Definition

An ordered set of vectors $\{v_1, \ldots, v_p\} \in \mathbb{R}^n$ is said to be *linearly independent* if the vector equation

$$x_1 v_1 + \ldots + x_p v_p = 0$$

has only the trivial solution. The (ordered) set $\{v_1, \ldots, v_p\}$ is said to be *linearly dependent* if there exist weights $c_1, \ldots, c_p$, not all zero, such that

$$c_1 v_1 + \ldots + c_p v_p = 0$$

An ordered set of vectors remains linearly independent (or dependent) even if the order is changed. Therefore it is acceptable to say that a set of vectors is linearly independent (or dependent) without actually mentioning the order, unless that particular ordering is actually used elsewhere.

Linear Independence of Matrix Columns

The columns of a matrix A are linearly independent if and only if the equation $\boxed{Ax = 0}$ has only the trivial solution.

Scalar Multiples

A set of two vectors $\{v_1, v_2\}$ is linearly dependent if at least one of the vectors is a multiple of the other. The set is linearly independent if and only if neither of the vectors is a multiple of the other.

Characterization of Linearly Dependent Sets

An indexed set $S = \{v_1, \ldots, v_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent and $v_1 \neq 0$, then some v_j (with $j > 1$) is a linear combination of the preceding vectors, $v_1, \ldots, v_{j-1}$.

Theorem

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{v_1, \ldots, v_p\}$ *in* $\mathbb{R}^n$ *is linearly dependent if* $p > n$.

Let $v_1, \ldots, v_p \in \mathbb{R}^n$.

$\{v_1, \ldots, v_p\}$ is l.d. if $p > n$

Let $\underbrace{V = [v_1 \ v_2 \ \ldots \ v_p]}$

This is an $n \times p$ matrix.

$$Vx = 0.$$

has a non-trivial solution if and only if the system has at least one free variable

$\Longleftrightarrow$ V has at least one column which is not a pivot column.

Let V' be the RREF of V.

V' is an $n \times p$ matrix.

Since V' has fewer rows than

columns and each pivot must occupy at most row. There must be columns which don't have pivots.

look this up

Pigeonhole principle

$\therefore$ V has a column which is not a pivot column

$\Rightarrow$ $Vx = 0$ has free variables

$\Rightarrow$ $Vx=0$ has a non trivial solution. $\square$

Theorem

If a set $S = \{v_1, \ldots, v_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.

Proof: Without loss of generality,

we may assume that

$v_1 = 0$.

Let $c_1 = 1$, $c_2 = 0$, $\cdots$, $c_p = 0$

Then $c_1 v_1 + c_2 v_2 + \cdots + c_p v_p$

$= c_1 v_1 = 0$. $\therefore \{v_1, \ldots, v_p\}$ is l.d.

Abstract Vector Spaces

Vector space structure

Definition

A *vector space (real vector space)* is a nonempty set V of objects, called *vectors*, on which are defined two operations, called *addition* and *multiplication by scalars (real numbers)*, subject to the ten axioms (or rules) listed below. The axioms must hold for all vectors u, v, and w in V and for all scalars c and d.

- The sum of u and v, denoted by $u + v$, is in V. Closure
- $u + v = v + u$ Commutativity
- $u + (v + w) = (u + v) + w$ ← Associativity
- There is a *zero* vector 0 in V such that $u + 0 = u$. ← Additive identy
- For each u in V , there is a vector $-u$ in V such that $u + (-u) = 0$.

Additive inverse

Addition is a binary operation

$$V \times V \longrightarrow V$$

$$\mathbb{R} \times V \longrightarrow V$$

$$(c, u) \longrightarrow cu$$

- The scalar multiple of u by c , denoted by cu, is in V .
- $c(u+v) = cu + cv$
- $(c+d)u = cu + du$

distributive laws

- $c(du) = (cd)u$
- $1u = u$

Please note that instead of "real vector spaces" we can also talk about "complex vector spaces". In this case we would assume that the scalars are complex. The rest of the definition would remain the same. We will come back to this later in the course.

Examples We've Seen Before

1. Euclidean spaces $\mathbb{R}^n$
2. $M_{m,n}(\mathbb{R})$ - the set of all $m \times n$ matrices having real entries

Examples of Vector Spaces - contd.

$a_n = \frac{1}{n}$ $\quad c_0 = 0$ $\quad \mathbb{Z} \longrightarrow \mathbb{R}$ generalization

$a_{-n} = -\frac{1}{n}$

$\mathbb{R}^n$

Let $\mathbb{S}$ be the space of all doubly infinite sequences of numbers (usually written in a row rather than a column):

$$\underline{\{y_k\}} = (\ldots, y_{-2}, y_{-1}, y_0, y_1, y_2, \ldots)$$

If $\{z_k\}$ is another element of $\mathbb{S}$, then the sum $\{y_k\} + \{z_k\}$ is the sequence $\{y_k + z_k\}$ formed by adding corresponding terms of $\{y_k\}$ and $\{z_k\}$. The scalar multiple $c\{y_k\}$ is the sequence $\{cy_k\}$.

$\mathbb{S}$ is sometimes called the space of *(discrete-time) signals*.

