Theorem 7.3 Chebyshev Inequality

For an arbitrary random variable Y and constant $c > 0$,

$$P\left[|Y - \mu_Y| \geq c\right] \leq \frac{\mathrm{Var}[Y]}{c^2}.$$

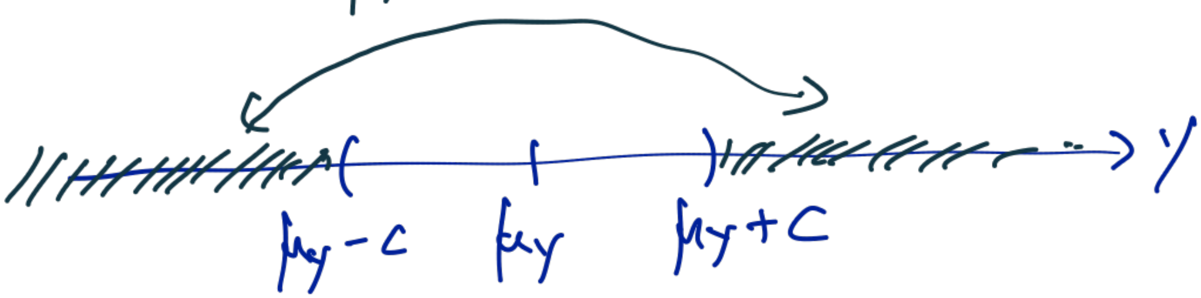
Define $Z \triangleq |Y - \mu_Y|^2$

$$\hookrightarrow P[Z \geq c^2] \leq \frac{E[Z]}{c^2}$$

$$= \frac{E\left[|Y - \mu_Y|^2\right]}{c^2}$$

$$= \frac{\mathrm{Var}[Y]}{c^2}$$

$|Y - \mu_Y| \geq c$

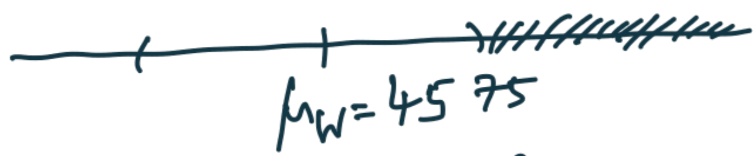


Quiz 7.2

$P[|Y-\mu_Y| \ge c] \le \dfrac{Var[Y]}{c^2}$

Elevators arrive randomly at the ground floor of an office building. Because of a large crowd, a person will wait for time W in order to board the third arriving elevator. Let X_1 denote the time (in seconds) until the first elevator arrives and let X_i denote the time between the arrival of elevator $i-1$ and i. Suppose X_1, X_2, X_3 are independent uniform $(0, 30)$ random variables. Find upper bounds to the probability W exceeds 75 seconds using

$$P[W>75] = P[W>(\sqrt{75})^2] \le \frac{E[W]}{75} = \frac{45}{75}$$



The event of interest: $\{W-\mu_W > 30\}$

For the Cheby ineq, the event was $|W-\mu_W| \ge c$

$$P[W-\mu_W > 30] \le P[|W-\mu_W| > 30] \le \frac{Var[W]}{c^2}$$

Problem 7.2.4

In a game with two dice, the event *snake eyes* refers to both dice showing one spot. Let R denote the number of dice rolls needed to observe the third occurrence of *snake eyes*. Find

(a) the upper bound to $P[R \geq 250]$ based on the Markov inequality,

(b) the upper bound to $P[R \geq 250]$ based on the Chebyshev inequality,

(c) the exact value of $P[R \geq 250]$.

