ECE 113- Basic Electronics

Lecture week 8: DC response in RLC circuit, AC response in circuits

Dr. Ram Krishna Ghosh, Assistant Professor

Office: B601, Research and Development Block

Email: rkghosh@iiitd.ac.in

INDRAPRASTHA INSTITUTE *of*
INFORMATION TECHNOLOGY
DELHI

Second order Circuits

$$C\frac{d^2v}{dt^2}+\frac{1}{R}\frac{dv}{dt}+\frac{1}{L}v=0$$

~~Initial condition is 0~~

- Combination of resistive elements and two energy storage elements (capacitor(s) and/or inductor(s))
- ‘Second-order’ refers to the order of the differential equation describing the circuit

Solution of a Linear Homogeneous 2nd order ordinary differential equation

- General form for second-order, (linear) ordinary differential equation:

$$\frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = f(t)$$

✓x(t) is a circuit quantity (voltage, current)

✓a0, a1 are constants, functions of the circuit elements

✓f(t) is a forcing function, usually the source voltage or current

✓For source free case: f(t) = 0

Solution of a Linear Homogeneous 2nd order ordinary differential equation

- This equation is often rewritten:

$$\frac{d^2x}{dt^2} + 2\alpha\frac{dx}{dt} + \omega_n^2 x = f(t)$$

$= 0$

$$\alpha = \frac{a_1}{2}$$

$$\omega_n = \sqrt{a_0}$$

  ✓ α is called damping factor
  ✓ ω_n is called the natural frequency
  ✓ $\omega_d^2 = \omega_n^2 - \alpha^2$ is called the damped frequency

$x = Ae^{mt}$

$m^2Ae^{mt} + 2\alpha mAe^{mt} + \omega_n^2Ae^{mt} = 0$

- Characteristic equation:

$$\underline{m^2 + 2\alpha m + \omega_n^2 = 0}$$

$\Rightarrow (m^2 + 2\alpha m + \omega_n^2) \times Ae^{mt} = 0$

  ✓ Quadratic roots:

$$m_1, m_2 = \frac{-2\alpha \pm 2\sqrt{\alpha^2 - \omega_n^2}}{2} = -\alpha \pm j\omega_d$$

Solution of a Linear Homogeneous 2nd order ordinary differential equation

Solution process:

1. Find the solution to the homogeneous equation
   - The solution is called the natural response (independent of the source applied)
   - Has the form

$$x_n(t) = A_1 e^{m_1 t} + A_2 e^{m_2 t}$$

   - Nature depends on m_1 and m_2

Solution of a Linear Homogeneous 2nd order ordinary differential equation

2. Look for a solution to the forced response
3. The complete solution is

$$x(t) = x_n(t) + x_f(t)$$

4. Use initial conditions to determine the constants.

Series RLC circuit

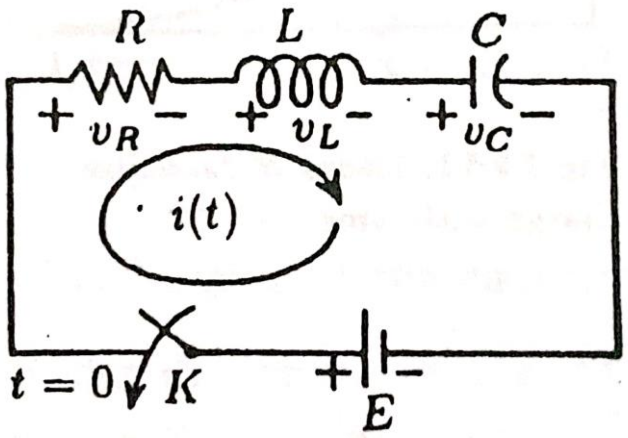


Fig.14.12. A series *RLC* circuit

$v_R = Ri$, $v_L = L\frac{di}{dt}$, and $v_C = q/C$

The KVL equation for the circuit is

$$v_R + v_L + v_C = E \quad \text{or,} \quad Ri + L\frac{di}{dt} + \frac{q}{C} = E$$

Since $i = dq/dt$ and $di/dt = d^2q/dt^2$, Eq. can be written as

$$\frac{d^2q}{dt^2} + 2b\frac{dq}{dt} + \omega_0^2 q = \frac{E}{L}$$

where $2b = R/L$, and $\omega_0^2 = 1/(LC)$

The complete solution of this equation is

$$\boxed{q = e^{-bt}\left(C_0 e^{n_0 t} + D_0 e^{-n_0 t}\right) + \frac{E}{L\omega_0^2}}$$

where C_0 and D_0 are constants and $n_0 = \sqrt{b^2 - \omega_0^2}$.

The second term on the on the right-hand side of the eq. is equal to *EC*, the final steady charge on the capacitor when it is charged by a battery of emf *E*. Taking $EC = q_0$, we can then write as,

$$q = e^{-bt}\left(C_0 e^{n_0 t} + D_0 e^{-n_0 t}\right) + q_0$$

To determine the constants, C_0 and D_0, we note that when $t=0$, $q=0$. Therefore, $C_0 + D_0 + q_0 = 0$

Again, at $t=0$, $i = dq/dt = 0$. So, differentiating w.r.t t and using this condition, we can get

$$n_0(C_0 - D_0) - b(C_0 + D_0) = 0.$$

Solving these Eqs. For C_0 and D_0 gives,

$$C_0 = -\frac{q_0}{2}\left(1 + \frac{b}{n_0}\right) \quad \text{and} \quad D_0 = -\frac{q_0}{2}\left(1 - \frac{b}{n_0}\right)$$

Putting these values of C_0 and D_0 , we get,

$$\boxed{q = -\frac{q_0}{2} e^{-bt}\left[\left(1 + \frac{b}{n_0}\right) e^{n_0 t} + \left(1 - \frac{b}{n_0}\right) e^{-n_0 t}\right] + q_0}$$

where

$$2b = R/L$$

$$\omega_0^2 = 1/(LC)$$

$$n_0 = \sqrt{b^2 - \omega_0^2}$$

$$q_0 = EC$$

There are three cases of interest: $b^2 > \omega_0^2$, $b^2 = \omega_0^2$, $b^2 < \omega_0^2$

Case 1: $b^2 > \omega_0^{\,2}$

Here, $b^2 > \omega_0^{\,2}$ means $(R^2/4L^2) > (1/LC)$ and $n_0 = \sqrt{b^2 - \omega_0^{\,2}}$ is real

The solution of q shows that in this case the capacitor charge increases exponentially and approaches q_0 asymptotically.

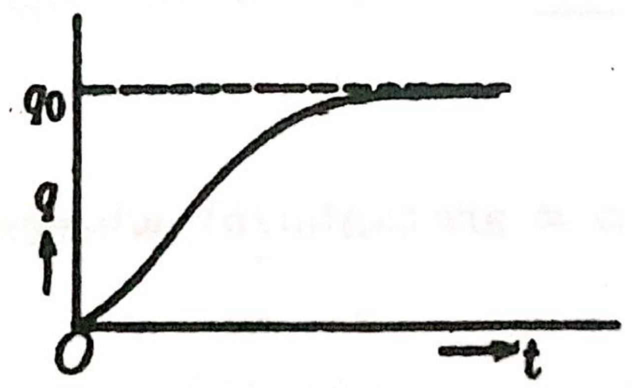


Fig.14.13. Variation of q with t in the overdamped case

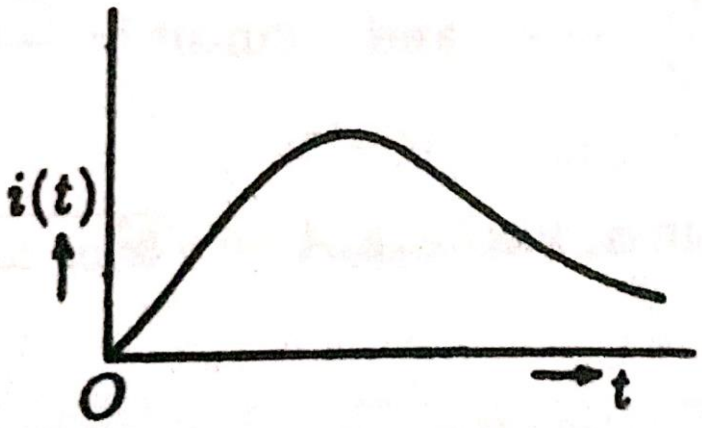


Fig.14.14. Current response for the overdamped case

The Current is given by

$$i(t) = \frac{dq}{dt} = \frac{E}{n_0 L} e^{-bt} \sinh n_0 t,$$

The circuit is said to be overdamped in this case

Case 2: $b^2 = \omega_0{}^2$

Here, $b^2 = \omega_0{}^2$ means $\left(\frac{R^2}{4L^2}\right) = (1/LC)$ and $n_0 = \sqrt{b^2 - \omega_0{}^2} = 0$

In this case, the solution of the KVL differential equation is

$$q = (A_1 + A_2 t)e^{-bt} + \frac{E}{L\omega_0^2}$$

where A_1 and A_2 are constants. With the conditions that at $t = 0$, $q = 0$, and $dq/dt = 0$, we can get

$$A_1 = -\frac{E}{L\omega_0^2} \quad \text{and} \quad A_2 = -\frac{bE}{L\omega_0^2}$$

Therefore, $\boxed{q = q_0\left[1 - (1 + bt)e^{-bt}\right]}$ current, $\boxed{i(t) = \frac{E}{L} t e^{-\omega_0 t}}$

The circuit is now said to be critically damped

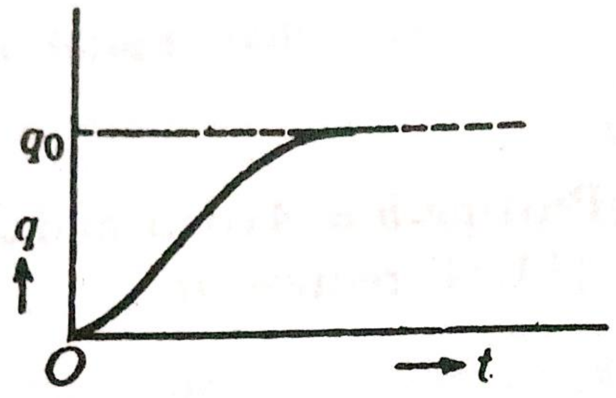


Fig.14.15. Variation of q with t for the critically damped case

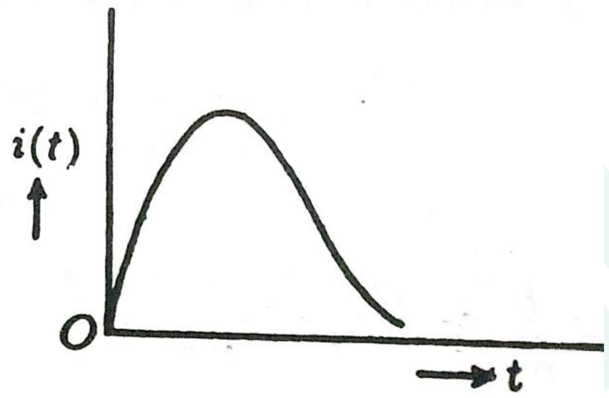


Fig.14.16. Current response for the critically damped case

Case 3: $b^2 < \omega_0{}^2$

Here, $b^2 < \omega_0{}^2$ means $\left(\frac{R^2}{4L^2}\right) < (1/LC)$ and $n_0 = \sqrt{b^2 - \omega_0{}^2}$ is imaginary. We can write,
$n_0 = j\sqrt{\omega_0{}^2 - b^2} = j\omega$; then we can write the solution of charge q as,

$$q = q_0\left[1 - \frac{e^{-bt}}{\omega}(\omega\cos\omega t + b\sin\omega t)\right]$$

where we have used the relationships $\cos\omega t = \dfrac{e^{j\omega t} + e^{-j\omega t}}{2}$ and $\sin\omega t = \dfrac{e^{j\omega t} + e^{-j\omega t}}{2j}$

Putting $b = A\cos\alpha$ and $\omega = A\sin\alpha$; we have $A = \sqrt{b^2 + \omega^2}$ and $\alpha = \arctan(\omega/b)$; therefore

$$q = q_0\left[1 - \frac{\sqrt{b^2+\omega^2}}{\omega}e^{-bt}\sin(\omega t + \alpha)\right]$$

$$= q_0\left[1 - \frac{\omega_0 e^{-bt}}{\sqrt{\omega_0^2 - b^2}}\sin\left(\sqrt{\omega_0^2 - b^2}\,t + a\right)\right]$$

The circuit is now said to be underdamped or oscillatory

Contd..

The capacitor charge oscillates about the final value of q_0 (=EC) with decaying amplitude. The angular frequency of the damped sinusoid is $\omega = \sqrt{{\omega_0}^2 - b^2}$.

The rate of decay of the amplitude with time is determined by the term $e^{-bt}(= e^{-R/_{2L}t})$, the associated log decrement being RT/2L, where T is the period of oscillation.

The frequency of oscillation of the capacitor charge is

$$f = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{\sqrt{\omega_0^2 - b^2}}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

The current in the circuit is given by,

$$i = \frac{dq}{dt} = \frac{q_0\omega_0^2}{\sqrt{\omega_0^2 - b^2}} e^{-bt} \sin\sqrt{\omega_0^2 - b^2}\,t$$

The current is oscillatory with the same frequency as the charge oscillates

Contd..

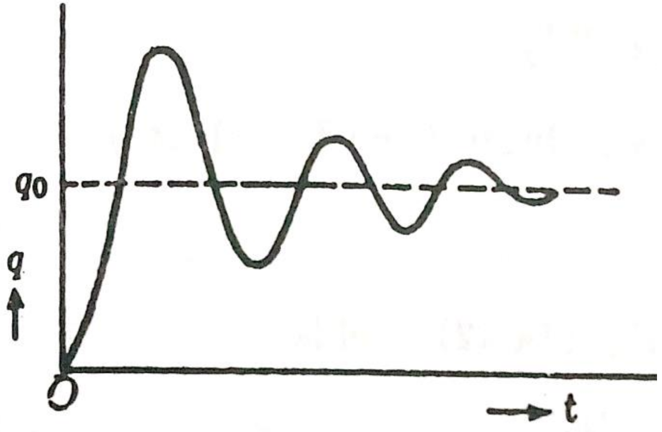


Fig.14.17. Variation of q with t for the underdamped case

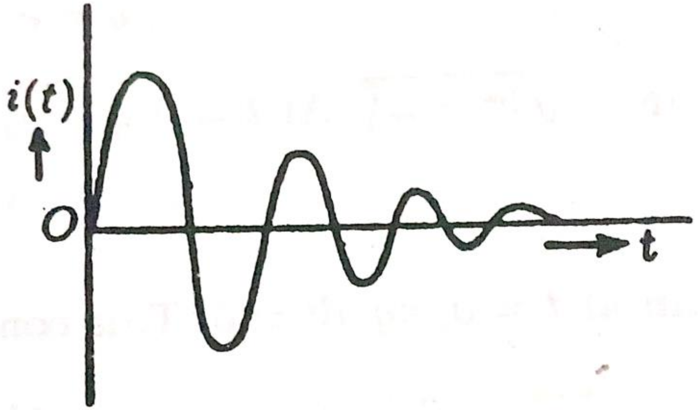


Fig.14.18. Current response for the underdamped case

The critical damping considered in case 2 is basically the transition from non-oscillatory behaviour considered in case 1 to the oscillatory behaviour as discussed here.

When R = 0 (which may happen at very low temperatures where the materials become superconducting), then b = 0 and $\omega=\omega_0$, so the charge and current (in this case 3) reduces to

$$q = q_0(1 - \cos\omega_0 t)$$

$$\text{and} \quad i = q_0\omega_0 \sin\omega_0 t.$$

The charge and current oscillations are now undamped or sustained with the angular frequency $\omega_0=1/\sqrt{LC}$. Therefore, ω_0 is called undamped natural frequency of the circuit. The charge q oscillates about q_0 with an amplitude q_0, so that the maximum and the minimum values of q are $2q_0$ and 0, respectively.

The oscillatory and non-oscillatory variation of current in series RLC has important application in radio-communication systems.

Discharging of the capacitor

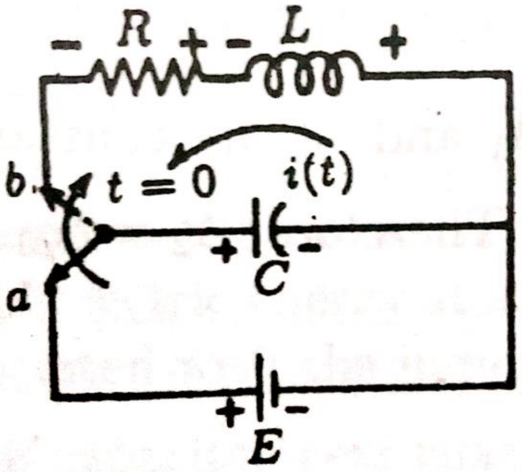


Fig.14.19. Discharging of a capacitor in a series RLC circuit

$$Ri + L\frac{di}{dt} + \frac{q}{C} = 0$$

$$\frac{d^2q}{dt^2} + 2b\frac{dq}{dt} + \omega_0^2 q = 0$$

where $2b = R/L$ and $\omega_0^2 = 1/\sqrt{LC}$

The solution of the differential equation is

$$\boxed{q = e^{-bt}\left(C_0 e^{n_0 t} + D_0 e^{-n_0 t}\right) \quad \text{where } n_0 = \sqrt{b^2 - \omega_0^2}}$$

At $t = 0, q = q_0 = EC$, so that we obtain

$$C_0 + D_0 = q_0.$$

Again at $t = 0$, $dq/dt = 0$. This condition, yields

$$n_0(C_0 - D_0) = b(C_0 + D_0)$$

Contd..

Solving for C_0 and D_0 we obtain

$$C_0 = \frac{q_0}{2}\left(1 + \frac{b}{n_0}\right) \quad \text{and} \quad D_0 = \frac{q_0}{2}\left(1 - \frac{b}{n_0}\right)$$

Substituting these values , we have

$$\boxed{q = \frac{q_0}{2} e^{-bt}\left[\left(1 + \frac{b}{n_0}\right)e^{n_0 t} + \left(1 - \frac{b}{n_0}\right)^{-n_0 t}\right]}$$

where $n_0 = \sqrt{b^2 - \omega_0^2}$

$2b = R/L$ and $\omega_0^2 = 1/\sqrt{LC}$

Again, there are three cases of interest: $b^2 > \omega_0{}^2$, $b^2 = \omega_0{}^2$, $b^2 < \omega_0{}^2$

over damped | critically damped | under damped

Case 1: $b^2 > \omega_0{}^2$

Here, $b^2 > \omega_0{}^2$ means $(R^2/4L^2) > (1/LC)$ and $n_0 = \sqrt{b^2 - \omega_0{}^2}$ is real and the capacitor charge will die out with time. This is overdamped case.

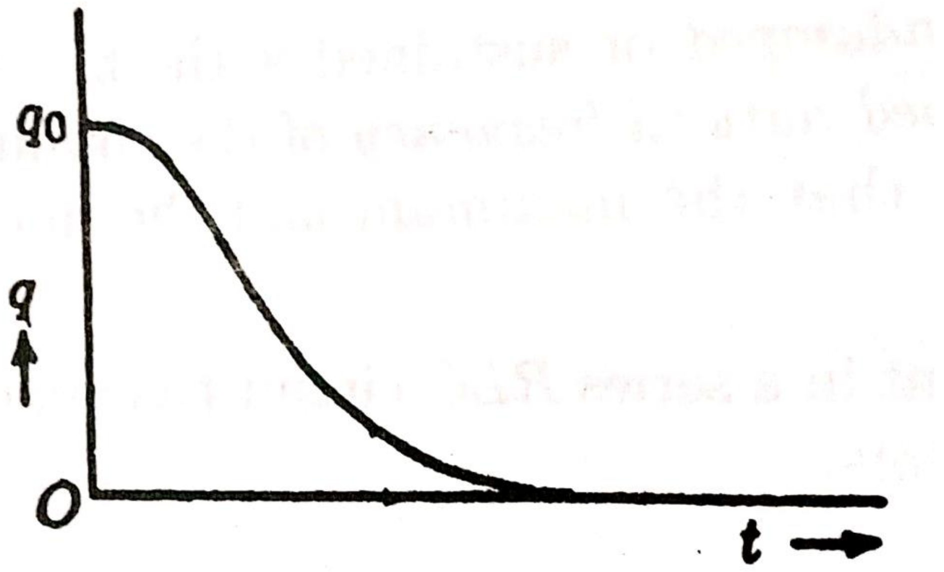


Fig.14.20. Capacitor discharge in the overdamped case

Case 2: $b^2 = \omega_0{}^2$

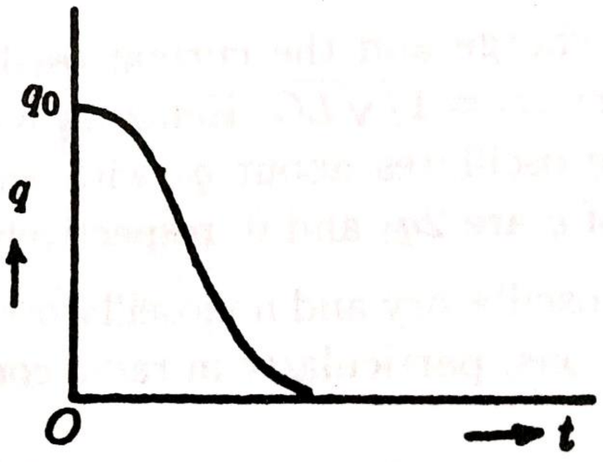


Fig.14.21. Capacitor discharge in the critically damped case

Here, $b^2 = \omega_0{}^2$ means $\left(\frac{R^2}{4L^2}\right) = (1/LC)$ and $n_0 = \sqrt{b^2 - \omega_0{}^2} = 0$

In this case, the solution of the differential equation is

$$q = (A_1 + A_2 t)e^{-bt}$$

where A_1 and A_2 are constants.

