Graphs: Minimum Spanning Trees (Kruskal’s algorithm)

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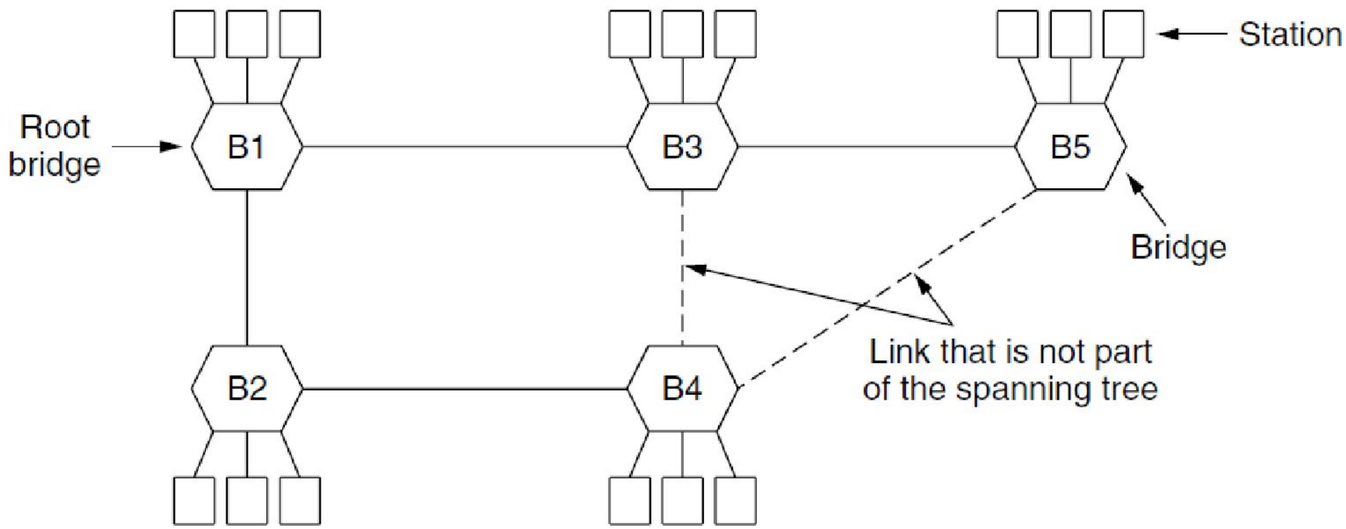
Some of the slides are from https://courses.cs.washington.edu/courses/cse373/22sp/, or from those prepared by Si Dong, M.T. Goodrich and R. Tamassia (reference??)

Outline

- Graphs:
  - Undirected graphs
  - Directed graphs
  - (Directed) acyclic graphs (or DAGs)
  - Sparse graphs
  - Weighted graphs
- Graph applications
- Representation of graphs:
  - Adjacency matrix
  - Linked lists
- Algorithms:
  - Traversal algorithms:
    - BFS
    - DFS
  - Topological sort
  - Minimum spanning trees
  - Dijkstra's Shortest path
    - One-to-one
    - One-to-many
    - Many-to-many

Spanning tree

- Spanning Tree applied to network of routers or bridges



Spanning tree

- Spanning Tree problem:

  Consider:

  a connected, undirected graph G = (V, E)

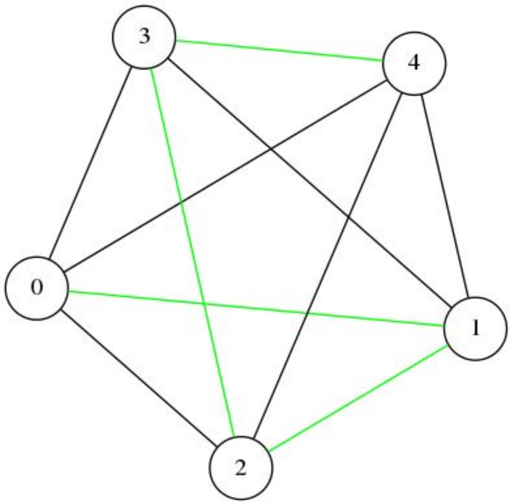
  Objective:

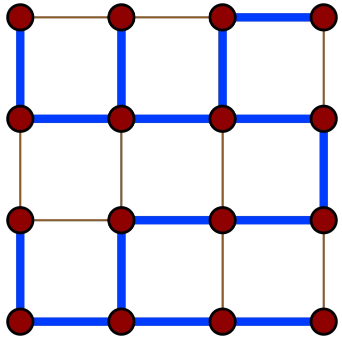
  compute a spanning tree G1 = (V, T), where

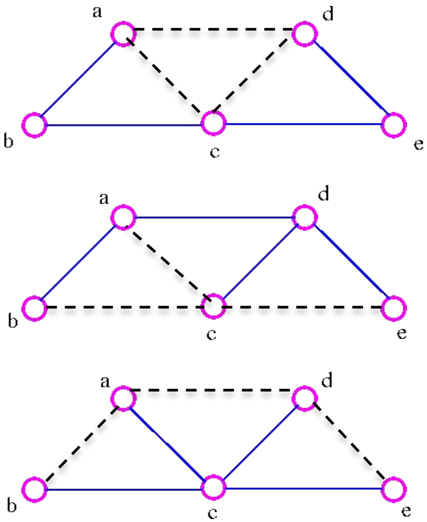
  1. G1 is a sub-graph of G, i.e. T is subset of E, while all vertices in V are in G1
  2. G1 is connected -- all vertices in V are reachable from every other vertex in V but using edges in T only,
  3. there are no cycles in G1

Spanning tree

- Spanning Tree: examples:





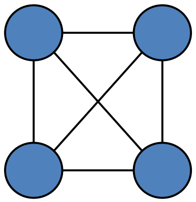


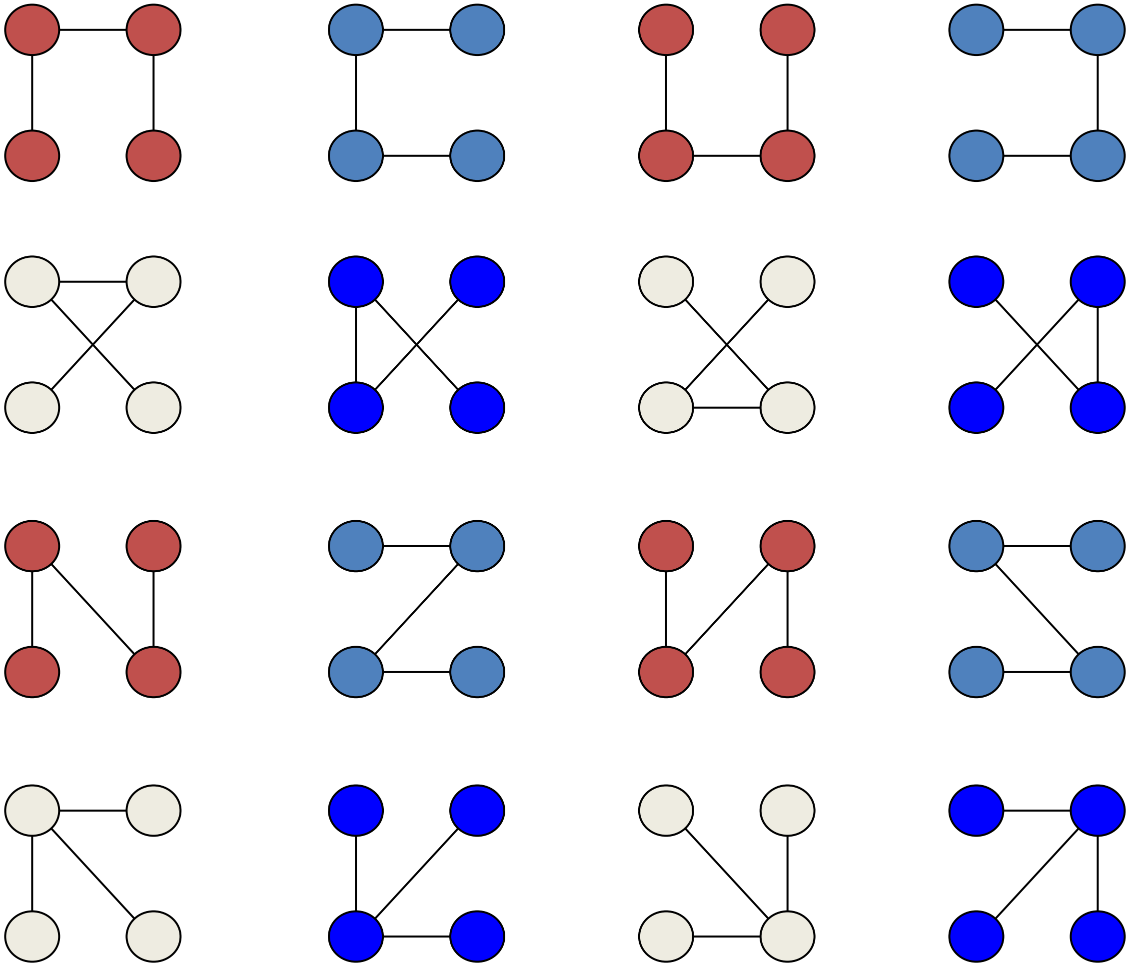
Spanning tree

- Spanning Tree: examples:

All possible spanning trees

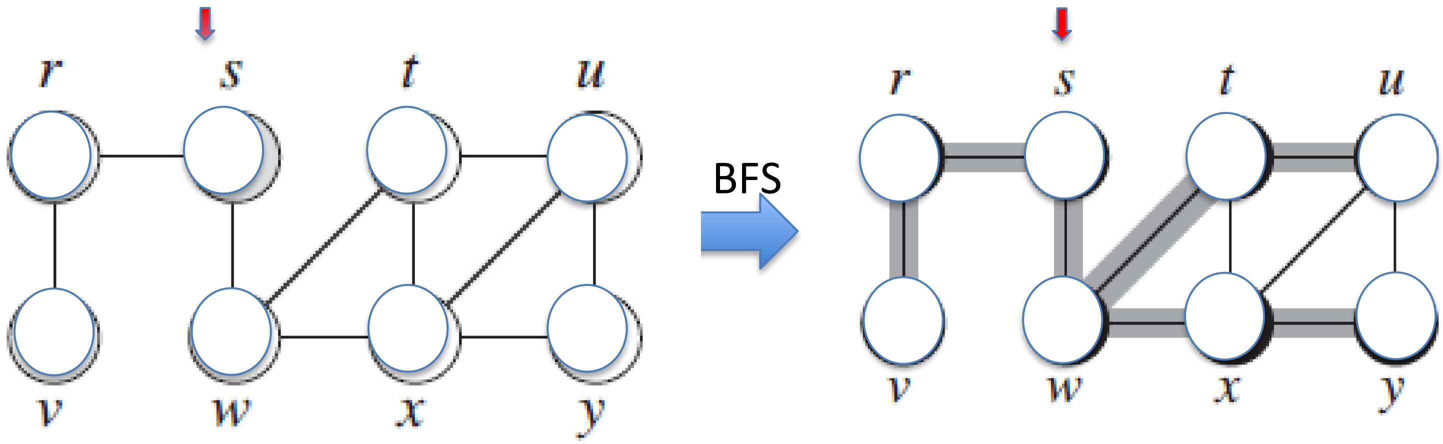
Complete Graph

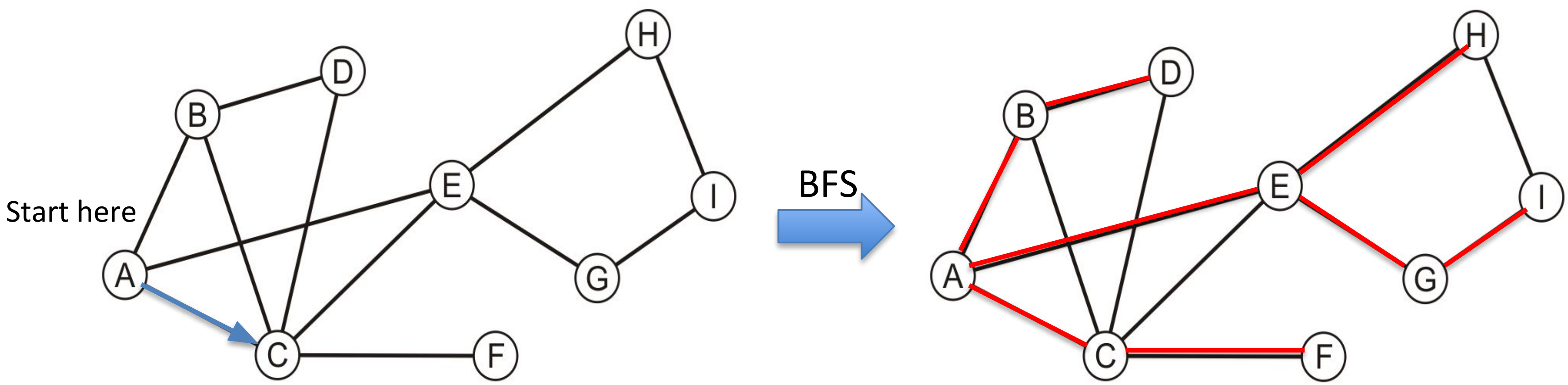




Spanning tree

- Spanning Tree: Use BFS to compute a spanning tree:



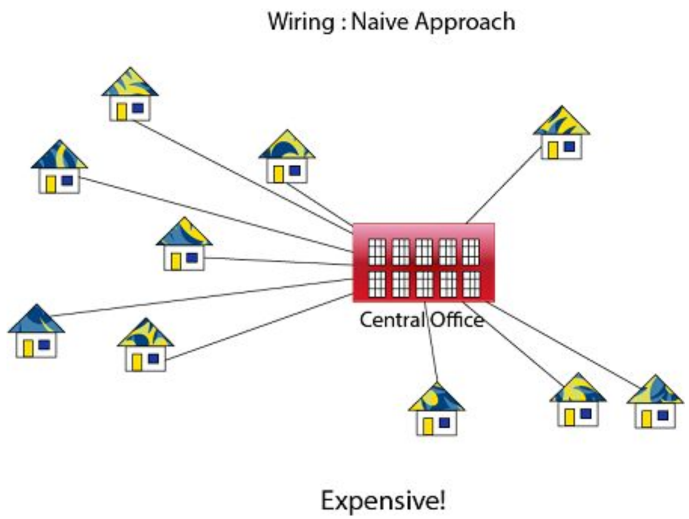


Minimum spanning tree

- Minimum spanning trees == minimum-weight spanning trees
- Applications:
  - routing wires on printed circuit boards
  - Planning sewer pipe layout
  - Road network planning
  - metro train network
  - telephone lines to a set of houses

Minimum spanning tree

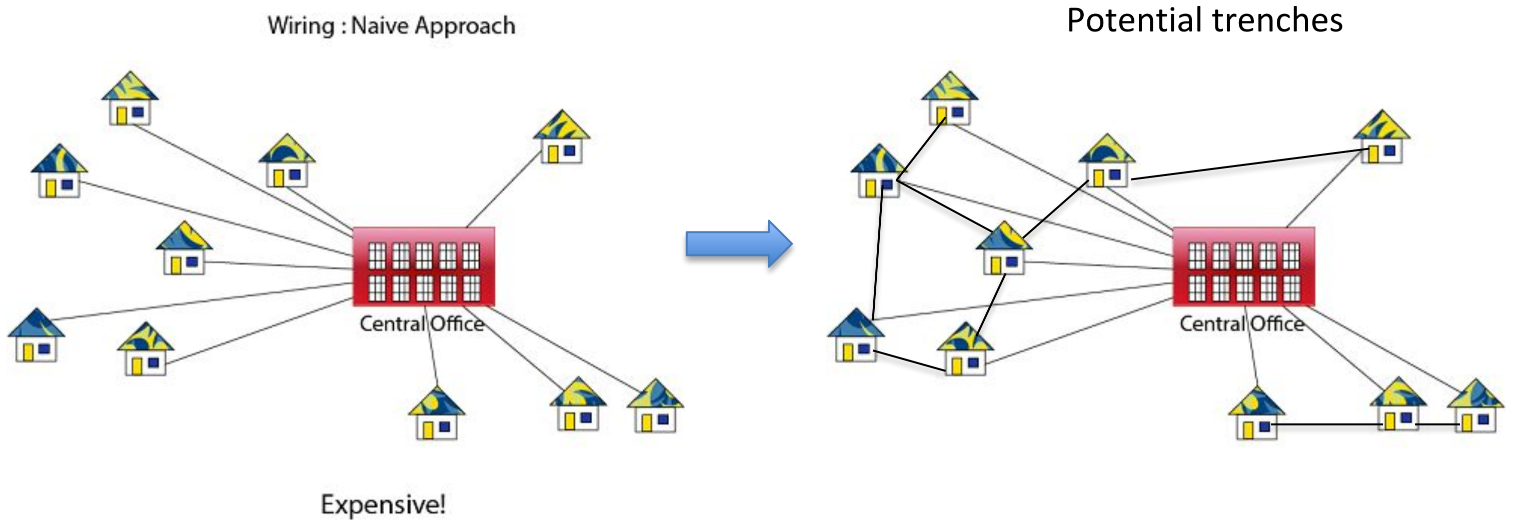
- MST, applied to cabling landline phones to homes



