Definition

Given vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots \mathbf{v_p} \in \mathbb{R}^{\mathbf{n}}$ and given scalars $c_1, c_2, \ldots, c_p$, the vector $\mathbf{y}$ defined by

$$\mathbf{y} = c_1\mathbf{v_1} + \ldots + c_p\mathbf{v_p}$$

is called a *linear combination* of $\mathbf{v_1}, \ldots, \mathbf{v_p}$ with *weights* or *coefficients* $c_1, \ldots, c_p$.

This is well defined because of associativity of vector addition.

Definition

If $\mathbf{a_1}, \ldots, \mathbf{a_n}$ are the columns of A then

$$A\mathbf{x} := [\mathbf{a}_1 \quad \mathbf{a}_2 \ldots \quad \mathbf{a}_n] \begin{bmatrix} x_1 \\ . \\ . \\ . \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + \ldots + x_n\mathbf{a}_n$$

A New Perspective

This matrix product has the property that if $\mathbf{b_1}, \ldots, \mathbf{b_p}$ are the columns of B then

$$AB = [A\mathbf{b}_1 \quad A\mathbf{b}_2 \quad \ldots \quad A\mathbf{b_p}]$$

Each column of AB is a linear combination of the columns of A using weights from the corresponding column of B.

Similarly, each row of AB is a linear combination of the rows of B using weights from the corresponding row of A. In other words

$$\boxed{row_i(AB) = row_i(A)B.}$$

$$B = [b_1 \ \underline{b_2} \ \cdots \ b_p]$$

$$AB = [\underline{Ab_1} \ \underline{Ab_2} \ \cdots \ Ab_p]$$

Practice Question 1

Suppose the first two columns, $\mathbf{b_1}$ and $\mathbf{b_2}$, of B are equal. What can you say about the columns of AB (if AB is defined)? Why?

Practice Question 2

Suppose the second column of B is all zeros. What can you say about the second column of AB?

Elementary Matrices in Perspective

Row Replacement

The operation $R_i \to \underline{R_i + cR_j}$ is achieved via left multiplicaton by a matrix of the form

$$\begin{array}{r c} & \\ & \\ & \\ ith \to & E = \\ & \\ & \\ & \end{array} \begin{bmatrix} 1 & 0 & \cdots & & & & & 0 \\ 0 & 1 & 0 & \cdots & & & & 0 \\ & & \ddots & & & & & \\ 0 & \cdots & 0 & 1 & \cdots & 0 & c & 0 \\ & & & & \ddots & & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{bmatrix} \qquad (i < j)$$

$$\underset{m\times n}{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & \cdots & \cdots & a_{mn} \end{bmatrix}} \underset{n\times p}{\begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{bmatrix}} = \begin{bmatrix} a_{11}R_1 + a_{12}R_2 + \cdots + a_{1n}R_n \\ \\ a_{m1}R_1 + \cdots + a_{mn}R_n \end{bmatrix}$$

$$\begin{array}{rc} & \begin{array}{ccc} & i,i & \end{array} \\ \text{ith row} & \underset{n\times n}{\begin{bmatrix} 1 & & & \\ & 1 & c & \\ & & \ddots & \\ & O & & 1 \end{bmatrix}} \\ & \text{jth column} \end{array} \underset{n\times p}{\begin{bmatrix} R_1 \\ \vdots \\ \hline \vdots \\ R_n \end{bmatrix}} \quad R_i$$

$$\underline{a_{i1}}R_1 + \underline{a_{i2}}R_2 + \cdots + a_{in}R_n$$

$$= a_{ii}R_i + a_{ij}R_j = R_i + cR_j$$

or

$$E = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & c & & 1 & \\ & & & & & 1 \end{bmatrix} \qquad (i > j)$$

The matrix has a c as its ij-th entry and otherwise looks like the $m \times m$ identity matrix.

Row Interchange

$$
E = \begin{array}{cl}
\begin{bmatrix}
1 & & & & & \\
& \ddots & & & & \\
& & 0 & & 1 & \\
& & & \ddots & & \\
& & 1 & & 0 & \\
& & & & & 1
\end{bmatrix}
&
\begin{array}{l}
\\
\\
\leftarrow \text{ith} \\
\\
\leftarrow \text{jth} \\
\\
\end{array}
\end{array}
$$

Row Scaling

$$E = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ 0 & - & c & - & - & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix} \leftarrow i\text{-th row}$$

Invertible Matrices

$AB - AC = 0$ $\quad$ $AB = BA = I_n$

$A(B - C) = 0$ $\quad$ $AC = CA = I_n$

How do I know $B = C$?

