Q1

Complex Eigenvalues

$$p(\lambda) = \begin{vmatrix} 2-\lambda & -3 \\ 1 & -1-\lambda \end{vmatrix} = (\lambda-2)(\lambda+1)+3$$
$$= \lambda^2 - \lambda + 1$$

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which sends $\mathbf{x} \mapsto A\mathbf{x}$, where

$$A = \begin{bmatrix} 2 & -3 \\ 1 & -1 \end{bmatrix}. \qquad \lambda = \frac{1 \pm \sqrt{3}i}{2}.$$

Let $\mathcal{B}$ be a basis of $\mathbb{R}^2$ such that $[T]_{\mathcal{B}}$ is a counter-clockwise rotation by an angle of θ.

$\shortparallel$

Identify a correct choice of θ from the following options.

$\cos 60 \pm i \sin 60$

Correct answers: 60 degrees, -60 degrees

$$\lambda_1 = e^{i\pi/3}$$
$$\lambda_2 = e^{-i\pi/3}$$

Q2

$p(\lambda) = (\lambda - 4)(\lambda + 2) + 12$

$= \lambda^2 - 2\lambda + 4$

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which sends $\mathbf{x} \mapsto A\mathbf{x}$, where

$$A = \begin{bmatrix} 4 & -6 \\ 2 & -2 \end{bmatrix}.$$

$\lambda = \dfrac{2 \pm \sqrt{-12}}{2}$

$1 \pm \sqrt{3}i$

Let $\mathcal{B}$ be a basis of $\mathbb{R}^2$ such that T is the composite of a 2D rotation and a scaling transformation with respect to $\mathcal{B}$-coordinates.

Identify a correct change-of-coordinates matrix $P_{\mathcal{B}}$ from the following options:

$Av = \lambda v$

$\rightarrow P_{\mathcal{B}} = \begin{bmatrix} \mathrm{Re}\, v & \mathrm{Im}\, v \end{bmatrix}$

$1 - \sqrt{3}\, i$

$15 - 5\sqrt{3}\, i$

10

Correct answers: options 1 and 3

1 $\begin{bmatrix} 15 & -5\sqrt{3} \\ 10 & 0 \end{bmatrix}$

2 $\begin{bmatrix} 3/2 & 1 \\ -\sqrt{3}/2 & 0 \end{bmatrix}$

3 $\begin{bmatrix} 15 & 5\sqrt{3} \\ 10 & 0 \end{bmatrix}$

4 $\begin{bmatrix} 3/2 & 1 \\ \sqrt{3}/2 & 0 \end{bmatrix}$

$$\begin{pmatrix} 4 & -6 \\ 2 & -2 \end{pmatrix} \begin{bmatrix} 15 - 5\sqrt{3}\, i \\ 10 \end{bmatrix} = \begin{bmatrix} -20\sqrt{3}\, i \\ 10 - 10\sqrt{3}\, i \end{bmatrix}$$

Q3

$p(x) = (x+1)(x-2)(x-1)(x-3)$

$p(A) = 0.$

$(A+I)(A-2I)(A-I)$
$(A-3I)$
$= 0.$

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Let $W = \underline{\mathrm{Span}\{A^4, A^3, A^2, A, I\}} \subset M_{4\times4}(\mathbb{R})$.

What is the dimension of W?

Correct answer: 4

$A^4 \in \mathrm{Span}\{A^3, A^2, A, I\}$

$c_1 A^3 + c_2 A^2 + c_3 A + c_4 I = 0.$

$$c_1 a_{jj}^3 + c_2 a_{jj}^2 + c_3 a_{jj} + c_4 = 0.$$

$\Rightarrow$ $-1, 2, 1$ and 3 are roots of a cubic poly $\times$.

Q4

Let A be a 3×3 matrix such that the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ reflects every vector in $\mathbb{R}^3$ across the plane

$$2x + 3y + 5z = 0.$$

$w \mapsto w$

Correct answers: 1 and 3

$v \mapsto -v$

1. A is diagonalizable
2. A is not diagonalizable
3. The eigenvalues of A are 1 and -1
4. The eigenvalues of A are 2 and 3
5. 1 is the only real eigenvalue of A

$Av = -v$

$Aw = w$

Review:Orthogonal Sets

Definition

Let V be a vector space with an inner product $\langle ., . \rangle$. A set of vectors $\{v_1, \ldots, v_p\}$ in V is said to be an *orthogonal set* if

$$\langle v_i, v_j \rangle = 0, \quad \text{for every } i \neq j, \quad \text{where } i, j \in \{1, \ldots, p\}.$$

Theorem

Let V be a vector space of dimension n. Let $\langle ., . \rangle$ be an inner product defined on V. Let $S = \{v_1, \ldots, v_p\}$ be an orthogonal set of non-zero vectors in an inner product space $(V, \langle ., . \rangle)$. Then S is linearly independent set and hence is a basis for Span S.