At $t = 0$, $q = 0$, so, $A_1 = q_0$

Again, at $t = 0$, $i = dq/dt = 0$, therefore, $A_2 = bq_0$

Therefore,

$$q = q_0(1 + bt)e^{-bt}$$

This is critically damped case. The resistance in this case is $R = 2\sqrt{L/C}$ which is known as critical damping resistance.

The current is $i = \dfrac{dq}{dt} = -q_0 b^2 t e^{-bt}$

The critically damped current has its maximum value at time t_0 which can be obtained by $di/dt = 0$. Therefore, $t_0 = 1/b = 2L/R$

Check for the solution of 2nd ODE with repeated root at: https://tutorial.math.lamar.edu/classes/de/RepeatedRoots.aspx

Case 3: $b^2 < \omega_0{}^2$

Here, $b^2 < \omega_0{}^2$ means $\left(\frac{R^2}{4L^2}\right) < (1/LC)$ and $n_0 = \sqrt{b^2 - \omega_0{}^2}$ is imaginary. We can write, $n_0 = j\sqrt{\omega_0{}^2 - b^2} = j\omega$; then we can write the solution of charge q as,

$$q = \frac{q_0}{\omega} e^{-bt} (\omega \cos \omega t + b \sin \omega t)$$

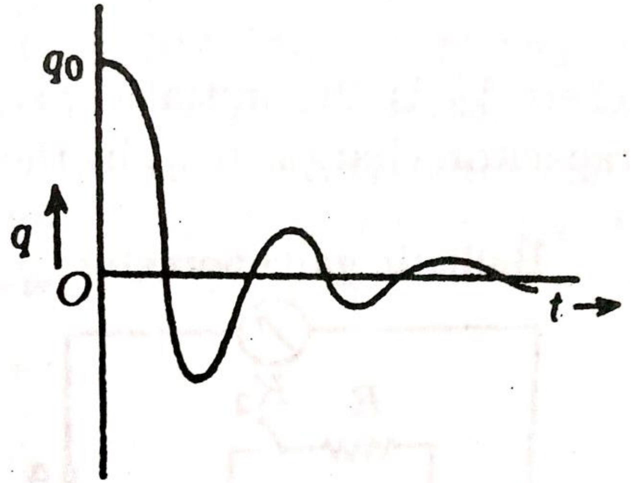


Fig.14.22. Oscillatory discharge of the capacitor

Similar to earlier case (i.e. in case of charging), we can also write in the following form

$$q = \frac{q_0 \omega_0}{\omega} e^{-bt} \sin(\omega t + \alpha)$$ where

$$\alpha = \arctan(\omega/b)$$

The instantaneous current *i(t)* is given by $i = \frac{dq}{dt} = -\frac{q_0}{\omega} \omega_0^2 e^{-bt} \sin \omega t$

Both the current and charge oscillate with decreasing amplitude with time with angular frequency ω. The rate of diminution of amplitude is governed by the term e^{-bt} and the corresponding log decrement is RT/2L.

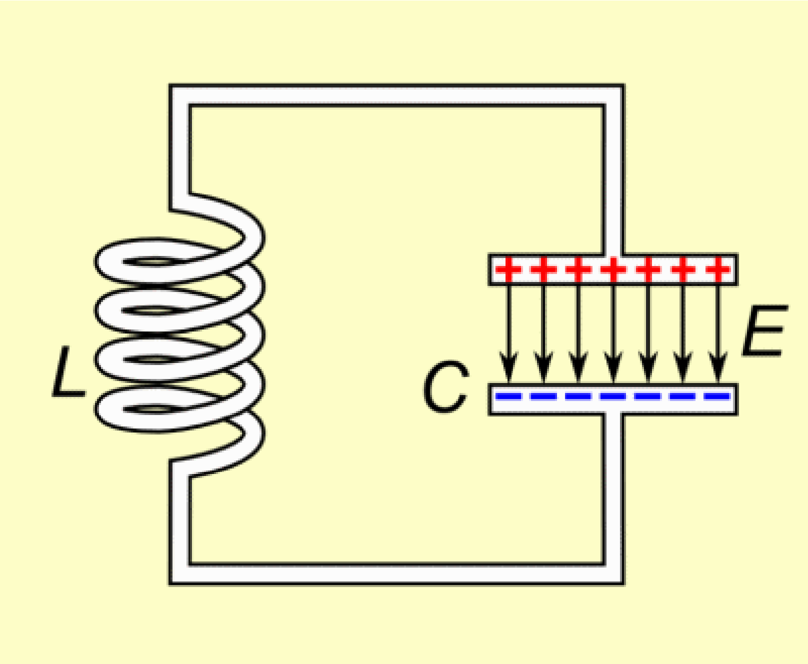
Undamped charge and current

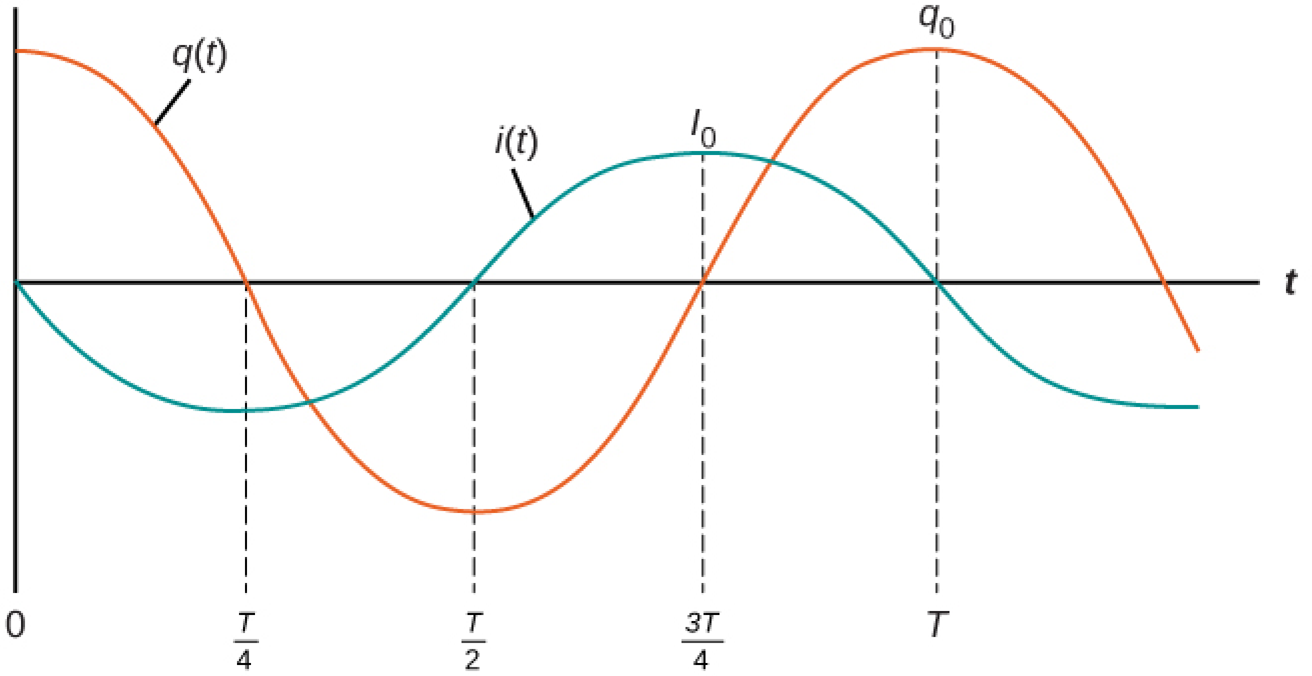
If $R = 0$, then the charge and current expression reduces to

$$q = q_0 \cos \omega_0 t,$$

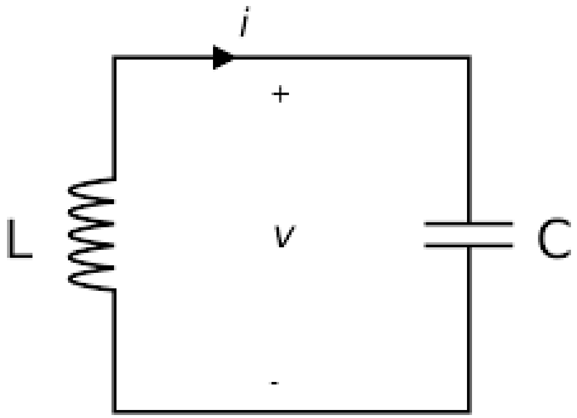
$$\text{and} \quad i = -q_0 \omega_0 \sin \omega_0 t$$

the charge and current oscillations are now undamped with angular frequency $\omega_0 = 1/\sqrt{LC}$





LC circuit natural response



By Kirchhoff's voltage law,

$$V_C + V_L = 0.$$

Likewise, by Kirchhoff's current law,

$$I_C = I_L.$$

$$V_L(t) = L\frac{\mathrm{d}I_L}{\mathrm{d}t},$$

$$I_C(t) = C\frac{\mathrm{d}V_C}{\mathrm{d}t}.$$

the second order differential equation

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}I(t) + \frac{1}{LC}I(t) = 0$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}I(t) + \omega_0^2 I(t) = 0.$$

The parameter ω_0,

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

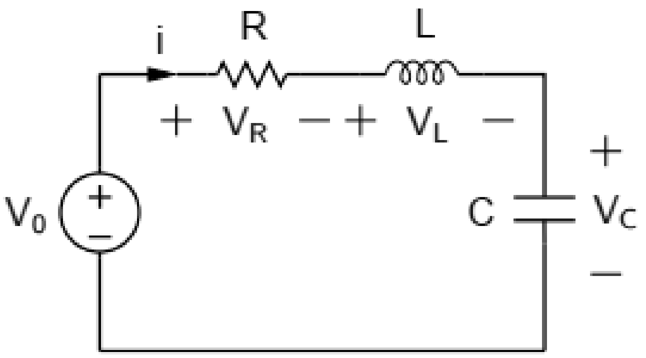
Solution

$$I(t) = Ae^{+j\omega_0 t} + Be^{-j\omega_0 t}$$

Homework: Do the analysis DC transient response of forced LC circuit (both series and parallel) and check the behavior

Another approach of series RLC circuit: using current

Step response



KVL: $V_R + V_L + V_C = V_0 \Rightarrow i\,R + L\,\frac{di}{dt} + \frac{1}{C}\int i\,dt = V_0$

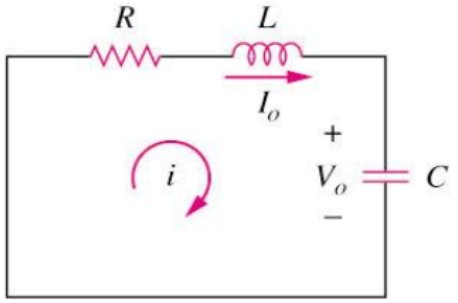
Differentiating w. r. t. t, we get,

$$R\,\frac{di}{dt} + L\,\frac{d^2 i}{dt^2} + \frac{1}{C}\,i = 0.$$

i.e., $\frac{d^2 i}{dt^2} + \frac{R}{L}\,\frac{di}{dt} + \frac{1}{LC}\,i = 0\,,$

a second-order ODE with constant coefficients.

Source free response



- The solution of the source-free series RLC circuit is called as the natural response of the circuit.
- The circuit is excited by the energy initially stored in the capacitor and inductor.

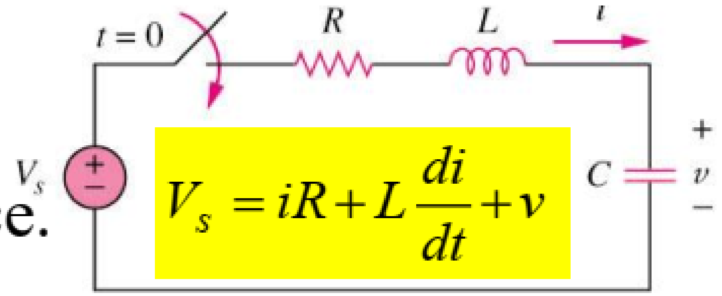
The 2nd order of expression

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0$$

Both are homogeneous 2nd order ODE equations

Another approach of series RLC circuit: using voltage

The step response is obtained by the sudden application of a dc source.



$$V_s = iR + L\frac{di}{dt} + v$$

$$i = C\frac{dv}{dt}$$

$$V_s = RC\frac{dv}{dt} + LC\frac{d^2v}{dt^2} + v$$

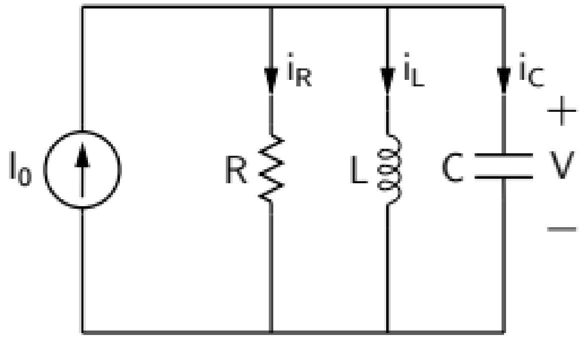
The 2nd order of expression

$$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{v}{LC} = \frac{V_s}{LC}$$

This is inhomogeneous 2nd order ODE equations

Another approach of parallel RLC circuit: using voltage

Step response



KCL: $i_R + i_L + i_C = I_0 \Rightarrow \frac{1}{R} V + \frac{1}{L} \int V\,dt + C \frac{dV}{dt} = I_0$

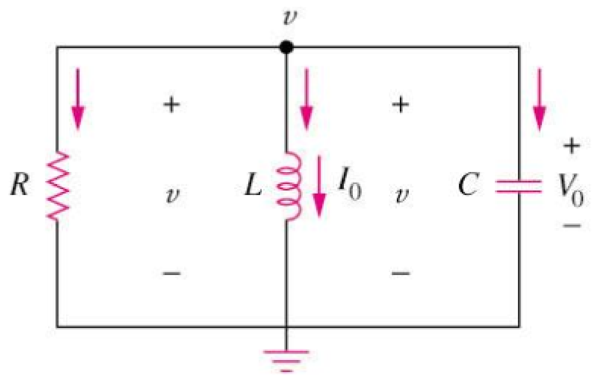
Differentiating w. r. t. t, we get,

$$\frac{1}{R}\frac{dV}{dt} + \frac{1}{L} V + C \frac{d^2V}{dt^2} = 0.$$

i.e., $\frac{d^2V}{dt^2} + \frac{1}{RC}\frac{dV}{dt} + \frac{1}{LC} V = 0$,

a second-order ODE with constant coefficients.

Source free response



Let $i(0) = I_0 = \frac{1}{L}\int_{-\infty}^{0} v(t)dt$

$v(0) = V_0$, Apply KCL to the top node:

$$\frac{v}{R} + \frac{1}{L}\int_{-\infty}^{t} v dt + C\frac{dv}{dt} = 0$$

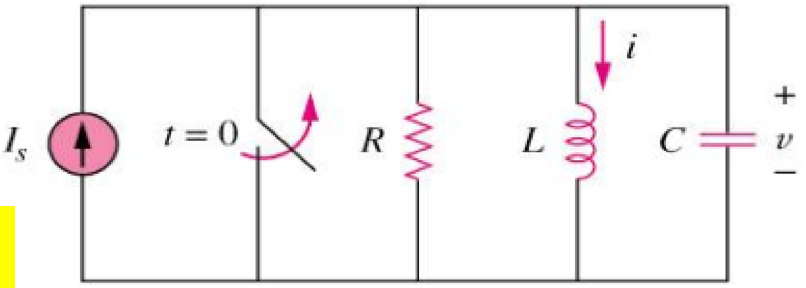
Taking the derivative with respect to t and dividing by C

The 2nd order of expression $\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0$

Both are homogeneous 2nd order ODE equations

Another approach of series RLC circuit: using current

The step response is obtained by the sudden application of a dc source.



$$C\frac{dv}{dt}+\frac{v}{R}+i=I_s$$

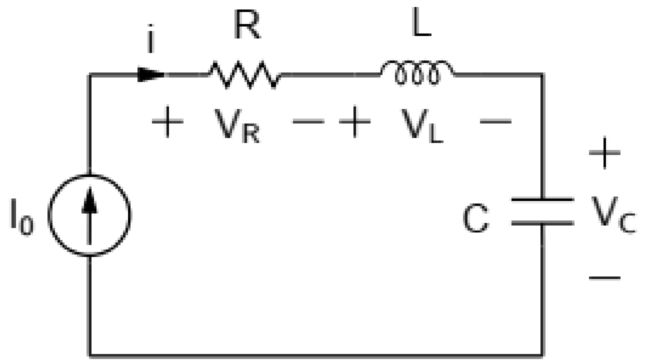
$$v=L\frac{di}{dt}$$

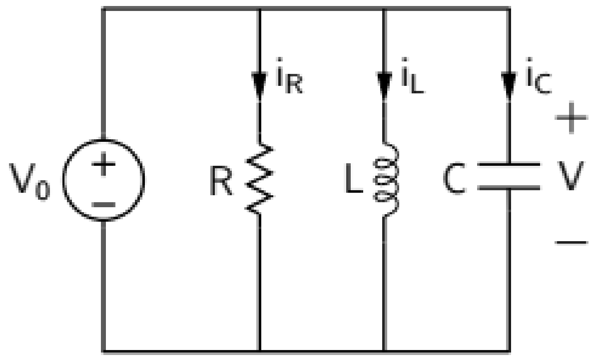
$$LC\frac{d^2i}{dt^2}+\frac{L}{R}\frac{di}{dt}+i=I_s$$

The 2nd order of expression

$$\frac{d^2i}{dt^2}+\frac{1}{RC}\frac{di}{dt}+\frac{i}{LC}=\frac{I_s}{LC}$$

This is inhomogeneous 2nd order ODE equations





* A series RLC circuit driven by a constant current source is trivial to analyze. Since the current through each element is known, the voltage can be found in a straightforward manner.

$$V_R = i\,R,\; V_L = L\,\frac{di}{dt},\; V_C = \frac{1}{C}\int i\,dt\,.$$

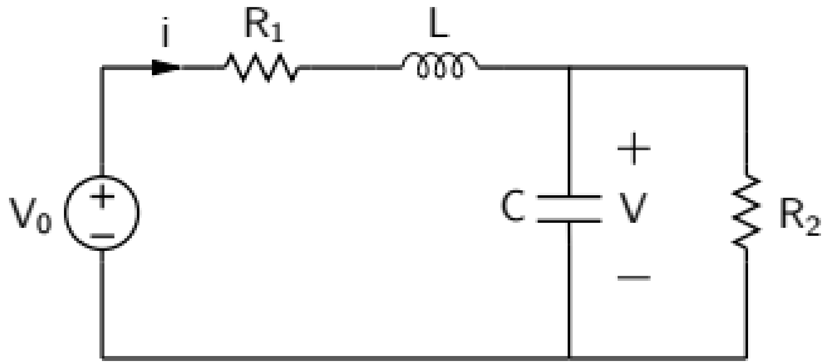
* A parallel RLC circuit driven by a constant voltage source is trivial to analyze. Since the voltage across each element is known, the current can be found in a straightforward manner.

$$i_R = V/R,\; i_C = C\,\frac{dV}{dt},\; i_L = \frac{1}{L}\int V\,dt\,.$$

* The above equations hold even if the applied voltage or current is not constant, and the variables of interest can still be easily obtained without solving a differential equation.

