Review - Inner Product

$+ : V \times V \to V$

Definition

Let V be a real vector space. An *inner product* $\boxed{\langle ., . \rangle : V \times V \to \mathbb{R}}$ is a function satisfying the following conditions:

1. $\langle \mathbf{v} + \mathbf{w}, \mathbf{z} \rangle = \langle \mathbf{v}, \mathbf{z} \rangle + \langle \mathbf{w}, \mathbf{z} \rangle$, for every $\mathbf{v}, \mathbf{w}, \mathbf{z} \in V$
2. $\langle c\mathbf{v}, \mathbf{w} \rangle = c\langle \mathbf{v}, \mathbf{w} \rangle$ for every $c \in \mathbb{R}$ and $\mathbf{v}, \mathbf{w} \in V$
3. $\langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{w}, \mathbf{v} \rangle$, for every $\mathbf{v}, \mathbf{w} \in V$
4. $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ for every $\mathbf{v} \in V$, and $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ holds if and only if $\mathbf{v} = \mathbf{0}$

Review: Generalization of Length

Definition

Let V be a vector space and let $\langle ., . \rangle$ be an inner product on V. The *length (or norm)* of a vector $v \in V$ is defined by

$$\|v\| = \sqrt{\langle v, v \rangle}$$

Review: Unit Vectors

A vector whose length is 1 is called a *unit vector*.

If we divide a nonzero vector v by its length, we obtain a unit vector $\hat{v}$, because if

$$\hat{v} = \frac{1}{\|v\|} v,$$

then

$$\|\hat{v}\| = \left\| \frac{1}{\|v\|} v \right\| = \frac{1}{\|v\|} \|v\| = 1$$

The process of creating $\hat{v}$ from v is called *normalizing* v, and we say that $\hat{v}$ is in the *same direction* as v.

Notation

We often say

$$\hat{v} = \frac{v}{\|v\|}$$

Review: Cauchy-Schwarz Inequality

Proposition

Let V be a vector space with an inner product $\langle ., . \rangle$. Let $u, v \in V$. Then

$$|\langle u, v \rangle| \leq \|u\| \|v\|$$

When $u \neq 0, v \neq 0$,

$$\left| \left\langle \frac{u}{\|u\|}, \frac{v}{\|v\|} \right\rangle \right| \leq 1$$

Review: Distance and Orthogonality

Definition

Let V be a vector space with an inner product $\langle .,. \rangle$. The *distance* $d(u, v)$ between vectors u and $v \in V$ is defined as

$$d(u, v) = \|u - v\|$$

Definition

Two vectors u and v in an inner product space $(V, \langle .,. \rangle)$ are said to be *orthogonal (to each other)* if

$$\langle u, v \rangle = 0.$$

Orthogonal Sets

Definition

Let V be a vector space with an inner product $\langle ., . \rangle$. A set of vectors $\{v_1, \ldots, v_p\}$ in V is said to be an *orthogonal set* if

$$\langle v_i, v_j \rangle = 0, \quad \text{for every } i \neq j, \quad \text{where } i, j \in \{1, \ldots, p\}.$$

Theorem

Let V be a vector space of dimension n. Let $\langle ., . \rangle$ be an inner product defined on V. Let $S = \{v_1, \ldots, v_p\}$ be an orthogonal set of non-zero vectors in an inner product space $(V, \langle ., . \rangle)$. Then S is linearly independent set and hence is a basis for $\operatorname{Span} S$.

Proof: Let

$$c_1 v_1 + c_2 v_2 + \cdots + c_p v_p = 0 \quad (1)$$

where $c_1, \ldots, c_p \in \mathbb{R}$.

For $j \in \{1, \ldots, p\}$

$$\langle c_1 v_1 + \cdots + c_p v_p, v_j \rangle$$

$$= c_1 \langle v_1, v_j \rangle + c_2 \langle v_2, v_j \rangle + \cdots + c_p \langle v_p, v_j \rangle$$

$$= \cancel{\sum_{\substack{i=1 \\ i \neq j}}^{r} c_i \langle v_i, v_j \rangle} + c_j \langle v_j, v_j \rangle$$

$$= c_j \langle v_j, v_j \rangle \quad \text{—— } (2)$$

From (1) & (2), we get

$$c_j \langle v_j, v_j \rangle = 0.$$

Since $v_j \neq 0$,

$\langle v_j, v_j \rangle \neq 0$.

$\therefore \quad c_j = 0$.

$\therefore \quad c_1 = c_2 = \cdots = c_p = 0$.

