Definition

A *pivot position* in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A. A *pivot column* is a column of A that contains a pivot position.

Solutions of Linear Systems

Definition

The variables corresponding to the pivot columns of the an augmented matrix of a linear system are called *basic variables*. The remaining variables are called *free variables*.

Using Row Reduction to Solve a Linear System

1. Write the augmented matrix of the system.
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.
3. Continue row reduction to obtain the reduced echelon form.
4. Write the system of equations corresponding to the matrix obtained in step 3.
5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

Vectors as Ordered Lists or n-tuples

$[a_1 \; . \; . \; . \; a_n]$

$\mathbb{R}^n \ni a = \underbrace{(a_1, \ldots, a_n)}_{n \text{ components}}$

$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$

We will **temporarily** use the word "vector" to refer to an ordered list of numbers.

Definition

The set of all n-tuples of real numbers is called $\mathbb{R}^n$.

Elements of $\mathbb{R}^n$ are usually **represented** as $n \times 1$ column vectors ($n \times 1$ matrices).

The vector whose entries are all zero is called the **zero vector** and is denoted by $\mathbf{0}$.

Equality of vectors in $\mathbb{R}^n$ and the operations of scalar multiplication and vector addition in $\mathbb{R}^n$ are defined entry by entry just as in $\mathbb{R}^2$.

Algebraic Properties of $\mathbb{R}^n$

For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^{\mathbf{n}}$ and all scalars c and d,

- $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
- $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$.
- $\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$ (where $-\mathbf{u}$ denotes $(-1)\mathbf{u}$)
- $c(\mathbf{u} + \mathbf{v}) = \mathbf{cu} + \mathbf{cv}$
- $(c + d)\mathbf{u} = \mathbf{cu} + \mathbf{du}$
- $c(d\mathbf{u}) = (\mathbf{cd})\mathbf{u}$
- $1\mathbf{u} = \mathbf{u}$

x_{i1} $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

Notation

If $\mathbf{x}$ is an $n \times 1$ column vector then we denote its i-th entry as x_i.

Definition

Given vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots \mathbf{v_p} \in \mathbb{R}^\mathbf{n}$ and given scalars $c_1, c_2, \ldots, c_p$, the vector $\mathbf{y}$ defined by

$$\mathbf{y} = c_1\mathbf{v_1} + \ldots + c_p\mathbf{v_p}$$

is called a *linear combination* of $\mathbf{v_1}, \ldots, \mathbf{v_p}$ with *weights* or *coefficients* $c_1, \ldots, c_p$.

$u + (v + w)$
$= (u + v) + w$
$\|$
$u + v + w$

This is well defined because of associativity of vector addition.

$u_1 + u_2 + \cdots + u_p$

$u = (3, 4, 5)$ $\quad w = (0, 0, 1)$

$v = (1, 2, 3)$

$-3u + v + w$

$= -3(3, 4, 5) + (1, 2, 3) + (0, 0, 1)$

$= (-9, -12, -15) + (1, 2, 3) + (0, 0, 1)$

$= (-8, -10, -11)$

$m \times r \quad , \quad r \times 1$

$m \times 1$

Definition

If $\mathbf{a_1}, \ldots, \mathbf{a_n}$ are the columns of A then $\quad , \; x \in \mathbb{R}^n$

$$\underline{A\mathbf{x}} := [\mathbf{a}_1 \quad \mathbf{a}_2 \ldots \quad \mathbf{a}_n] \begin{bmatrix} x_1 \\ . \\ . \\ . \\ x_n \end{bmatrix} = \underbrace{x_1\mathbf{a}_1 + \ldots + x_n\mathbf{a}_n}_{\sum_{j=1}^{n} x_j a_j}$$

The above definition agrees with the definition of $A\mathbf{x}$, when $\mathbf{x}$ is viewed as a column matrix, as the $\boxed{k, 1\text{-th}}$ entry of the column matrix $A\mathbf{x}$ is the same as the k-th entry of $A\mathbf{x}$, viewed as a vector in $\mathbb{R}^m$.