A general RLC circuit

A general *RLC* circuit (with one inductor and one capacitor) also leads to a second-order ODE. As an example, consider the following circuit:



$$V_0 = R_1\, i + L\,\frac{di}{dt} + V \tag{1}$$

$$i = C\,\frac{dV}{dt} + \frac{1}{R_2}\,V \tag{2}$$

Substituting (2) in (1), we get

$$V_0 = R_1\left[CV' + V/R_2\right] + L\left[CV'' + V'/R_2\right] + V\,, \tag{3}$$

$$V''\left[LC\right] + V'\left[R_1C + L/R_2\right] + V\left[1 + R_1/R_2\right] = V_0\,. \tag{4}$$

General method to solve 2nd order ODE

Consider the second-order ODE with constant coefficients,

$$\frac{d^2 y}{dt^2} + a\,\frac{dy}{dt} + b\,y = K\ (\text{constant})\,.$$

The general solution $y(t)$ can be written as,

$$y(t) = y^{(h)}(t) + y^{(p)}(t)\,,$$

where $y^{(h)}(t)$ is the solution of the homogeneous equation,

$$\frac{d^2 y}{dt^2} + a\,\frac{dy}{dt} + b\,y = 0\,,$$

and $y^{(p)}(t)$ is a particular solution.

Since $K =$ constant, a particular solution is simply $y^{(p)}(t) = K/b$.

In the context of RLC circuits, $y^{(p)}(t)$ is the steady-state value of the variable of interest, i.e.,

$$y^{(p)} = \lim_{t\to\infty} y(t),$$

which can be often found by inspection.

Contd..

For the homogeneous equation,

$$\frac{d^2y}{dt^2} + a\,\frac{dy}{dt} + b\,y = 0\,,$$

we first find the roots of the associated *characteristic equation*,

$$r^2 + a\,r + b = 0\,.$$

Let the roots be r_1 and r_2. We have the following possibilities:

* r_1, r_2 are real, $r_1 \neq r_2$ ("overdamped")
  $y^{(h)}(t) = C_1 \exp(r_1 t) + C_2 \exp(r_2 t)\,.$
* r_1, r_2 are complex, $r_{1,2} = \alpha \pm j\omega$ ("underdamped")
  $y^{(h)}(t) = \exp(\alpha t)\,[C_1 \cos(\omega t) + C_2 \sin(\omega t)]\,.$
* $r_1 = r_2 = \alpha$ ("critically damped")
  $y^{(h)}(t) = \exp(\alpha t)\,[C_1\,t + C_2]\,.$

TABLE Summary of Relevant Equations for Source-Free *RLC* Circuits

Type	Condition	Criteria	α	ω_0	Response
Parallel Series	Overdamped	$\alpha > \omega_0$	$\frac{1}{2RC}$ $\frac{R}{2L}$	$\frac{1}{\sqrt{LC}}$	$A_1e^{s_1t} + A_2e^{s_2t}$, where $s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega^2}$
Parallel Series	Critically damped	$\alpha = \omega_0$	$\frac{1}{2RC}$ $\frac{R}{2L}$	$\frac{1}{\sqrt{LC}}$	$e^{-\alpha t}(A_1t + A_2)$
Parallel Series	Underdamped	$\alpha < \omega_0$	$\frac{1}{2RC}$ $\frac{R}{2L}$	$\frac{1}{\sqrt{LC}}$	$e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$, where $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$

Examples 1

1. Determine the complete response of the circuit of Figure 23. Given R = 5 k; C = 1 μF; L = 1 H; Vs = 25 V.

Ans:

Assumptions: The capacitor has been charged (through a separate circuit, not shown) prior to the switch closing, such that $v_C(0) = 5$ V.

Analysis:

1. Apply KVL to determine the circuit differential equation:

$$V_S - v_C(t) - v_R(t) - v_L(t) = 0$$

$$V_S - \frac{1}{C}\int_{-\infty}^{t} i\,dt - iR - L\frac{di}{dt} = 0$$

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i = \frac{1}{L}\frac{dV_S}{dt} = 0 \qquad t > 0$$

We note that the above equations that we have chosen the series (inductor) current as the variable in the differential equation; we also observe that the DC forcing function is zero, because the capacitor acts as an open circuit in the steady state, and the current will therefore be zero as $t \to \infty$.

2. We determine the characteristic polynomial by substituting s for d/dt:

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

$$s_{1,2} = -\frac{L}{2R} \pm \frac{1}{2}\sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}$$

$$= -2{,}500 \pm \sqrt{(5{,}000^2 - 4 \times 10^6}$$

$$s_1 = -208.7; \quad s_2 = -4{,}791.3$$

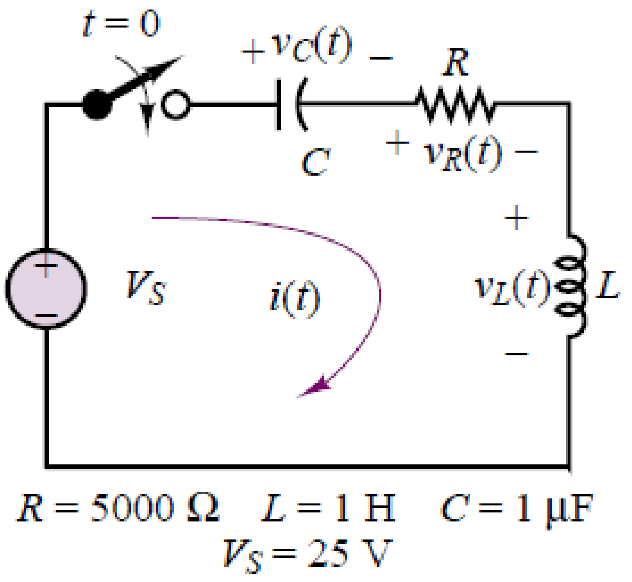


Figure 23

Contd..

These are real, distinct roots, therefore we have an overdamped circuit with natural response given by

$$i_N(t) = K_1 e^{-208.7t} + K_2 e^{-4,791.3t}$$

3. The forced response is zero, as stated earlier, because of the behavior of the capacitor as $t \to \infty$: $F = 0$.
4. The complete solution is therefore equal to the natural response:

$$i(t) = i_N(t) = K_1 e^{-208.7t} + K_2 e^{-4,791.3t}$$

5. The initial conditions for the energy storage elements are $v_C(0^+) = 5$ V; $i_L(0^+) = 0$ A.
6. To evaluate the coefficients K_1 and K_2, we consider the initial conditions $i_L(0^+)$ and $di_L(0^+)/dt$. The first of these is given by $i_L(0^+) = 0$, as stated above. Thus,

$$i(0^+) = 0 = K_1 e^0 + K_2 e^0$$

$$K_1 + K_2 = 0$$

$$K_1 = -K_2$$

To use the second initial condition, we observe that

$$\frac{di}{dt}(0^+) = \frac{di_L}{dt}(0^+) = \frac{1}{L} v_L(0^+)$$

Contd..

and we note that the inductor voltage *can* change instantaneously; i.e., $v_L(0^-) \neq v_L(0^+)$. To determine $v_L(0^+)$ we need to apply KVL once again at $t = 0^+$:

$$V_S - v_C(0^+) - v_R(0^+) - v_L(0^+) = 0$$
$$v_R(0^+) = i(0^+)R = 0$$
$$v_C(0^+) = 5$$

Therefore

$$v_L(0^+) = V_S - v_C(0^+) - v_R(0^+) = 25 - 5 - 0 = 20 \text{ V}$$

and we conclude that

$$\frac{di}{dt}(0^+) = \frac{1}{L} v_L(0^+) = 20$$

Now we can obtain a second equation in K_1 and K_2,

$$\frac{di}{dt}(0^+) = 20 = -208.7K_1e^0 - 4.791.3K_2e^0$$

and since

$$K_1 = -K_2$$
$$20 = 208.7K_2 - 4.791.3K_2$$
$$K_1 = 4.36 \times 10^3$$
$$K_2 = -4.36 \times 10^{-3}$$

Contd..

Finally, the complete solution is:

$$i(t) = 4.36 \times 10^{-3} e^{-208.7t} - 4.36 \times 10^{-3} e^{-4,791.3t} \text{ A}$$

To compute the desired quantity, that is, $v_C(t)$, we can now simply integrate the result above, remembering that the capacitor initial voltage was equal to 5 V:

$$v_C(t) = \frac{1}{C}\int_0^t i(t)dt + v_C(0)$$

$$\frac{1}{C}\int_0^t i(t)dt = 10^6 \left(\int_0^t \left(4.36 \times 10^{-3} e^{-208.7t} - 4.36 \times 10^{-3} e^{-4,791.3t}\right) dt \right)$$

$$= \frac{10^6 \times 4.36 \times 10^{-3}}{(-208.7)} \left[e^{-208.7t} - 1\right]$$

$$- \frac{10^6 \times 4.36 \times 10^{-3}}{(-4,791.3)} \left[e^{-4,791.3t} - 1\right]$$

$$= -20.9e^{-208.7t} + 20.9 + 0.9e^{-4,791.3t} - 0.9 \qquad t > 0$$

$$= 20 - 20.9e^{-208.7t} + 0.9e^{-4,791.3t}$$

$$v_C(t) = 25 - 20.9e^{-208.7t} + 0.9e^{-4,791.3t} \text{ V}$$

Contd..

- The capacitor voltage is plotted in Figure 24.

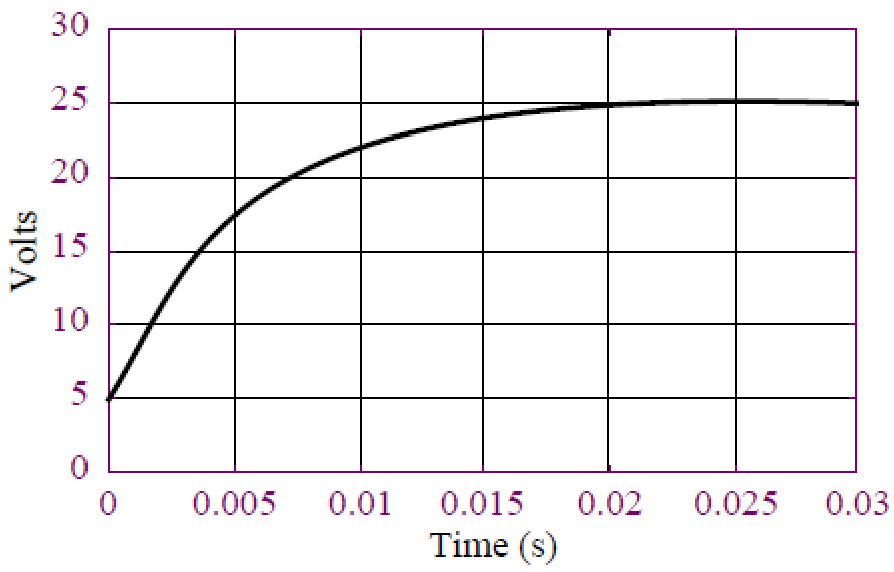


Figure 24

Contd..

2. We determine the characteristic polynomial by substituting s for d/dt:

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$

$$s_{1,2} = -\frac{1}{2RC} \pm \frac{1}{2}\sqrt{\left(\frac{1}{RC}\right)^2 - \frac{4}{LC}}$$

$$= -500 \pm \frac{1}{2}\sqrt{(1,000)^2 - 10^6}$$

$$s_1 = -500 \qquad s_2 = -500$$

These are real, repeated roots, therefore we have a critically damped circuit with natural response given by equation 5.64:

$$v_N(t) = K_1 e^{-500t} + K_2 t e^{-500t}$$

3. The forced response is zero, as stated earlier, because of the behavior of the inductor as $t \to \infty$: $F = 0$.
4. The complete solution is therefore equal to the natural response:

$$v(t) = v_N(t) = K_1 e^{-500t} + K_2 t e^{-500t}$$

Contd..

5. The initial conditions for the energy storage elements are: $v_C(0^+) = 0$ V; $i_L(0^+) = 0$ A.
6. To evaluate the coefficients K_1 and K_2, we consider the initial conditions $v_C(0^+)$ and $dv_C(0^+)/dt$. The first of these is given by $v_C(0^+) = 0$, as stated above. Thus,

$$v(0^+) = 0 = K_1 e^0 + K_2 \times 0 e^0$$

$$K_1 = 0$$

To use the second initial condition, we observe that

$$\frac{dv}{dt}(0^+) = \frac{dv_C(0^+)}{dt} = \frac{1}{C} i_C(0^+)$$

and we note that the capacitor current *can* change instantaneously; i.e., $i_C(0^-) \neq i_C(0^+)$. To determine $i_C(0^+)$ we need to apply KCL once again at $t = 0^+$:

$$I_S - i_L(0^+) - i_R(0^+) - i_C(0^+) = 0$$

$$i_L(0^+) = 0; \quad i_R(0^+) = \frac{v(0^+)}{R} = 0;$$

Therefore

$$i_C(0^+) = I_S - 0 - 0 - 0 = 5 \text{ A}$$

Contd..

and we conclude that

$$\frac{dv}{dt}(0^+) = \frac{1}{C} i_C(0^+) = 5$$

Now we can obtain a second equation in K_1 and K_2,

$$i_C(t) = C\frac{dv}{dt} = C\left[K_1(-500)e^{-500t} + K_2 e^{-500t} + K_2(-500)te^{-500t}\right]$$

$$i_C(0^+) = C\left[K_1(-500)e^0 + K_2 e^0 + K_2(-500)(0)e^0\right]$$

$$5 = C\left[K_1(-500) + K_2\right]$$

$$K_2 = \frac{5}{C} = 2.5 \times 10^6$$

Finally, the complete solution is:

$$v(t) = 2.5 \times 10^6 te^{-500t} \text{ V}$$

Contd..

A plot of the voltage response of this critically damped circuit is shown in Figure 25.

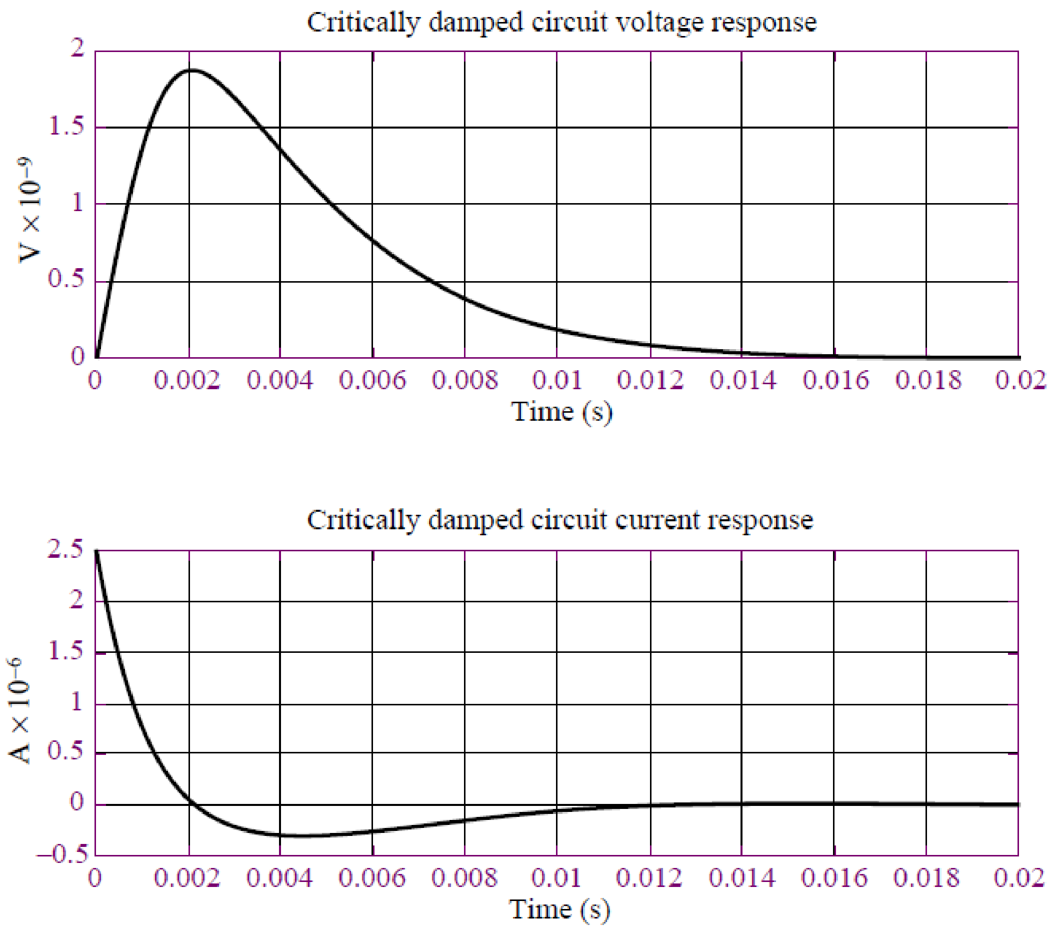


Figure 25

Examples 2

2. Determine the complete response of *v(t)* in the circuit of Figure 24. Given I_S= 5 A; *R* = 500 ; *C* = 2 μF; *L* = 2 H. Ans:

Assumptions: The capacitor voltage and inductor current are equal to zero at $t = 0^+$.

Analysis:

1. Apply KCL to determine the circuit differential equation:

$$I_S - i_L(t) - i_R(t) - i_C(t) = 0$$

$$I_S - \frac{1}{L}\int_{-\infty}^{t} v(t)dt - \frac{v(t)}{R} - C\frac{dv(t)}{dt} = 0$$

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = \frac{1}{C}\frac{dI_S}{dt} = 0 \qquad t > 0$$

We note that the DC forcing function is zero, because the inductor acts as a short circuit in the steady state, and the voltage across the inductor (and therefore across the parallel circuit) will be zero as $t \to \infty$.

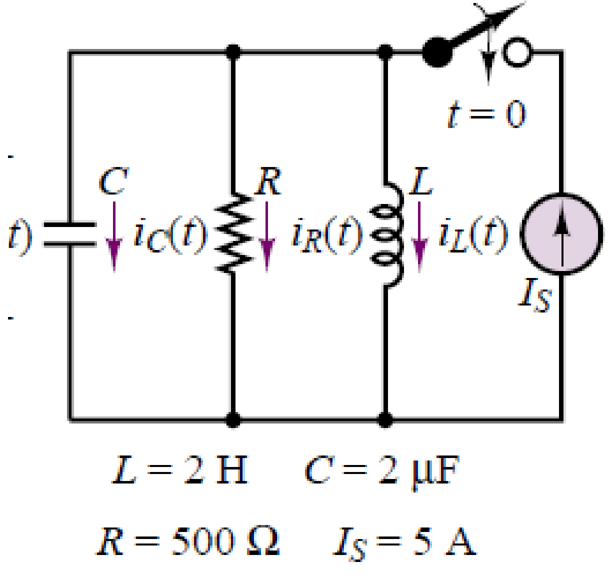


Figure 24

Contd..

2. We determine the characteristic polynomial by substituting s for d/dt:

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$

$$s_{1,2} = -\frac{1}{2RC} \pm \frac{1}{2}\sqrt{\left(\frac{1}{RC}\right)^2 - \frac{4}{LC}}$$

$$= -500 \pm \frac{1}{2}\sqrt{(1,000)^2 - 10^6}$$

$$s_1 = -500 \qquad s_2 = -500$$

These are real, repeated roots, therefore we have a critically damped circuit with natural response given by equation 5.64:

$$v_N(t) = K_1 e^{-500t} + K_2 t e^{-500t}$$

3. The forced response is zero, as stated earlier, because of the behavior of the inductor as $t \to \infty$: $F = 0$.
4. The complete solution is therefore equal to the natural response:

$$v(t) = v_N(t) = K_1 e^{-500t} + K_2 t e^{-500t}$$

Contd..

5. The initial conditions for the energy storage elements are: $v_C(0^+) = 0$ V; $i_L(0^+) = 0$ A.
6. To evaluate the coefficients K_1 and K_2, we consider the initial conditions $v_C(0^+)$ and $dv_C(0^+)/dt$. The first of these is given by $v_C(0^+) = 0$, as stated above. Thus,

$$v(0^+) = 0 = K_1 e^0 + K_2 \times 0 e^0$$

$$K_1 = 0$$

To use the second initial condition, we observe that

$$\frac{dv}{dt}(0^+) = \frac{dv_C(0^+)}{dt} = \frac{1}{C} i_C(0^+)$$

and we note that the capacitor current *can* change instantaneously; i.e., $i_C(0^-) \neq i_C(0^+)$. To determine $i_C(0^+)$ we need to apply KCL once again at $t = 0^+$:

$$I_S - i_L(0^+) - i_R(0^+) - i_C(0^+) = 0$$

$$i_L(0^+) = 0; \quad i_R(0^+) = \frac{v(0^+)}{R} = 0;$$

Therefore

$$i_C(0^+) = I_S - 0 - 0 - 0 = 5 \text{ A}$$

Contd..

and we conclude that

$$\frac{dv}{dt}(0^+) = \frac{1}{C} i_C(0^+) = 5$$

Now we can obtain a second equation in K_1 and K_2,

$$i_C(t) = C\frac{dv}{dt} = C\left[K_1(-500)e^{-500t} + K_2 e^{-500t} + K_2(-500)te^{-500t}\right]$$

$$i_C(0^+) = C\left[K_1(-500)e^0 + K_2 e^0 + K_2(-500)(0)e^0\right]$$

$$5 = C\left[K_1(-500) + K_2\right]$$

$$K_2 = \frac{5}{C} = 2.5 \times 10^6$$

Finally, the complete solution is:

$$v(t) = 2.5 \times 10^6 te^{-500t} \text{ V}$$

Contd..

