# Review - Inner Product

**Definition**

Let $V$ be a real vector space. An *inner product* $\langle .,. \rangle : V \times V \to \mathbb{R}$ is a function satisfying the following conditions:

1. $\langle \mathbf{v} + \mathbf{w}, \mathbf{z} \rangle = \langle \mathbf{v}, \mathbf{z} \rangle + \langle \mathbf{w}, \mathbf{z} \rangle$, for every $\mathbf{v}, \mathbf{w}, \mathbf{z} \in V$
2. $\langle c\mathbf{v}, \mathbf{w} \rangle = c\langle \mathbf{v}, \mathbf{w} \rangle$ for every $c \in \mathbb{R}$ and $\mathbf{v}, \mathbf{w} \in V$
3. $\langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{w}, \mathbf{v} \rangle$, for every $\mathbf{v}, \mathbf{w} \in V$
4. $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ for every $\mathbf{v} \in V$, and $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ holds if and only if $\mathbf{v} = \mathbf{0}$

# Review

**Example**

Let $V = C[a, b]$ the space of continuous real-valued functions defined on interval $[a, b]$. Define

$$\langle f, g\rangle := \int_a^b f(t)g(t)\,dt\,.$$

## Proposition

Let $V$ be a vector space and let $\langle ., . \rangle$ be an inner product on $V$. Let $u_1, \ldots, u_n, w \in V$ and $c_1, \ldots, c_n \in \mathbb{R}$. Then

$$\langle c_1 u_1 + \cdots + c_n u_n, w \rangle = c_1 \langle u_1, w \rangle + \cdots + c_n \langle u_n, w \rangle$$

The proof is a routine verification using induction and is left as an exercise.

## Definition

Let $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n$. The *length (or norm)* of $\mathbf{v}$ is defined by

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

# Generalization of Length

## Definition

Let $V$ be a vector space and let $\langle .,. \rangle$ be an inner product on $V$. The *length (or norm)* of a vector $v \in V$ is defined by

$$\underline{\|v\| = \sqrt{\langle v, v \rangle}}$$

## Example

$V = C[0, 2\pi]$

$$\langle f, g \rangle = \int_0^{2\pi} f(t)g(t)\,dt\,.$$

$$\underline{\|\sin x\|} = \sqrt{\int_0^{2\pi} \sin^2 x\,dx} = \underbrace{\sqrt{\int_0^{2\pi} \frac{1-\cos 2x}{2}\,dx}} = \sqrt{\pi}$$

# Unit Vectors

$v = (x, y)$

$\hat{v} = \left( \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right)$

A vector whose length is 1 is called a *unit vector*.

If we divide a nonzero vector $v$ by its length, we obtain a unit vector $\hat{v}$, because if

$$\hat{v} = \frac{1}{\|v\|} v,$$

then

$$\|\hat{v}\| = \left\| \frac{1}{\|v\|} v \right\| = \frac{1}{\|v\|} \|v\| = 1$$

The process of creating $\hat{v}$ from $v$ is called *normalizing* $v$, and we say that $\hat{v}$ is in the *same direction* as $v$.

## Notation

We often say

$$\hat{v} = \frac{v}{\|v\|}$$

# Cauchy-Schwarz Inequality

← Very Important

**Proposition**

Let $V$ be a vector space with an inner product $\langle ., . \rangle$. Let $u, v \in V$. Then