Review

Definition

Let $(V, \langle ., . \rangle)$ be an inner product space. An orthogonal set which is also a basis for V is called an *orthogonal basis* for V.

Definition

Let $(V, \langle ., . \rangle)$ be an inner product space. An orthogonal set of unit vectors in V is called an *orthonormal set*.

Definition

Let $(V, \langle ., . \rangle)$ be an inner product space. An orthonormal set which is also a basis for V is called an *orthonormal basis* for V.

Review

Definition

Let $(V, \langle ., . \rangle)$ be an inner product space. Let $v \in V$ be a nonzero vector. For any vector $u \in V$, the *orthogonal projection of u onto v* is defined as

$$\text{proj}_v u := \frac{\langle u, v \rangle}{\langle v, v \rangle} v = \langle u, \hat{v} \rangle \hat{v},$$

and the vector

$$u - \text{proj}_v u$$

is called the *component of u which is orthogonal to v*.

Review

Theorem

Let $(V, \langle ., . \rangle)$ be an inner product space. Let $\mathcal{B} = \{v_1, \dots, v_n\}$ be an orthogonal basis for V. Let $v \in V$ be any vector. Then the $\mathcal{B}$-coordinates of v are:

$$[v]_{\mathcal{B}} = \left(\frac{\langle v, v_1 \rangle}{\langle v_1, v_1 \rangle}, \frac{\langle v, v_2 \rangle}{\langle v_2, v_2 \rangle}, \dots, \frac{\langle v, v_n \rangle}{\langle v_n, v_n \rangle} \right).$$

Corollary

Let $(V, \langle ., . \rangle)$ be an inner product space. Let $\mathcal{B} = \{\hat{v}_1, \dots, \hat{v}_n\}$ be an orthonormal basis for V. Let $v \in V$ be any vector. Then the $\mathcal{B}$-coordinates of v are:

$$[v]_{\mathcal{B}} = \left(\langle v, \hat{v}_1 \rangle, \dots, \langle v, \hat{v}_n \rangle \right).$$

The Gram-Schmidt Process

Theorem

Given a basis $\{v_1, \ldots, v_p\}$ *for an inner product space* $(V, \langle ., . \rangle)$*, define*

$$w_1 = v_1 \implies v_1 \in \operatorname{span}\{w_1\}$$

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 \implies v_2 \in \operatorname{Span}\{w_1, w_2\}$$

$$\vdots$$

$$w_p = v_p - \frac{\langle v_p, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_p, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2 - \cdots - \frac{\langle v_p, w_{p-1} \rangle}{\langle w_{p-1}, w_{p-1} \rangle} w_{p-1}$$

$\implies v_p \in \operatorname{span}\{w_1, \ldots, w_p\}$

Then $\{w_1, \ldots, w_p\}$ *is an orthogonal basis for* V*. In addition*

$$\operatorname{Span}\{w_1, \ldots, w_k\} = \operatorname{Span}\{v_1, \ldots, v_k\} \quad \text{for } 1 \le k \le p$$

Proof:

We show that the sets $\{w_1, \ldots, w_k\}$ are orthogonal, using induction on k. For $k = 1$, there is nothing to show.

Assume that $\{w_1, \ldots, w_l\}$ is an orthogonal set, for every $1 \leq l < k$. We show that $\{w_1, \ldots, w_k\}$ is orthogonal.

Let i and j be positive integers such that $i \neq j$ and $i, j \leq k$. If i and j are strictly less than k then the vectors w_i and w_j are orthogonal, by the induction hypothesis.

