Review

has real entries

Definition

An *eigenvector* of an $n \times n$ matrix A is a nonzero vector $\mathbf{x} \in \mathbb{R}^n$ (or in $\mathbb{C}^n$) such that $\boxed{A\mathbf{x} = \lambda\mathbf{x}}$ for some $\lambda \in \mathbb{R}$ (or $\lambda \in \mathbb{C}$). A number λ is called an *eigenvalue* of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an $\mathbf{x}$ is called an *eigenvector corresponding to* λ.

Theorem

λ is an eigenvalue of A if and only if $\det(A - \lambda I) = 0$.

roots $\lambda \rightarrow$ $p(\lambda) = \det(A - \lambda I)$

Review

Definition

$p(\lambda) = \det(A - \lambda I)$ is called the *characteristic polynomial* of A.
The equation

$$\det(A - \lambda I) = 0$$

is called the *characteristic equation* of A.

Review

$\text{Null}(A - \lambda I)$

$\{x \in \mathbb{R}^n \mid (A - \lambda I)x = 0\}$ or $\{x \in \mathbb{C}^n \mid (A-\lambda I)x = 0\}$

Definition

If λ is an eigenvalue of A then the nullspace of $A - \lambda I$ is called the *eigenspace* corresponding to λ. When $\lambda \in \mathbb{R}$, the set

$$\{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \lambda\mathbf{x}\}$$

is called the *real λ-eigenspace*, and the set

$$\{\mathbf{x} \in \mathbb{C}^n \mid A\mathbf{x} = \lambda\mathbf{x}\}$$

is called the *complex λ-eigenspace*.

$\mathbb{R} \subset \mathbb{C}$

$\boxed{2 \times 2}$

Review

$\det(B - \lambda I) = \det(A - \lambda I)$

Definition

Two $n \times n$ matrix A and B are said to be *similar* if there exists an invertible matrix P such that $B = P^{-1}AP$.

Theorem

Similar matrices have the same characteristic polynomial and therefore the same eigenvalues, **with the same multiplicities**.

Review

Theorem

The eigenvalues of a triangular matrix are the entries on its main diagonal.

Diagonalization

$P = [b_1 \ \cdots \ b_n]$

Definition

A square matrix A is said to be *diagonalizable* if A is similar to a diagonal matrix, that is, if $A = PDP^{-1}$ for some invertible matrix P and some diagonal matrix D.

Theorem

Let A be an $n \times n$ matrix. Then A is diagonalizable if and only if there exists a basis of $\mathbb{R}^n$ consisting of eigenvectors of A.

(In other words, an $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors. This statement is called the Diagonalization Theorem in your course textbook.)

Recall

$$[T]_\beta = P_\beta^{-1} A P_\beta$$

Proposition

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Let A be the standard matrix of T. Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be any ordered basis of $\mathbb{R}^n$. Let

$$P_{\mathcal{B}} = [\mathbf{b}_1 \quad \cdots \quad \mathbf{b}_n]$$

be the change-of-coordinates matrix from $\mathcal{B}$ to the standard basis in $\mathbb{R}^n$. Then the $\mathcal{B}$-matrix of T is $P_{\mathcal{B}}^{-1} A P_{\mathcal{B}}$.

$$\boxed{x \longmapsto Ax}$$

Proof of Diagonalization Theorem:

Let us assume that A is diagonalizable, and prove that there exists a basis of $\mathbb{R}^n$ which consists of eigenvectors of A.

As A is diagonalizable,

$\exists$ an invertible matrix P and a diagonal matrix D such that

$$PAP^{-1} = D.$$

Let $b_1, \ldots, b_n$ be the columns of P^{-1}, i.e. $P^{-1} = [b_1 \; b_2 \; \cdots \; b_n]$.

Consider the linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$T(x) = Ax \quad , \forall x \in \mathbb{R}^n.$$

Let $\beta = \{b_1, \ldots, b_n\}$.
Since P^{-1} is invertible, the columns

of P^{-1} are lin. independent and therefore β is a basis of $\mathbb{R}^n$.

$$[T]_\beta = P A P^{-1} = D.$$

We know that

$$[T]_\beta = \left[[T(b_1)]_\beta \; \vdots \; \cdots \; [T(b_n)]_\beta \right]$$

Let $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$,

where $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$.

$$[T(b_j)]_\beta = (0, \ldots, \lambda_j, 0 \ldots 0)$$

$\Rightarrow T(b_j) = \lambda_j b_j$. $\therefore$ $\{b_1, \ldots, b_n\}$ is is a basis of eigenvectors.

Converse: We assume that $\mathbb{R}^n$ has a basis of eigenvectors of A, and prove that A is diagonalizable.

Let $\mathcal{B} = \{b_1, \ldots, b_n\}$ be a basis of $\mathbb{R}^n$ such that

$$A b_j = \lambda_j \cdot b_j \,, \quad \forall j = 1, \ldots, n.$$

where $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$.

