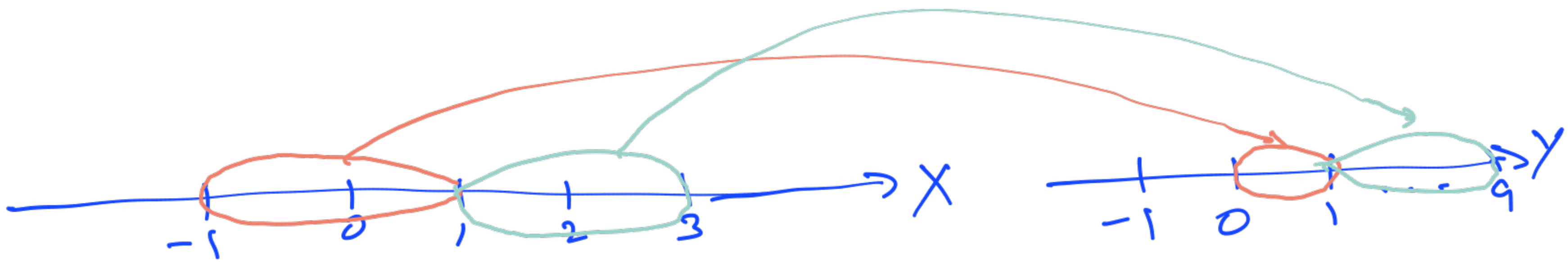
$Y = X^2$

Example 3.26 Problem

$\{Y \le 0.5\} = \{-\sqrt{0.5} \le X \le \sqrt{0.5}\}$

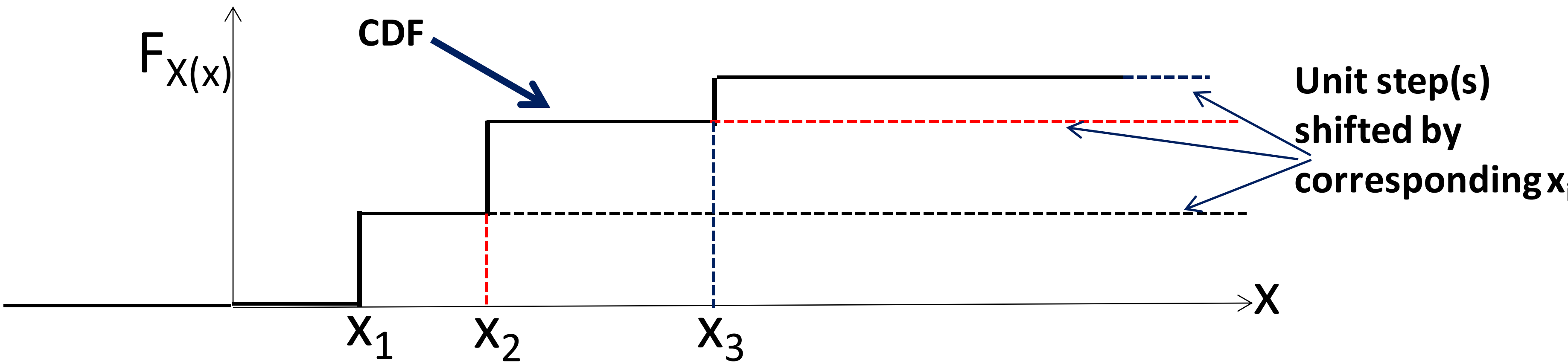
Suppose X is uniformly distributed over $[-1, 3]$ and $Y = X^2$. Find the CDF $F_Y(y)$ and the PDF $f_Y(y)$.

