Mathematical Models Of Control Systems

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Contents

1 From Differential Equations to State Space

Time-varying control system: system in which one or more system parameters may vary as a function of time

State of a system: set of variables whose values, together with input, will provide future state and output of the system

State vector: column vector consisting of the state variables

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

Input vector: column vector consisting of the input variables

$$\mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{pmatrix}$$

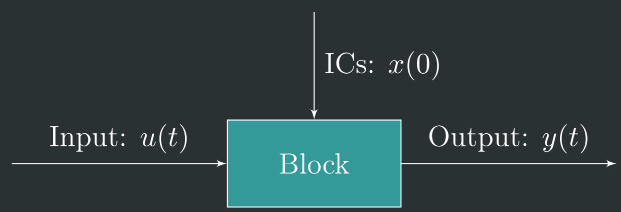
State differential equation: representation of the system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

where $\mathbf{A}$ is an $n \times n$ matrix and $\mathbf{B}$ is a $n \times m$ matrix.

Output equation: relation of outputs to state variables and input signals

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$



State space representation uses the state differential equation and output equation.

$$\begin{cases} \dot{\mathbf{x}}(t) = f(x(t), u(t)) \\ \mathbf{y}(t) = h(x(t), u(t)) \end{cases}$$

$$\mathbf{x}(t) \in \mathbb{R}^n, \mathbf{u}(t) \in \mathbb{R}^m, \mathbf{y}(t) \in \mathbb{R}^p, f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m, h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$$

2 Linearization

A pair $(\bar{x}, \bar{u}) \in \mathbb{R}^n \times \mathbb{R}^m$ is an **equilibrium point** if $f(\bar{x}, \bar{u}) = 0$.

Note that $\mathbf{x}(t) \equiv \bar{x}, \mathbf{u}(t) \equiv \bar{u}, \mathbf{y}(t) \equiv \bar{y} = h(\bar{x}, \bar{u})$ is a solution to the state space representation.

Consider some small deviation from an equilibrium point $\delta\mathbf{x}$ and $\delta\mathbf{u}$. By defining these variations of state and output as $\delta\mathbf{x}(t) = \mathbf{x}(t) - \bar{x}$ and $\delta\mathbf{y}(t) = \mathbf{y}(t) - \bar{y}$, the state space form becomes (with Taylor expansion)

$$\begin{cases} \dot{\delta\mathbf{x}}(t) = f(x, u) = \frac{\partial f}{\partial x}(\bar{x}, \bar{u})\delta x(t) + \frac{\partial f}{\partial u}(\bar{x}, \bar{u})\delta u(t) + \mathcal{O}(\|\delta x(t)\|^2) + \mathcal{O}(\|\delta u(t)\|^2) \\ \mathbf{y}(t) = h(x, u) - \bar{y} = \frac{\partial h}{\partial x}(\bar{x}, \bar{u})\delta x(t) + \frac{\partial h}{\partial u}(\bar{x}, \bar{u})\delta u(t) + \mathcal{O}(\|\delta x(t)\|^2) + \mathcal{O}(\|\delta u(t)\|^2) \end{cases}$$

This can be represented with matrices as (dropping the deltas):

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{cases}$$

which is the linearization around $(\bar{x}, \bar{u})$.

With the definition from the Taylor expansion, the matrices can be found from the Jacobian matrix. For example:

$$\mathbf{A} = \frac{\partial f}{\partial x}(\bar{x}, \bar{u}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\bar{x}, \bar{u}) & \dots & \frac{\partial f_1}{\partial x_n}(\bar{x}, \bar{u}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(\bar{x}, \bar{u}) & \dots & \frac{\partial f_n}{\partial x_n}(\bar{x}, \bar{u}) \end{pmatrix}$$

3 Laplace Transform

The **Laplace Transform** for a function $f(t)$ is

$$F(s) = \int_{0^-}^{\infty} f(t)e^{-st}\, dt = \mathcal{L}\{f(t)\}$$

Some Laplace Tranform properties:

- Laplace Transform is linear
- $\mathcal{L}\{f(t-\tau)\} = e^{-\tau s}F(s)$
- $\mathcal{L}\{e^{\alpha t}f(t)\} = F(s-\alpha)$
- $\mathcal{L}\left\{\frac{d}{dt}f(t)\right\} = sF(s) - f(0^-)$
- $\mathcal{L}\left\{tf(t)\right\} = -\frac{d}{ds}F(s)$
- $\mathcal{L}\left\{f(t) * g(t)\right\} = F(s)G(s)$

The **Final Value Theorem** states that

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s)$$

where all poles of $Y(s)$ lie in the left-hand side of the plane and $Y(s)$ must not have > 1 pole at the origin.