Definition

An $n \times n$ matrix A is defined to be *invertible* if there exists an $n \times n$ matrix B such that $AB = BA = I_n$

The inverse of an $n \times n$ matrix A is unique, if it exists. It is denoted by A^{-1}.

Uniqueness follows as a consequence of the associative law.

An invertible matrix is also called a *nonsingular* matrix. A matrix which is not invertible is called a *singular* matrix.

A matrix A is invertible if and only if $\det A \neq 0$.

$$\rightarrow \quad A(B-C) = 0$$

$$\Rightarrow \quad BA(B-C) = 0$$

$$\Rightarrow \quad I_n(B-C) = 0$$

$$\Rightarrow \quad B - C = 0$$

$$\Rightarrow \quad B = C$$

$$EF = 0$$

$$AB - AC = 0$$

$$AB = AC \leftarrow$$

$$BAB = BAC$$

$$IB = IC$$

$$B = C$$

Theorem

a. *If A is an invertible matrix, then A^{-1} is invertible and*

$$(A^{-1})^{-1} = A$$

b. *If A and B are $n \times n$ invertible matrices, then so is AB, and the inverse of AB is the product of the inverses of A and B in the reverse order. That is,*

$$(AB)^{-1} = B^{-1}A^{-1}$$

c. *If A is an invertible matrix, then so is A^T, and the inverse of A^T is the transpose of A^{-1}. That is,*

$$(A^T)^{-1} = (A^{-1})^T$$

(1) TP : $(A^{-1})^{-1} = A$.

By definition of A^{-1} $\longrightarrow$ $A\underbrace{A^{-1}} = A^{-1}A = I$

$\therefore (A^{-1})^{-1} = A \longrightarrow$ obvious from definition.

(2) TP : $(AB)^{-1} = B^{-1}A^{-1}$

$$(B^{-1}A^{-1})AB = B^{-1}(A^{-1}A)B$$

$$= B^{-1} I B$$

$$= B^{-1}B = I .$$

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$$

$$= A I A^{-1}$$

$$= AA^{-1} = I$$

(c) TP: $(A^T)^{-1} = (A^{-1})^T$

$$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I.$$

$$A^T (A^{-1})^T = (A^{-1}A)^T = I^T = I.$$

$$\therefore (A^T)^{-1} = (A^{-1})^T$$

$Ax = b$

$\Rightarrow A^{-1}Ax = A^{-1}b \Rightarrow Ix = A^{-1}b \Rightarrow x = A^{-1}b$

If A is an invertible $n \times n$ matrix, then for each $\mathbf{b} \in \mathbb{R}^{\mathbf{n}}$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.

Theorem

An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}.

Claim:

A is invertible iff the RREF of A is invertible.

A square matrix in reduced echelon form can only be invertible if it is the identity matrix.

First we assume that the RREF of A is invertible.

Let A' be the RREF of A.

There exists a sequence of row operations which takes A to A'.

$\therefore$ There exists a sequence of elementary matrices $E_1, \ldots, E_m$, where $m \in \mathbb{N}$, such that

$$A' = \underbrace{E_m E_{m-1} \cdots E_2 E_1} A$$

$$A' = E_2 E_1 A$$
$$E_2^{-1} A' = E_2^{-1} E_2 E_1 A$$
$$E_1^{-1} E_2^{-1} A' = E_1^{-1} E_1 A = A$$

lemma

[Proof that elementary matrix is invertible - left as an exercise].

$$A = \underbrace{E_1^{-1} E_2^{-1} \cdots E_m^{-1}} A'$$

$\therefore$ A is a product of invertible matrices.

$\Rightarrow$ A is invertible.

Conversely, assume that A is invertible.

Suppose A' is the RREF of A.

~~A~~ before $A' = E_m E_{m-1} \cdots E_1 A$

where $E_1, \ldots, E_m$ are elementary

matrices.

$\therefore$ A' is a product of invertible matrices

$\Rightarrow$ A' is invertible.

Theorem: A product of invertible matrices is invertibles. Prove this as an exercise

TP: A product A invertible matrices is invertible, and the inverse is obtained by multiplying the inverses in reverse order.

Let $A_1, A_2, \dots, A_m$ be invertible matrices.

If $m=2$, then nothing to show.

Assume that the theorem is

true for $m = k$.

Then for $m = k+1$,

$$(A_1 \cdots A_m)^{-1} = [\underbrace{(A_1 \cdots A_{m-1})}\underbrace{A_m}]^{-1}$$

$$= A_m^{-1} \underbrace{(A_1 \cdots A_{m-1})^{-1}}$$

$$= A_m^{-1} A_{m-1}^{-1} \cdots A_1^{-1}$$, by ind. hyp.