Matrix of a Linear Transformation

Proposition (M)

Let V, W be vector spaces. Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be an ordered basis for V and $\mathcal{C} = \{w_1, \ldots, w_m\}$ be an ordered basis for W. Let $T : V \to W$ be a linear transformation. There exists a unique $m \times n$ matrix A such that

$$[T(v)]_{\mathcal{C}} = A[v]_{\mathcal{B}}, \quad \text{for every } v \in V.$$

Further, we have

$$A = [[T(v_1)]_{\mathcal{C}} \quad \ldots \quad [T(v_n)]_{\mathcal{C}}]$$

This unique matrix A is called is called the *matrix of T* with respect to the bases $\mathcal{B}$ and $\mathcal{C}$, and is denoted by $[T]_{\mathcal{B},\mathcal{C}}$.

Example

2×3 , 3×4

$\mathbb{R}^6 \to \mathbb{R}^8$

$V = M_{2\times 3}$, the vector space of all 2×3 matrices having real entries, and

$W = M_{2\times 4}$, the vector space of all 2×4 matrices having real entries.

Let

3×4

$$A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 2 & 1 & 2 & 3 \\ 8 & -1 & 1 & 2 \end{bmatrix}$$

$M_{2\times 3} \to M_{2\times 4}$

and let $T : V \to W$ be defined by

$$\boxed{T(B) = BA, \forall B \in M_{2\times 3}}$$

$\dim M_{2\times 3} = 6$

$\dim M_{2\times 4} = 8$

Let $B_1, B_2 \in M_{2\times 3}$.

$$T(B_1 + B_2) = (B_1 + B_2)A$$
$$= B_1 A + B_2 A$$
$$= T(B_1) + T(B_2).$$

Let $c \in \mathbb{R}$, $B \in M_{2\times 3}$

$$T(cB) = cBA$$

$$= cT(B)$$

Yes T is a l.t. ✓

Let's fin a basis for $M_{2\times 3}$.

$$\beta = \left\{ \overset{E_1}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}, \overset{E_2}{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}, \overset{E_3}{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}, \underset{E_4}{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}, \underset{E_5}{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}, \underset{E_6}{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \right\}$$

$$A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 2 & 1 & 2 & 3 \\ 8 & -1 & 1 & 2 \end{bmatrix}$$

$$T(E_1) = E_1 A$$

Let's fin a basis for $M_{2\times 4}$.

$$\to \mathcal{F} = \Bigg\{ \underbrace{\overset{F_1}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}, \overset{F_2}{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}, \overset{F_3}{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}, \ldots \ldots, \underset{F_8}{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}}_{8 \text{ such matrices.}} \Bigg\}$$

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{\mathcal{F}}$$

$$[T]_{\beta, \mathcal{C}_\varphi} = \begin{bmatrix} 1 & 2 & 5 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 3 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 3 & 2 \end{bmatrix}$$

8 X 6 matrix

$$T(E_1) = E_1 A \qquad [T(E_1)]_{\mathcal{C}_\varphi} = ?$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 2 & 1 & 2 & 3 \\ 8 & -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 \end{bmatrix}$$

Change of coordinates

Theorem

Let $\mathcal{B} = \{b_1, \ldots, b_n\}$ and $\mathcal{C} = \{c_1, \ldots, c_n\}$ be ordered bases of a vector space V. Then there is a unique $n \times n$ matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ such that

$$[x]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[x]_{\mathcal{B}}, \quad \forall x \in V.$$

The columns of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ are the $\mathcal{C}$-coordinate vectors of the vectors in the basis $\mathcal{B}$. That is,

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = [[b_1]_{\mathcal{C}} \quad [b_2]_{\mathcal{C}} \quad \cdots [b_n]_{\mathcal{C}}]$$

Proof: We apply Proposition (M) to the identity map.

