Standard 2nd Order System, PI Controllers, and Extra Poles

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Contents

1 Standard Second Order System

The standard second order system has a transfer function given by

$$H(s) = \frac{\omega^2}{s^2 + 2\xi\omega s + \omega^2}$$

where $\omega > 0$ and $\xi \geq 0$.

The poles of $H(s)$ are $s_\pm = -\xi\omega \pm \omega\sqrt{\xi^2 - 1}$, and so forms of the solution depend on the poles.

1.1 Case 1: Overdamped Case

For 2 real distinct poles, $\sqrt{\xi^2 - 1}$ is real and more than 0. This means $\xi > 1$ and this means the standard second order system can be decomposed as a sum of 2 first order systems with poles that have negative real parts.

This means that for this case any bounded input will give a bounded system response.

The system's impulse response is $h(t) = A_+ e^{s_+ t} + A_- e^{s_- t}$ where the A terms are the coefficients of the corresponding poles in the PF decomposition.

1.2 Case 2: Critically Damped Case

For 1 real repeated pole, $xi = 1$ and the repeated pole is at $s_{root} = -\omega$, which has a negative real part.

The system's impulse response is $h(t) = \omega^2 t e^{-\omega t}$.

1.3 Case 3: Underdamped Case

For 2 complex conjugate poles with a non-zero real component, $0 < \xi < 1$ and poles are at

$$s_\pm = -\xi\omega \pm \omega j \sqrt{1 - \xi^2}$$

which have negative real parts.

The system's impulse response is

$$h(t) = \frac{\omega}{\sqrt{1 - \xi^2}} e^{-\xi\omega t} \sin((\omega\sqrt{1 - \xi^2})t)$$

If ω increases, the poles move further from the origin and the response is faster.

if ξ increases, θ decreases and there is more damping.

1.4 Case 4: Undamped Case

For 2 complex conjugate poles with a zero real component, $\xi = 0$ and poles are at

$$s_\pm = \pm\omega j$$

which have 0 real parts and so the system may not be well behaved.

The system's impulse response is

$$h(t) = \omega \sin(\omega t)$$

1.5 Step Response

The step response for $\xi \geq 0$ other than $\xi = 1$ is

$$1 - \frac{1}{\sqrt{1-\xi^2}} e^{-\xi\omega t} \sin\left((\omega\sqrt{1-\xi^2})t + \theta\right)$$

where $\theta = \cos^{-1}\xi$.

At $\xi = 1$ this is a t scaled exponential.

2 PI Controllers

P controllers can get to goal quickly but miss the exact value. I controllers can get to exact velocity but too slowly.

P and I controllers can be used together and give a system with transfer function

$$H_{RV}(s) = \frac{\frac{k_i}{m}\left(\frac{k_p}{k_i}s + 1\right)}{s^2 + \left(\frac{b+k_p}{m}\right)s + \frac{k_i}{m}}$$