# Average of a Discrete Random Variable

- **Def 2.14** The expected value of X is

$$E[X] = \mu_X = \sum_{x \in S_X} x P_X(x).$$

$$P_X(x) = \begin{cases} 9/10 & x = 0 \\ 1/10 & x = 10 \end{cases}$$

$$P_Y(x) = \begin{cases} \frac{999}{1000} & x = 0 \\ \frac{1}{1000} & x = 10^6 \end{cases}$$

$E[X] =$

$E[Y] =$

# Average of a Discrete Random Variable

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Is E[X] a random variable?

# Average of a Discrete Random Variable

- **Def 2.14** The expected value of X is

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$x(1), x(2), \ldots, x(n)$

Thinking like a frequentist...

$$\underbrace{\frac{1}{n} \sum_{i=1}^{n} x(i)}_{\downarrow} = \underbrace{\frac{1}{n} \sum_{x \in S_X} (n(x))\, x}$$

$$\lim_{n \to \infty} \frac{1}{n} \sum_{x \in S_X} n(x)\, x$$

# Expected Value

- Consider n trials of an experiment.
- We obtain x(1), x(2), …, x(n)

- The empirical average is

$$m_n = \frac{1}{n}\sum_{i=1}^{n} x(i)$$

# Expected Value

- For a range $S_{X,}$ write as a sum over the range space
- Say $N_x$ is the number of times the value x in $S_X$ occurs during the n trials

# Expected Value

$$m_n = \frac{1}{n} \sum_{x \in S_X} N_x x$$

As $n \to \infty$, $N_x/n \to P_X(x)$,
and we say that $m_n \to E[X]$.

We used the relative frequency definition of probability to link empirical averages obtained from multiple experiment trials with the expected value of the RV X.

# Some Examples of E[X]

- **Def 2.5:** X is a Bernoulli(p) RV if the PMF of X has the form

$$P_X(x) = \begin{cases} 1-p & x=0, \\ p & x=1, \\ 0 & \text{otherwise.} \end{cases}$$

where $0 < p < 1$.

- E[X] = ?
  - p

# Some Examples of E[X]

- Geometric(p)

$$P_X(x) = \begin{cases} p(1-p)^{x-1} & x = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{aligned} E[X] &= \sum_{x=1}^{\infty} x P_X(x) \\ &= \sum_{x=1}^{\infty} x p (1-p)^{x-1} \\ &= p \sum_{x=1}^{\infty} x q^{x-1} \\ &= \frac{1}{p}. \end{aligned}$$

We define q=1-p

# E[X] Examples

- When X is Geometric(p)
  - $E[X] = 1/p$
- A 0.1 probability of a faulty device =>
  - On an average the $10^{th}$ device will be the first to be found faulty
  - This does not mean that every $10^{th}$ device will be faulty. In fact $P[X=10] = 0.03$!
- Please see Theorem 2.6 and 2.7 for E[.] of other RVs

$X_n \sim \text{Binomial}(n, p)$

$\Rightarrow Y_i \sim \text{Bern}(p)$

$Y_1 = y_1 = 0$
$Y_2 = y_2 = 1$
$Y_3 = y_3 = 0$
$Y_4 = y_4 = 1$

$\frac{0}{1} \ \frac{1}{2} \ \frac{0}{3} \ \frac{1}{4} \quad n = 4$

## Problem 2.5.10

Let binomial random variable $X_n$ denote the number of successes in $n$ Bernoulli trials with success probability $p$. Prove that $E[X_n] = np$. Hint: Use the fact that $\sum_{x=0}^{n-1} P_{X_{n-1}}(x) = 1$.