Courtesy www.javatpoint.com

Minimum spanning tree

- MST, applied to cabling landline phones to homes

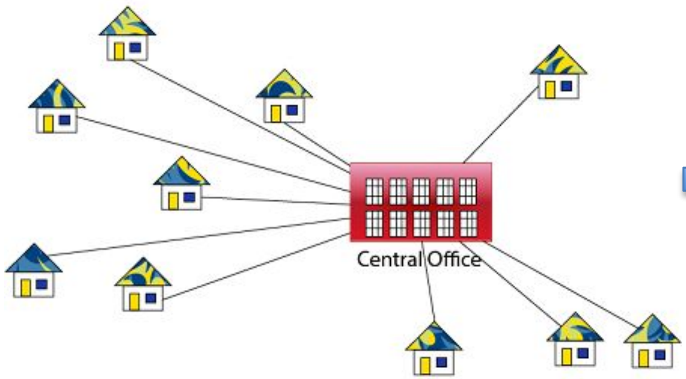


Courtesy www.javatpoint.com

Minimum spanning tree

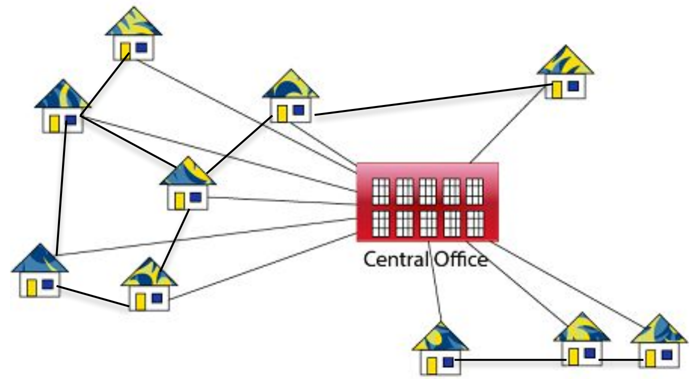
- MST, applied to cabling landline phones to homes

Wiring : Naive Approach

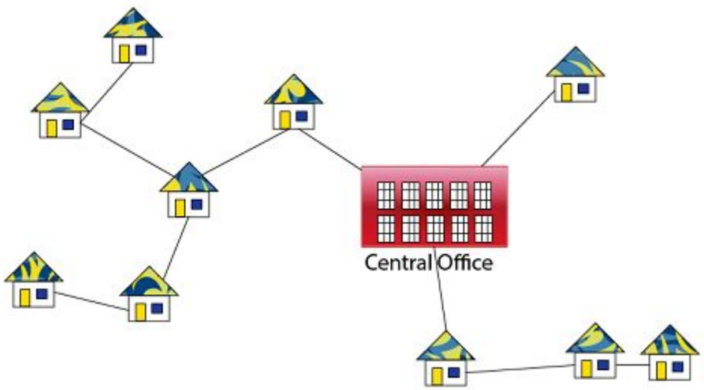


Expensive!

Potential trenches



Wiring : Better Approach



Minimize the total length of wire connecting the customers

Courtesy www.javatpoint.com

Minimum spanning tree

- Minimum-weight Spanning Tree problem:

  Consider:

  a connected, undirected graph G = (V, E), with weights w(u, v) associated with each edge, (u, v) in E

  Objective:

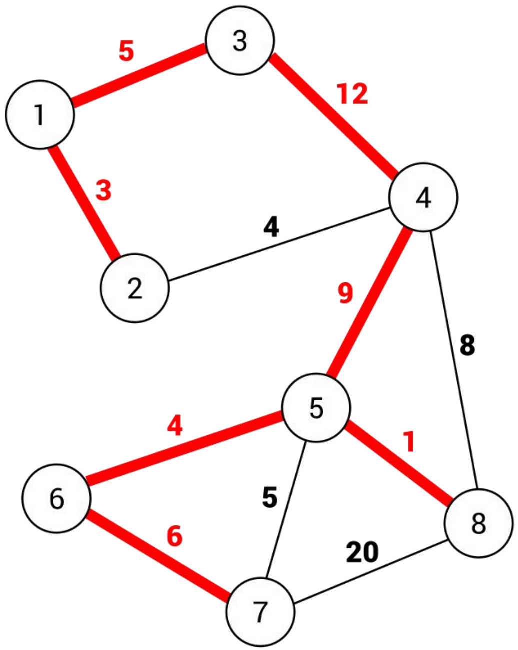
  compute a spanning tree G1 = (V, T), with minimize sum of weight of edges in T, viz.

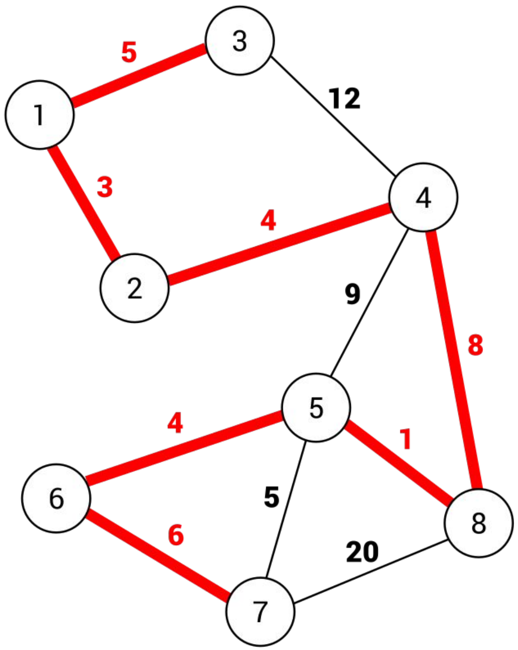
$$w(T) = \sum_{(u,v)\in T} w(u,v)$$

  (Note: G1 = (V, T) is a spanning tree of G, and $T \subseteq E$)

Minimum spanning tree

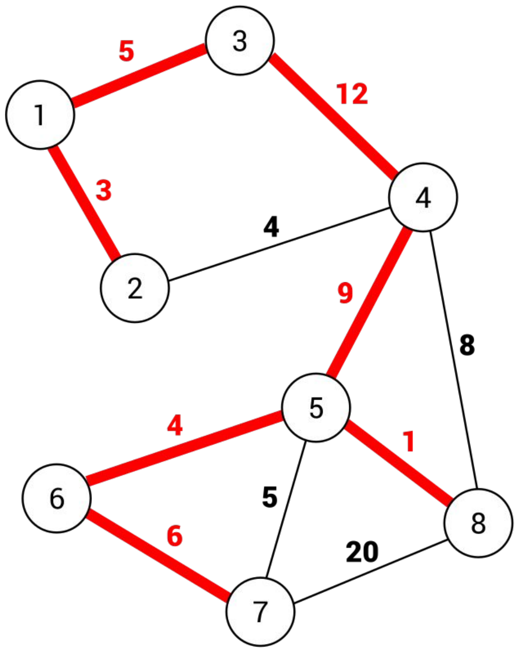
- Minimum spanning Tree problem:

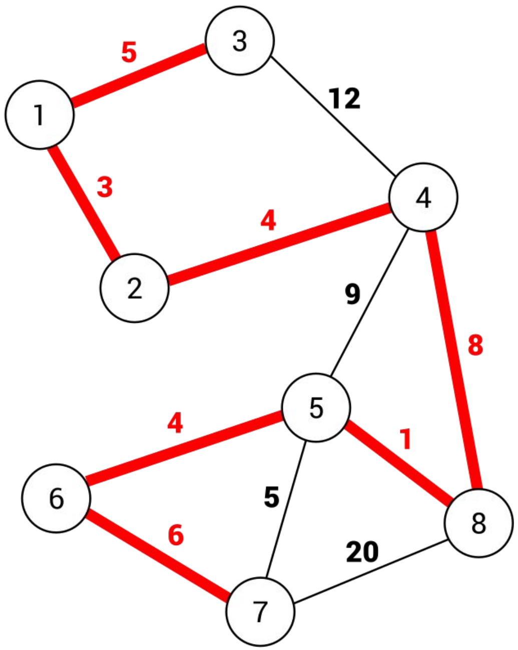




Minimum spanning tree

- Minimum spanning Tree problem:

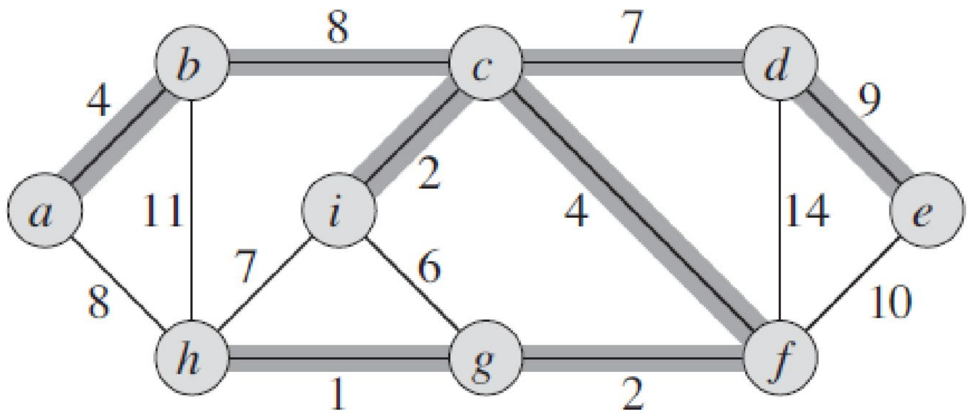




- What is the w(T) in each case?
- Can we still do better? Possibly.

Minimum spanning tree

- Minimum spanning Tree problem:



- Minimum weight = 37
- Not a unique minimum spanning tree – replace edge (b, c) with (a, h)

Minimum spanning tree

- Two algorithms:
  - Kruskal’s algorithm
  - Prim’s algorithm
- Time complexity is O(E logV)
  - May be improved – but that is for later

Kruskal's minimum spanning tree

Kruskal's algorithm:

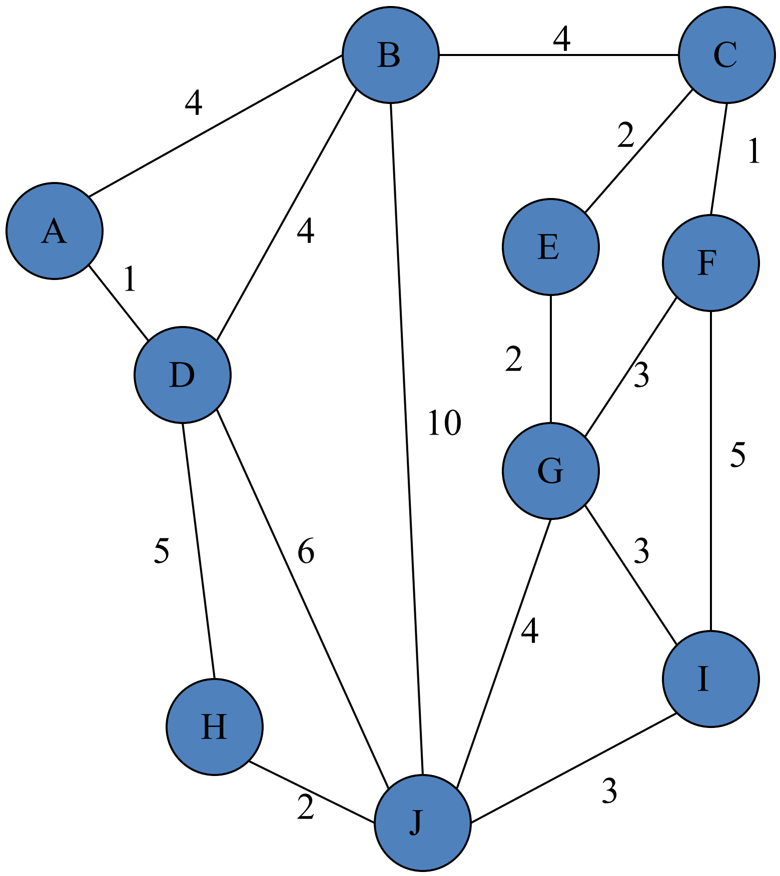
- Start with all vertices but no edges in the spanning tree
- Repeatedly add the cheapest edge that does not create a cycle

Prim's algorithm:

- Start with any one vertex in the spanning tree
- Repeatedly add the cheapest edge, and the NEW node it leads to
  - the new vertex is not in the spanning tree

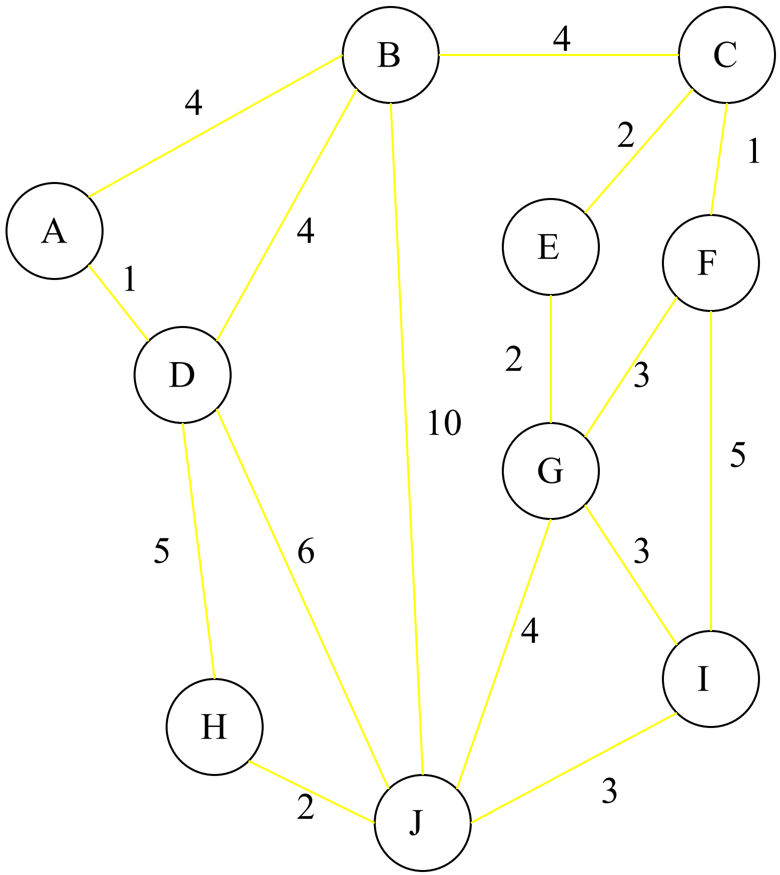
Kruskal’s minimum spanning tree

Graph, G

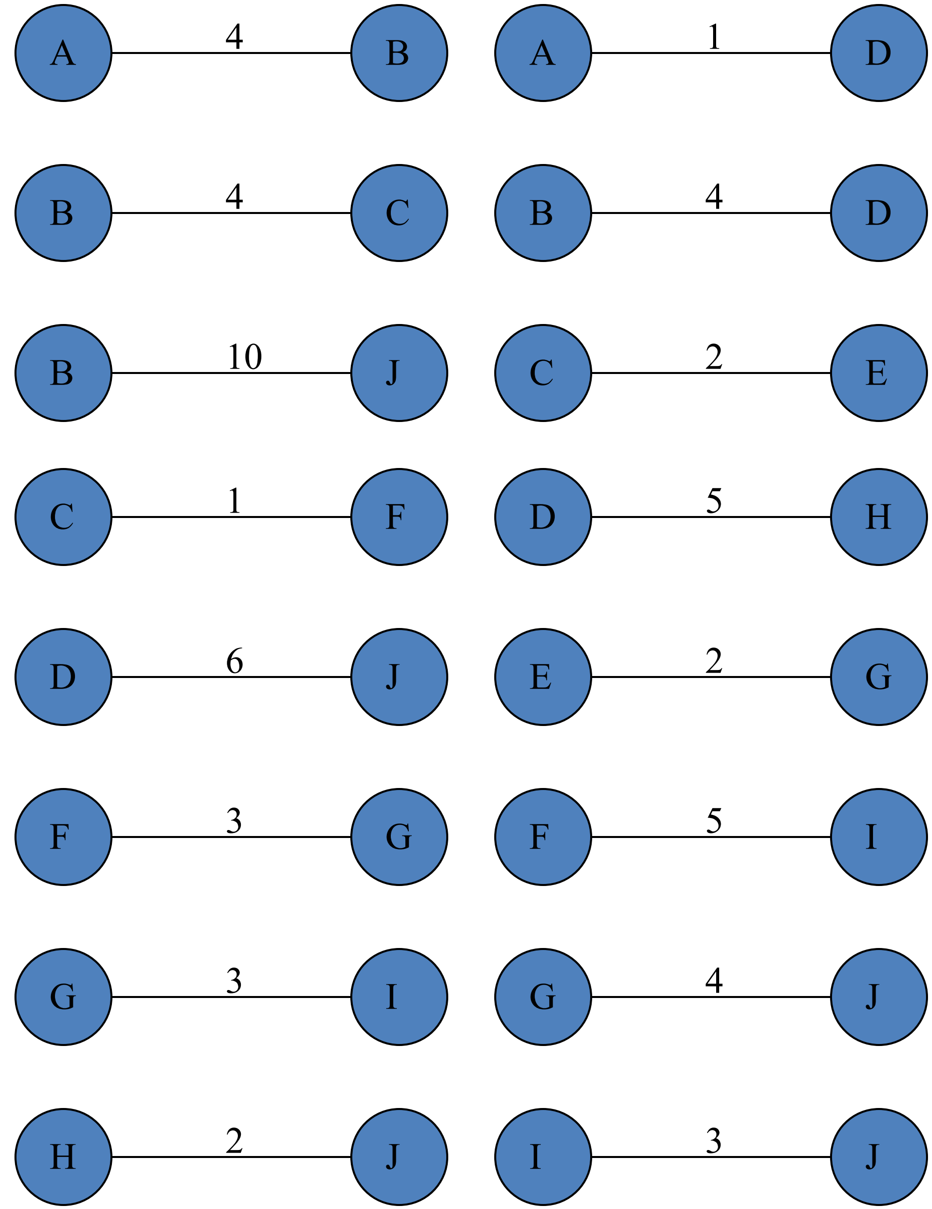


Kruskal’s minimum spanning tree

Current state of G1 = (V, T),
Initially T = []



