Proposition

Let $\{v_1, \ldots, v_n\}$ be a linearly independent set in a vector space V. If $w \notin Span(\{v_1, \ldots, v_n\})$ then the set $\{v_1, \ldots, v_n, w\}$ is linearly independent.

Spanning Set Theorem, p. 212 of course textbook

Theorem

Let V be a vector space. Let $S = \{v_1, v_2, \ldots, v_p\}$ be a set in V and let $H = Span\{v_1, v_2, \ldots, v_p\}$.

1. *If one of the vectors in S, say v_k, is a linear combination of the remaining vectors in S, then the set formed from S by removing v_k still spans H.*
2. *If $H \neq \{0\}$, some subset of S is a basis for H.*

Proposition

Let $S = \{v_1, \dots, v_n\}$ be an ordered set of vectors in V, let $w \in V$ be any vector, and let $S' = \{w, v_1, \dots, v_n\}$. Then $\operatorname{Span} S = \operatorname{Span} S'$ if and only if $w \in \operatorname{Span} S$.

Characterization of Linearly Depen

Let V be a vector space. An indexed
or more vectors is linearly dependent i
one of the vectors in S is a linear com

In fact, if S is linearly dependent and
$j > 1$) is a linear combination of the p
$v_1, \ldots, v_{j-1}$.

Theorem

Let V be a finite dimensional vector space. Let $\mathcal{B}_1 = \{b_1, \ldots, b_n\}$ be a basis of V. Let $\mathcal{B}_2 = \{v_1, \ldots, v_n\}$ be any other linearly independent subset of V. Then $\mathcal{B}_2$ is also a basis of V.

Pf: Consider the set $\{v_1, b_1, \ldots, b_n\}$.
This is a linearly dependent set, because $v_1 \in \text{Span}\{b_1, \ldots, b_n\} = V$

Proposition from Feb 3rd quoted in this proof:

Characterization of Linearly Dependent Sets

Let V be a vector space. An indexed set $S = \{v_1, \ldots, v_p\}$ of two or more vectors is linearly dependent in V if and only if at least one of the vectors in S is a linear combination of the others.

In fact, if S is linearly dependent and $v_1 \neq 0$, then some v_j (with $j > 1$) is a linear combination of the preceding vectors, $v_1, \ldots, v_{j-1}$.

$\therefore$ By Proposition quoted from Feb 3rd (see previous page)

$\exists\, i_1 \in \{1, \ldots, n\}$ such that

$b_{i_1} \in \operatorname{Span}\{v_1, b_1, \ldots, b_{i_1 - 1}\}$.

Put $S_1 = \{v_1, b_1, \ldots, b_n\} \setminus \{b_{i_1}\}$.

Then $V = \operatorname{Span} S_1$.

Having constructed $S_1, \dots, S_j$, we construct S_{j+1} as follows:

Since $v_{j+1} \in \operatorname{span} S_j$, the set

$$\{v_{j+1}, v_j, \dots, v_1, b_1, \dots, b_n\} \setminus \{b_{i_1}, b_{i_2}, \dots, b_{i_j}\}$$

$$= (S_j \cup \{v_{j+1}\})$$

is linearly dependent.

By Proposition from Feb 3rd, $\exists\, i_{j+1}$, such that $b_{i_{j+1}} \in \operatorname{Span}\{v_{j+1}, \ldots, v_1, b_1, \ldots, b_{i_{j+1}-1}\} \setminus \{b_{i_1}, \ldots, b_{i_j}\}$

Put $S_{j+1} = \{v_{j+1}, \ldots, v_1, b_1, \ldots, b_n\} \setminus \{b_{i_1}, \ldots, b_{i_{j+1}}\}$.

$\operatorname{Span} S_{j+1} = V$.

Clearly $S_n = \{v_n, \ldots, v_1\}$,

and by construction, $\operatorname{Span} S_n = V$.

$\therefore$ S_n is a basis of V. $\square$

β_2 //

Second iteration:

$S_1 = \{v_1, b_1, \ldots, b_n\} \setminus \{b_{i_1}\}$.

