Recursion and Recurrences

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IIIT, Delhi

Summer Semester,
2nd May, 2022

Recursion: A Recap

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Example:

```
#include <stdio.h>

int main(void) {
    printf(" The universe is never ending! ");
    main();
    return 0; }
```

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Example:

```
int sum(int n) {
   if (n <= 1)
      return n;
   else
      return (n + sum(n - 1)); }
```

Example: sum(4)

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Function call	Value returned		
sum(1)	1		
sum(2)	$2 + \text{sum}(1)$	or	$2 + 1$
sum(3)	$3 + \text{sum}(2)$	or	$3 + 2 + 1$
sum(4)	$4 + \text{sum}(3)$	or	$4 + 3 + 2 + 1$

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- The base case is considered,
- then working out from the base case, the other cases are considered.

Recursion (Cont.)

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Example: $\mathrm{sum}()$

- $\mathrm{sum(n)} = n + (n-1) + \cdots + 1 = n + \mathrm{sum(n-1)}$.
- The variable n is reduced by 1 each time until
- the base case with $n = 1$ is reached.

Examples: Factorial

$$0\,! = 1, \quad n\,! = n(n-1)\cdots 3\cdot 2\cdot 1 \quad \text{for } n > 0$$

or equivalently,

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- **Base Case:** $0! = 1$ and $1! = 1$.
- **Recursive Case:** $n! = n\cdot(n-1)!$.

Factorial: Recursive Version

```
int RecFactorial (int n) {        /* recursive version */
   if (n <= 1)
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      return (n * RecFactorial (n - 1)); }
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- Factorial function grows very rapidly!
- RecFactorial(n) runs only a few values of n (upto $n = 12$!!).
- For $n > 12$, incorrect values are returned.
- This type of programming error is common!!

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Take Away: Functions that are logically correct can return incorrect values if the logical operations in the body of the function are beyond the integer precision available to the system!!

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int IterFactorial (int n) {        /* iterative version */
    int product = 1;

    for ( ; n > 1; - -n)
        product *= n;
    return product; }
```

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IterFactorial(n): Takes only 1 function call.

Efficiency Considerations

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- Takes care of its bookkeeping by stacking arguments and variables for each invocation.
- *This stacking of arguments, while invisible to the user, is still costly in time and space.*
- On some machines a simple recursive call with one integer argument can require eight 32-bit words on the stack.

Fibonacci Sequence

Fibonacci sequence is defined recursively as

$$f_1 = 1, \quad f_2 = 1, \quad f_{i+1} = f_i + f_{i-1} \quad \text{for } i = 1, 2, \ldots$$

Every element ($i \geq 3$) is the sum of it's previous two elements.

The sequence begins as $1, 1, 2, 3, 5, \ldots$

Fibonacci Sequence

Consider the following sequence:

$$
\begin{aligned}
2/1 &= 2.0 \quad \text{(bigger)}\\
3/2 &= 1.5 \quad \text{(smaller)}\\
5/3 &= 1.67 \quad \text{(bigger)}\\
8/5 &= 1.6 \quad \text{(smaller)}\\
13/8 &= 1.625 \quad \text{(bigger)}\\
21/13 &= 1.615 \quad \text{(smaller)}\\
34/21 &= 1.619 \quad \text{(bigger)}\\
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Note:

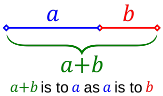
- This sequence seem to be converging!
- It converges to the *golden ratio*.

