Def 3.1 Cumulative Distribution Function

- The CDF of a RV X is $F_X(x) = P[X \leq x]$
- Properties of a CDF are the same for all kinds of random variables
  - Discrete,
  - Continuous, and
  - Mixed.

Theorem 3.1

For any random variable X,

(a) $F_X(-\infty) = 0$ $\rightarrow$ $P[X \leq -\infty]$

(b) $F_X(\infty) = 1$ $P[X \leq \infty]$

(c) $P[x_1 < X \leq x_2] = F_X(x_2) - F_X(x_1)$

CDF Discrete vs. Continuous

- The CDF of a discrete RV contained jumps
- The size of the jump (change) was the *probability of the RV at the point of jump (given by the PMF)*
- The larger the jump, the larger the accumulation of probability at the point in the range space

CDF Discrete vs. Continuous

$d(x) = \frac{1}{2} a x^2$

The CDF of a continuous RV is continuous

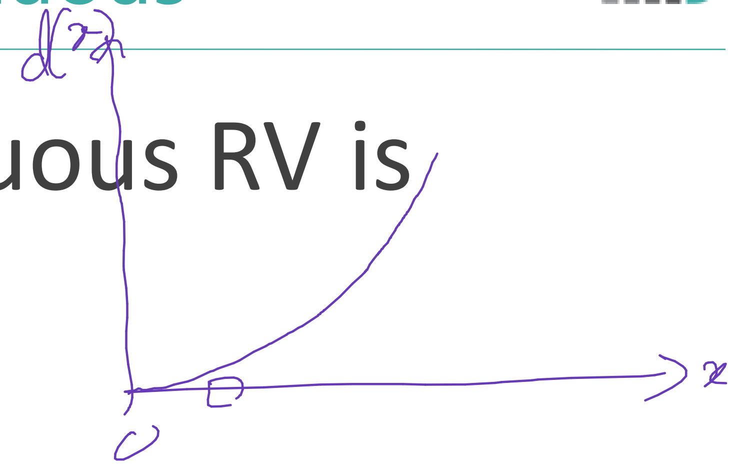
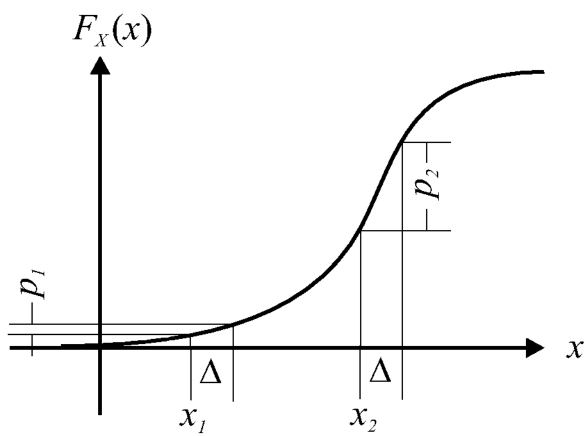


Figure 3.2



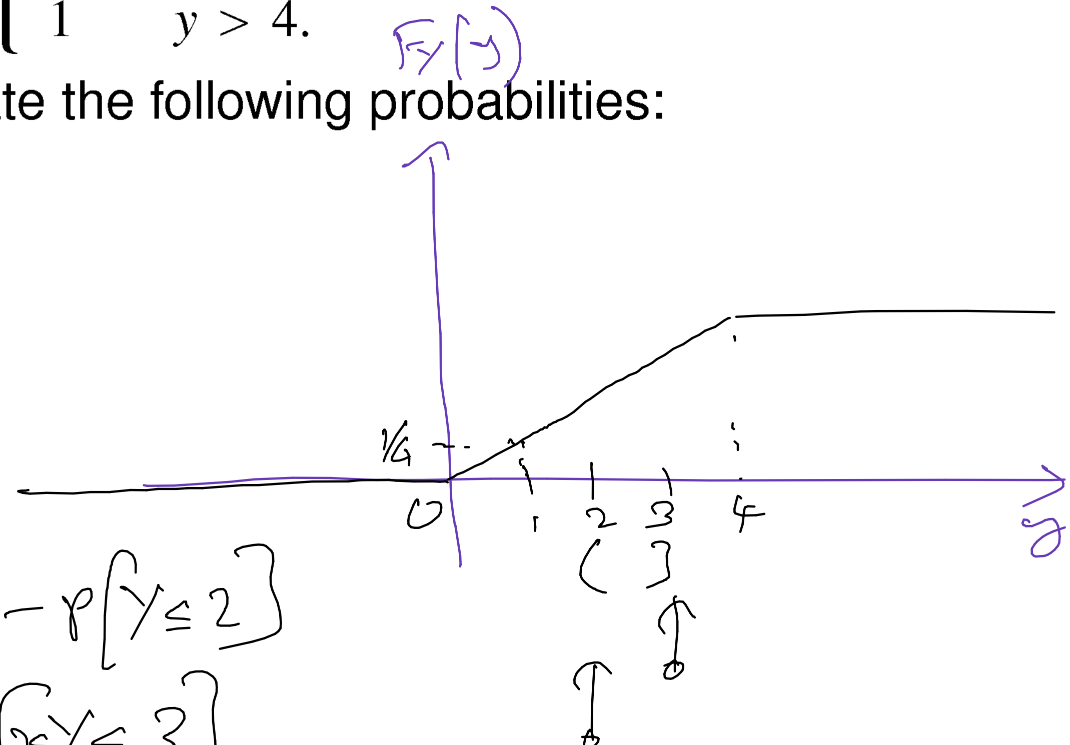
The graph of an arbitrary CDF $F_X(x)$.

