Linear System Theory

Arnav Gupta

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1 Stability

For the initial value problem with $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ and $\mathbf{x}(t_0) = x_0$, the unique solution is

$$\mathbf{x}(t) = e^{A(t-t_0)}x_0$$

Note that the matrix exponential comes from the Taylor series

$$e^M = \sum_{k=0}^{\infty} \frac{M^k}{k!}$$

where $M^0 = I$.

A system in state-space form is:

- stable if, for every initial condition $\mathbf{x}(t_0) = x_0 \in \mathbb{R}^n$, $\mathbf{x}(t) = e^{A(t-t_0)}x_0$ is uniformly bounded

- asymptotically stable if, in addition, for every initial condition $\mathbf{x}(t_0) = x_0 \in \mathbb{R}^n$, $\mathbf{x}(t) \to 0$ as $t \to \infty$
- unstable otherwise

A system is asymptotically stable if and only if all eigenvalues of A have negative real part.

A **Bounded-Input-Bounded-Output (BIBO) stable** system is one with bounded response to a bounded input.

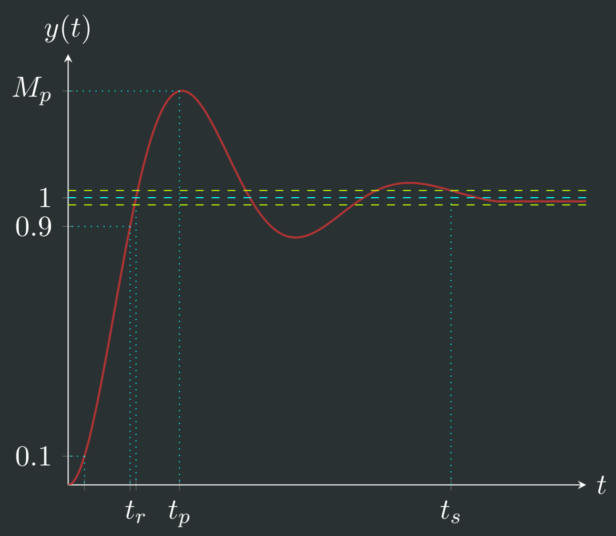
A system is BIBO stable if and only if all poles of all entries of its transfer function have negative real part.

Asymptotic stability implies BIBO stability.

2 Performance

Important performance metrics to describe the behaviour of a system with a unit step signal are:

- **steady-state gain**: $\lim_{t\to\infty} y(t)$
- **rise time**: time to go from 10% to 90% of steady-state
- **peak time**: time at which peak is achieved
- **overshoot percentage**: $OS\% = \frac{|C_{\max}|-C_{ss}}{C_{ss}} \times 100\%$
- **settling time**: smallest time such that $\forall t \geq T$, $\frac{|C_{ss}-y(t)|}{|C_{ss}|} \leq \varepsilon$



2.1 First Order Systems

Steady-state gain is μ.

Rise-time is 2τ.

Peak time and overshoot would not make sense for a first order system.

Settling time is $-\tau \ln(\varepsilon)$. For $\varepsilon = 0.05, T_s \approx 3\tau$. For $\varepsilon = 0.02, T_s \approx 4\tau$. For $\varepsilon = 0.01, T_s \approx 5\tau$.

2.2 Second Order Systems

Steady-state gain is μ.

Rise-time is

$$\frac{2.16\zeta + 0.16}{\omega_n}$$

Peak time is

$$\frac{\pi}{\sqrt{1-\zeta^2\omega_n}}$$

Overshoot percentage is

$$e^{\frac{-\zeta\pi}{1-\zeta^2}}$$

Settling time is

$$-\frac{\ln(\varepsilon)}{\zeta\omega_n}$$

3 Lower Order Approximations

Higher order systems often have a few poles that are much closer to the imaginary axis than others, these are **dominant poles**.

For a system with transfer function:

$$G(s) = G_{fast}(s)G_{slow}(s)$$

where the poles of $G_{slow}(s)$ are more dominant, the fast components can be approximated by a static transfer function whose value is equal to steady-state gain, $G_{fast}(0)$.

4 System Identification

Assume a structure of the model

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

which gives

$$y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_1\dot{y}(t) + a_0 y(t) = b_m u^{(m)}(t) + \cdots + b_1 u(t) + b_0 u(t)$$

For any time t_k, set

$$y^{(n)}(t_k) = b_m u^{(m)}(t_k) + \cdots + b_0 u(t_k) + a_{n-1}\left(-y^{(n-1)}(t_k)\right) + \cdots + a_0(-y(t_k))$$

which can be represented as:

$$d^T(t_k)\theta = \begin{pmatrix} u^{(m)}(t_k) & \cdots & u(t_k) & -y^{(n-1)}(t_k) & \cdots & -y(t_k) \end{pmatrix} \begin{pmatrix} b_m \\ \vdots \\ b_0 \\ a_{n-1} \\ \vdots \\ a_0 \end{pmatrix}$$

Consider times $t_1, \ldots, t_N$, where $N \gg n+m+1$.

$$Y = D\theta \implies \begin{pmatrix} y^{(n)}(t_1) \\ \vdots \\ y^{(n)}(t_N) \end{pmatrix} = \begin{pmatrix} d^T(t_1) \\ \vdots \\ d^T(t_N) \end{pmatrix} \theta$$

By minimizing $\|Y - D\theta\|^2$, θ can be found that best approximates the system:

$$\theta^* = (D^T D)^{-1} D^T Y = D^\dagger Y$$