Unsorted list of edges (by weight)



Kruskal's minimum spanning tree

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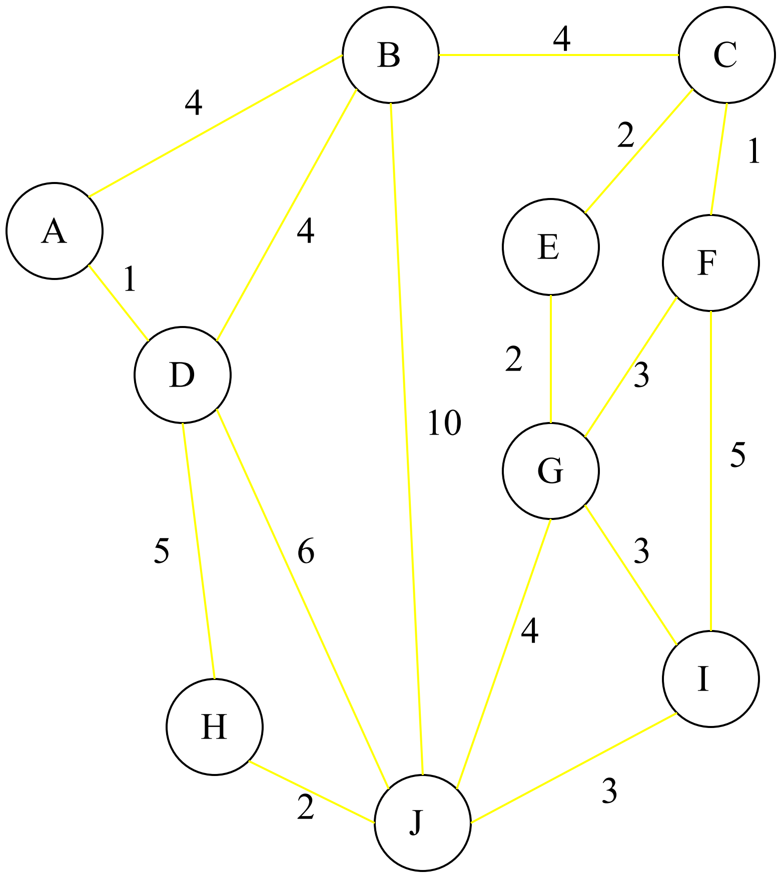
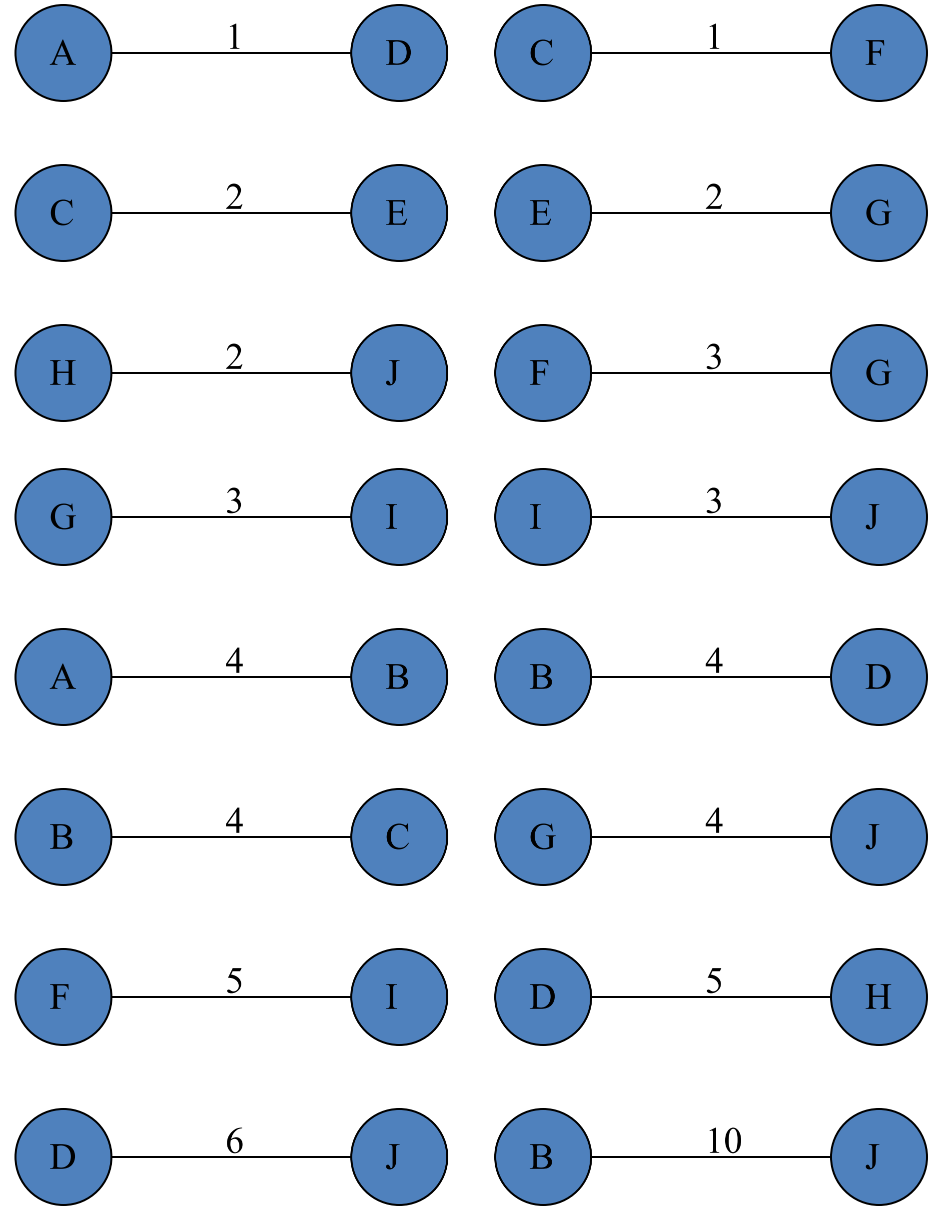
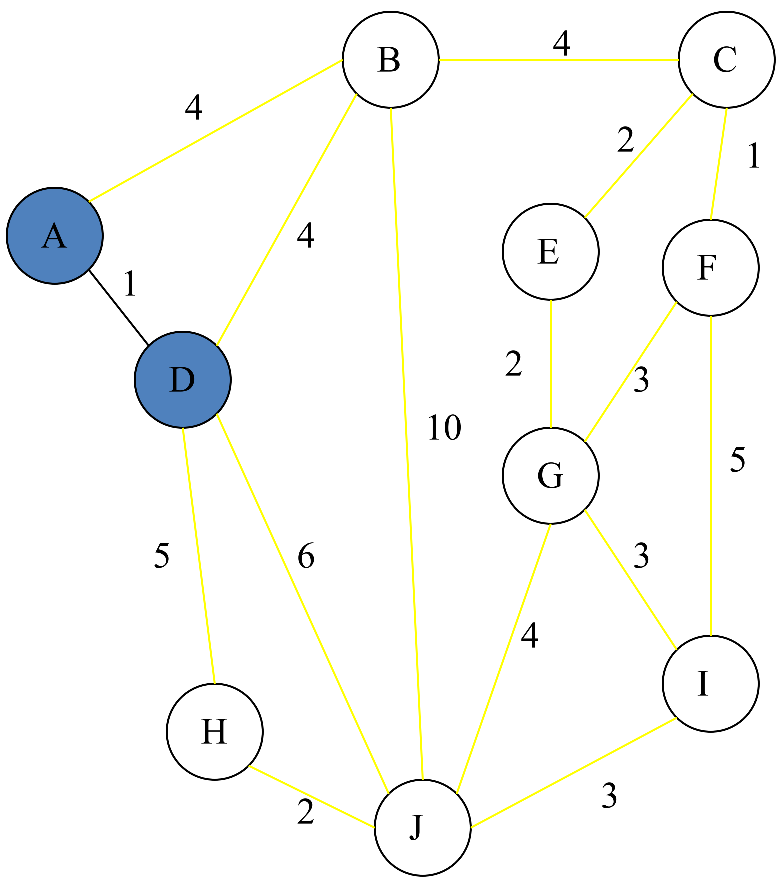


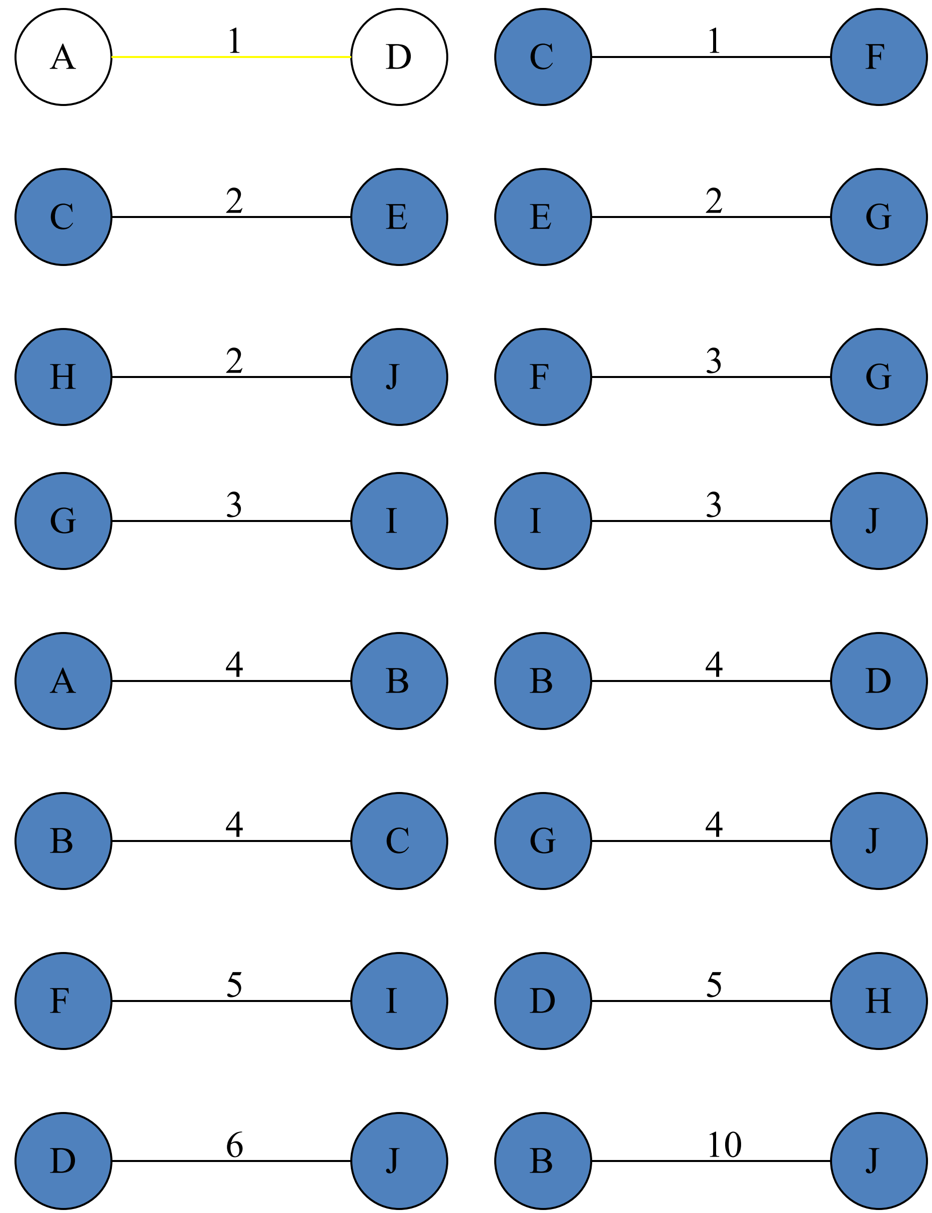
Sorted list of edges (by weight)



Kruskal's minimum spanning tree

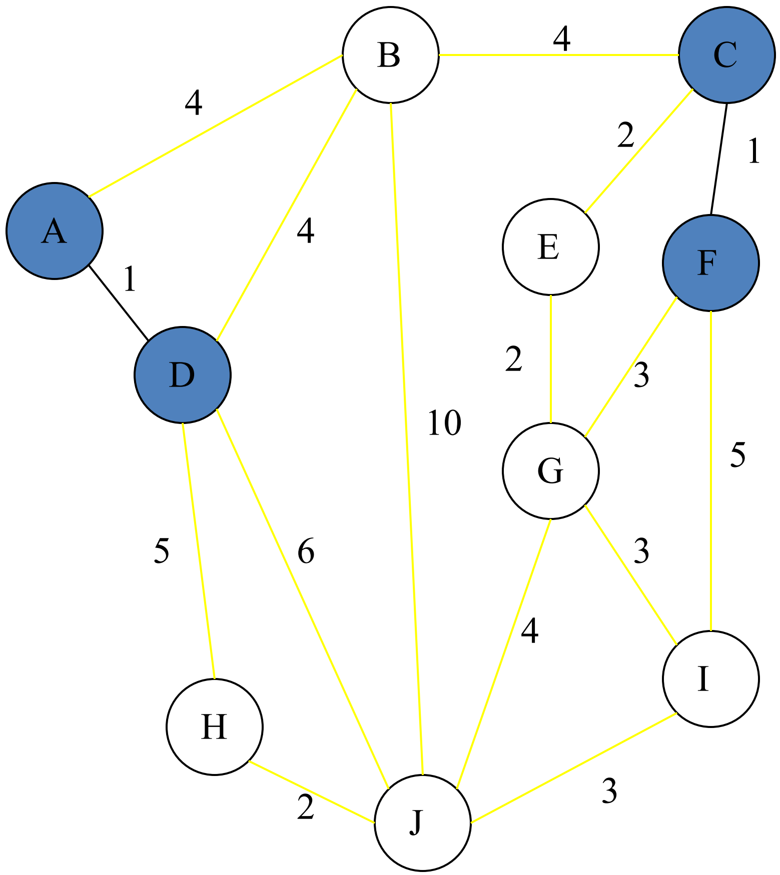
Current state of G1 = (V, T),
Add (a, d) to T