The **Initial Value Theorem** states that

$$\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s)$$

4 Transfer Function

For a single-input-single-output (SISO) system, the **transfer function** is

$$G(s) = \frac{\mathcal{L}\{y(t)\}}{\mathcal{L}\{u(t)\}} = \frac{Y(s)}{U(s)}$$

where Laplace transforms are taken assuming zero initial conditions of the state.

For a general LTI system, the transfer function $G(s)$ is given by

$$G(s) = C(sI - A)^{-1}B + D$$

With the transfer function, the output to an input signal can be found by:

1. Compute $U(s)$.
2. Compute $Y(s) = G(s)U(s)$.
3. Compute $y(t) = \mathcal{L}^{-1}\{Y(s)\}$.

A transfer function $G(s)$ is **real rational** if it can be expressed as

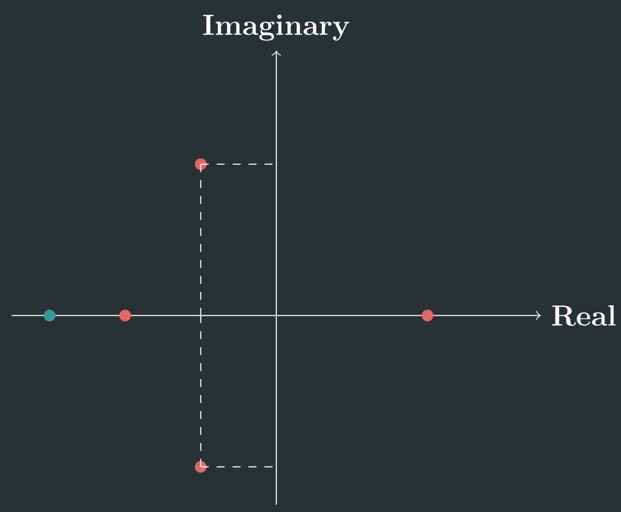
$$G(s) = \frac{b_m s^m + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

for real coefficients $a_i \in \mathbb{R}, \forall i \in [0, n-1]$, and $b_i \in \mathbb{R}, \forall i \in [0, m]$.

$G(s)$ is **proper** if $\lim_{s\to\infty} G(s)$ exists in $\mathbb{C}$ (such that $n \geq m$). $G(s)$ is **strictly proper** if $\lim_{s\to\infty} G(s) = 0$ (such that $n > m$).

$p \in \mathbb{C}$ is a **pole** of $G(s)$ if $\lim_{s\to p} |G(s)| = \infty$. $z \in \mathbb{C}$ is a **zero** of $G(s)$ if $\lim_{s\to z} |G(s)| = 0$.

For rational transfer function, poles and zeros are the roots of the denominator and numerator respectively.

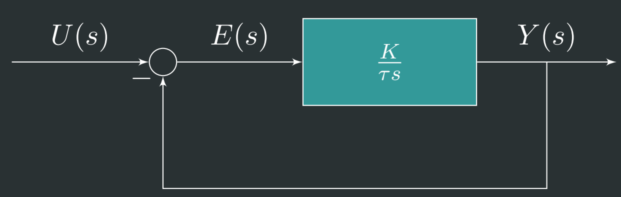


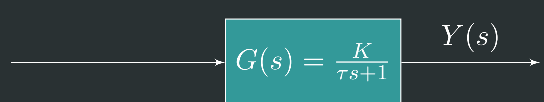
Poles in Red, Zero in Blue

First order systems have transfer functions of the form:

$$G_1(s) = \frac{K}{1 + \tau s}$$

where K is the **steady-state gain** of the system and τ is the **time constant** of the system.



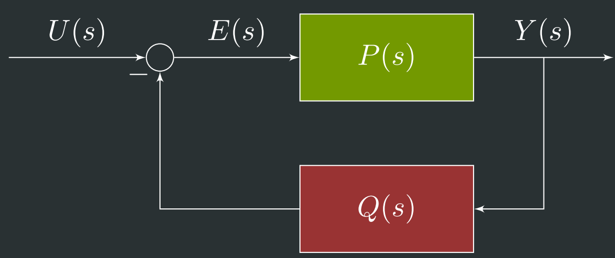


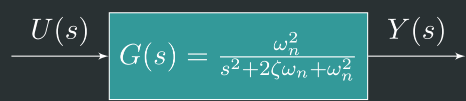
Second order systems have transfer functions of the form:

$$G_2(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2}$$

where ω_n is the **natural frequency** of the system and ζ is the **damping ratio** of the system. This has roots at $-\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$.

When $\zeta > 1$, the roots are real and the system is **overdamped**. When $\zeta < 1$, the roots are real and the system is **underdamped**. When $\zeta = 1$, the roots are real and the system is **critically damped**.





The **Laplacian** matrix is the degree matrix minus the adjacency matrix.

$$L = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

A system where $\delta\dot{x} = \delta u$ is a **single integrator** system, since the system output is simply an integral of the input signal.