Reduced Row Echelon Form

Definition

If a matrix in echelon form satisfies the following additional conditions, then it is in *reduced echelon form (or reduced row echelon form)*:

1. The leading entry in each nonzero row is 1.
2. Each leading 1 is the only nonzero entry in its column.

Uniqueness of the Reduced Row Echelon Form (RREF)

Theorem

Each matrix is row equivalent to one and only one reduced echelon matrix.

Definition

If a matrix A is row equivalent to an echelon matrix U , we call U *an echelon form (or row echelon form)* of A ; if U is in reduced echelon form, we call U *the reduced echelon form* of A.

(The proof of the theorem will not be covered in class. Interested students may look at the Proofs document posted in GC.)

Therefore the **pivot positions are the same in any echelon form of a given matrix**.

Definition

A *pivot position* in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A. A *pivot column* is a column of A that contains a pivot position.

The Row Reduction Algorithm

Forward Phase

1. Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.
2. Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.
3. Use row replacement operations to create zeros in all positions below the pivot.
4. Cover (or ignore) the row containing the pivot position and cover all rows, if any, above it. Apply steps 1–3 to the submatrix that remains. Repeat the process until there are no more nonzero rows to modify.

Backward Phase - Step 5

Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by a scaling operation.

Solutions of Linear Systems

Definition

The variables corresponding to the pivot columns of the an augmented matrix of a linear system are called *basic variables*. The remaining variables are called *free variables*.

Example

not a pivot column

$$\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\hookrightarrow$ not pivot column

Basic variables - x_1, x_2
Free variable - x_3

$$\begin{cases} x_1 - 5x_3 = 1 \\ x_2 + x_3 = 4 \end{cases}$$

The existence of free variables (in a consistent system) indicates that there are infinitely many solutions. These solutions are expressed in *parametric form*, using the free variables as parameters.

For example, the matrix we just saw corresponds to the equation

$$\begin{aligned} x_1 - 5x_3 &= 1 \\ x_2 + x_3 &= 4 \end{aligned}$$

and the solution set in parametric form is:

$$\begin{cases} x_1 = 1 + 5x_3 \\ x_2 = 4 - x_3 \\ x_3 \text{ is free} \end{cases}$$

This solution is called a *general solution* of the system because it gives an explicit description of all solutions.

$$
\begin{array}{ccc|c}
x_1 & x_2 & x_3 & \\
\end{array}
$$

$$
\longrightarrow \left[\begin{array}{ccc|c}
1 & 0 & 0 & 2 \\
0 & 5 & 3 & 7 \\
0 & 0 & 0 & 5
\end{array}\right]
$$

$$x_1 = 2$$

$$5x_2 + 3x_3 = 7$$

$$\boxed{0 = 5}$$

Two Fundamental Questions Revisited

Existence and Uniqueness Theorem

A linear system is consistent if and only if the rightmost column of the augmented matrix is *not* a pivot column—that is, if and only if an echelon form of the augmented matrix has *no* row of the form

$$\begin{bmatrix} 0 & \dots & 0 & b \end{bmatrix} \text{ with } b \text{ nonzero}$$

If a linear system is consistent, then the solution set contains either

1. a unique solution, when there are no free variables, or
2. infinitely many solutions, when there is at least one free variable.

Using Row Reduction to Solve a Linear System

1. Write the augmented matrix of the system.
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.
3. Continue row reduction to obtain the reduced echelon form.
4. Write the system of equations corresponding to the matrix obtained in step 3.
5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

Example

Find the general solution of the system

$$\begin{aligned} x_1 - 2x_2 - x_3 + 3x_4 &= 0 \\ -2x_1 + 4x_2 + 5x_3 - 5x_4 &= 3 \\ 3x_1 - 6x_2 - 6x_3 + 8x_4 &= 2 \end{aligned}$$

The augmented matrix is

$$\left[\begin{array}{rrrr|r} 1 & -2 & -1 & 3 & 0 \\ -2 & 4 & 5 & -5 & 3 \\ 3 & -6 & -6 & 8 & 2 \end{array}\right]$$

We bring this to echelon form.

$$\begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ -2 & 4 & 5 & -5 & 3 \\ 3 & -6 & -6 & 8 & 2 \end{bmatrix} \xrightarrow{R_2 \to R_2 + 2R_1} \begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 3 & -6 & -6 & 8 & 2 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - 3R_1} \begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & -3 & -1 & 2 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