A plot of the voltage response of this critically damped circuit is shown in Figure 25.

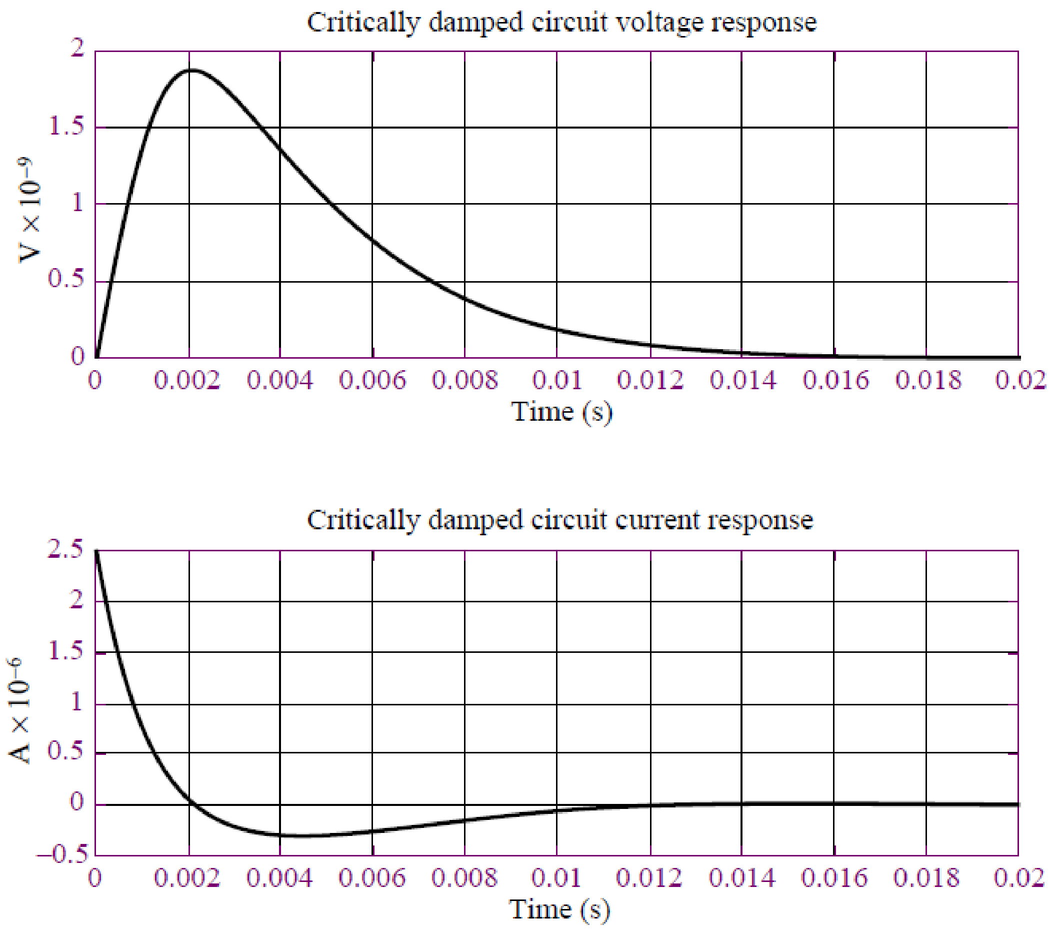


Figure 25

Examples 3

3. Determine the complete response of the circuit of Figure 26. Given $R = 10$; $C = 10\ \mu$F; $L = 5$ mH.

Ans:

Assumptions: No energy is stored in the capacitor and inductor before the switch closes; i.e., $v_C(0^-) = 0$ V; $i_L(0^-) = 0$ A.

Analysis: Since the load voltage is given by the expression $v_{\text{load}} = Ri_L(t)$, we shall solve for the inductor current.

1. Apply KVL to determine the circuit differential equation:

$$V_B - v_L(t) - v_C(t) - v_R(t) = 0$$

$$V_B - L\frac{di_L}{dt} - \frac{1}{C}\int_{-\infty}^{t} i_L dt - i_L R = 0$$

$$\frac{d^2 i_L}{dt^2} + \frac{R}{L}\frac{di_L}{dt} + \frac{1}{LC}i_L = \frac{1}{L}\frac{dV_B}{dt} = 0 \qquad t > 0$$

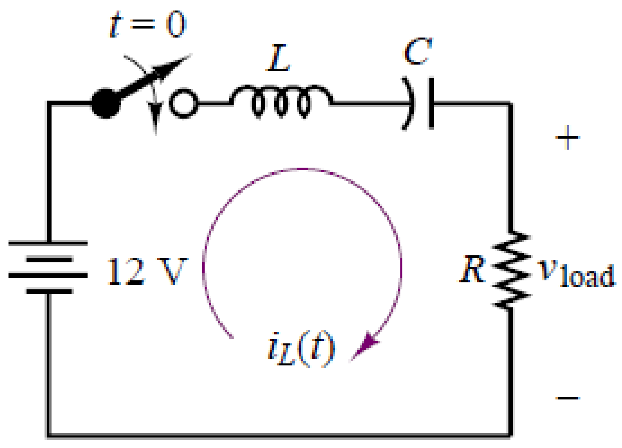


Figure 26

Contd..

2. We determine the characteristic polynomial by substituting s for d/dt:

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

$$s_{1,2} = -\frac{L}{2R} \pm \frac{1}{2}\sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}$$

$$= -1{,}000 \pm j4359$$

These are complex conjugate roots, therefore we have an underdamped circuit with natural response given by equation 5.65:

$$i_{LN}(t) = K_1 e^{(-1{,}000+j4{,}359)t} + K_2 e^{(-1{,}000-j4{,}359)t}$$

3. The forced response is zero, as stated earlier, because of the behavior of the capacitor as $t \to \infty$: $F = 0$.
4. The complete solution is therefore equal to the natural response:

$$i_L(t) = i_{LN}(t) = K_1 e^{(-1{,}000+j4{,}359)t} + K_2 e^{(-1{,}000-j4{,}359)t}$$

Contd..

5. The initial conditions for the energy storage elements are $v_C(0^+) = 0$ V; $i_L(0^+) = 0$ A.
6. To evaluate the coefficients K_1 and K_2, we consider the initial conditions $i_L(0^+)$ and $di_L(0^+)/dt$. The first of these is given by $i_L(0^+) = 0$, as stated above. Thus,

$$i(0^+) = 0 = K_1 e^0 + K_2 e^0$$

$$K_1 + K_2 = 0$$

$$K_1 = -K_2$$

To use the second initial condition, we observe that

$$\frac{di_L}{dt}(0^+) = \frac{1}{L} v_L(0^+)$$

and we note that the inductor voltage *can* change instantaneously; i.e., $v_L(0^-) \neq v_L(0^+)$. To determine $v_L(0^+)$ we need to apply KVL once again at $t = 0^+$:

$$V_S - v_C(0^+) - v_R(0^+) - v_L(0^+) = 0$$

$$v_R(0^+) = i_L(0^+)R = 0; \quad v_C(0^+) = 0$$

Contd..

Therefore

$$v_L(0^+) = V_S - 0 - 0 = 12 \quad \text{V}$$

and

$$\frac{di_L}{dt}(0^+) = \frac{v_L(0^+)}{L} = 2,400$$

Now we can obtain a second equation in K_1 and K_2,

$$\frac{di}{dt}(0^+) = (-1,000 + j4,359)K_1e^0 - (-1,000 - j4,359)K_2e^0$$

and since

$$K_1 = -K_2$$

$$2,400 = K_1\left[(-1,000 + j4,359) - (-1,000 - j4,359)\right]$$

$$K_1 = \frac{2,400}{j8,718} = -j0.2753$$

$$K_2 = -K_1 = j0.2753$$

Contd..

Note that K_1 and K_2 are complex conjugates. Finally, the complete solution is:

$$v_{\text{Load}}(t) = Ri_L(t) = 10\left(-j0.2753e^{(-1,000+j4,359)t} + j0.2753e^{(-1,000-j4,359)t}\right)$$

$$= 2.753e^{-1,000t}\left(-je^{j4,359t} + je^{-j4,359t}\right)$$

$$= 5.506e^{-1,000t}\sin(4,359t)\ \text{V}$$

The output voltage of the circuit is shown in Figure 27.

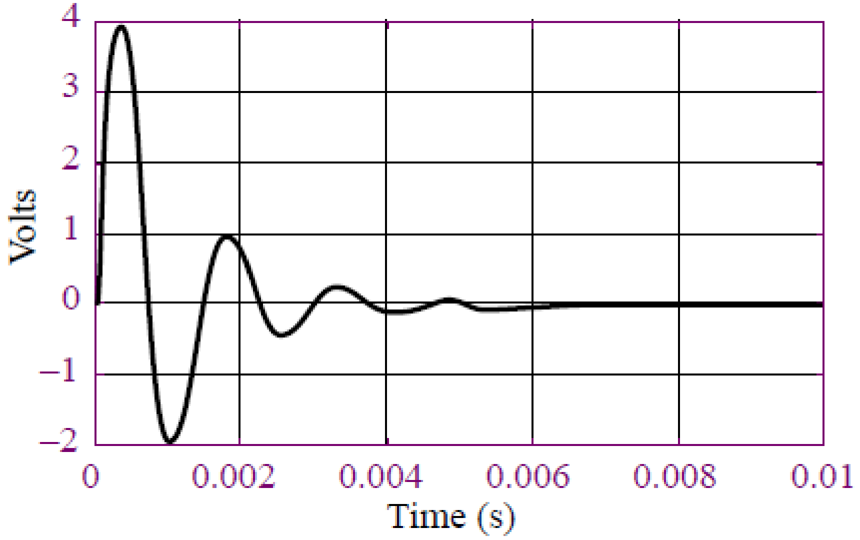


Figure 27

Example

Complete Solution of First-Order Circuit

Problem

Determine an expression for the capacitor voltage in the circuit of Figure 13.

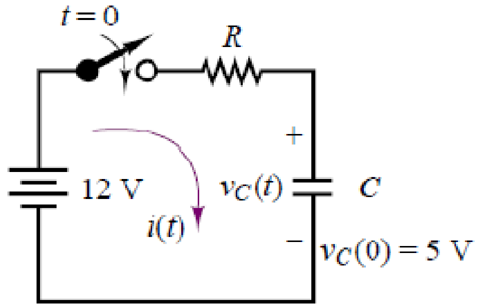


Fig. 13

Solution

Known Quantities: Initial capacitor voltage; battery voltage, resistor and capacitor values.

Find: Capacitor voltage as a function of time, $v_C(t)$, for all t .

Given Data: $v_C(t = 0-) = 5$ V; R = 1 k; C = 470 μF; $V_B = 12$ V.

The capacitor had previously been charged to an initial voltage of 5 V.

So $v_C(t) = 5$ V for $t < 0$

At t = 0 the switch closes, and the circuit is described by the following differential equation, obtained by application of KVL:

$$V_B - RC\frac{dv_C(t)}{dt} - v_C(t) = 9 \qquad t > o$$

$$\frac{dv_C(t)}{dt} - \frac{1}{RC}v_C(t) = \frac{1}{RC}V_B \qquad t > o \qquad (1)$$

Example

We have general first order differential equation as:

$$\frac{dx_F(t)}{dt} - \frac{1}{\tau}x_F(t) = f(t) \qquad (2)$$

Comparing (1) and (2), we get

$x = v_C \qquad \tau = RC \qquad f(t) = \frac{1}{RC}V_B \qquad t > 0$ s

The natural response of the circuit is therefore of the form:

$x_N(t) = v_{CN}(t) = K\, e^{-t/\tau} = K\, e^{-t/RC} \qquad t > 0$ s,

while the forced response is of the form:

$x_F(t) = v_{CF}(t) = \tau_f(t) = V_B \qquad t > 0$ s.

Thus, the complete response of the circuit is given by the expression

$x(t) = v_C(t) = v_{CN}(t) + v_{CF}(t) = K\, e^{-t/RC} + VB \qquad t > 0$ s

Now that we have the complete response, we can apply the initial condition to determine the value of the constant K. At time $t = 0$,

$v_C(0) = 5 = K\, e^{-0/RC} + V_B$

$K = 5 - 12 = -7$ V

We can finally write the complete response with numerical values:

$v_C(t) = -7\, e^{-t/0.47} + 12\, V \qquad t > 0$

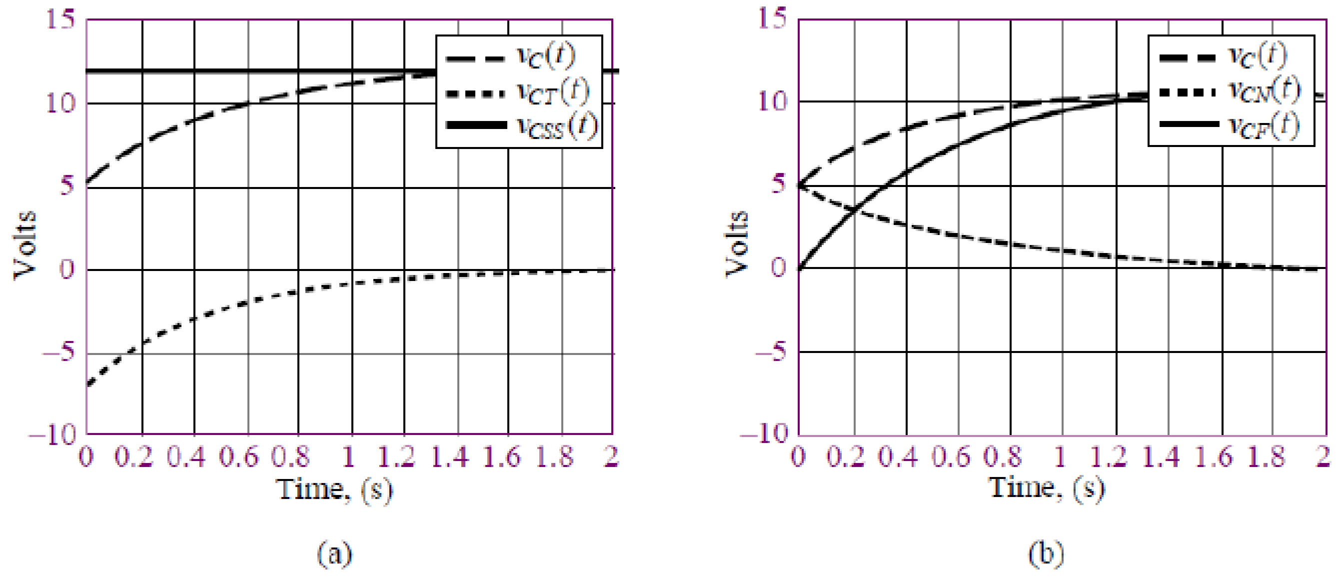
$= v_CT(t) + v_{CSS}(t)$

$= 12\,(1 - e^{-t/0.47}) + 5\, e^{-t/0.47} \qquad t > 0$

$= v_{CF}(t) + v_{CN}(t)$

Example

The complete response described by the above equations is shown graphically in Fig. 14.



Fig, 14 (a) Complete, transient, and steady-state responses of the circuit
(b) Complete, natural, and forced responses of the circuit.

Problems

1. An approximate circuit representation of a DC motor consists of series *RL* circuit, shown in Figure 15. Apply the first-order circuit solution methodology just described to this approximate DC motor equivalent circuit to determine the transient current. Given $i_L(0\text{-}) = 0\text{A}$, R = 4 Ω, L = 0.1 H and $V_B = 50$ V.
2. Determine the motor voltage for all time in the simplified electric motor circuit model shown in Figure 16. The motor is represented by the series *RL* circuit in the shaded box. Given $R_B = 2\ \Omega$; $R_S = 20\ \Omega$; $R_m = 0.8\ \Omega$; $L = 3$ H; $V_B = 100$ V.
3. The circuit of Figure 17 has a switch that can be used to connect and disconnect a battery. The switch has been open for a very long time. At $t = 0$, the switch closes, and then at $t = 50$ ms, the switch opens again. Assume that $R1 = R2 = 1{,}000\ \Omega$,$R3 = 500\ \Omega$, and $C = 25\ \mu\text{F}$.

   a. Determine the capacitor voltage as a function of time.

   b. Plot the capacitor voltage from $t = 0$ to $t = 100$ ms.

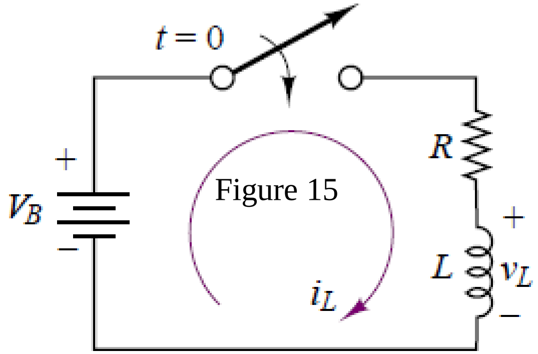


Figure 15

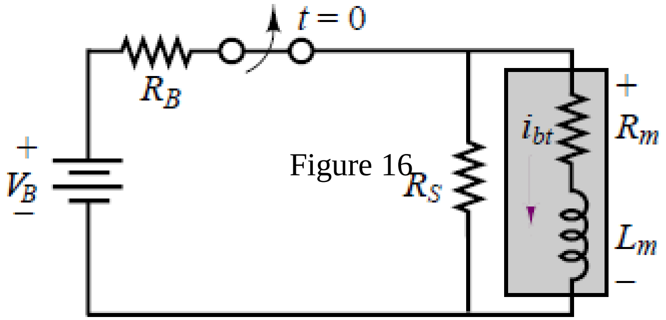


Figure 16

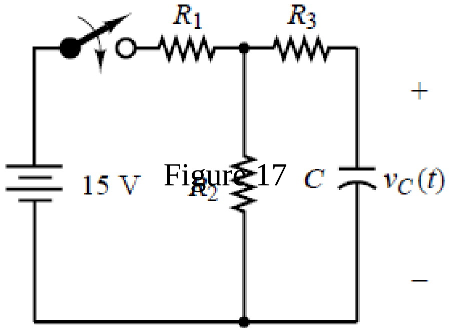


Figure 17

Problems

2. Just before the switch is opened at t = 0, the current through the inductor is 1.70 mA in the direction shown in Figure P.2. Did steady-state conditions exist just before the switch was opened? Given L = 0.9 mH, V_S = 12 V, R_1 =

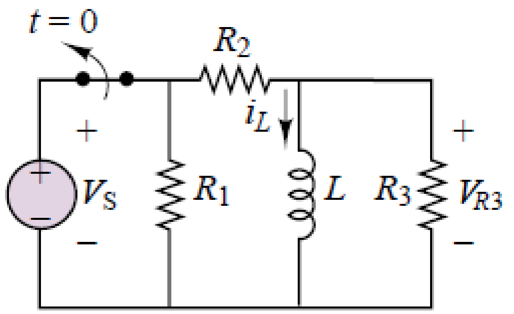


Figure P.2

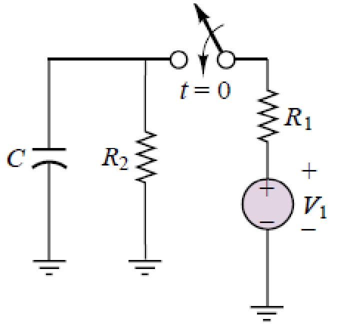


Figure P.3

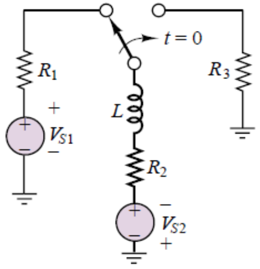


Figure P.4

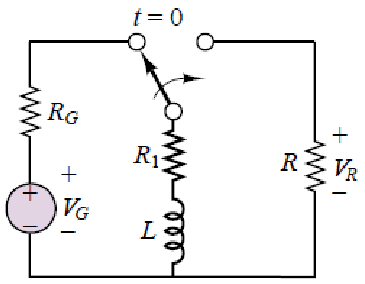


Figure P.5

3. Determine the current through the capacitor just before and just after the switch is closed in Figure P.3. Assume steady-state conditions for $t < 0$. Given V_1 = 12 V, C = 150 micro F, R_1 = 400 mohm, R_2 = 2.2 k.

4. Determine $i_{R3}(t)$ for $t > 0$ in Figure P.4. Given V_{S1} = 23 V, V_{S2} = 20 V, L = 23 mH, R_1 = 0.7 ohm, R_2 = 13 ohm and R_3 =0.33 k.

