Definition

Let V and W be vector spaces and let $T : V \to W$ be a linear transformation. The *range* of T is defined as the set

$$\text{range}(T) = \{w \in W \mid w = T(v), \text{ for some } v \in V\}$$

Proposition

range(T) is a subspace of W.

$Ax = b$

$Ax = 0$

V

$Tv = w$

Definition

Let V and W be vector spaces and let $T : V \to W$ be a linear transformation. We define the *kernel* of T to be the set

$$\ker T = \{v \in V \mid T(v) = 0\}.$$

Proposition

$\ker T$ is a subspace of V.

$Tv = 0$

Proposition

Let V and W be vector spaces. Let $T : V \to W$ be a linear transformation.

T is 1-1 (injective) if and only if $\ker T = \{0\}$.

Pf: We first assume that T is 1-1 and prove that $\ker T = \{0\}$.

$\rightarrow \ker T = \{ v \in V \mid Tv = 0 \}$

We first show that $\ker T \subset \{0\}$.

Let $v \in \ker T$ be any vector.

By definition, $Tv = 0$ — ①

We also know that $T(0) = 0$ — ②

Since T is 1-1, ① & ② $\Rightarrow v = 0$.

$\Rightarrow \ker T \subset \{0\}$. — 3

We already know that $\{0\} \subset \ker T$, because $T(0) = 0$. — 4

From 3 & 4, $\ker T = \{0\}$.

Conversely, let us assume that $\ker T = \{0\}$, and prove that T is 1-1.

Let $v, w \in V$, such that $Tv = Tw$.

$\Rightarrow \underline{Tv - Tw} = 0$

we will show that $v = w$.

$$\Rightarrow \quad T(v-w) = 0$$

$$\Rightarrow \quad v - w \in \ker T = \{0\}.$$

$$\Rightarrow \quad v - w = 0$$

$$\Rightarrow \quad v = w.$$

$\therefore$ T is 1-1

$\square$.

$$T(v-w)$$
$$= T(v) + T(-w)$$
$$= T(v) + T(-1 \cdot w)$$
$$= T(v) + (-1)T(w)$$
$$= T(v) - T(w)$$
$$= 0.$$

Not expected
on an exam.

Example

V = set of all continuously differentiable functions on $\mathbb{R}$

$= \mathcal{C}^1(\mathbb{R})$

$W = \mathcal{C}(\mathbb{R})$.

$T : V \to W$ is the map $Tf = f'$

$\text{Ker}\, T =$ set of all constant functions.

$f' = 0 \iff f$ is a constant

$f' = g' \iff f - g$ is a constant

$c_1, \ldots, c_n \in \mathbb{R}, \; v_1, \ldots, v_n \in V$

$$T(c_1 v_1 + \cdots + c_n v_n)$$

$$= c_1 T(v_1) + \cdots + c_n T(v_n)$$

$\downarrow$

has been proved.

$\rightarrow c_1 v_1 + \cdots + c_n v_n = 0$

$0 = T(0) = T(c_1 v_1 + \cdots + c_n v_n)$
$= c_1 T(v_1) + \cdots + c_n T(v_n)$

A 1-1 transformation preserves linear independence.

Proposition

Let V and W be vector spaces. Let $T : V \to W$ be a 1-1 linear transformation. A

Then if $\{v_1, \ldots, v_n\}$ is a linearly independent subset of V then $\{T(v_1), \ldots, T(v_n)\}$ is a linearly independent subset of W. B

$A \Rightarrow B$

Linear dependence relations are always preserved under linear maps.

Proof: We need to show that

$$c_1 T(v_1) + \cdots + c_n T(v_n) = 0$$

has only the trivial solution

$$c_1 = \cdots = c_n = 0.$$

Suppose if possible that

$$c_1 T(v_1) + \cdots + c_n T(v_n) = 0 \quad \text{--- (1)}$$

for some $c_1, \ldots, c_n \in \mathbb{R}$.

Since T is linear,

$$c_1 T(v_1) + \cdots + c_n T(v_n)$$

$$= T(c_1 v_1 + \cdots + c_n v_n) \quad \text{---} (2)$$

From (1) & (2), we obtain $\underline{T(c_1 v_1 + \cdots + c_n v_n) = 0}$.