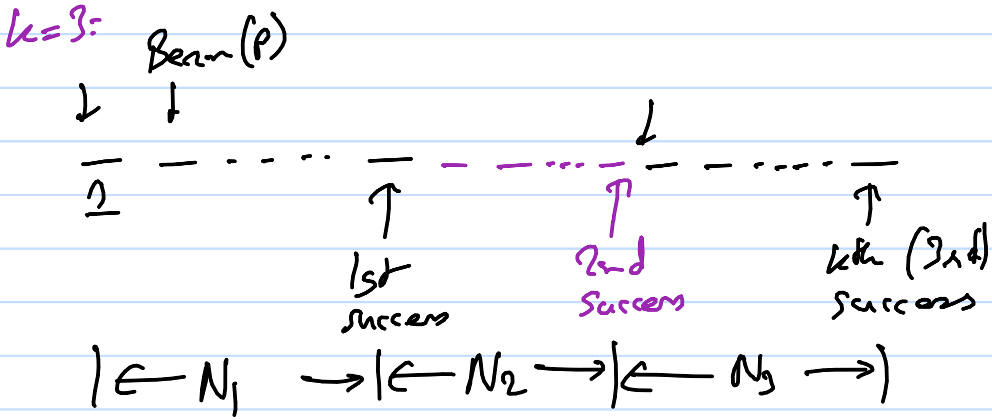
$$X_n = \sum_{i=1}^{n} Y_i$$

Given the properties of the expectation operator $E[\cdot]$

$$E[X_n] = E\left[\sum_{i=1}^{n} Y_i\right]$$

$$= \sum_{i=1}^{n} E[Y_i] = \sum_{i=1}^{n} p = np.$$

$$X \sim \text{Pascal}(k, p)$$

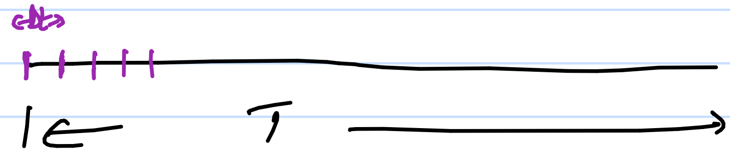



$$\left.\begin{array}{l} N_1 \sim \\ N_2 \sim \\ N_3 \sim \end{array}\right. \text{Geom}(p)$$

$$X(k, p) = \sum_{i=1}^{k} N_i \quad , \quad N_i \sim \text{Geom}(p)$$

$$E[X] = \sum_{i=1}^{k} E[N_i] = \sum_{i=1}^{k} \frac{1}{p} = \frac{k}{p}$$

Your experiment involves observing arrival over time T sec




$$T = n\Delta t$$

$$P[X_n = x] = {}^{n}C_x \, p^x (1-p)^{n-x}$$

Let $\boxed{np = \lambda T}$

$$P(X_n = x) = {}^{n}C_x \left(\frac{\lambda T}{n}\right)^x \left(1 - \frac{\lambda T}{n}\right)^{n-x}$$

$$\lim_{n\to\infty} P[X_n = x] = \lim_{n\to\infty} \frac{n!}{x!\,(n-x)!} \frac{(\lambda T/n)^x}{(1-\lambda T/n)^x} \left(1 - \frac{\lambda T}{n}\right)^n$$

$$= \lim_{n\to\infty} \frac{(\lambda T)^x}{x!} \frac{(n(n-1)(n-2)\cdots 1)}{n^x\,(n-x)!} \boxed{\frac{1}{\left(1-\frac{\lambda T}{n}\right)^x}} \left(1-\frac{\lambda T}{n}\right)^n$$

$\hookrightarrow 1$

$$= \lim_{n\to\infty} \frac{(\lambda T)^x}{x!} \boxed{\frac{n(n-1)\cdots(n-x+1)}{n^x}} \left(1 - \frac{\lambda T}{n}\right)^n$$

$Y \sim \text{Poisson}(\lambda T)$

$$P(Y = x) = \frac{e^{-\lambda T}(\lambda T)^x}{x!}$$

# Theorem 2.8

Perform $n$ Bernoulli trials. In each trial, let the probability of success be $\alpha/n$, where $\alpha > 0$ is a constant and $n > \alpha$. Let the random variable $K_n$ be the number of successes in the $n$ trials. As $n \to \infty$, $P_{K_n}(k)$ converges to the PMF of a Poisson $(\alpha)$ random variable.

# The Properties of the E[.] Operator

- E[.] is a linear operator

$$E[cX+bY] = E[cX] + E[bY] = cE[X] + bE[Y]$$

- Every linear operator has two properties:

$$\hat{A}(f+g) = \hat{A}f + \hat{A}g \qquad (1)$$
$$\hat{A}(cf) = c\hat{A}f \qquad (2)$$

where $\hat{A}$ is the operator, $c$ is a scalar, $f$ and $g$ are functions.

$$E[cX] = cE[X]$$

where $c$ is a scalar

$$E[X+Y] = E[X] + E[Y]$$

$$E\left[\sum_{i=1}^{\infty} X_i\right] = \sum_{i=1}^{\infty} E[X_i]$$

See http://vergil.chemistry.gatech.edu/notes/quantrev/node14.html

# The Properties of the E[.] Operator

- Scalar Multiplication of a Random Variable
  - $E[aX] = a\,E[X]$
  - $E[aX^2] = a\,E[X^2]$ (Note that $X^2$ is a RV)

- $E[X + X^2] = E[X] + E[X^2]$

- $E[(X + c)^2] = E[X^2 + c^2 + 2cX] = E[X^2] + E[c^2] + 2E[cX]$
  $= E[X^2] + c^2 + 2cE[X]$