5. The circuit of Figure P.5 is a simple model of an automotive ignition system. The switch models the "points" that switch electrical power to the cylinder when the fuel-air mixture is compressed. *R* is the resistance between the electrodes (i.e., the "gap") of the spark plug. Determine the value of *L* and *R1* so that the voltage across the spark plug gap just after the switch is changed is 23 kV and so that this voltage will change exponentially with a time constant τ = 13 ms. Given V_G = 12 V, R_G = 0.37 ohm and R = 1.7 k.

Problems

6. For the circuit shown in Figure P.6, assume that switch S_1 is always open and that switch S_2 closes at $t = 0$.
   a. Find the capacitor voltage, $v_c(t)$, at $t = 0+$.
   b. Find time constant, τ, for $t \geq 0$.
   c. Find an expression for $v_c(t)$ and sketch the function and evaluate its value for $t = 0, \tau, 2\tau, 5\tau$ and 10τ.
7. In the circuit shown in Figure P.7, $V_{S1} = 12$ V, $V_{S2} = 9$ V, $R_{S1} = 130\ \Omega$, $R_{S2} = 290\ \Omega$, $R_1 = 1.1$ kΩ, $R_2 = 700\ \Omega$, $L = 17$ mH and $C = 0.35\ \mu$F. Assume that DC steady-state conditions exist for $t < 0$. Determine the voltage across capacitor and the current through the inductor and R_{S2} and I approaches infinity.
8. At $t < 0$, the circuit shown in Figure P.8 is at steady state and the voltage across capacitor is + 7 V. The switch is changed as shown at $t = 0$ and $V_s = 12$ V, $C = 3300\ \mu$F, $R_1 = 9.1$ k, $R_2 = 4.3$ k, $R_3 = 4.3$ k, L – 16 mH. Determine the initial voltage across R_2 just after switch is changed,
9. Assume the circuit of Figure P.9 initially stores no energy, Switch S_1 is open and S_2 is closed. Switch S_1 is closed at $t = 0$ and S_2 is opened at $t = 5$ s. Determine an expression for the capacitor voltage for $t \geq 0$.

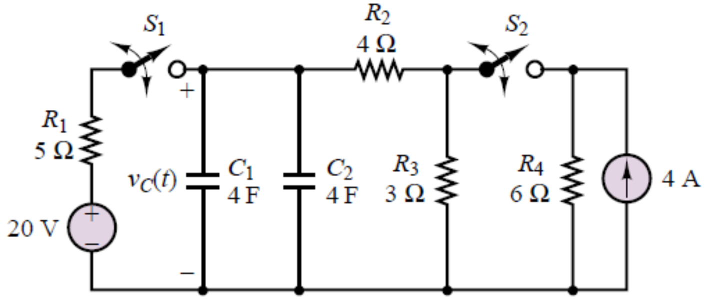


Figure P.6

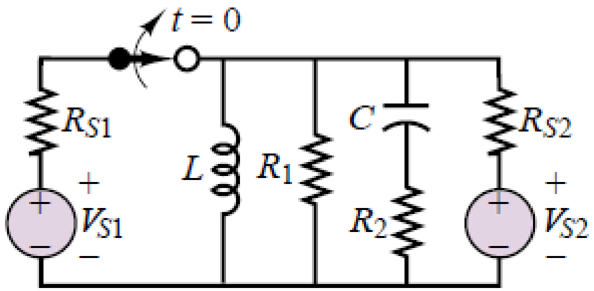


Figure P.7

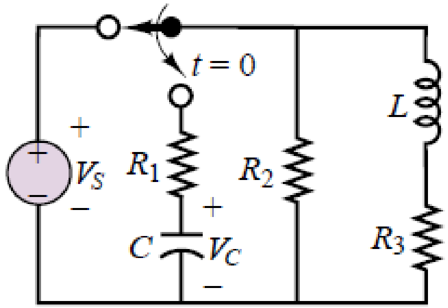


Figure P.8

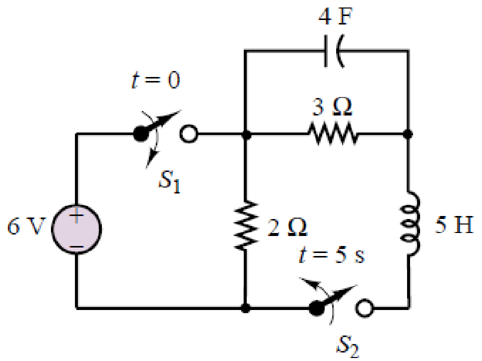


Figure P.9

Problems

10. Find the capacitor voltage $v_c(t)$ and the current $i(t)$ in 200 ohm resistor of Figure P.10 for all time.

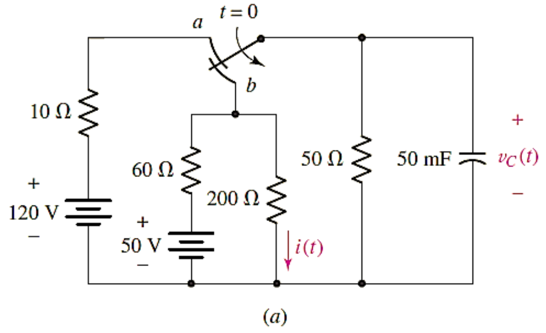


Figure P.10

11. Find an expression for $v_c(t)$ valid for $t > 0$ in circuit of Figure P.11.

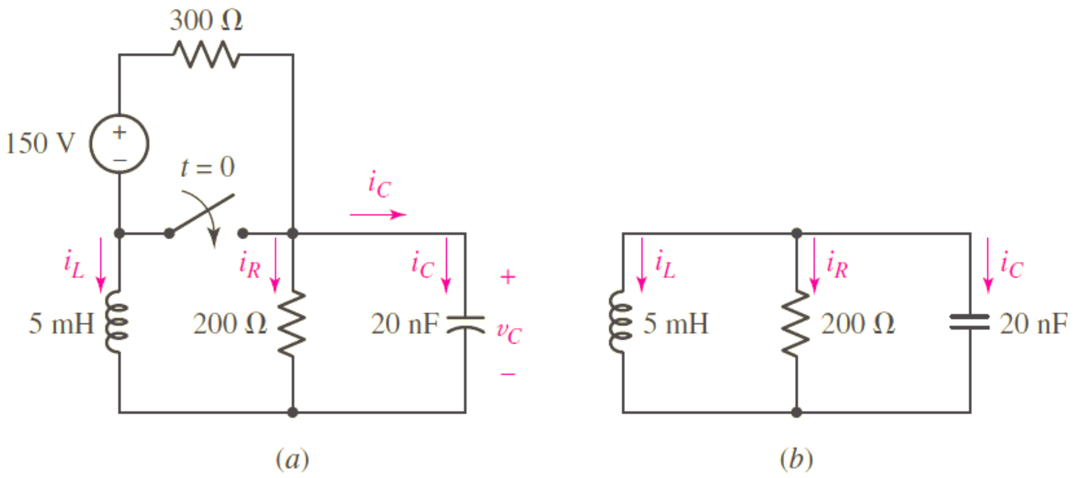


Figure P.11

Overdamped Parallel RLC circuit

1. The circuit of Fig. 1 reduces to a simple parallel RLC circuit after t = 0. Determine an expression for the resistor current i_R valid for all time.

If the circuit after $t > 0$ is overdamped, we expect a response of the form

$$i_R(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}, \qquad t > 0 \qquad [18]$$

For $t > 0$, we have a parallel RLC circuit with $R = 30\ \text{k}\Omega$, $L = 12$ mH, and $C = 2$ pF. Thus, $\alpha = 8.333 \times 10^6\ \text{s}^{-1}$ and $\omega_0 = 6.455 \times 10^6$ rad/s. We do therefore expect an overdamped response, with $s_1 = -3.063 \times 10^6\ \text{s}^{-1}$ and $s_2 = -13.60 \times 10^6\ \text{s}^{-1}$.

To determine numerical values for A_1 and A_2, we first analyze the circuit at $t = 0^-$, as drawn in Fig. 9.6*b*. We see that $i_L(0^-) = i_R(0^-) = 4/32 \times 10^3 = 125\ \mu\text{A}$, and $v_C(0^-) = 4 \times 30/32 = 3.75$ V.

In drawing the circuit at $t = 0^+$ (Fig. 9.6*c*), we only know that $i_L(0^+) = 125\ \mu\text{A}$ and $v_C(0^+) = 3.75$ V. However, by Ohm's law we can calculate that $i_R(0^+) = 3.75/30 \times 10^3 = 125\ \mu\text{A}$, our first initial condition. Thus,

$$i_R(0) = A_1 + A_2 = 125 \times 10^{-6} \qquad [19]$$

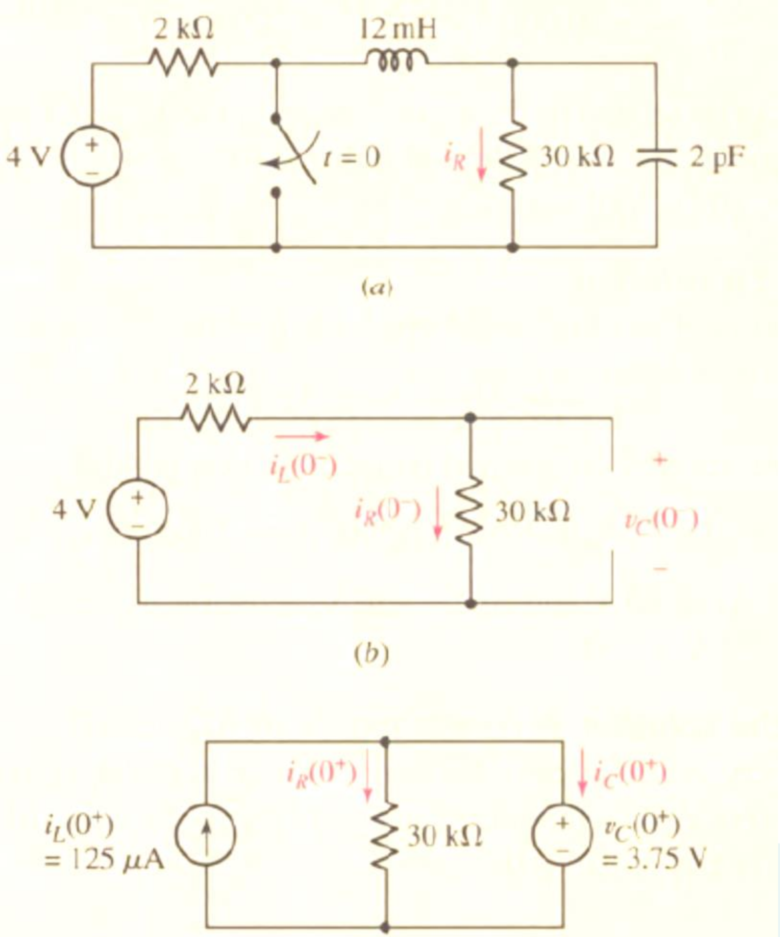


Fig. 1

Overdamped Parallel RLC circuit

How do we obtain a *second* initial condition? If we multiply Eq. [18] by 30×10^3, we obtain an expression for $v_C(t)$. Taking the derivative and multiplying by 2 pF yields an expression for $i_C(t)$:

$$i_C = C\frac{dv_C}{dt} = (2 \times 10^{-12})(30 \times 10^3)(A_1 s_1 e^{s_1 t} + A_2 s_2 e^{s_2 t})$$

By KCL,

$$i_C(0^+) = i_L(0^+) - i_R(0^+) = 0$$

Thus,

$$-(2 \times 10^{-12})(30 \times 10^3)(3.063 \times 10^6 A_1 + 13.60 \times 10^6 A_2) = 0 \qquad [20]$$

Solving Eqs. [19] and [20], we find that $A_1 = 161.3\ \mu\text{A}$ and $A_2 = -36.34\ \mu\text{A}$. Thus,

$$i_R = \begin{cases} 125\ \mu\text{A}, & t < 0 \\ 161.3e^{-3.063\times10^6 t} - 36.34e^{-13.6\times10^6 t}\ \mu\text{A}, & t > 0 \end{cases}$$

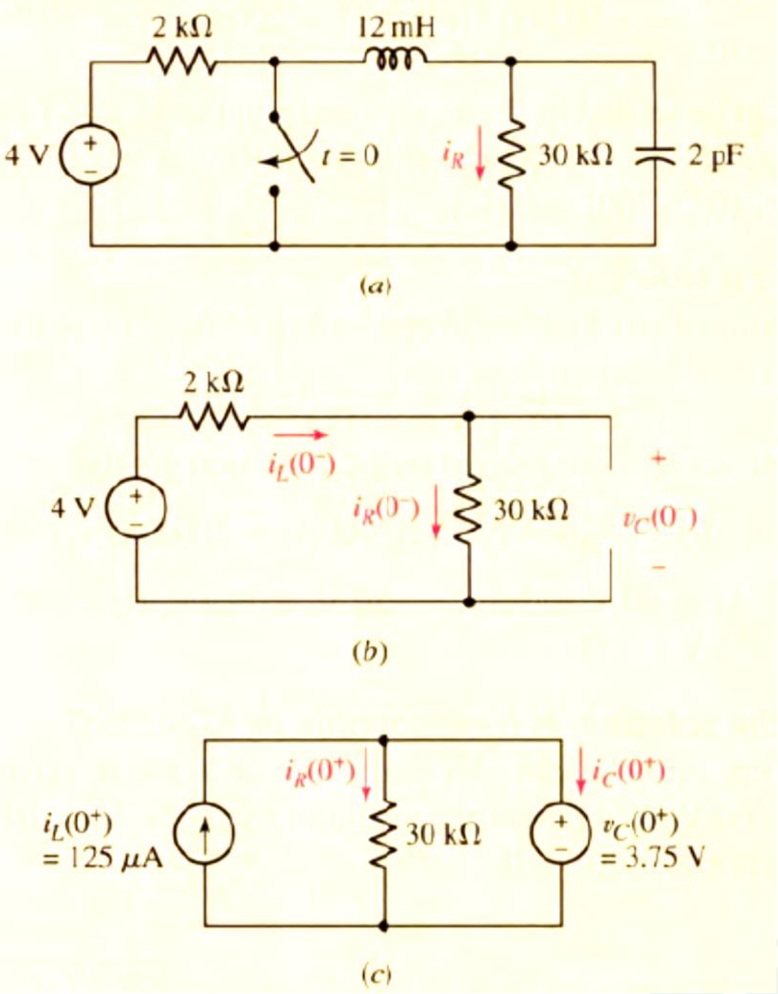


Fig. 1

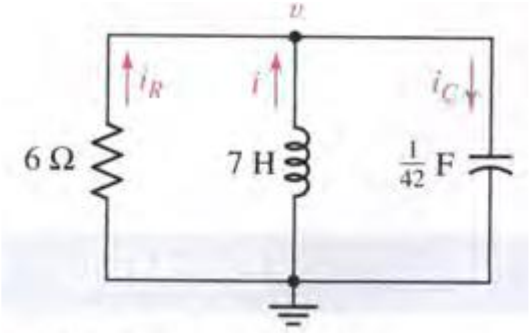
Overdamped Parallel RLC circuit

2. For the circuit shown in Fig. 2, show that the output voltage v(t) is given as $v(t) = 84(e^{-t} - e^{-6t})$ V. The voltage across the capacitor v(t) =0 and current through inductor i(t) = 10 A for t < 0.

We define new term resonant frequency **(ω_0)** = $\frac{1}{\sqrt{LC}}$ and exponential damping factor **(α)** = $\frac{1}{2RC}$. The constant α is a measure how rapidly the natural response decays to its final value.

For the given circuit, we have

$$\alpha = 3.5 \qquad \omega_0 = \sqrt{6} \qquad (\text{all } s^{-1})$$
$$s_1 = -1 \qquad s_2 = -6$$



- Fig. 2

And the general form of natural response is $v(t) = A_1 e^{-t} + A_2 e^{-6t}$

and $\frac{dv}{dt} = -A_1 e^{-t} - 6A_2 e^{-6t}$

The constants A_1 and A_2 is obtained by using the initial conditions.

At $t = 0$

$v(t) = 0$ and $i(t) = 10$ A, thus $A_1 + A_2 = 0$

At $t = 0$, $\frac{dv}{dt} = -A_1 - 6A_2$

One can evaluate $\frac{dv}{dt}$ from the expression $i_C = C\frac{dv}{dt}$

Overdamped Parallel RLC circuit

One can evaluate $\frac{dv}{dt}$ from the expressio $i_C = C\frac{dv}{dt}$

and $$-i_C(0) + i(0) + i_R(0) = 0$$

That is $$\left.\frac{dv}{dt}\right|_{t=0} = \frac{i_C(0)}{C} = \frac{i(0) + i_R(0)}{C} = \frac{i(0)}{C} = 420 \text{ V/s}$$

$$v(t) = 84(e^{-t} - e^{-6t}) \text{ V}$$

Output voltage

The output is maximum at *tm = 0.358 s and vm = 48.9 V*

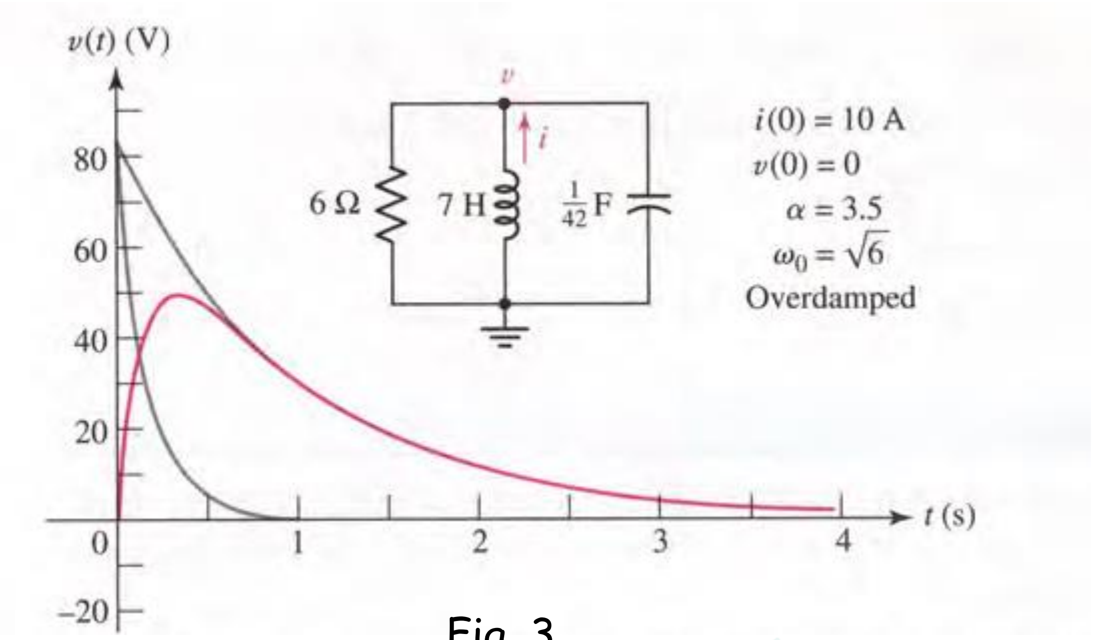


Fig. 3

Settling time

- Time required to reduce the response magnitude to less than 1% of the maximum response magnitude is called the settling time.
- For the previous problem, find maximum V, time when V is max and the settling time.

Settling time Example

For $t > 0$, the capacitor current of a certain source-free parallel *RLC* circuit is given by $i_c(t) = 2e^{-2t} - 4e^{-t}$ A. Sketch the current in the range $0 < t < 5$ s, and determine the settling time.

We have $i_c(t) = 2e^{-2t} - 4e^{-t}$ A

This is plotted in Fig. 4.

and $\frac{di_c}{dt} = -4e^{-2t} - 4e^{-t} = 0$ at $t = tm$.

