Probability and Random Processes

Sums of RV

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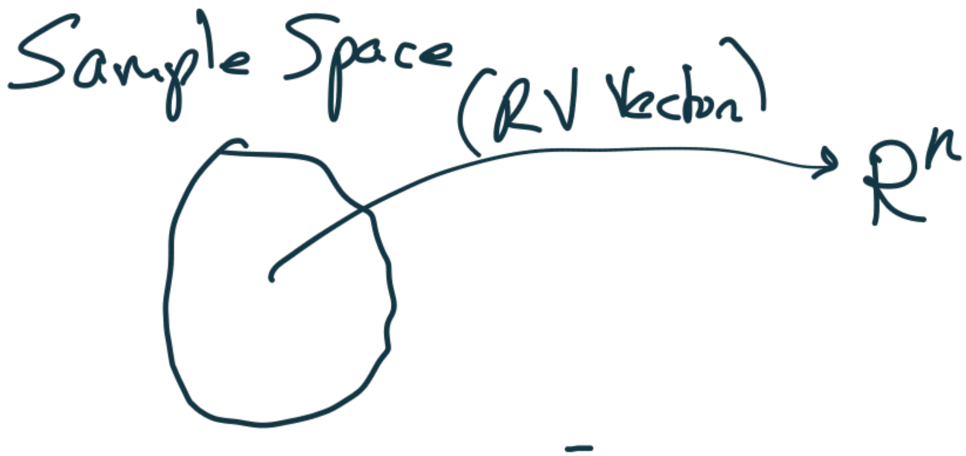
Expected Value of Sum is Sum of Expected Values

Theorem 6.1

For any set of random variables $X_1, \ldots, X_n$, the expected value of $W_n = X_1 + \cdots + X_n$ is

$$E[W_n] = E[X_1] + E[X_2] + \cdots + E[X_n].$$

- We earlier proved that $E[W_2] = E[X_1] + E[X_2]$
  - Use induction



Expected Value of Sum is Sum of Expected Values

- The theorem is valid for any joint distribution of the random variables $X_1, X_2, \ldots, X_n$

- The variables do not need to be independent! Can have ANY joint distribution

- Therefore, a very powerful theorem!

Variance of a Sum Of Random Variables

Theorem 6.2

The variance of $W_n = X_1 + \cdots + X_n$ is

$$\text{Var}[W_n] = \sum_{i=1}^{n} \text{Var}[X_i] + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \text{Cov}\left[X_i, X_j\right].$$

- This is also the ________ of the elements of the Covariance Matrix of the random variables X_1, X_2, ..., X_n

$$\begin{bmatrix} \text{Var}[X_1] & \text{Cov}[X_1,X_2] & \cdots & \text{Cov}[X_1,X_n] \\ & \text{Var}[X_2] & & \\ & & \ddots & \\ \text{Cov}[X_n,X_1] & \cdots & \cdots & \text{Var}[X_n] \end{bmatrix}_{n \times n}$$