$\{v_2, v_1, b_1, \ldots b_{i_2}, b_n\} \setminus \{b_{i_1}\}$

$v_2 \in \operatorname{Span}\{v_1, b_1, \ldots, b_n\} \setminus \{b_{i_1}\}$

linearly dependent.

$S_2 = \{v_2, v_1, b_1, \ldots, b_n\} \setminus \{b_{i_1}, b_{i_2}\}$

Theorem

Let V be a finite dimensional vector space. Any two bases of V must have the same cardinality.

Pf: Suppose $\beta_1 = \{\underbrace{v_1, \dots v_n}, \dots v_m\}$

and $\beta_2 = \{w_1, \dots, w_n\}$ are

two bases for V. Suppose that if possible $m > n$.

By proposition proved earlier,

$\{v_1, \ldots, v_n\}$ is also a basis

for V.

$\therefore \quad v_{n+1} \in \operatorname{Span}\{v_1, \ldots, v_n\}$

$\Rightarrow \{v_1, \ldots, v_m\}$ is linearly dependent. $\times$.

Hence $m \leq r$. Clearly by the same argument $r \leq m$. So $m = r$. $\blacksquare$

Definition

The cardinality of a basis of a vector space V is called the *dimension* of V.

$S = \{v_1, \cdots, v_m\}$

Theorem

Let V be a finite-dimensional vector space. Any linearly independent subset of V can be extended to a basis for V.

Corollary

Let W be a proper subspace of a vector space V. If W is finite dimensional then

$$\dim W < \dim V$$

Proof: Since V is finite dimensional, say $\dim V = n$, there exists a basis of V, say $\beta_1 = \{b_1, \dots, b_n\}$.

Clearly $m \leq n$. If $m = n$, then nothing to show.
Assume $m < n$
Let $\underbrace{S_0 = \{v_1, \dots, v_m\}}$. Choose

Since $\operatorname{Span}\{v_1, \dots, v_m\} \neq V$,
$\exists\, b_{i_1} \in \beta_1$ such that

$b_{i_1} \notin \operatorname{Span}\{v_1, \dots, v_m\}$.

$S_1 = \{v_1, \dots, v_m, b_{i_1}\}$. where $j \leq m$,

Having constructed S_j, construct S_{j+1} as follows.

Since $\{v_1, \dots, v_m, b_{i_1}, \dots, b_{i_j}\}$ has less than n elements, $\operatorname{span}\{v_1, \dots, v_m, b_{i_1}, \dots, b_{i_j}\} \neq V$. $\therefore\ \exists\, b_{i_{j+1}} \in \beta_1 \setminus \{b_{i_1}, \dots, b_{i_j}\}$

such that $b_{i_{j+1}} \notin \operatorname{Span}\{v_1, \dots, v_m, b_{i_1}, \dots, b_{i_j}\}$.

Then $S_{j+1} = \{v_1, \ldots, v_m, b_{i_1}, \ldots, b_{i_{j+1}}\}$

Then $\underbrace{S_{n-m} = \{v_1, \ldots, v_m, b_{i_1}, \ldots, b_{i_{n-m}}\}}$ has n linearly independent vectors. Hence S_{n-m} is a basis for V.

Proof of Corollary: Let W be a proper subspace of V.

Let $\dim V = n$.

Claim: $\dim W < n$.

Let S be the collection of all linearly independent subsets of W.

If S has an element, β which

has cardinality n, then

β must be a basis of V.

$\Rightarrow V = W$ $\cancel{*}$.

$\Rightarrow$ All linearly independent subsets of W have fewer than n elements.

Choose some element of S which

has maximum cardinality,

say β', with m elements, where $m < n$.

Then β' is a basis of W,

because $\leftarrow$ $v \in W \setminus \operatorname{Span} \beta'$

$v \in W$ and $v \notin \operatorname{Span} \beta'$

$\Rightarrow \beta' \cup \{v\} \subseteq W$ ✗

$\hookrightarrow$ linearly independent set.

$\therefore \dim W = m < n_{\square}$.