Frequency Response And Bode Plots

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1 Frequency Response

The steady state response of the system to a sinusoidal input signal. The resulting output signal for a linear system is sinusoidal in the steady state, and differs from the input in the amplitude and phase signal.

Specifically, the **frequency response** of a system is the complex-valued function

$$G : \omega \in \mathbb{R} \mapsto G(j\omega) \in \mathbb{C}$$

which can be expressed as:

$$|G| : \omega \in \mathbb{R} \mapsto |G(j\omega)| \in \mathbb{R}$$

$$\angle G : \omega \in \mathbb{R} \mapsto \angle G(j\omega) \in \mathbb{R}$$

The transfer function in the frequency domain becomes

$$G(j\omega) = |G(j\omega)|e^{j\phi(\omega)}$$

where $\phi(\omega) = \angle G(j\omega)$ comes from the zeros and poles of the transfer function.

For an system $Y(s) = G(s)U(s)$, where $u(t) = \mathcal{U}\sin(\omega t)$:

$$y(t) = \text{transient} + \mathcal{U}|G(j\omega)|\sin(\omega t + \angle G(j\omega))$$

where the transient part goes to 0 and the sinusoidal part is the steady state response.

2 Bode Plots

Bode plots are a graphical representation of a system's frequency response with plots for the magnitude and phase along a logarithmic scale for ω.

The magnitude is representation with

$$|G(j\omega)|_{dB} = 20\log|G(j\omega)|$$

Consider the general transfer function:

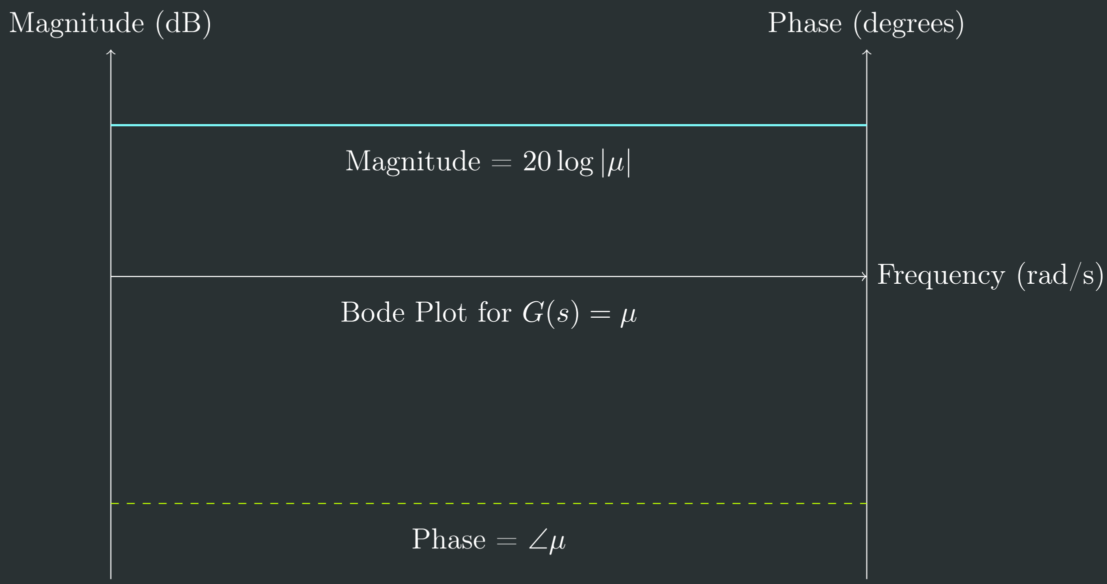
$$G(s) = \frac{\mu\prod_i(1+T_i s)\prod_i\left(1+\frac{2\xi_i}{\alpha_{n,i}}s+\frac{s^2}{\alpha_{n,i}^2}\right)}{s^\rho\prod_i(1+\tau_i s)\prod_i\left(1+\frac{2\zeta_i}{\omega_{n,i}}s+\frac{s^2}{\omega_{n,i}^2}\right)}$$

where μ is the gain, ρ is the number of poles if $\rho > 0$ or zeros if $\rho < 0$ at the origin, T_i, τ_i are the time constants of the real zeros and poles respectively, ξ_i, ζ_i are the damping constants of the complex conjugate zeros and poles, and $\alpha_{n,i}, \omega_{n,i}$ are the natural frequencies of the complex conjugate zeros and poles.

