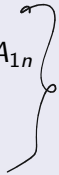


Let A be an $n \times n$ matrix (where $n \geq 2$). A_{ij} denotes the $n-1 \times n-1$ submatrix formed by deleting the i -th row and j -th column of A , for $1 \leq i, j \leq n$.

Definition

For $n \geq 2$, the determinant of an $n \times n$ matrix $A = (a_{ij})$ is the sum of n terms of the form $\pm a_{1j} \det A_{1j}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \dots, a_{1n}$ are from the first row of A . In symbols,

$$\begin{aligned} \det A &= a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n} \\ &= \sum_{j=1}^n (-1)^{1+j} a_{1j} \det A_{1j} \end{aligned}$$


The determinant of a 1×1 matrix is the single entry of that matrix.

Definition

Let $A = (a_{ij})$ be an $n \times n$ matrix (where $n \geq 2$). The (i, j) -cofactor of A is the number C_{ij} given by

$$\underline{C_{ij} = (-1)^{i+j} \det A_{ij}}$$

same formula as previous page $\leftarrow \det A = a_{11}C_{11} + a_{12}C_{12} + \dots + a_{1n}C_{1n}$

Theorem

The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column. The expansion across the i -th row using the cofactors is

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \dots + a_{in}C_{in}$$

The cofactor expansion down the j -th column is

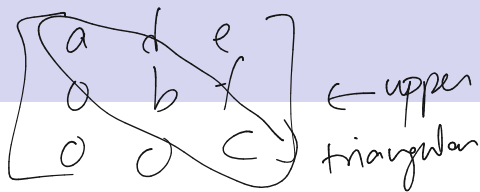
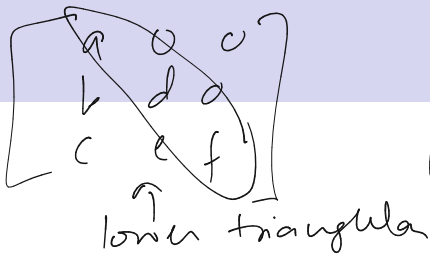
$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \dots + a_{nj}C_{nj}.$$

A little group theory (namely the fact that S_n is generated by transpositions and that cosets corresponding to transpositions partition S_n into permutations of the set $\{1, \dots, n\}$), and some induction can be used to establish that

$$\det A = \sum_{\pi \in S_n} \text{sign}(\pi) \prod_{j=1}^n a_{j, \pi(j)}$$

(12)
 $a_{12}a_{21}a_{33} \dots a_{nn}$

strictly out of syllabus



Theorem

If A is a triangular matrix, then $\det A$ is the product of the entries on the main diagonal of A .

Idea: Use definition and induction - lower triangular case.

For the upper triangular case, expand along first column instead.

Proof: A lower triangular.

$$\det \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - 0 \cdot a_{21} = a_{11}a_{22}.$$

Assume that the result

holds true for $n = k \geq 2$.

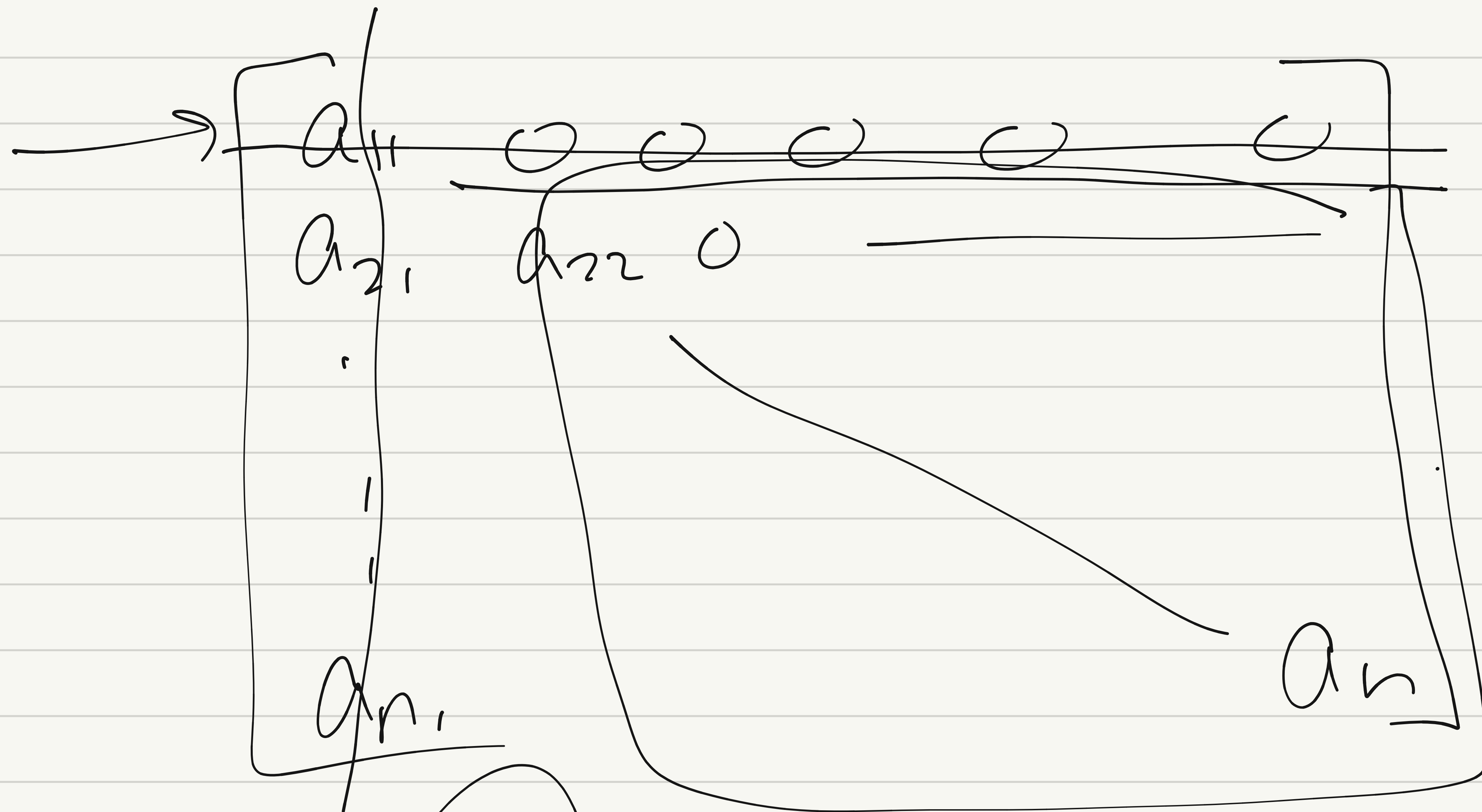
We show that the result holds true when $n = k + 1$.

