Invertible Matrices

Definition

An $n \times n$ matrix A is defined to be *invertible* if there exists an $n \times n$ matrix B such that $AB = BA = I$.

The inverse of an $n \times n$ matrix A is unique, if it exists. It is denoted by A^{-1}.

An invertible matrix is also called a *nonsingular* matrix. A matrix which is not invertible is called a *singular* matrix.

Lemma

A is invertible iff the RREF of A is invertible.

Lemma

A square matrix A in reduced echelon form is invertible iff it is the identity matrix, i.e. $A = I$.

Theorem

An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}.

Statement: An $n \times n$ matrix A in reduced echelon form is invertible $\Leftrightarrow$ $A = I$.

$(\Leftarrow)$: Assume $A = I$.
Clearly A is in reduced echelon form and is also invertible.

$\boxed{(\Rightarrow)}$: Assume A is in reduced echelon form and that A is invertible, i.e. A^{-1} exists.

Then the system of equations

$\underline{Ax = 0}$ has a $\underline{\text{unique solution}}$,

because

$A^{-1}Ax = \boxed{x = 0}$.

$\therefore$ $Ax = 0$ has no free variables.

$\therefore$ By Uniqueness Existence Thm,

$\rightarrow$ A has no non-pivot columns.

$\therefore$ All columns of A are pivot columns.

$\rightarrow$ $\therefore$ $A = I$.

$$\begin{bmatrix} A & 0 \end{bmatrix}$$

coeff. matrix

<u>Statement</u>: Any sequence of elementary row ops that reduce A to I also reduce I to A^{-1}.

Let $E_1, \ldots E_m$ be the elementary matrices involved in reducing A to I,

i.e. $$\underbrace{E_m E_{m-1} \cdots E_1}A = I$$

$$\Rightarrow E_m E_{m-1} \cdots E_1 = A^{-1} \text{ (mult on right by } A^{-1}\text{)}$$

$$\Rightarrow E_m E_{m-1} \cdots E_1 I = A^{-1}$$

Algorithm for Finding A^{-1}

Row reduce the augmented matrix

$$[A \quad I].$$

If A is row equivalent to I , then $[A \quad I]$ is row equivalent to

$$[I \quad A^{-1}].$$

Otherwise, A does not have an inverse.

Example

Find the inverse of

$$\begin{bmatrix} 1 & -2 & -1 \\ -1 & 5 & 6 \\ 5 & -4 & 5 \end{bmatrix}$$

if it exists.

$$\left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ -1 & 5 & 6 & 0 & 1 & 0 \\ 5 & -4 & 5 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \to R_2 + R_1$

$$\to \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 5 & -4 & \underline{5} & 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - 5R_1$

$$\to \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 0 & \underline{6} & \underline{10} & -5 & \underline{0} & 1 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 0 & 0 & 0 & -7 & -2 & 1 \end{array}\right]$$

Invertible Matrix Theorem (some parts)

Theorem

Let A be an $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

- **a.** *A is an invertible matrix.*
- **b.** *A is row equivalent to the $n \times n$ identity matrix.*
- **c.** *A has n pivot positions.*
- **d.** *The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*
- **e.** *The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.*
- **f.** *There is an $n \times n$ matrix C such that $CA = I$.*
- **g.** *There is an $n \times n$ matrix D such that $AD = I$.*
- **h.** *A^T is an invertible matrix.*

Proof of Theorem (parts listed)

$A \Rightarrow B$

$\neg B \Rightarrow \neg A$

We already know that (a) is equivalent to (b), and that (a) is equivalent to (h).

$\boxed{0}$ $(a) \Leftrightarrow (b)$

We will show that

1. (b) $\iff$ (c)
2. (a) $\implies$ (d)
3. (d) $\implies$ (c)
4. (a) $\implies$ (e) : $x = A^{-1}b$
5. (e) $\implies$ (c) (equivalently, not (c) $\implies$ not (e))
6. (e) holds if and only if (g) holds.
7. (h) holds if and only if (f) holds.

(b) $\implies$ (c) is obvious.

(c) $\implies$ (b): If there are n pivot positions, they must be on the diagonal.

(a) $\implies$ (d): Multiply both sides by A^{-1}.

(d) $\implies$ (c): By the Existence and Uniqueness theorem, a consistent system has a unique solution if and only if there are no free variables.

$$\begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow \text{ith entry}$$

$e_1 \dots e_n \rightarrow$ columns of the identity matrix I_n

(a) $\implies$ (e): Multiply both sides by A^{-1}.

not (c) $\implies$ not (e): If A does not have n pivot positions, then the RREF of A must have at least one row which does not contain a pivot. This can only happen if it is a row of zeros.

Let A' be the RREF of A. Suppose the i-th row of A' consists of zeros. Then the equation

$$A'\mathbf{x} = \mathbf{e}_i$$

$A' = \underbrace{E_m \cdots E_1}_{} A = EA$

has no solution.

has no soln.

$EAx = e_1$

$Ax = E^{-1}e_1$ has no soln.

Let $E^{-1} e_1 = b$.

$$\Rightarrow \underbrace{Ax = b \text{ has no solution.}}$$

Suppose if possible that

$A' x = e_i$ has a solution,

say y, i.e. $A' y = e_i$.

$y = (y_1, \dots, y_n)$

Let's call the columns of A'

$a_1', a_2', \dots, a_r'$.

$$A'y = y_1 a_1' + y_2 a_2' + \dots + y_n a_n'$$

The ith entry of $A'y$ is

$$y_1 (a_1')_i + \dots + y_n (a_n')_i = 0.$$

The ith entry of e_i is 1.

$\therefore$ $A'x = e_i$ has no solution.

Suppose y is a solution of $Ax = 0$ $\Rightarrow$ $Ay = 0$

$\Rightarrow$ $y = A^{-1} \cdot 0 = 0$.