Theorem 6.3

When $X_1, \ldots, X_n$ are uncorrelated,

$$\mathrm{Var}[W_n] = \mathrm{Var}[X_1] + \cdots + \mathrm{Var}[X_n].$$

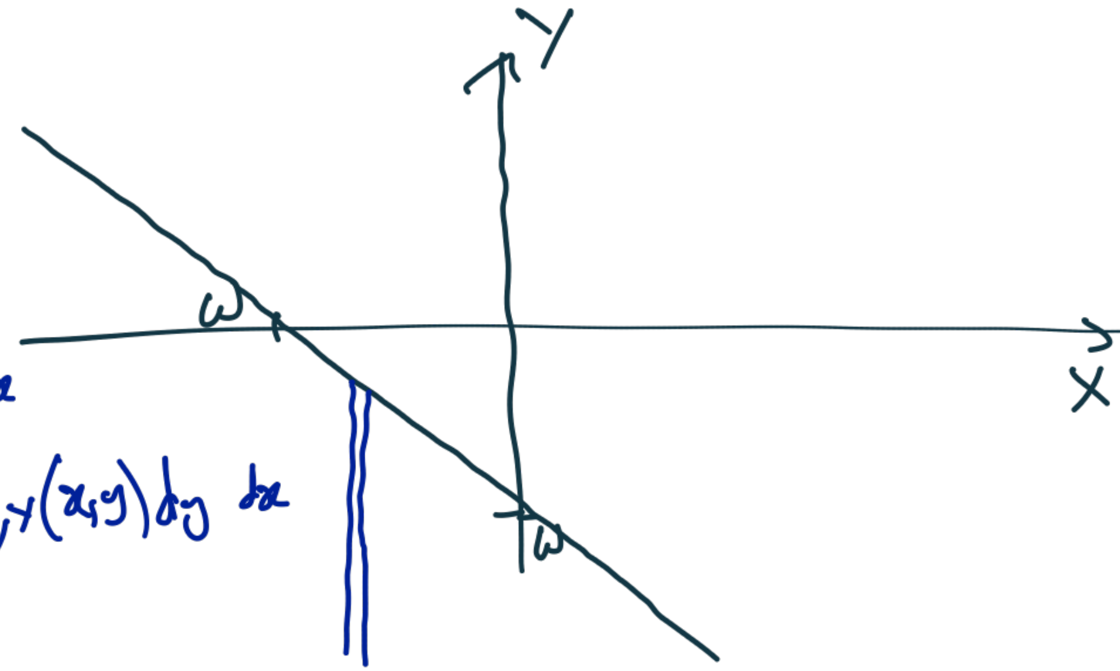
Theorem 6.4

The PDF of $W = X + Y$ is

$$f_W(w) = \int_{-\infty}^{\infty} f_{X,Y}(x, w - x)\, dx = \int_{-\infty}^{\infty} f_{X,Y}(w - y, y)\, dy.$$

$$P[W \le w] = ?$$

$$\hookrightarrow P[X+Y \le w] = \int_{-\infty}^{\infty} \int_{-\infty}^{w-x} f_{X,Y}(x,y)\, dy\, dx$$



PDF of Sum of Two Random Variables

Theorem 6.5

When X and Y are independent random variables, the PDF of $W = X + Y$ is

$$f_W(w) = \int_{-\infty}^{\infty} f_X(w - y) f_Y(y)\, dy = \int_{-\infty}^{\infty} f_X(x) f_Y(w - x)\, dx.$$

Moment Generating Function

Moment Generating Function

Definition 6.1 (MGF)

For a random variable X, the moment generating function (MGF) *of X is*

$$\phi_X(s) = E\left[e^{sX}\right].$$

- What about the continuous case?
- One-to-one mapping between MGF of a RV and its PDF
  - See table in textbook

Theorem 6.6

A random variable X with MGF $\phi_X(s)$ has nth moment

$$\phi_X(s) = E[e^{sX}]$$

$$E[X^n] = \left.\frac{d^n \phi_X(s)}{ds^n}\right|_{s=0}.$$

$$\frac{d}{ds} E[e^{sX}] = E[Xe^{sX}]$$

$$E[Xe^{sX}]\Big|_{s=0} = E[X]$$

Example 6.5 Problem

X is an exponential random variable with MGF $\phi_X(s) = \lambda/(\lambda - s)$. What are the first and second moments of X? Write a general expression for the nth moment.

$$\frac{d}{ds}\frac{\lambda}{\lambda - s} = \frac{+\lambda}{(\lambda - s)^2}$$

At $s=0$, $\frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$

$$E[X^n] = \frac{n!}{\lambda^n}$$

$$\frac{d^2}{ds^2}\frac{\lambda}{\lambda - s} = \frac{2\lambda}{(\lambda - s)^3}$$

MGF of a Scaled and Shifted RV

Theorem 6.7

The MGF of $Y = aX + b$ is $\phi_Y(s) = e^{sb}\phi_X(as)$.

