L2 Space, Inner Product on L2, Computing Fourier Coefficients

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1 L2 Space

A complex valued function f is in $L^2([a,b])$ if

$$\int_a^b |f(x)|^2 dx$$

exists and is finite.

f is in the class L^2 if

$$\int_{-\infty}^b |f(x)|^2 dx$$

exists and is finite.

L2 space forms a vector space.

2 Fourier Coefficients

2.1 Inner Product for L2

If f and g are complex valued function in $L^2([a,b])$, then the **standard inner product** is

$$\langle f, g \rangle = \frac{1}{b-a} \int_a^b f(t)\overline{g(t)}\, dt$$

If $f, g \in L^2([a,b])$, then $\langle f, g \rangle$ exists and is finite.

The set of complex exponentials $\left\{ e^{\frac{2\pi n}{\tau} jt} \mid n \in \{0, \pm 1, \pm 2, \dots\} \right\}$ is an orthonormal basis for a subspace of $L^2([-\tau/2, \tau/2])$.

2.2 Fourier Series

If $f \in L^2([-\tau/2, \tau/2])$, then the Fourier series in complex form of $f(t)$ is

$$\sum_{n=-\infty}^{\infty} c_n e^{\frac{2\pi n}{\tau} jt}$$

where c_n are found by projecting f into the basis of complex exponentials.

If $f \in L^2([-\tau/2, \tau/2])$, then the Fourier coefficients c_n of $f(t)$ are

$$c_n = \left\langle f(t), e^{\frac{2\pi n}{\tau} jt} \right\rangle$$

If f is real valued, then $c_n = \overline{c_{-n}}$.