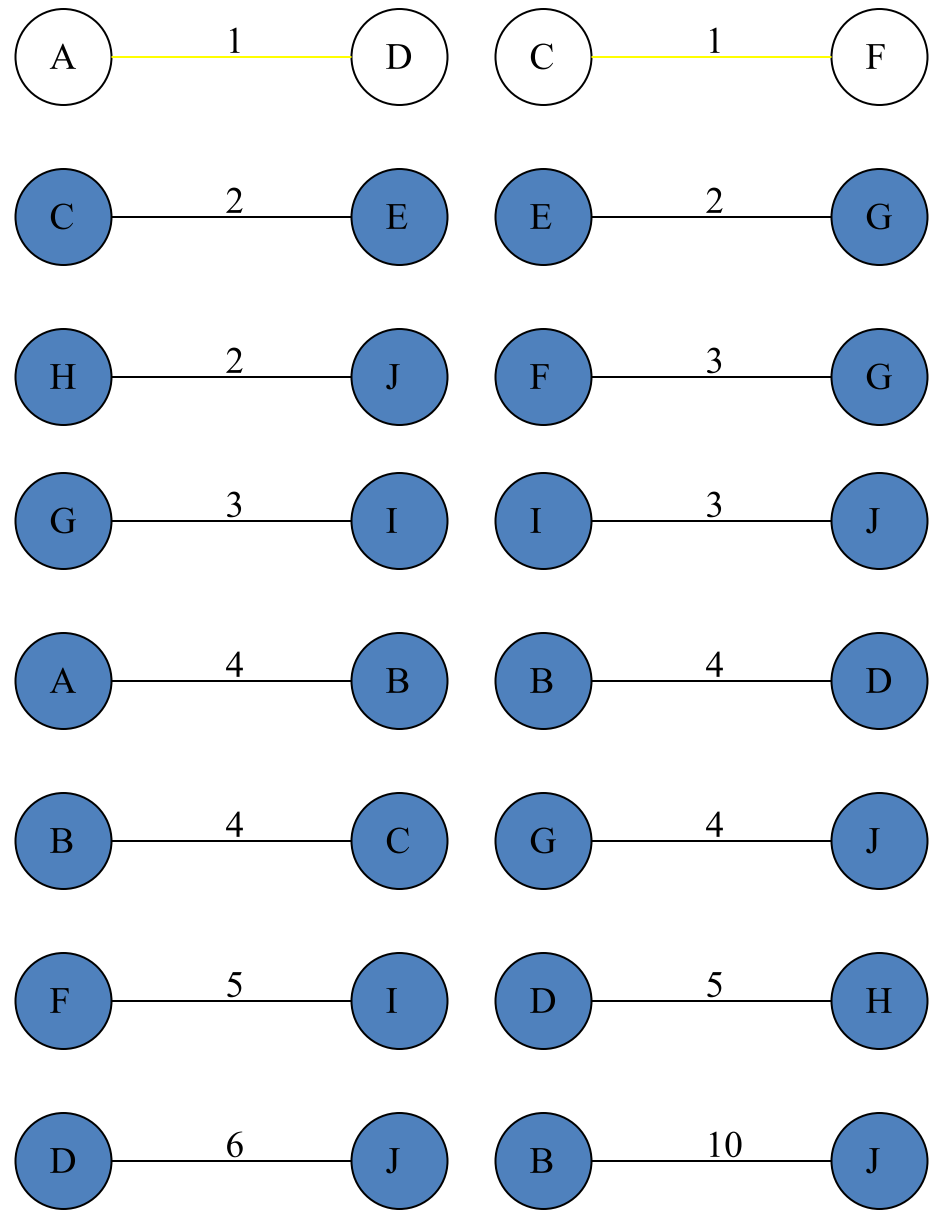




Kruskal's minimum spanning tree

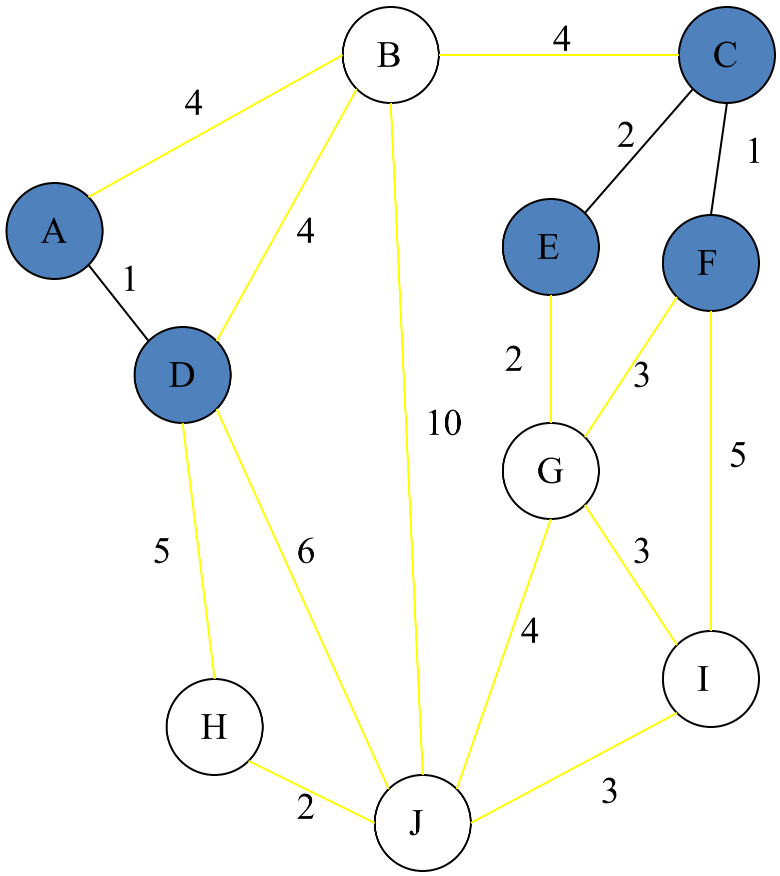
Current state of G1 = (V, T),
Add (c, f) to T

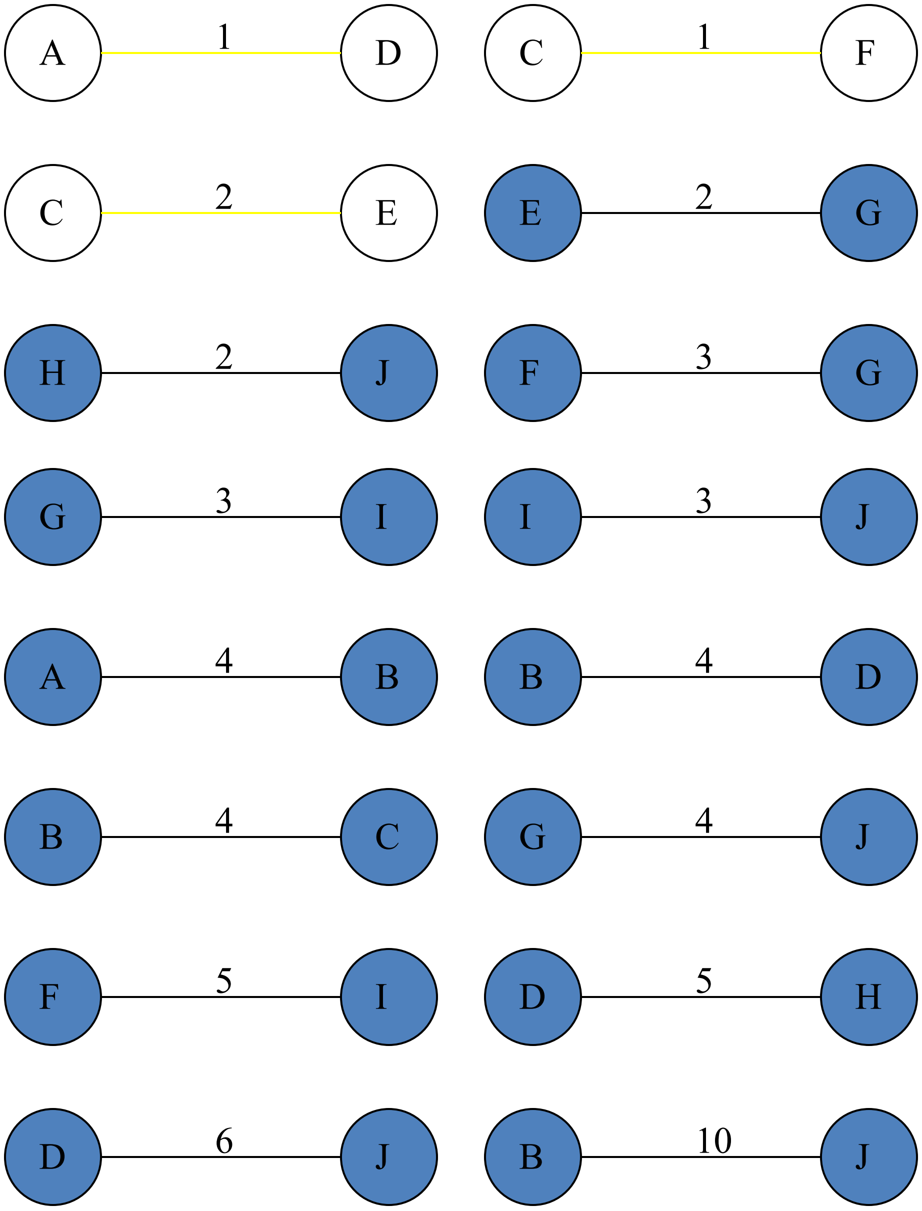




Kruskal's minimum spanning tree

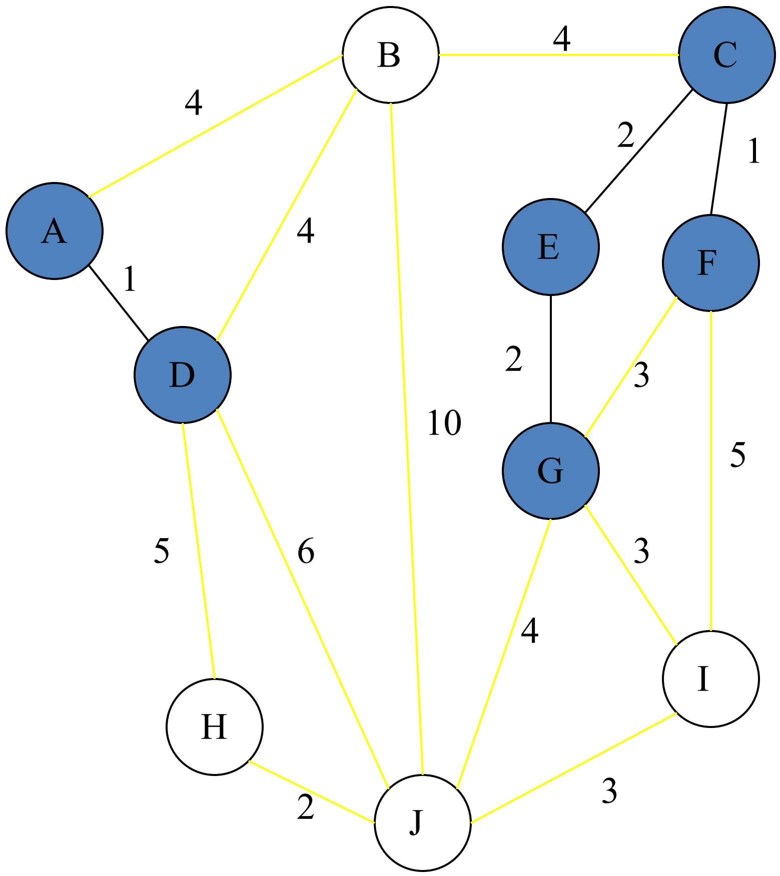
Current state of G1 = (V, T),
Add (c, e) to T

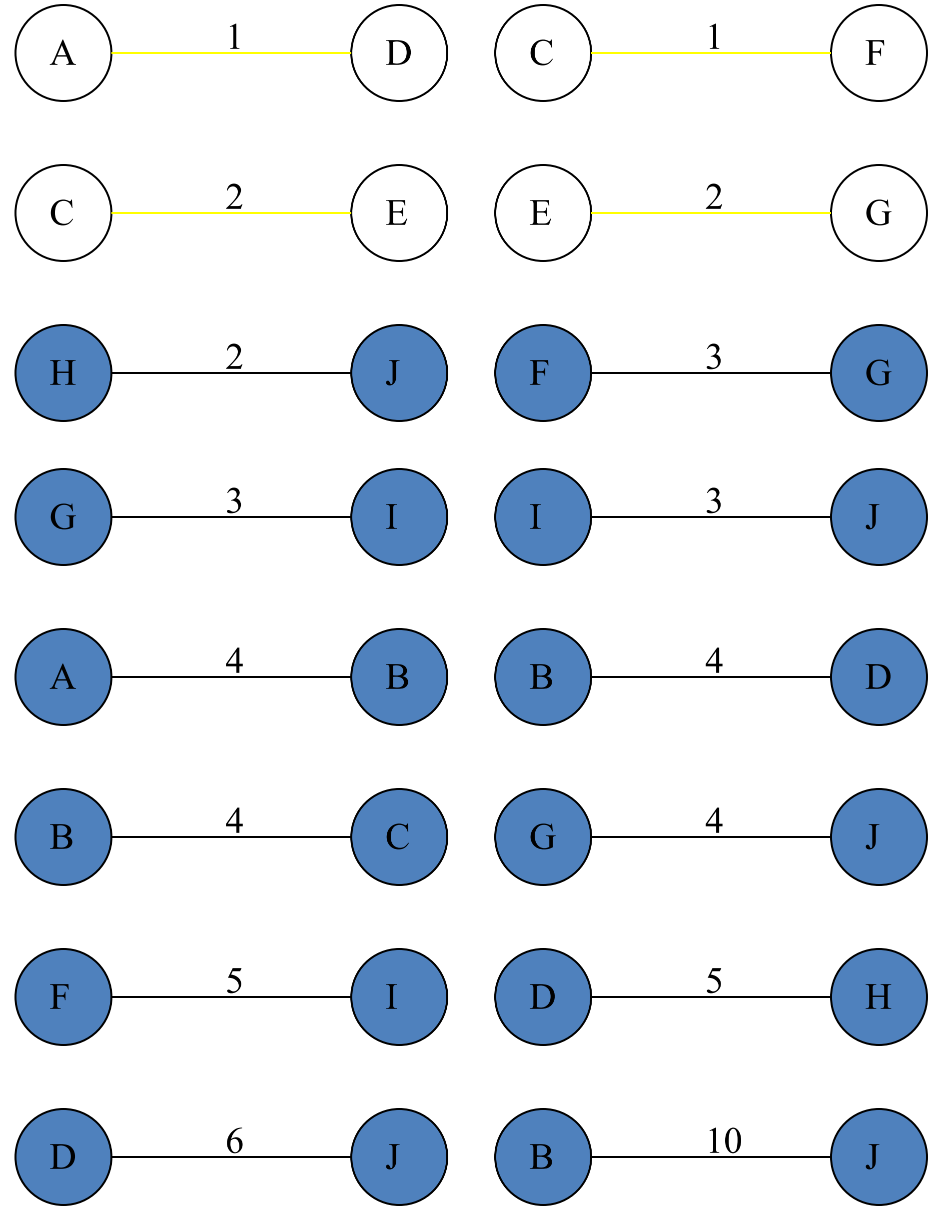




Kruskal's minimum spanning tree

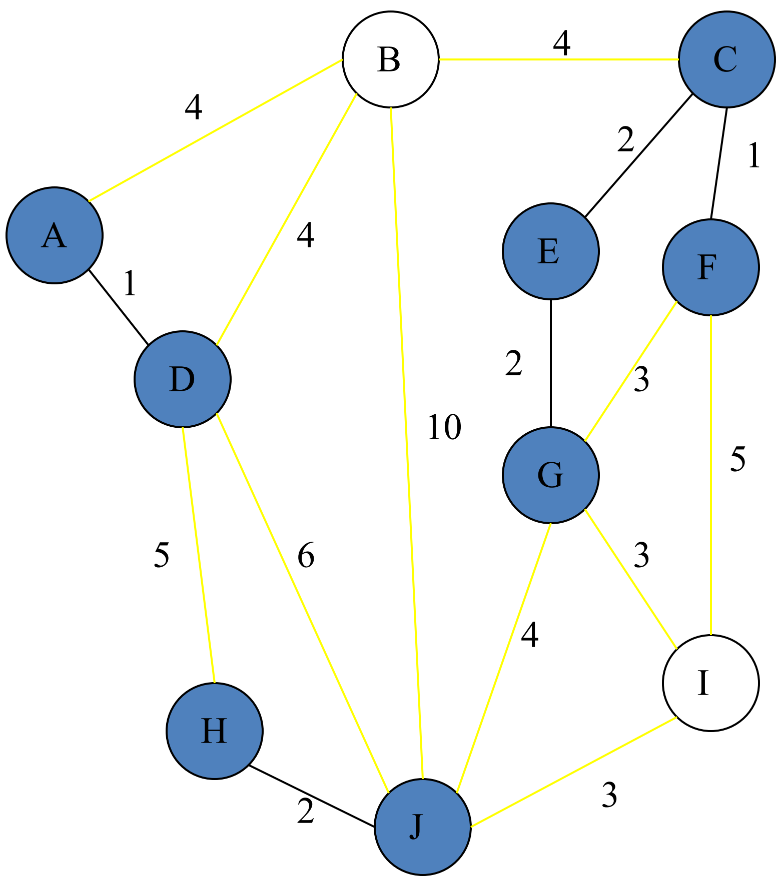
Current state of G1 = (V, T),
Add (e, g) to T

