

Let  $A$  be an  $n \times n$  matrix (where  $n \geq 2$ ).  $A_{ij}$  denotes the  $n - 1 \times n - 1$  submatrix formed by deleting the  $i$ -th row and  $j$ -th column of  $A$ , for  $1 \leq i, j \leq n$ .

### Definition

For  $n \geq 2$ , the determinant of an  $n \times n$  matrix  $A = (a_{ij})$  is the sum of  $n$  terms of the form  $\pm a_{1j} \det A_{1j}$ , with plus and minus signs alternating, where the entries  $a_{11}, a_{12}, \dots, a_{1n}$  are from the first row of  $A$ . In symbols,

$$\begin{aligned}\det A &= a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n} \\ &= \sum_{j=1}^n (-1)^{1+j} a_{1j} \det A_{1j}\end{aligned}$$

The determinant of a  $1 \times 1$  matrix is the single entry of that matrix.

## Definition

Let  $A = (a_{ij})$  be an  $n \times n$  matrix (where  $n \geq 2$ ). The  $(i, j)$ -cofactor of  $A$  is the number  $C_{ij}$  given by

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

## Theorem

*The determinant of an  $n \times n$  matrix  $A$  can be computed by a cofactor expansion across any row or down any column. The expansion across the  $i$ -th row using the cofactors is*

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

*The cofactor expansion down the  $j$ -th column is*

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}.$$

## Theorem

*If  $A$  is a triangular matrix, then  $\det A$  is the product of the entries on the main diagonal of  $A$ .*

## Proposition

Let  $E$  be an  $n \times n$  elementary matrix. Then

- 1  $\det E = c$ , when  $E$  corresponds to scaling a row by a nonzero scalar  $c$
- 2  $\det E = 1$ , when  $E$  corresponds to a row replacement operation
- 3  $\det E = -1$ , when  $E$  corresponds to a row interchange operation

## Proposition

Let  $A$  be an  $n \times n$  matrix having two identical rows. Then the determinant of  $A$  is zero.

Idea(s) behind proof:

First consider the case where the identical rows are adjacent.

Expand.

Next consider the non-adjacent case. Use induction.

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## Continued from Tuesday's Lecture

If  ~~$n=1$~~ , then there is nothing to show. We have also shown that the result is true when  $n=2$ . Therefore let us assume that the result holds true for every  $n < k$  and show that it holds true for  $n=k$  (where  $k$  is an integer greater than 2). *wlog, let  $i < j$ .*

Let  $i$  and  $j$  be the rows of  $A$  which are identical. If these rows are adjacent, then there is nothing to show. So assume that there exists  $l$  such that  $i < l < j$ . Expanding across row  $l$  we get

$$\det A = (-1)^{l+1} a_{l1} \det A_{l1} + \dots + (-1)^{l+n} a_{ln} \det A_{ln}.$$

*submatrices*

Each of the ~~minors~~  $A_{lm}$  ( $m=1, \dots, n$ ) has at least two identical rows. So by the induction hypothesis,

$$\det A_{lm} = 0, \text{ for } m=1, \dots, n.$$

Hence  $\det A = 0$ .

## Theorem

Let  $A$  be an  $n \times n$  matrix. Let  $E$  be an  $n \times n$  elementary matrix. Then

$$\det(EA) = \det(E) \det(A)$$

For row replacement and row scaling, expand along the appropriate row.

For row interchange,  $R_i \longleftrightarrow R_j$ , is equivalent to the following sequence to row operations:

1.  $R_j \rightarrow R_i + R_j$  : let this correspond to elementary matrix  $E_1$
2.  $R_i \rightarrow R_i - R_j$  : let this correspond to elementary matrix  $E_2$
3.  $R_j \rightarrow R_i + R_j$  : let this correspond to elementary matrix  $E_3$
4.  $R_i \rightarrow -R_i$  : let this correspond to elementary matrix  $E_4$

## Proof

$$(EA)_{ik} = a_{ik} + ca_{jk}$$

Let  $A = (a_{ij})$ .