Hence let us assume, without loss of generality, that $i = k$ and $j < k$. Then

$$\begin{aligned} \langle w_i, w_j \rangle &= \langle w_k, w_j \rangle \\ &= \left\langle v_k - \sum_{m=1}^{k-1} \frac{\langle v_k, w_m \rangle}{\langle w_m, w_m \rangle} w_m, w_j \right\rangle \end{aligned}$$

$$
\begin{aligned}
&= \langle v_k, w_j \rangle - \sum_{m=1}^{k-1} \frac{\langle v_k, w_m \rangle}{\langle w_m, w_m \rangle} \langle w_m, w_j \rangle \\
&= \langle v_k, w_j \rangle - \frac{\langle v_k, w_j \rangle}{\langle w_j, w_j \rangle} \langle w_j, w_j \rangle \\
&= 0.
\end{aligned}
$$

Clearly

$$v_k \in \operatorname{Span}\{w_1, \ldots, w_k\}, \qquad \text{for } 1 \leq k \leq p$$

Hence

$$\{v_1, \ldots, v_k\} \subset \operatorname{Span}\{w_1, \ldots, w_k\}, \qquad \text{for } 1 \leq k \leq p$$

So,

$$\operatorname{Span}\{v_1, \ldots, v_k\} \subset \operatorname{Span}\{w_1, \ldots, w_k\}, \qquad \text{for } 1 \leq k \leq p$$

Since

$$k = \dim \operatorname{Span}\{v_1, \ldots, v_k\} = \dim \operatorname{Span}\{w_1, \ldots, w_k\} \leq k,$$

it follows that

$$\operatorname{Span}\{v_1, \ldots, v_k\} = \operatorname{Span}\{w_1, \ldots, w_k\}, \qquad \text{for } 1 \leq k \leq p.$$

In particular $\{w_1, \ldots, w_p\}$ is an orthogonal basis for V.

Definition

A matrix A is said to be *orthogonally diagonalizable* if there exists an orthogonal matrix P (with $P^{-1} = P^T$), and a diagonal matrix D such that

$$A = PDP^T = PDP^{-1}$$

Example

Find a diagonal matrix D, and an orthogonal matrix P such that $A = PDP^{-1}$, where A is given below.

$$A = \begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & -1 \\ 1 & -1 & 5 \end{bmatrix}.$$

Recall that the eigenvalues of A were found to be $\lambda_1 = 6, \lambda_2 = 3$,

$x \mapsto Ax$

and that the bases for the eigenspaces were calculated to be

$$\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\} \text{ is a basis for the 6-eigenspace, and}$$

$$\mathcal{B}_2 = \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\} \text{ is a basis for the 3-eigenspace.}$$

Thus $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$. We apply the Gram-Schmdit process to get an orthonormal basis of eigenvectors.

$$\begin{bmatrix}1\\1\\0\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\0\\1\end{bmatrix}$$

Gram-Schmidt process:

$$\mathbf{v}_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix}1\\1\\0\end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix}-1\\1\\1\end{bmatrix}$$

6 3

Step 1: $\mathbf{w}_1 = \mathbf{v}_1$

Step 2: $\mathbf{w}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \begin{bmatrix}\frac{1}{2}\\1\\-\frac{1}{2}\end{bmatrix}$

Step 3: $\mathbf{w}_3 = \mathbf{v}_3 - \frac{\mathbf{v}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{v}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = \mathbf{v}_3$

The next step in calculating P is to normalize the vectors $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$.

$$\hat{\mathbf{w}}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \hat{\mathbf{w}}_2 = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{bmatrix}, \hat{\mathbf{w}}_3 = \begin{bmatrix} -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

Thus

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}, \quad D = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

HW: check $AP = PD$.

Caution

The Gram-Schmidt process gives us a *basis of eigenvectors*, only because matrix A is symmetric. If we had started off with a non-symmetric matrix we would **not** have obtained a set of eigenvectors after applying this algorithm.

Theorem

An $n \times n$ matrix A is orthogonally diagonalizable if and only if A is a symmetric matrix.

We will only prove one implication, because we don't have much time left. Students can look up the proof of the other implication in the Proofs Document uploaded in GC at the beginning of the semester.

We prove that if A is orthogonally diagonizable then $A = A^T$.

Proof: Suppose A is orthogonally diagonalizable.

$\Rightarrow \exists$ an orthogonal matrix P and a diagonal matrix D such that $A = PDP^{-1} = PDP^T$

$\therefore A^T = (PDP^T)^T = (P^T)^T D^T P^T = PDP^T = A.$

$v = (v_1, \dots, v_n)$

$\bar{v}^T v = \sum_{i=1}^{n} \bar{v}_i v_i = \sum_{i=1}^{n} |v_i|^2$

Theorem

Let A be an $n \times n$ symmetric matrix having real entries. Then A has real eigenvalues.