Quiz 3.1

The cumulative distribution function of the random variable Y is

$$F_Y(y) = \begin{cases} 0 & y < 0, \\ y/4 & 0 \le y \le 4, \\ 1 & y > 4. \end{cases} \tag{3.9}$$

Sketch the CDF of Y and calculate the following probabilities:



(1) $P[Y \le -1]$

(2) $P[Y \le 1]$

(3) $P[2 < Y \le 3]$ $= P[Y \le 3] - P[Y \le 2]$

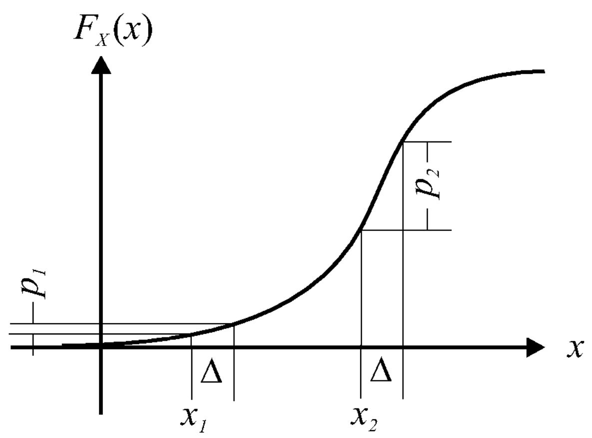
$P[2 \le Y \le 3]$

$=$

(4) $P[Y > 1.5]$

Example CDF of a continuous RV

Figure 3.2



$P[x_2 < X \le x_2 + \Delta]$

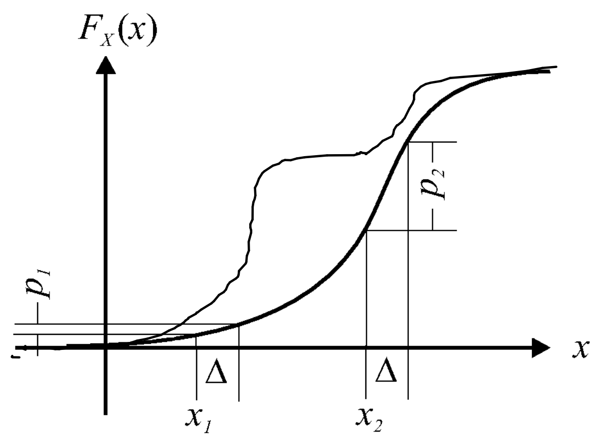
$P[x_1 < X \le x_1 + \Delta]$

The graph of an arbitrary CDF $F_X(x)$.

- No jumps anymore
- Instead, we have rates of accumulation of probability

Information in a CDF

Figure 3.2



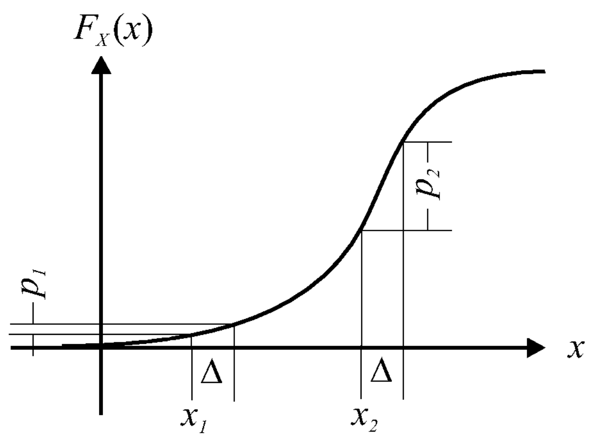
The graph of an arbitrary CDF $F_X(x)$.

$$\underbrace{F_X(x_2+\Delta) - F_X(x_2)}_{P[x_2 < X \le x_2 + \Delta]} > \underbrace{F_X(x_1+\Delta) - F_X(x_1)}_{P[x_1 < X \le x_1 + \Delta]}$$

- What is the LHS and the RHS?

