Exponential(λ)

Definition 3.6 Exponential Random Variable

X *is an* exponential (λ) random variable *if the PDF of* X *is*

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0, \\ 0 & \textit{otherwise}, \end{cases}$$

where the parameter $\lambda > 0$.

- The inter-arrival time between two packet arrivals at a server may be modeled as an exp RV
- Received power may also be modeled as an exp RV

Exponential(λ)

Definition 3.6 *Exponential Random Variable*

X *is an* exponential (λ) random variable *if the PDF of* X *is*

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0, \\ 0 & \textit{otherwise}, \end{cases}$$

where the parameter $\lambda > 0$.

- What is the average inter-arrival time?

$E[Y] = P_0$

Problem 3.4.1

Radars detect flying objects by measuring the power reflected from them. The reflected power of an aircraft can be modeled as a random variable Y with PDF

$$f_Y(y) = \begin{cases} \frac{1}{P_0} e^{-y/P_0} & y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where $P_0 > 0$ is some constant. The aircraft is correctly identified by the radar if the reflected power of the aircraft is larger than its average value. What is the probability $P[C]$ that an aircraft is correctly identified?

$P[Y > P_0] = e^{-1}$.

$$P[X>x \mid X>t] = \begin{cases} \dfrac{P[X>x, X>t]}{P[X>t]} & x>t \\ 0 & x \le t \end{cases}$$

$$= \begin{cases} \dfrac{P[X>x]}{P[X>t]} & x>t \\ 0 & x \le t. \end{cases}$$

$$= \frac{e^{-\lambda x}}{e^{-\lambda t}} = e^{-\lambda(x-t)}$$

Suppose the additional time you will wait, given that the bus didn't arrive for time t, is Y.

$$P[Y>y \mid X>t] = P[X>y+t \mid X>t] = e^{-\lambda y}, \quad y \ge 0$$

Exponential and Geometric RV(s)

Theorem 3.9

If X is an exponential (λ) random variable, then $K = \lceil X \rceil$ is a geometric (p) random variable with $p = 1 - e^{-\lambda}$.

- Proof is similar to the case of Uniform distribution

X is an exponential RV that models the time to arrival of a bus at the bus stop. $\rightarrow P[X > x] = e^{-\lambda x}$, $x \geq 0$.

Suppose you wait for time t and no bus arrives.

How would you revise your belief about the time it takes for a bus to arrive at the bus stop?

$\hookrightarrow P[X > x \mid X > t] =$

Example 3.14 Problem

Phone company A charges $\$0.15$ per minute for telephone calls. For any fraction of a minute at the end of a call, they charge for a full minute. Phone Company B also charges $\$0.15$ per minute. However, Phone Company B calculates its charge based on the exact duration of a call. If T, the duration of a call in minutes, is an exponential $(\lambda = 1/3)$ random variable, what are the expected revenues per call $E[R_A]$ and $E[R_B]$ for companies A and B?

Erlang(n,λ)

Definition 3.7 Erlang Random Variable

X is an Erlang (n, λ) *random variable if the PDF of X is*

$$f_X(x) = \begin{cases} \dfrac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!} & x \geq 0, \\ 0 & \textit{otherwise.} \end{cases}$$

where the parameter $\lambda > 0$, and the parameter $n \geq 1$ is an integer.

- The waiting time between n-occurrences of an event
- The waiting time between n customer arrivals
- n => Sum of n independent Exp(λ) Random Variables

Theorem 3.10

If X is an Erlang (n, λ) random variable, then

$$E[X] = \frac{n}{\lambda}, \qquad \operatorname{Var}[X] = \frac{n}{\lambda^2}.$$

Quiz 3.4

Continuous random variable X has $E[X] = 3$ and $\text{Var}[X] = 9$. Find the PDF, $f_X(x)$, if

(1) X has an exponential PDF,

(2) X has a uniform PDF.

