Functions of a Random Variable

- You are measuring the power consumed by your appliance

- The power observed is a random variable, say W and is in watts
  - The randomness in the observed power can be due to measurement errors introduced by your equipment

- The power in dB, $Y = 10 * \log10(W)$ is also a random variable and is a function of the RV W.
  - We say that Y is derived from W

Derived Random Variable

- **Def 2.15** Each sample value *y* of a derived random variable Y is a **mathematical function** g(x) of a sample value *x* of another random variable X.
  - The notation Y=g(X) is used to describe the relationship of the two random variables.

$$S_X \xrightarrow{g(.)} S_Y$$

<u>Mathematical function:</u>
One-to-one or a many-to-one relation

g(.) maps the range of X into the range of Y

Given PMF of X

- Calculate PMF of Y
  - Think the event {Y = y} in terms of the range space of X

Given: $P_X(x),\ x \in \mathbb{R}$

Calculate $\underbrace{P_Y(y),}\ y \in \mathbb{R}$

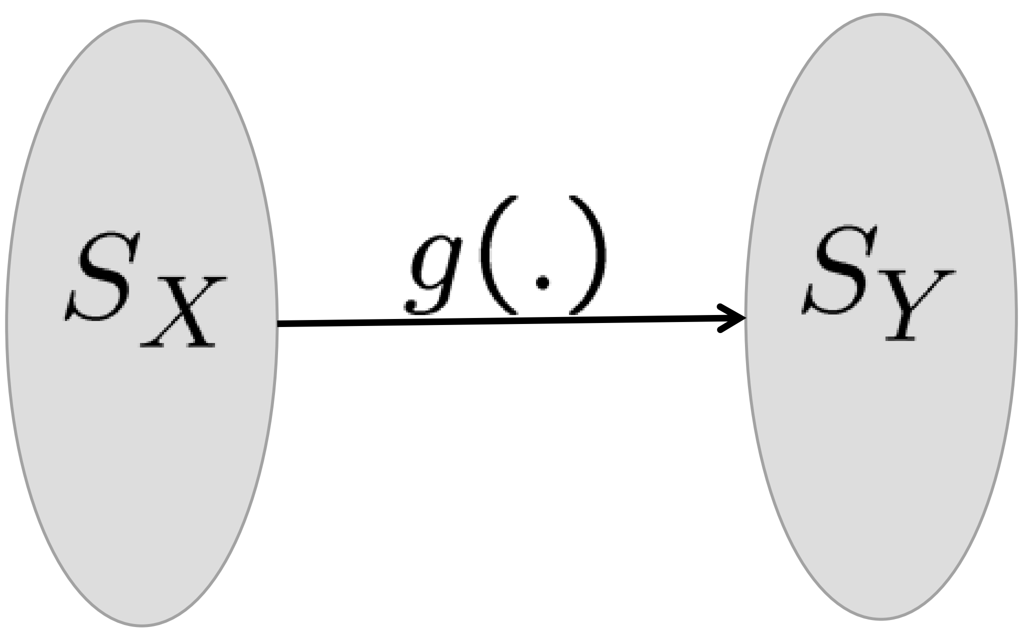
$$P[Y=y] = \sum_{x \in S_X : g(x) = y} P[X=x]$$

$$\{Y=y\} = \{x : \underbrace{x \in S_X,\ g(x)=y}\}$$

$P_X(x) \text{ -> } P_Y(y)$

- **Theorem 2.9** For a discrete random variable X, the PMF of Y=g(X) is

$$P_Y(y) = \sum_{x:g(x)=y} P_X(x)$$



Given PMF of X

- Calculate E[Y]

$$E[Y] = \sum_{y \in S_Y} y\, \underline{P[Y=y]}$$

$$= \sum_{y \in S_Y} y \sum_{x : g(x) = y} P[X=x]$$

$$= \sum_{y \in S_Y} \sum_{x : g(x) = y} g(x) P[X=x]$$

$$= \sum_{x \in S_X} g(x)\, p[X=x] = E[g(X)]$$

Derived Random Variable

- **Theorem 2.10** Given a random variable X with PMF $P_X(x)$ and the derived random variable $Y=g(X)$, the expected value of Y is

$$E[Y] = \mu_Y = \sum_{x \in S_X} g(x) P_X(x).$$

More on the E[.] Operator

- Consider the RV $Y = g(X) = X - E[X] = X - \mu_X$
- The expected value of X is subtracted from each number in S_X. The resulting range is the range S_Y of the random variable Y
- What is E[Y]?
- Using First principles