$\Rightarrow c_1v_1 + \cdots + c_nv_n \in \ker T.$

Since T is 1-1, $\ker T = \{0\}$.

$\therefore c_1v_1 + \cdots + c_nv_n = 0.$

Since $v_1, \ldots, v_n$ are linearly independent,

$c_1 = c_2 = \cdots = c_n = 0$. $\square$

What is the converse? Is it true?

If $v_1, \ldots, v_n$ linearly independent $\Rightarrow T(v_1), \ldots, T(v_n)$ linearly independent for every collection of linearly independent sets then T is 1-1.

HW: Is the converse true? If so, prove it.

$Ax = 0$. $\boxed{Ax = b}$

Theorem (Rank-Nullity Theorem)

Let V, W be vector spaces. Let $T : V \to W$ be a linear transformation. Then

$$\dim \ker T + \dim \operatorname{range} T = \dim V.$$

$\ker T \subseteq V$

If V is infinite dimensional then least one of the summands on left hand side is infinity (i.e. either ker T *or* range T*, or both, must be infinite dimensional).*

Idea: Any linearly independent subset of a vector space can be extended to a basis for the whole space (Theorem covered on February 12th).

The aforementioned Theorem also holds for infinite-dimensional vector spaces. Interested students may refer to Theorem 5.25 of Proofs Document.

Proof: Let us consider the case when V is finite dimensional.

Let $\dim \operatorname{Ker} T = n$.

$\dim V = m$.

Since $\operatorname{Ker} T \subset V$, $n \leq m$.

Let $\{v_1, \dots, v_n\}$ be a basis for $\ker T$.

We can extend this to a basis of V, by adding <u>$m-n$</u> elements.

So let $\{v_1, \ldots, v_n, v_{n+1}, \ldots, v_m\}$

be a basis of V.

Claim: $\{T(v_{n+1}), \ldots, T(v_m)\}$ is a basis of Range T.

Let us show that

$\underbrace{T(v_{n+1}), \ldots, T(v_m)}$ are linearly independent.

Suppose if possible that

$\rightarrow c_1 T(v_{n+1}) + \ldots + c_{n-m} T(v_m) = 0.$

for some $\underline{c_1, \ldots, c_{n-m} \in \mathbb{R}}$.

Since T is linear,

$$T(\underbrace{c_1 v_{r+1} + c_2 v_{r+2} + \cdots + c_{n-m} v_m}) = 0$$

$$\therefore \boxed{c_1 v_{n+1} + \cdots + c_{n-m} v_m} \in \underline{\ker T}.$$

$\operatorname{span}\{v_1, \ldots, v_r\}$

$\therefore \exists$ scalars $d_1, \ldots, d_n$ such that

$$c_1 v_{n+1} + \cdots + c_{n-m} v_m$$

$$= \underline{d_1 v_1 + \cdots + d_n v_n}.$$

$$\Rightarrow \left\{ d_1 v_1 + \cdots + d_n v_n - c_1 v_{n+1} - \cdots - c_{n-m} v_m = 0 \right\}$$

Since $v_1, \ldots, v_m$ are lin. indep, we obtain $d_1 = \cdots = d_r = \underbrace{c_1 = \cdots = c_{n-m} = 0}$

$\therefore \{T(v_{n+1}), \ldots, T(v_m)\}$

is linearly independent.

Next we show that

$$\operatorname{Span}\{T(v_{n+1}), \ldots, T(v_m)\}$$

$$= \operatorname{Range} T$$

Let $w \in \text{Span}\{T(v_{n+1}), \dots, T(v_m)\}$

$\Rightarrow w = c_1 T(v_{n+1}) + \dots + c_{n-m} T(v_m)$

for some $c_1, \dots, c_{n-m} \in \mathbb{R}$.

Since T is linear,

$\Rightarrow w = T(c_1 v_{n+1} + \dots + c_{n-m} v_m)$

$\therefore$ $w \in$ Range T.

$\Rightarrow$ $\cancel{S}$pan $\{T(v_{n+1}), \ldots, T(v_m)\}$

$\subset$ Range T. —— ①

Next let $w \in$ Range T

$\Rightarrow \exists\ \underline{v \in V}$ such that $Tv = w$.