$\therefore \quad \{v_1, \ldots, v_p\}$ is a l.i. set. $\square$

Definition

Let $(V, \langle ., . \rangle)$ be an inner product space. An orthogonal set which is also a basis for V is called an *orthogonal basis* for V.

Definition

Let $(V, \langle ., . \rangle)$ be an inner product space. An orthogonal set of unit vectors in V is called an *orthonormal set*.

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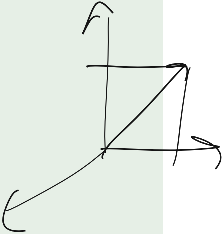
Let $(V, \langle ., . \rangle)$ be an inner product space. An orthonormal set which is also a basis for V is called an *orthonormal basis* for V.

Example

$V = \mathbb{R}^3$. Let $\langle ., . \rangle$ be the usual dot product.



Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

Then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal basis of $\mathbb{R}^3$,

$$\hat{\mathbf{v}}_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \quad \hat{\mathbf{v}}_2 = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \quad \hat{\mathbf{v}}_3 = \begin{bmatrix} -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

and $\{\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2, \hat{\mathbf{v}}_3\}$ is an orthonormal basis of $\mathbb{R}^3$.

Example

$V = C[-\pi, \pi]$ with the inner product defined by

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)g(t)\,\mathrm{dt}$$

Let $W = \text{Span}\{1, \cos t, \cos 2t, \cos 3t\}$. Then

$$\left\{\frac{1}{\sqrt{2}}, \cos t, \cos 2t, \cos 3t\right\}$$

is an orthonormal basis of W.

$f_0 = 1$

$f_1 = \cos t$

$f_2 = \cos 2t$

$f_3 = \cos 3t$

$$\langle f_0, f_1 \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos t \, dt$$

$$= \frac{\sin t}{\pi} \Big|_{-\pi}^{\pi} = 0 .$$

$$\langle f_0, f_2 \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos 2t \, dt$$

$$= \frac{\sin 2t}{2\pi} \Big|_{-\pi}^{\pi} = 0$$

Similarly $\langle f_0, f_2 \rangle$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} \cos 3t \, dt = 0.$$

$$\langle f_1, f_2 \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \underbrace{2 \cos t \cos 2t}\, dt$$

$$= \frac{1}{2\pi \cdot \pi} \int^{\pi} \left(\cos 3t + \cos t \right) dt$$

$$= \frac{1}{2\pi}\left[\frac{\sin 3t}{3} + \sin t\right]\Big|_{-\pi}^{\pi}$$

$$= 0 \;.$$

$\langle f_1, f_3 \rangle = 0$ and $\langle f_2, f_3 \rangle = 0$

(HW) complete this part.

$$\langle f_0, f_0 \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} dt$$

$$= \frac{2\pi}{\pi} = 2.$$

$$\|f_0\| = \sqrt{2}.$$

$f_1 = \cos t$

$$\langle f_1, f_1 \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos^2 t \, dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{1+\cos 2t}{2}\right)dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} dt + \cancel{\frac{1}{2\pi}\cdot\int_{-\pi}^{\pi}\cos 2t\,dt}$$

$$= 1$$

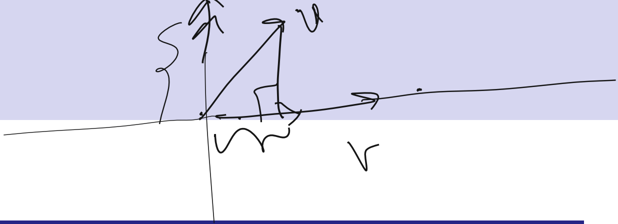
(HW) Check: $\cos 2t$ and $\cos 3t$ are also unit vectors.

$$\mathcal{B} = \{b_1, \ldots, b_n\} \quad V$$

$$v = c_1 b_1 + \cdots + c_n b_n$$

$$v \longmapsto (c_1, \ldots, c_n)$$

Question: If we have an orthonormal basis, how do we find coordinates with respect to this basis? How would go about answering this question in $\mathbb{R}^3$?



Definition

Let $(V, \langle ., . \rangle)$ be an inner product space. Let $v \in V$ be a nonzero vector. For any vector $u \in V$, the *orthogonal projection of u onto v* is defined as

$$\operatorname{proj}_v u := \boxed{\frac{\langle u, v \rangle}{\langle v, v \rangle}} v = \langle u, \hat{v} \rangle \hat{v},$$

and the vector

$$u - \operatorname{proj}_v u$$

is called the *component of u which is orthogonal to v*.