Proof: Let λ be an eigenvalue of A, and let $\mathbf{v} \in \mathbb{C}^n$ be a corresponding eigenvector, i.e.

$$A\mathbf{v} = \lambda\mathbf{v} \quad \Rightarrow \overline{Av} = \overline{\lambda v} \quad \Rightarrow (\overline{Av})^T = (\overline{\lambda v})^T$$

Then

$$(\overline{A\mathbf{v}})^T A\mathbf{v} = (\overline{\lambda\mathbf{v}})^T \lambda\mathbf{v} = \bar{v}^T \bar{\lambda}^T \lambda v$$

Hence

$$\bar{\mathbf{v}}^T \bar{A}^T A\mathbf{v} = |\lambda|^2 \bar{\mathbf{v}}^T \mathbf{v}$$

So

$$|\lambda|^2 \bar{\mathbf{v}}^T \mathbf{v} = \bar{\mathbf{v}}^T A^T \lambda\mathbf{v} = \bar{\mathbf{v}}^T \lambda A\mathbf{v} = \lambda^2 \bar{\mathbf{v}}^T \mathbf{v}$$

$(\bar{A})^T = A^T = A$

Since $v \neq 0$, $\overline{v}^T v = \sum_{i=1}^{n} |v_i|^2 \neq 0$

$\therefore$ we cancel $\overline{v}^T v$ on both sides

Hence

$$|\lambda|^2 = \lambda^2$$

If $\lambda = 0$, then λ is real. So suppose $\lambda \neq 0$. Then

$$\left(\frac{\lambda}{|\lambda|}\right)^2 = 1$$

Hence

$$\underline{\lambda = \pm|\lambda|.}$$

So λ is real.

Theorem

If an $n \times n$ matrix A is symmetric, then any two eigenvectors from different eigenspaces are orthogonal.

Proof:

Let λ and μ be distinct eigenvalues of A, and let $\mathbf{v}$ and $\mathbf{w}$ be the corresponding eigenvectors, i.e.

$$A\mathbf{v} = \lambda\mathbf{v}, \quad A\mathbf{w} = \mu\mathbf{w}$$

Then

$$\begin{aligned}
\lambda\mathbf{v} \cdot \mathbf{w} = \lambda\mathbf{v}^T\mathbf{w} &= (\lambda\mathbf{v})^T\mathbf{w} \\
&= (A\mathbf{v})^T\mathbf{w} \\
&= \mathbf{v}^T A^T \mathbf{w} \\
&= \mathbf{v}^T A\mathbf{w} = \mathbf{v}^T \mu\mathbf{w} = \mu\mathbf{v} \cdot \mathbf{w}
\end{aligned}$$

$$(\lambda v - \mu v) \cdot w = 0.$$

Thus $(\lambda - \mu)\mathbf{v} \cdot \mathbf{w} = 0$.

As $\lambda \neq \mu$, this can only happen if

$$\mathbf{v} \cdot \mathbf{w} = 0.$$

$\square$

The Spectral Theorem for Symmetric Matrices

Theorem

An $n \times n$ symmetric matrix A has the following properties:

geometric multiplicity

1. *A has n real eigenvalues, counting multiplicities.*
2. *The dimension of the eigenspace for each eigenvalue λ equals the multiplicity of λ as a root of the characteristic equation.* → algebraic multiplicity
3. *The eigenspaces are mutually orthogonal, in the sense that eigenvectors corresponding to different eigenvalues are orthogonal.*
4. *A is orthogonally diagonalizable.*

(1) Consequence of a fact that a polynomial of degree n with real coefficients has exactly n complex roots → fundamental theorem of algebra

(2) Follows from statement (2) of eigenvalue theorem (see next page).

Theorem

Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \ldots, \lambda_p$.

1. *For $1 \leq k \leq p$, the dimension of the λ_k-eigenspace is less than or equal to the multiplicity of the eigenvalue λ_k.*
2. *The matrix A is diagonalizable if and only if the sum of the dimensions of the distinct eigenspaces equals n, and this happens if and only if the dimension of the λ_k-eigenspace equals the multiplicity of λ_k, for each $k = 1, \ldots, p$.*
3. *If A is diagonalizable and $\mathcal{B}_k$ is a basis for the eigenspace corresponding to λ_k for each k, then the total collection of vectors in the sets $\mathcal{B}_1, \ldots, \mathcal{B}_p$ forms an eigenvector basis for $\mathbb{R}^n$.*

Interested students can look up the Proofs document for a proof.