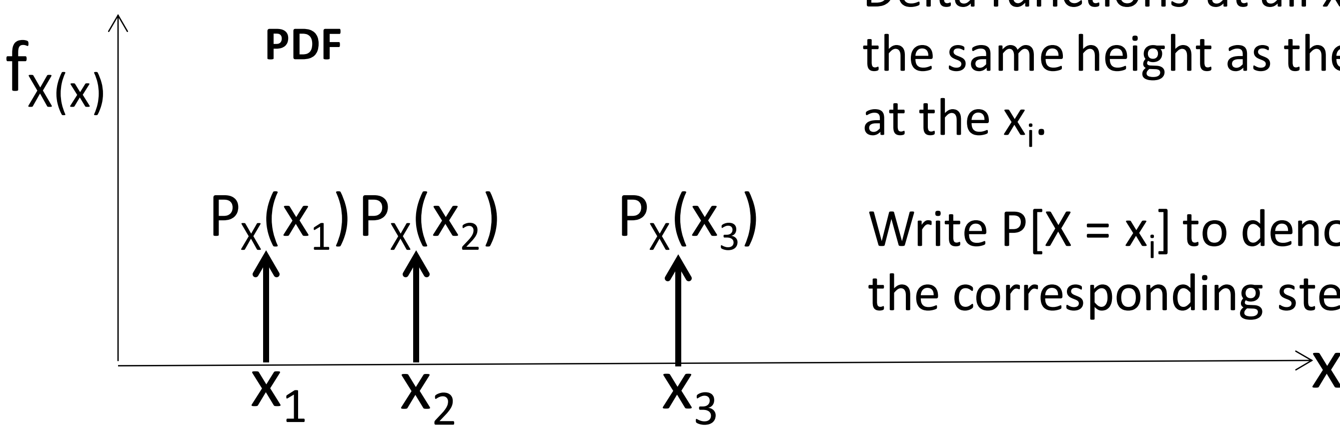


$$P[Y \le y] = \begin{cases} 0 & y < 0 \\ P[X \le \sqrt{y}] - P[X \le -\sqrt{y}] & 0 \le y \le 1 \\ P[X \le \sqrt{y}] & y > 1 \end{cases}$$

PDF of a Discrete RV

- Having defined the unit step, we can now define a pdf for a discrete random variable!





Delta functions at all x_i are shown of the same height as they all are infinite at the x_i.

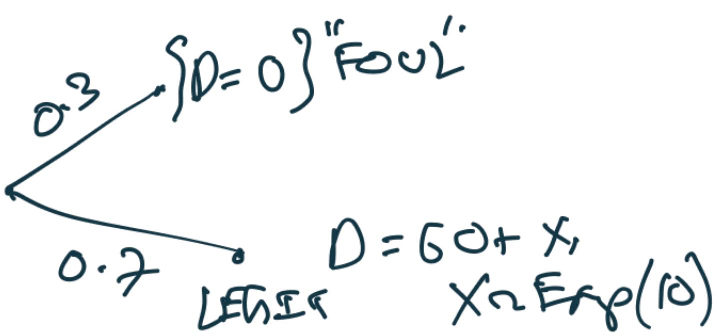
Write $P[X = x_i]$ to denote the size of the corresponding step in the CDF.

- Jumps in CDF correspond to delta functions in the pdf!

Problem 3.6.8

With probability 0.7, the toss of an Olympic shot-putter travels $D = 60 + X$ feet, where X is an exponential random variable with expected value $\mu = 10$. Otherwise, with probability 0.3, a foul is committed by stepping outside of the shot-put circle and we say $D = 0$. What are the CDF and PDF of random variable D?

$$P[D \le d] = \begin{cases} 0 & d < 0 \\ 0.3 & d = 0 \\ 0.3 & 0 < d < 60 \\ 0.3 + 0.7\,P[X \le d-60] & d \ge 60 \end{cases}$$



$$P[D \le d] = P[D \le d \mid FOUL]\,P[FOUL] + P[D \le d \mid LEGIT]\,P[LEGIT]$$

Problem 3.6.9

X

For 70% of lectures, Professor Y arrives on time. When Professor Y is late, the arrival time delay is a continuous random variable uniformly distributed from 0 to 10 minutes. Yet, as soon as Professor Y is 5 minutes late, all the students get up and leave. (It is unknown if Professor Y still conducts the lecture.) If a lecture starts when Professor Y arrives and always ends 80 minutes after the scheduled starting time, what is the PDF of T, the length of time that the students observe a lecture.

$$P[T \le t] = \begin{cases} 0.3\,P[X \ge 5] & t \le 0 \end{cases}$$

$$P[T \le t] = \begin{cases} 0.3\,P[X > 5] & 0 \le t < 75 \\ 0.3\,P[X \ge 80 - t] & 75 \le t < 80 \\ 1 & t \ge 80 \end{cases}$$

Problem 3.5.10

In mobile radio communications, the radio channel can vary randomly. In particular, in communicating with a fixed transmitter power over a "Rayleigh fading" channel, the receiver signal-to-noise ratio Y is an exponential random variable with expected value γ. Moreover, when $Y = y$, the probability of an error in decoding a transmitted bit is $P_e(y) = Q(\sqrt{2y})$ where $Q(\cdot)$ is the standard normal complementary CDF. The average probability of bit error, also known as the bit error rate or BER, is

$$\overline{P}_e = E[P_e(Y)] = \int_{-\infty}^{\infty} Q(\sqrt{2y}) f_Y(y)\, dy.$$

Find a simple formula for the BER $\overline{P}_e$ as a function of the average SNR γ.