We first consider the case where  $E$  corresponds to row replacement, say  $R_i \rightarrow R_i + cR_j$ , where  $c \in \mathbb{R}$  (or  $c \in \mathbb{C}$ , if  $A$  has complex entries). Expanding along the  $i$ -th row,

$$\begin{aligned} \det(EA) &= (-1)^{i+1}(a_{i1} + ca_{j1}) \det EA_{i1} + \dots + (-1)^{i+n}(a_{in} + ca_{jn}) \det EA_{in} \\ &\rightarrow = (-1)^{i+1}(a_{i1} + ca_{j1}) \det A_{i1} + \dots + (-1)^{i+n}(a_{in} + ca_{jn}) \det A_{in} \\ &= \det A + c((-1)^{i+1}a_{j1} \det A_{i1} + \dots + (-1)^{i+n}a_{jn} \det A_{in}) \end{aligned}$$

But the expression

becomes zero.

$$(-1)^{i+1}a_{j1} \det A_{i1} + \dots + (-1)^{i+n}a_{jn} \det A_{in}$$

is nothing but the determinant of the matrix obtained by replacing the  $i$ -th row of  $A$  by the  $j$ -th row of  $A$ . We know that any matrix having identical rows has zero determinant.



$$\begin{array}{c}
 \text{jth row} \\
 \text{ith} \rightarrow
 \end{array}
 \rightarrow
 \begin{bmatrix}
 a_{i1} & \dots & a_{in} \\
 a_{j1} & \dots & a_{jn} \\
 \cancel{a_{i1}} & \dots & \cancel{a_{in}} \\
 a_{n1} & \dots & a_{nn}
 \end{bmatrix}
 \xrightarrow{(-1)^{i+j}}
 \begin{bmatrix}
 A_{i1} & \dots & A_{in} \\
 a_{j1} & \dots & a_{jn} \\
 \cancel{a_{i1}} & \dots & \cancel{a_{in}} \\
 a_{n1} & \dots & a_{nn}
 \end{bmatrix}
 \xrightarrow{(-1)^{i+j}}
 \begin{bmatrix}
 A_{i1} & \dots & A_{in} \\
 a_{j1} & \dots & a_{jn} \\
 \cancel{a_{i1}} & \dots & \cancel{a_{in}} \\
 a_{n1} & \dots & a_{nn}
 \end{bmatrix}$$

$$(-1)^{j+1} a_{j1} A_{j1} + \dots + (-1)^{n+j} a_{jn} A_{jn}$$

So

$$(-1)^{i+1} a_{j1} \det A_{i1} + \dots + (-1)^{i+n} a_{jn} \det A_{in} = 0.$$

Hence  $\det EA = \det A$ . Since  $\det E = 1$ ,

$$\boxed{\det EA = \det E \det A.} \quad \checkmark$$

Next, suppose  $E$  corresponds to scaling, say  $R_i \rightarrow cR_i$  for some  $c \in \mathbb{R}$  (or  $c \in \mathbb{C}$ , if  $A$  has complex entries). Then expanding along the  $i$ -th row, we get

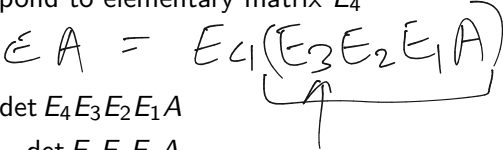
$$\det(EA) = (-1)^{i+1} \underline{ca_{i1}} \det A_{i1} + \dots + (-1)^{i+n} ca_{in} \det A_{in} = \underline{c \det A}.$$

Next, suppose  $E$  corresponds to row interchange, say  $R_i \longleftrightarrow R_j$ .

