Review

Definition

An *eigenvector* of an $n \times n$ matrix A is a nonzero vector $\mathbf{x} \in \mathbb{R}^n$ (or in $\mathbb{C}^n$) such that $\boxed{A\mathbf{x} = \lambda\mathbf{x}}$ for some $\lambda \in \mathbb{R}$ (or $\lambda \in \mathbb{C}$). A number λ is called an *eigenvalue* of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an $\mathbf{x}$ is called an *eigenvector corresponding to* λ.

Theorem

λ is an eigenvalue of A if and only if $\det(A - \lambda I) = 0$.

Definition

$p(\lambda) = \det(A - \lambda I)$ is called the *characteristic polynomial* of A. The equation

$$\det(A - \lambda I) = 0$$

is called the *characteristic equation* of A.

Complex Eigenvalues of 2 × 2 matrices.

Definition

Let A be an $n \times n$ matrix. A number $\lambda \in \mathbb{C}$ is called a *complex eigenvalue* of A if there exists a nonzero vector $\mathbf{x} \in \mathbb{C}^n$ such that $A\mathbf{x} = \lambda\mathbf{x}$.

Abuse of Notation

Every real number is a complex number. However, we often use the word "complex eigenvalue" to refer to an eigenvalue which is in $\mathbb{C} \setminus \mathbb{R}$.

I will however, try to explicitly mention that an eigenvalue is of the form $a + bi$ where $b \neq 0$, when dealing with such eigensvalues.

$$\begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix} = (\cos\theta - \lambda)^2 + \sin^2\theta$$

$$= \cos^2 + \lambda^2 - 2\cos\theta\lambda + \sin^2\theta$$

$$= \lambda^2 - 2\cos\theta\lambda + 1$$

The eigenvalues of a (non-trivial) 2×2 rotation matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

are complex.

Any 2×2 matrix which has complex eigenvalues is similar to the product of a scaling and a rotation.

$$p(\lambda) = \lambda^2 - 2\cos\theta\lambda + 1$$

roots: $\dfrac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} = \cos\theta \pm i\sin\theta = e^{\pm i\theta}$

