A linear transformation is a *mapping* (or a *function*) which *preserves the structure of a vector space*. It *respects linearity* (essentially, it takes "flat" objects to "flat" objects.)

Definition

Let V, W be vector spaces. A function $T : V \to W$ is said to be a *linear transformation* if

(i) $T(v + w) = T(v) + T(w), \quad \forall v, w \in V$

(ii) $T(cv) = cT(v) \quad \forall v \in V, c \in \mathbb{R}$

Coordinates with respect to a Basis

$V \longleftrightarrow \mathbb{R}^n$

Definition

Suppose $\mathcal{B} = \{b_1, \ldots, b_n\}$ is a basis for V and $x \in V$. The *coordinates of x relative to* $\mathcal{B}$(or the $\mathcal{B}$*-coordinates of* x) are the weights $c_1, \ldots, c_n$ such that

$$x = c_1 b_1 + \ldots + c_n b_n.$$

The vector $(c_1, \ldots, c_n) \in \mathbb{R}^n$ is denoted by $[x]_{\mathcal{B}}$, and is called the *coordinate vector of x relative to* $\mathcal{B}$ or the $\mathcal{B}$*-coordinate vector of* x. The mapping

$$x \to [x]_{\mathcal{B}}$$

is called the *coordinate mapping (determined by* $\mathcal{B}$*)*.

Theorem

Let $\mathcal{B} = \{b_1, \ldots, b_n\}$ *be a basis for a vector space* V*. The coordinate mapping*

$$x \to [x]_{\mathcal{B}}$$

is an invertible linear transformation from V *to* $\mathbb{R}^n$*.*

Any linear transformation can be completely determined by what is does to a fixed basis, in the sense that,

Proposition

If $\mathcal{B} = \{b_1, \ldots, b_n\}$ is a basis for a vector space V, and if $S, T : V \to W$ are linear transformations, then $S = T$ iff

$$S(b_i) = T(b_i), \quad \forall i = 1, \ldots, n.$$

$a_1 = Ae_1$

Rough

$T(e_1) = Ae_1$

$I_n = [e_1 \; \cdots \; e_n]$

$\cancel{T(e_2)} = Ae_2$

Proposition

Every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation. In other words, there exists a unique $m \times n$ matrix A such that

$$T(\mathbf{x}) = A\mathbf{x}, \quad \forall \mathbf{x} \in \mathbb{R}^n$$

A is called the *standard matrix for the linear transformation* T.

Ax

$A = [a_1 \; \cdots \; a_n]$

$x = (x_1, \ldots, x_n)$

$= x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$

Proof :

$$\boxed{\begin{aligned} x &= (x_1, x_2, x_3, \ldots) \\ &= x_1\hat{i} + x_2\hat{j} + x_3\hat{k} \end{aligned}}$$

Define $A = [T(e_1) \; T(e_2) \; \cdots \; T(e_n)]$.

Claim : $T(x) = Ax, \; \forall x \in \mathbb{R}^n$.

Let $\underline{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

Since $x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$

We apply T on both sides to obtain

$$T(x) = T(x_1 e_1 + \cdots + x_n e_n)$$

$$= x_1 T(e_1) + x_2 T(e_2) + \cdots + x_n T_n(e_n)$$

$$= Ax. \qquad \square$$

The Change-of-coordinate in $\mathbb{R}^n$

$$P_\beta \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = P_\beta [x]_\beta .$$

Definition

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be an ordered basis of $\mathbb{R}^n$. The matrix

$$P_{\mathcal{B}} = [\mathbf{b}_1 \quad \ldots \quad \mathbf{b}_n]$$

formed using the basis vectors $\mathbf{b}_1, \ldots, \mathbf{b}_n$ as columns, is called the *change-of-coordinates* matrix from $\mathcal{B}$ to the standard basis in $\mathbb{R}^n$.