↳①

$\therefore \exists$ scalars $d_1, \ldots, d_m$ such that

$$v = d_1 v_1 + \cdots + d_m v_m \quad (2)$$

from (1) & (2)

$$T(d_1 v_1 + \cdots + d_m v_m) = w.$$

$$\Rightarrow \underbrace{d_1 T(v_1) + \cdot + d_n T(v_n)}_{} + d_{n+1} T(v_{n+1}) + \cdots + d_m T(v_m) = w.$$

Since $v_1, \ldots, v_n \in \ker T$,

$$\Rightarrow \underbrace{d_{n+1} T(v_{n+1}) + \cdots + d_m T(v_m)}_{} = w.$$

$\Rightarrow w \in \mathrm{Span}\{T(v_{n+1}), \ldots, T(v_m)\}$

$\therefore \mathrm{Span}\{T(v_{n+1}), \ldots T(v_m)\}$

$= \mathrm{Range}\, T.$

$\therefore T(v_{n+1}), \ldots, T(v_m)$ is a basis of Range T.

$\therefore \quad \dim \operatorname{Ker} T + \dim \operatorname{Range} T$

$= \dim V$;

if V is finite dimensional.

Next we consider the case where V is infinite-dimensional.

If $\ker T$ is infinite dimensional, then nothing to show.

Suppose $\ker T$ is finite dimensional, say $\dim \ker T = r$.
Let $\{v_1, \dots, v_n\}$ be a basis of $\ker T$.

Then $\{v_1, \ldots, v_n\}$ can be extended to a basis of V.,

Read up the proof. Proofs document.

Say $\{v_1, \ldots, v_n, v_{n+1}, \ldots\}$

(v_n) is a linearly independent infinite sequence of vectors such that

$$V = \operatorname{span}\{v_n : n \in \mathbb{N}\}.$$

Suppose if possible that the range T is finite dimensional.

and $\dim \operatorname{Range} T = m$.

Let $w_1, \ldots, w_m$ be a basis of Range T.

Let $v_{n+1}, \ldots, v_{n+m} \in V$ such that $T(v_{n+1}) = w_1, \ldots, T(v_{n+m}) = w_m$.

Claim: $v_1, \ldots, v_{n+m}$ is a basis of V.

Let $v \in V$. Let $Tv = w$.

Since $w \in \text{Range } T$,

$\exists\ c_1, \ldots, c_m$ such that
$w = c_1 w_1 + \cdots + c_m w_m$.