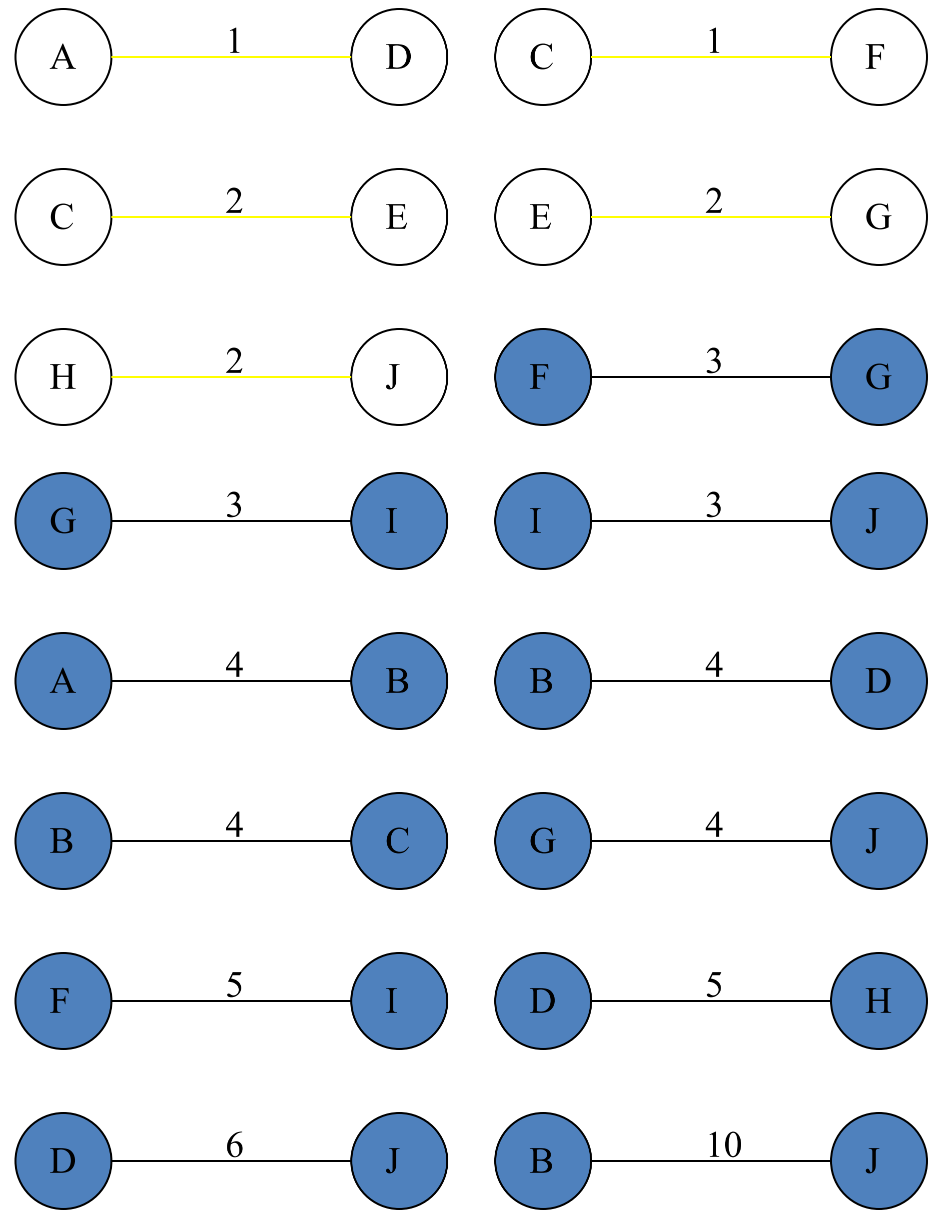




Kruskal's minimum spanning tree

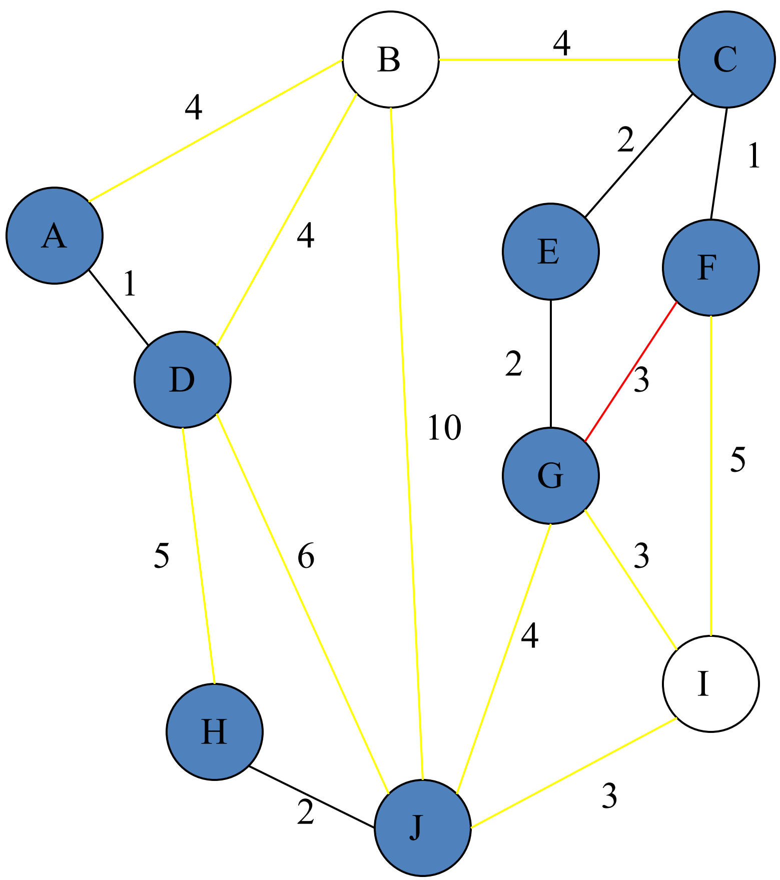
Current state of G1 = (V, T),
add (h, j) to T

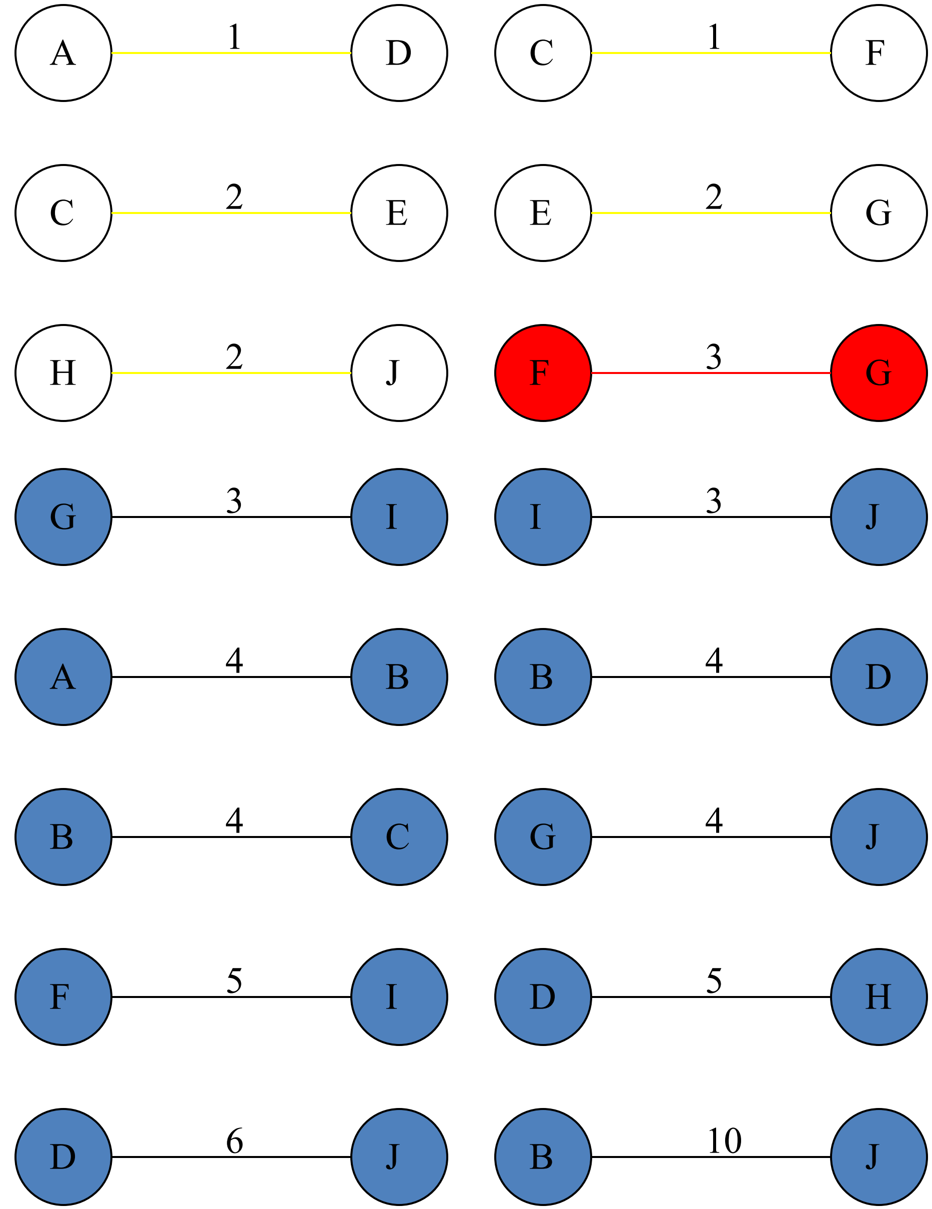




Kruskal’s minimum spanning tree

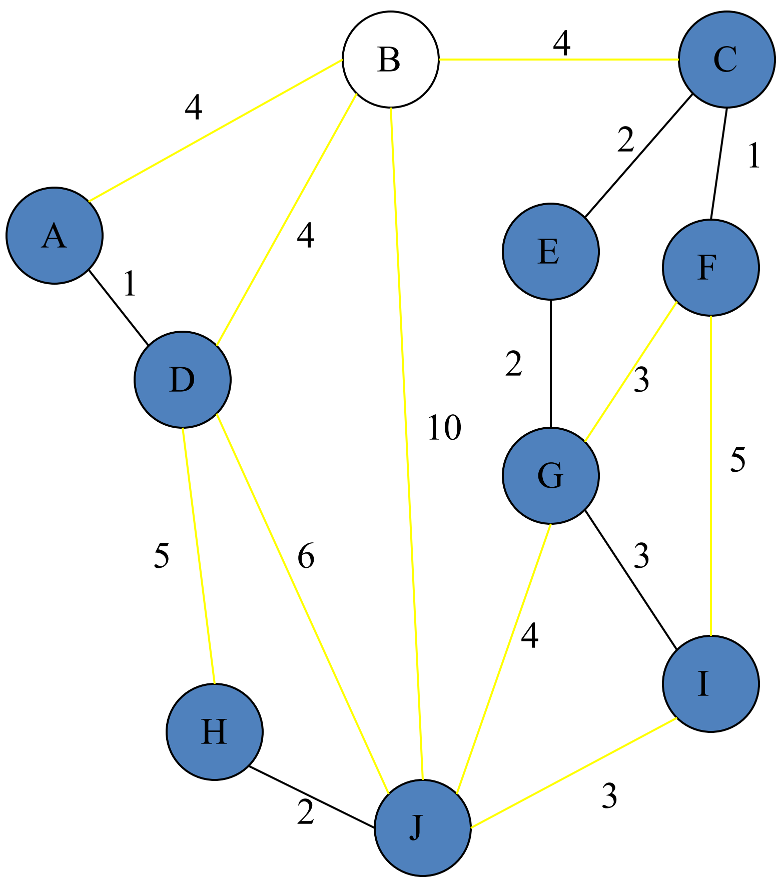
Current state of G1 = (V, T),
(f, g) forms a cycle No change

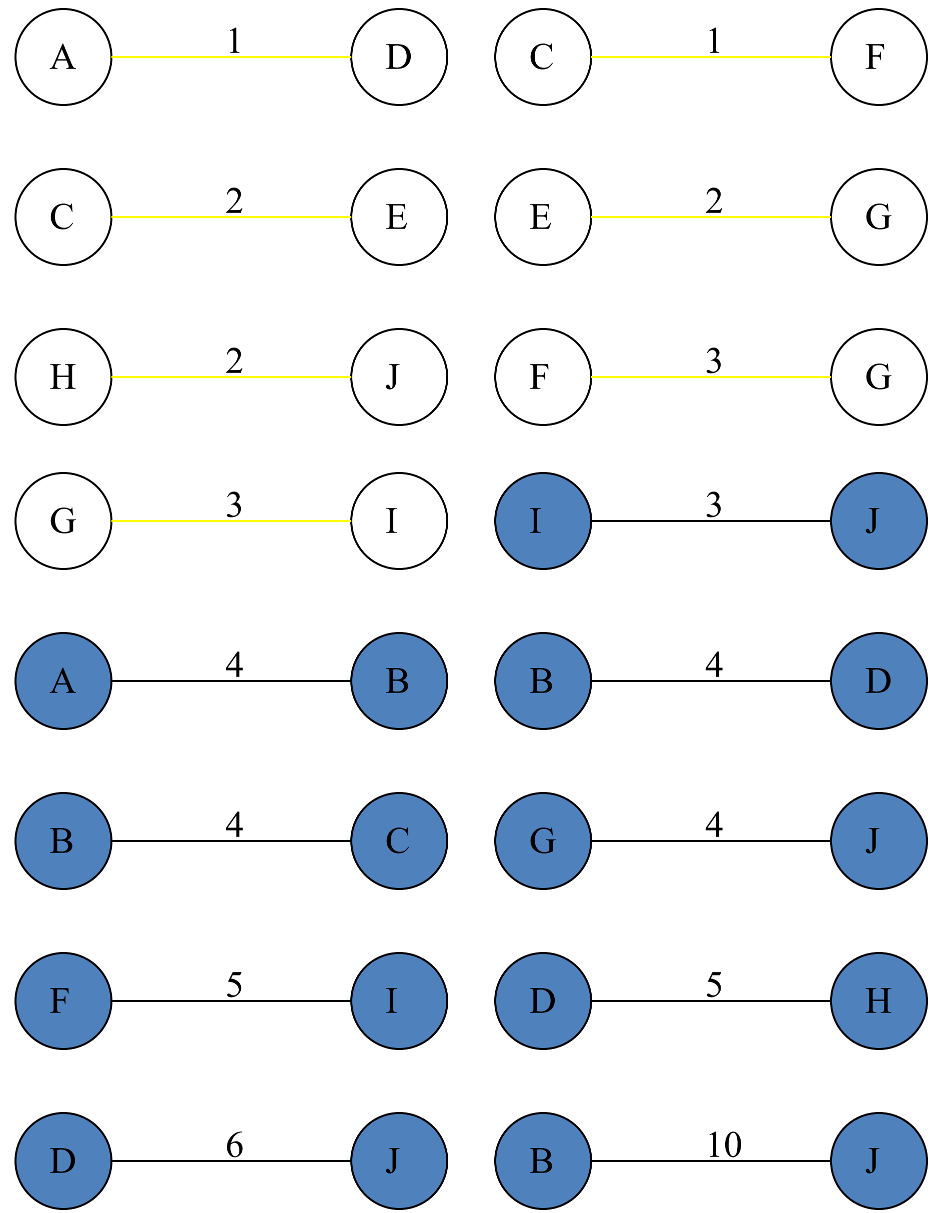




Kruskal's minimum spanning tree

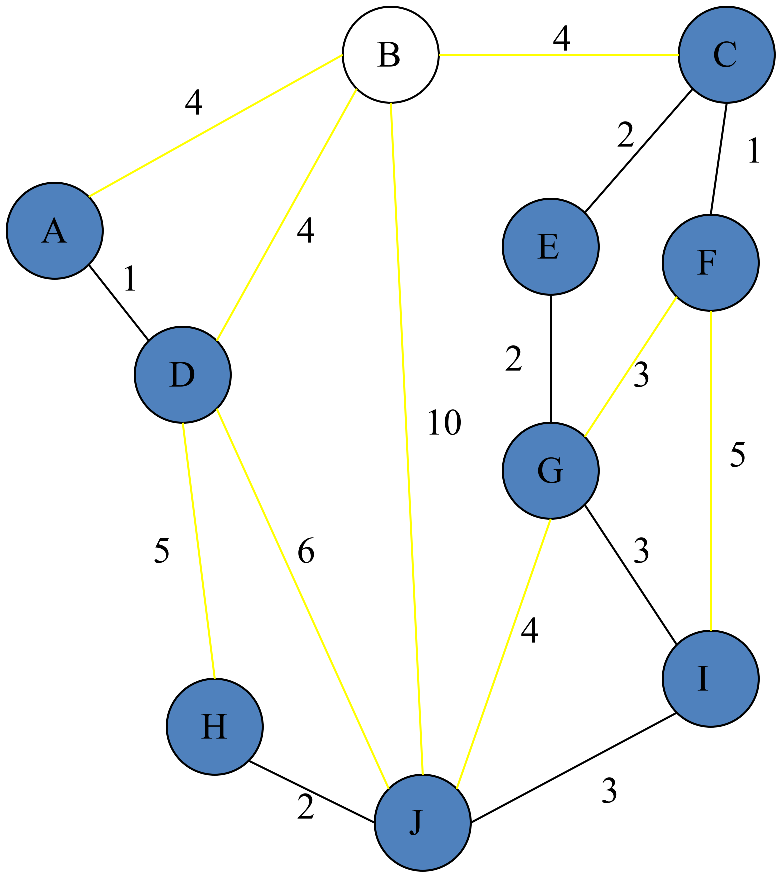
Current state of G1 = (V, T),
add (g, i) to T