Pf Let $T : V \to V$ be the identity map, i.e. $T(v) = v$, $\forall v \in V$.

$$[T]_{\beta, \mathcal{E}} = \begin{bmatrix} [T(b_1)]_{\mathcal{E}} & [T(b_2)]_{\mathcal{E}} & \cdots & [T(b_n)]_{\mathcal{E}} \end{bmatrix}$$

$$= \begin{bmatrix} [b_1]_{\mathcal{E}} & [b_2]_{\mathcal{E}} & \cdots & [b_n]_{\mathcal{E}} \end{bmatrix}.$$

nothing but $P_{\beta \leftarrow \mathcal{E}}$.

Definition

The matrix $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ in the above theorem is called the *change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$*.

Let's look at examples over matrices, function spaces.

Example :

Let V = the set of solutions of the 2nd order ODE

$$y'' - y = 0. \qquad \boxed{y'' = y}$$

general solution: $c_1 e^{x} + c_2 e^{-x}$

$$\beta = \{e^x, e^{-x}\}.$$

$$\varphi = \{\sinh x, \cosh x\}.$$

$y'' = y.$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\sinh x = \frac{e^x - e^{-x}}{2}.$$

$$\frac{d}{dx}(\cosh x) = \frac{d}{dx}\left(\frac{e^{x}+e^{-x}}{2}\right)$$

$$= \frac{1}{2}\left(e^{x} - e^{-x}\right)$$

$$= \sinh x$$

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^{x}-e^{-x}}{2}\right) = \frac{1}{2}\left(e^{x}+e^{-x}\right)$$

$$= \cosh x$$

$$\underset{\mathcal{C} \leftarrow \beta}{P} = \begin{bmatrix} [e^{x}]_{\mathcal{C}} & [e^{-x}]_{\mathcal{C}} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\mathcal{C} = \{ \underline{\sinh x}, \underline{\cosh x} \}$$

$$\sinh = \frac{e^{x} - e^{-x}}{2} \qquad \cosh x = \frac{e^{x} + e^{-x}}{2}$$

$$e^{x} = \frac{e^{x} - e^{-x}}{2} + \frac{e^{x} + e^{-x}}{2} \Rightarrow [e^{x}]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$e^{-x} = \cosh x - \sinh x$$

$$= \frac{e^{x}+e^{-x}}{2} - \frac{e^{x}-e^{-x}}{2}.$$

$$= (-1)\sinh x + (1)\cosh x$$

$$[e^{-x}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Special case : $V = \mathbb{R}^n$

$$\underset{\varphi \leftarrow \beta}{P} = \underline{\underset{\varphi}{P^{-1}}}\ \underline{\underline{\underset{\beta}{P}}}$$

$$\underbrace{S_2 \circ S_1^{-1}}$$

$S_1 \longrightarrow$ coordinate with respect to basis β.

$S_2 \longrightarrow$ coordinate wrt φ,

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "S_1^{-1}"] \arrow[rr, bend right] & V \arrow[r, "S_2"] & \mathbb{R}^n \end{tikzcd}$$

Proposition

Let V be a finite dimensional vector space. Let $\mathcal{B}, \mathcal{C}$ be bases for V. Let $T : V \to V$ be a linear transformation. Then the matrix of T with respect to $\mathcal{B}$ and $\mathcal{C}$ are similar to each other. If $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ is the change-of-basis matrix from $\mathcal{B}$ to $\mathcal{C}$, then

$$[T]_{\mathcal{B}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}^{-1} [T]_{\mathcal{C}} \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}.$$

We will look at examples when we look at the eigenvalue/eigenvector concept.

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}^{-1} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$$

Proof

Let $v \in V$. Then

$$[T(v)]_{\mathcal{B}} = [T]_{\mathcal{B}}[v]_{\mathcal{B}}$$

and

$$[T(v)]_{\mathcal{C}} = [T]_{\mathcal{C}}[v]_{\mathcal{C}}$$

Now,

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[T(v)]_{\mathcal{B}} = [T]_{\mathcal{C}}\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[v]_{\mathcal{B}}$$

Hence

$$[T]_\beta [v]_\beta = \quad [T(v)]_{\mathcal{B}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}^{-1}[T]_{\mathcal{C}}\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[v]_{\mathcal{B}}$$

As the choice of v was arbitrary,

$$[T]_{\mathcal{B}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}^{-1}\,[T]_{\mathcal{C}}\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}.$$

$\boxed{\text{col } A}$ $\qquad A \qquad m \times n$

$\text{nul } A \qquad T : \mathbb{R}^n \to \mathbb{R}^m$

$x \mapsto Ax$

Let me emphasize this point once more - any n-dimensional real vector space V can be looked at as $\mathbb{R}^n$, using a coordinate map, once we fix a basis.

