Homework 5
CS161
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① ⓐ $P \Rightarrow Q, \neg Q \Rightarrow \neg P$

$P \Rightarrow Q \equiv \neg P \vee Q$

$\neg Q \Rightarrow \neg P \equiv Q \vee \neg P$

W	P	Q	$\neg P \vee Q$	$Q \vee \neg P$
w_1	0	0	1	1
w_2	0	1	1	1
w_3	1	0	0	0
w_4	1	1	1	1

$M(P \Rightarrow Q) = \{w_1, w_2, w_4\}$

$M(\neg Q \Rightarrow \neg P) = \{w_1, w_2, w_4\}$

Since $M(\Delta) = M(\alpha)$, the pair of sentences are equivalent

(b) $P \Leftrightarrow \neg Q$, $((P \wedge \neg Q) \vee (\neg P \wedge Q))$

$P \Leftrightarrow \neg Q \equiv (P \Rightarrow \neg Q) \wedge (\neg Q \Rightarrow P)$
$\equiv (\neg P \vee \neg Q) \wedge (Q \vee P)$

W	P	Q	$Q \vee P$	$\neg P \vee \neg Q$	$\neg P \wedge Q$	$P \wedge \neg Q$
w_1	0	0	0	1	0	0
w_2	0	1	1	1	1	0
w_3	1	0	1	1	0	1
w_4	1	1	1	0	0	0

$M(Q \vee P) = \{w_2, w_3, w_4\}$
$M(\neg P \vee \neg Q) = \{w_1, w_2, w_3\}$ $\rightarrow$ $M(P \Leftrightarrow \neg Q) = M(Q \vee P) \cap M(\neg P \vee \neg Q)$
$= \{w_2, w_3\}$

$M(P \wedge \neg Q) = \{w_2\}$
$M(\neg P \wedge Q) = \{w_3\}$ $\longrightarrow$ $M((P \wedge \neg Q) \vee (\neg P \wedge Q)) = M(P \wedge \neg Q) \cup M(\neg P \wedge Q)$
$= \{w_2, w_3\}$

$\therefore$ Since $M(P \Leftrightarrow \neg Q) = M((P \wedge \neg Q) \vee (\neg P \wedge Q))$, they are equivalent

2. Valid: $M(\alpha)$ = All worlds

Unsatisfiable: $M(\alpha) = \emptyset$

(a) $(Smoke \Rightarrow Fire) \Rightarrow (\neg Smoke \Rightarrow \neg Fire)$

Smoke $\rightarrow$ S

Fire $\rightarrow$ F

$(\neg S \vee F) \Rightarrow (S \vee \neg F)$

S	F	$\neg S \vee F$	$S \vee \neg F$	$(\neg S \vee F) \Rightarrow (S \vee \neg F)$
0	0	1	1	1
0	1	1	0	0
1	0	0	1	1
1	1	1	1	1

∴ Not valid but is satisfiable

(b) $(Smoke \Rightarrow Fire) \Rightarrow ((Smoke \vee Heat) \Rightarrow Fire)$

Smoke $\rightarrow$ S Heat $\rightarrow$ H

Fire $\rightarrow$ F

If hypothesis is false, then the statement is true.

If hypothesis is true, then the conclusion must be true for the statement to be true.

$(\neg S \vee F) \Rightarrow (\neg(S \vee H) \vee F)$

$(\neg S \vee F) \Rightarrow ((\neg S \wedge \neg H) \vee F)$

$(\neg S \vee F) \Rightarrow (\neg S \vee F) \wedge (\neg H \vee F)$

S	F	H	$(\neg S \vee F) \Rightarrow (\neg S \vee F) \wedge (\neg H \vee F)$
0	0	0	T
0	0	1	F
0	1	0	T
0	1	1	T
1	0	0	T
1	0	1	T
1	1	0	T
1	1	1	T

1 if both sides are true,
0 otherwise

Not valid because $M(\alpha)$ does not equal all worlds, but satisfiable because $M(\alpha) \neq \emptyset$.