$c_n = a_n + b_n$

$\underline{a_1, a_2, a_3 \ldots - - -}$

Examples of Vector Spaces - contd.

For $n \geq 0$, the set $\mathbb{P}_n$ of of polynomials of degree at most n consists of all polynomials of the form

$$p(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n \qquad (*)$$

where the coefficients $a_0, \ldots, a_n$ are real numbers and t is a variable which takes real values. If all the coefficients are zero, p is called the *zero polynomial*.

The *degree* of a nonzero polynomial p is defined as the highest power of t in $(*)$ whose coefficient is not zero.

Examples of Vector Spaces - contd.

Let D be a set

Let V be the set of all real valued function defined on a set D.

D can be any set you like.

$V = \{ f \mid f : D \to \mathbb{R}$ is a function$\}$

If $x \in D$, $f, g \in V$

$(f+g)(x) = f(x) + g(x)$

$D = \{a, b, c\}$.

$f : D \to \mathbb{R}$. $\quad g : D \to \mathbb{R}$

$f(a) = 1$, $f(b) = 2$, $f(c) = 5$

$g(a) = 0.5$ $\quad g(b) = \pi$ $\quad g(c) = e^{-7}$

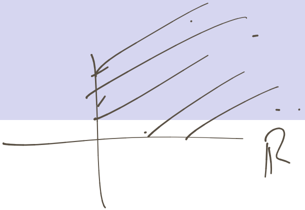
$h = f + g$

$h(c) = f(c) + g(c)$.

$h(a) = f(a) + g(a)$.

$h(b) = f(b) + g(b)$

Practice Questions



Why are the following sets *not* vector spaces?

- Let V be the first quadrant in the xy-plane, i.e.

$$V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \;:\; x \geq 0, y \geq 0 \right\}$$

$c = -1$

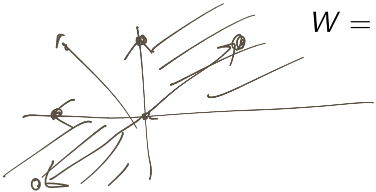
$v = (1,1)$

$cv = (-1,-1)$

$\notin$ first quadrant

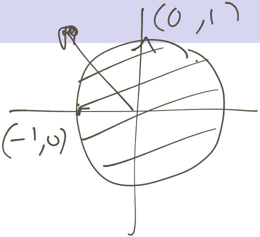
- Let W be the union of the first and third quadrants in the xy-plane, i.e.

$$W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \;:\; xy \geq 0 \right\}$$



$v_1 = (0,1)$

$v_2 = (-1,0)$



Why are the following sets *not* vector spaces?

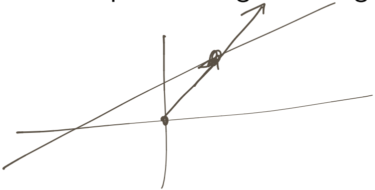
- Let D be the unit disk in the xy-plane, i.e.

$$D = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \; : \; x^2 + y^2 \leq 1 \right\}$$

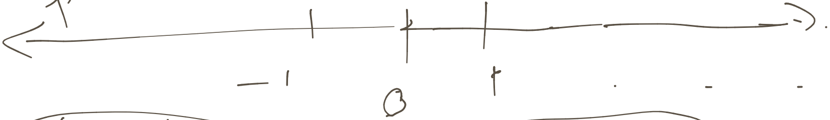
$v_1 = (0,1)$

$v_2 = (-1,0)$

- A line in $\mathbb{R}^2$ (or $\mathbb{R}^3$) which does not pass through the origin



Vector space $\to \mathbb{R}$ $(-\infty, \infty)$

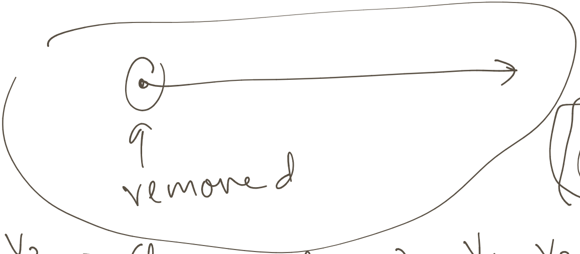


$$e^{\log v_1 + \log v_2} = e^{\log v_1} \cdot e^{\log v_2} = v_1 v_2$$

What is addition? Scalar multiplication?

$x \mapsto e^x$

$\boxed{(0, \infty)}$



$$v_1 \oplus v_2 = e^{(\log v_1 + \log v_2)} = v_1 v_2$$

$v_1, v_2 \in (0, \infty)$

$$V = (0, \infty)$$

$$v_1 \oplus v_2 := v_1 v_2 \quad , \forall v_1, v_2 \in V.$$

$$c \circledast v := \underset{\uparrow}{\text{answer for yourself}} \quad , \forall c \in \mathbb{R}, v \in V$$

$$\left.\begin{aligned} x+y &= 10 \\ 3x+3y &= 30 \end{aligned}\right\}$$

$$\begin{aligned} x+y &= 10 \\ x+y &= 5 \end{aligned}$$

$$\begin{aligned} x+y &= 4 \\ x-y &= 5 \end{aligned}$$

What about scalar

multiplication?

$\longrightarrow$ Think about

this.