Information in a CDF

Figure 3.2



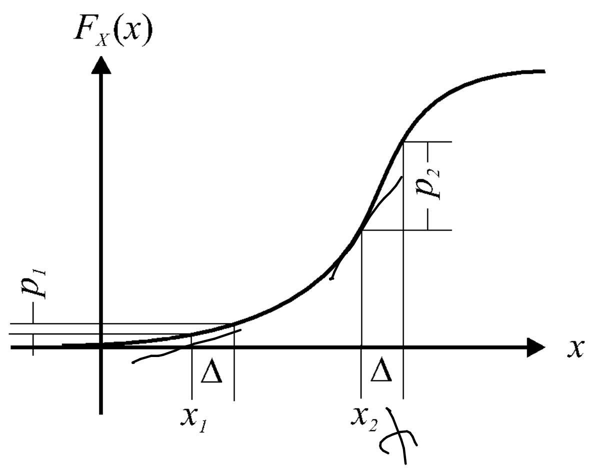
The graph of an arbitrary CDF $F_X(x)$.

$$F_X(x_2+\Delta) - F_X(x_2) > F_X(x_1+\Delta) - F_X(x_1)$$

- Rate of accumulation of probability in the region right of x_2 is greater than in the region right of x_1

Information in a CDF

Figure 3.2



The graph of an arbitrary CDF $F_X(x)$.

- As the interval Δ becomes smaller the rates of accumulation -> slope of the CDF

Probability Density Function

- Greater **the slope of the CDF** at any point x, the more the likelihood of occurrence of the interval around x

Probability Density Function

Definition 3.3 (PDF)

The probability density function *(PDF) of a continuous random variable* X *is*

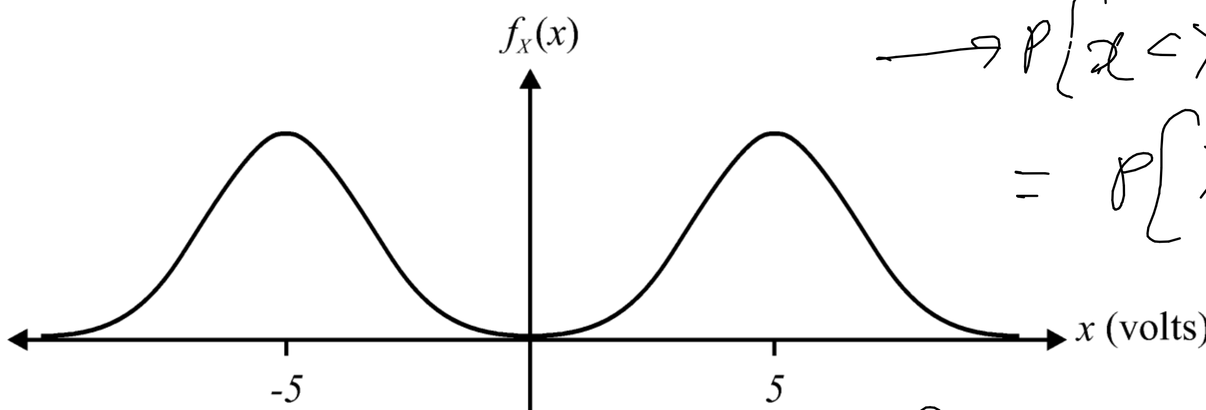
$$f_X(x) = \frac{dF_X(x)}{dx}.$$

Note the lowercase f

Example PDF

$g(x+\delta x) \approx g(x) + (\delta x)\, g'(x)$

Figure 3.3



$\rightarrow P[x < X \le x+\delta x]$

$= P[X \le x+\delta x]$

$- P[X \le x]$

$F_X(x+\delta x) \approx F_X(x) + (\delta x)\dfrac{dF_X(x)}{dx}$

δx

$= F_X(x+\delta x)$

$- F_X(x)$

$\approx f_X(x)\, \delta x.$

The PDF of the modem receiver voltage X.