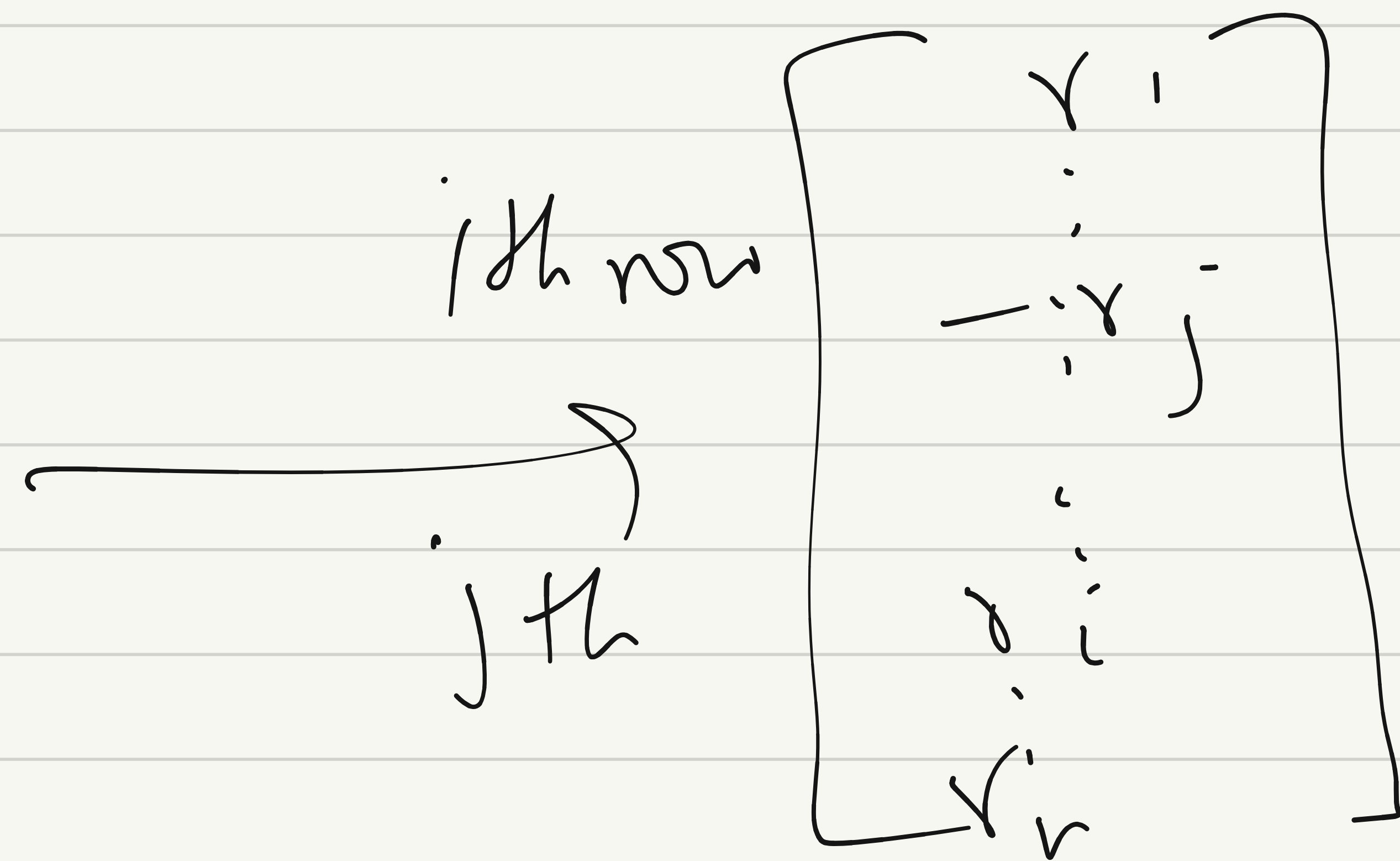
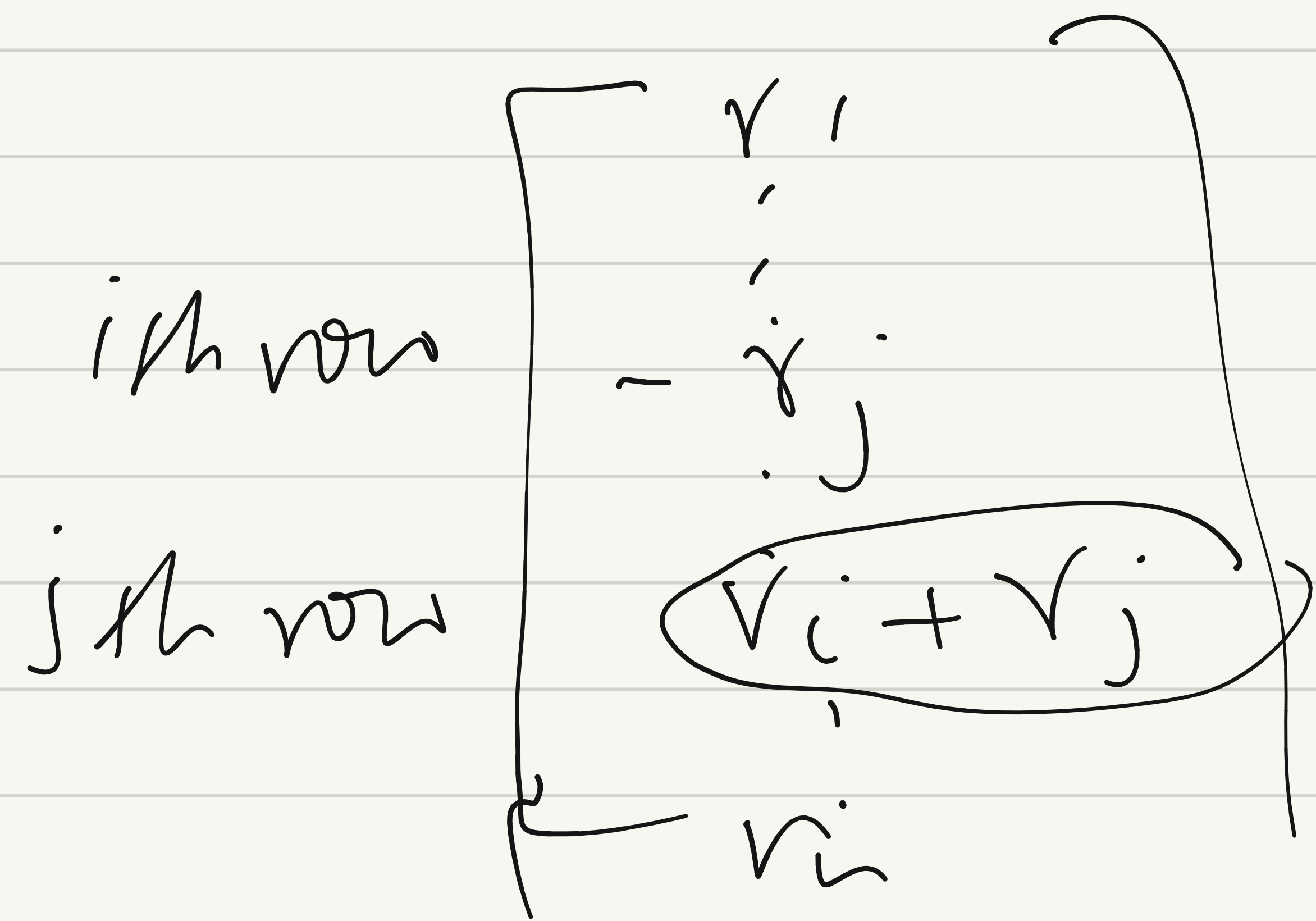