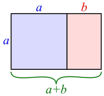
Golden Ratio

$$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.6180339887498948482$$

- It is a special number.
- Couple of ways to visually understand it are with

a *line segment* *Golden rectangles*





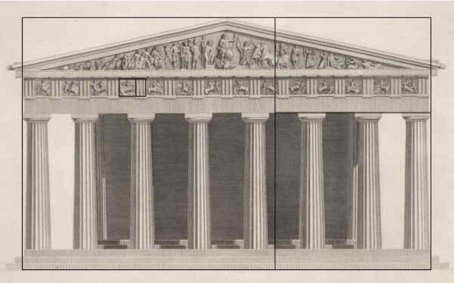
- It is an *irrational number* that is a root of the quadratic equation

$$x^2 - x - 1 = 0$$

Golden Ratio

- Reciprocal of φ or φ^{-1}:
  - $f_n/f_{n+1} \to 0.618$ as $n \to \infty$.
  - This is the reciprocal of φ: $1/1.618 = 0.618$.
  - It is highly unusual for the decimal representation of the fractional part of a number and its reciprocal to be exactly the same.
  - This only adds to the mystique of the Golden Ratio and leads us to ask: What makes it so special?

- **Some examples:**



The ancient temple in Greece fits almost precisely into a golden rectangle.

- **Some examples:**

1 : 1.618

Butterflies.

Recursive Fibonacci Sequence: Function Calls

```
int RecFibonacci (int n) {
    if (n <= 1)
        return n;
    else
        return (RecFibonacci(n - 1) + RecFibonacci(n - 2)); }
```

Recursive Fibonacci Sequence: Function Calls

Value of n	Value of RecFibonacci(n)	Number of function calls required to recursively compute RecFibonacci(n)
0	0	1
1	1	1
2	1	3
......		
23	28657	92735
24	46368	150049
......		
42	267914296	866988873
43	433494437	1402817465

Requires a large number of function calls even for moderate values of n.

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- It is seductive to use recursion.
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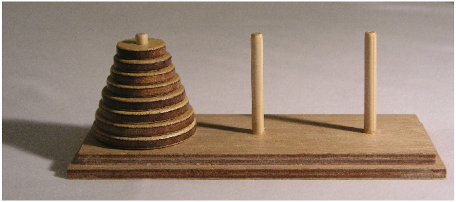
- The inefficiencies, however, are often of little consequence - as in the case of the quicksort algorithm.
- For many applications, recursive code is easier to write, understand, maintain.
- These reasons often prescribe its use.

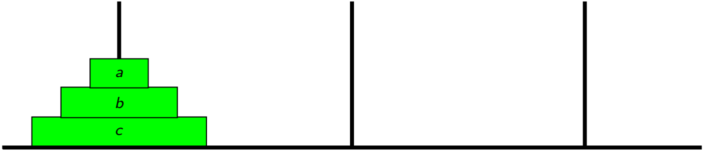
Towers of Hanoi

Towers of Hanoi: Problem Statement



- There are three towers.
- n disks of decreasing radius are placed on the 1^{st} tower.
- Move all of the disks from the 1^{st} tower to the 3^{rd} tower.
- **Condition:** At no moment of time can a larger disk be placed on top of smaller disks.
- The remaining tower can be used to temporarily hold disks.

Towers of Hanoi: Solution for $n = 3$



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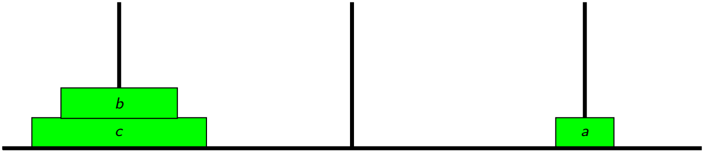
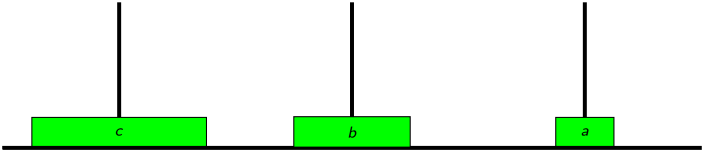
Step 1: Move disks a to tower 3.



Towers of Hanoi: Solution for $n = 3$

Step 1: Move disks a to tower 3.

Step 2: Move disks b to tower 2.