The change-of-coordinates matrix takes a coordinate vector with respect to the $\mathcal{B}$ basis and tranforms it to standard coordinates. So if $\mathbf{x}$ is a vector in $\mathbb{R}^n$, then

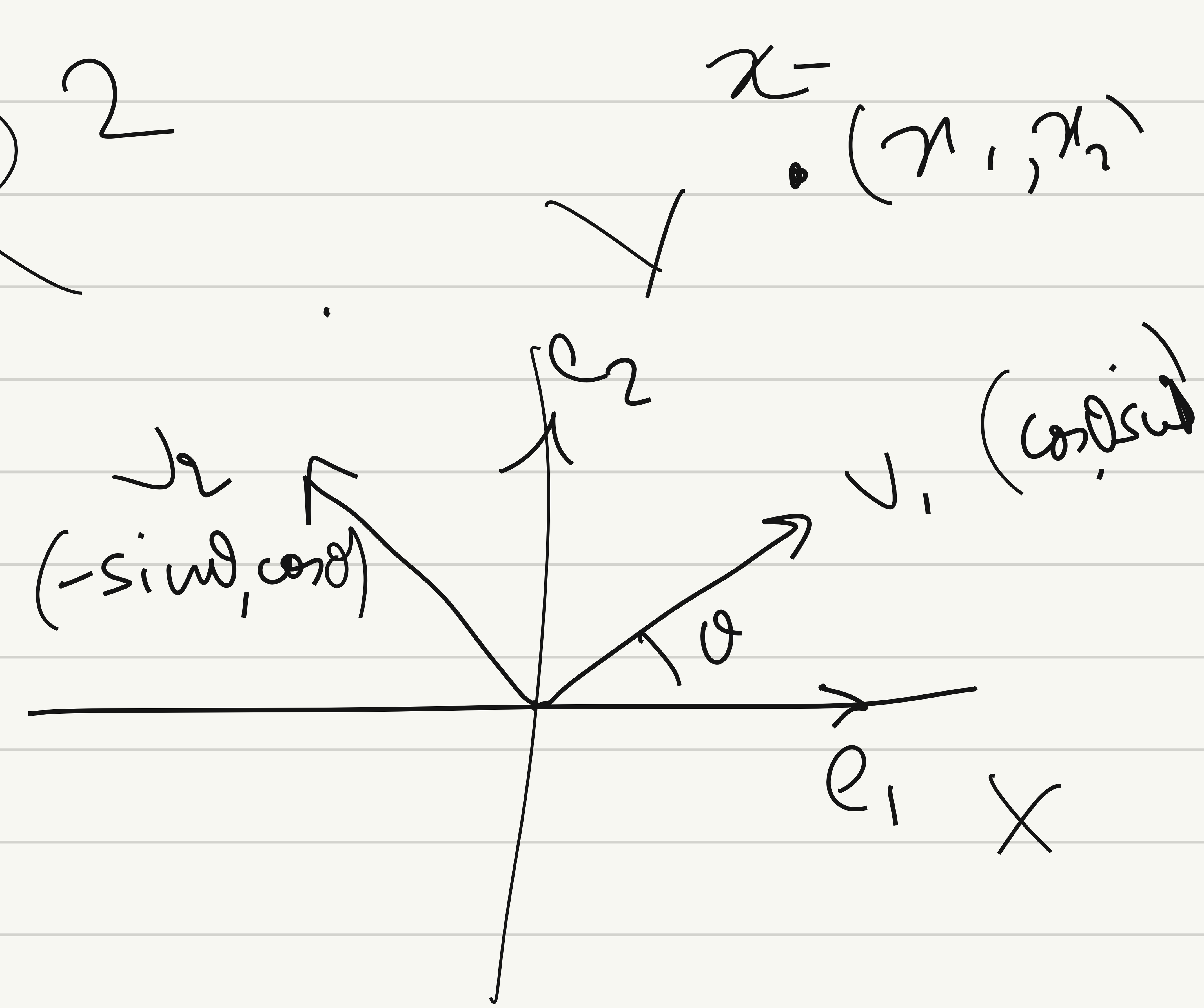
$$\mathbf{x} = P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}.$$

$[X]_\beta = (c_1, \ldots, c_n)$

$X = \boxed{c_1 b_1 + c_2 b_2 + \cdots + c_n b_n}$

Example :

$\mathbb{R}^2$.



$$\beta = \{ v_1 , v_2 \} .$$

$$x = (x_1 , x_2) \in \mathbb{R}^2$$

standard coordinates .

