Fourier Series, Parseval and Dirichlet Theorems, Fourier Transform

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1 Integrating and Differentiating Fourier Series

Integrals and derivatives for Fourier series do not follow from linearity.

The Fourier series of a PWC1 τ periodic L^2 function $f(t)$ can be term-by-term integrated to give a convergent series that **uniformly converges** over any finite interval. If f_p is in L^2 and

$$f_p(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{2\pi njt}{\tau}}$$

then

$$\int_a^t f_p(t)\,dt = \sum_{n=-\infty}^{\infty} c_n \int_a^t e^{\frac{2\pi njt}{\tau}}\,dt$$

and the convergence is at least pointwise. Here the RHS might not be a Fourier series.

Let f be a PWC1 τ periodic function, that is continuous, and satisfies $f(-\tau/2) = f(\tau/2)$. The series for f can be term-by-term differentiated to give a pointwise convergent series that **pointwise converges** to $f_p'(t)$ for all t such that $f_p''(t)$ exists. Explicitly, if f satisfies the above and

$$f_p(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{2\pi n j t}{\tau}}$$

then

$$f_p'(t) = \sum_{n=-\infty}^{\infty} c_n \frac{d}{dt} e^{\frac{2\pi n j t}{\tau}}$$

and the convergence is pointwise for all points where $f_p''(t)$ exists.

2 Parseval and Dirichlet Theorems

Dirichlet's Theorem: if f is PWC1 and τ periodic, then the Fourier series of $f(t)$ converges pointwise to $f_p(t)$

Dirichlet's Theorem gives that the average error for truncated Fourier series goes to zero in the limit.

Parseval's Theorem: if f is $L^2[-\tau/2, \tau/2]$ and τ periodic, then

$$\frac{1}{\tau} \int_{-\tau/2}^{\tau/2} |f(t)|^2 dt = \sum_{n=-\infty}^{\infty} |c_n|^2$$

Real-Valued Parseval's Theorem: if f is a real valued PWC1 and τ periodic, then

$$\frac{1}{\tau} \int_{-\tau/2}^{\tau/2} |f(t)|^2 dt = c_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} \left(c_n^2 + s_n^2\right)^2$$

where c_n and s_n are the Fourier cosine and sine coefficients.

3 Fourier Transform

Fourier series require τ periodic functions.

If $f \in L^1$, then the Fourier transform of f is

$$F(\omega) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t) e^{-\omega t j}\, dt$$

and the inverse Fourier transform of $F(\omega)$ is

$$f(t) = \mathcal{F}\{F(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{\omega tj}\,d\omega$$

Fourier transform is the two-sided Laplace transform where $s = \omega j$.