Definition 4.14 Conditional Expected Value of a Function

For continuous random variables X and Y and any y such that $f_Y(y) > 0$, the conditional expected value *of $g(X, Y)$ given $Y = y$ is*

$$E\left[g(X, Y)|Y = y\right] = \int_{-\infty}^{\infty} g(x, y) f_{X|Y}(x|y)\, dx.$$

Conditional Expected Value

Definition 4.15 Conditional Expected Value

If we evaluate at $Y=y$, we get $E[X|Y=y]$

The conditional expected value $E[X|Y]$ *is a function of random variable* Y *such that if* $Y = y$ *then* $E[X|Y] = E[X|Y = y]$.

Conditional Expected Value

Definition 4.15 Conditional Expected Value

The conditional expected value $E[X|Y]$ is a function of random variable Y such that if $Y = y$ then $E[X|Y] = E[X|Y = y]$.

Note that E[X|Y] is a random variable

- It is a function of the RV _Y_?

$E[X|Y]$: First calculate $E[X|Y=y]$ using $f_{X|Y}(x|y)$

Conditional Expected Value

Definition 4.15 *Conditional Expected Value*

The conditional expected value $E[X|Y]$ is a function of random variable Y such that if $Y = y$ then $E[X|Y] = E[X|Y = y]$.

$$\int_{y \in S_Y} \left[\int_{x \in S_X} x \, f_{X|Y}(x|z) \, dx \right] f_Y(z) \, dz$$

- E[X|Y=y] is a function of ___

some $g(Y)$

- E[E[X|Y]] is the expectation of a function of RV ___
- E[E[X|Y=1]] = _____ ?

$$E[X|Y=1] = \int_{x \in S_X} x \, f_{X|Y}(x|1) \, dx$$

Consider

$$\int_{y \in S_Y} \left[\int_{x \in S_X} h(x,y)\, f_{X|Y}(x|y)\, dx \right] f_Y(y)\, dy$$

$$= \int_{y \in S_Y} \int_{x \in S_X} h(x,y)\, \underbrace{f_{X|Y}(x|y)\, f_Y(y)}_{f_{X,Y}(x,y)}\, dx\, dy$$

$$= \int_{y \in S_Y} \int_{x \in S_X} h(x,y)\, f_{X,Y}(x,y)\, dx\, dy = E[h(X,Y)]$$

$$\therefore \; E[h(X,Y)] = E\big[E[h(X,Y) \mid Y]\big]$$

Example 4.20 Problem

For random variables X and Y in Example 4.5, we found in Example 4.19 that the conditional PDF of X given Y is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} 1/(1-y) & y \le x \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4.109}$$

Find the conditional expected values $E[X|Y=y]$ and $E[X|Y]$.

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Find the conditional expected values $E[X|Y=y]$ and $E[X|Y]$.

- E[X|Y=y] = (1+y)/2
- E[X|Y] = (1+Y)/2

Independent Random Variables

Definition 4.16 *Independent Random Variables*

Random variables X *and* Y *are* independent *if and only if*

Discrete: $P_{X,Y}(x, y) = P_X(x) P_Y(y)$

Continuous: $f_{X,Y}(x, y) = f_X(x) f_Y(y)$.

If we have n discrete RV(s), we must evaluate the joint PMF over N^n values, assuming each RV has N values in their range spaces.

$P_{X,Y,Z}(x,y,z) = P_X(x) P_Y(y) P_Z(z) \quad \forall\, x,y,z.$

$f_{X,Y,Z}(x,y,z) = f_X(x) f_Y(y) f_Z(z) \quad \forall\, x,y,z$

- For independent X and Y, $P_{X|Y}(x|y)$ = ?

- What about $F_{X,Y}(x,y)$?

$\rightarrow F_X(x) F_Y(y)$

Independent RVs

Theorem 4.27

For independent random variables X and Y,

$$\int \int \underbrace{g(x)h(y) f_{X,Y}(x,y)}_{f_X(x) f_Y(y)} \, dx \, dy = E[g(X)] \, E[h(Y)]$$

(a) $E[g(X)h(Y)] =$,

(b) $r_{X,Y} = E[XY] =$,

(c) $\text{Cov}[X, Y] =$,

(d) $\text{Var}[X + Y] =$,

(e) $E[X|Y = y] =$,

(f) $E[Y|X = x] =$.

Example 4.24 Problem

$$f_{U,V}(u,v) = \begin{cases} 24uv & u \geq 0, v \geq 0, u+v \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4.133}$$

Are U and V independent?