Birthday Problem (Example 3.5 RN)

- Assume that birthdays occur uniformly over all 365 days of the year. In a group of k people what is the probability that no two people have the same birthday?
- HW: Plot this probability as a function of k
- Probability is the ratio of the number of favorable outcomes to total number of possibilities, which is

$$\frac{(365)(364)\cdots(365-k+1)}{(365)^k}$$

Coincidences

- Generalization of the birthday problem
- You have n possible events
  - n=365 in birthday problem
- In a set of k events, the probability that no two events are the same is

$$\frac{(n)_k}{n^k} = \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{k-1}{n}\right)$$

- Fix k and let n -> ∞
  - The probability goes to 1!
- Note as n becomes large for a fixed k, it does not matter whether you permute with or without replacement
  - Even when you permute with replacement the probability of picking the same item from a large number n of items is small for small k

Monty Hall Problem

http://mathworld.wolfram.com/MontyHallProblem.html

- Assume that a room is equipped with three doors. Behind two are goats, and behind the third is a shiny new car. You are asked to pick a door, and will win whatever is behind it. Let's say you pick door 1.
- Before the door is opened, however, someone who knows what's behind the doors (Monty Hall) opens *one of the other* two doors, revealing a goat, and asks you if you wish to change your selection to the third door (i.e., the door which neither you picked nor he opened). The Monty Hall problem is deciding whether you do.

$$P[\text{Shiny car behind } 3 \mid \text{You picked } 1 \ \& \ \text{MH opened } 2]$$

$$= \frac{P[\text{MH opened } 2, \text{Shiny car behind } 3 \mid \text{You picked } 1]}{P[\text{MH opened } 2 \mid \text{You picked } 1]}$$

$$p[\text{Car behind } 3 \mid \text{You picked } 1 \text{ \& MH opened } 2]$$

$$= \frac{p[\text{Car behind } 3, \text{ You picked } 1, \text{ MH opened } 2]}{p[\text{You picked } 1 \text{ \& MH opened } 2]}$$

$$= \frac{p[\text{MH opened } 2 \mid \text{You picked } 1, \text{ Car behind } 3] \; p[\text{You picked } 1, \text{ Car behind } 3]}{p[\text{You picked } 1, \text{ MH opened } 2]}$$

$$= \frac{p[\text{Car Behind } 3 \mid \text{You picked } 1] \; p[\text{You picked } 1]}{p[\text{MH opened } 2 \mid \text{You picked } 1] \; p[\text{You picked } 1]}$$

$$= \frac{1/3}{1/2} = 2/3.$$

Now consider

$$p[\text{Car behind } 1 \mid \text{You picked } 1 \text{ \& MH opened } 2]$$

$$= 1 - p[\text{Car behind } 2 \cup \text{Car behind } 3 \mid \text{You picked } 1, \text{ MH opened } 2]$$

$$= 1 - P[\text{Car behind } 2 \mid \text{You picked } 1, \text{MH opened } 2]$$

$$- P[\text{Car behind } 3 \mid \text{You picked } 1, \text{MH opened } 2]$$

$$= 1 - 0 - 2/3 = 1/3$$

Changing your choice of door increases your chance of winning the car.

Of course, there is still a 33.33% chance that the car is behind 1!

If you always choose to swap your chosen door with the one that MH didn't open, on an average you will win $(2/3)n$ shiny cars at the end of playing n such games hosted by Monty Hall.

Summary For a RV X

- CDF is given by $F_X(x) = P[X <= x]$
  - Nondecreasing
  - Starts at 0 and ends at 1
  - Is continuous for a continuous RV
  - Contains steps for a discrete RV. Step size at x is the PMF at x, $P_X(x) = P[X=x]$
- PDF $f_X(x)$ is the derivative of $F_X(x)$
  - Always >= 0
  - It is the slope of the CDF at x
  - Area under PDF is 1

$$\begin{aligned} P[x_1 \le X \le x_2] &= P[X \le x_2] - P[X < x_1] \\ &= F_X(x_2) - F_X(x_1) \\ &= \int_{x_1}^{x_2} f_X(x)\,dx \end{aligned}$$

Summary For a RV X

- Moments of RV X

$$E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x)\,dx$$

- When conditioning on an event E

$$P_X(x) = P_{X|B}(x)P[B] + P_{X|B^c}(x)P[B^c]$$

$$F_X(x) = F_{X|B}(x)P[B] + F_{X|B^c}(x)P[B^c]$$

$$f_X(x) = f_{X|B}(x)P[B] + f_{X|B^c}(x)P[B^c]$$

$$E[g(x)] = E[g(x)|B]P[B] + E[g(x)|B^c]P[B^c]$$