$Y = g(X)$

$Y = X + E[X]$

$E[Y] = E[X + E[X]]$

$= E[X] + E[E[X]]$

$= E[X] + E[X]$

$= 2E[X]$

More on the E[.] Operator

- Consider the RV $Y = g(X) = X - E[X] = X - \mu_X$
- The expected value of X is subtracted from each number in S_X. The resulting range is the range S_Y of the random variable Y
- What is E[Y]?
- Using properties of E[]

The E[.] Operator

- **Theorem 2.12** For any random variable X,

$$E[aX + b] = aE[X] + b.$$

- Proof is similar to the previous theorem

Variance

$$E\left[\left(X - E[X]\right)^2\right]$$

- **Def 2.16** Variance of X is

$$Var[X] = E[(X - \mu_X)^2]$$

Think of the relative frequency interpretation of variance (which is also the empirical estimate)

$$Var(X) = E\left[(X - \mu_X)^2\right]$$

$$= E\left[X^2 + \mu_X^2 - 2\mu_X X\right]$$

$$= E\left[X^2\right] + E\left[\mu_X^2\right] - E\left[2\mu_X X\right]$$

- Standard deviation is just the square-root of Variance
  - Values of the RV within the SD of the Expected Value are referred to as its *typical values*

$$E\left[X^2\right] + \mu_X^2 - 2\mu_X E\left[X\right] = E\left[X^2\right] - \left(E\left[X\right]\right)^2$$

Properties of Variance

- **Theorem 2.13**

$$Var[X] = E[X^2] - (E[X])^2$$

$$Var[aX + b] = ?$$ $E[(ax+b)^2] - (E[ax+b])^2$

$= a^2 Var[x]$

- **Theorem 2.14**

$$Var[aX + b] = a^2 Var[X]$$

  - Offsetting a RV by a constant does not change its variance!

•Show both results using the properties of the E[.] operator
•Also, show using first principles, that is by expanding E[g(x)]

