Let A be an $n \times n$ matrix (where $n \geq 2$). A_{ij} denotes the $n-1 \times n-1$ submatrix formed by deleting the i-th row and j-th column of A, for $1 \leq i, j \leq n$.

Definition

For $n \geq 2$, the determinant of an $n \times n$ matrix $A = (a_{ij})$ is the sum of n terms of the form $\pm a_{1j} \det A_{1j}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of A. In symbols,

$$\begin{aligned}\det A &= a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n} \\ &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}\end{aligned}$$

The determinant of a 1×1 matrix is the single entry of that matrix.

Definition

Let $A = (a_{ij})$ be an $n \times n$ matrix (where $n \geq 2$). The (i, j)-*cofactor of* A is the number C_{ij} given by

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

Theorem

The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column. The expansion across the i-th row using the cofactors is

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

The cofactor expansion down the j-th column is

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}.$$

Theorem

If A is a triangular matrix, then $\det A$ is the product of the entries on the main diagonal of A.

Proposition

Let E be an $n \times n$ elementary matrix. Then

1. $\det E = c$, when E corresponds to scaling a row by a nonzero scalar c
2. $\det E = 1$, when E corresponds to a row replacement operation
3. $\det E = -1$, when E corresponds to a row interchange operation

Proposition

Let A be an $n \times n$ matrix having two identical rows. Then the determinant of A is zero.

Idea(s) behind proof:

First consider the case where the identical rows are adjacent. Expand.

Next consider the non-adjacent case. Use induction.

Continued from Tuesday's Lecture

~~If $n = 1$, then there is nothing to show.~~ We have also shown that the result is true when $n = 2$. Therefore let us assume that the result holds true for every $n < k$ and show that it holds true for $n = k$ (where k is an integer greater than 2). wlog, let $i<j$.

Let i and j be the rows of A which are identical. If these rows are adjacent, then there is nothing to show. So assume that there exists l such that $i < l < j$. Expanding across row l we get

$$\det A = (-1)^{l+1} a_{l1} \det A_{l1} + \ldots + (-1)^{l+n} a_{ln} \det A_{ln}.$$

submatrices

Each of the ~~minors~~ A_{lm} $(m = 1, \ldots, n)$ has at least two identical rows. So by the induction hypothesis,

$$\det A_{lm} = 0, \text{ for } m = 1, \ldots, n.$$

Hence $\det A = 0$.

Theorem

Let A be an $n \times n$ matrix. Let E be an $n \times n$ elementary matrix. Then

$$\det(EA) = \det(E)\det(A)$$

For row replacement and row scaling, expand along the appropriate row.

For row interchange, $R_i \longleftrightarrow R_j$, is equivalent to the following sequence to row operations:

1. $R_j \rightarrow R_i + R_j$: let this correspond to elementary matrix E_1
2. $R_i \rightarrow R_i - R_j$: let this correspond to elementary matrix E_2
3. $R_j \rightarrow R_i + R_j$: let this correspond to elementary matrix E_3
4. $R_i \rightarrow -R_i$: let this correspond to elementary matrix E_4

Proof

$(EA)_{ik} = a_{ik} + c a_{jk}$

Let $A = (a_{ij})$.

We first consider the case where E corresponds to row replacement, say $R_i \to R_i + cR_j$, where $c \in \mathbb{R}$ (or $c \in \mathbb{C}$, if A has complex entries). Expanding along the the i-th row,

$$
\begin{aligned}
&\det(EA)\\
&= (-1)^{i+1}(a_{i1} + ca_{j1}) \det EA_{i1} + \ldots + (-1)^{i+n}(a_{in} + ca_{jn}) \det EA_{in}\\
&= (-1)^{i+1}(a_{i1} + ca_{j1}) \det A_{i1} + \ldots + (-1)^{i+n}(a_{in} + ca_{jn}) \det A_{in}\\
&= \det A + \underbrace{c((-1)^{i+1}a_{j1} \det A_{i1} + \ldots + (-1)^{i+n}a_{jn} \det A_{in})}_{\text{becomes zero.}}
\end{aligned}
$$

But the expression

$$(-1)^{i+1}a_{j1} \det A_{i1} + \ldots + (-1)^{i+n}a_{jn} \det A_{in}$$

is nothing but the determinant of the matrix obtained by replacing the i-th row of A by the j-th row of A. We know that any matrix having identical rows has zero determinant.