Quiz 4.10(A)

Random variables X and Y in Example 4.1 and random variables Q and G in Quiz 4.2 have joint PMFs:

$P_{X,Y}(x,y)$	$y=0$	$y=1$	$y=2$
$x=0$	0.01	0	0
$x=1$	0.09	0.09	0
$x=2$	0	0	0.81

$P_{Q,G}(q,g)$	$g=0$	$g=1$	$g=2$	$g=3$	
$q=0$	0.06	0.18	0.24	0.12	$\rightarrow 0.6$
$q=1$	0.04	0.12	0.16	0.08	$\rightarrow 0.4$
	0.10	0.3	0.4	0.2	

$P[X=1, Y=0] = 0.09$

$P[X=1] = 0.18$

$P[Y=0] = 0.1$

(1) Are X and Y independent?

(2) Are Q and G independent?

Quiz 4.10(B)

Random variables X_1 and X_2 are independentand identically distributed with probability density function

$$f_X(x) = \begin{cases} 1 - x/2 & 0 \leq x \leq 2, \\ 0 & \text{otherwise.} \end{cases} \quad (4.144)$$

(1) What is the joint PDF $f_{X_1,X_2}(x_1, x_2)$?

(2) Find the CDF of $Z = \max(X_1, X_2)$.

$$P[Z \leq z] = P[\max(X_1, X_2) \leq z]$$
$$= P[X_1 \leq z, X_2 \leq z]$$
$$= P[X_1 \leq z]\, P[X_2 \leq z] = (P[X \leq z])^2$$

Problem 4.10.13

X and Y are independent random variables with PDFs

$$f_X(x) = \begin{cases} 2x & 0 \le x \le 1, \\ 0 & \text{otherwise}, \end{cases}$$

$$f_Y(y) = \begin{cases} 3y^2 & 0 \le y \le 1, \\ 0 & \text{otherwise}. \end{cases}$$

Let $A = \{X > Y\}$.

(a) What are $E[X]$ and $E[Y]$?

(b) What are $E[X|A]$ and $E[Y|A]$?

$$E[X|A] = \int_{x \in A} x \, f_{X|A}(x)\, dx$$

$$\cancel{f_{X|A}(x) = \begin{cases} \dfrac{f_X(x)}{P[A]} & x \in A \\ 0 & \text{otherwise} \end{cases}}$$

Because $A \subset S_X \times S_Y$

We need to calculate the conditional joint PDF

$$f_{X,Y|A}(x,y) = \begin{cases} \dfrac{f_{X,Y}(x,y)}{P[A]} & (x,y) \in A \\ 0 & \text{otherwise} \end{cases}$$

$$E[X|A] = E[g(X,Y)|A] \quad \left\{ \text{Define } g(X,Y) = X. \right.$$

$$= \int_{y \in S_Y} \int_{x \in S_X} f_{X,Y|A}(x,y)\, g(x,y)\, dx\, dy$$

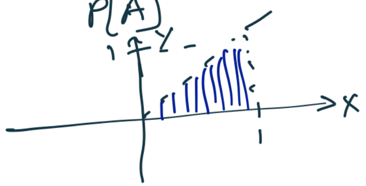
$$= \int_y \int_x f_{X,Y|A}(x,y)\, x\, dx\, dy$$

Q) What is $\displaystyle\int_{y \in S_Y} f_{X,Y|A}(x,y)\, dy$?