Once we fix bases for an n-dimensional vector space V and an m-dimensional vector space W, any linear transformation $T : V \to W$ can be thought of in terms of an $m \times n$ matrix which is uniquely determined by the choice of the two fixed bases.

So we can generalize what we've learned about matrices and fundamental subspaces to the context of abstract vector spaces.

$T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$T(x) = Ax$,

What is the column space of a matrix A, if we look at the matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$?

What is col A?

range of T.

$T(x) = (x, 0)$ or $T(x) = (x, x)$

or $T(x) = (x, (x)$

$T : \mathbb{R} \to \mathbb{R}^2$

$\mathbb{R}$

Definition

Let V and W be vector spaces and let $T : V \to W$ be a linear transformation. The *range* of T is defined as the set

$$\text{range}(T) = \{w \in W \mid w = T(v), \text{ for some } v \in V\}$$

Proposition

range(T) is a subspace of W.

The range of a function can't be empty
so Range$(T) \neq \phi$:

Pf: Let $w_1, w_2 \in$ Range(T).

$\Rightarrow \exists\, v_1 \in V \wedge v_2 \in V$ such

that

$$T(v_1) = w_1$$

$$T(v_2) = w_2.$$

$$\Rightarrow \quad \underline{w_1 + w_2} = T(v_1) + T(v_2)$$

$$= T(v_1 + v_2).$$

$$\Rightarrow w_1 + w_2 \in \text{Range}(T).$$

Since w_1, w_2 were arbitrary, Range(T) is closed under vector addition.

Let $c \in \mathbb{R}$, $w \in \text{Range}(T)$.

Then $\exists\, v \in V$ such that

$w = T(v)$.

$\therefore\ cw = cT(v) = T(cv)$

$\therefore\ cw \in \text{Range}(T)$.

$\therefore$ Range(T) is a subspace of W.

Recall $\text{Nul } A = \{x \in \mathbb{R}^r \mid Ax = 0\}$

Let p be a solution of $Ax = b$.

If w is a solution of $Ax = b$ then w is of the form $p + v_h$.

Recall that solutions of the system $A\mathbf{x} = \mathbf{b}$ are translations of the nullspace of A (the solution space of $A\mathbf{x} = \mathbf{0}$).

In the case of a linear transformation we use a similar approach to find all the pre-images of a vector in the range.

$w \in \text{Range}(T)$

Definition

Let V and W be vector spaces and let $T : V \to W$ be a linear transformation. We define the *kernel* of T to be the set

$$\ker T = \{v \in V \mid T(v) = 0\}.$$

Null space of T.

$x + y + z = 0.$

$x + y + z = 2$

Proposition

ker T is a subspace of V.

Pf: (i) Since $T(0)=0$, $\Rightarrow$ $0 \in \ker T$. $\therefore \ker T \neq \emptyset$.

(ii) Let $v_1, v_2 \in \ker T$.

$$\Rightarrow \quad T(v_1) = 0$$
$$T(v_2) = 0$$

$$\Rightarrow \quad T(v_1 + v_2) = T(v_1) + T(v_2)$$
$$= 0 \, .$$

(iii) Let $v \in \ker T$, $c \in \mathbb{R}$.
$T(cv) = cT(v) = 0 \Rightarrow cv \in \ker T$.

Example

Let V be the vector space of all differentiable real-valued functions defined on $\mathbb{R}$.
Let W be the vector space of all real-valued functions defined on $\mathbb{R}$.

Let $T : V \to W$ be the differentiation map, sending $f(x) \mapsto f'(x)$.

What is the kernel of T? If $f'(x) = g'(x)$ what can you say about f and g?

Ker T = all constant function.