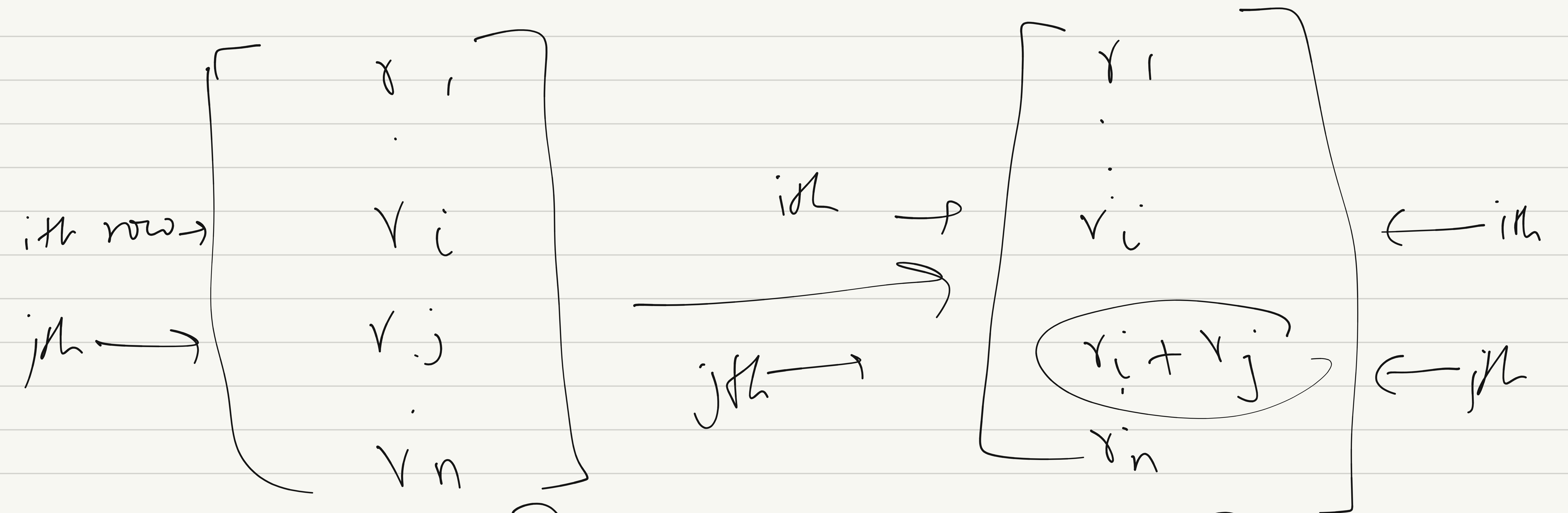
This is equivalent to the following sequence to row operations:

1.  $R_j \rightarrow R_i + R_j$  : let this correspond to elementary matrix  $E_1$
2.  $R_i \rightarrow R_i - R_j$  : let this correspond to elementary matrix  $E_2$
3.  $R_j \rightarrow R_i + R_j$  : let this correspond to elementary matrix  $E_3$
4.  $R_i \rightarrow -R_i$  : let this correspond to elementary matrix  $E_4$

Therefore

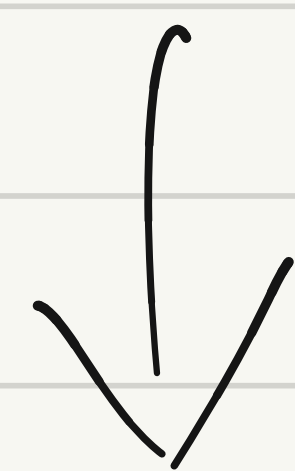
$$\begin{aligned} EA &= E_4(E_3E_2E_1A) \\ \det EA &= \det E_4E_3E_2E_1A \\ &= -\det E_3E_2E_1A \\ &= -\det E_2E_1A \\ &= -\det E_1A \\ &= -\det A \end{aligned}$$


Hence  $\det EA = \det E \det A$ .



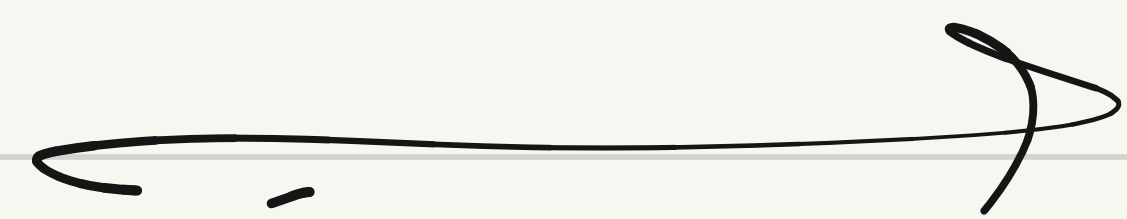
$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Swap  
 $R_1$  and  $R_3$