$$\boxed{re^{i\theta} = r\cos\theta + i(r\sin\theta)}$$

$re^{\mp i\theta}$

$\mathbb{C}$ A : 2×2 matrix

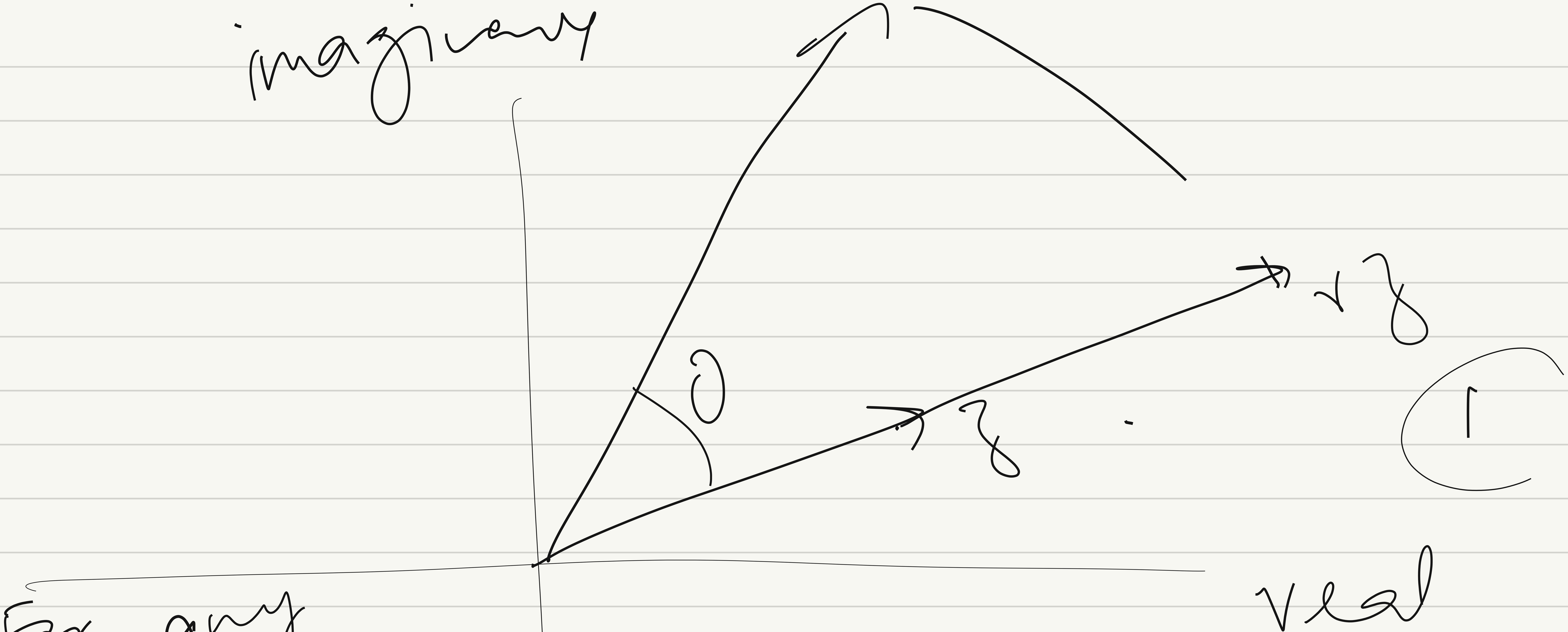
T $\boxed{x \longmapsto Ax}$

Fix $\lambda \in \mathbb{C}$

$\mathbb{C} \xrightarrow{f} \mathbb{C}$

$\underline{z} \longmapsto \lambda z$

$f(z) = \lambda z$



For any

$z \in \Gamma$

$\boxed{e^{i\theta} z}$

$$f(z) = \lambda z$$

$$\lambda = re^{i\theta}$$

$$|\lambda| = r$$

$$\arg \lambda = \theta.$$

$\mathbb{R}^2 \to \mathbb{R}^2 \qquad x \mapsto Ax$

Theorem

If A is a 2×2 matrix and $\lambda = re^{-i\theta}$ is a complex eigenvalue of A, and $\mathbf{v} \in \mathbb{C}^2$ is a complex eigenvector of A corresponding to λ, then

$$P^{-1}AP = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix},$$

where $P = [Re(\mathbf{v}) \quad Im(\mathbf{v})]$.

Any complex number represents a scaling and a rotation. So we extract that information from the eigenvalues. Because of some strange coincidence which I don't know how to explain, the same scaling ~~and~~ but OPPOSITE (negative of arg λ) rotation happen if we look at the coordinate system determined by the real and imaginary parts of the complex eigenvectors.

$$\begin{bmatrix} a + bi \\ c + di \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} + i \begin{bmatrix} b \\ d \end{bmatrix}$$

$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix}$$

Proof

Let v be eigenvector of A corresponding to $\lambda = re^{-i\theta}$.

$$\therefore \quad Av = \lambda v.$$

$$\lambda v = AP \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$= A \begin{bmatrix} Re(v) & Im(v) \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$— \quad 1$$

$$v = Re(v) + i\,Im(v)$$

$$v = \begin{bmatrix} Re(v) & Im(v) \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$v = P \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$\lambda v = \begin{bmatrix} Re\,v & Im(v) \end{bmatrix} \begin{bmatrix} \lambda \\ \lambda i \end{bmatrix}$$

$$\overline{A v} = \overline{\lambda v}$$

$$\Rightarrow \bar{A}\,\bar{v} = \bar{\lambda}\,\bar{v}$$

$$\Rightarrow AP\begin{bmatrix} 1 \\ -i \end{bmatrix} = A\begin{bmatrix} Re(v) & Im(v) \end{bmatrix}\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$= \bar{\lambda}\bar{v} \quad \text{--- } 2$$

$$\bar{v} = Re(v) - i\,Im(v)$$

$$= \begin{bmatrix} Re(v) & Im(v) \end{bmatrix}\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$\bar{\lambda}\bar{v} = \begin{bmatrix} Re(v) & Im(v) \end{bmatrix}\begin{bmatrix} \lambda \\ \lambda i \end{bmatrix}$$

From (1) & (2)

$$AP\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} = A\begin{bmatrix} \mathrm{Re}(v) & \mathrm{Im}(v) \end{bmatrix}\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$$

$$= \begin{bmatrix} \mathrm{Re}(v) & \mathrm{Im}(v) \end{bmatrix}\begin{bmatrix} \lambda & \bar{\lambda} \\ \lambda i & -\bar{\lambda} i \end{bmatrix}$$

$$= P\begin{bmatrix} \lambda & \bar{\lambda} \\ \lambda i & -\bar{\lambda} i \end{bmatrix}$$

$$AP = P\begin{bmatrix} \lambda & \bar{\lambda} \\ \lambda i & -\bar{\lambda} i \end{bmatrix}\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}^{-1}$$