Consider $P = [b_1 \cdots b_n]$

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the mapping defined by $T(x) = Ax$, $\forall x \in \mathbb{R}^n$.

Then

$$[T]_\beta = \left[[T(b_1)]_\beta \quad \cdots \quad [T(b_n)]_\beta \right]$$

$$= \left[[Ab_1]_\beta \quad \cdots \quad [Ab_n]_\beta \right]$$

$$= \left[[\lambda_1 b_1]_\beta \quad \cdots \quad [\lambda_n b_n]_\beta \right]$$

$$= \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & 0 & \ddots & \vdots \\ 0 & \vdots & 0 \ \ddots & \lambda_n \end{bmatrix}$$

We know $[T]_B$ is similar to A.

$\therefore$ A is diagonalizable.

$$\begin{vmatrix} 5-\lambda & 1 & 1 \\ 1 & 5-\lambda & -1 \\ 1 & -1 & 5-\lambda \end{vmatrix}$$

Example

Diagonalize the following matrix, if possible.

$$A = \begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & -1 \\ 1 & -1 & 5 \end{bmatrix}.$$

$$p(\lambda) = (6-\lambda)(\lambda^2 - 9\lambda + 18), \quad \lambda_1 = 6, \lambda_2 = 3$$

$\det(A - \lambda I) = 0$

$$\dim\left(\mathrm{Null}\left(\begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & -1 \\ 1 & -1 & 5 \end{bmatrix} - 3I\right)\right)$$

$$= \dim\left(\mathrm{Null}\left(\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\right)\right)$$

$$= 1.$$

$$\dim\left(\mathrm{Null}\,(A - 6I)\right)$$

$$= \dim\left(\mathrm{Null}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix}\right)$$

$$= \underline{2}$$

Solve $(A - 3I)x = 0$.

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 0 & -3 & 3 \end{bmatrix} \xrightarrow{R_2 \to R_2 - \frac{1}{2}R_1}$$

$$\longrightarrow \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3/2 & -3/2 \\ 0 & -3 & 3 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + \cdots} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_2 = x_3$ $\quad$ $x_1 = -\frac{1}{2}x_2 - \frac{1}{2}x_3 = -x_3$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \quad v_1$$

$$\begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & -1 \\ 1 & -1 & 5 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \qquad x_1 = x_2 + x_3$$

$R_3 \to R_3 + R_1$

$R_2 \to R_2 + R_1$

$R_1 \to -R_1$

$$\begin{array}{c} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \quad\;\; x_2 \quad x_3 \end{array}$$

$$\text{Null}(A - 6I)$$

$$\begin{bmatrix} y_1 \\ y_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$AP = \begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & -1 \\ 1 & -1 & 5 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 6 & 6 \\ 3 & 6 & 0 \\ 3 & 0 & 6 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} -3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}}_{D}$$

$$AP = PD$$

$$\Rightarrow P^{-1}AP = D.$$

<u>Proposition</u>: If $\lambda \neq \mu$ are eigenvalues of A and $Av = \lambda v$ and $Aw = \mu w$, then v and w are lin. independent.

Suppose if possible that $v = cw$, where $c \neq 0$

for some scalar $c \in \mathbb{R}$.

$$\therefore \quad \lambda v = Av$$

$$\lambda c w \overset{\parallel}{=} A(cw)$$

$$= cAw$$

$$= c\mu w$$

$$\Rightarrow (\lambda - \mu) c w = 0 \Rightarrow \lambda = \mu \text{ ✗.}$$

Example

Diagonalize the following matrix, if possible.

$$A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

$$p(\lambda) = -\lambda^3 - 3\lambda^2 + 4, \quad \lambda_1 = 1, \lambda_2 = -2$$

DIY .

The Question

How do we know whether a given square matrix is diagonalizable?

In other words, when does an *eigenvector basis* exist?

The following results provide a *sufficient* condition.

$\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

Theorem

If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are eigenvectors that correspond to real *distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly independent.* (*)

Idea: Induction

Corollary

An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Proof: Let us assume that

that $(*)$ holds for all

$1 \leq r \leq k$, where $k < n$.

We show that $(*)$ holds

for $r = k+1$.

Let $\lambda_1, \ldots, \lambda_{k+1}$ be distinct eigenvalues of A and let $v_1, \ldots, v_{k+1} \in \mathbb{R}^n$ such that $A v_j = \lambda_j v_j$, for $j = 1, \ldots, k+1$

Suppose

$$c_1 v_1 + c_2 v_2 + \cdots + c_{k+1} v_{k+1} = 0 \quad (1)$$

for some $c_1, \ldots, c_{k+1} \in \mathbb{R}$.

