Expected Values

Theorem 4.15

The variance of the sum of two random variables is

$$\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] + 2E[(X-\mu_X)(Y-\mu_Y)].$$

$Z \triangleq$

$$\mathrm{Var}(Z) = E[Z^2] - (E[Z])^2$$
$$= E[(X+Y)^2] - (E[X+Y])^2$$

Definition 4.4 Covariance

The covariance *of two random variables* X *and* Y *is*

$$\mathrm{Cov}[X,Y] = E[(X-\mu_X)(Y-\mu_Y)].$$

First & 2nd order descriptions of a pair of R.V.s

$E[X]$, $E[Y]$, $\mathrm{Cov}[X,Y]$, $\mathrm{Var}[X]$, $\mathrm{Var}[Y]$

$$\begin{bmatrix} E[X_1] \\ E[X_2] \\ \vdots \\ E[X_n] \end{bmatrix}_{n\times 1}$$

$X_1, X_2, X_3, \dots, X_n$

Cov Matrix:

$$\begin{bmatrix} E[X_1^2] & E[X_1X_2] & \cdots & E[X_1X_n] \\ \vdots & \ddots & & \\ \vdots & & \ddots & \\ E[X_nX_1] & & \cdots & E[X_n^2] \end{bmatrix}_{n\times n}$$

Expected Values

Definition 4.5 Correlation

The correlation *of* X *and* Y *is* $r_{X,Y} = E[XY]$

Theorem 4.16

$$E[(X-\mu_X)(Y-\mu_Y)] = E[XY - \mu_X Y - \mu_Y X + \mu_X \mu_Y]$$

$$= E[XY] - \mu_X E[Y] - \mu_Y E[X] + \mu_X \mu_Y$$

$$= E[XY] - \mu_X \mu_Y - \mu_Y \mu_X + \mu_X \mu_Y$$

$$= E[XY] - \mu_X \mu_Y$$

(a) $\text{Cov}[X, Y] = r_{X,Y} - \mu_X \mu_Y$.

(b) $\text{Var}[X + Y] = \text{Var}[X] + \text{Var}[Y] + 2\,\text{Cov}[X, Y]$.

(c) If $X = Y$, $\text{Cov}[X, Y] = \text{Var}[X] = \text{Var}[Y]$ and $r_{X,Y} = E[X^2] = E[Y^2]$.

VS Problem

Example 4.12 Problem

For the integrated circuits tests in Example 4.1, we found in Example 4.3 that the probability model for X and Y is given by the following matrix.

$P_{X,Y}(x, y)$	$y = 0$	$y = 1$	$y = 2$	$P_X(x)$
$x = 0$	0.01	0	0	
$x = 1$	0.09	0.09	0	
$x = 2$	0	0	0.81	
$P_Y(y)$				

(4.73)

Find $r_{X,Y}$ and $\mathrm{Cov}[X, Y]$.

More Definitions!

Definition 4.6 Orthogonal Random Variables

Random variables X *and* Y *are* orthogonal *if* $r_{X,Y} = 0$.

$E[XY]$

Experiment trials:

$(x_1, y_1) \longrightarrow x_1 y_1$

$(x_2, y_2) \quad x_2 y_2$

$(x_3, y_3) \quad x_3 y_3$

$\vdots$

$(x_n, y_n) \quad x_n y_n$

Mean of products in

$$\frac{1}{n} \sum_{i=1}^{n} x_i y_i$$

More Definitions!

Definition 4.7 Uncorrelated Random Variables

Random variables X *and* Y *are* uncorrelated *if* $\mathrm{Cov}[X, Y] = 0$.

More Definitions!

Definition 4.8 *Correlation Coefficient*

The correlation coefficient *of two random variables* X *and* Y *is*

$$\rho_{X,Y} = \frac{\mathrm{Cov}\,[X, Y]}{\sqrt{\mathrm{Var}[X]\,\mathrm{Var}[Y]}} = \frac{\mathrm{Cov}\,[X, Y]}{\sigma_X \sigma_Y}.$$

Correlation Coefficient

Theorem 4.17

$$-1 \le \rho_{X,Y} \le 1.$$

- Let W = X – aY, for any constant a
- We have Var[W] = Var[X] – 2a Cov[X,Y] + a^2 Var[Y]
- Use the fact that Var[W] >= 0
- Let a = σ_x/σ_y to get the upper bound
- Let a = - σ_x/σ_y to get the lower bound