$$\phi_Y(s) = E\left[e^{sY}\right] = E\left[e^{s(aX+b)}\right]$$

$$= E\left[e^{saX} e^{sb}\right]$$

$$= e^{sb} E\left[e^{(sa)X}\right]$$

$$= e^{sb} \phi_X(as)$$

Theorem 6.8: MGF of Sum of Independent Random Variables

For a set of independent random variables $X_1, \ldots, X_n$, the moment generating function of $W = X_1 + \ldots + X_n$ is $E[e^{sW}] = E[e^{s(X_1+X_2+\cdots+X_n)}] =$

$$\phi_W(s) = \phi_{X_1}(s)\phi_{X_2}(s)\ldots\phi_{X_n}(s).$$

$E[e^{sX_1} e^{sX_2} \cdots e^{sX_n}]$

Given indep.

$(E[e^{sX_1}] E[e^{sX_2}] \cdots E[e^{sX_n}])$

When $X_1, \ldots, X_n$ are iid, each with MGF $\phi_{X_i}(s) = [\phi_X(s)]^n$,

$$\phi_W(s) = (\phi_X(s))^n.$$

independent & identically distributed

If $X_1, X_2, \ldots, X_n$ are identical, then $X_1, X_2, \ldots, X_n$ are all distributed as some RV X. $f_{X_1}(x) = f_{X_2}(x) \ldots = f_{X_n}(x)$ $\forall x$.

Independent RV(s):

$$f_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n) = f_{X_1}(x_1)\, f_{X_2}(x_2) \ldots f_{X_n}(x_n)$$

$$\forall (x_1, x_2, \ldots, x_n) \in R^n$$

iid:

$$f_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n) = f_X(x_1)\, f_X(x_2) \ldots f_X(x_n)$$

$$\forall (x_1, x_2, \ldots, x_n) \in R^n$$

Example 6.6 Problem

J and K are independent random variables with probability mass functions

$$P_J(j) = \begin{cases} 0.2 & j = 1, \\ 0.6 & j = 2, \\ 0.2 & j = 3, \\ 0 & \text{otherwise,} \end{cases} \qquad P_K(k) = \begin{cases} 0.5 & k = -1, \\ 0.5 & k = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{6.40}$$

Find the MGF of $M = J + K$? What are $E[M^3]$ and $P_M(m)$?

Sum of Independent Poisson Random Variables is Poisson Distributed

Theorem 6.9

If $K_1, \ldots, K_n$ are independent Poisson random variables, $W = K_1 + \cdots + K_n$ is a Poisson random variable.

- The PMF of Poisson RV is

$$P_{K_i}(x) = \begin{cases} \alpha_i^x e^{-\alpha_i}/x! & x = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

- The MGF is

$$\phi_{K_i}(s) = e^{(\alpha_i)(e^s - 1)}$$

Sum of Independent Poisson Random Variables is Poisson Distributed

Theorem 6.9

If $K_1, \ldots, K_n$ are independent Poisson random variables, $W = K_1 + \cdots + K_n$ is a Poisson random variable.

- The MGF of the sum of *n* independent RVs is...

Sum of Independent Gaussian Random Variables is Gaussian Distributed

Theorem 6.10

The sum of n independent Gaussian random variables $W = X_1 + \cdots + X_n$ is a Gaussian random variable.

- The MGF of a Gaussian is

$$\phi_{X_i}(s) = e^{s\mu_i + s^2\sigma_i^2/2}$$

- Can you show the above?

Sum of Independent Exponentials

Theorem 6.11

If $X_1, \ldots, X_n$ are iid exponential (λ) random variables, then $W = X_1 + \cdots + X_n$ has the Erlang PDF

$$f_W(w) = \begin{cases} \frac{\lambda^n w^{n-1} e^{-\lambda w}}{(n-1)!} & w \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

- The MGF of an exponential is

$$\phi_{X_i}(s) = \left(\frac{\lambda}{\lambda - s}\right)$$

Quiz 6.4(B)