Thus the system is inconsistent.

$$x_1 - 3x_2 - 5x_3 = 0$$
$$x_2 + x_3 = 3$$

Example

Find the general solution of the linear system whose augmented matrix is

$$\begin{bmatrix} 1 & -3 & -5 & 0 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

is is. is not pivot column

Let us reduce this to RREF.

$$\xrightarrow{R_1 \to R_1 + 3R_2} \begin{bmatrix} 1 & 0 & -2 & 9 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

x_1, x_2 basic

x_3 - free

Let us write the system of equations.

$$x_1 - 2x_3 = 9$$
$$x_2 + x_3 = 3$$

$(x_1, x_2, x_3) = (9, 3, 0) + x_3(2, -1, 1)$

Rewrite expressing basic variables in terms of free variables (if any).

$$x_1 = 9 + 2x_3$$
$$x_2 = 3 - x_3$$
$$x_3 \text{ free}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

If a system has two equations and three unknowns, is it always consistent?

$$x_1 + x_2 + x_3 = 0$$
$$x_1 + x_2 + x_3 = 1$$

not consistent

Vectors as Ordered Lists or n-tuples

(x, y, z, t)

We will look at vector spaces later.

We will **temporarily** use the word "vector" to refer to an ordered list of numbers.

Please review what you were taught about vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ in high school.

Definition

The set of all n-tuples of real numbers is called $\mathbb{R}^n$.

Elements of $\mathbb{R}^n$ are usually **represented** as $n \times 1$ column vectors ($n \times 1$ matrices).

$\begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$

$(5, 6, 7) \leftarrow$ 3-tuple

$(2, 3)$
$\downarrow$
2-tuple

$\mathbf{0} \qquad (0,0,0)$

$$(5,3,2) \neq (2,3,5)$$

The vector whose entries are all zero is called the **zero vector** and is denoted by $\mathbf{0}$.

Equality of vectors in $\mathbb{R}^n$ and the operations of scalar multiplication and vector addition in $\mathbb{R}^n$ are defined entry by entry just as in $\mathbb{R}^2$.

$$(1,2,3)+(5,0,7) = (6,2,10)$$

$$(1,0) \neq (0,1)$$

Algebraic Properties of $\mathbb{R}^n$

$u = (u_1, u_2, \ldots, u_n)$

$v = (v_1, v_2, \ldots, v_n)$

$u + v$

$= (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n)$

For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^{\mathbf{n}}$ and all scalars c and d,

- $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ → commutativity $\quad v + u = (v_1 + u_1, \ldots, v_n + u_n)$
- $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ → associativity
- $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$
- $\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$ (where $-\mathbf{u}$ denotes $(-1)\mathbf{u}$)
- $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
- $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
- $c(d\mathbf{u}) = (cd)\mathbf{u}$
- $1\mathbf{u} = \mathbf{u}$

distributive laws

additive identity

$u_j + v_j = v_j + u_j$

$\forall j = 1 \ldots n$

$u + v + w$

$u + 0 = (u_1 + v_2 + \cdots + u_n) + (0, 0, 0 \ldots 0)$

$$u = (u_1, \ldots, u_n)$$

$$-u = (-u_1, \ldots, -u_n)$$

$$= -1 \cdot u$$

$$u + (-u) = (u_1, \ldots, u_n)$$

$$+ (-u_1, \ldots, -u_n)$$

$$= (u_1 - u_1, \ldots, u_n - u_n)$$

$$= (0, \ldots, 0)$$

$$= 0$$

$$x_1 + 3x_2 = f$$

$$\underline{c}\,x_1 + \underline{d}\,x_2 = g$$

$$\boxed{x_2 = \frac{g - cf}{d - 3c}}$$

$$\begin{bmatrix} 1 & 3 & f \\ c & d & g \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & f \\ 0 & d-3c & g-cf \end{bmatrix}$$

$$\cancel{d - 3c = 0} \Rightarrow \cancel{\boxed{g - cf = 0} \Rightarrow c = \frac{g}{f}}$$

$$\rightarrow d - 3c \neq 0 \Rightarrow \text{system consistent}$$

$$Ax = b$$

$$\begin{bmatrix} A & b \end{bmatrix}$$

$$\begin{cases} 2x - 2y = 2 \\ \underline{x - y = 1} \end{cases}$$