(c) $((Smoke \land Heat) \Rightarrow Fire) \Leftrightarrow ((Smoke \Rightarrow Fire) \lor (Heat \Rightarrow Fire))$

Smoke $\rightarrow$ S Fire $\rightarrow$ F
Heat $\rightarrow$ H

$$((S \land H) \Rightarrow F) \Leftrightarrow ((S \Rightarrow F) \lor (H \Rightarrow F))$$

$$(\neg(S \land H) \lor F) \Leftrightarrow ((\neg S \lor F) \lor (\neg H \lor F))$$

$$(\neg S \lor \neg H \lor F) \Leftrightarrow (\neg S \lor F \lor \neg H)$$

S	F	H	$(\neg S \lor \neg H \lor F) \Leftrightarrow (\neg S \lor F \lor \neg H)$
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

Both valid and satisfiable since $M(\alpha)$ = all worlds and $M(\alpha) \neq 0$.

③ Mythical → Myth   Horned → H
Immortal → I   Magical → Mag
Mammal → Mam

ⓐ ① $Myth \Rightarrow I$
② $\neg Myth \Rightarrow (\neg I \wedge Mam)$
③ $(Mam \vee I) \Rightarrow H$
④ $H \Rightarrow Mag$

ⓑ ① $Myth \Rightarrow I \equiv \neg Myth \vee I$
② $\neg Myth \Rightarrow (\neg I \wedge Mam) \equiv Myth \vee (\neg I \wedge Mam)$
$\equiv (Myth \vee \neg I) \wedge (Myth \vee Mam)$
③ $(Mam \vee I) \Rightarrow H \equiv \neg(Mam \vee I) \vee H \equiv (\neg Mam \wedge \neg I) \vee H$
$\equiv (\neg Mam \vee H) \wedge (\neg I \vee H)$
④ $H \Rightarrow Mag \equiv \neg H \vee Mag$

$$\boxed{\text{CNF: } (\neg Myth \vee I) \wedge (Myth \vee \neg I) \wedge (Myth \vee Mam) \wedge (\neg Mam \vee H) \wedge (\neg I \vee H) \wedge (\neg H \vee Mag)}$$

ⓒ Δ: CNF
1. $\neg Myth \vee I$
2. $Myth \vee \neg I$
3. $Myth \vee Mam$
4. $\neg Mam \vee H$
5. $\neg I \vee H$
6. $\neg H \vee Mag$

Ⓐ Prove that the unicorn is mythical, i.e., $\alpha = Myth$
∴ We want to prove $\Delta \wedge \neg\alpha$ is unsatisfiable
7. $\neg Myth$ (assuming the opposite of the conclusion)
8. $\neg I$ (2,7)
9. Mam (3,7)
10. H (4,9)
11. Mag (6,10)
12. $\neg Myth$ (1,8) → supports the assumption made in 7

∴ There are no contradictions and no more resolutions that need to be performed. This means that $\Delta \wedge \neg\alpha$ ($\alpha = Myth$) is satisfiable. Therefore, Δ is insufficient to conclude that a unicorn is mythical.

(B) Prove that the unicorn is magical → $\Delta \wedge \neg Mag$ is unsatisfiable

7. $\neg Mag$ (assuming the opposite of the desired conclusion)
8. $\neg H$ (6,7)
9. $\neg Mam$ (4,8)
10. $\neg I$ (5,8)
11. $Myth$ (3,9)
12. $\neg Myth$ (1,10

11 and 12 pose a contradiction. Therefore, since $\Delta \wedge \neg Mag$ is unsatisfiable, the unicorn must indeed be magical.

(C) Prove that the unicorn is horned → $\Delta \wedge \neg H$ is unsatisfiable

7. $\neg H$ (assuming the opposite of the desired conclusion)
8. $\neg Mam$ (4,7)
9. $\neg I$ (5,7)
10. $Myth$ (3,8)
11. $\neg Myth$ (1,9)

10 and 11 pose a contradiction. Therefore, since $\Delta \wedge \neg H$ is unsatisfiable, the unicorn must be horned.