$$= \left(-\frac{1}{2i}\right) P \begin{bmatrix} \lambda & \bar{\lambda} \\ \lambda i & -\bar{\lambda} i \end{bmatrix}\begin{bmatrix} -i & -1 \\ -i & 1 \end{bmatrix}$$

$$= \left(-\frac{1}{2i}\right) P \begin{bmatrix} -i(\lambda+\bar{\lambda}) & \bar{\lambda}-\lambda \\ \lambda-\bar{\lambda} & -\lambda i-\bar{\lambda} i \end{bmatrix}$$

$$= P \begin{bmatrix} \mathrm{Re}(\lambda) & \mathrm{Im}(\lambda) \\ -\mathrm{Im}(\lambda) & \mathrm{Re}(\lambda) \end{bmatrix}$$

$$= P \begin{bmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{bmatrix}$$

$$\underline{P^{-1}AP} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$$

In general: $\lambda = a+bi$

$$\underline{\lambda - \bar{\lambda}}$$

$$= a+bi - (a-bi)$$

$$= 2bi$$

$$\lambda = re^{-i\theta}$$

$$= r\cos\theta - ir\sin\theta$$

Inner Product in $\mathbb{R}^n$

$|\hat{u} \cdot \hat{v}| = |\cos\theta|$

Definition

If $\mathbf{u} = (u_1, \ldots, u_n), \mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n$ then the *inner product* or the *dot product* of $\mathbf{u}$ and $\mathbf{v}$ is the real number

$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{n} u_i v_i$$

$\sum_{i=1}^{n} u_i^2 = u \cdot u$

bilinear

$\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

Theorem

Let $\mathbf{u}, \mathbf{v}$, *and* $\mathbf{w}$ *be vectors in* $\mathbb{R}^n$, *and let* c *be a scalar. Then*

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ — symmetry
2. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
3. $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$

   linearity
4. $\mathbf{u} \cdot \mathbf{u} \geq 0$, *and* $\mathbf{u} \cdot \mathbf{u} = 0$ *if and only if* $\mathbf{u} = \mathbf{0}$ ← positive definite

verification is HW

Inner Product in other Vector Spaces

Definition

Let V be a real vector space. An *inner product* $\langle .,. \rangle : V \times V \to \mathbb{R}$ is a function satisfying the following conditions:

1. $\langle \mathbf{v} + \mathbf{w}, \mathbf{z} \rangle = \langle \mathbf{v}, \mathbf{z} \rangle + \langle \mathbf{w}, \mathbf{z} \rangle$, for every $\mathbf{v}, \mathbf{w}, \mathbf{z} \in V$
2. $\langle c\mathbf{v}, \mathbf{w} \rangle = c\langle \mathbf{v}, \mathbf{w} \rangle$ for every $c \in \mathbb{R}$ and $\mathbf{v}, \mathbf{w} \in V$
3. $\langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{w}, \mathbf{v} \rangle$, for every $\mathbf{v}, \mathbf{w} \in V$
4. $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ for every $\mathbf{v} \in V$, and $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ holds if and only if $\mathbf{v} = \mathbf{0}$

Example

Let $V = C[a,b]$ the space of continuous real-valued functions defined on interval $[a,b]$. Define

$$\langle f, g \rangle := \int_a^b f(t)g(t)\,dt\,.$$

(1) $f, g, h \in C[a,b]$

$$\langle f+g, h \rangle = \int_a^b (f(t)+g(t))\,h(t)\,dt$$

$$= \int_a^b \left(f(t)h(t) + g(t)h(t) \right) dt$$

$$= \underbrace{\int_a^b f(t)h(t)\,dt}_{} + \underbrace{\int_a^b g(t)h(t)\,dt}_{}$$

$$= \langle f, h \rangle + \langle g, h \rangle$$

2. $c \in \mathbb{R}$, $f, g \in V$.

$$\langle cf, g \rangle = \int_a^b c\, f(t) g(t)\, dt$$

$$= c \int_a^b f(t) g(t)\, dt$$

$$= c \langle f, g \rangle .$$

3 $f, g \in V$.

$$\langle f, g \rangle = \int_a^b f(t)g(t)\,dt$$

$$= \int_a^b g(t)f(t)\,dt$$

$$= \langle g, f \rangle$$

④ If $f = 0$

then $\langle f, f \rangle = \int_a^b 0 \, dt$

$= 0$.

Conversely, if $\langle f, f \rangle = 0$

$\Rightarrow \int_a^b (f(t))^2 \, dt = 0$

$\Rightarrow f = 0$

<u>Fact from calculus:</u>

If the definite integral on $[a,b]$ of a non-negative <u>continuous</u> function is zero then the function must be zero.