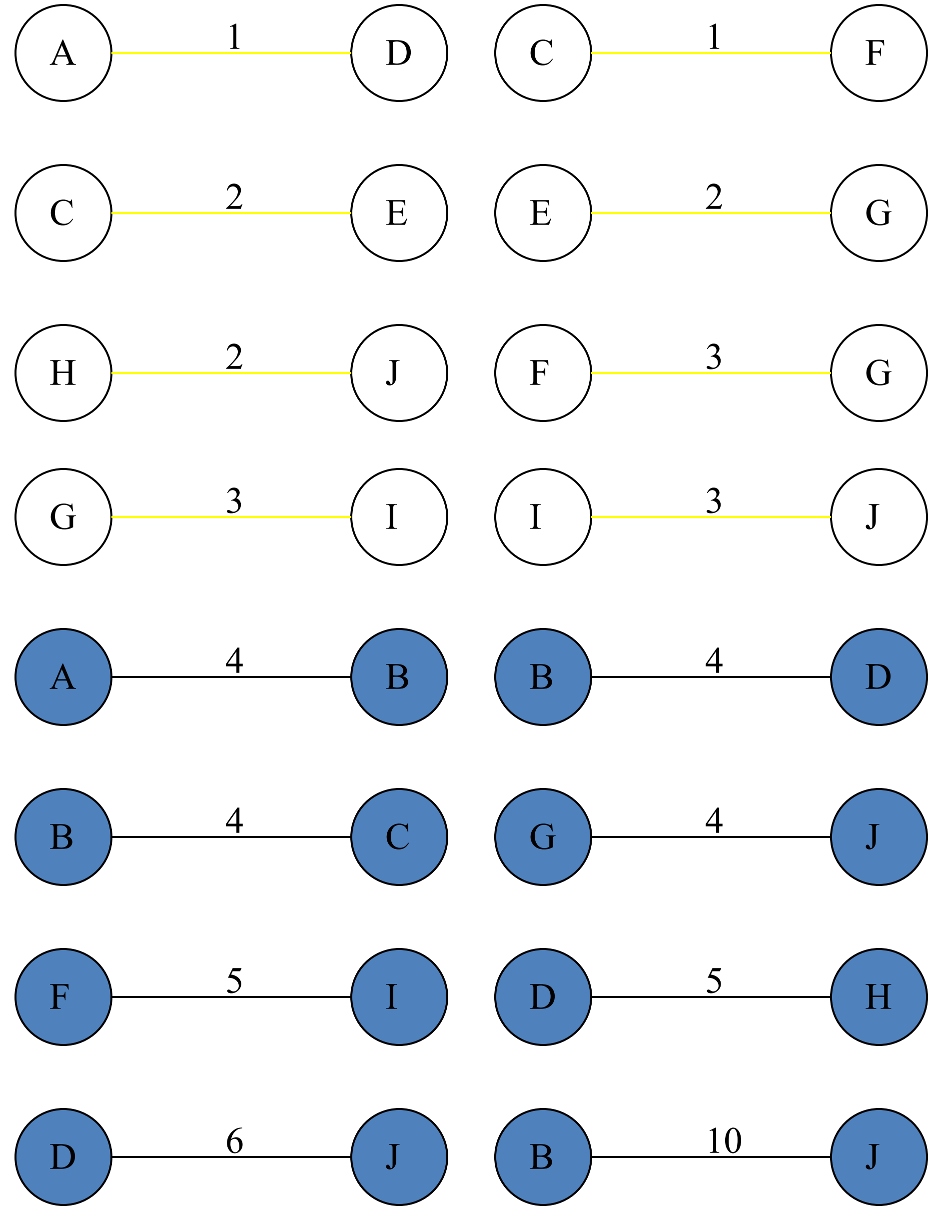




Kruskal’s minimum spanning tree

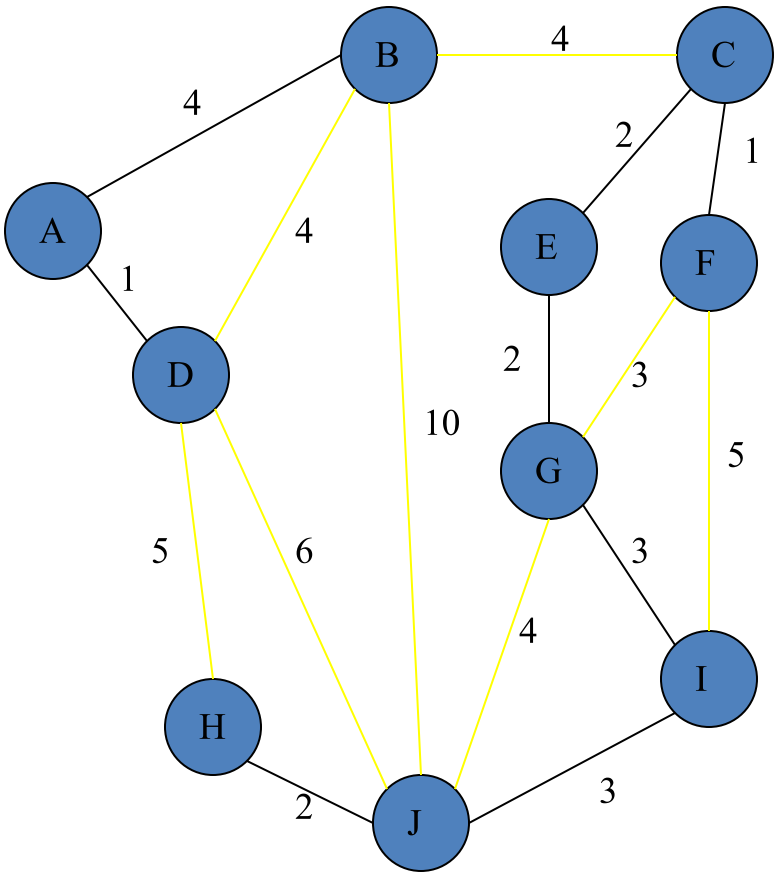
Current state of G1 = (V, T),
add (j, i) to T

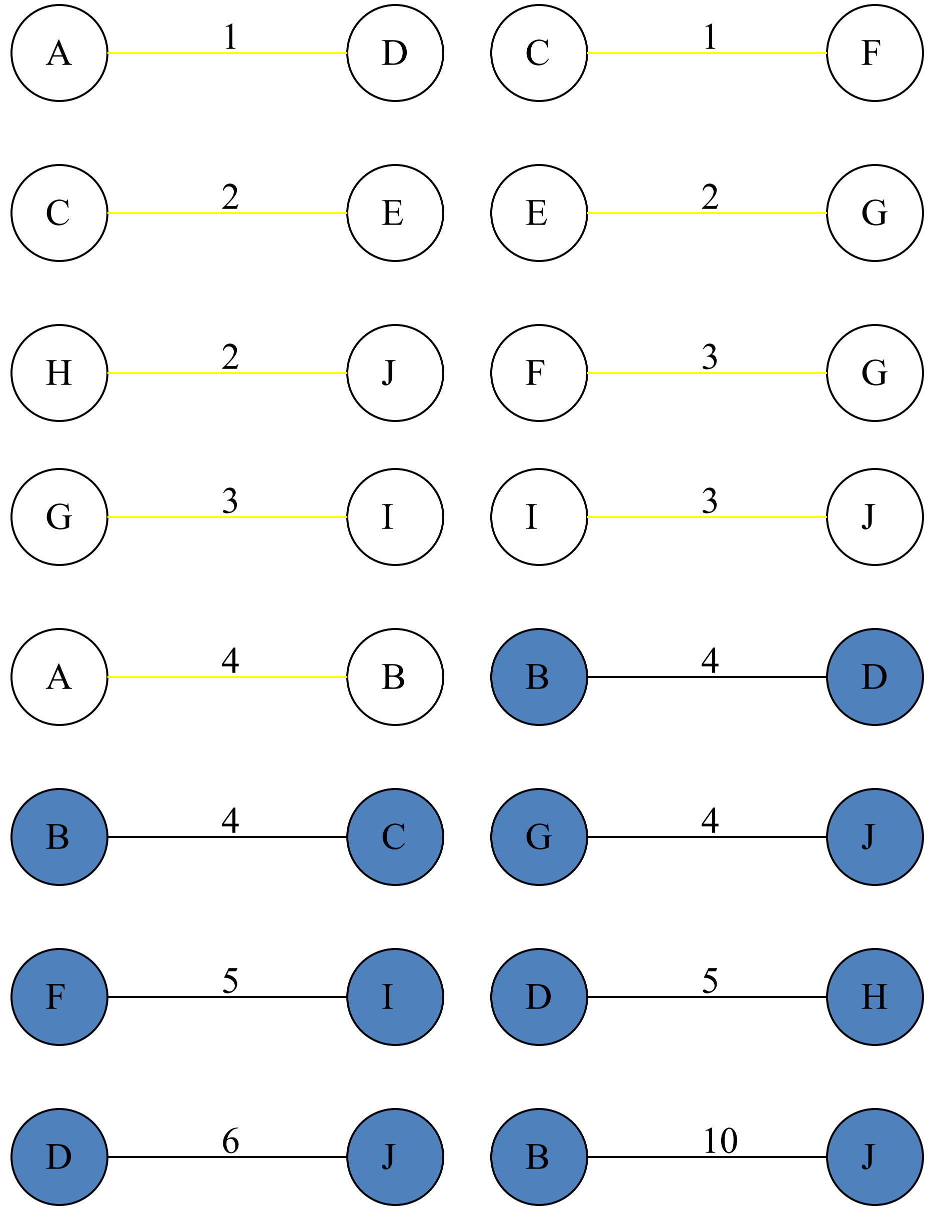




Kruskal’s minimum spanning tree

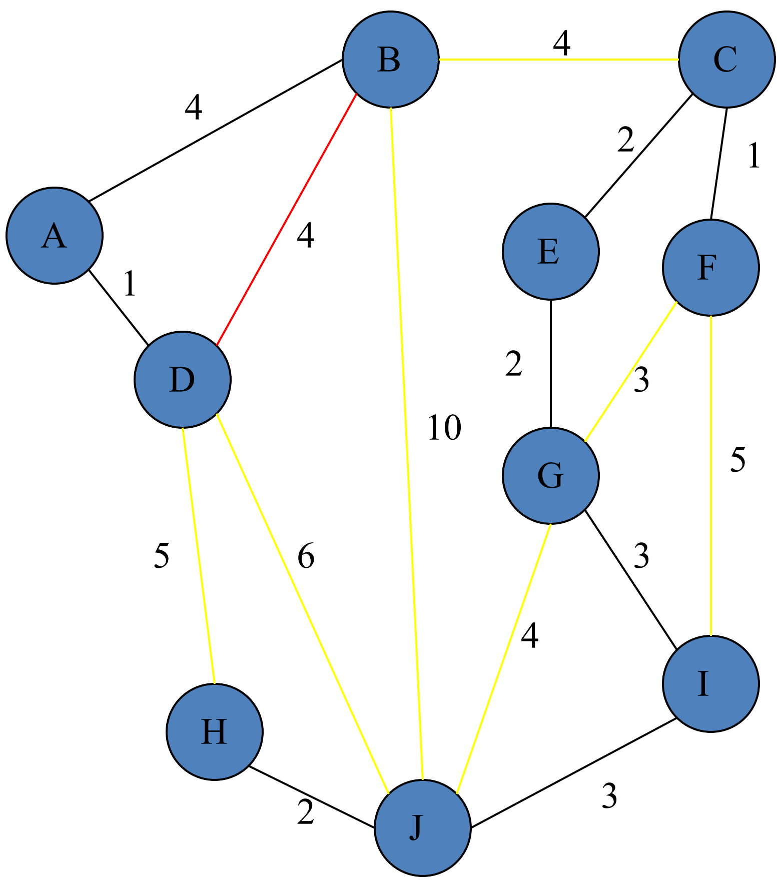
Current state of G1 = (V, T), add (a, b) to T

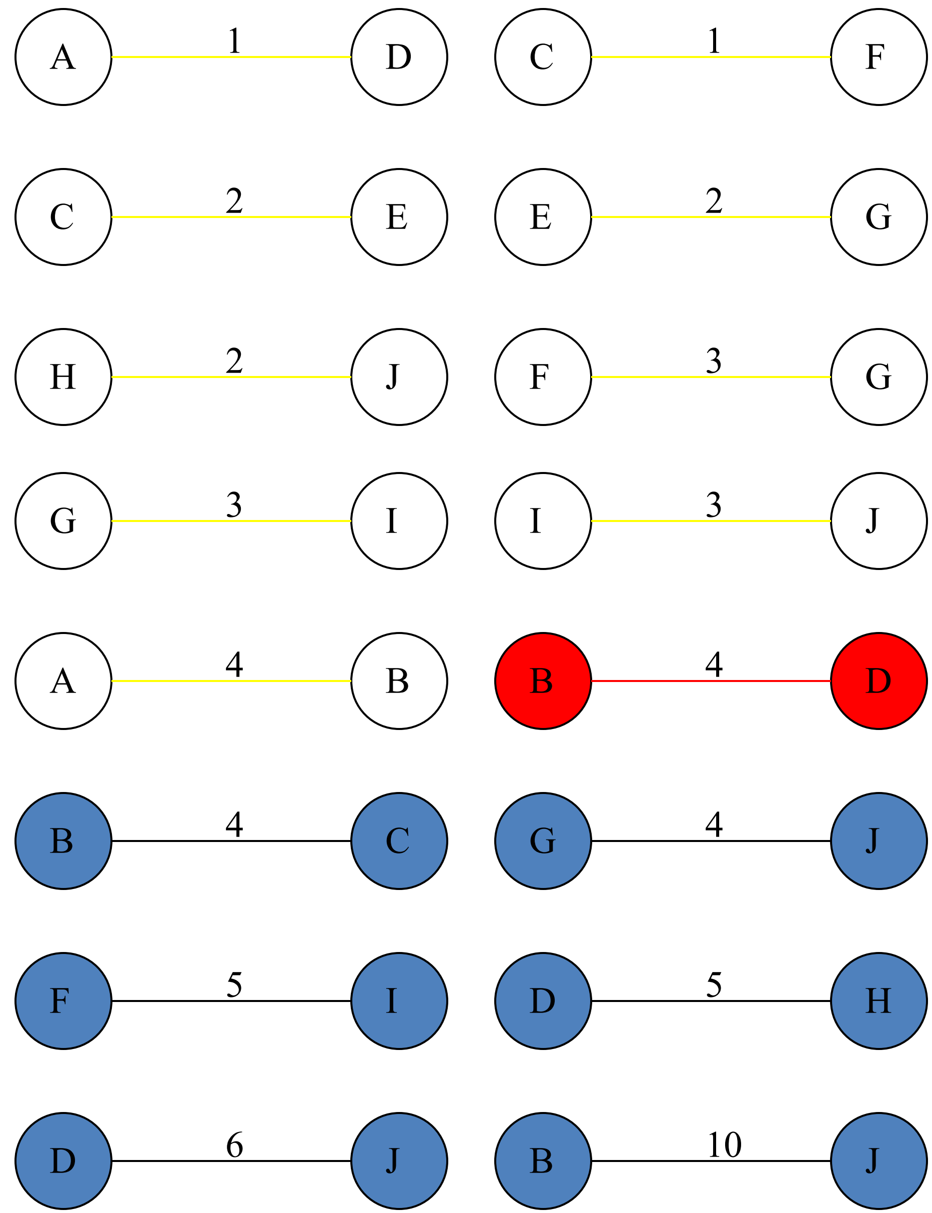




Kruskal’s minimum spanning tree

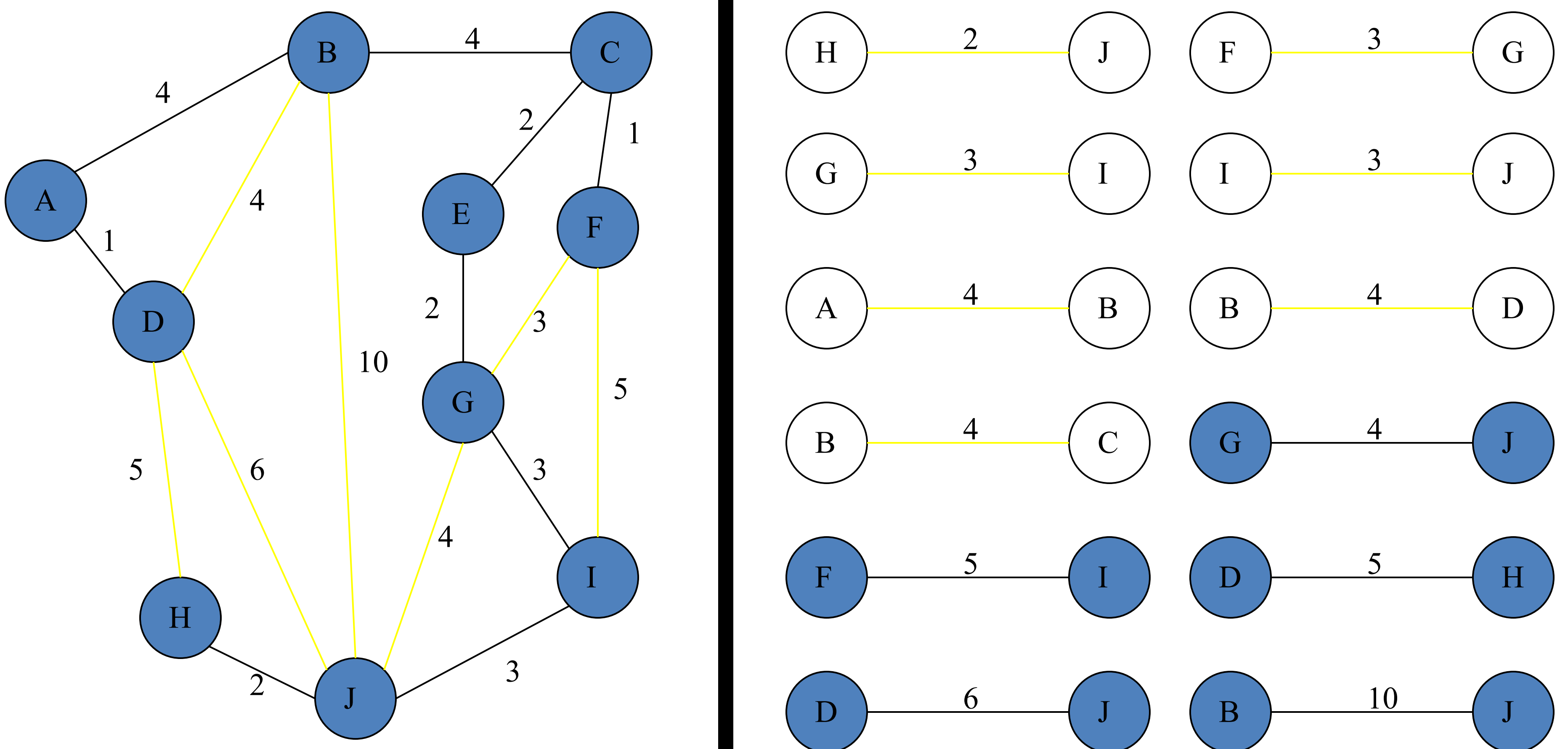
Current state of G1 = (V, T),
(b, d) forms a cycle ⍰ no change





