

Linear System Theory

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1 Stability

For the initial value problem with $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ and $\mathbf{x}(t_0) = x_0$, the unique solution is

$$\mathbf{x}(t) = e^{A(t-t_0)}x_0$$

Note that the matrix exponential comes from the Taylor series

$$e^M = \sum_{k=0}^{\infty} \frac{M^k}{k!}$$

where $M^0 = I$.

A system in state-space form is:

- stable if, for every initial condition $\mathbf{x}(t_0) = x_0 \in \mathbb{R}^n$, $\mathbf{x}(t) = e^{A(t-t_0)}x_0$ is uniformly bounded

- asymptotically stable if, in addition, for every initial condition $\mathbf{x}(t_0) = x_0 \in \mathbb{R}^n$, $\mathbf{x}(t) \rightarrow 0$ as $t \rightarrow \infty$
- unstable otherwise

A system is asymptotically stable if and only if all eigenvalues of A have negative real part.

A **Bounded-Input-Bounded-Output (BIBO) stable** system is one with bounded response to a bounded input.

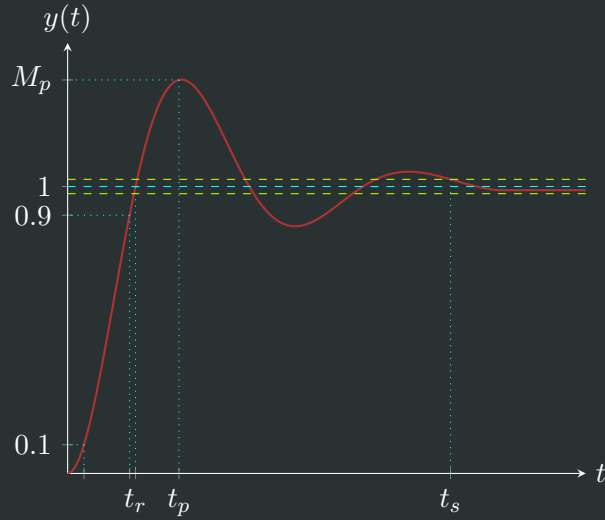
A system is BIBO stable if and only if all poles of all entries of its transfer function have negative real part.

Asymptotic stability implies BIBO stability.

2 Performance

Important performance metrics to describe the behaviour of a system with a unit step signal are:

- **steady-state gain:** $\lim_{t \rightarrow \infty} y(t)$
- **rise time:** time to go from 10% to 90% of steady-state
- **peak time:** time at which peak is achieved
- **overshoot percentage:** $OS\% = \frac{|C_{\max}| - C_{ss}}{C_{ss}} \times 100\%$
- **settling time:** smallest time such that $\forall t \geq T, \frac{|C_{ss} - y(t)|}{|C_{ss}|} \leq \varepsilon$



2.1 First Order Systems

Steady-state gain is μ .

Rise-time is 2τ .

Peak time and overshoot would not make sense for a first order system.

Settling time is $-\tau \ln(\varepsilon)$. For $\varepsilon = 0.05$, $T_s \approx 3\tau$. For $\varepsilon = 0.03$, $T_s \approx 4\tau$. For $\varepsilon = 0.01$, $T_s \approx 5\tau$.

2.2 Second Order Systems

Steady-state gain is μ .

Rise-time is

$$\frac{2.16\zeta + 0.16}{\omega_n}$$

Peak time is

$$\frac{\pi}{\omega_n \sqrt{1 - \zeta^2}}$$

Overshoot percentage is

$$e^{\frac{-\zeta\pi}{1-\zeta^2}}$$

Settling time is

$$-\frac{\ln(\varepsilon)}{\zeta\omega_n}$$

3 Lower Order Approximations

Higher order systems often have a few poles that are much closer to the imaginary axis than others, these are **dominant poles**.

For a system with transfer function:

$$G(s) = G_{fast}(s)G_{slow}(s)$$

where the poles of $G_{slow}(s)$ are more dominant, the fast components can be approximated by a static transfer function whose value is equal to steady-state gain, $G_{fast}(0)$.

4 System Identification

Assume a structure of the model

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_ms^m + \dots + b_1s + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0}$$

which gives

$$y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \dots + a_1\dot{y}(t) + a_0y(t) = b_mu^{(m)}(t) + \dots + b_1u(t) + b_0u(t)$$

For any time t_k , set

$$y^{(n)}(t_k) = b_mu^{(m)}(t_k) + \dots + b_0u(t_k) + a_{n-1} \left(-y^{(n-1)}(t_k) \right) + \dots + a_0(-y(t_k))$$

which can be represented as:

$$d^T(t_k)\theta = \begin{pmatrix} u^{(m)}(t_k) & \dots & u(t_k) & -y^{(n-1)}(t_k) & \dots & -y(t_k) \end{pmatrix} \begin{pmatrix} b_m \\ \vdots \\ b_0 \\ a_{n-1} \\ \vdots \\ a_0 \end{pmatrix}$$