$\{ e_1, e_2 \}$

standard coordinate

$$[x]_\beta = ?$$

$$\rightarrow \quad P_\beta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \leftarrow \text{rotation by } \theta$$

$$x = (x_1, x_2)$$

$$x = P_\beta [x]_\beta$$

$$\rightarrow [x]_\beta = P_\beta^{-1} x.$$

$$P_\beta^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$\uparrow$$

rotation by $-\theta$.

$$[x]_\beta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= (x_1\cos\theta + x_2\sin\theta,$$
$$-x_1\sin\theta + x_2\cos\theta)$$

$$T(x) = [x]_\beta \quad \begin{cases} x = P_\beta (x)_\beta \\ [x]_\beta = P_\beta^{-1} x. \end{cases}$$

$$\overset{\shortparallel}{P_\beta^{-1} x}$$

Proposition

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be an ordered basis of $\mathbb{R}^n$. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the coordinate transformation which sends

$$\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}.$$

The change-of-coordinates matrix $P_{\mathcal{B}}$ is the standard matrix of the inverse T^{-1} of the coordinate transformation.

The standard matrix of the coordinate transformation T is $P_{\mathcal{B}}^{-1}$.

$$\boxed{T(x) = P_\beta^{-1} x.}$$

$$T^{-1}[X]_\beta = x$$

$$\shortparallel$$

$$P_\beta [X]_\beta$$

<u>Proof</u> : Let $x \in \mathbb{R}^n$.

We know that

$$x = P_\beta [x]_\beta .$$

$$\therefore \quad [x]_\beta = P_\beta^{-1} x .$$

$$\therefore \quad T(x) = \boxed{P_\beta^{-1}} x$$

$\Rightarrow P_\beta^{-1}$ is the standard matrix of x.

Let $y \in \mathbb{R}^n$, where

$y = [x]_\beta$, i.e. $y = T(x)$ for some $x \in \mathbb{R}^n$.

$\Rightarrow \quad T^{-1} y = x$. ——— ①

Since $y = [x]_\beta$,

$x = P_\beta [x]_\beta$ ——— ②

① & ② $\Rightarrow T^{-1} y = P_\beta y \Rightarrow P_\beta$ is the standard matrix of T^{-1}

Matrix of a Linear Transformation

$T: V \to V.$

$\beta = \{v_1, \ldots, v_n\}$

Proposition (M)

Let V, W be vector spaces. Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be an ordered basis for V and $\mathcal{C} = \{w_1, \ldots, w_m\}$ be an ordered basis for W. Let $T : V \to W$ be a linear transformation. There exists a unique $m \times n$ matrix A such that

$$[T(v)]_{\mathcal{C}} = A[v]_{\mathcal{B}}, \quad \text{for every } v \in V.$$

$[T]_{\beta, C}$

Further, we have

$$A = [[T(v_1)]_{\mathcal{C}} \quad \ldots \quad [T(v_n)]_{\mathcal{C}}]$$

Particular Case: The $\mathcal{B}$-matrix

Definition

Let $T : V \to V$ be a linear transformation from a vector space to itself. Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be an ordered basis for V. There is a unique matrix $[T]_{\mathcal{B}}$, which we call the $\mathcal{B}$-matrix of T such that

$$[T(v)]_{\mathcal{B}} = [T]_{\mathcal{B}}[v]_{\mathcal{B}}, \quad \forall v \in V.$$

