Analysis Of Feedback Control Systems

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1 Block Diagrams

For series interconnections, multiple transfer functions.

For parallel interconnections, add transfer functions.

For feedback interconnections, transfer function is

$$\frac{G(s)}{1+G(s)}$$

2 Stability of Interconnected Systems

Series and parallel systems are stable if both original transfer functions are stable (assuming no cancellations).

For feedback, stability is unknown simply based off stability of original transfer function.

3 Routh-Hurwitz Criterion

For a closed-loop system, the transfer function tends to be of the form

$$F(s) = \frac{L(s)}{1 + L(s)}$$

To determine if such a system is stable, it must be known whether the roots of $1 + L(s)$ are in the left half of the complex plane $\mathbb{C}^-$ (preferably without direct computation).

For any polynomial $\pi(s) = s^n + \cdots + a_1 s + a_0$ for $a_i \in \mathbb{R}$, $\pi(s)$ is **Hurwitz** if all its roots are in $\mathbb{C}^-$.

A closed loop system is stable if $1 + L(s)$ is Hurwitz.

Note that a polynomial $\pi(s)$ can be factored into

$$\pi(s) = (s - \lambda_1) \cdots (s - \lambda_r)(s - \mu_1)(s - \bar{\mu}_1) \cdots (s - \mu_p)(s - \bar{\mu}_p)$$

where $\lambda_1, \ldots, \lambda_r$ are real roots and $\mu_1, \bar{\mu}_1, \cdots \mu_p, \bar{\mu}_p$ are complex conjugate pairs of roots.

If $\pi(s)$ is Hurwitz, $\pi(s)$ has strictly positive coefficients.

3.1 Routh's Algorithm

The first step is to build the following table:

$$\begin{array}{c|c|c|c|c}
s^n & 1 & a_{n-2} & a_{n-4} & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\
s^{n-2} & r_{2,0} & r_{2,1} & r_{2,2} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
s^2 & r_{n-2,0} & r_{n-2,1} & r_{n-2,2} & \cdots \\
s^1 & r_{n-1,0} & r_{n-1,1} & r_{n-1,2} & \cdots \\
s^0 & r_{n,0} & & &
\end{array}$$

where each r is the negative inverse of the value above it, multiplied by the determinant of the 2×2 square above it. For examples,

$$r_{2,0} = -\frac{1}{a_{n-1}} \begin{vmatrix} 1 & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}$$

This table stops along each row once 0 is reached. The process is terminated when a 0 is reached in the first column.

The next step is the use the **Routh-Hurwitz Criterion**:

1. $\pi(s)$ is Hurwitz $\iff$ all elements in the Routh array (first column of table) have the same sign
2. If the Routh array has no zeros, then
   (a) the number of sign changes is the number of bad roots (non-negative real parts)
   (b) no roots exist on the imaginary axis

4 Nyquist Criterion

The transfer function $G(s)$ is a mapping from s to $G(s)$, which is $\mathbb{C} \to \mathbb{C}$.

The **Nyquist contour** goes along the real axis and then in a circular fashion on the postive real side. The **Nyquist plot** is the image of the Nyquist contour through $L(s)$, the open loop transfer function of the closed loop system.

If $L(s)$ has poles with 0 real, part, the Nyquist plot goes around them.

The Nyquist plot has symmetry with the real line, that is

$$L(-j\omega) = \overline{L(j\omega)}$$

If $L(s)$ is strictly proper $L(j\omega) \to 0$ as $|\omega| \to \infty$.

Let p be the number of poles of $L(s)$ with positive real part. Let N be the number of loops the Nyquist plot makes around the point $-1 \in \mathbb{C}$, with > 0 if counter-clockwise and < 0 if clockwise. N is undefined if the Nyquist plot goes through -1.

By the **Nyquist Criterion**, the closed loop system is stable if and only if $N = P$.

If N is undefined, the closed loop system may be stable or unstable.

If N is well defined and $N \neq P$, the closed loop system is unstable.

$P - N$ is the number of poles of the closed loop system with positive real part.

$L(s)$ is stable for $P = 0$. Further, if:

- $|L(j\omega) < 1 \; \forall \omega \implies$ the closed loop system is stable
- $|\angle L(j\omega) < 1 \; \forall \omega \implies$ the closed loop system is stable

5 Bode Criterion

Let $L(s)$ be the open loop transfer function of a closed-loop system. Assume $L(s)$ is stable and the Nyquist plot of $L(s)$ intersects the negative real axis only once. The distance from -1 to $L(j\omega\pi)$ gives the gain margin.

On a Bode plot, the **gain margin** K_m occurs when the phase hits $-180°$ and is the distance from the frequency axis to $|L(j\omega\pi)|$. The gain margin is positive if $|L(j\omega\pi)|_{dB} < 0_{dB}$ (which indicates a stable system) and is otherwise negative. The gain margin represents the maximum multiplicative factor on the gain of $L(s)$ at ω_π that the system can tolerate before becoming unstable.

$$K_m = \frac{1}{|L(j\omega_\pi)|}$$

where ω_π is the frequency such that $\angle L(j\omega_\pi) = -180°$.

Let $L(s)$ be the open loop transfer function of a closed-loop system. Assume $L(s)$ is stable and the Nyquist plot of $L(s)$ intersects the unit circle once once from outside to inside. The frequency at which $|L(j\omega)| = 1, |L(j\omega)|_{dB} = 0_{dB}$ is the **crossover frequency**.

The **phase margin** ϕ_m is the distance between $\angle L(j\omega)$ and $-180°$. Specifically, this is the frequency at which the magnitude $|L(j\omega)|$ goes to 0. The phase margin is positive if $180° - |\angle L(j\omega_c)| > 0$ and negative otherwise.

The **Bode Criterion** states that if $L(s)$ has no poles with positive real parts and $|L(j\omega)|_{dB}$ crosses the 0_{dB} axis only once from above to below, then

$$\mu > 0, \phi_m > 0 \iff F(s) = \frac{L(s)}{1 + L(s)} \text{ stable}$$

The closed loop system $F(s)$ is stable for $K_m > 0, \phi_m > 0$.