$$R_3 \rightarrow R_1 + R_3 \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 8 & 10 & 12 \end{bmatrix} \xrightarrow{R_1 \rightarrow R_1 - R_3} \begin{bmatrix} -7 & -8 & -9 \\ 4 & 5 & 6 \\ 8 & 10 & 12 \end{bmatrix}$$

$$R_3 \rightarrow R_1 + R_3$$



$$\begin{bmatrix} -7 & -8 & -9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}$$

$$R_1 \rightarrow -R_1$$



$$\begin{bmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}$$

$$B = EA$$
$$\det B = \det E \det A$$

### Corollary

Let  $A$  be a square matrix.

- a** If a multiple of one row of  $A$  is added to another row to produce a matrix  $B$ , then  $\det B = \det A$
- b** If two rows of  $A$  are interchanged to produce  $B$ , then  $\det B = -\det A$ .
- c** If one row of  $A$  is multiplied by  $k$  to produce  $B$ , then  $\det B = k \cdot \det A$ .

## Theorem

*Let  $A$  and  $B$  be  $n \times n$  matrices. Then*

$$\det AB = \det A \det B.$$



### Lemma

Let  $E_1, \dots, E_p$  be a sequence of  $n \times n$  elementary matrices. Then

$$\det \left( \prod_{i=1}^p E_i \right) = \prod_{i=1}^p \det E_i.$$

Proof: Exercise.

$$\det(E_1 E_2) = \det E_1 \det E_2$$

→ use induction.

## Theorem

*Let  $U$  be an echelon form of  $A$  obtained by row replacement and row interchange operations. Let  $r$  be the number of row interchanges involved in the row reduction process. Then*

$$\det A = \begin{cases} (-1)^r \cdot \left( \text{product of pivots in } U \right) & \text{when } A \text{ is invertible} \\ 0 & \text{when } A \text{ is not invertible} \end{cases}$$

## Corollary

*A square matrix  $A$  is invertible if and only if  $\det A \neq 0$ .*

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## Proof

$$U = E_p \cdots E_1 A$$

→ A square echelon matrix is upper triangular, and has all pivot positions on the main diagonal.

Therefore  $\det U$  equals the product of the diagonal entries of  $U$ .

If  $A$  is invertible, then all the diagonal entries of  $U$  are pivots and therefore  $\det U$  equals the product of pivots in  $U$ .

If  $A$  is not invertible, then at least one diagonal entry of  $U$  is zero and therefore  $\det U = 0$ .

Since  $U$  is obtained from  $A$  using  $r$  row interchanges and some row replacement operations, it follows that

$$\det U = \underline{(-1)^r} \det A.$$

Thus

$$\det A = \begin{cases} (-1)^r \cdot \left( \begin{array}{l} \text{product of} \\ \text{pivots in } U \end{array} \right) & \text{when } A \text{ is invertible} \\ 0 & \text{when } A \text{ is not invertible} \end{cases}$$

$$AB = E_1 \cdots E_p E_{p+1} \cdots E_q$$

Back to the proof of  $\det AB = \det A \det B$ .

We first consider the case where both  $A$  and  $B$  are invertible.

There exist elementary matrices  $E_1, \dots, E_p$  such that

$$A = E_1 E_2 \cdots E_p$$

and  $E_{p+1}, \dots, E_q$  such that

$$B = E_{p+1} \cdots E_q.$$

Hence by Lemma stated earlier,

$$\det AB = \det \left( \prod_{i=1}^q E_i \right) = \det \left( \prod_{i=1}^p E_i \right) \det \left( \prod_{i=p+1}^q E_i \right) = \det A \det B.$$

$$AD = I \Rightarrow A \text{ is invertible}$$

$$\Rightarrow A \left( B (AB)^{-1} \right) = I$$

Next, suppose either  $A$  or  $B$  or both, are singular.

