Real Fourier Series Convergence

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1 Real Fourier Series

1.1 Periodic Functions

A function f defined on $\mathbb{R}$ is τ **periodic** if for all $t \in \mathbb{R}$

$$f(t) = f(t + \tau)$$

Generally pick the smallest value of τ such that the above holds.

The theorem for Fourier Coefficients for Series in Complex Form holds for τ periodic functions, and can integrate over 1 period.

When finding the Fourier series without knowing the domain for f and knowing that f is τ periodic, first find the τ period of f and do the computation over a period of f.

1.2 Fourier Sinusoidal Series

A function f is **even** if $f(-t) = f(t)$. A function f is **odd** if $f(-t) = -f(t)$.

For a real valued τ periodic function $f \in L^2([-\tau/2, \tau/2])$:

- if f is even, then the Fourier series can be simplified to a sum of cosine waves: Fourier cosine series
- if f is odd, then the Fourier series can be simplified to a sum of sine waves: Fourier sine series

If f is a real valued function that is in $L^2([-\tau/2, \tau/2])$, then

- if f is even, then the Fourier cosine series for f is

$$\sum_{n=0}^{\infty} c_n \cos\left(\frac{2\pi n}{\tau} t\right)$$

  where

$$c_n = \begin{cases} \langle f(t), 1 \rangle & n = 0 \\ 2\left\langle f(t), \cos\left(\frac{2\pi n}{\tau} t\right)\right\rangle & n > 0 \end{cases}$$

- if f is odd, then the Fourier sine series for f is

$$\sum_{n=1}^{\infty} s_n \sin\left(\frac{2\pi n}{\tau} t\right)$$

  where

$$s_n = 2\left\langle f(t), \sin\left(\frac{2\pi n}{\tau} t\right)\right\rangle$$

If f is real, then it can be decomposed into even and odd functions as follows:

$$f_{even}(t) = \frac{f(t) + f(-t)}{2} \quad \text{and} \quad f_{odd}(t) = \frac{f(t) - f(-t)}{2}$$

with $f(t) = f_{even}(t) + f_{odd}(t)$.

Every real valued function in $L^2([-\tau/2, \tau/2])$ admits a real valued Fourier series with some sin and/or cos terms.

2 Convergence of Fourier Series

2.1 Types of Convergence

If $f_1, f_2, \ldots, f_n, \ldots$ is a sequence of L^2 functions defined on $[a, b]$, then:

- the sequence **converges in the $L^2([a,b])$ norm**, or **converges in the mean**, or **converges almost everywhere**, to f if

$$\lim_{n\to\infty}\sqrt{\int_a^b |f_n(x)-f(x)|^2\,dx}=0$$

  which is when the average error goes to 0

- the sequence **pointwise converges** to f if for any $x\in[a,b]$

$$\lim_{n\to\infty}(f_n(x)-f(x))=0$$

  which is when the error at each point goes to 0

- the sequence **uniformly converges** to f if

$$\lim_{n\to\infty}\max_{[a,b]}|f_n(x)-f(x)|=0$$

  which is when the maximum error converges to 0

  - if the maximum does not exist, then replace it with the smallest upper bound (called the sup)

2.2 Fourier Series Convergence

A function f is **Piecewise C^1** (PWC1) on the interval $[a,b]$ if there is a finite partition $a=t_0<t_1<\cdots<t_k=b$ such that:

- f' exists on each interval (t_i,t_{i+1})
- f' is continuous on each interval (t_i,t_{i+1})
- f and f' are bounded on each interval (t_i,t_{i+1})

The **periodic extension** of a function f defined on $[a,b]$ is the $b-a$ periodic function f_p such that

- $f_p(t)=f(t)$ for $t\in(a,b)$ where $f(t)$ is continuous
- $f_p(t)=\frac{f(t^-)+f(t^+)}{2}$ for $t\in(a,b)$ where $f(t)$ is not continuous
- $f_p(a)=\frac{f(a)+f(b)}{2}=f_p(b)$

Let f_p be the periodic extension of a function $f\in L^2([-\tau/2,\tau/2])$:

- the Fourier series of f converges in the L^2 norm to f and f_p on any finite subinterval of $[-\tau/2,\tau/2]$

- if f_p is piecewise C^1, then the Fourier series of f converges pointwise to f_p for all $x \in \mathbb{R}$
- if f_p is piecewise C^1 and continuous, then the Fourier series of f converges uniformly to f_p on any finite interval of $\mathbb{R}$

Gibbs Phenomenon: for an $L^2([a, b])$ function f with periodic extension f_p, if f_p is not continuous at some point t_0, then <u>truncated</u> Fourier series of f will have growing oscillations near the point t_0

- these oscillations do not appear in the infinite sum