Let $X_1, \ldots, X_n$ be independent Gaussian random variables with $E[X_i = 0]$ and $\text{Var}[X_i] = i$. Find the PDF of

$$W = \alpha X_1 + \alpha^2 X_2 + \cdots + \alpha^n X_n. \tag{6.54}$$

$$E[e^{sW}] = E[e^{s(\alpha X_1 + \alpha^2 X_2 + \cdots + \alpha^n X_n)}]$$

$$= E[e^{s\alpha X_1} e^{s\alpha^2 X_2} \cdots e^{s\alpha^n X_n}]$$

$$= \underbrace{E[e^{s\alpha X_1}]}_{} E[e^{s\alpha^2 X_2}] \cdots E[e^{s\alpha^n X_n}]$$

$$\phi_{X_1}(\alpha s)\ \phi_{X_2}(\alpha^2 s) \cdots \phi_{X_n}(\alpha^n s)$$

Problem 6.8.4

In a subway station, there are exactly enough customers on the platform to fill three trains. The arrival time of the nth train is $X_1 + \cdots + X_n$ where $X_1, X_2, \ldots$ are iid exponential random variables with $E[X_i] = 2$ minutes. Let W equal the time required to serve the waiting customers.

Find $P[W > 20]$.

Central Limit Theorem

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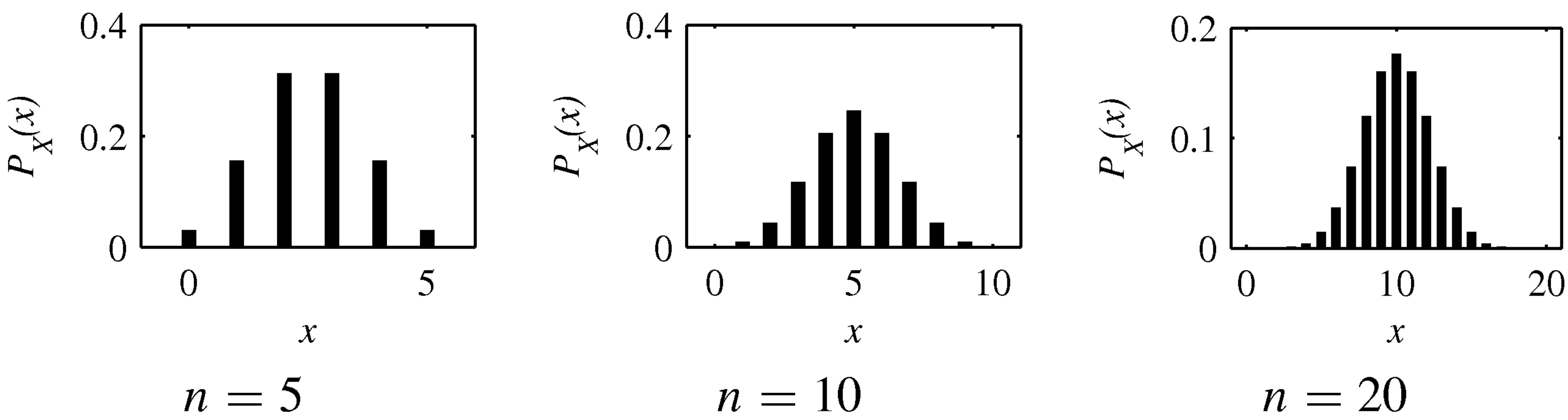
Central Limit Theorem

Figure 6.1



The PMF of the X, the number of heads in n coin flips for $n = 5, 10, 20$. As n increases, the PMF more closely resembles a bell-shaped curve.

- Which PMF are we talking about?

Central Limit Theorem

Theorem 6.14 Central Limit Theorem

Given $X_1, X_2, \ldots$, a sequence of iid random variables with expected value μ_X and variance σ_X^2, the CDF of $Z_n = \underline{(\sum_{i=1}^{n} X_i - n\mu_X)/\sqrt{n\sigma_X^2}}$ has the property

$$\lim_{n\to\infty} F_{Z_n}(z) = \Phi(z).$$

$W_n \triangleq \sum_{i=1}^{n} X_i$ $\qquad$ $\lim_{n\to\infty} \mathrm{Var}[W_n] = ?$