Then

$$A(c_1 v_1 + \cdots + c_{k+1} v_{k+1}) = 0$$

$$\Rightarrow c_1 A v_1 + \cdots + c_{k+1} A v_{k+1} = 0$$

$$\Rightarrow \underline{c_1 \lambda_1 v_1} + c_2 \lambda_2 v_2 + \overset{c_3 \lambda_3 v_3 + \cdots +}{\wedge} + c_{k+1} \lambda_{k+1} v_{k+1} = 0 \quad \text{--- (2)}$$

Multiplying (1) by λ_1, we get

$$\underline{\lambda_1 c_1 v_1} + \underline{\lambda_1} c_2 v_2 + \lambda_1 c_3 v_3 + \cdots + \lambda_1 c_{k+1} v_{k+1} = 0 \quad \text{--- (3)}$$

Subtracting ② from ③

we obtain

$$\underline{c_2(\lambda_1-\lambda_2)}v_2+\underline{c_3(\lambda_1-\lambda_3)}v_3$$

$$+\cdots+\underline{c_{k+1}(\lambda_1-\lambda_{k+1})}v_{k+1}=0$$

The induction hypothesis implies that $v_2,\ldots,v_{k+1}$ are lin. independent.

$$\therefore \quad c_j \underbrace{(\lambda_1 - \lambda_j)} = 0$$

for $j = 2, \ldots, k+1$.

Since $\lambda_1, \ldots, \lambda_{k+1}$ are distinct

$$\lambda_1 - \lambda_j \neq 0 \text{ for } j = 2, \ldots, k+1$$

$\therefore \ c_j = 0$ for $j = 2, \ldots, k+1$.

$\therefore$ From ① we obtain

$$c_1 v_1 = 0 .$$

Since $v_1 \neq 0$

$\therefore$ $c_1 = 0$.

$\therefore$ $c_1 = c_2 = \cdots = c_{k+1} = 0$.

$\therefore$ $\{v_1, \ldots, v_{k+1}\}$ is lin indep.

What can we say when the eigenvalues are NOT distinct?

1 – eigenspace $I_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

Theorem

Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \ldots, \lambda_p$.

1. *For $1 \leq k \leq p$, the dimension of the λ_k-eigenspace is less than or equal to the multiplicity of the eigenvalue λ_k.*
2. *The matrix A is diagonalizable if and only if the sum of the dimensions of the distinct eigenspaces equals n, and this happens if and only if the dimension of the λ_k-eigenspace equals the multiplicity of λ_k, for each $k = 1, \ldots, p$.*
3. *If A is diagonalizable and $\mathcal{B}_k$ is a basis for the eigenspace corresponding to λ_k for each k, then the total collection of vectors in the sets $\mathcal{B}_1, \ldots, \mathcal{B}_p$ forms an eigenvector basis for $\mathbb{R}^n$.*

Interested students can look up the Proofs document for a proof.

Cayley Hamilton Theorem (for Diagonalizable Matrices)

Theorem

Let A be a diagonalizable $n \times n$ matrix and let $p(x)$ be the characteristic polynomial of A. Then $p(A) = 0$. In other words, if

$$p(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0,$$

then

$$A^n + c_{n-1}A^{n-1} + \cdots + c_1 A + c_0 I = 0.$$

Remark

This theorem holds for ALL $n \times n$ matrices but we will only prove it for diagonalizable matrices. Students interested in the general cases should look up *Algebra* by Michael Artin.

Proof: $\exists$ an invertible matrix P and a diagonal matrix D such that

$$P^{-1}AP = D. \Rightarrow A = PDP^{-1}$$

We already know that

$$p(\lambda) = \det(A - \lambda I) = \det(D - \lambda I)$$

Let $d_{11}, \ldots, d_{nn}$ be the diagonal entries of D.

Then $\det(D - \lambda I)$

$$= (d_{11} - \lambda)(d_{22} - \lambda) \cdots (d_{nn} - \lambda)$$

$$\therefore \quad p(\lambda) = \prod_{j=1}^{n} (d_{jj} - \lambda)$$

$$P(D) = \prod_{j=1}^{n} \left(d_{jj} I - D \right)$$

$$= \prod_{j=1}^{n} \begin{bmatrix} d_{jj} - d_{11} & & & & 0 \\ & d_{jj} - d_{22} & & & \\ & & d_{jj} - d_{jj} & & \\ & & & \ddots & \\ 0 & & & & d_{jj} - d_{nn} \end{bmatrix} = 0$$

Let $p(\lambda) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_0$

$$p(D) = D^n + c_{n-1} D^{n-1} + \cdots + c_0 I \quad \text{— (2)}$$

$$p(A) = A^n + c_n A^{n-1} + \cdots + c_0 I$$
$$= (PDP^{-1})^n + c_n (PDP^{-1})^{n-1} + \cdots + c_0 PP^{-1}$$

Explanation:

$$PD^2P^{-1} = PDP^{-1}PDP^{-1}$$

$$= (PDP^{-1})^2$$

$$\therefore p(A) = PD^nP^{-1} + \cdots + P(c_0 I)P^{-1}$$

$$= P(D^n + c_{n-1}D^{n-1} + \cdots + c_0 I)P^{-1}$$

$$= P\,p(D)\,P^{-1} = 0.$$