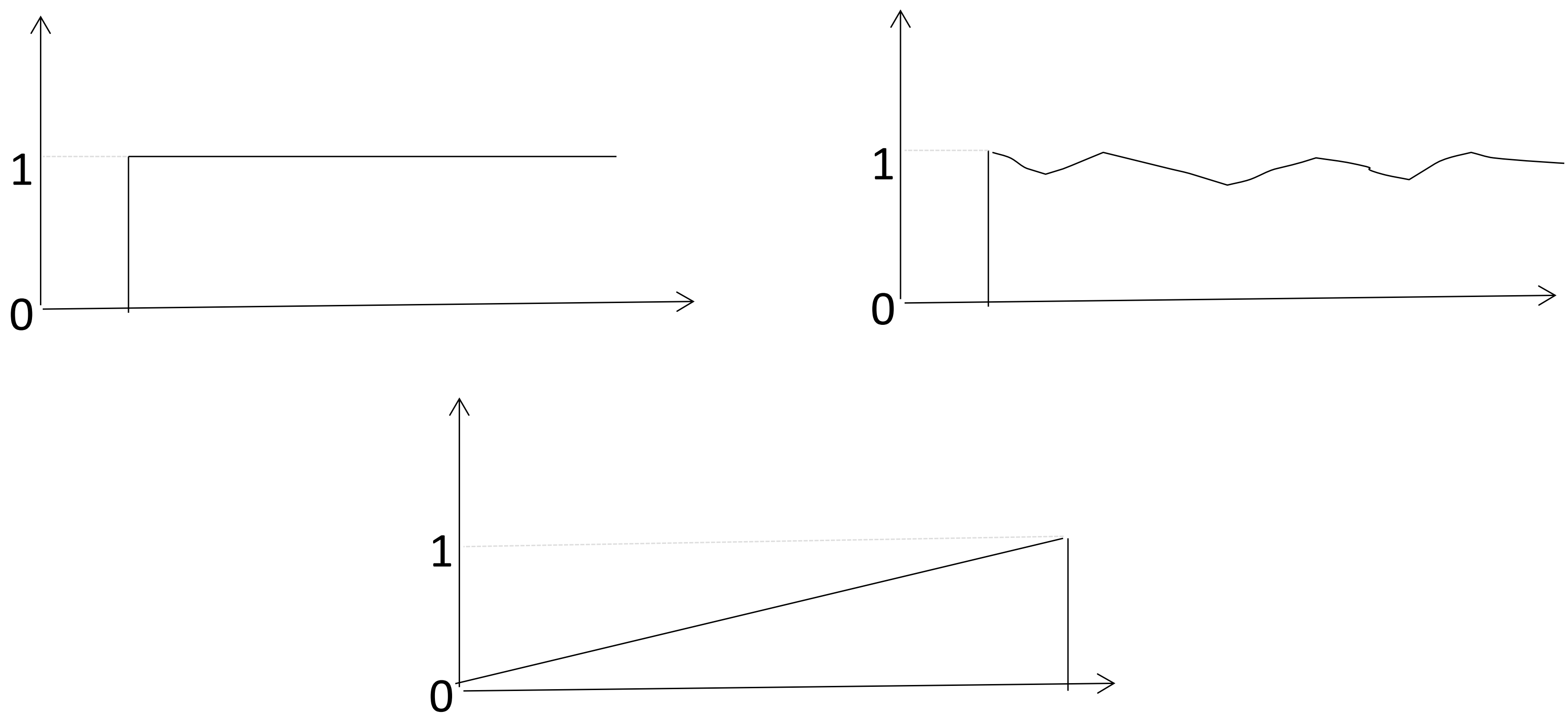
Moments

- For random variable X
  - The n^{th} moment is $E[X^n]$
  - The n^{th} central moment is $E[(X - E[X])^n]$

VVS Word Problems

- X is a discrete uniform RV with range S_X = {1,2,3,4,5,…,100}. Find E[X] and E[X^2].

- Which are valid CDFs?



VVS Word Problems

- $Y = X + 10$
- Express $Var[Y]$ in terms of $Var[X]$

- $Y = 10X + 10$
- Express $Var[Y]$ in terms of $Var[X]$

Quiz 2.6

Monitor three phone calls and observe whether each one is a voice call or a data call. The random variable N is the number of voice calls. Assume N has PMF

$E[N] = 0.3(1+2+3)$

$= 1.8$

$$P_N(n) = \begin{cases} 0.1 & n = 0, \\ 0.3 & n = 1, 2, 3, \\ 0 & \text{otherwise.} \end{cases} \tag{2.75}$$

Voice calls cost 25 cents each and data calls cost 40 cents each. T cents is the cost of the three telephone calls monitored in the experiment.

$T = 25N + 40(3-N)$

(1) Express T as a function of N.

$E[T] = 25E[N] + 120$
$-40E[N]$

$S_T = \{$

(2) Find $P_T(t)$ and $E[T]$.

$\hookrightarrow P[T = t]$

Conditional PMF of a Discrete RV

- The PMF of a RV X gives the probability of the event {X=x}
- PMF $P_X(x) = P[X=x]$
- The PMF of a RV conditioned on an event B is $P[X=x|B]$

$$\sum_{x \in B} P[X=x] = P[B]$$

- **Def 2.19** Given the event B, with P[B] > 0, the conditional probability mass function of X is $P_{X|B}(x) = P[X=x|B]$.

$$= \frac{P[X=x, B]}{P[B]} = \begin{cases} \frac{P[X=x]}{P[B]} & x \in B \\ 0 & \text{otherwise} \end{cases}$$

Theorem 2.16

- A RV X resulting from an experiment with event space $B_1, B_2, \ldots, B_m$ has PMF __________?
- What properties do $B_1, B_2, \ldots, B_m$ satisfy?

$$P_X(x) = \sum_{i=1}^{m} P_{X|B_i}(x) P[B_i].$$

VS Word Problem (RY)

Example 2.38 Problem

Let X denote the number of additional years that a randomly chosen 70 year old person will live. If the person has high blood pressure, denoted as event H, then X is a geometric $(p = 0.1)$ random variable. Otherwise, if the person's blood pressure is regular, event R, then X has a geometric $(p = 0.05)$ PMF with parameter. Find the conditional PMFs $P_{X|H}(x)$ and $P_{X|R}(x)$.