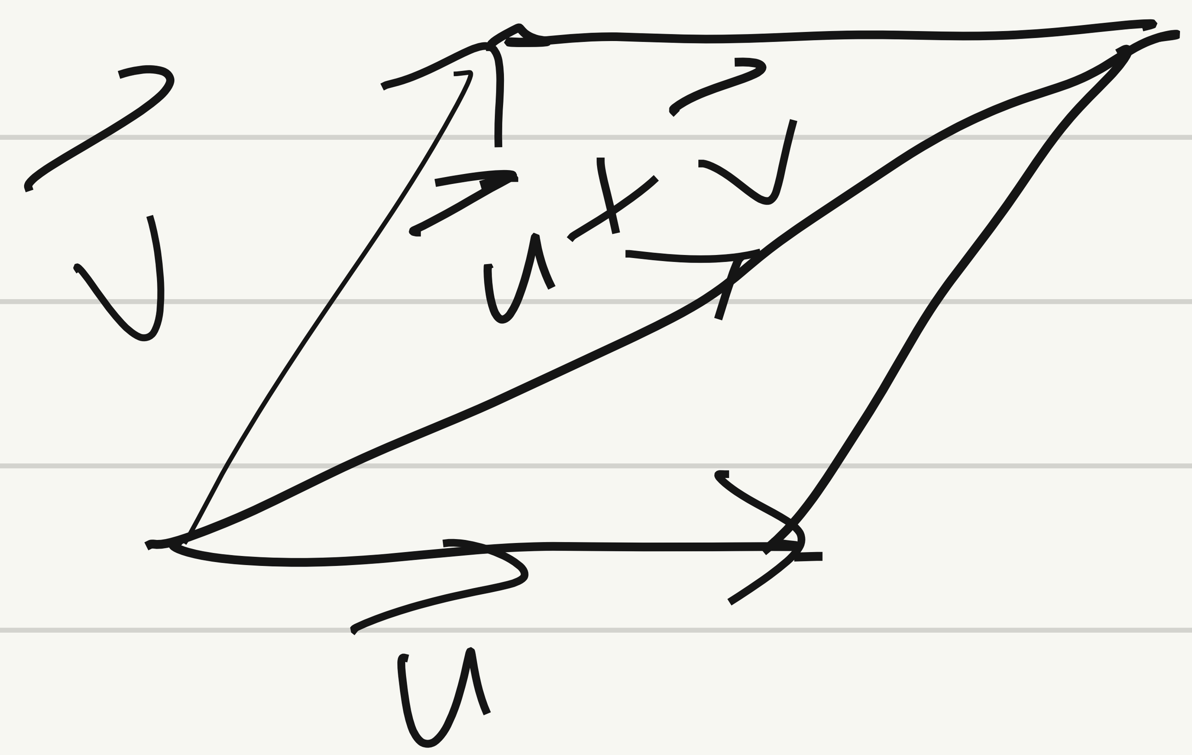
$$|\langle u, v \rangle| \leq \|u\| \|v\|$$