Correlation Coefficient

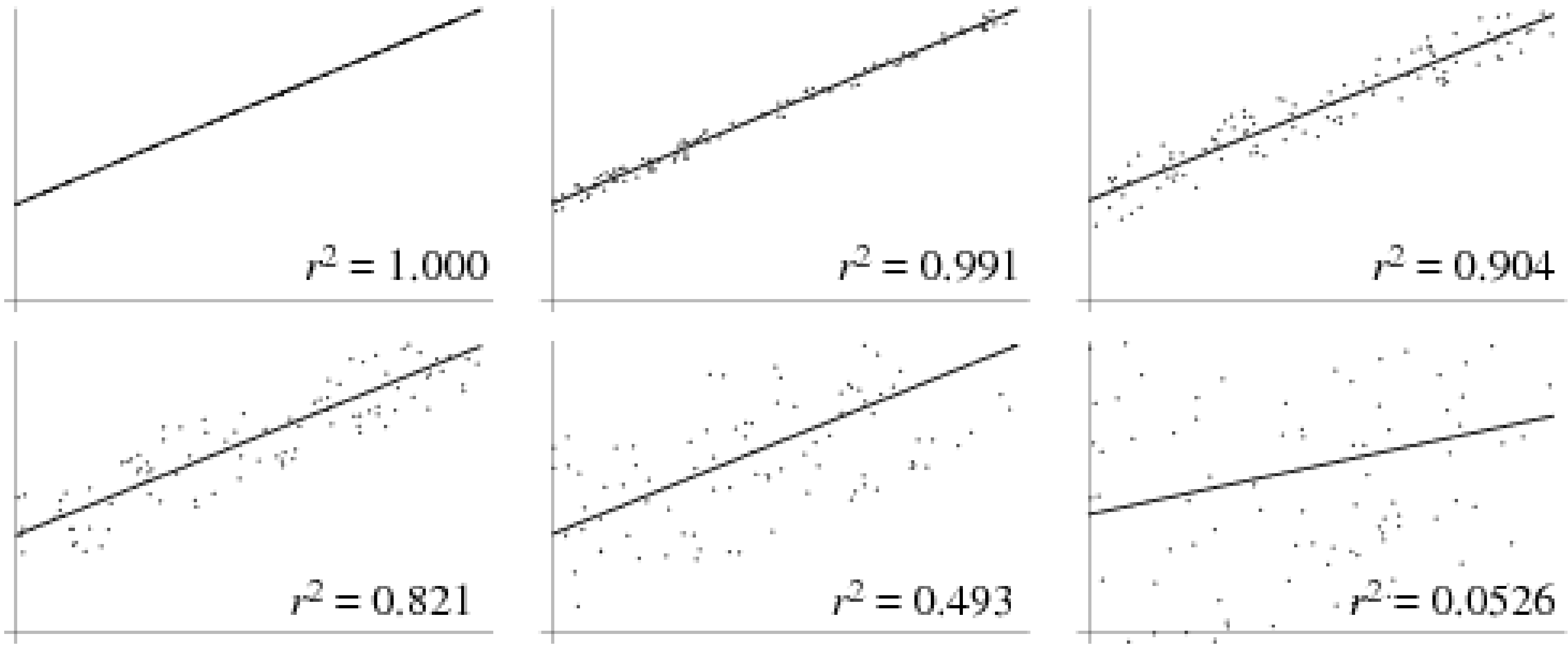
- A positive correlation coefficient implies that when X is high wrt E[X], even Y tends to be high wrt E[Y]
  - When we observe high values of X it is likely that Y too is high

Correlation Coefficient

- A negative coefficient implies that when X is low, Y is likely to be high and vice-versa

- If X and Y are uncorrelated then no such trend is observed

Examples (Scatter Plots)



Weisstein, Eric W. "Correlation Coefficient." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/CorrelationCoefficient.html

Y is an Affine Function of X

Theorem 4.18

If X and Y are random variables such that $Y = aX + b$,

$$\rho_{X,Y} = \begin{cases} -1 & a < 0, \\ 0 & a = 0, \\ 1 & a > 0. \end{cases}$$

Conditioning on an Event

Definition 4.9 ***Conditional Joint PMF***

For discrete random variables X *and* Y *and an event,* B *with* $P[B] > 0$*, the* conditional joint PMF *of* X *and* Y *given* B *is*

