor**: The RelativeFactor is the price (divided by 100) of the ValueNote when its Cumulative Rate equals the Standardized Value Rate of the DeliveryContract, calculated using the price calculation method and Cumulative Rate convention established in Question 1.B.

Let's explore one potential approach to price this DeliveryContract. This method assumes that the deliverable ValueNotes' Effective Rates are perfectly correlated and can be modeled using a single normal random variable W_t. Under this approach, each ValueNote's Effective Rate (ER_t^i) involves following geometric Brownian motion:

$$ER_t^i = \bar{ER}_t^i e^{\sigma_i W_t + \left(-\frac{\sigma_i^2}{2}\right)t}$$

Where:

$W_t \sim N(0,\ t) = \sqrt{t}z$, representing a normal distribution with mean 0 and variance t.

σ_i represents the volatility parameter for i^{th} deliverable.

$\bar{ER}_t^i$ is the risk-adjusted effective rate for the i^{th} deliverable ValueNote on the expiration date calculated using the ValueNote's forward price on expiration date given by the below quadratic equation.

$$\bar{ER}_T^{i\,2}\left(\frac{1}{2}VP_T^{i}{''}\left(\tilde{ER}_T^i\right)e^{\sigma_i^2 T}\right) + \bar{ER}_T^i\left(VP_T^{i}{'}\left(\tilde{ER}_T^i\right) - VP_T^{i}{''}\left(\tilde{ER}_T^i\right)\tilde{ER}_T^i\right) + \frac{1}{2}VP_T^{i}{''}\left(\tilde{ER}_T^i\right)\tilde{ER}_T^{i\,2} - VP_T^{i}{'}\left(\tilde{ER}_T^i\right)\tilde{ER}_T^i = 0$$

Here $\tilde{ER}_T^i$ is the Effective Rate for i^{th} deliverable ValueNote on expiration date corresponding to the forward price calculated by:

$$\tilde{VP}_t = (1 + \tilde{r}t)\left(VP_0 - \sum_{t_j \le t \le t_{j+1}} \frac{VF_j}{(1 + \tilde{r}t_j)}\right)$$

Here $\tilde{r}$ is the risk-free interest rate which will be provided to you in the input.

Using this effective rate model, we can determine the forward price of each ValueNote at expiration date T_{ED}. This allows us to identify the most economical ValueNote for any given value of z. The price of the DeliveryContract can be expressed as the expected value of the minimum Price-to-RelativeFactor ratio among all deliverable ValueNotes:

$$DeliveryContract\ Price\ =\ E[\left(\frac{VP_i(ER^i_{T_{ED}})}{RF_i}\right); i\ =\ 1,\ ...,\ n]$$

Where:

RF_i is the RelativeFactor of the i^{th} deliverable ValueNote.

VP_i is the price of the i^{th} deliverable ValueNote at expiration date T_{ED} as described in Question 1.

n is the number of deliverable ValueNotes in the basket.

Let's consider a range of $z \in [-\ 3,\ 3]$ within which we can reasonably approximate the Price-to-RelativeFactor ratio function as a quadratic function for each deliverable ValueNote. To construct these quadratic approximations, we'll generate 2000 equally spaced points across this range and calculate the Price-to-RelativeFactor ratio at each point. For each deliverable ValueNote i, we then find coefficients a_i, b_i, c_i that minimize the normally weighted mean squared error. This weighting ensures points closer to the mean have greater influence on the fit.

Once we have the quadratic approximations for all deliverable ValueNotes, we can calculate the intersection points between each pair of quadratic functions by finding the roots of their differences. After sorting these intersection points along the z-axis, we can examine each interval between intersection points to determine which ValueNote has the lowest Price-to-RelativeFactor ratio. This ValueNote becomes the most economical for that range of z values.

Tasks for question 2

Q2.1) Define a DeliveryContract and implement calculations to a) Create a basket of n ValueNotes with different characteristics b) Calculate the RelativeFactor for each deliverable ValueNote.

Q2.2) Use the pricing theory provided in the question 2 to Calculate the price of the DeliveryContract.

Q2.3) Implement the calculation of delivery probabilities for all ValueNotes in the DeliveryContract.

Q2.4) Derive and implement analytical expressions for:

a) $\frac{\partial DeliveryContractPrice}{\partial \sigma_i}$, the sensitivity of the DeliveryContract price to changes in the volatility of effective rate each deliverable ValueNote.

b) $\frac{\partial DeliveryContractPrice}{\partial VP_{o_i}}$, the sensitivity of the DeliveryContract price to changes in the today price of each deliverable ValueNote

Note: The solution must show complete analytical derivation. Any numerical approximation methods (such as using vol or price shocks) will result in zero marks for the entire pricing problem.

Q2.5) [BONUS] Alternative Pricing Methods (10 extra marks). Propose and implement an alternative method for pricing the DeliveryContract. This could include using Monte Carlo simulation with correlated paths, implementing more sophisticated interpolation techniques like cubic splines instead of quadratic approximation, developing an AI/ML model for this, or your own innovative approach. Briefly explain your method's theoretical foundation and demonstrate its advantages and limitations compared to the given approach.