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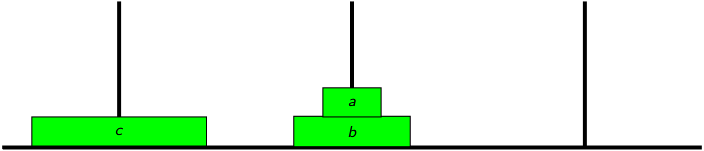

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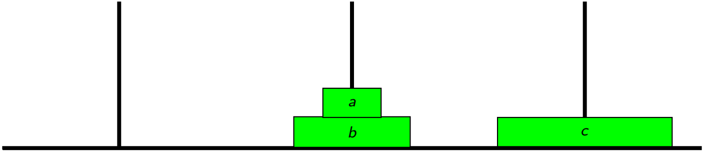

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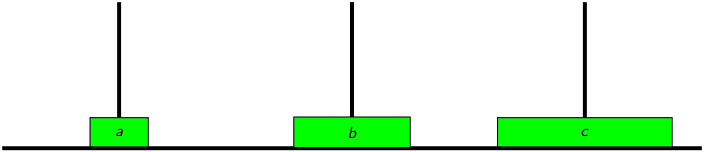

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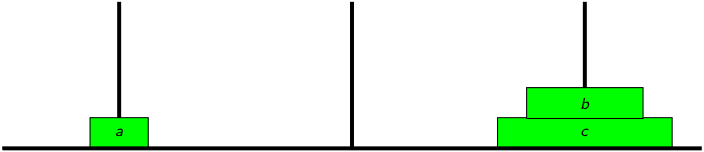

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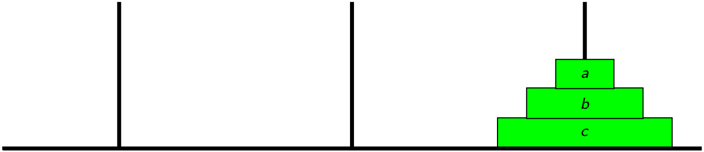

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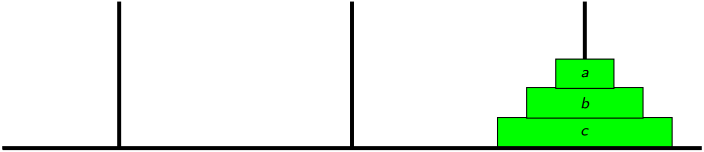

Homework:

- Write a recursive algorithm that solves the Towers of Hanoi problem for n disks.
- Implement your algorithm in C.

Recurrences

Recurrence

Definition

A **recurrence relation** is an equation that expresses each element of a sequence $\{a_n\}_{n=0}^{\infty}$ as a function of the preceding ones, i.e,

$$a_n = \psi(a_0, a_1, \ldots, a_{n-1}).$$

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  - The number of instructions in one instance of function call depends on the number of instructions executed when recursive calls are made.
  - In such cases it is easier for us to express it as some *recurrence relation* of the times/space complexity.
- Appears frequently in the analysis of algorithms.

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Outline

- We briefly discuss few useful technique for solving recurrences.

- Present general solutions of two classes of recurrences that are among the most common recurrences involved in analyzing algorithms.

Intelligent Guesses

- Guessing a solution may seem like a nonscientific method!
- But, keeping our pride aside, it works very well for a wide class of recurrence relations.
- It works even better when we are *not* trying to find the exact solution, but only an upper bound.
- **Why guess?** Proving a certain bound is valid is easier than deriving that bound.

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- **Note:** By definition we need $n-2$ steps to compute $F(n)$.
- Would be more convenient to have an explicit (or closed-form) expression for $F(n)$.
  - It would enable us to compute $F(n)$ quickly.
  - We can also compare $F(n)$ with other known functions.

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- $\therefore$ $F(n) = \mathcal{O}((a_1)^n)$.
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Note: This idea, can be used to solve recurrences of the form

$F(n) = b_1F(n-1) + b_2F(n-2) + \cdots + b_kF(n-k) \quad$ (k constant).

Books Consulted

1. *Introduction to Algorithms: A Creative Approach* by Udi Manber.

2. *Introduction to Algorithms* by Thomas H Cormen, Charles E Leiserson, Ronald L Rivest, Clifford Stein.

Thank You for your kind attention!