2.1 Constant Gain

The constant gain μ has

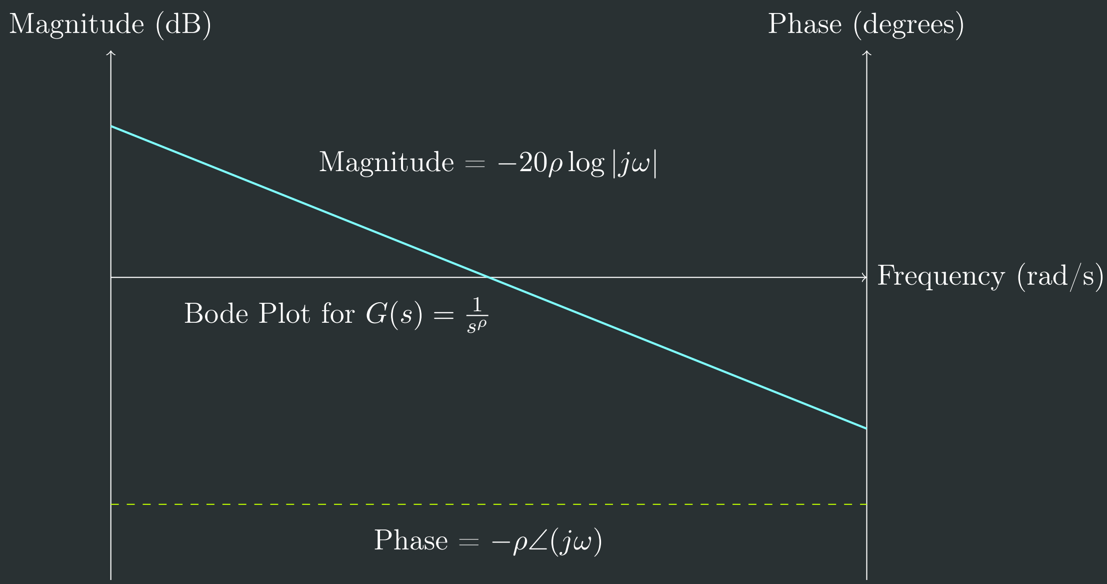
$$|G(j\omega)|_{dB} = 20\log|\mu|, \angle G(j\omega) = \angle\mu$$



2.2 Poles/Zeros at Origin

Poles/Zeros at the origin $j\omega$ have:

$$|G(j\omega)|_{dB} = -20\rho \log |j\omega|, \angle G(j\omega) = \rho \angle (j\omega)$$



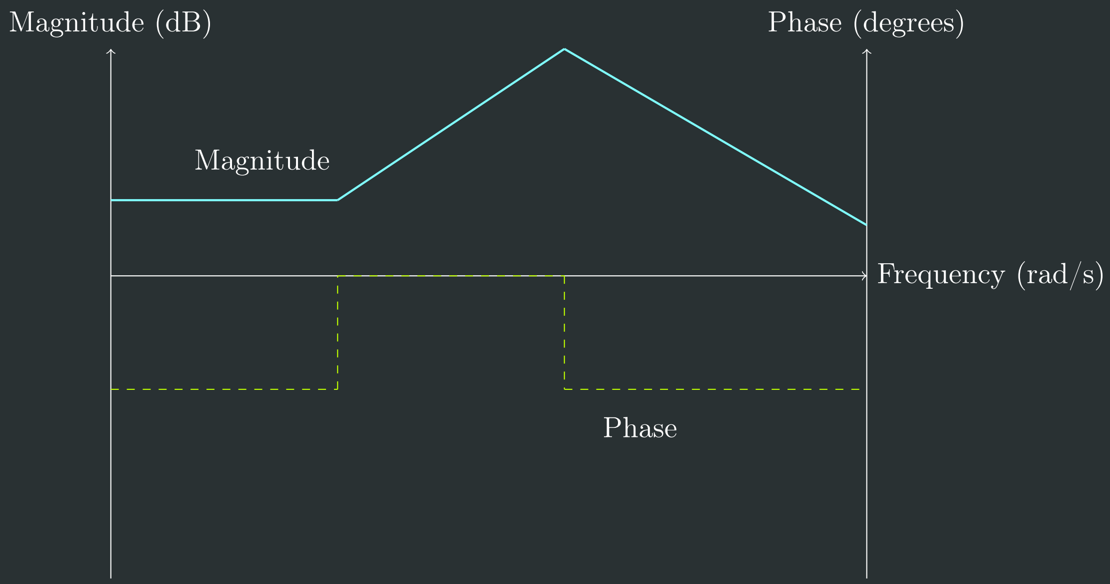
2.3 Poles/Zeros on Real Axis

Zeros on the real axis have:

$$|G(j\omega)|_{dB} = \sum_i 20\log|1 + j\omega T_i|, \ \angle G(j\omega) = \sum_i \angle(1 + j\omega T_i)$$