$$\Rightarrow Tv = c_1 T(v_{n+1}) + \cdots + c_m T(v_{n+m})$$

$$\Rightarrow Tv - c_1 T(v_{n+1}) - \cdots - c_m T(v_{n+m})$$

$$= 0$$

$$\Rightarrow T(v - c_1 v_{n+1} - c_2 v_{n+2} \cdots - c_m v_{n+m}) = 0$$

$$\Rightarrow v - c_1 v_{n+1} - \cdots - c_m v_{n+m} \in \ker T.$$

$$\Rightarrow v - c_1 v_{n+1} + \cdots + c_m v_{n+m}$$

$$\in \operatorname{Span}\{v_1, \ldots, v_n\}.$$

$$\therefore\ v \in \operatorname{Span}\{v_1, \ldots, v_{n+m}\}$$

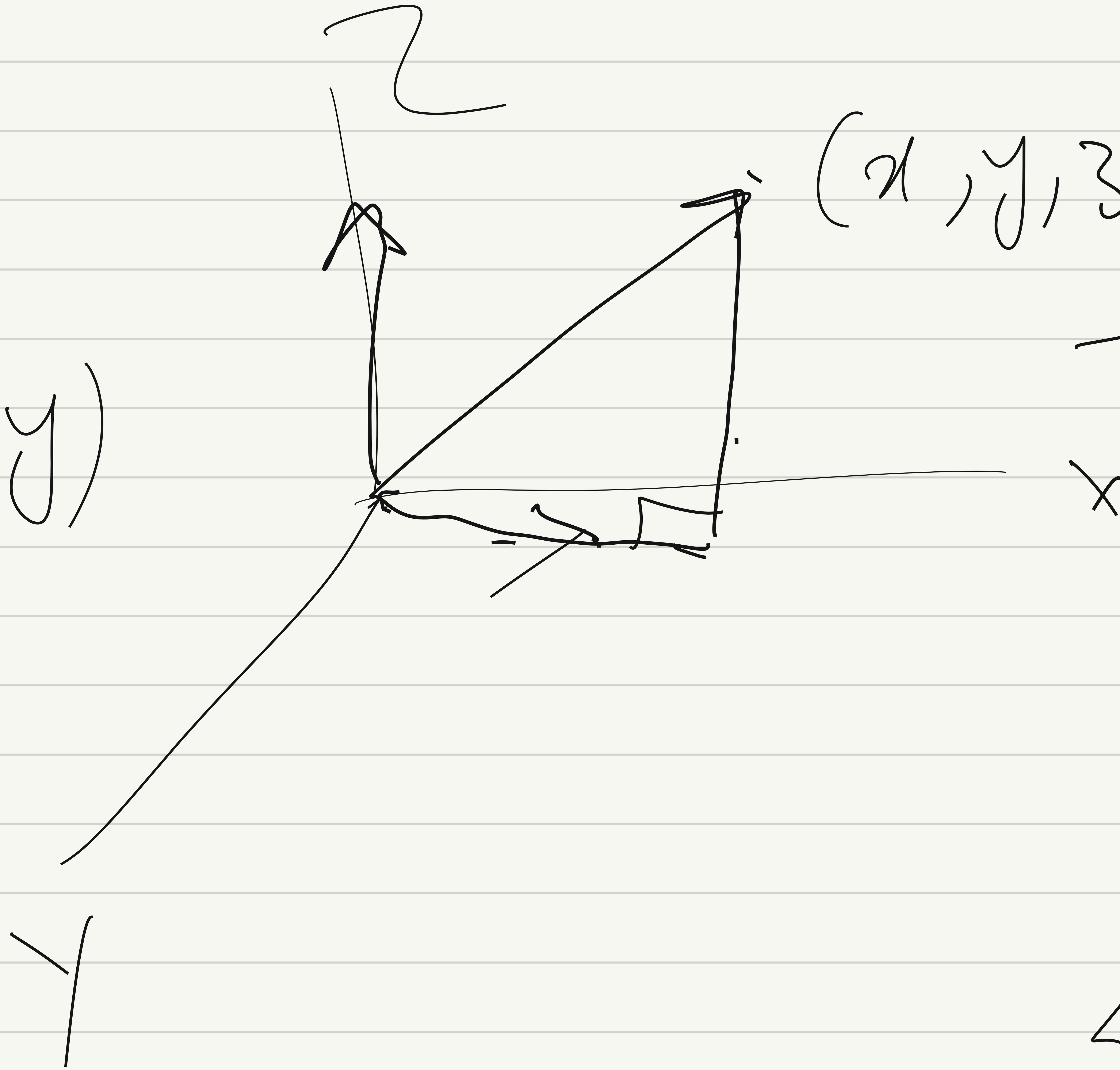
finite dim

$$\Rightarrow V \subset \operatorname{Span}\{v_1, \ldots, v_{n+m}\}$$

$\therefore$ Range T is ~~infinite~~ dimensional.