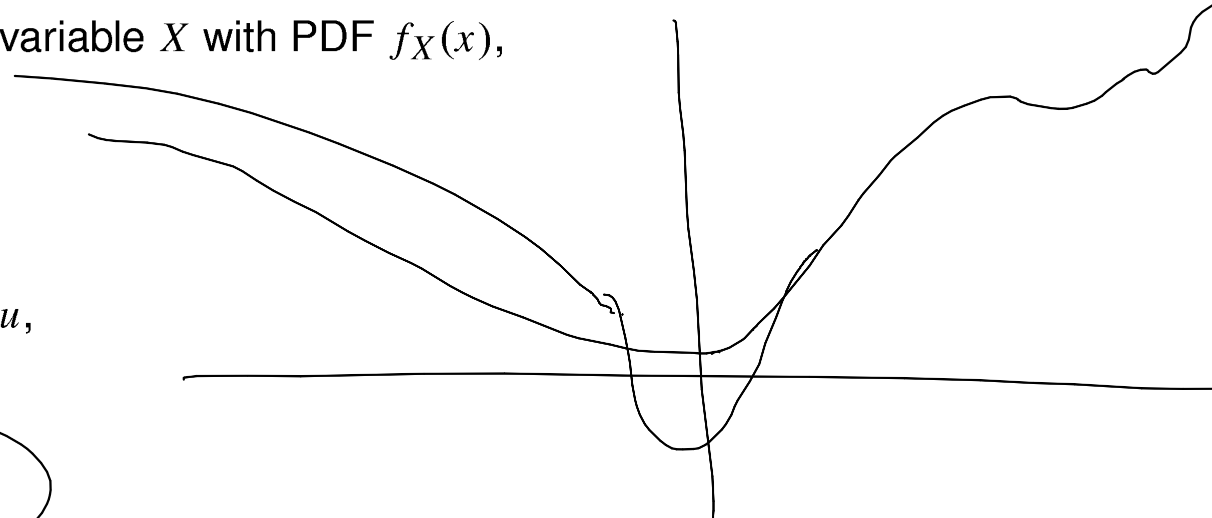
$= F_X(x) + (\delta x)\, f_X(x)$

- You are sending 0(s) and 1(s) mapped to -5 and +5
- Receiver sees the above PDF
- Which regions are most likely to be seen by the receiver and why?

Properties of a PDF

Theorem 3.2

For a continuous random variable X with PDF $f_X(x)$,



(a) $f_X(x) \geq 0$ for all x,

(b) $F_X(x) = \int_{-\infty}^{x} f_X(u)\, du$,

(c) $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$.

- Why?

Properties of a PDF

Theorem 3.2

For a continuous random variable X with PDF $f_X(x)$,

(a) $f_X(x) \geq 0$ for all x,

(b) $F_X(x) = \int_{-\infty}^{x} f_X(u)\, du$,

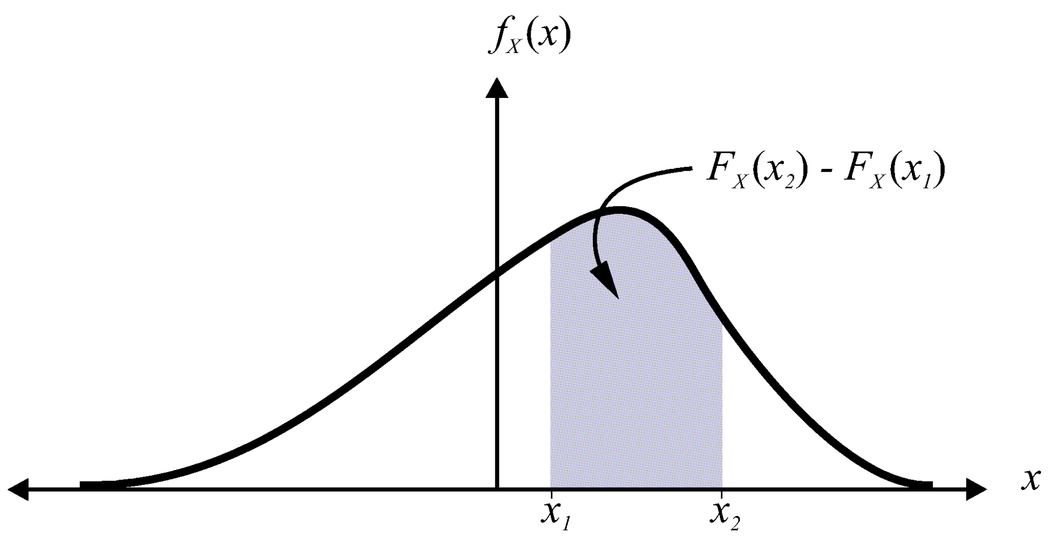
(c) $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$.

