$P[H_0] + P[H_1] + P[H_2] = 1$

$P[H_0] = P[H_1] = P[H_2] = 1/3$

$P[F] + P[V] = 1$

$P[F] = 5/12$

# Problem 1.4.6

$P[V] = 7/12$

Suppose a cellular telephone is equally likely to make zero handoffs ($H_0$), one handoff ($H_1$), or more than one handoff ($H_2$). Also, a caller is either on foot ($F$) with probability $5/12$ or in a vehicle ($V$).

(a) Given the preceding information, find three ways to fill in the following probability table:

| | $H_0$ | $H_1$ | $H_2$ |
|---|---|---|---|
| $F$ | | | |
| $V$ | | | |

(b) Suppose we also learn that $1/4$ of all callers are on foot making calls with no handoffs and that $1/6$ of all callers are vehicle users making calls with a single handoff. Given these additional facts, find all possible ways to fill in the table of probabilities.

$P[FH_0] = 1/4$

$P[VH_1] = 1/6$

$$P[H_0] + P[H_1] + P[H_2] = 1$$

$$P[H_0] = P[H_1] = P[H_2] = 1/3$$

$$P[F] + P[V] = 1$$

$$P[F] = 5/12$$

$$P[V] = 7/12$$

| | $H_0$ | $H_1$ | $H_2$ |
|---|---|---|---|
| F | $P[FH_0]$ | $P[FH_1]$ | $P[FH_2]$ |
| V | $P[VH_0]$ | $P[VH_1]$ | $P[VH_2]$ |

| | | |
|---|---|---|
| $P[F]P[H_0]$ | $P[F]P[H_1]$ | $P[F]P[H_2]$ |
| $P[V]P[H_0]$ | $P[V]P[H_1]$ | $P[V]P[H_2]$ |

$\longrightarrow$ $0$ $\quad 1/3 \quad 5/12 - 1/3 \rightarrow 5/12$

$1/3 \quad 0 \quad 7/12 - 1/3 \rightarrow 7/12$

$\downarrow$

# Conditional Probability

- What is the probability that it is raining right now?
- The number you state is an expression of your belief
- What if I tell you that it is very cloudy outside?
  - Does this change your belief about rain?
- If A is the event {rain} and B is the event {cloudy sky}, then P[A] is your belief in absence of any other information
- P[A|B] is your belief after you are told that B has occurred

$$P\left[\frac{A}{B}\right]$$

# Conditional Probability

- P[A] is called the a priori probability of event A
- It is the prior probability of A, prior to you knowing other facts
- P[A|B] is the probability of A given that B has occurred.
  - It is the conditional probability of A given B
- What is P[A|A]?
- Roll of a die: What is the probability of the outcome {2}?
- Let E be the event that an even number was obtained. What is P[{2}|E]?

# Conditional Probability

- **Definition 1.6**

$$P[A|B] = \frac{P[AB]}{P[B]}$$

If A and B are mutually exclusive,
then P[AB] = ?

P[A|B] = ?

# Theorem 1.9

A conditional probability measure $P[A|B]$ has the following properties that correspond to the axioms of probability.

Axiom 1: $P[A|B] \geq 0$.

Axiom 2: $P[B|B] = 1$.

Axiom 3: If $A = A_1 \cup A_2 \cup \cdots$ with $A_i \cap A_j = \phi$ for $i \neq j$, then

$$P\left[A|B\right] = P\left[A_1|B\right] + P\left[A_2|B\right] + \cdots$$

# Law of Total Probability (Theorem 1.10)

For an event space $\{B_1, \ldots, B_m\}$ with $P[B_i] > 0$ for all $i$, $P[A] = \sum_{i=1}^{i=m} P[A|B_i]P[B_i]$

Why this?

Proof…?

$Q$ is the event that a resistor is within 50Ω of the nominal value.

$$Q = (Q \cap B_1) \cup (Q \cap B_2) \cup (Q \cap B_3),$$

where $\{B_1, B_2, B_3\}$ is an event space

# Example 1.19 Problem

A company has three machines $B_1$, $B_2$, and $B_3$ for making 1 kΩ resistors. It has been observed that 80% of resistors produced by $B_1$ are within 50 Ω of the nominal value. Machine $B_2$ produces 90% of resistors within 50 Ω of the nominal value. The percentage for machine $B_3$ is 60%. Each hour, machine $B_1$ produces 3000 resistors, $B_2$ produces 4000 resistors, and $B_3$ produces 3000 resistors. All of the resistors are mixed together at random in one bin and packed for shipment. What is the probability that the company ships a resistor that is within 50 Ω of the nominal value?

$$P(Q) = P[QB_1] + P[QB_2] + P[QB_3]$$

$$= P[Q|B_1]\,P[B_1] + \cdots$$

$P[Q|B_1] = 0.8$

$P[Q|B_2] = 0.9$

$P[Q|B_3] = 0.6$

# Bayes' Theorem – The Most Important Theorem Ever!

$$P[B|A] = \frac{P[A|B]P[B]}{P[A]}$$

Proof follows from the definition of conditional probability

# Quiz 1.5

Monitor three consecutive phone calls going through a telephone switching office. Classify each one as a voice call $(v)$ if someone is speaking, or a data call $(d)$ if the call is carrying a modem or fax signal. Your observation is a sequence of three letters (each one is either $v$ or $d$). For example, three voice calls corresponds to $vvv$. The outcomes $vvv$ and $ddd$ have probability $0.2$ whereas each of the other outcomes $vvd$, $vdv$, $vdd$, $dvv$, $dvd$, and $ddv$ has probability $0.1$. Count the number of voice calls $N_V$ in the three calls you have observed. Consider the four events $N_V = 0$, $N_V = 1$, $N_V = 2$, $N_V = 3$. Describe in words and also calculate the following probabilities:

(1) $P[N_V = 2]$

(2) $P[N_V \geq 1]$

(3) $P[\{vvd\}|N_V = 2]$

(4) $P[\{ddv\}|N_V = 2]$

(5) $P[N_V = 2|N_V \geq 1]$

(6) $P[N_V \geq 1|N_V = 2]$

# Problem 1.5.6

Deer ticks can carry both Lyme disease and human granulocytic ehrlichiosis (HGE). In a study of ticks in the Midwest, it was found that 16% carried Lyme disease, 10% had HGE, and that 10% of the ticks that had either Lyme disease or HGE carried both diseases.

(a) What is the probability $P[LH]$ that a tick carries both Lyme disease ($L$) and HGE ($H$)?

(b) What is the conditional probability that a tick has HGE given that it has Lyme disease?

# Bayes' and Law of Total Probability

For an event space $\{B_1, \ldots, B_m\}$ with $P[B_i] > 0$ for all $i$,

$$P[B_i|A] = \frac{P[AB_i]}{P[A]} = \frac{P[A|B_i]P[B_i]}{P[A]}$$
$$= \frac{P[A|B_i]P[B_i]}{\sum_{i=1}^{m} P[A|B_i]P[B_i]}$$

We know the a priori probabilities of the $B_i$(s) and also the probability of an observable event A given $B_i$. Having seen A (the effect) we want to know the chance that a certain $B_i$ is the cause (that caused A)