Poles on the real axis have:

$$|G(j\omega)|_{dB} = -\sum_i 20\log|1 + j\omega \tau_i|, \ \angle G(j\omega) = -\sum_i \angle(1 + j\omega \tau_i)$$



Bode Plot for Zero and Pole on Real Axis

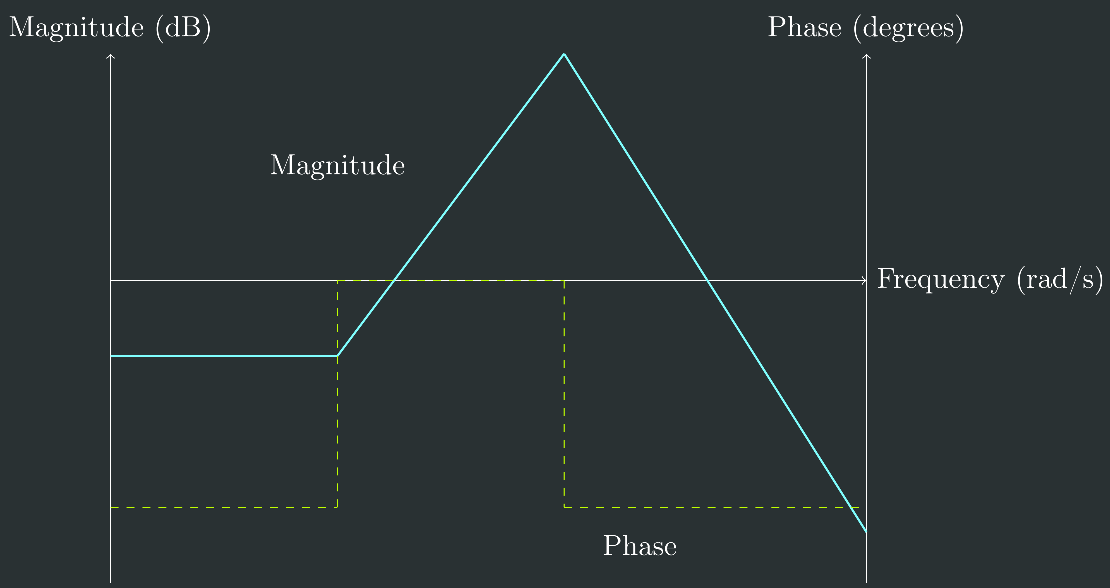
2.4 Complex Conjugate Poles/Zeros

Complex conjugate zeros have:

$$|G(j\omega)|_{dB} = \sum_i 20\log\left|1 + \frac{2j\xi_i\omega}{\alpha_{n,i}} - \frac{\omega^2}{\alpha_{n,i}^2}\right|, \ \angle G(j\omega) = \sum_i \angle\left(1 + \frac{2j\xi_i\omega}{\alpha_{n,i}} - \frac{\omega^2}{\alpha_{n,i}^2}\right)$$

Complex conjugate poles have:

$$|G(j\omega)|_{dB} = -\sum_i 20\log\left|1 + \frac{2j\xi_i\omega}{\alpha_{n,i}} - \frac{\omega^2}{\alpha_{n,i}^2}\right|, \ \angle G(j\omega) = -\sum_i \angle\left(1 + \frac{2j\xi_i\omega}{\alpha_{n,i}} - \frac{\omega^2}{\alpha_{n,i}^2}\right)$$



Bode Plot for Conjugate Complex Zeros and Poles

The **bandwidth** frequency occurs when the logarithmic gain is -3 dB, which is at $\omega = 1/tau$ for poles and zeros on the real axis.

As ζ goes from 1 to 0, overshoot increases for the magnitude plot and the steepness increases for the phase plot.

The **resonant frequency** ω_r occurs at $\omega_r = \omega_n\sqrt{1-2\zeta^2}$.