By iteration we have *ts = 5.296 s.*

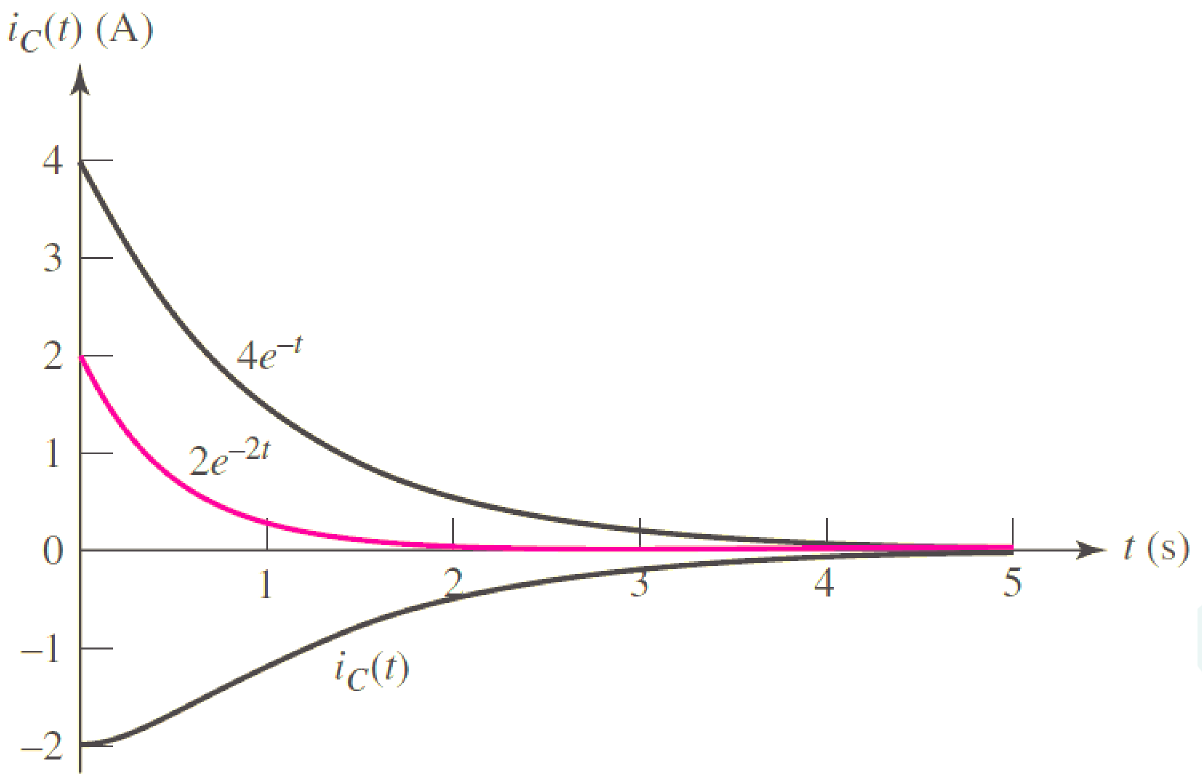


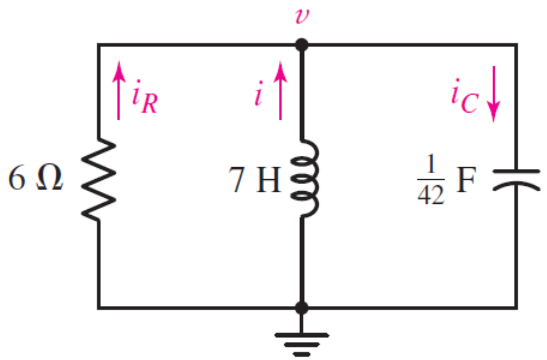
Fig. 4

Critically Damped Parallel R L C

In the circuit shown, find the value of resistance (a replacement for 6 ohm resistance) such that this circuit response will be critically damped. It is given that V(0) = 0 V and i(0) = 10 A.

Ans:

A circuit is said to be critically damped if following conditions are satisfied:



$$\alpha = \omega_0$$

Where $\omega_0 = \frac{1}{\sqrt{L\,C}}$ and $\alpha = \frac{1}{2\,R\,C}$

Thus $LC = 4R^2C^2$

or $L = 4R^2C$

For critical damping R = $\sqrt{\frac{L}{4\,C}}$ Ohm $= \frac{7\sqrt{6}}{2}$ = 8.57 ohm

The following equation describes the behaviour of the circuit

$$\frac{d^2v}{dt^2} + 2\alpha\frac{dv}{dt} + \alpha^2 v = 0$$

The solution to above differential equations is following:

$$v(t) = 420te^{-2.45t}\ \mathrm{V}$$

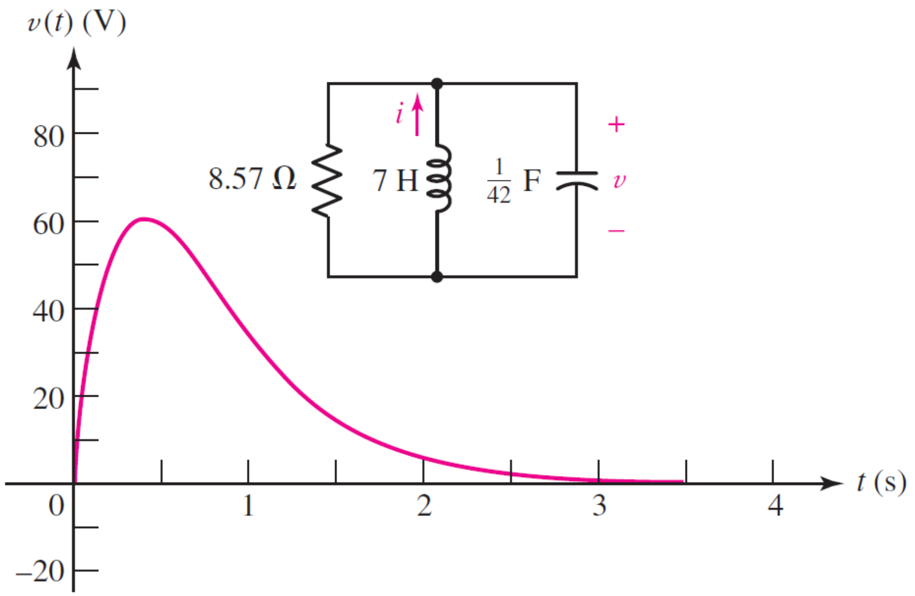
Critically Damped Parallel R L C

The output voltage is given as $v(t) = 420te^{-2.45t}$ V

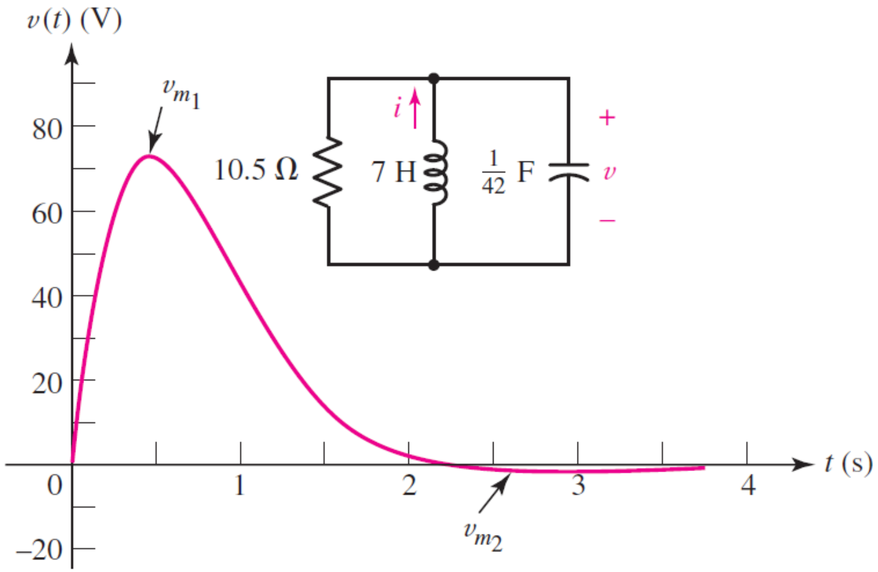
At t = ∞, the value of *v(t)* can be obtained and its equal to *0.*

As described earlier this function will increase and will have a peak and then start decreasing exponentially. The peak will occur at t_m *= 0.406 s and the peak voltage* v_m *would be 63.1 V. Settling time* t_s *will be 3.12 s.*

The v(t) versus time plot is shown below.



Underdamped parallel RLC Circuits



- The form of underdamped response will be given as

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

- Where

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$\sqrt{\alpha^2 - \omega_0^2} = \sqrt{-1}\sqrt{\omega_0^2 - \alpha^2} = j\sqrt{\omega_0^2 - \alpha^2}$$

- And

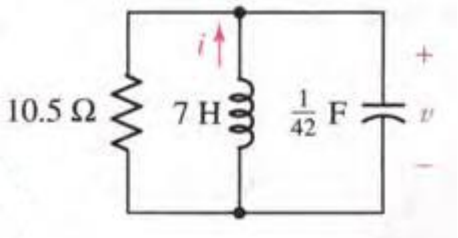
$$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

- The response may be written as $v(t) = e^{-\alpha t}(A_1 e^{j\omega_d t} + A_2 e^{-j\omega_d t})$

$$v(t) = e^{-\alpha t}\left\{(A_1 + A_2)\left[\frac{e^{j\omega_d t} + e^{-j\omega_d t}}{2}\right] + j(A_1 - A_2)\left[\frac{e^{j\omega_d t} - e^{-j\omega_d t}}{j2}\right]\right\}$$

$$v(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$$

Underdamped parallel RLC Circuits



- We consider the circuit shown.
- We have

$$\alpha = \frac{1}{2RC} = 2\ \mathrm{s}^{-1}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \sqrt{6}\ \mathrm{s}^{-1}$$

- And

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{2}\ \mathrm{rad/s}$$

- The response is now

$$v(t) = e^{-2t}(B_1 \cos \sqrt{2}t + B_2 \sin \sqrt{2}t)$$

- We have *v (0) = 0* and

$$v(t) = B_2 e^{-2t} \sin \sqrt{2}t$$

- and

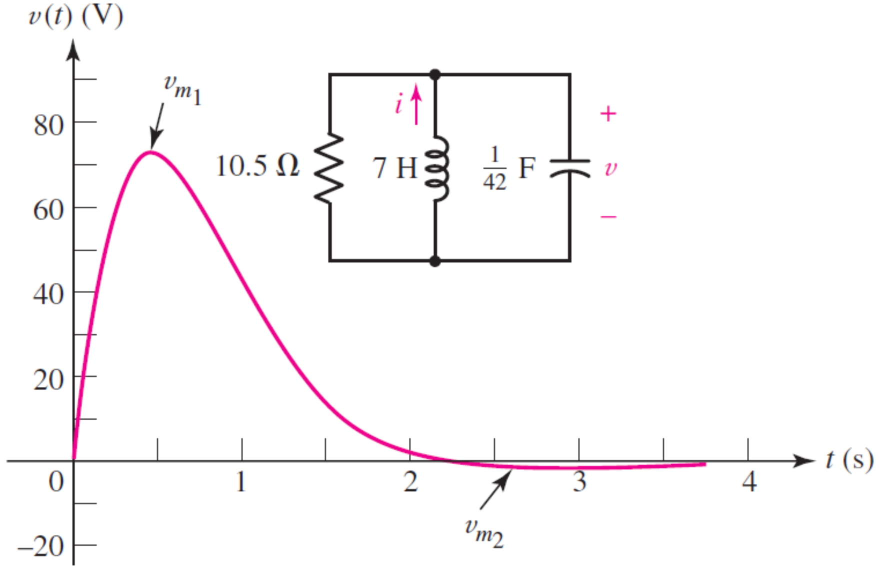
$$\frac{dv}{dt} = \sqrt{2}B_2 e^{-2t} \cos \sqrt{2}t - 2B_2 e^{-2t} \sin \sqrt{2}t$$

- At t = 0,

$$\left.\frac{dv}{dt}\right|_{t=0} = \sqrt{2}B_2 = \frac{i_C(0)}{C} = 420$$

- And

$$v(t) = 210\sqrt{2}e^{-2t} \sin \sqrt{2}t$$



Underdamped parallel RLC Circuits

$$v(t) = 210\sqrt{2}e^{-2t}\sin\sqrt{2}t$$

- The response of this circuit looks like in Fig. a. For different values of R > 10.5 ohm, we find oscillations around v = 0 (Fig. b).

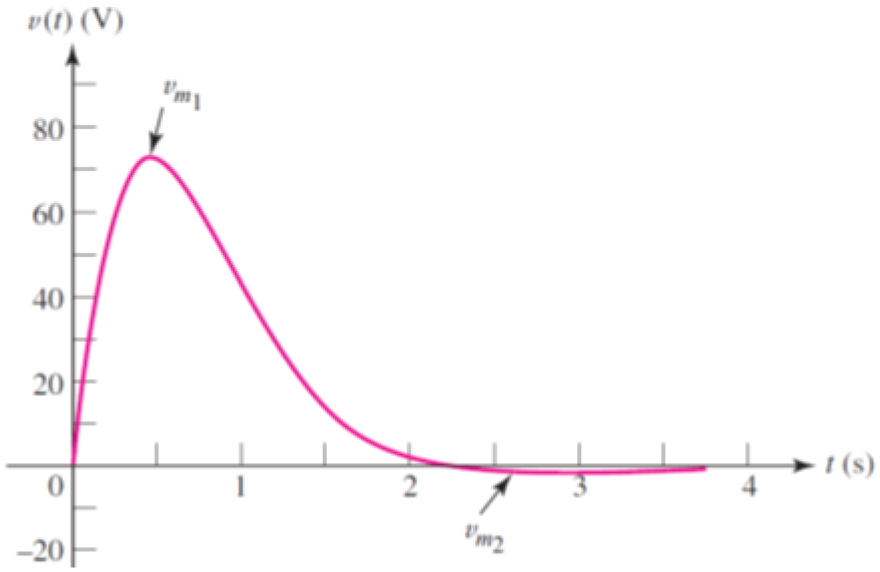


Fig. a

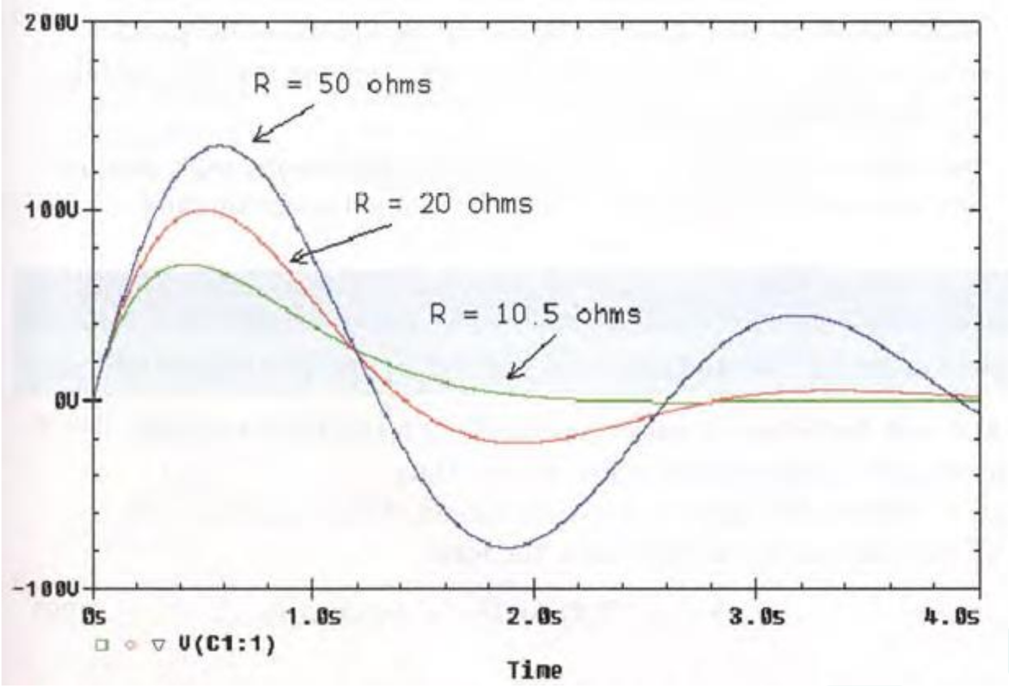


Fig. b

Example

Determine *$i_L(t)$* for the circuit 9.17(a) and plot the waveform.

At $t = 0$, both the 3 A source and the 48 Ω resistor are removed, leaving the circuit shown in Fig. 9.17*b*. Thus, $\alpha = 1.2\ \text{s}^{-1}$ and $\omega_0 = 4.899$ rad/s. Since $\alpha < \omega_0$, the circuit is *underdamped*, and we therefore expect a response of the form

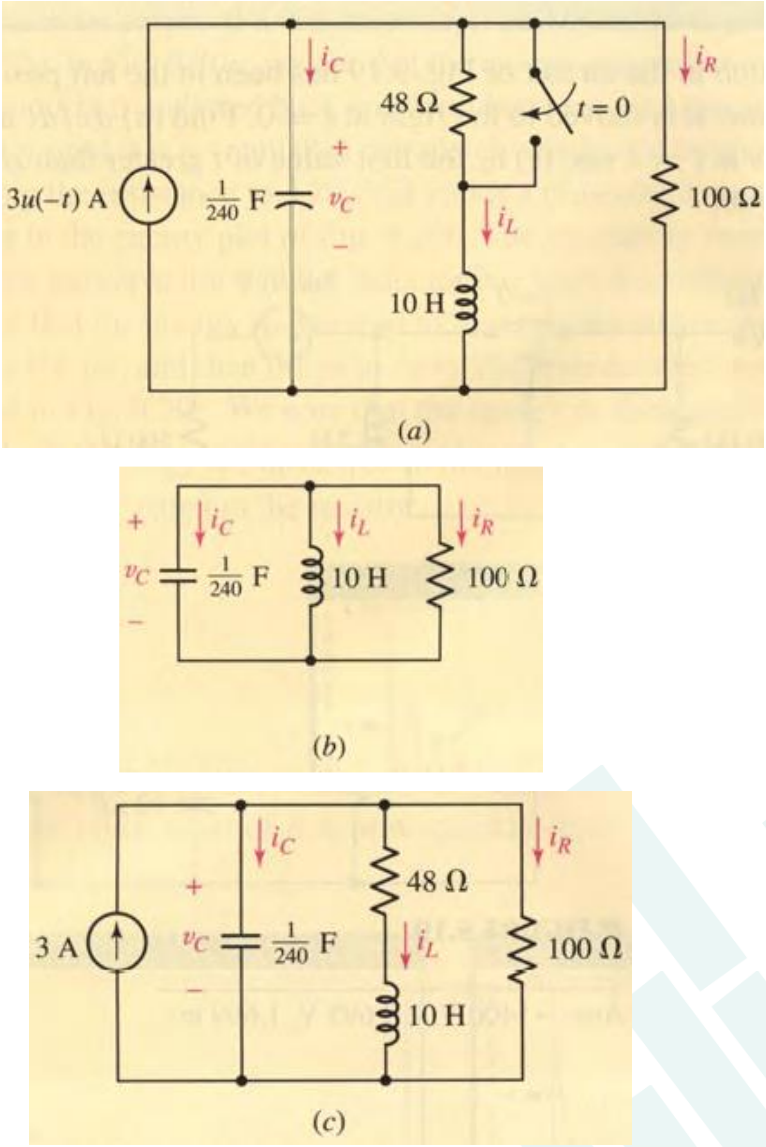
$$i_L(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t) \qquad [28]$$

where $\omega_d = \sqrt{\omega_0^2 - \alpha^2} = 4.750$ rad/s. The only remaining step is to find B_1 and B_2.

Fig. 9.17*c* shows the circuit as it exists at $t = 0^-$. We may replace the inductor with a short circuit and the capacitor with an open circuit; the result is $v_C(0^-) = 97.30$ V and $i_L(0^-) = 2.027$ A. Since neither quantity can change in zero time, $v_C(0^+) = 97.30$ V and $i_L(0^+) = 2.027$ A.

Substituting $i_L(0) = 2.027$ into Eq. [28] yields $B_1 = 2.027$ A. To determine the other constant, we first differentiate Eq. [28]:

$$\frac{di_L}{dt} = e^{-\alpha t}(-B_1\omega_d \sin \omega_d t + B_2\omega_d \cos \omega_d t) - \alpha e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t) \qquad [29]$$



Example

and note that $v_L(t) = L(di_L/dt)$. Referring to the circuit of Fig. 9.17*b*, we see that $v_L(0^+) = v_C(0^+) = 97.3$ V. Thus, multiplying Eq. [29] by $L = 10$ H and setting $t = 0$, we find that

$$v_L(0) = 10(B_2\omega_d) - 10\alpha B_1 = 97.3$$

Solving, $B_2 = 2.561$ A, so that

$$i_L = e^{-1.2t}(2.027 \cos 4.75t + 2.561 \sin 4.75t) \text{ A}$$

which we have plotted in Fig. 9.18.

