Problem 2.8.10

Let random variable X have PMF $P_X(x)$. We wish to guess the value of X before performing the actual experiment. If we call our guess $\hat{x}$, the expected square of the error in our guess is

$$e(\hat{x}) = E\left[(X - \hat{x})^2\right]$$

$(X - \hat{x})^2$

Show that $e(\hat{x})$ is minimized by $\hat{x} = E[X]$.

First order necessary condition requires that

$$\left.\frac{de(\hat{x})}{d\hat{x}}\right|_{\hat{x}=\hat{x}^*} = 0$$

$$e(\hat{x}) = E\left[(X-\hat{x})^2\right] = E\left[X^2 + \hat{x}^2 - 2\hat{x}X\right] = E[X^2] + \hat{x}^2 - 2\hat{x}E[X]$$

$$\frac{de(\hat{x})}{d\hat{x}} = 2\hat{x} - 2E[X].$$

Set to 0

$\Rightarrow \hat{x} = E[X]$

If N is discrete Uniform$(1,100)$:

$$P_{N|B}(n) = P[N=n \mid B]$$

$$= \begin{cases} \dfrac{P[N=n]}{P[N\geq 20]} & n \in \{20, 21, \ldots, 100\} \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \dfrac{1/100}{81/100} = 1/81 & n \in \{20, \ldots, 100\} \\ 0 & \text{otherwise} \end{cases}$$

Suppose X is the no. of additional tests needed.

$$P[X=1] = 1/81$$

$$P[X=2] = P[N=21 \mid N \geq 20] = \frac{1}{81}$$

$$\vdots$$

$$\Rightarrow \quad X \sim \text{Discrete Unif}(1, 81)$$

When $N \sim \text{Geom}(p)$, $X \sim \text{Geom}(p)$.

This property is called the memoryless property.

Geometric is the only memoryless distribution among discrete RVs.

Problem 2.9.6

Select integrated circuits, test them in sequence until you find the first failure, and then stop. Let N be the number of tests. All tests are independent with probability of failure $p = 0.1$. Consider the condition $B = \{N \geq 20\}$.

(a) Find the PMF $P_N(n)$.

(b) Find $P_{N|B}(n)$, the conditional PMF of N given that there have been 20 consecutive tests without a failure.

(c) What is $E[N|B]$, the expected number of tests given that there have been 20 consecutive tests without a failure?

$$P_{N|B}(n) = P[N=n|B]$$

$$= \frac{P[N=n, N \geq 20]}{P[N \geq 20]} = \begin{cases} \frac{P[N=n]}{P[N \geq 20]} & n \in B \\ 0 & \text{otherwise} \end{cases}$$

$n \geq 20$

$$\frac{(1-p)^{n-1} p}{\sum_{n=20}^{\infty} (1-p)^{n-1} p} = (1-p)^{n-20} p$$

Problem 2.9.7

Every day you consider going jogging. Before each mile, including the first, you will quit with probability q, independent of the number of miles you have already run. However, you are sufficiently decisive that you never run a fraction of a mile. Also, we say you have run a marathon whenever you run at least 26 miles.

(a) Let M equal the number of miles that you run on an arbitrary day. What is $P[M > 0]$? Find the PMF $P_M(m)$.

(b) Let r be the probability that you run a marathon on an arbitrary day. Find r.

(c) Let J be the number of days in one year (not a leap year) in which you run a marathon. Find the PMF $P_J(j)$. This answer may be expressed in terms of r found in part (b).

(d) Define $K = M - 26$. Let A be the event that you have run a marathon. Find $P_{K|A}(k)$.

Assignments

- Clearly state all the RV(s), range space of the RV(s), PMF, event space that you may use to solve the problem
  - Note that event of interest must lie in the event space/ range space
  - It is fine to start with a layman description of the event. However, you must specify the event using proper mathematical notation
- Problems 2.2.6 & 2.2.7
- Problem 2.3.7
- Problem 2.3.11
- Problem 2.3.13
- Problems 2.5.1 – 2.5.8
- 2.5.9
- 2.4.1 and 2.6.1
- 2.6.5, 2.6.6, 2.7.5, 2.7.6
- 2.7.9
- 2.8.9
- 2.9.6
- 2.9.7