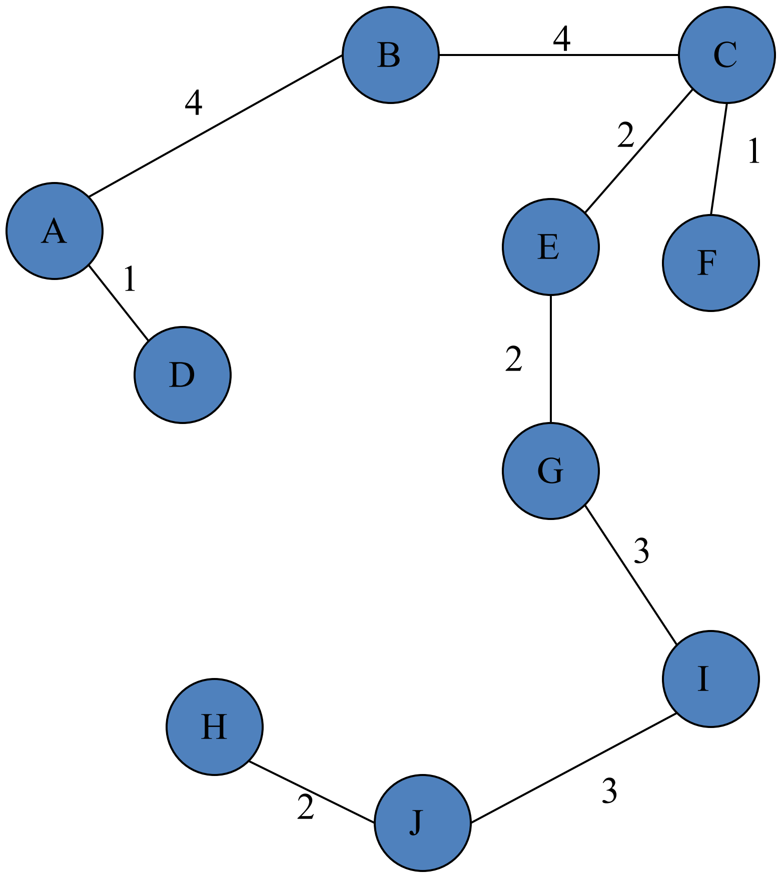
Kruskal's minimum spanning tree

Current state of G1 = (V, T),
add (b, c) to T

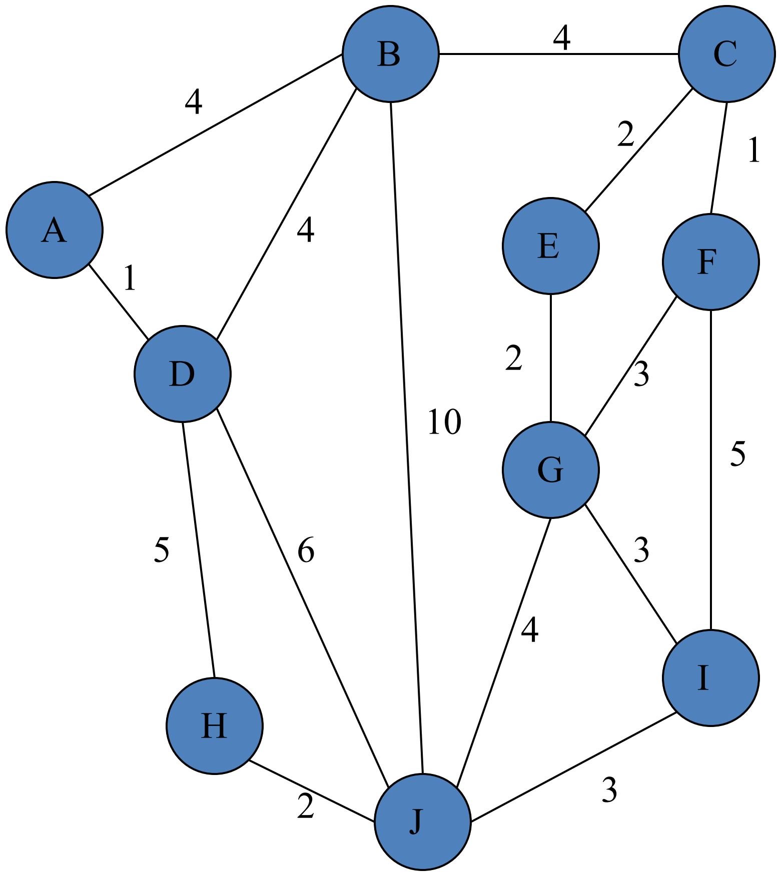


Kruskal's minimum spanning tree

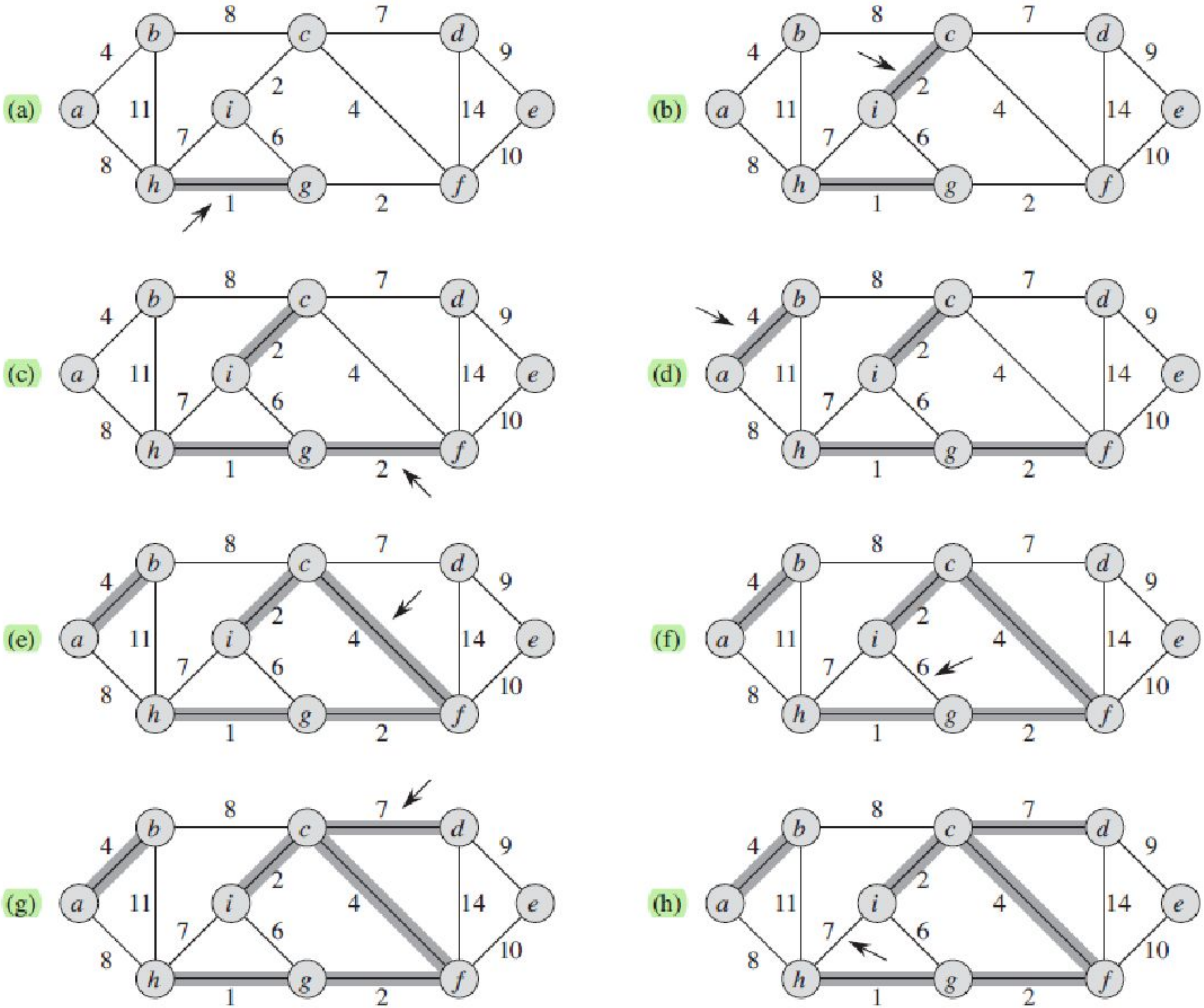
Current state of G1 = (V, T),
We have a minimum spanning tree



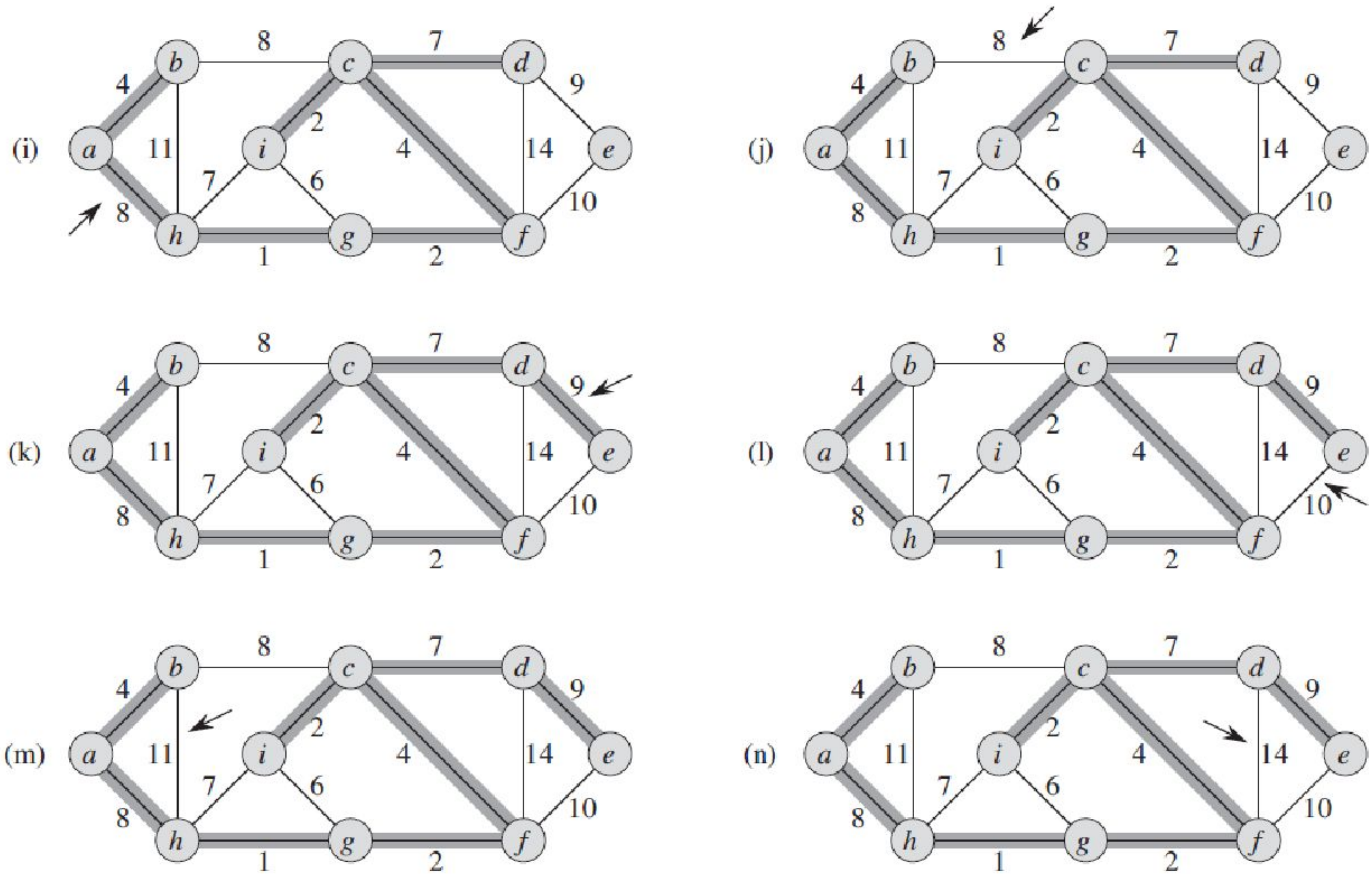
Complete graph



Kruskal’s minimum spanning tree



Kruskal’s minimum spanning tree



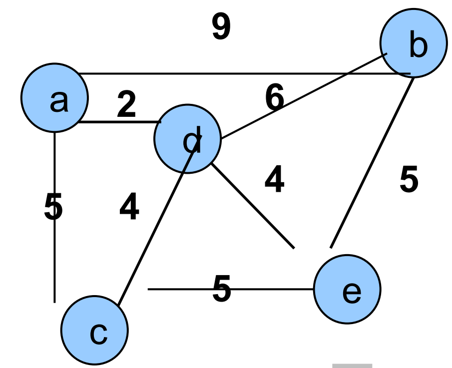
Kruskal's minimum spanning tree

Kruskal's algorithm on G = (V, E), with weights of edges in array W = [w(e)]

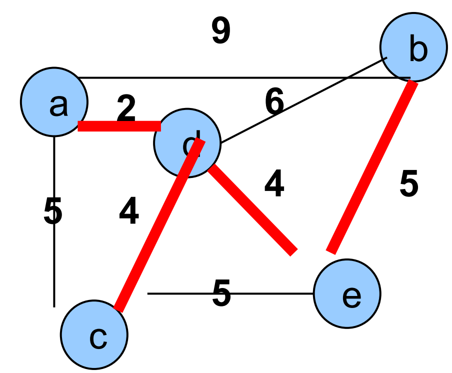
```
function MST-Kruskal(G, W)
T = Φ
for each vertex v ε V
        Make-Set(v)                 //created |V| sets each with one vertex
                                    //each set is identified by a specific member of the set
sort edges in E into non-decreasing order by weight w(e)
                                    //instead, partially sort the edges using a (min) binary heap
for each edge (u, v) in E           //in non-decreasing order of weight w(e)
                                    //Or stop after one has added |V|-1 edges
        if Find-Set(u) ≠ Find-set(v)
                T = T U {(u, v)}    //add edge (u, v) to T
                Union(u, v)         //merge two sets that contain vertices u and v
        delete edge e               //delete edge e from sorted list or from min heap
return T
```

Kruskal's minimum spanning tree

function MST-Kruskal(G, W)
T = Φ
for each vertex v ε V
    Make-Set(v)
sort edges in E into non-decreasing order by weight w(e)
for each edge (u, v) in E
    if Find-Set(u) ≠ Find-set(v)
        T = T U {(u, v)}
        Union(u, v)
    delete edge e
return T



Kruskal's algorithm



State of computation

Initial:	set of sets = [{a}, {b}, {c}, {d}, {e}]	T = []
++ edge (a, d):	set of sets = [{a, d}, {b}, {c}, {e}]	T = [(a,d)]
++ edge (d, c):	set of sets = [{a, d, c}, {b}, {e}]	T = [(a,d), (d, c)]
++ edge (d, e):	set of sets = [{a, d, c, e}, {b}]	T = [(a,d), (d, c), (d, e)]
++ edge (e, b):	set of sets = [{a, d, c, e, b}]	T = [(a,d), (d, c), (d, e), (e, b)]

Sets, and their representation

Study how to create sets, and manage them:

Operations on sets:

- Make-Set(v)
- Find-Set(u)
- Union(u, v)

Sets, and their representation

Consider the universe of symbols, U = {1, 2, …, N} or U = {red, green, blue, … }

And now consider one or more sets, S1, S2, etc. the Union of which is the universe U
For example S1 = {1, 7, 8, 9}, S2 = {2, 5, 10}, S3 = {3, 4, 6}
Note S1, S2, S3 are disjoint, and together they cover the entire universe, viz. U = S1 U S2 U S3

Equivalently, the universe U is portioned into multiple sets, S1, S2, etc.