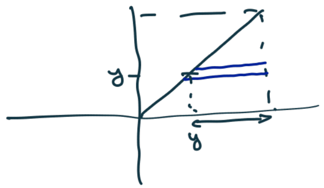
$$\hookrightarrow = f_{X|A}(x)$$

$$\int_{y \in S_Y} \int_{x \in S_X} f_{X,Y|A}(x,y)\, x\, dx\, dy$$

$$= \underbrace{\int \int}_{(x,y) \in A} x \frac{f_{X,Y}(x,y)}{P[A]}\, dx\, dy$$



$$= \int_0^1 \int_y^1 x \, \frac{f_{X,Y}(x,y)}{P[A]} \, dx \, dy$$



Given that X & Y are ind RV,

$$f_{X,Y}(x,y) = f_X(x) \, f_Y(y)$$

$P[A] = ?$

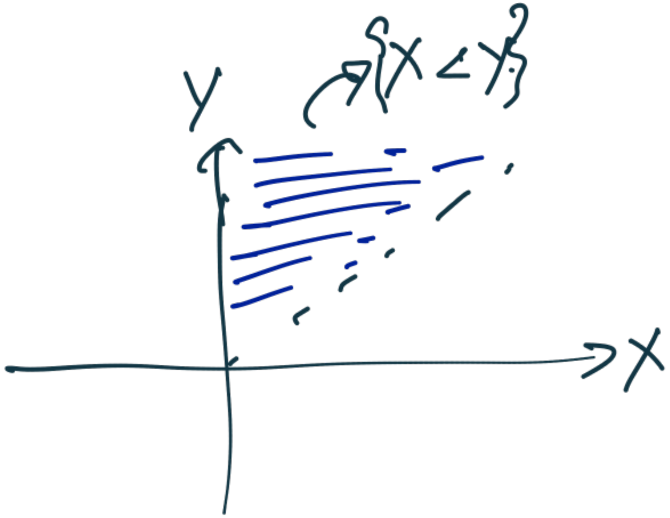
$$\iint_{(x,y)\in A} f_{X,Y}(x,y) \, dx \, dy$$

A Shorter Exponential

Let $X \sim Exp(\lambda)$ and $Y \sim Exp(\mu)$. Assume that X and Y are independent random variables. Find $\boxed{f_{X|X<Y}(x).}$

$f_X(x) = \lambda e^{-\lambda x}, \ x \geq 0$

$f_Y(y) = \mu e^{-\mu y}, \ y \geq 0.$



$$P[X<Y] = \frac{\lambda}{\lambda+\mu}$$

$$f_{X|X<Y}(x) = \int_x^{\infty} f_{X,Y|X<Y}(x,y)\, dy$$

$$= \int_x^{\infty} \frac{\lambda e^{-\lambda x} \mu e^{-\mu y}}{(\lambda/(\lambda+\mu))}\, dy$$

$$\frac{\lambda e^{-\lambda x}(\lambda+\mu)}{\lambda} \int_x^{\infty} \mu e^{-\mu y}\, dy$$

$$= (\lambda+\mu) e^{-\lambda x} e^{-\mu x} = (\lambda+\mu) e^{-(\lambda+\mu)x}$$

Problem 4.9.14

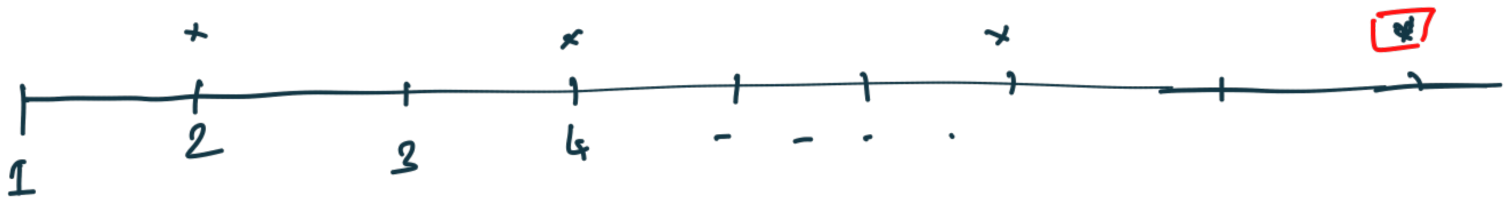
Suppose you arrive at a bus stop at time 0 and at the end of each minute, with probability p, a bus arrives, or with probability $1 - p$, no bus arrives. Whenever a bus arrives, you board that bus with probability q and depart. Let T equal the number of minutes you stand at a bus stop. Let N be the number of buses that arrive while you wait at the bus stop.

(a) Identify the set of points (n, t) for which $P[N = n, T = t] > 0$.

(b) Find $P_{N,T}(n, t)$.

(c) Find the marginal PMFs $P_N(n)$ and $P_T(t)$.

(d) Find the conditional PMFs $P_{N|T}(n|t)$ and $P_{T|N}(t|n)$.



Problem 4.9.15

Each millisecond at a telephone switch, a call independently arrives with probability p. Each call is either a data call (d) with probability q or a voice call (v). Each data call is a fax call with probability r. Let N equal the number of milliseconds required to observe the first 100 fax calls. Let T equal the number of milliseconds you observe the switch waiting for the first fax call. Find the marginal PMF $P_T(t)$ and the conditional PMF $P_{N|T}(n|t)$. Lastly, find the conditional PMF $P_{T|N}(t|n)$.