$$\int_0^\infty \left[99 + 10(x-20)^+\right] \frac{1}{\tau} e^{-x/\tau}\, dx$$

$$= \underbrace{\int_0^\infty 99 \frac{1}{\tau} e^{-x/\tau}\, dx}_{} + 10 \int_0^\infty (x-20)^+ \frac{1}{\tau} e^{-x/\tau}\, dx$$

$$= 99 + 10 \int_{20}^\infty (x-20) \frac{1}{\tau} e^{-x/\tau}\, dx.$$

Problem 3.4.9

$y^+ \triangleq \begin{cases} y, & y \geq 0 \\ 0 \end{cases}$

Long-distance calling plan A offers flat rate service at 10 cents per minute. Calling plan B charges 99 cents for every call under 20 minutes; for calls over 20 minutes, the charge is 99 cents for the first 20 minutes plus 10 cents for every additional minute. (Note that these plans measure your call duration exactly, without rounding to the next minute or even second.) If your long-distance calls have exponential distribution with expected value τ minutes, which plan offers a lower expected cost per call?

X is the R.V.

$R_A = 10X.$ $\quad R_B = 99 + 10(X-20)^+ = g(X).$

$E[R_A] = E[10X] = 10E[X] = 10\tau.$

$E[R_B] = E[g(X)] = \int_{-\infty}^{\infty} g(x) \frac{1}{\tau} e^{-x/\tau} dx$

$= \int_{-\infty}^{\infty} 99 + 10(x-20)^+ \frac{1}{\tau} e^{-x/\tau} dx$

Gaussian(μ,σ)

- The Bell-shaped distribution
- The Normal distribution
- Parameters are the mean μ and standard deviation is σ
  - Variance is σ^2
  - If X is Gaussian we often write X is $N[\mu, \sigma^2]$

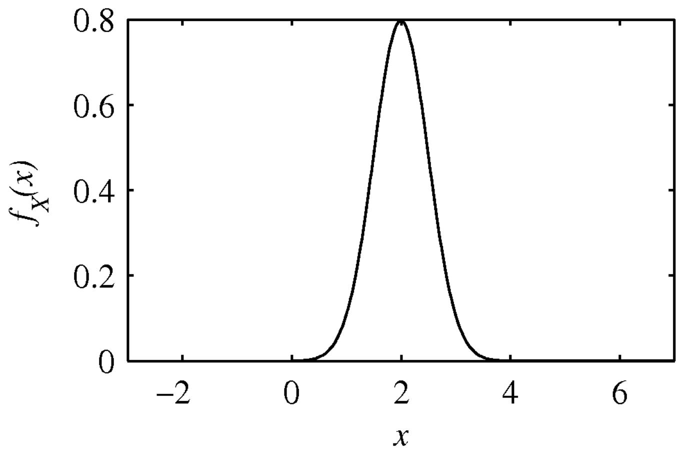
Definition 3.8 Gaussian Random Variable

X is a Gaussian (μ, σ) random variable if the PDF of X is

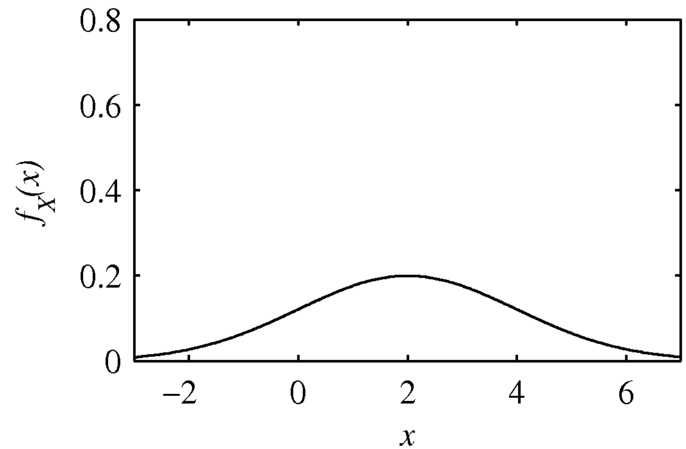
$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2},$$

where the parameter μ can be any real number and the parameter $\sigma > 0$.

Figure 3.5



(a) $\mu = 2, \sigma = 1/2$



(b) $\mu = 2, \sigma = 2$

Two examples of a Gaussian random variable X with expected value μ and standard deviation σ.