$$
\begin{array}{l}
\\
j\text{th row} \\
i\text{th} \\
\\
\end{array}
\begin{bmatrix}
a_{11} & - & - & - & a_{1n} \\
a_{j1} & - & - & - & a_{jn} \\
\cancel{a_{j1}} & - & - & - & \cancel{a_{jn}} \\
a_{n1} & - & - & - & a_{nn}
\end{bmatrix}
\qquad
\overset{A}{\begin{bmatrix}
a_{11} & \cdots & a_{1n} \\
\vdots & & \\
a_{j1} & \cdots & a_{jn} \\
\cancel{a_{i1}} & - & \cancel{a_{in}} \\
a_{n1} & - & a_{nn}
\end{bmatrix}}
$$

$$(-1)^{j+1} a_{j1} A_{j1} + \cdots + (-1)^{n+j} a_{jn}$$

So

$$(-1)^{i+1} a_{j1} \det A_{i1} + \ldots + (-1)^{i+n} a_{jn} \det A_{in} = 0.$$

Hence $\det EA = \det A$. Since $\det E = 1$,

$$\det EA = \det E \det A.$$

Next, suppose E corresponds to scaling, say $R_i \to cR_i$ for some $c \in \mathbb{R}$ (or $c \in \mathbb{C}$, if A has complex entries). Then expanding along the i-th row, we get

$$\det(EA) = (-1)^{i+1} c a_{i1} \det A_{i1} + \ldots + (-1)^{i+n} c a_{in} \det A_{in} = c \det A.$$

Next, suppose E corresponds to row interchange, say $R_i \longleftrightarrow R_j$.

This is equivalent to the following sequence to row operations:

1. $R_j \to R_i + R_j$: let this correspond to elementary matrix E_1
2. $R_i \to R_i - R_j$: let this correspond to elementary matrix E_2
3. $R_j \to R_i + R_j$: let this correspond to elementary matrix E_3
4. $R_i \to -R_i$: let this correspond to elementary matrix E_4

Therefore

$EA = E_4(E_3E_2E_1A)$

$$\begin{aligned}
\det EA &= \det E_4E_3E_2E_1A \\
&= -\det E_3E_2E_1A \\
&= -\det E_2E_1A \\
&= -\det E_1A \\
&= -\det A
\end{aligned}$$

Hence $\det EA = \det E \det A$.

$$
\begin{array}{r}
\\ \\ \text{ith row} \rightarrow \\ \text{jth} \rightarrow \\ \\ \\
\end{array}
\begin{bmatrix} r_1 \\ \vdots \\ r_i \\ r_j \\ \vdots \\ r_n \end{bmatrix}
\longrightarrow
\begin{array}{r}
\\ \\ \text{ith} \rightarrow \\ \text{jth} \rightarrow \\ \\
\end{array}
\begin{bmatrix} r_1 \\ \vdots \\ r_i \\ r_i + r_j \\ r_n \end{bmatrix}
\begin{array}{l}
\\ \\ \leftarrow \text{ith} \\ \leftarrow \text{jth} \\ \\
\end{array}
$$

$$
\begin{array}{r}
\\ \\ \text{ith row} \\ \text{jth row} \\ \\ \\
\end{array}
\begin{bmatrix} r_1 \\ \vdots \\ -r_j \\ r_i + r_j \\ \vdots \\ r_n \end{bmatrix}
\longrightarrow
\begin{array}{r}
\\ \\ \text{ith row} \\ \\ \text{jth} \\ \\ \\
\end{array}
\begin{bmatrix} r_1 \\ \vdots \\ -r_j \\ \vdots \\ r_i \\ \vdots \\ r_n \end{bmatrix}
$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Swap R_1 and R_3

$$\downarrow$$

$$R_3 \to R_1 + R_3 \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 8 & 10 & 12 \end{bmatrix} \xrightarrow{R_1 \to R_1 - R_3} \begin{bmatrix} -7 & -8 & -9 \\ 4 & 5 & 6 \\ 8 & 10 & 12 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_1 + R_3} \begin{bmatrix} -7 & -8 & -9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}$$

$$\xrightarrow{R_1 \to -R_1} \begin{bmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}$$

$B = EA$

$\det B = \det E \det A$

Corollary

Let A be a square matrix.

- **a** *If a multiple of one row of A is added to another row to produce a matrix B, then $\det B = \det A$*
- **b** *If two rows of A are interchanged to produce B, then det $B = -\det A$.*
- **c** *If one row of A is multiplied by k to produce B, then det $B = k \cdot \det A$.*

Theorem

Let A and B be $n \times n$ matrices. Then

$$\det AB = \det A \det B.$$

Lemma

Let $E_1, \ldots, E_p$ be a sequence of $n \times n$ elementary matrices. Then

$$\det\left(\prod_{i=1}^{p} E_i\right) = \prod_{i=1}^{p} \det E_i .$$

Proof: Exercise.

$\det(E_1 E_2) = \det E_1 \det E_2$

$\to$ use induction.

Theorem

Let U be an echelon form of A obtained by row replacement and row interchange operations. Let r be the number if row interchanges involved in the row reduction process. Then

$$\det A = \begin{cases} \underbrace{(-1)^r}_{} \cdot \begin{pmatrix} \textit{product of} \\ \textit{pivots in } U \end{pmatrix} & \textit{when } A \textit{ is invertible} \\ 0 & \textit{when } A \textit{ is not invertible} \end{cases}$$

Corollary

A square matrix A is invertible if and only if $\det A \neq 0$.

Proof

$U = E_p \cdots E_1 A$

A square echelon matrix is upper triangular, and has all pivot positions on the main diagonal.

Therefore $\det U$ equals the product of the diagonal entries of U.

If A is invertible, then all the diagonal entries of U are pivots and therefore $\det U$ equals the product of pivots in U.