Question how do we represent them, and carry out operations efficiently

- Make-Set(v)
- Find-Set(u) ≠ Find-set(v)
- Union(u, v)

Sets, and their representation

For example, U = {1, 2, …, 10} and disjoints sets S1 = {1, 7, 8, 9}, S2 = {2, 5, 10}, S3 = {3, 4, 6}
Note S1, S2, S3 are disjoint, and together they cover the entire universe, viz. U = S1 U S2 U S3

Here is one way to represent the disjoint sets that makes it efficient to carry out operations:

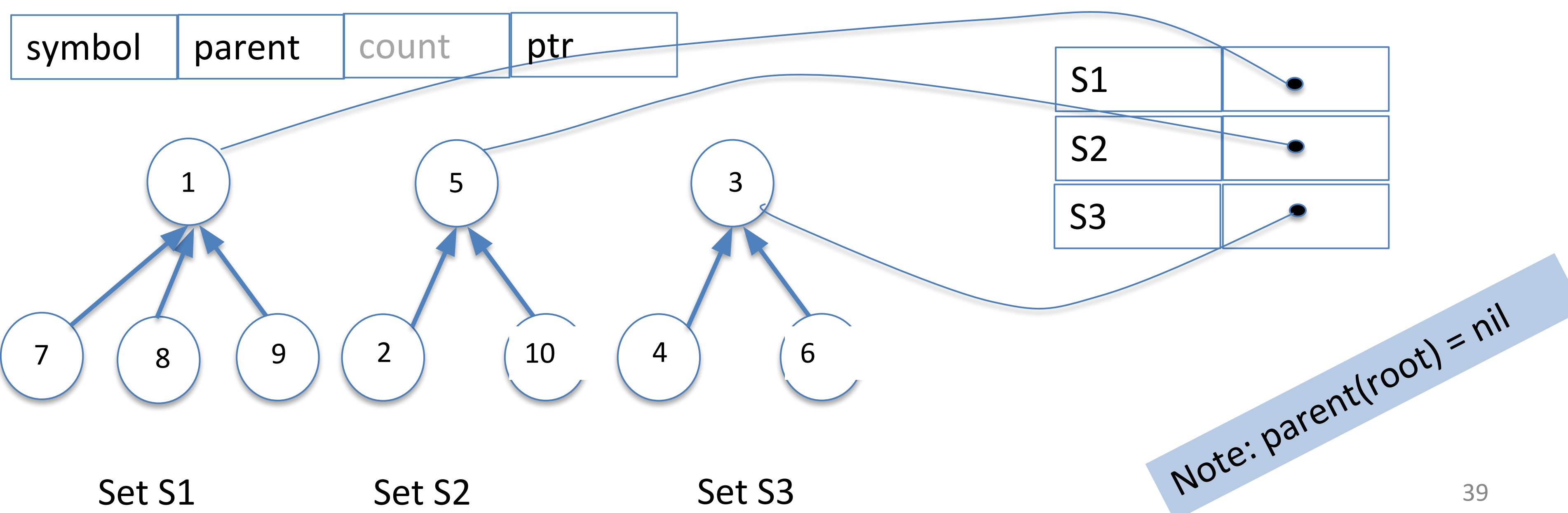
- Make-Set(v)
- Find-Set(u) ≠ Find-set(v)
- Union(u, v)

That nodes point to their parents will have significance to “Union” and “Find

Node structure

symbol	parent	count	ptr

Collection of sets

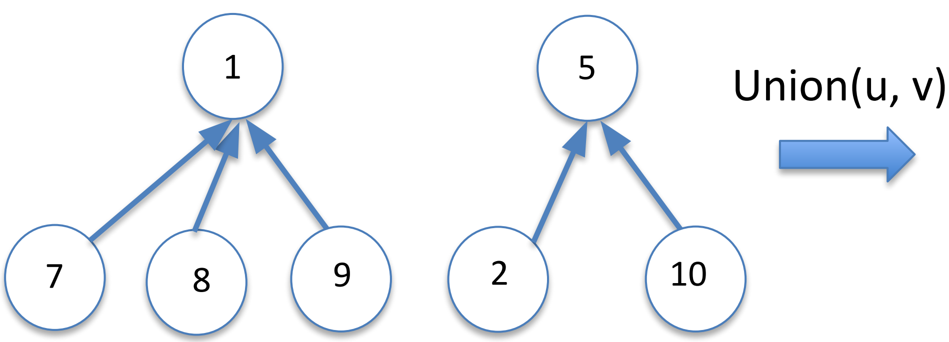


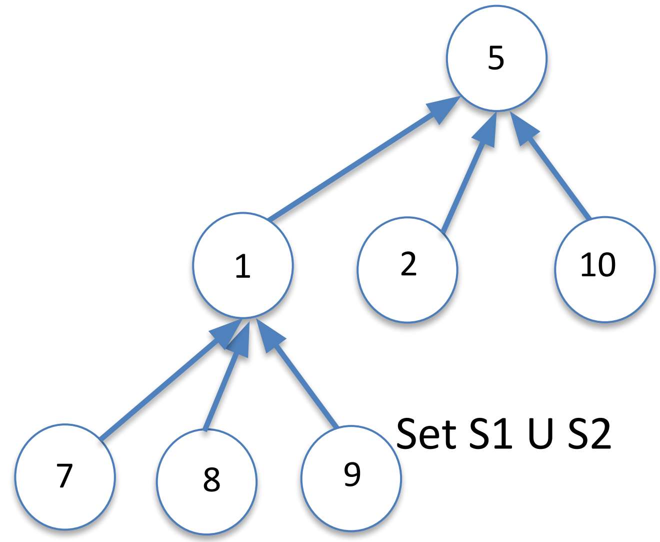
Sets, and their representation

For example, U = {1, 2, …, 10} and disjoints sets S1 = {1, 7, 8, 9}, S2 = {2, 5, 10}, S3 = {3, 4, 6}
Note S1, S2, S3 are disjoint, and together they cover the entire universe, viz. U = S1 U S2 U S3

Consider operation Union(u, v)

- To be sure consider Union(1, 5)





Set S1

Set S2

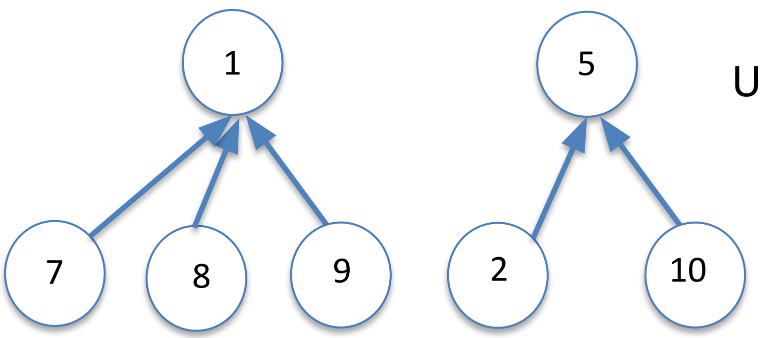
```
function Union (u, v)
parent(u) = v
return(v)
```

Sets, and their representation

For example, U = {1, 2, …, 10} and disjoints sets S1 = {1, 7, 8, 9}, S2 = {2, 5, 10}, S3 = {3, 4, 6}
Note S1, S2, S3 are disjoint, and together they cover the entire universe, viz. U = S1 U S2 U S3

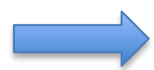
Consider operation Union(u, v)

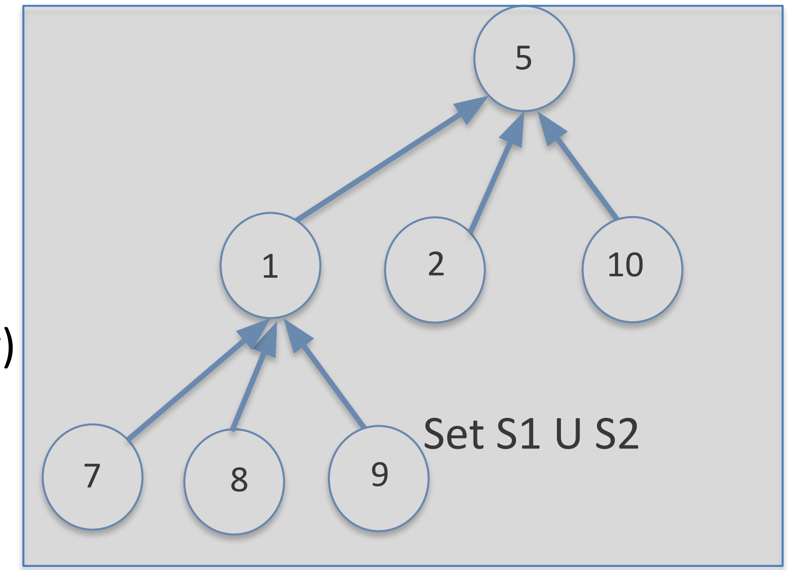
- To be sure consider Union(1, 5)



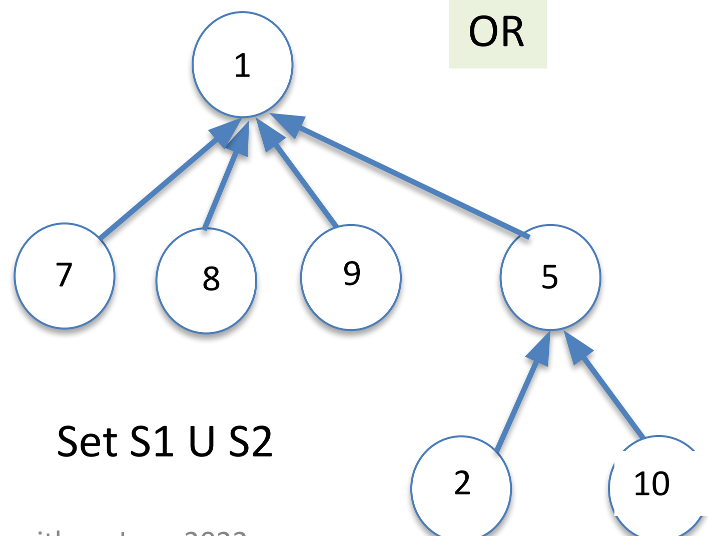
Set S1   Set S2

Union(u, v)





OR



Set S1 U S2

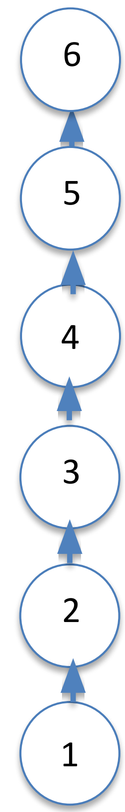
function Union (u, v)
~~parent(u) = v~~ parent(v) = u
return(v)

Sets, and their representation

In the worst case the height of tree will be O(n), where n is the number of symbols

For example, U = {1, 2, …, 6}, S1={1}, S2={2}, S3={3}, S4={4}, S5={5}, S6={6}, and consider

Union(1, 2)
Union(2, 3)
Union(3, 4)
Union(4, 5)
Union(5, 6)



While Union(u,v) is efficient, or O(1), a Find(u) operation will be complex, or O(n) in the worst case, where n = |U|

⍰ Form the union so as to minimize the height of resulting tree

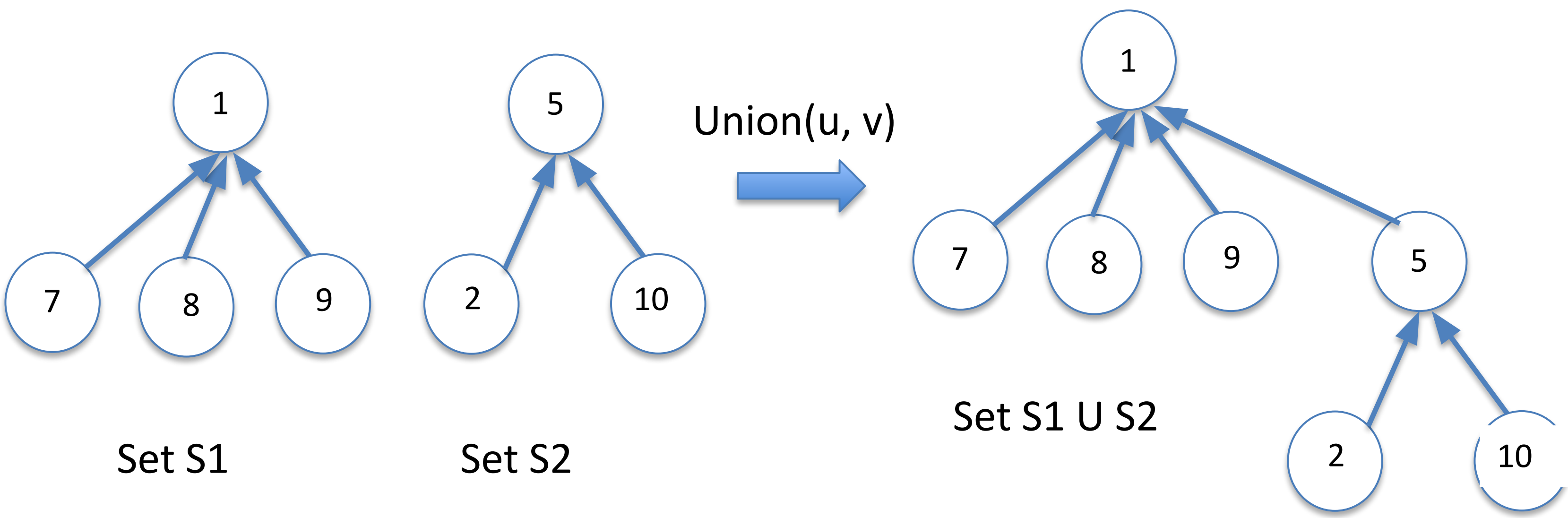
function Union (u, v)
parent(u) = v
return(v)

Sets, and their representation

Another approach where we maintain count of symbols in set (or sub-tree):

Node structure			
	symbol	parent	count

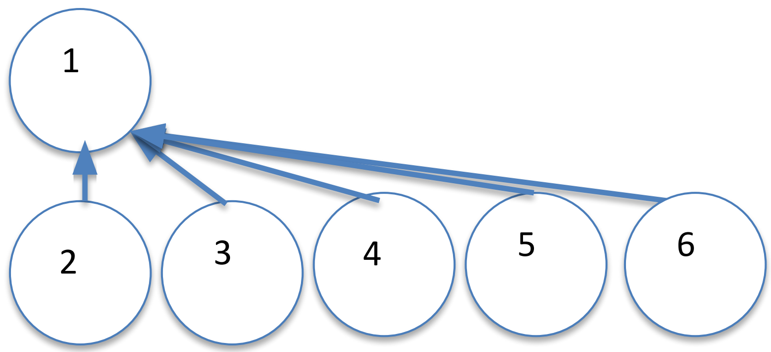
```
function Union (u, v)
if count(u) < count(v)
then parent(u) = v
        count(v) = count(v) + count(u)
else parent(v) = u
        count(u) = count(u) + count(v)
return(v)
```



Sets, and their representation

For example, U = {1, 2, …, 6}, S1={1}, S2={2}, S3={3}, S4={4}, S5={5}, S6={6}, and consider

Union(1, 2)
Union(2, 3)
Union(3, 4)
Union(4, 5)
Union(5, 6)



function Union (u, v)
if count(u) < count(v)
then parent(u) = v
    count(v) = count(v) + count(u)
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Sets, and their representation

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Node structure

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else parent(v) = u
     count(u) = count(u) + count(v)
return(v)
```

- Every node in resulting tree has level ≤ floor($(\log_2 n) + 1$
- Find(u) runs in time $O(\log_2 n)$

```
function find(u)
temp = u
while parent(temp) ≠ nil do
     temp = parent(temp)
return(temp)
```

Sets, and their representation

Time complexity

Union operation: O(1)

```
function Union (u, v)
if count(u) < count(v)
then parent(u) = v
        count(v) = count(v) + count(u)
else parent(v) = u
        count(u) = count(u) + count(v)
return(v)
```

Find operation: $O(\log_2 N)$

```
function find(u)
temp = u
while parent(temp) ≠ nil do
        temp = parent(temp)
return(temp)
```

Kruskal’s minimum spanning tree

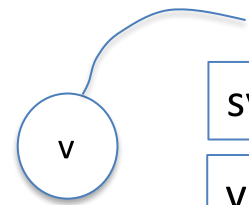
Kruskal’s algorithm on G = (V, E), with weights of edges in array W = [w(e)]

```
function MST-Kruskal(G, W)
T = Φ
for each vertex v ε V
      Make-Set(v)                    //created |V| sets each with one vertex
                                     //each set is identified by a specific member of the set
sort edges in E into non-decreasing order by weight w(e)
                                     //instead, partially sort the edges using a (min) binary heap
for each edge (u, v) in E            //in non-decreasing order of weight w(e)
                                     //Or stop after one has added |V|-1 edges
      if Find-Set(u) ≠ Find-set(v)
            T = T U {(u, v)}         //add edge (u, v) to T
            Union(u, v)              //merge two sets that contain vertices u and v
      delete edge e                  //delete edge e from sorted list or from min heap
return T
```

Kruskal’s minimum spanning tree

Make-Set(v)

```
Function Make-Set(v)
      X = getnode()
      X.Symbol = v
      X.Parent = nil
      X.Count = 1
Return X
```



X

symbol	parent	count
v	nil	1

Kruskal’s minimum spanning tree

Time complexity of Kruskal’s algorithm on G = (V, E), with weights of edges in array W = [w(e)]

Let n = |V|, m = |E|

```
function MST-Kruskal(G, W)
T = Φ                                     O(1)
for each vertex v ε V                     O(n)
      Make-Set(v)          //cremated |V| sets each with one vertex
                           //each set is identified by a specific member of the set
sort edges in E into non-decreasing order by weight w(e)          O(m)
                           //instead, partially sort the edges using a (min) binary heap
for each edge (u, v) in E  //in non-decreasing order of weight w(e)     O(m log m) = O(m log n)
                           //Or stop after one has added |V|-1 edges
      if Find-Set(u) ≠ Find-set(v)
            T = T U {(u, v)}      //add edge (u, v) to T
            Union(u, v)           //merge two sets that contain vertices u and v
      delete edge e        //delete edge e from sorted list or from min heap
return T
```

⍰ Time complexity of Kruskal’s algorithm: O(E log E) = O(E log V)

Q&A