$$\det A = \sum_{j=1}^{k+1} (-1)^{1+j} a_{1j} \det A_{1j} = a_{11} \det A_{11}$$

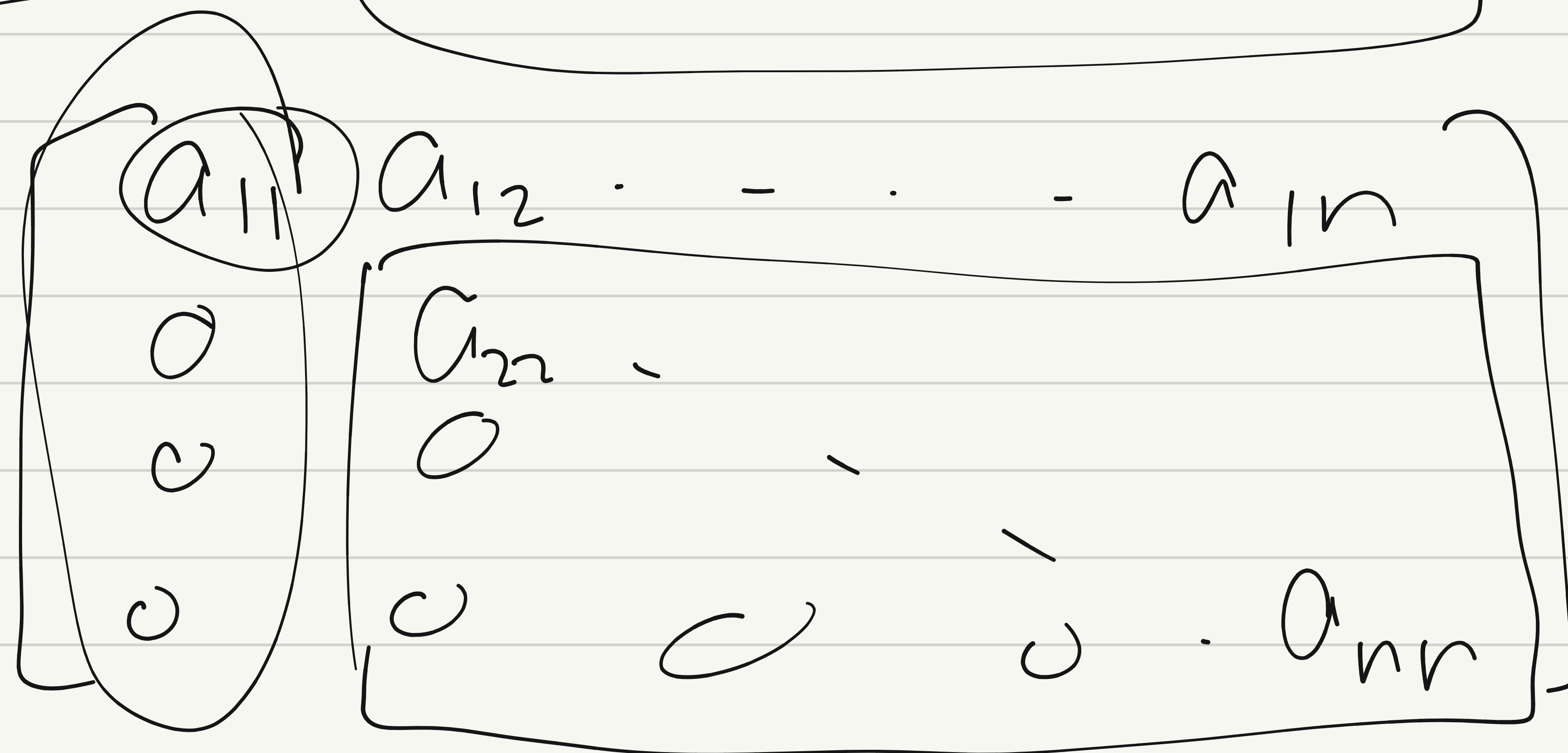
$$\det A_{11} = a_{22} a_{33} \cdots a_{k+1, k+1}, \text{ by induction hypothesis.}$$

