$\{0\}$, V $\rightarrow$ trivial subspaces of V

Definition

A *subspace* of a vector space V is a subset H of V that has two properties:

a. H is closed under vector addition. That is, for each u and v in H, the sum $u + v$ is in H.

b. H is closed under multiplication by scalars. That is, for each u in H and each scalar c, the vector cu is in H.

Examples

- The *zero subspace* $\{\mathbf{0}\}$ consisting of only the zero vector, is a subspace of every vector space.
- Lines in $\mathbb{R}^2$ passing through the origin, are subspaces of $\mathbb{R}^2$.
- Planes in $\mathbb{R}^3$ passing through the origin, are subspaces of $\mathbb{R}^3$.
- Solutions of any homogeneous linear system $Ax = 0$. If A is an $m \times n$ matrix then these solutions sets are subspaces of $\mathbb{R}^n$.

Let A be an $m \times n$ matrix.

$$W = \{ x \in \mathbb{R}^n \mid Ax = 0 \}.$$

(i) Let $v, w \in W$.

$\Rightarrow Av = 0$, $Aw = 0$.

Then $A(\underline{v+w}) = 0$

$\Rightarrow v + w \in W$

$v \in W \Rightarrow Av = 0$

(ii) Let $v \in W$, $c \in \mathbb{R}$.

$A(cv) = cAv = 0$

$\Rightarrow cv \in W$.

(i) & (ii) $\Rightarrow$ W is a subspace of $\mathbb{R}^n$.

Definition

Let $v_1, \ldots, v_p$ be distinct elements of a vector space V, and let $c_1, \ldots, c_p$ be scalars. The vector

$$c_1 v_1 + \ldots + c_p v_p$$

is called a *linear combination* of the vectors $v_1, \ldots, v_p$.

Definition

Let S be a nonempty subset of a vector space V. The set of all elements of V that can be expressed as linear combinations of elements of S is called the *span* of S, and is denoted by $\mathit{Span}\, S$. If S is the empty set, we define $\mathit{Span}\, S$ to be the singleton set $\{\mathbf{0}\}$.

We saw that Span S is a subspace of V, if S is a finite set.

Need to show: Span S is a subspace of V when S is an infinite set.

Claim: If $S_1 \subset S_2 \subset V$ then

$$\operatorname{Span} S_1 \subset \operatorname{Span} S_2.$$

Let $v \in \operatorname{Span} S_1$. Then $v = c_1 u_1 + \cdots + c_n u_n$ for some $c_1, \ldots, c_n \in \mathbb{R}$, and some $u_1, \ldots, u_n \in S_1$.

Clearly $u_1, \ldots, u_n \in S_2$.

$\therefore \quad c_1 u_1 + c_2 u_2 + \ldots + c_n u_n$

$\in \operatorname{Span} S_2$.

$\Rightarrow \operatorname{Span} S_1 \subset \operatorname{Span} S_2$

Now consider the case when S is an infinite subset of V.

Let $v, w \in \operatorname{Span} S$.

$\Rightarrow \quad v = c_1 u_1 + \cdots + c_n u_n$

for some scalars $c_1, \ldots, c_n$

and vectors $\underline{u_1, \ldots, u_n} \in S$.

Also

$w = d_1 v_1 + \cdots + d_m v_m$

for some scalars $d_1, \ldots, d_m$ and some vectors $v_1, \ldots, v_m \in S$.

Consider $A = \{u_1, \ldots, u_n\}$
$\cup \{v_1, \ldots, v_m\}$

Clearly A is a finite set,
Also, $\underline{\text{Span} A} \subset \text{Span}\, S$.
because $A \subset S$.
Since A is finite, Span A is a subspace of V.

$\Rightarrow \quad v+w \in \operatorname{Span} A \subset \operatorname{Span} S$

$\Rightarrow \quad v+w \in \operatorname{Span} S$

Closure of Span S under scalar multiplication

— complete this ~~for~~ homework.

$\therefore$ Span S is a subspace of V.

Proposition

Let $S \subset V$. Then $Span\, S$ is a subspace of V.

Proved on earlier pages.

Example

$c \sin x + d \cos x$

Consider the set V of all solutions of the differential equation

$$y'' + y = 0$$

The set $W = \text{Span}\{\sin x\}$ is a subspace of V.