$$\begin{array}{cc} \in \mathbb{R}^3 \; a_1 & a_2 \\ \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} & \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} \end{array} \quad \overset{x}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} \in \mathbb{R}^2 = \begin{bmatrix} 6 \\ 6 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 \\ 3 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 3 \end{bmatrix}$$

$$A = [a_1 \ a_2 \ \cdots \ a_n] = \begin{bmatrix} a_{11} & \cdots & a_{1r} \\ a_{21} & \cdots & a_{2r} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kr} \\ \vdots & & \vdots \\ a_{m1} & & a_{mr} \end{bmatrix}$$

a_1 $\quad$ $\underline{m \times r}$

$$X = \underset{\underline{n \times 1}}{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}$$

$$Ax = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n .$$

$$(Ax)_k = x_1 \underline{(a_1)_k} + \cdots + x_r (a_n)_k$$

$$= x_1 a_{k1} + \cdots + x_n a_{kn}$$

$= a_{k1} x_1 + \cdots + a_{kn} x_n$

This is the same as

$a_{k1} x_{11} + \cdots + a_{kn} x_{n1}$

if x is viewed as a $n \times 1$ matrix

$(Ax)_{k1}$

↳ when A & x are viewed as matrices

$$A = \begin{array}{ccc} a_1 & a_2 & a_3 \\ \end{array}$$
$$A = \begin{bmatrix} 4 & 1 & 6 \\ 3 & 0 & 1 \\ 2 & 5 & 9 \end{bmatrix} \qquad x = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \begin{array}{l} \leftarrow x_1 \\ \leftarrow x_2 \\ \leftarrow x_3 \end{array}$$

$$Ax = 1 \cdot a_1 + 0 \cdot a_2 + 2a_3$$

$$Ax = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 12 \\ 2 \\ 18 \end{bmatrix}$$

$$= \begin{bmatrix} 16 \\ 5 \\ 20 \end{bmatrix}$$

$A = [a_1 \; . \; . \; . \; a_n]$

Theorem

If A is an $m \times n$ matrix, with columns $\mathbf{a_1}, \ldots, \mathbf{a_n}$, and if $\mathbf{b}$ is in $\mathbb{R}^m$, the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

has the same solution set as the vector equation

$$\boxed{x_1\mathbf{a}_1 + \ldots + x_n\mathbf{a}_n = \mathbf{b}}$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$[\mathbf{a_1} \quad \mathbf{a_2} \ldots \quad \mathbf{a_n} \quad \mathbf{b}]$$

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Homogeneous Linear Systems

$\rightarrow x_1 + x_2 = 0$
$\rightarrow 3x_1 + 5x_2 = 0$
$(0,0)$

$x_1 + x_2 = 0$
$2x_1 + 2x_2 = 0$

Definition

A system of linear equations is said to be *homogeneous* if it can be written in the form

$$A\mathbf{x} = \mathbf{0},$$

where A is an $m \times n$ matrix.

Can a homogeneous system of equations ever be inconsistent?

The answer is no. The zero vector is always a solution.

$[A \quad 0]$

What about non-trivial solutions?

Fact

The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the equation has at least one free variable.

Why?

If there are no free variables, then there must be a unique solution, which we already know to be the zero vector.

Conversely, if there is a free variable then there are infinitely many solutions.

Parametric Vector Form

Whenever the solution set of a linear system is described explicitly in terms of a linear combination of vectors with variable and/or fixed weights, we say that the solution is in *parametric vector form*.

For a homogeneous system, the weights are all variable.

Example

$$\begin{aligned} 2x_1 + 3x_2 - 4x_3 + x_4 &= 0 \\ x_2 - 3x_3 + 2x_4 &= 0 \end{aligned}$$

$$\begin{bmatrix} 2 & 3 & -4 & 1 & 0 \\ 0 & 1 & -3 & 2 & 0 \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{2} R_1} \begin{bmatrix} 1 & 3/2 & -2 & 1/2 & 0 \\ 0 & 1 & -3 & 2 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 - \frac{3}{2}R_2} \begin{bmatrix} 1 & 0 & 5/2 & -5/2 & 0 \\ 0 & 1 & -3 & 2 & 0 \end{bmatrix}$$

reduced echelon form.

The general solution is

$$\begin{aligned} x_1 &= -\frac{5}{2}x_3 + \frac{5}{2}x_4 \\ x_2 &= 3x_3 - 2x_4 \\ x_3 &\text{ free} \\ x_4 &\text{ free} \end{aligned}$$

general solution

In parametric vector form,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \mathbf{x} = x_3 \begin{bmatrix} -5/2 \\ 3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 5/2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$