4. Deterministic → OR gate has only one valid input

Figure 1 is decomposable because each AND gate has different atoms on either side as inputs. It is not smooth because the atoms on either side of the OR gate are not the same, for example, there is an OR with (A) and ($\neg A \wedge \neg B$) as inputs. It is deterministic as there is only one input that is true on either side of the OR gate. Eg. the first OR, ($\neg A \wedge B$) $\vee$ ($A \wedge \neg B$), only one can be true because A and $\neg A$ exist in each of the clauses. Therefore, if A=1 and B=1, the ($\neg A \wedge B$) is true and ($A \wedge \neg B$) is false. This is true for all the OR gates in Figure 1.

Figure 2 is decomposable because each AND gate has different atoms on either side. It is smooth because atoms on either side of the OR gate are the same. Eg, for the first OR, we have ($\neg A \wedge B$) $\vee$ ($\neg A \wedge B$), which are the same. It is not deterministic because either side of the OR gates are the same. Therefore, they will both be true or both be false at the same time. There will never be an instance where only one the two inputs are true.

⑤ $w(A, \neg B, C) = w(A)\,w(\neg B)\,w(C)$

WMC: Added weight of its satisfying assignments (i.e. models)

$w(A) = 0.2 \quad w(\neg A) = 0.8$
$w(B) = 0.4 \quad w(\neg B) = 0.6$
$w(C) = 0.6 \quad w(\neg C) = 0.4$
$w(D) = 0.8 \quad w(\neg D) = 0.2$

ⓐ

A	B	$\neg A \wedge B$	$\neg B \wedge A$	$(\neg A \wedge B) \vee (\neg B \wedge A)$
0	0	0	0	0
0	1	1	0	1 → (
1	0	0	1	1
1	1	0	0	0

$$\therefore \text{WMC} = w(\neg A, B) + w(A, \neg B)$$
$$= w(\neg A)w(B) + w(A)\,w(\neg B)$$
$$= (0.8)(0.4) + (0.2)(0.6)$$
$$= 0.32 + 0.12 = \boxed{0.44}$$

ⓑ NNF circuit : $(\neg A \wedge B) \vee (\neg B \wedge A)$

$$= (\neg A * B) + (\neg B * A)$$
$$= (0.8 \times 0.4) + (0.6 \times 0.2)$$
$$= 0.32 + 0.12 = \boxed{0.44}$$

The count on the root and WMC are equivalent.

ⓒ

O_1: $((A \wedge \neg B) \vee (\neg A \wedge B))$
O_2: $((\neg D \wedge \neg C) \vee (C \wedge D))$
O_3: $((\neg A \wedge \neg B) \vee (A \wedge B))$
O_4: $((C \wedge \neg D) \vee (\neg C \wedge D))$

Left-to-right

A_1: $O_1 \wedge O_2 = (((A \wedge \neg B) \vee (\neg A \wedge B)) \wedge ((\neg D \wedge \neg C) \vee (C \wedge D)))$

A_2: $O_3 \wedge O_4 = (((\neg A \wedge \neg B) \vee (A \wedge B)) \wedge ((C \wedge \neg D) \vee (\neg C \wedge D)))$

Root: $A_1 \vee A_2$

$$= (((A \wedge \neg B) \vee (\neg A \wedge B)) \wedge ((\neg D \wedge \neg C) \vee (C \wedge D))) \vee (((\neg A \wedge \neg B) \vee (A \wedge B)) \wedge ((C \wedge \neg D) \vee (\neg C \wedge D)))$$

Using the count on the root formula:

$$(((0.2 * 0.6) + (0.8 * 0.4)) * ((0.2 * 0.4) + (0.8 * 0.6))) +$$
$$(((0.8 * 0.6) + (0.2 * 0.4)) * ((0.6 * 0.2) + (0.4 * 0.8)))$$
$$= ((0.12 + 0.32) * (0.08 + 0.48)) + ((0.48 + 0.08) * (0.12 + 0.32))$$
$$= (0.44)(0.56) + (0.56)(0.44) = 2(0.44)(0.56) = \boxed{0.4928}$$