By Corollary stated earlier,  $\det A = 0$  or  $\det B = 0$  (or both).  
Hence  $\det A \det B = 0$ .

$AB$  invertible  $\Rightarrow \underline{AB(AB)^{-1} = I} \Rightarrow A$  is invertible, and  
 $\underline{(AB)^{-1}AB = I} \Rightarrow B$  is invertible (by the Invertible Matrix Theorem).

Therefore  $AB$  is not invertible.

Hence  $\det AB = 0$ . Therefore  $\det AB = \det A \det B$ .

$Ax = b$ .  
New for you

$$A = [a_1 \dots a_i \dots a_n]$$

For any  $n \times n$  matrix  $A$  and any  $\mathbf{b}$  in  $\mathbb{R}^n$ , let  $A_i(\mathbf{b})$  be the matrix obtained from  $A$  by replacing column  $i$  by the vector  $\mathbf{b}$ .

$$\rightarrow A_i(\mathbf{b}) = [\mathbf{a}_1 \quad \dots \quad \mathbf{b} \quad \dots \quad \mathbf{a}_n]$$

↑  
i-th column

### Theorem (Cramer's Rule)

Let  $A$  be an invertible  $n \times n$  matrix. For any  $\mathbf{b}$  in  $\mathbb{R}^n$  or  $\mathbb{C}^n$ , the unique solution  $\mathbf{x}$  of  $A\mathbf{x} = \mathbf{b}$  has entries given by

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}, \quad i = 1, 2, \dots, n$$

Idea: We express  $A_i \mathbf{b}$  as the product of  $A$  and a matrix. What matrix could this be?

$$\underline{\underline{Ax = b.}}$$

$$\left[ \begin{array}{ccccccc} a_1 & a_2 & \cdot & b & \cdot & \cdot & a_n \end{array} \right]$$

$\uparrow$                        $\uparrow$   
 $i$ th                       $i$ th

$$= \left[ \begin{array}{ccccccc} a_1 & \cdot & \cdot & \cdot & \cdot & \cdot & a_n \end{array} \right]$$

$$\left[ \begin{array}{cc|c|c} 1 & 0 & x_1 & 0 \\ 0 & 1 & \cdot & \cdot \\ \cdot & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & x_n & 1 \end{array} \right]$$

$\downarrow$                        $\checkmark$   
 $i$ th

$$\det(A_i(b)) = \det(A) \det(I_i(x))$$

$$I_i(\mathbf{x}) = \begin{bmatrix} 1 & 0 & \cdots & x_1 & \cdots & 0 \\ 0 & 1 & \cdots & x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_n & \cdots & 1 \end{bmatrix}$$

Handwritten annotations: "ith row" with an arrow pointing to the  $i$ -th row of the matrix; "column" with an arrow pointing to the  $i$ -th column of the matrix; a circle around the entry  $x_i$  in the  $i$ -th row and  $i$ -th column.

### Lemma

Let  $I$  be the  $n \times n$  identity matrix, and let  $\mathbf{x} = (x_1, \dots, x_n)$  be any vector in  $\mathbb{R}^n$ , or in  $\mathbb{C}^n$ . Then

$$\det I_i(\mathbf{x}) = x_i.$$

$i$ ,  $i$ th entry of  $I_i(\mathbf{x}) = x_i$



$$c(dv) = (cd)v.$$

Axler:

$M_n(\mathbb{R})$   
 $\uparrow$   
 ring

$$a \in F$$

$$v \in V$$

$$av = 0$$

$$F = \mathbb{R} \text{ on } \mathbb{C}$$

$$\text{If } a \text{ fixed } \overline{ab} = 0 \Rightarrow a = 0 \text{ or } b = 0$$

Prove:  $a = 0$  or  $v = 0$ .

case (i) If  $a = 0$ , nothing to show.

case (ii):  $a \neq 0$ .  $av = 0$   
 $v = \frac{1}{a} \cdot av = a^{-1}(av) = a^{-1} \cdot 0 = 0$