Also $V = \text{Span}\{\sin x, \cos x\}$.

In general, solution sets of n-th order ODEs can be expressed as the span of n linearly independent solutions, but this is something you will cover in your other classes.

Homework

$\mathbb{R}^n$

Characterization of Linearly Dependent Sets

An indexed set $S = \{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others.

In fact, if S is linearly dependent and $\mathbf{v_1} \neq \mathbf{0}$, then some $\mathbf{v_j}$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v_1}, \ldots, \mathbf{v_{j-1}}$.

A very similar proof to the one we saw on earlier works in the context of abstract vector spaces. Please adapt the proof which was presented for the $\mathbb{R}^n$ case as homework.

Fundamental Subspaces of matrix A.

Definition

The *null space* of an $m \times n$ matrix A written as *Nul* A, is the set of all solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$. In set notation,

$$Nul\ A = \underline{\{\mathbf{x} \in \mathbb{R}^n \ : \ A\mathbf{x} = \mathbf{0}\}}$$

Proposition

Nul A is a subspace of $\mathbb{R}^n$.

proof on previous slide.

Definition

The *column space* of an $m \times n$ matrix A, written as $Col\, A$, is the set of all linear combinations of the columns of A. If

$$A = [\mathbf{a}_1 \quad \dots \quad \mathbf{a}_n],$$

then

$$Col\, A = Span\underline{\{\mathbf{a}_1, \dots, \mathbf{a}_n\}}.$$

Proposition

$Col\, A$ is a subspace of $\mathbb{R}^m$.

Proof : Span of any subset of $\mathbb{R}^m$ is a subspace of $\mathbb{R}^m$.

We will revisit this when we look at SVD

Singular Value Decomposition

Definition

The *row space* of an $m \times n$ matrix A, written as *Row* A, is the set of all linear combinations of the rows of A.

$$Row\ A = \underline{Col\ A^T}$$

Thus *Row* A is a subspace of $\mathbb{R}^n$.

Nul A, *Col* A and *Row* A are called the *Fundamental Subspaces* associated with the matrix A. $\rightarrow$ David C. Lay

Some sources also consider $\underline{Nul\ A^T}$ as a fundamental subpace associated with A.

Gilbert Strang

Definition

Let V be a vector space. Let S be an infinite subset of V. We say S is a *linearly independent* set if every finite subset of S is linearly independent.

~~**Note: The above definition is given only for the sake of completeness.** You will NOT be asked any question on an exam which is based on infinite linearly independent sets.~~

Definition

Let V be a vector space. A set of vectors $\mathcal{B} \subset V$ is said to be a *basis* of V if

(i) $\mathcal{B}$ is linearly independent.

(ii) $\mathcal{B}$ spans V.

Whenever $\mathcal{B}$ is a finite set, we say V is *finite dimensional*.

Claim: Columns of the $n \times n$ Identity matrix I_n form a basis of $\mathbb{R}^n$.

$\mathbb{R}^3$ $\quad$ columns of I_3

$(1, 0, 0) \leftarrow e_1$

$(0, 1, 0) \leftarrow e_2$

$(0, 0, 1) \leftarrow e_3$

Pf: Let $v \in \mathbb{R}^n$, so

$v = (v_1, \ldots, v_n)$,

where $v_1, \ldots, v_n \in \mathbb{R}$.

$v = v_1 e_1 + v_2 e_2 + \ldots + v_n e_n \in \text{Span}\{e_1, \ldots, e_n\}$

$c_1(1, 0, \cdots, 0) = (c_1, 0, \cdots, 0)$
$c_2(0, 1, \cdots 0) = (0, c_2, \cdots 0)$

$\Rightarrow \mathbb{R}^n = \operatorname{span}\{e_1, \cdots, e_n\}$

Claim: $\{e_1, \cdots, e_n\}$ is a linearly independent set

Suppose $\underbrace{c_1e_1 + \cdots + c_ne_n}_{c_1(1,0,\cdots,0) + c_2(0,1,0\cdots0) + \cdots + c_n(0,\cdots,0,1)} = 0$,
where $c_1, \cdots, c_n \in \mathbb{R}$.

$\Rightarrow (c_1, \cdots, c_n) = 0 \Rightarrow c_1 = c_2 = \cdots = c_n = 0$.
$\therefore \{e_1, \cdots, e_n\}$ is a l.i. set.