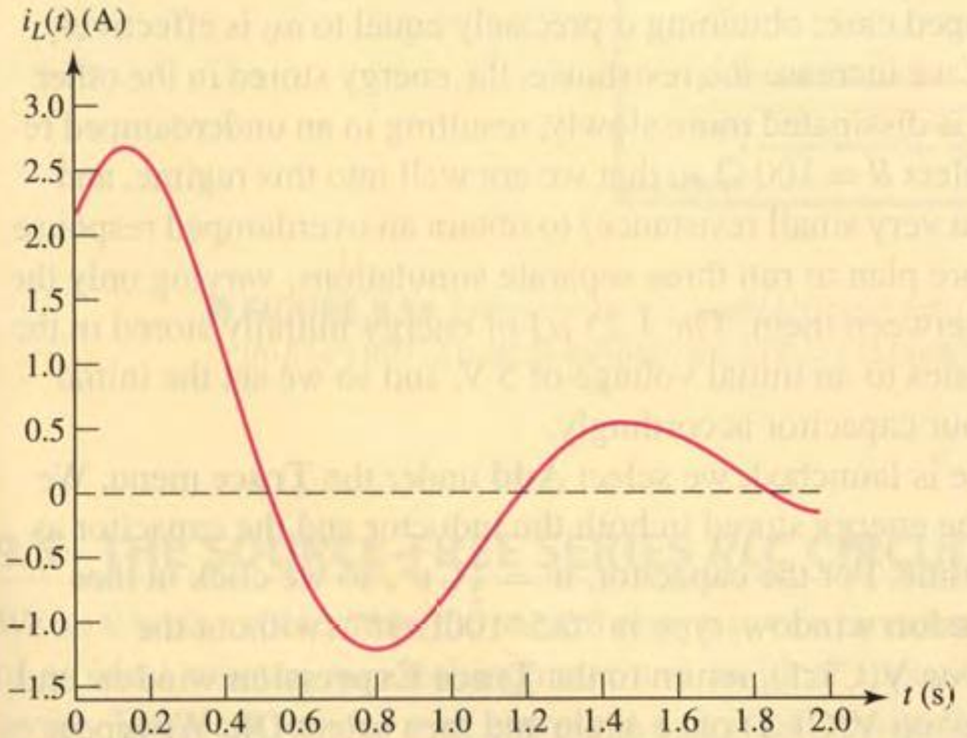


FIGURE 9.18 Plot of $i_L(t)$, showing obvious signs of being an underdamped response.

Example

Determine the initial conditions in the circuit of the Fig. P.0, and also find values at t = 0+ for the first derivatives of i_L and v_C. Then find $v_C(t)$

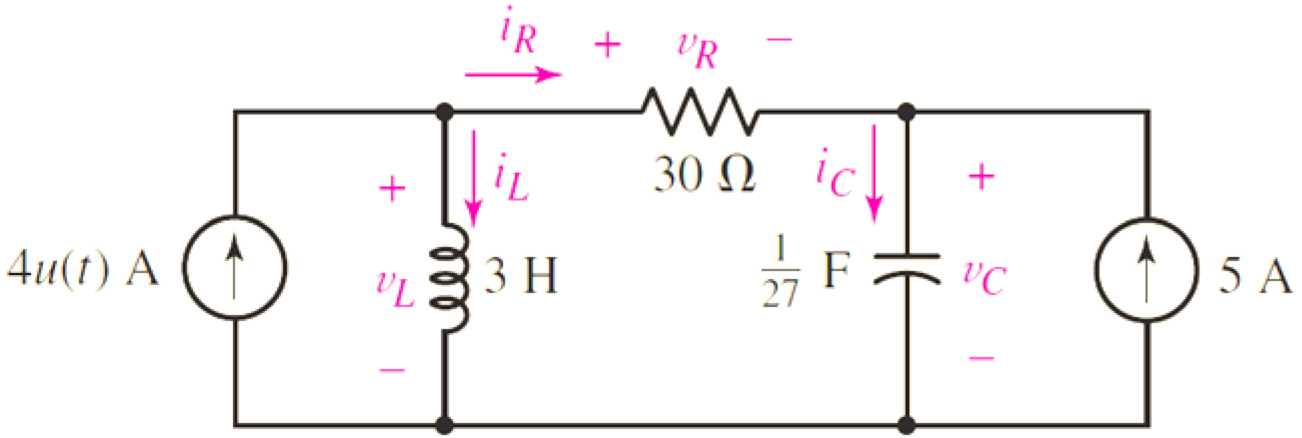


Fig. P.0

Example

Determine the initial conditions in the circuit of the Fig. P.a, and also find values at t = 0+ for the first derivatives of i_L and v_C. Then find $v_C(t)$

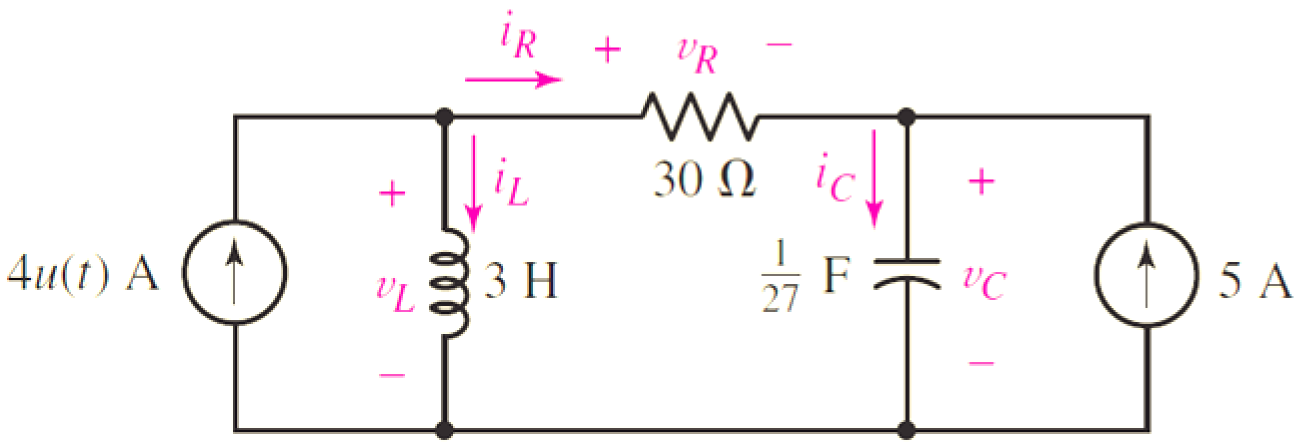


Fig. P.a

Our objective is to find values of each current ant voltage at both t $= 0^{-}$ $and\ t = 0^{+}$. The initial values of the derivatives can be easily found.

Example

At $t = 0^{-}$, only the right hand source is active. It is shown in fig. P.b. The circuit seems to be in this condition forever and all currents and voltages are constant. Thus a dc current through inductor requires zero voltage across it:

$$v_L(0^{-}) = 0$$

And dc voltage across the capacitor requires zero current through it:

$$i_C(0^{-}) = 0$$

Applying Kirchhoff's current law to the right node, we get

$$i_R(0^{-}) = -5\ A$$

This gives

$$v_R(0^{-}) = -150\ V$$

Applying Kirchhoff's voltage law around left hand mesh, we get

$$v_C(0^{-}) = 150\ V$$

Inductor current $\quad i_L(0^{-}) = 5\ A$

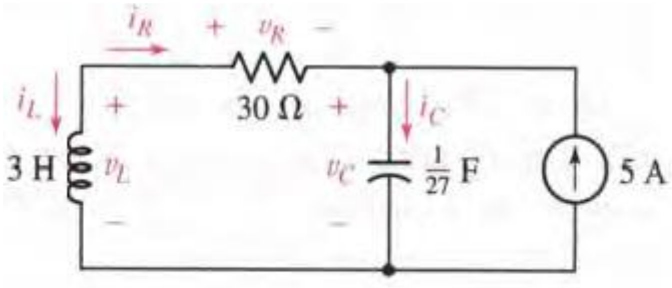


Fig. P.b

Example

$t = 0^+$, During $t = 0^-$ to $t = 0^+$, left hand current source is active and many of the current and voltage values at $t = 0^-$ will change abruptly. Corresponding circuit is shown in fig. P.c.

The inductor current and capacitor voltage can not change instantly and both of these remains constant during switching interval. Thus

$$i_L(0^+) = 5 \text{ A} \qquad \text{and} \qquad v_C(0^+) = 150 \text{ V}$$

We have at left hand node

$$i_R(0^+) = -1 \text{ A} \qquad \text{and} \qquad v_R(0^+) = -30 \text{ V}$$

$$\text{So} \quad i_C(0^+) = 4 \text{ A} \qquad \text{and} \qquad v_L(0^+) = 120 \text{ V}$$

We have our six initial values at $t = 0^-$ and six more at $t = 0^+$, only the capacitor voltages and inductor current remains unchanged from $t = 0^-$ values.

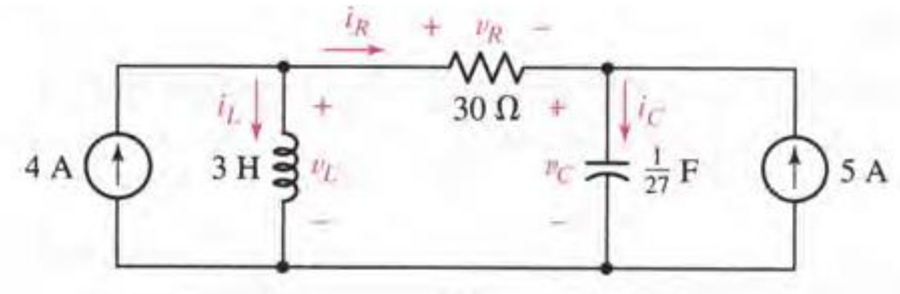


Fig. P.b

Example

In order to determine $v_C(t)$, the circuit appears as a series RLC circuits and s_1 = - 1 and s_2 = - 9.

The forced response is obtained by drawing the dc equivalent. The forced response is 150 V.

$$v_C(t) = 150 + A\,e^{-t} + \mathrm{B}\,e^{-9t}$$

$$v_C(0^+) = 150 = 150 + A + \mathrm{B}$$

A + B = 0

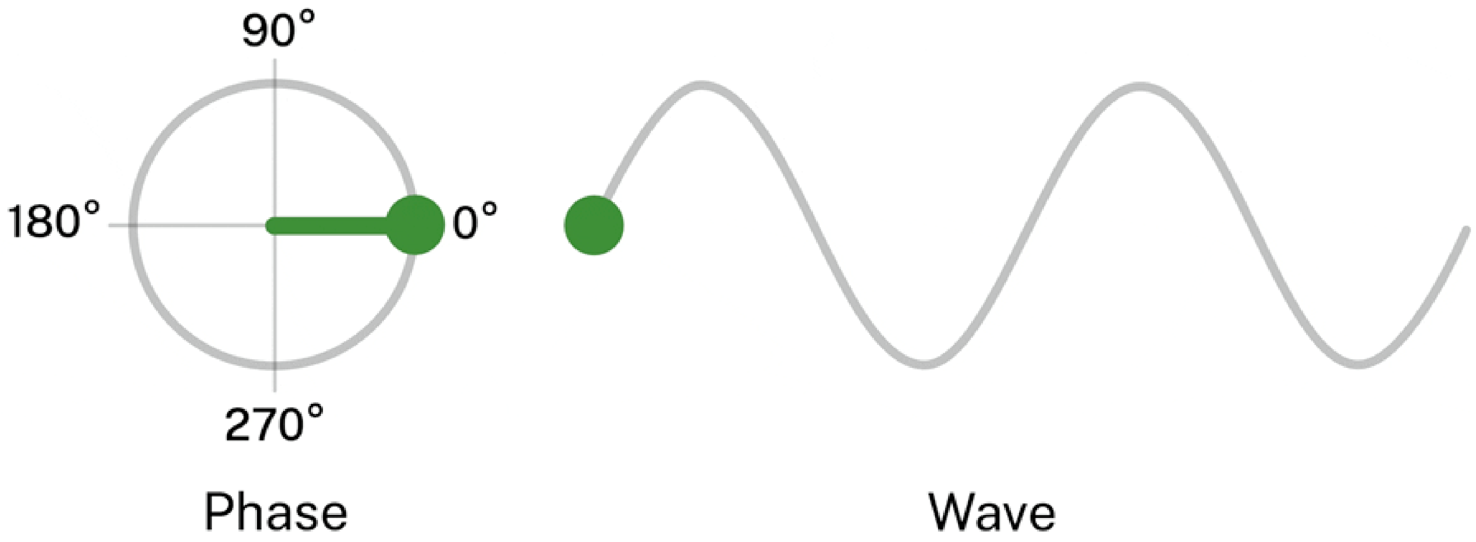
$$\frac{dv_C}{dt} = -A\,e^{-t} - 9\mathrm{B}\,e^{-9t}$$

$$\frac{dv_C}{dt} \text{ at } t = 0^+ = 108 = -\mathrm{A} - 9\mathrm{B}$$

We have A = 13.5 and B = -13.5

And complete response $v_C(t) = 150 + 13.5\,(e^{-t} - e^{-9t})$ V

AC response of circuits



Alternating signal is an electric signal which periodically reverses direction

TIME-DEPENDENT SIGNAL SOURCES

- This section will consider sources that generate time-varying voltages and currents and, in particular, sinusoidal sources.
- Fig. 1 shows the convention employed to denote time-dependent signal sources.
- Most important classes of time-dependent signals is that of periodic signals.

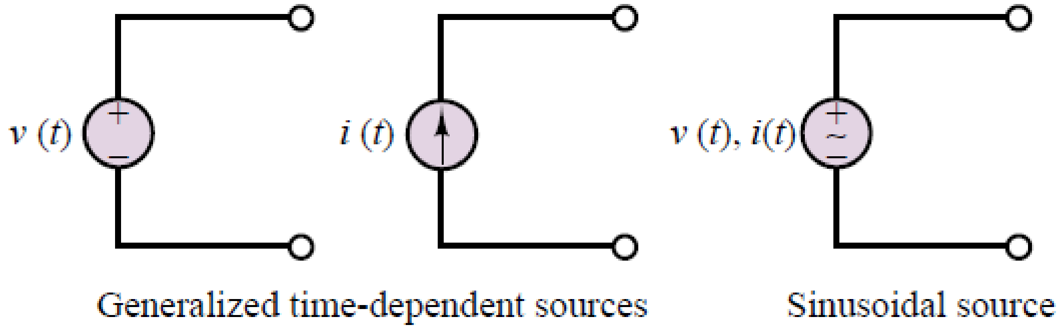


Fig. 1 Time-dependent signal sources

- These signals appear frequently in practical applications and are a useful approximation of many physical phenomena.
- A periodic signal x(t) is a signal that satisfies the following equation:
- $x(t) = x(t + nT)$ $\qquad$ $n = 1, 2, 3,$
- Where T is the time period of x(t).

TIME-DEPENDENT SIGNAL SOURCES

- Fig. 2 shows a number of the periodic waveforms that are typically encountered in the study of electrical circuits.
- Waveforms such as the sine, triangle, square, pulse, and saw-tooth waves are provided in the form of voltages (or, less frequently, currents) by commercially available signal generators. Such instruments allow for selection of the waveform peak amplitude, and of its period.

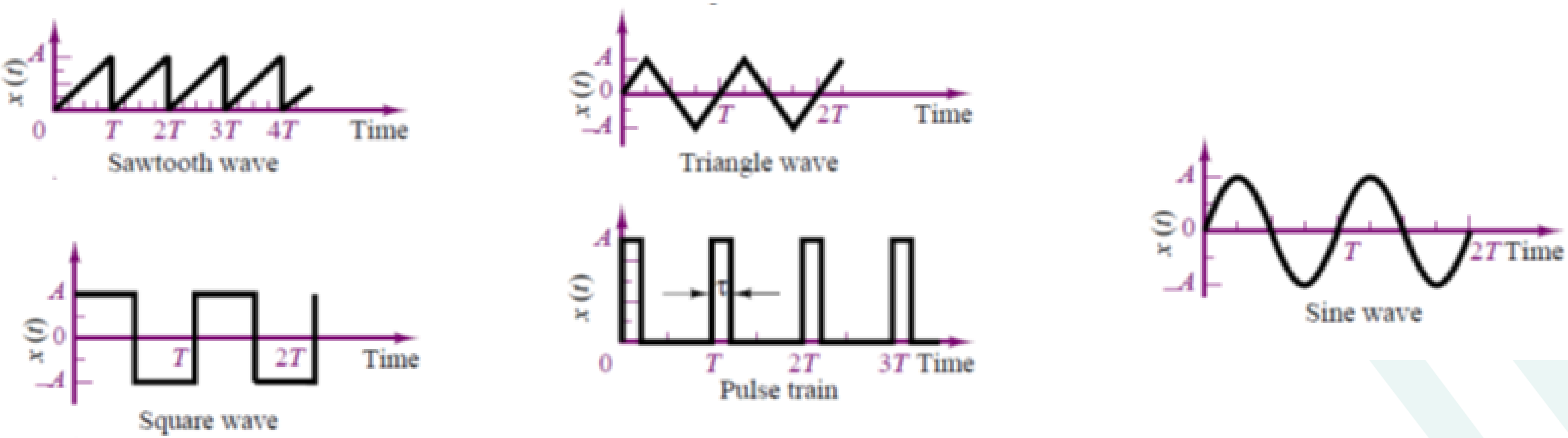
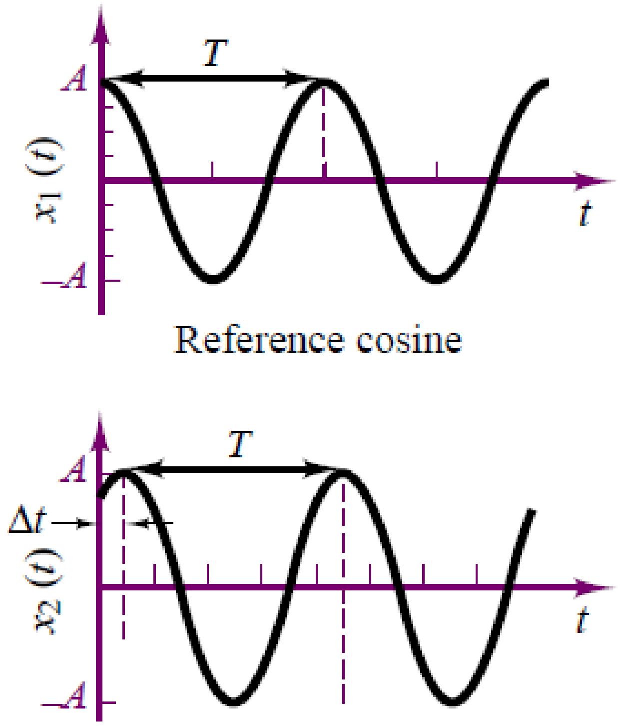


Fig. 2

- These signals appear frequently in practical applications and are a useful approximation of many physical phenomena.