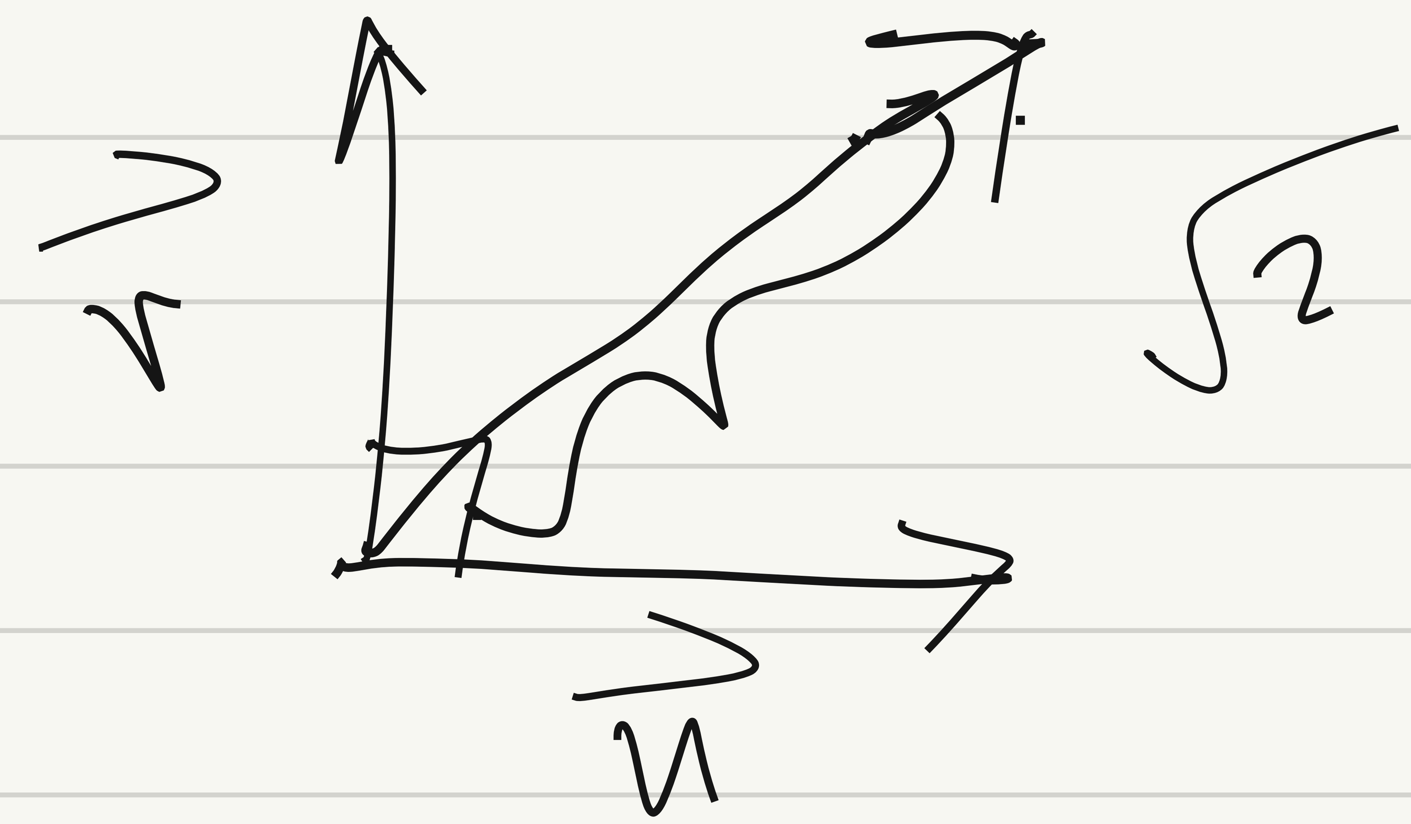
Idea behind proof:

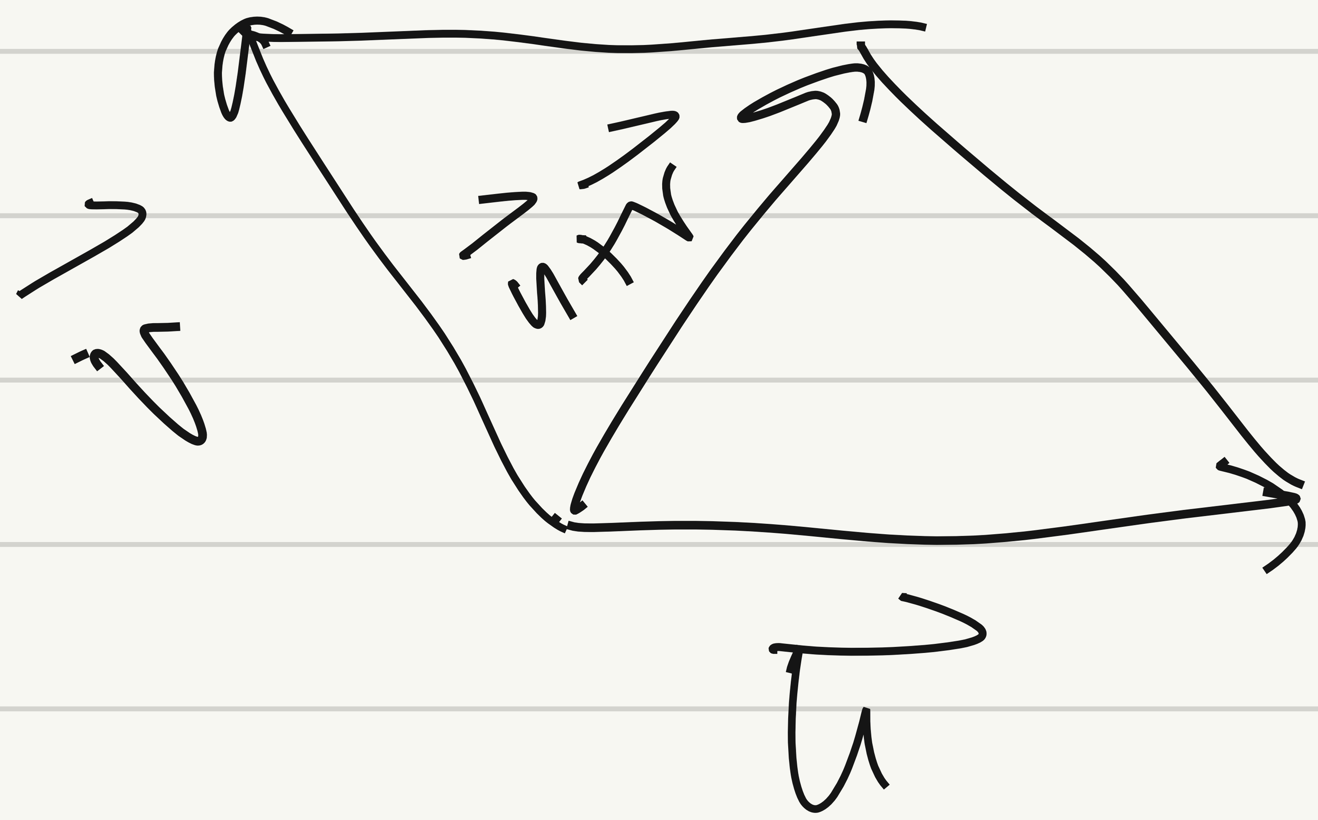
We first prove the statement for unit vectors.

Information about the angle between two vectors can be derived from the sum and difference between the two vectors.

If two unit vectors are orthogonal then their sum has square length 2. If the two vectors are close together, this number is more than 2 and if it is less, then it's less than 2.




$$|\hat{u}\cdot\hat{v}| = |\cos\theta|$$

$$\boxed{|u\cdot v| = \|u\|\,\|v\|\,|\cos\theta|}$$

**Proof**: First we consider the case when $u$ and $v$ are unit vectors,

i.e. $\|u\| = 1$, $\|v\| = 1$.

$$\langle u+v, u+v \rangle = \langle u, u+v \rangle + \langle v, u+v \rangle$$

$$= \langle u, u \rangle + \langle u, v \rangle + \langle v, u \rangle + \langle v, v \rangle$$

$$= 1 + 2\langle u, v \rangle + 1$$

$$= 2 + 2\langle u, v \rangle$$

Since $\langle u+v, u+v \rangle \geq 0$,

$$2 + 2\langle u, v \rangle \geq 0.$$

$$1 \geq -\langle u, v \rangle \quad \text{—— } (1)$$

$$\langle u-v, u-v \rangle = \langle u, u-v \rangle - \langle v, u-v \rangle$$

$$= \langle u, u \rangle - \langle u, v \rangle - \langle v, u \rangle + \langle v, v \rangle$$

$$= 2 - 2\langle u, v \rangle.$$

Since $\langle u-v, u-v \rangle \geq 0$,

$$2 - 2\langle u, v \rangle \geq 0 .$$

$$\Rightarrow \quad 1 \geq \langle u, v \rangle \quad \text{——} \quad (2)$$

From (1) & (2), we get

$$|\langle u, v \rangle| \leq 1 .$$

Now consider the case where

$u$ and $v$ are not unit

vectors.

If $u = 0$, then

$$\langle u, v\rangle = \langle 0, v\rangle = \langle 0 \cdot 0, v\rangle$$
$$= 0\langle 0, v\rangle$$
$$= 0.$$

$\|u\|\|v\| = 0.$

If $v = 0$, same argument holds.

So assume that $u \neq 0$, $v \neq 0$.