Further, $[T]_{\mathcal{B}}$ is obtained using the formula

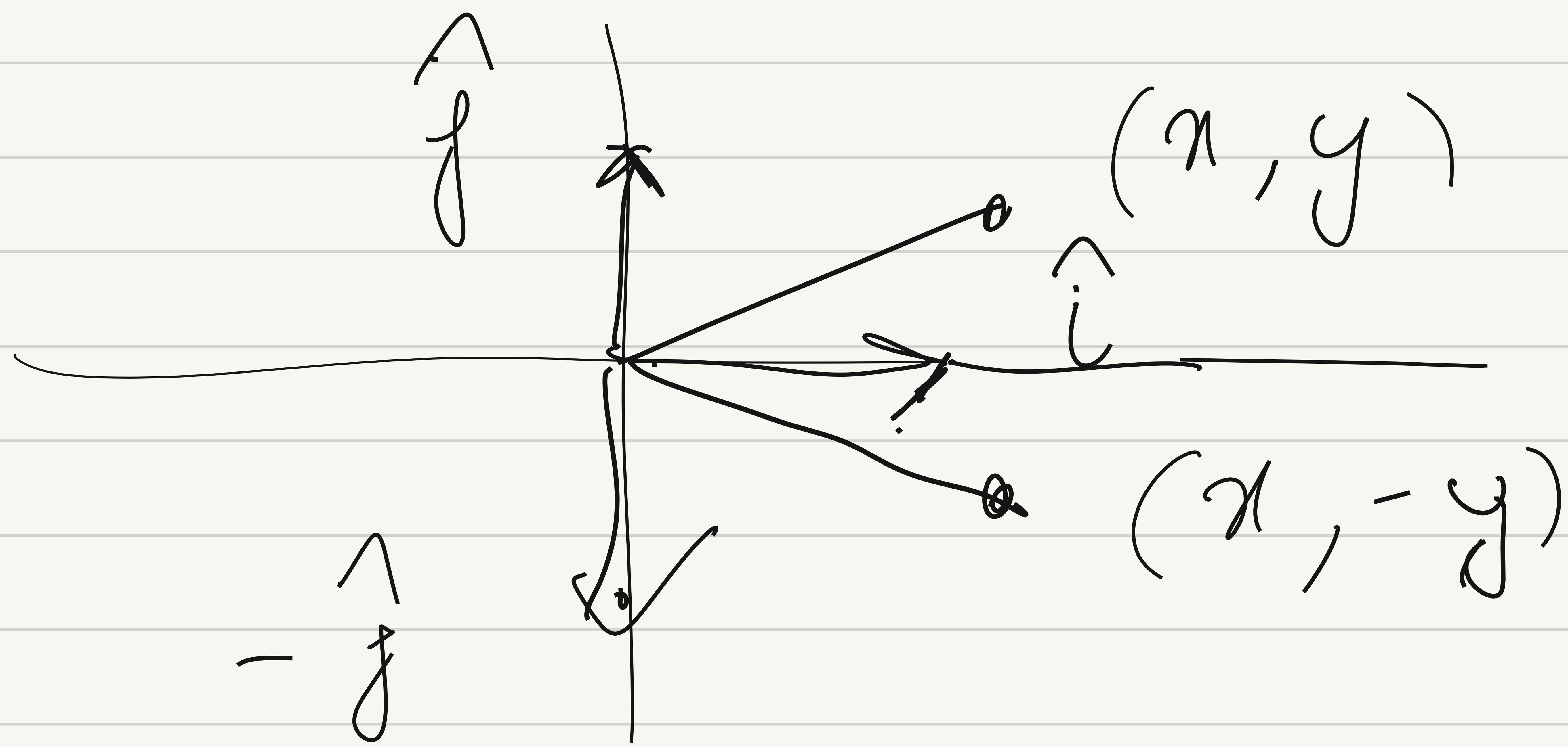
$$[T]_{\mathcal{B}} = [[T(v_1)]_{\mathcal{B}} \quad \ldots \quad [T(v_n)]_{\mathcal{B}}]$$