TIME-DEPENDENT SIGNAL SOURCES

- A periodic signal x(t) is a signal that satisfies the following equation:
  - ➢ $x(t) = x(t + nT)$ $n = 1, 2, 3,$
  - ➢ Where T is the time period of x(t).
  - ➢ Frequency of the signal is given by 1/T Hz



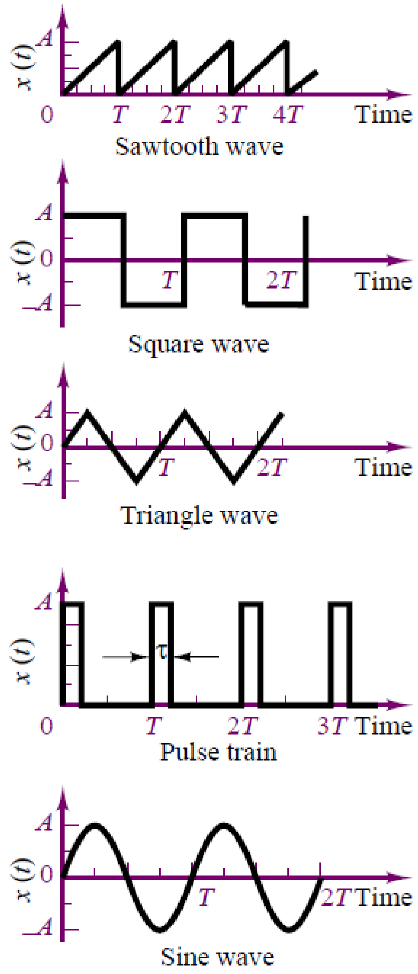


Fig. 2

TIME-DEPENDENT SIGNAL SOURCES

- Fig. 3 shows a sinusoidal waveform and its relevant parameters.
- A generalized sinusoid is defined as follows:

$$X(t) = A\cos(\omega t + \varphi)$$

where A is the peak amplitude

ω is the frequency in rad/s and

Φ is the phase.

$$X1(t) = A\cos(\omega t)$$

$$X2(t) = A\cos(\omega t + \varphi).$$

Natural frequency $f = 1/T$ (cycles/s or Hz)

$$\omega = 2\pi f \text{ (radians/s)}$$

$$\varphi = 2\pi(\Delta T/T) \text{ (radians)}$$

$$= 360(\Delta T/T) \text{ (degree)}$$

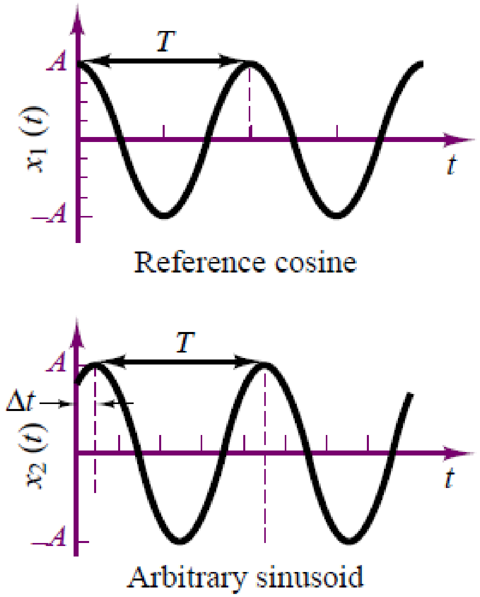
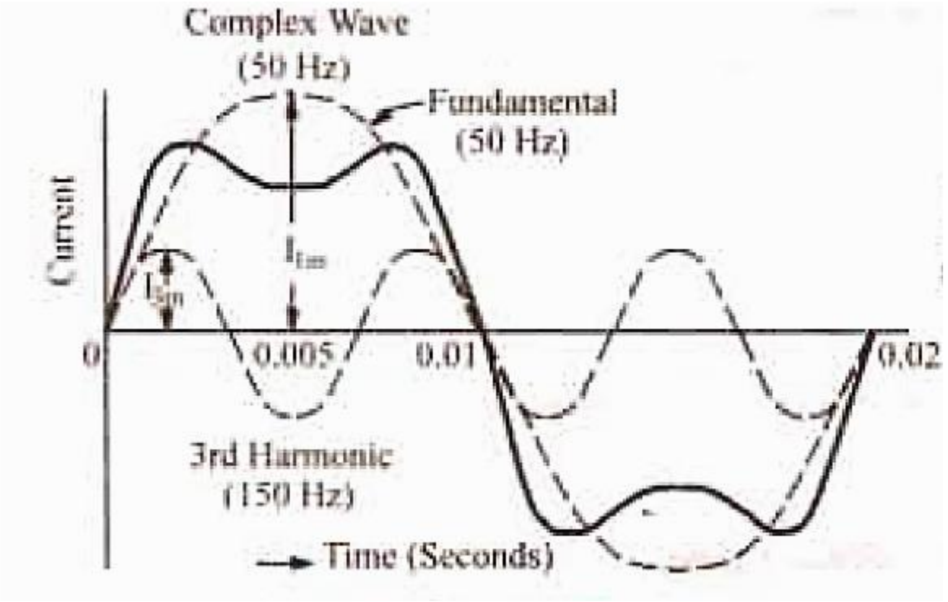
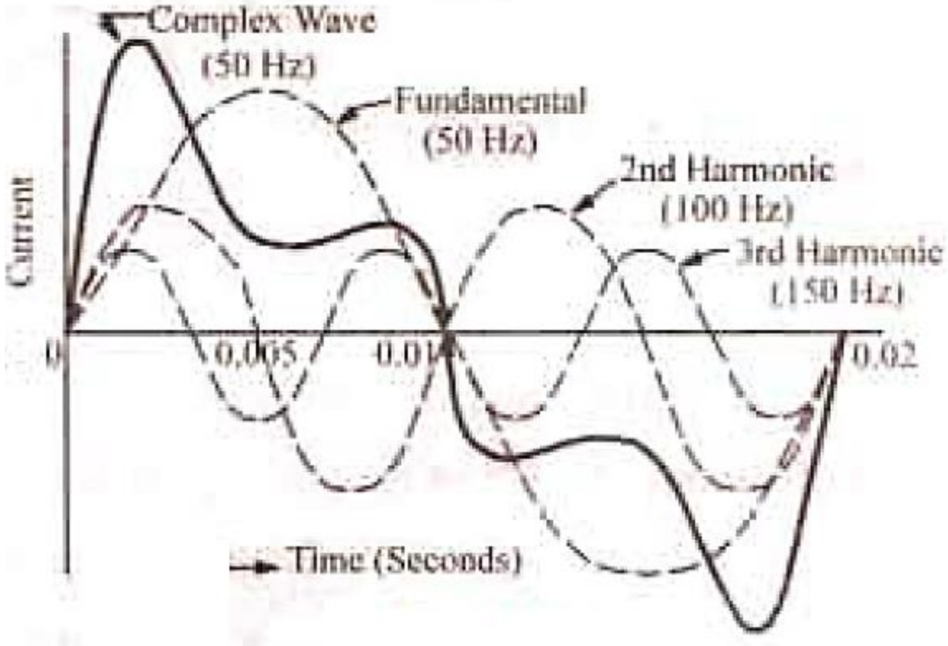


Fig. 3 Sinusoidal signal

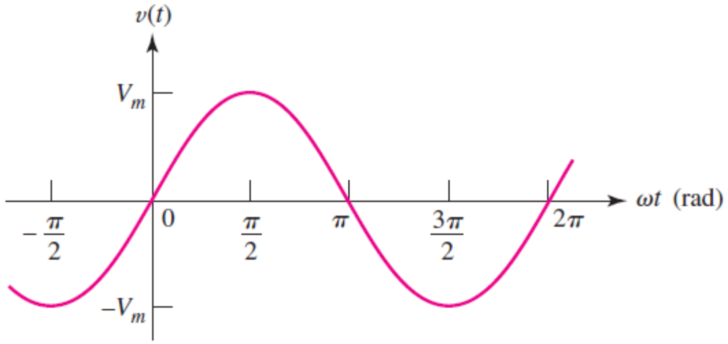
Fig. 2

Complex Waves





Sinusoidal input and Lead and Lag

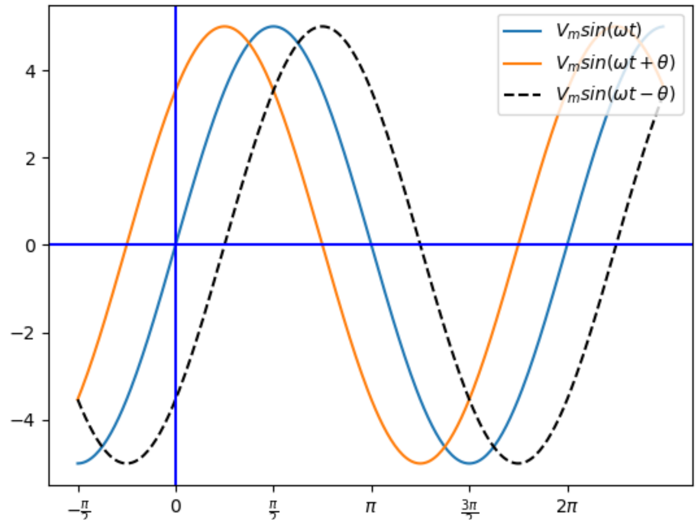


$v_r(t) = V_0 \sin(\omega t)$

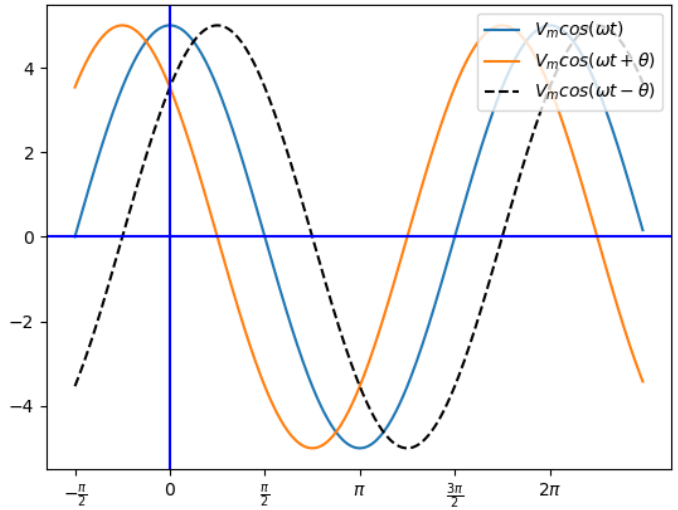
$v(t) = V_0 \sin(\omega t + \theta)$ with a phase θ

If θ is positive, the voltage v(t) attains its zero value earlier than $v_r(t)$. Here, v(t) is said lead $v_r(t)$. If θ is negative, the opposite situation arises and v(t) is said to lag $v_r(t)$.

v(t)



v(t)



Average and RMS Values

- The most common types of measurements are the average (or DC) value of a signal waveform—which corresponds to just measuring the mean voltage or current over a period of time—and the root-mean-square (or rms) value, which takes into account the fluctuations of the signal about its average value.
- The operation of computing the average value of a signal corresponds to integrating the signal waveform over some (presumably, suitably chosen) period of time. We define the time-averaged value of a signal x(t) as

$$\textbf{Average of } \boldsymbol{x(t)} = \boldsymbol{< x(t) > = \frac{1}{T}\int_0^T x(t')\, dt'}, \quad \text{where } T \text{ is period of signal}$$

- A useful measure of the voltage of an AC waveform is the **Effective or root-mean-square, or rms, value of the signal, x(t), defined as follows:**

$$x_{rms} = \sqrt{\frac{1}{T}\int_0^T x^2(t)\, dt'}$$

$$So,\ I_{rms} = \sqrt{\frac{1}{T}\int_0^T i^2(t)\, dt'}$$

The average power in a resistance R for a periodic current i(t) of period T is

$$P = \frac{1}{T}\int_0^T Ri^2(t)dt = I_{\mathrm{rms}}^2 R,$$

The effective or rms current may be defined as the constant current which will produce the same power in a resistor as that produced on the average by the actual periodic current

Average and RMS Values

- Compute the rms value of the sinusoidal current $i(t) = I_0 sin(\omega t)$

$$I_{\text{rms}}^2 = I_{\text{eff}}^2 = \frac{I_0^2}{T}\int_0^T \sin^2 \omega t dt$$

$$= \frac{I_0^2}{\omega T}\int_0^{2\pi} \sin^2 \omega t d(\omega t) = \frac{I_0^2}{2\pi}\int_0^{2\pi} \frac{1-\cos 2\omega t}{2} d(\omega t) = \frac{I_0^2}{2}$$

$$I_{\text{rms}} = \frac{I_0}{\sqrt{2}} = 0.707 I_0.$$

Root Mean Squared

Root Square

$$V_{RMS} \equiv \sqrt{\left(V^2\right)_{avg}} = \frac{1}{\sqrt{2}} V_p = 0.707V_p$$

Mean

- Similarly for voltage $v(t) = V_0 sin(\omega t)$, we can show

$$V_{\text{rms}} = \frac{V_0}{\sqrt{2}} = 0.707 V_0$$

- When the rms voltage, measured by voltmeter, is 100 V and the voltage is sinusoidal, the time varying voltage is given by $\text{v(t)} = 100\sqrt{2}\,\sin(wt)$

Question: why is the shock from the AC mains of 220 V (rms) is more severe than the shock from the 220 V DC mains?

Average and RMS Values

The average value of a sinusoidal voltage or current over a full cycle is zero. However, over a half cycle the average will not be zero. This average for a current $i(t) = I_0 sin(\omega t)$ is given by

$$
\begin{aligned}
I_{\mathrm{av}} &= \frac{2}{T}\int_0^{T/2} I_0 \sin \omega t dt = \frac{2}{\omega T}\int_0^{\pi} I_0 \sin \omega t d(\omega t) \\
&= \frac{2I_0}{2\pi} \cos \omega t \Big|_{\omega t=\pi}^{0} = \frac{2I_0}{\pi} = 0.637\ I_0 .
\end{aligned}
$$

Form Factor: The ratio of the rms value to the average value otver a half cycle period is defined to be the form factor of the periodic waveform . For a sinusoidal current, form factor is

$$\text{Form factor} = \frac{I_0}{\sqrt{2}} \Big/ \frac{2I_0}{\pi} = 1.11$$

homeowork: Identify the type of waveforms shown in Fig. P.1. Compute the average and rms values of these waveforms:

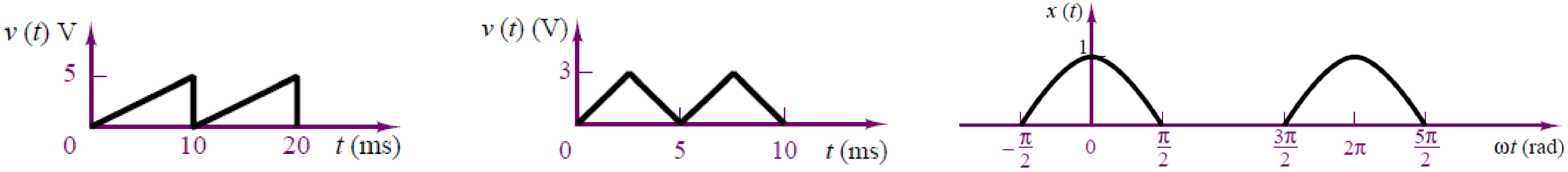


Fig. P.1

Power in AC circuits

Suppose $\boldsymbol{v(t) = V_0 sin(\omega t)}$ is the applied voltage, and the corresponding current is $\boldsymbol{i(t) = I_0 sin(\omega t - \theta)}$

The instantaneous power is

$$\begin{aligned} p(t) &= v(t)i(t) = V_0 I_0 \sin\omega t \sin(\omega t - \theta) \\ &= \frac{V_0 I_0}{2}[\cos\theta - \cos(2\omega t - \theta)]. \end{aligned}$$

The average power over a time period T is

$$P = \int_0^T p(t)dt \Big/ \int_0^T dt, \quad \rightarrow \quad P = \frac{V_0 I_0}{2}\cos\theta = \frac{V_0}{\sqrt{2}}\frac{I_0}{\sqrt{2}}\cos\theta = V_{\mathrm{rms}} I_{\mathrm{rms}} \cos\theta$$

Note:
integral of $\cos(2\omega t - \theta)$ = 0 for a complete cycle

- The term $\cos\theta$ is called "***power factor***". The product $V_{rms}I_{rms}$ is known as "***apparent power***".

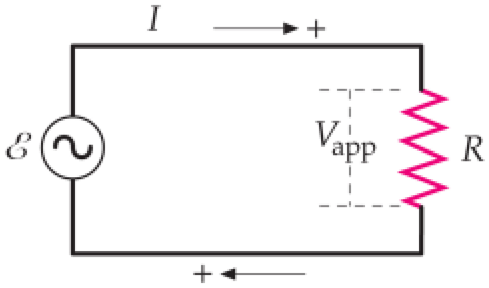
- **Real power = apparent power × power factor**

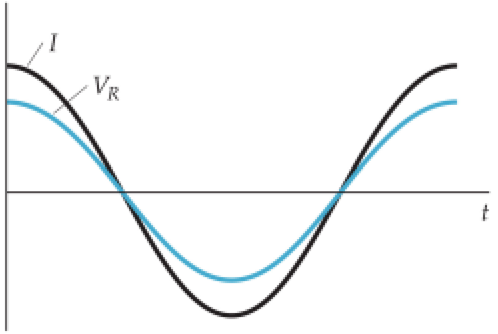
- Since, $cos\theta \leq 1$, the power dissipation in a AC circuit is less than the product $V_{rms}I_{rms}$ except when $cos\theta = 1$
- For $cos\theta = 1$, both current and voltage are in same phase and we have resonance. Unity factor (i.e. $cos\theta = 1$) and resonance are thus synonymous.
- P = 0 for $\theta = \frac{\pi}{2}$. Current is then called ***wattless.*** This case occurs when the circuit is purely inductive or purely capacitive.

AC response of a resistance

Suppose $\boldsymbol{v(t) = V_0 \sin(\omega t)}$ is the applied across a resistance R, from Ohm's Law, instantaneous current will be

$$i(t) = \frac{v(t)}{R} = \frac{V_0}{R}\sin\omega t$$





Current and voltage both are in same frequency and phase. The amplitude of current is $I_0 = V_0/R$

The instantaneous power is $$p(t) = v(t)i(t) = \frac{V_0^2}{R}\sin^2\omega t = \frac{V_0^2}{2R}(1-\cos 2\omega t)$$

The average power over a time period T is $$P = \frac{\int_0^T p(t)dt}{\int_0^T dt} = \frac{V_0^2}{2R} = \frac{I_0^2 R}{2}$$

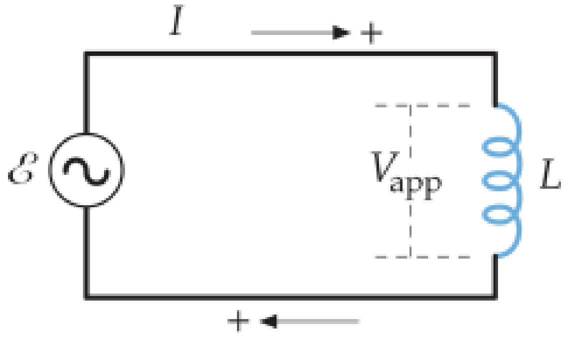
If $W(t)$ is the instantaneous energy, then we have $p(t) = dW/dt$. Then the energy supplied to the resistance in N complete cycle is

$$W_N = \int_0^{NT} p(t)dt = \frac{V_0^2}{2R}NT$$

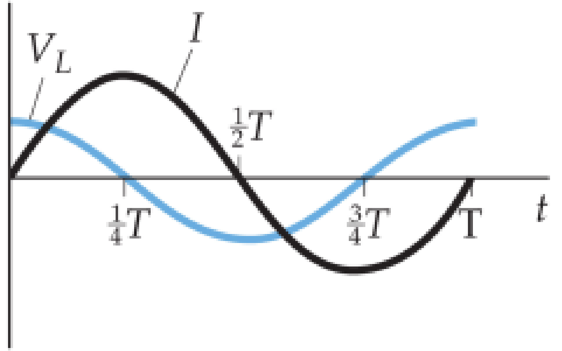
This energy is dissipated as Joule heat

AC response of a inductance

If $\boldsymbol{i(t) = I_0 \sin(\omega t)}$, then
$$v(t) = L\frac{di}{dt} = L\frac{d}{dt}(I_0 \sin \omega t) = \omega L I_0 \cos \omega t$$
$$= \omega L I_0 \sin\left(\omega t + \frac{\pi}{2}\right).$$



- Voltage *v(t)* across an inductance leads the current *i(t)* by 90°. When the phase difference between two quantities is 90°, they are said to be ***quadrature***.
- The peak value of voltage is $V_0 = \omega L I_0$ *or* $I_0 = V_0/\omega L$. The quantity ωL is called ***inductive reactance***. The unit of ωL is ohm. The inverse of inductive reactance i.e. $1/\omega L$ is known as ***inductive susceptance***.



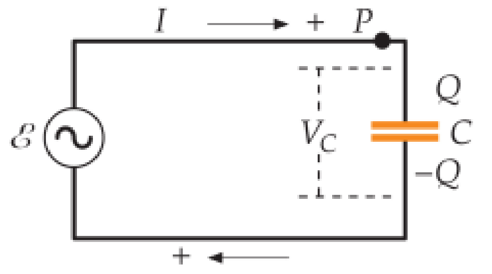
The instantaneous power is $$p(t) = v(t)i(t) = \omega L I_0^2 \sin \omega t \cos \omega t = \frac{1}{2} I_0^2 \omega L \sin 2\omega t$$

The energy input to the inductor in a time interval t_1 is $$W = \int_0^{t_1} p(t)dt = \frac{\omega L I_0^2}{2}\int_0^{t_1} \sin 2\omega t dt = \frac{\omega L I_0^2}{4}(1 - \cos 2\omega t_1)$$

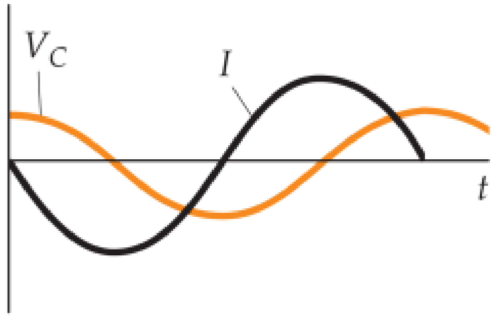
The average power over a full time period T is 0 since integral of sin(ωt) ; inductor doesn't dissipate energy, the store energy

AC response of a capacitance

If $v(t) = V_0 sin(\omega t)$ is applied across a capacitor, then the instantaneous current is



$$i(t) = \frac{dq}{dt} = C\frac{dv}{dt} = C\frac{d}{dt}(V_0 \sin \omega t)$$
$$= \omega C V_0 \cos \omega t = \omega C V_0 \sin\left(\omega t + \frac{\pi}{2}\right)$$



- Current $i(t)$ across leads the voltage $v(t)$ by 90°. When the phase difference between two quantities is 90°, they are said to be ***quadrature***.
- The peak value of voltage is $I_0 = \omega C V_0$ *or* $V_0 = I_0/\omega C$. The quantity $1/\omega C$ is called ***capacitive reactance***. The unit of ωL is ohm. The inverse of inductive reactance i.e. ωC is known as ***capacitive susceptance***.

The instantaneous power is

$$p(t) = v(t)i(t) = \omega C V_0^2 \sin \omega t \cos \omega t$$
$$= \frac{\omega C V_0^2}{2} \sin 2\omega t.$$

The energy supplied from then souce in a time interval t_1 is

$$W = \int_0^{t_1} p(t)dt = \frac{\omega C V_0^2}{2} \int_0^{t_1} \sin 2\omega t dt$$
$$= \frac{1}{4} C V_0^2 (1 - \cos 2\omega t_1).$$

The average power over a full time period T is 0 since integral of sin(ωt) ; capacitor doesn't dissipate energy, the store energy