Then $\dfrac{u}{\|u\|}$ and $\dfrac{v}{\|v\|}$ are unit vectors.

$$\therefore \quad \left| \left\langle \frac{u}{\|u\|}, \frac{v}{\|v\|} \right\rangle \right| \leq 1$$

$$\Rightarrow \frac{1}{\|u\|\|v\|} |\langle u, v \rangle| \leq 1$$

$$\Rightarrow |\langle u, v \rangle| \leq \|u\|\|v\|.$$

# Distance

**Definition**

For $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, the distance between $\mathbf{u}$ and $\mathbf{v}$, written as $\text{dist}(\mathbf{u}, \mathbf{v})$, is the length of the vector $\mathbf{u} - \mathbf{v}$. That is,

$$\text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

We generalize this to other inner product spaces.

**Definition**

Let $V$ be a vector space with an inner product $\langle ., . \rangle$. The *distance* $d(u, v)$ between vectors $u$ and $v \in V$ is defined as

$$d(u, v) = \|u - v\|$$

# Orthogonality

Fourier transform

## Definition

Two vectors $u$ and $v$ in an inner product space $(V, \langle ., . \rangle)$ are said to be *orthogonal (to each other)* if

$$\langle u, v \rangle = 0.$$

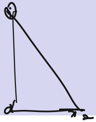
## Example

$V = C[0, 2\pi]$ and inner product as defined earlier.

The functions $\sin x$ and $\cos x$ are orthogonal, because

$$\langle \sin x, \cos x \rangle = \int_0^{2\pi} \sin x \cos x \, dx = \frac{1}{2} \int_0^{2\pi} \sin 2x \, dx = 0.$$

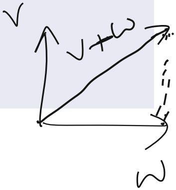
# Pythagoras Theorem



**Theorem**

*Let $(V, \langle ., . \rangle)$ be an inner product space. The vectors $v$ and $w$ are orthogonal if and only if*

$$\|v + w\|^2 = \|v\|^2 + \|w\|^2.$$




Proof:

$$\begin{aligned}
&\|v + w\|^2 - \|v\|^2 - \|w\|^2 \\
&\quad = \langle v + w, v + w\rangle - \langle v, v\rangle - \langle w, w\rangle \\
&\quad = \langle v, v + w\rangle + \langle w, v + w\rangle - \langle v, v\rangle - \langle w, w\rangle \\
&\quad = \langle v + w, v\rangle + \langle v + w, w\rangle - \langle v, v\rangle - \langle w, w\rangle \\
&\quad = \cancel{\langle v, v\rangle} + \langle w, v\rangle + \langle v, w\rangle + \cancel{\langle w, w\rangle} - \cancel{\langle v, v\rangle} - \cancel{\langle w, w\rangle} \\
&\quad = 2\langle v, w\rangle.
\end{aligned}$$

# The Triangle inequality

**Theorem**

*Let $(V, \langle ., . \rangle)$ be an inner product space, and let $v, w \in V$. Then*

$$\|v + w\| \leq \|v\| + \|w\|$$


Proof:

$$\|v + w\|^2 - \|v\|^2 - \|w\|^2 = 2\langle v, w \rangle.$$

By the Cauchy-Schwarz inequality,

$$|\langle v, w \rangle| \leq \|v\|\|w\|$$

$$2\langle v, w \rangle \leq 2|\langle v, w \rangle| \leq 2\|v\|\|w\|$$

So

$$\|v + w\|^2 - \|v\|^2 - \|w\|^2 \leq 2\|v\|\|w\|$$

Hence

$$\|v + w\|^2 \leq (\|v\| + \|w\|)^2$$

$$a_1, \ldots, a_n$$

$$\sum_{j=1}^{n} a_j^2$$

$$\frac{a_1^2}{x_1} + \frac{a_2^2}{x_2} + \cdots + \frac{a_n^2}{x_n} \geq \frac{(a_1 + \cdots + a_n)^2}{x_1 + \cdots + x_n}$$