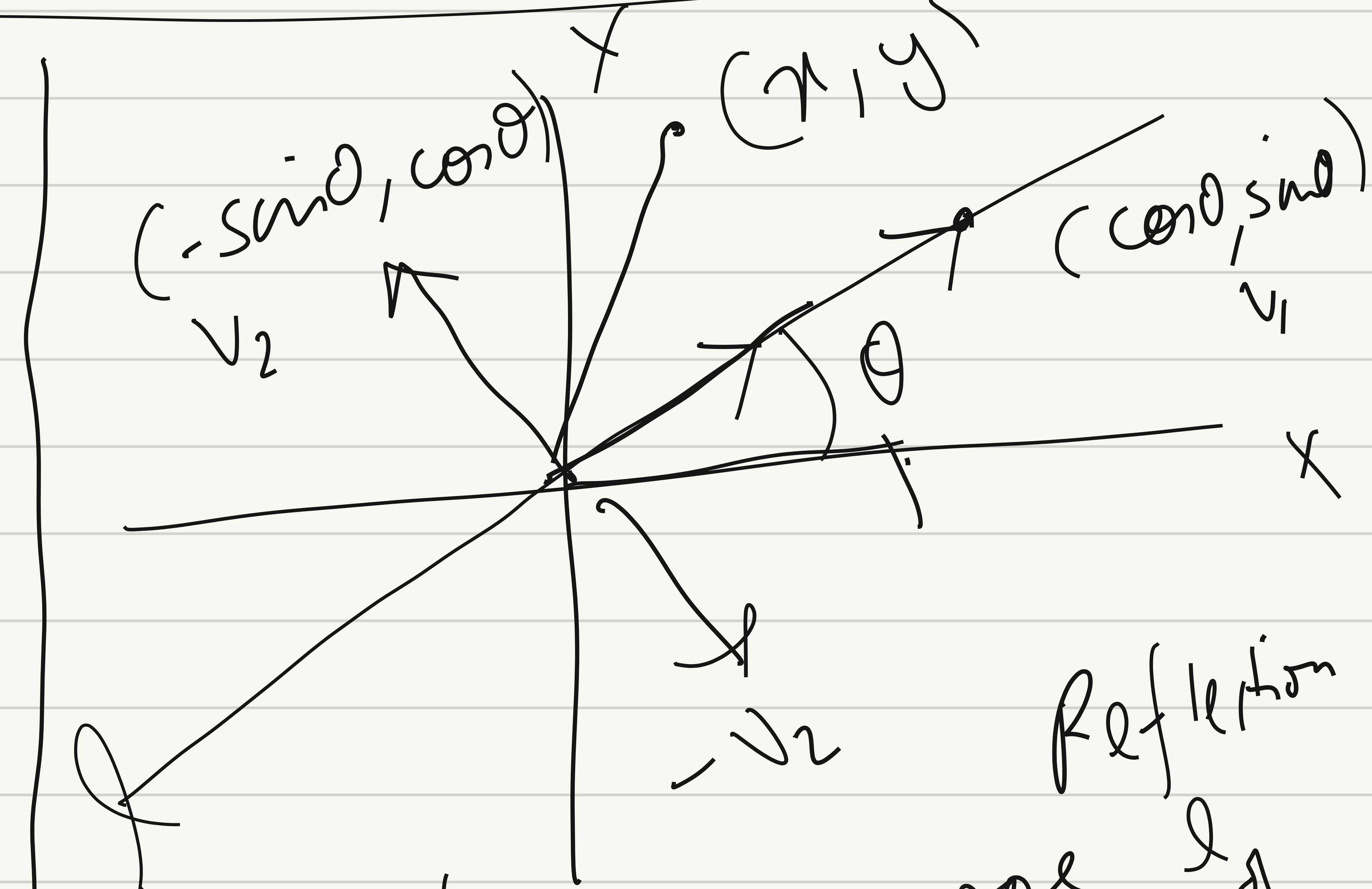
Before we prove this proposition, let us see a few applications.

Reflection in $\mathbb{R}^2$:

$\beta = \{v_1, \dots, v_n\}$

$$\boxed{[T]_\beta = \Big[\, [T(v_1)]_\beta \; \cdots\cdots \; [T(v_n)]_\beta \,\Big].}$$





Reflection across l makes angle of θ w/ x-axis

$v_1 \mapsto v_1$

$v_2 \mapsto -v_2.$

$\beta = \{v_1, v_2\}$ $\quad\quad$ $i_\beta = [v_1 \quad v_2]$

$$[T]_\beta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\boxed{\begin{aligned} &[T(v_2)]_\beta \\ &\quad \| \\ &[-v_2]_\beta \\ &\quad \| \\ &(0, -1) \end{aligned}}$$

$$T(v_1) = v_1$$

$$= 1 \cdot v_1 + 0 \cdot v_2 .$$

$$[T(v_1)]_\beta = [v_1]_\beta = (1, 0).$$

$$T(v_2) = -v_2 . = 0 \cdot v_1 + (-1) \cdot v_2 .$$

$$A = P_\beta [T]_\beta P_\beta^{-1}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta - \sin^2\theta & 2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \sin^2\theta - \cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

Proposition

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Let A be the standard matrix of T. Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be any ordered basis of $\mathbb{R}^n$. Let

$$P_{\mathcal{B}} = [\mathbf{b}_1 \quad \cdots \quad \mathbf{b}_n]$$

be the change-of-coordinates matrix from $\mathcal{B}$ to the standard basis in $\mathbb{R}^n$. Then the $\mathcal{B}$-matrix of T is $P_{\mathcal{B}}^{-1} A P_{\mathcal{B}}$.

Proof:

For every $\mathbf{x} \in \mathbb{R}^n$,

$$\begin{aligned} [T(\mathbf{x})]_{\mathcal{B}} &= [A\mathbf{x}]_{\mathcal{B}} \\ &= P_{\mathcal{B}}^{-1} A\mathbf{x} \\ &= P_{\mathcal{B}}^{-1} A P_{\mathcal{B}} [\mathbf{x}]_{\mathcal{B}}. \end{aligned}$$

$$[T]_{\beta} = P_{\beta}^{-1} A P_{\beta}$$

$$\boxed{A = P_{\beta} [T]_{\beta\beta} P_{\beta}^{-1}}$$