$$P_{X,Y|B}(x, y) = P[X = x, Y = y|B].$$

Conditioning on an Event

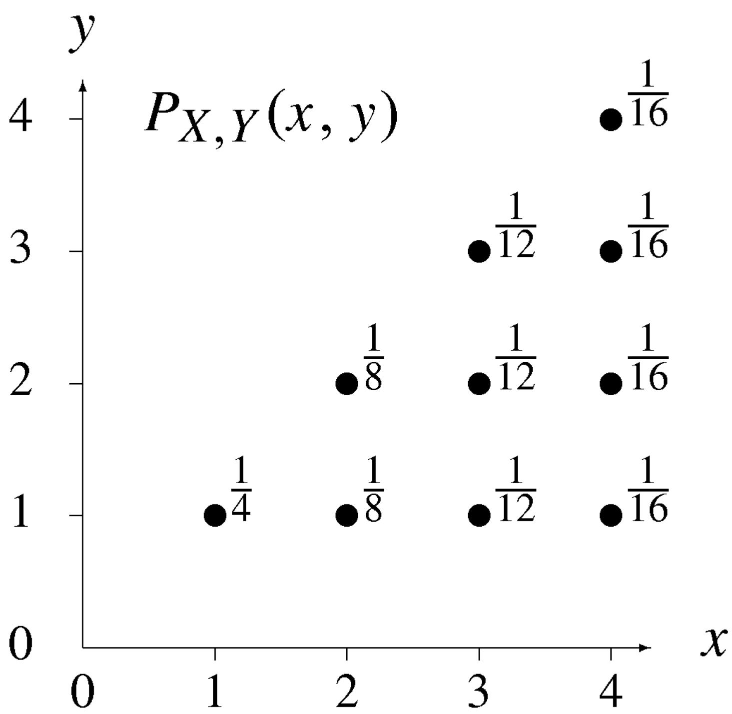
Theorem 4.19

For any event B, a region of the X, Y plane with $P[B] > 0$,

$$P_{X,Y|B}(x, y) = \begin{cases} \dfrac{P_{X,Y}(x, y)}{P[B]} & (x, y) \in B, \\ 0 & \text{otherwise.} \end{cases}$$

$$\hookrightarrow P[X=x, Y=y | B] = \frac{P[X=x, Y=y, B]}{P[B]} =$$

Example 4.13 Problem



Random variables X and Y have the joint PMF $P_{X,Y}(x, y)$ as shown. Let B denote the event $X + Y \leq 4$. Find the conditional PMF of X and Y given B.

Conditioning on an Event

Definition 4.10 Conditional Joint PDF

Given an event B with $P[B] > 0$, the conditional joint probability density function of X and Y is

$$f_{X,Y|B}(x,y) = \begin{cases} \dfrac{f_{X,Y}(x,y)}{P[B]} & (x,y) \in B, \\ 0 & \text{otherwise.} \end{cases}$$

$$P[x \le X < x+dx,\ y \le Y < y+dy \mid B] \approx f_{X,Y|B}(x,y)\,dx\,dy$$

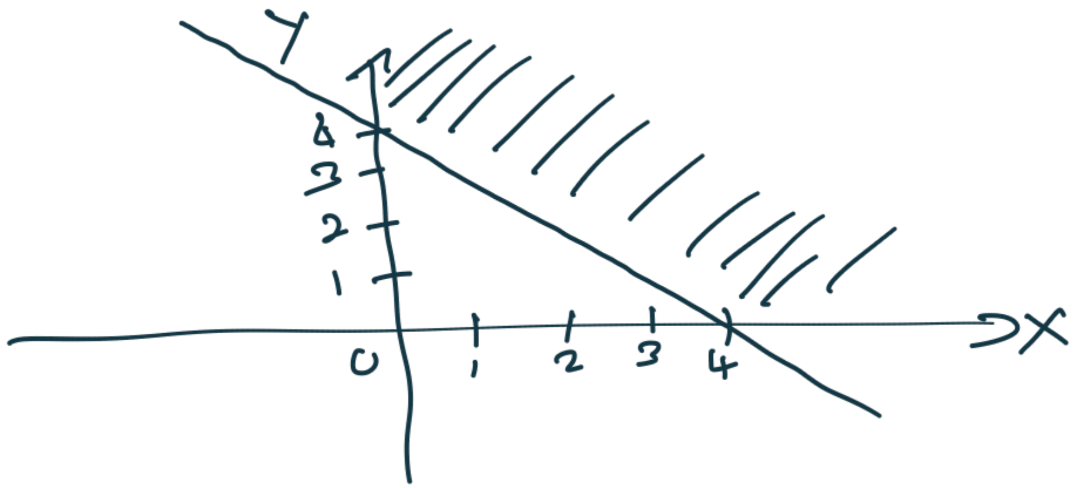
Conditioning on an Event

Example 4.14 Problem

X and Y are random variables with joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 1/15 & 0 \le x \le 5, 0 \le y \le 3, \\ 0 & \text{otherwise.} \end{cases} \tag{4.83}$$

Find the conditional PDF of X and Y given the event $B = \{X + Y \ge 4\}$.



$y = -x + 4$