Submission Format

Three files are expected from you put together in a .zip file, for it to be qualified as a valid submission. See details below for the files and their extensions. Please ensure that these files are zipped directly, and not any folder which contain these files. Partial submissions are accepted - you may submit solutions for any subset of the questions, and your submission will be evaluated based on those portions.

1. C++ code in a file with .cpp extention

You have the flexibility to choose your preferred coding approach. You may opt for either functional or modular, object-oriented programming style. Your choice should reflect the method you believe aligns best with the requirements of the problem.

Note: Please explicitly include the individual header files required.
Refrain from using #include <bits/stdc++.h> since we'll be using clang compiler to test your code.

2. A .csv file with outputs for the following questions:

The output must be entered in the .csv file according to the tables provided below:

Consider a basket of four ValueNotes (n = 4) as below:

Value Note Table	*VN1*	*VN2*	*VN3*	*VN4*
Notional (N)	100	100	100	100
Maturity ($M,\ in\ years$)	5	1.5	4.5	10
Value Rate ($VR,\ in\ \%$)	3.5	2	3.25	8
Payment Frequency (PF)	1	2	1	4

Effective rate volatility (σ_i in %)	1.5	2.5	1.5	5

The basket has the following parameters:

1. Standardized Value Rate: 5%
2. Expiration time: 3 months
3. Relative Factor convention: Cumulative
4. Effective rate convention: Cumulative
5. $\tilde{r}$, risk-free interest rate: 4%

For Q1, enter your output with regards to *VN1* in the following format:

	Linear	Cumulative	Recursive
Q 1.1 a) Price, for $ER_0 = 5\%$	Price if ER_0 is linear	Price if ER_0 is cumulative	Price if ER_0 is recursive
Q 1.1 b) Rate, for VP_o = 100	Linear effective rate from price	Cumulative effective rate from price	Recursive effective rate from price
Q 1.2 a) $\frac{\partial VP_0}{\partial ER_0}$ $ER_0 = 5\%$	Price sensitivity if ER_0 is linear	Price sensitivity if ER_0 is cumulative	Price sensitivity if ER_0 is recursive
Q 1.2 b) $\frac{\partial ER_0}{\partial VP_0}$ VP_o = 100	Linear effective rate sensitivity	Cumulative effective rate sensitivity	Recursive effective rate sensitivity
Q 1.3 a) $\frac{\partial^2 VP_0}{\partial ER_0^2}$ $ER_0 = 5\%$	Derivative of price sensitivity to linear effective rate	Derivative of price sensitivity to cumulative effective rate	Derivative of price sensitivity to recursive effective rate
Q 1.3 b) $\frac{\partial^2 ER_0}{\partial VP_0^2}$ VP_o = 100	Derivative of linear effective rate sensitivity to price	Derivative of cumulative effective rate sensitivity to price	Derivative of recursive effective rate sensitivity to price

For Q2, enter your output in the following format (append these 5 columns to the output of Q1):

Q 2.1	Q 2.2	Q 2.3	Q 2.4 a)	Q 2.4 b)
RF for *VN1*	Price of Delivery contract	Delivery probability for *VN1*	$\frac{\partial DeliveryContractPrice}{\partial \sigma_1}$ for $\sigma_1 = 2$	$\frac{\partial DeliveryContractPrice}{\partial VP_{o_1}}$ for $VP_{o_1} = 95$
RF for *VN2*		Delivery probability for *VN2*	$\frac{\partial DeliveryContractPrice}{\partial \sigma_2}$ for $\sigma_2 = 3$	$\frac{\partial DeliveryContractPrice}{\partial VP_{o_2}}$ for $VP_{o_2} = 97$

RF for *VN3*		Delivery probability for *VN3*	$\frac{\partial DeliveryContractPrice}{\partial \sigma_3}$ for $\sigma_3 = 4$	$\frac{\partial DeliveryContractPrice}{\partial VP_{o_3}}$ for $VP_{o_3} = 99$
RF for *VN4*		Delivery probability for *VN4*	$\frac{\partial DeliveryContractPrice}{\partial \sigma_4}$ for $\sigma_4 = 5$	$\frac{\partial DeliveryContractPrice}{\partial VP_{o_4}}$ for $VP_{o_4} = 100$

To calculate the values for 2.4 a) and 2.4 b), use the same parameters of ValueNotes as mentioned in the Value Note Table.

NOTE: We have provided an example csv file format. Please ensure that your output is in the exact same format.

3. A .docx file for the documentation.

Provide a documentation file in .docx format. This should comprehensively explain your implementation approach, include relevant mathematical derivations, justify your key assumptions, analyze your results, and discuss any challenges you faced. Explain your considerations for computational efficiency and demonstrate how you've validated the correctness of your implementation. Additionally, detail the design patterns and architectural elements you've incorporated, explaining how these choices enhance the code's scalability, maintainability, and production readiness. Include specific examples of how your design decisions would facilitate future enhancements or integration with larger systems.