If A is not invertible, then at least one diagonal entry of U is zero and therefore $\det U = 0$.

Since U is obtained from A using r row interchanges and some row replacement operations, it follows that

$$\det U = (-1)^r \det A.$$

Thus

$$\det A = \begin{cases} (-1)^r \cdot \begin{pmatrix} \text{product of} \\ \text{pivots in } U \end{pmatrix} & \text{when } A \text{ is invertible} \\ 0 & \text{when } A \text{ is not invertible} \end{cases}$$

$AB = (E_1 \cdots E_p E_{p+1} \cdots E_q)$

Back to the proof of $\det AB = \det A \det B$.

We first consider the case where both A and B are invertible.

There exist elementary matrices $E_1, \ldots, E_p$ such that

$$A = E_1 E_2 \ldots E_p$$

$\prod_{i=1}^{p} \det(E_i) \prod_{i=p+1}^{q} \det(E_i)$

and $E_{p+1}, \ldots, E_q$ such that

$$B = E_{p+1} \ldots E_q.$$

Hence by Lemma stated earlier,

$= \prod_{i=1}^{q} \det(E_i)$

$$\det AB = \det \left(\prod_{i=1}^{q} E_i \right) = \det \left(\prod_{i=1}^{p} E_i \right) \det \left(\prod_{i=p+1}^{q} E_i \right) = \det A \det B.$$

$AD = I \implies A$ is invertible

$A\underbrace{(B(AB)^{-1})} = I$

Next, suppose either A or B or both, are singular.

By Corollary stated earlier, $\det A = 0$ or $\det B = 0$ (or both). Hence $\det A \det B = 0$.

AB invertible $\implies AB(AB)^{-1} = I \implies A$ is invertible, and $(AB)^{-1}AB = I \implies B$ is invertible (by the Invertible Matrix Theorem).

Therefore AB is not invertible.

Hence $\det AB = 0$. Therefore $\det AB = \det A \det B$.

$Ax=b$.

New for you

$A = [a_1 \cdot a_i \cdot a_n]$

For any $n \times n$ matrix A and any $\mathbf{b}$ in $\mathbb{R}^n$, let $A_i(\mathbf{b})$ be the matrix obtained from A by replacing column i by the vector $\mathbf{b}$.

$$A_i(\mathbf{b}) = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{b} & \cdots & \mathbf{a}_n \end{bmatrix}$$

i-th column

Theorem (Cramer's Rule)

Let A be an invertible $n \times n$ matrix. For any $\mathbf{b}$ in $\mathbb{R}^n$ or $\mathbb{C}^n$, the unique solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ has entries given by

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}, \quad i = 1, 2, \ldots, n$$

Idea: We express $A_i\mathbf{b}$ as the product of A and a matrix. What matrix could this be?

$$Ax = b.$$

$$\begin{array}{ccc} & & \text{ith} \\ \left[\begin{array}{ccccccc} a_1 & a_2 & \cdots & b & \cdots & \cdots & a_n \end{array}\right] \\ & & \text{ith.} \end{array}$$

$$= \left[\begin{array}{ccc} a_1 & \cdots & a_n \end{array}\right] \begin{array}{c} \begin{array}{cccc} & & \text{ith} & \checkmark \end{array} \\ \left[\begin{array}{cccc} 1 & 0 & x_1 & 0 \\ 0 & 1 & \vdots & \vdots \\ \vdots & 0 & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \vdots & x_n & 1 \end{array}\right] \end{array}$$

$$\det(A_i(b)) = \det(A)\det(I_i(x))$$

$$\begin{array}{cc} & \begin{array}{ccccc} & & \downarrow \text{in column} & & \end{array} \\ I_i(x) = & \begin{bmatrix} 1 & 0 & x_1 & 0 & \cdots \\ 0 & 1 & x_2 & \vdots & \\ \vdots & \vdots & \vdots & & \\ 0 & \cdots & x_i & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & x_n & & 1 \end{bmatrix} \end{array} \quad \begin{array}{l} \\ \\ \\ \leftarrow i\text{th row} \\ \\ \\ \end{array}$$

Lemma

Let I be the $n \times n$ identity matrix, and let $\mathbf{x} = (x_1, \ldots, x_n)$ be any vector in $\mathbb{R}^n$, or in $\mathbb{C}^n$. Then

$$\det I_i(\mathbf{x}) = x_i.$$

i,ith entry $\quad I \quad x_i$

of $I_i(x) = x_i$

$c(dv) = (cd)v$.

Axler:

$M_n(\mathbb{R})$

$\uparrow$

ring

$a \in F$

$v \in V$

$F = \mathbb{R}$ or $\mathbb{C}$

In a field

$ab = 0$

$\Rightarrow a = 0$ or $b = 0$

$av = 0$

Prove: $a = 0$ or $v = 0$.

case (i) If $a = 0$, nothing to show.

case (ii): $a \neq 0$. $av = 0$

$\Rightarrow a^{-1}(av) = a^{-1} \cdot 0$

$= 0$

$v = 1v = (a^{-1}a)v$