$$\det A = a_{11} a_{22} \cdots a_{R+1, R+1}$$

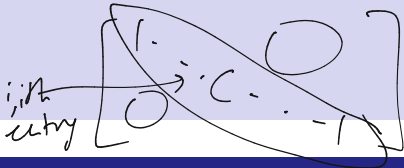
□.



Upper
triangular
case - similar.



$$R_i \rightarrow c R_i$$



Proposition

Let E be an $n \times n$ elementary matrix. Then

- 1 $\det E = c$, when E corresponds to scaling a row by a nonzero scalar c
- 2 $\det E = 1$, when E corresponds to a row replacement operation
- 3 $\det E = -1$, when E corresponds to a row interchange operation

In case of row replacement or row scaling, E is triangular ...

We show that $\det E = -1$, when E corresponds to a row interchange, using induction on n .

Base case .. what is n ?

E corresponds to replacing

R_i and R_j .

$$E = (e_{kl})$$

$$e_{kl} = \begin{cases} 1, & \text{if } k=l \\ 0, & \text{otherwise} \end{cases}$$

$$e_{kl} = \begin{cases} 1, & \text{if } k \neq i, l \neq j \\ & \text{and } k = l \\ 0 & \text{or if } k = i \text{ and } l = j \\ & \text{or if } k = j \text{ and } l = i \\ 0 & \text{otherwise} \end{cases}$$

$$R_1 \leftrightarrow R_3$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\det E = e_{11} \det E_{11} - e_{12} \det E_{12}$$

$$+ \dots + (-1)^{i+j} e_{ij} \det E_{ij}$$

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Back to earlier proof:

Base case: $n = 2$.
only possibility: interchange R_1 and R_2 .

$$\det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1.$$

Induction hypothesis: Assume
that the result holds for
 $2 \leq n < k$. We show that

the hypothesis holds for $n = K$.

Assume that the induction hypothesis is true for $2 \leq k < n$. We show that it holds when $n = k$.

Main idea: Expand along a row
which is not involved
in the row interchange.

If the interchange is $R_i \leftrightarrow R_j$
 then we expand along row
 where $l \neq i$ and $l \neq j$.

$$\det E = (-1)^{l+1} e_{l1} \det \underbrace{E_{l1}}_{l+n} + \dots + (-1)^{l+n} e_{ln} \det E_{ln}.$$

$$= (-1)^{l+l} e_{ll} \det \underbrace{E_{ll}}_{\det E_{ll}}$$

The matrix E_{ii} also an elementary matrix which corresponds to a row interchange.

\therefore By induction hypothesis.

$$\det E_{ii} = -1.$$

$$\therefore \det E = -1.$$

Suppose E corresponds to $R_i \leftrightarrow R_j$. We expand the determinant along row l , where $l \neq i, l \neq j$. Let $E = (a_{ij})$. Then

$$\det E = \det E_{ll}$$

Since E_{ll} is an $k - 1 \times k - 1$ elementary matrix corresponding to a row interchange operation, it follows by the induction hypothesis that

$$\det E_{ll} = -1.$$

Therefore $\det E = -1$.

Proposition

Let A be an $n \times n$ matrix having two identical rows. Then the determinant of A is zero.

Idea(s) behind proof:

First consider the case where the identical rows are adjacent.
Expand.

Next consider the non-adjacent case. Use induction.

Consider the case where
 R_i, R_{i+1} are identical.

Expanding along the i th row,

we get

$$\begin{aligned} \textcircled{1} \quad \left\{ \begin{aligned} \det A &= (-1)^{i+1} \det A_{i1} + \dots + (-1)^{i+r} \det A_{ir} \\ \text{along } i\text{-th row,} \end{aligned} \right. \\ \textcircled{2} \quad \left\{ \begin{aligned} \det A &= (-1)^{i+2} \det A_{i+1,1} + \dots + (-1)^{i+1+n} \det A_{i+1,n} \end{aligned} \right. \end{aligned}$$

Clearly $A_{ij} = A_{i+1,j}$.

\therefore (1) \rightarrow (2) \Rightarrow

$$\det A = - \det A$$

$$\Rightarrow \det A = 0.$$