- Why is (a) true?
  - CDF is a non-decreasing function and PDF is its slope
- Why (b)?
  - From definition of a PDF
- Why (c)?
  - The integral is $P[X < \infty] = F_X(\infty) = 1$.

Properties of a PDF

Theorem 3.3

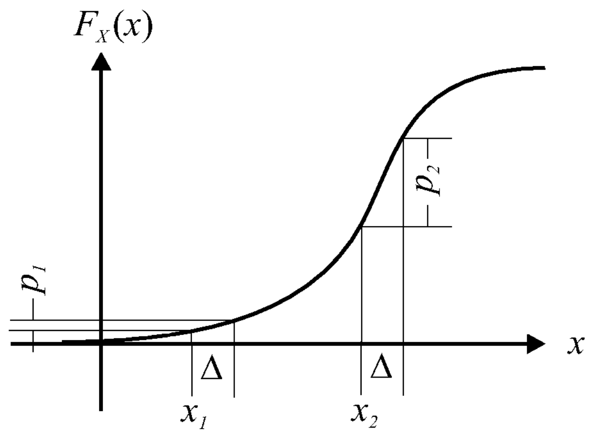
$$P[x_1 < X \leq x_2] = \int_{x_1}^{x_2} f_X(x)\,dx.$$



The PDF and CDF of X.

Properties of a PDF

Figure 3.2



The graph of an arbitrary CDF $F_X(x)$.

As the interval Δ becomes smaller:

$$P[x_2 < X \leq x_2 + \Delta] \approx \frac{F_X(x_2+\Delta) - F_X(x_2)}{\Delta}\Delta$$

Intervals and CRV(s)

- Consider the four different events
- A = (0,1)
- B = (0,1]
- C = [0,1)
- D = [0,1]
- They belong to the range space of a **continuous RV X**
- What can we say about P[A], P[B], P[C], and P[D]?

$P[A] = P[0 < X < 1]$ $\quad P[0 \le X \le 1 \mid X = 0.5]$

$= 1$

$P[B] = P[0 < X \le 1]$

$P[D] = P[0 \le X \le 1]$

x

$\cancel{P[0 \le x \le 1]}$

Problem on PDF

Quiz 3.2

Random variable X has probability density function

$$f_X(x) = \begin{cases} cxe^{-x/2} & x \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.25}$$

Sketch the PDF and find the following:

(1) the constant c

(2) the CDF $F_X(x)$

(3) $P[0 \leq X \leq 4]$

(4) $P[-2 \leq X \leq 2]$

(5) What is the mode of the random variable X?

Problem 3.1.3

The CDF of random variable W is

$$F_W(w) = \begin{cases} 0 & w < -5, \\ (w+5)/8 & -5 \le w < -3, \\ 1/4 & -3 \le w < 3, \\ 1/4 + 3(w-3)/8 & 3 \le w < 5, \\ 1 & w \ge 5. \end{cases}$$

$$\frac{(-3+5)}{8} = \frac{2}{8} = \frac{1}{4}$$

(a) What is $P[W \le 4]$? $\rightarrow$ $F_W(4)$

(b) What is $P[-2 < W \le 2]$? $\rightarrow$ $F_W(2) - F_W(-2)$

(c) What is $P[W > 0]$? $1 - F_W(0)$

(d) What is the value of a such that $P[W \le a] = 1/2$?

$$\frac{1}{4} + 3(w-3)/8 = \frac{1}{2}$$

Problem 3.2.4

For a constant parameter $a > 0$, a Rayleigh random variable X has PDF

$$f_X(x) = \begin{cases} a^2 x e^{-a^2x^2/2} & x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

What is the CDF of X?

$$\to F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx = \int_{-\infty}^{0} f_X(x)\,dx + \int_{0}^{\infty} f_X(x)\,dx$$

$$= \int_{0}^{x} f_X(x)\,dx$$

Expected Value

- **Def 3.4** The expected value of a continuous random variable X is

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

- Summation for the discrete case is replaced by integration for the continuous case
- **Theorem 3.4** The expected value of a function g(X) of a continuous random variable X is

$$E[g(X)] \quad \cancel{E[X]} = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

These Equalities are Valid for Both Continuous and Discrete RV(s)

Theorem 3.5

For any random variable X,

(a) $E[X - \mu_X] = \boxed{}$

(b) $E[aX + b] = \boxed{}$

(c) $\mathrm{Var}[X] = \boxed{}$

(d) $\mathrm{Var}[aX + b] = \boxed{}$

$E\left[(X - \mu_X)^2\right]$

$g(X) = (X - \mu_X)^2$