$R = X_1 + X_2 + X_3$, where $X_i \sim \text{Geom}(1/36)$ and independent

$$E[R] = (36)\,3 = 108.$$

$$\text{Var}[R] = \text{Var}[X_1 + X_2 + X_3]$$

$$= \text{Var}[X_1] + \text{Var}[X_2] + \text{Var}[X_3] = 3\,\text{Var}[X_1].$$

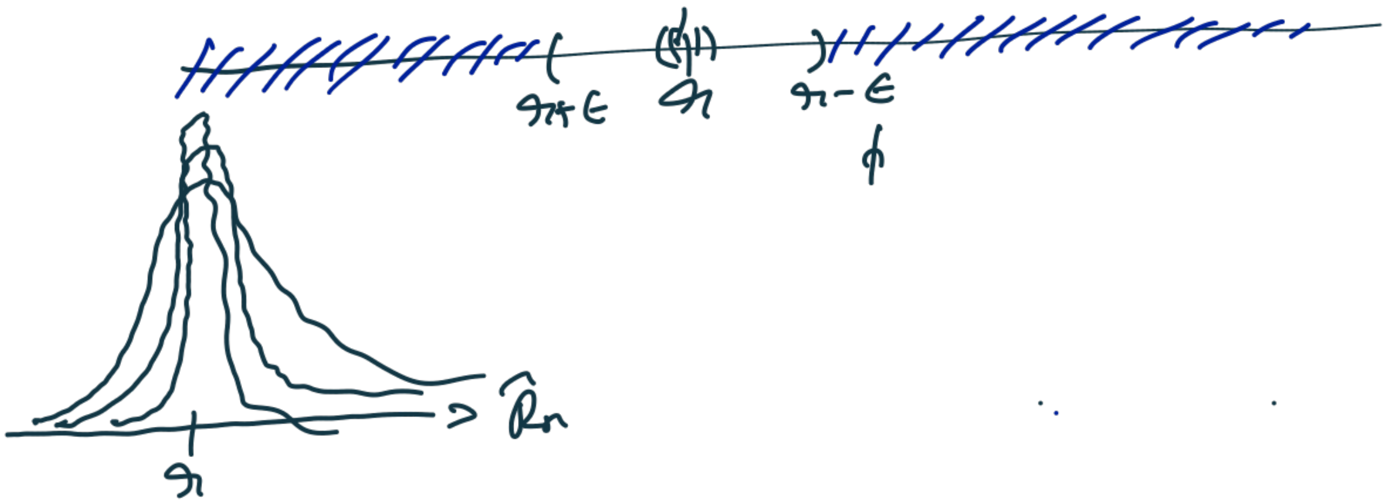
Section 7.3

Point Estimates of Model Parameters

Definition 7.2 Consistent Estimator

The sequence of estimates $\hat{R}_1, \hat{R}_2, \ldots$ *of the parameter* r *is* consistent *if for any* $\epsilon > 0$,

$$\lim_{n\to\infty} P\left[\left|\hat{R}_n - r\right| \geq \epsilon\right] = 0.$$



$n \to \infty$

Definition 7.3 Unbiased Estimator

An estimate, $\hat{R}$*, of parameter* r *is* unbiased *if* $E[\hat{R}] = r$*; otherwise,* $\hat{R}$ *is* biased.

Definition 7.4 Estimator

The sequence of estimators $\hat{R}_n$ *of parameter* r *is* asymptotically unbiased *if*

$$\lim_{n\to\infty} E[\hat{R}_n] = r.$$

Definition 7.5 Mean Square Error

The mean square error *of estimator* $\hat{R}$ *of parameter* r *is*

$$e = E\left[(\hat{R} - r)^2\right].$$

Theorem 7.4

Given:

If a sequence of unbiased estimates $\hat{R}_1, \hat{R}_2, \ldots$ of parameter r has mean square error $e_n = \mathrm{Var}[\hat{R}_n]$ satisfying $\lim_{n\to\infty} e_n = 0$, then the sequence $\hat{R}_n$ is consistent.

Show that:

$$\lim_{n\to\infty} P\left[|R_n - r| > \epsilon\right] = 0$$

Use the Cheby ineq:

$$P\left[|R_n - r| > \epsilon\right] \le \frac{\mathrm{Var}[R_n]}{\epsilon^2}$$

$$\lim_{n\to\infty} P\left[|R_n - r| > \epsilon\right] \le 0 \qquad \left[\because \text{ Given that } \lim_{n\to\infty} e_n = 0\right]$$