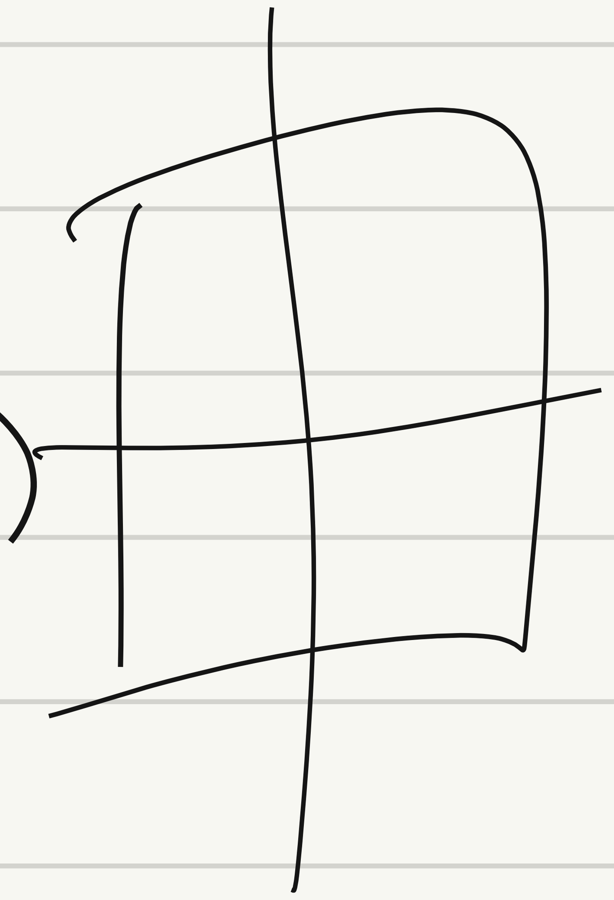
(x, y, z)

$\mapsto (x, y)$



(x, y, z)

$\dim \mathbb{R}^3$



$T : (x, y, z)$

$\mapsto (x, y, 0)$

Range:

$\{(x, y, 0) : x, y \in \mathbb{R}\}$

Kernel of T is the z-axis $\rightarrow$ 1-dimensional.

$\dim \text{Range}(T) = 2$.

$\dim V$ $\quad x \longmapsto Ax$ $= V$

$= \dim \ker T$

$+ \dim \text{Range } T$

$T : \mathbb{R}^n \longrightarrow \mathbb{R}^m = W$

Corollary

Let A be an $m \times n$ matrix. Then

$$\dim \operatorname{col} A + \dim \operatorname{nul} A = n.$$

Definition

Let A be an $m \times n$ matrix. The dimension of col A is called the *rank* of A.

column rank A

$\boxed{Ax = 0}$ $\qquad Ax = b$

$T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, maps $x \mapsto Ax$

$\ker T$

$$= \{ x \in \mathbb{R}^n \mid Tx = 0 \}$$

$$= \{ x \in \mathbb{R}^n \mid Ax = 0 \}$$

$$= \text{nul}(A).$$

$$\text{Range } T = \{ y \in \mathbb{R}^m \mid y = T(x) \text{ for some } x \in \mathbb{R}^n \}$$

$$= \{ y \in \mathbb{R}^m \mid y = Ax \text{ for some } x \in \mathbb{R}^n \}$$

$$= \text{col}(A).$$

Definition

Let A be an $m \times n$ matrix. The dimension of row A is called the *row rank* of A.

Theorem

The row rank of A equals the column rank of A.

Proposition

Let A be an $m \times n$ matrix. The rows containing pivots in the RREF of A are a basis for the row space of A.

Idea: Since row operations are reversible, the span of the rows remains the same when we perform elementary row operations.

<u>Lemma</u>: Let B be a matrix obtained by performing an elementary row operation on A.

Then $\operatorname{row} B = \operatorname{row} A$.

Let the rows of A be

$a_1^T, \ldots, a_m^T$,

$$A = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_n^T \end{bmatrix}$$

where $a_j^T \in \mathbb{R}^r \to \boxed{1 \times n \text{ matrix}}$

a_j is $n \times 1$ matrix

$\therefore$ Row A = Span$\{a_1^T, \ldots, a_m^T\}$.

Suppose I perform a row operation on A

and obtain a matrix B.

case (i) The row operation is of the row replacement type, say add c times the jth row of A to the ith row of A where $c \in \mathbb{R}$.

Then the rows of B

are $a_1^T, a_2^T, \ldots a_{i-1}^T, a_i^T + c a_j^T, a_{i+1}^T \ldots a_m^T$

ith row of B

$\Rightarrow$ rows of B are contained in the span $\{a_1^T, \ldots, a_m^T\}$

$\Rightarrow$ row $B \subset$ row A.

Since row operations

are reversible,

row $A \subset$ row B.

$\therefore$ row $A =$ row B.

~~Case (iii) :~~

Case(iii) : Do yourself.

As a consequence of this lemma

row A = row space of RREF of A.

Next we will show that rows containing pivots in RREF of A are a basis of the row space of the RREF of A.