A linear transformation is a *mapping* (or a *function*) which *preserves the structure of a vector space*. It *respects linearity* (essentially, it takes "flat" objects to "flat" objects.)

Definition

Let V, W be vector spaces. A function $T : V \to W$ is said to be a *linear transformation* if

(i) $T(v + w) = T(v) + T(w), \quad \forall v, w \in V$

(ii) $T(cv) = cT(v) \quad \forall v \in V, c \in \mathbb{R}$

Example

If A is an $m \times n$ matrix then the *matrix transformation* $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$T(\mathbf{x}) = A\mathbf{x}$$

is a linear transformation. Clearly,

(i) If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ then
$T(\mathbf{x}+\mathbf{y}) = A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y} = T(\mathbf{x}) + T(\mathbf{y})$

(ii) If $\mathbf{x} \in \mathbb{R}^n$ and $c \in \mathbb{R}$ then $T(c\mathbf{x}) = A(c\mathbf{x}) = cA\mathbf{x} = cT(\mathbf{x})$.

nobody does this $\longrightarrow$ $T(0_V) = 0_W$.

If the structure of a vector space is to be preserved, then at the very least, we need to preserve the identity.

Proposition

Let V and W be vector spaces. Let $T : V \to W$ be a linear transformation.

$$T(0) = 0.$$

Proof:

$0 = 0 + 0$ in V

$$T(0) = T(0 + 0) = T(0) + T(0).$$ in W

If we subtract $T(0)$ from both sides we get

$$T(0) = 0.$$

Coordinates with respect to a Basis

$(x,y) \in \mathbb{R}^2$ $(1,0)$ $(0,1)$

$(-x,-y)_\beta$ $\beta = \{(-1,0),(0,-1)\}$

Theorem (Unique Representation Theorem)

Let $\mathcal{B} = \{b_1, \ldots, b_n\}$ be a basis for a vector space V. Then for each $x \in V$, there exists a unique ordered n-tuple of scalars $\underbrace{(c_1, \ldots, c_n)}_{\mathbb{R}^n}$ such that

$$\boxed{x = c_1 b_1 + \ldots + c_n b_n}$$

$(x,y) = x(1,0) + y(0,1)$

$(x,y) = -x(-1,0) + (-y)(0,-1)$

$\cdot x \in V$

Suppose if possible that

$x = c_1 b_1 + \cdots + c_n b_n$ — (1)

and

$x = d_1 b_1 + \cdots + d_n b_n$ — (2)

where $c_1, \ldots, c_n, d_1, \ldots, d_n \in \mathbb{R}$.

subtracting (1) from (2)

$$0 = x - x = (d_1 - c_1) b_1 + \cdots + (d_n - c_n) b_n$$

$$d_1 - c_1 = d_2 - c_2 = \cdots = d_n - c_n = 0$$ coefficients.

are

$$\Rightarrow \quad d_j - c_j = 0 \quad \text{for every } j = 1, \ldots, n.$$

$$\Rightarrow \quad c_j = d_j, \quad \forall j = 1, \ldots, n$$

$V \longrightarrow \mathbb{R}^n$

Definition

Suppose $\mathcal{B} = \{b_1, \ldots, b_n\}$ is a basis for V and $x \in V$. The *coordinates of x relative to $\mathcal{B}$*(or the *$\mathcal{B}$-coordinates of x*) are the weights $c_1, \ldots, c_n$ such that

$$x = c_1 b_1 + \ldots + c_n b_n.$$

The vector $(c_1, \ldots, c_n) \in \mathbb{R}^n$ is denoted by $[x]_{\mathcal{B}}$, and is called the *coordinate vector of x relative to $\mathcal{B}$* or the *$\mathcal{B}$-coordinate vector of x*. The mapping

$$x \to [x]_{\mathcal{B}}$$

is called the *coordinate mapping (determined by $\mathcal{B}$)*.

Definition

Let V and W be vector spaces. A linear transformation $T : V \to W$ is said to be *invertible* if T is 1-1 and onto.

In other words, if a linear transformation also happens to be a bijection, it is called invertible.

Proposition

The inverse of any invertible linear transformation $T : V \to W$ is a linear transformation.

Proof: Let $T: V \to W$ be a linear transformation which is 1-1 and onto.

Let $T^{-1}: W \to V$ be the inverse of T.