Theorem

Let $(V, \langle ., . \rangle)$ be an inner product space. Let $\mathcal{B} = \{v_1, \dots, v_n\}$ be an orthogonal basis for V. Let $v \in V$ be any vector. Then the $\mathcal{B}$-coordinates of v are:

$$[v]_{\mathcal{B}} = \left(\frac{\langle v, v_1 \rangle}{\langle v_1, v_1 \rangle}, \frac{\langle v, v_2 \rangle}{\langle v_2, v_2 \rangle}, \dots, \frac{\langle v, v_n \rangle}{\langle v_n, v_n \rangle} \right).$$

Pf:

Let $v \in V$.

Suppose $v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$

where $c_1, \ldots, c_n \in \mathbb{R}$.

(This means $[v]_\beta = (c_1, \ldots, c_n)$.)

Let $j \in \{1, \ldots, n\}$.

$$\langle v, v_j \rangle$$

$$= \langle c_1 v_1 + c_2 v_2 + \cdots + c_n v_r, v_j \rangle$$

$$= c_1 \langle v_1, v_j \rangle + c_2 \langle v_2, v_j \rangle$$

$$+ \cdots + c_r \langle v_n, v_j \rangle$$

$$= c_j \langle v_j, v_j \rangle .$$

Since $v_j \neq 0$, we know that $\langle v_j, v_j \rangle \neq 0$.

$$\therefore \quad c_j = \frac{\langle v, v_j \rangle}{\langle v_j, v_j \rangle}.$$

Corollary

Let $(V, \langle .,. \rangle)$ be an inner product space. Let $\mathcal{B} = \{\hat{v}_1, \ldots, \hat{v}_n\}$ be an orthonormal basis for V. Let $v \in V$ be any vector. Then the $\mathcal{B}$-coordinates of v are:

$$[v]_{\mathcal{B}} = (\langle v, \hat{v}_1 \rangle, \ldots, \langle v, \hat{v}_n \rangle) .$$

Related to Fourier transform.

Example

$V = C[-\pi, \pi]$ with the inner product defined by

$$\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)g(t)\,\mathrm{dt}$$

$W = \mathrm{Span}\{1, \cos t, \cos 2t, \cos 3t\}$. Let $f \in W$. The coordinates of f with respect to the orthogonal basis
$f_0 = 1, f_1 = \cos t, f_2 = \cos 2t, f_3 = \cos 3t$ are

$\langle f_0, f_0\rangle = 2$

$$\frac{\langle f, f_0\rangle}{\langle f_0, f_0\rangle} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\mathrm{dt}$$

$$\langle f, f_n\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,\mathrm{dt}, \qquad \text{for } n = 1, 2, 3.$$

$n \leq m \qquad U = [u_1 \; \cdots \; u_n]$

Theorem

An $m \times n$ matrix U has orthonormal columns if and only if $U^T U = I$.

$u_1, \ldots, u_n \in \mathbb{R}^m$.

Proof:

Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ be the columns of U. Then the i, j-th entry of $U^T U$ is

$$(\text{row } i \text{ of } U^T).(\text{column } j \text{ of } U) = \mathbf{u}_i^T \mathbf{u}_j = \mathbf{u}_i \cdot \mathbf{u}_j$$

$\square$

And if things weren't confusing enough ...

Definition

An $n \times n$ matrix is said to be *orthogonal* if its columns form an **orthonormal set**.

$U^T U = I$

$P^T P = I$

Proposition

An $n \times n$ matrix P is orthogonal if and only if $P^T = P^{-1}$.

Proof:

By the theorem on the previous slide, P is orthogonal if and only if $P^T = P^{-1}$.

□

Example

$$P = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}.$$

$$P^T \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}$$

The Gram-Schmidt Process

Algorithm.

Theorem

Given a basis $\{v_1, \dots, v_p\}$ for an inner product space $(V, \langle ., . \rangle)$, define

$$w_1 = v_1$$

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1$$

$$w_3 = v_3 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_3, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2$$

$$w_p = v_p - \frac{\langle v_p, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_p, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2 - \cdots - \frac{\langle v_p, w_{p-1} \rangle}{\langle w_{p-1}, w_{p-1} \rangle} w_{p-1}$$

Then $\{w_1, \dots, w_p\}$ is an orthogonal basis for V. In addition

$$\text{Span}\{w_1, \dots, w_k\} = \text{Span}\{v_1, \dots, v_k\} \quad \textit{for } 1 \le k \le p$$