Claim: T^{-1} is a linear transformation

Say $w_1, w_2 \in W$.

$$\text{Let } T^{-1}(w_1) = v_1 \in V. \qquad \text{——} \quad (2)$$

$$\text{and } T^{-1}(w_2) = v_2 \in V. \qquad \text{——} \quad (3)$$

$$\Rightarrow T(v_1) = w_1, \quad T(v_2) = w_2.$$

$$\Rightarrow \underline{T(v_1+v_2)} = T(v_1) + T(v_2)$$

$$= \underline{w_1 + w_2}$$

$$\Rightarrow \underline{T^{-1}(w_1+w_2)} = v_1 + v_2 \qquad \text{——} \quad (1)$$

from (1), (2) & (3), we obtain

$$T^{-1}(w_1 + w_2) = T^{-1}(w_1) + T^{-1}(w_2).$$

Since w_1, w_2 were arbitrary, T^{-1} satisfies the 1st condition of the definition of a linear transformation

Let $w \in W$, and $c \in \mathbb{R}$.

want to show: $T^{-1}(cw) = cT^{-1}(w)$.

Let $T^{-1}(w) = v \in V$. — 2

$$\Rightarrow \quad T(v) = w.$$

$$\Rightarrow \quad cT(v) = cw$$

$$\Rightarrow \quad T(cv) = cw$$

$$\Rightarrow \quad cv = T^{-1}(cw) \quad \text{—— } 1$$

From 1 & 2, we get

$$T^{-1}(cw) = c\,T^{-1}(w).$$

$\therefore$ The 2nd condition in the definition of a linear transformation is satisfied. $\square$.

Caution

We do not multiply by a linear transformation.

A linear transformation is a function. We are applying it.

$V \hookrightarrow \mathbb{R}^n$

Theorem

Let $\mathcal{B} = \{b_1, \ldots, b_n\}$ be a basis for a vector space V. The coordinate mapping

$$x \to [x]_{\mathcal{B}}$$

is an invertible linear transformation from V to $\mathbb{R}^n$.

Proof : let $x_1, x_2 \in V$.

such that

① — $[x_1]_\beta = [x_2]_\beta$.

② — let $[x_1]_\beta = (c_1, \ldots, c_n) \in \mathbb{R}^r$

③ — and $[x_2]_\beta = (d_1, \ldots, d_n) \in \mathbb{R}^r$

$\therefore \quad x_1 = c_1 b_1 + \cdots + c_n b_n$ — (4)

$x_2 = d_1 b_1 + \cdots + d_n b_n$ — (5)

From (1), (2) & (3),

$(c_1, \ldots, c_n) = (d_1, \ldots, d_n)$

$\Rightarrow \quad c_j = d_j$ for $j = 1, \ldots, n$. — (6)

From (4), (5) & (6)

$x_1 = x_2$

$\therefore$ The mapping $x \longmapsto [x]_\beta$ is 1-1.

Let $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. ← arbitrary

constructed

Let $x = y_1 b_1 + y_2 b_2 + \cdots + y_n b_n \in V$

Then $[x]_\beta = (y_1, \ldots, y_n)$.

$[x]_\beta = y$. $\Rightarrow$ y is in the range of the mapping.

$\therefore$ The mapping $x \mapsto [x]_\beta$ is onto.

Any linear transformation can be completely determined by what is does to a fixed basis, in the sense that,

Proposition

If $\mathcal{B} = \{b_1, \ldots, b_n\}$ is a basis for a vector space V, and if $S, T : V \to W$ are linear transformations, then $S = T$ iff

$$\longrightarrow \boxed{S(b_i) = T(b_i)}, \quad \forall i = 1, \ldots, n.$$

Let's assume that $S(b_i) = T(b_i)$,

<u>Proof</u> : Let $x \in V$ for $i = 1, \ldots, n$

$$\boxed{\text{WTS: } S(x) = T(x)}$$

Let $x = c_1 b_1 + \cdots + c_n b_n$,

where $c_1, \ldots, c_n \in \mathbb{R}$.

Then $S(x) = S(c_1 b_1 + \cdots + c_n b_n)$

$$= S(c_1 b_1) + S(c_2 b_2) + \cdots + S(c_n b_n)$$

$$= c_1 S(b_1) + \cdots + c_n S(b_n)$$

$$= c_1 T(b_1) + \cdots + c_n T(b_n)$$

$$= T(c_1 b_1) + \cdots + T(c_n b_n)$$

$$= T(c_1 b_1 + \cdots + c_n b_n)$$

$$= T(x). \quad \therefore \; S = T.$$

other direction:

$$S = T$$

obviously

$$S(b_i) = T(b_i), \forall i = 1, \dots, r$$

Proposition

Every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation. In other words, there exists a unique $m \times n$ matrix A such that

$$T(\mathbf{x}) = A\mathbf{x}, \quad \forall \mathbf{x} \in \mathbb{R}^n$$

A is called the *standard matrix for the linear transformation* T.

Proposition

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be an ordered basis of $\mathbb{R}^n$. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the coordinate transformation which sends

$$\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}.$$

The change-of-coordinates matrix $P_{\mathcal{B}}$ is the standard matrix of the inverse T^{-1} of the coordinate transformation.

The standard matrix of the coordinate transformation T is $P_{\mathcal{B}}^{-1}$.