Invertible Matrix Theorem (some parts)

$I_n = [e_1 \ldots e_n]$

$Ax = e_i$

Theorem

Let A be an $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

- **a.** *A is an invertible matrix.*
- **b.** *A is row equivalent to the $n \times n$ identity matrix.*
- **c.** *A has n pivot positions.*
- **d.** *The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*
- **e.** *The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.*
- **f.** *There is an $n \times n$ matrix C such that $CA = I$.*
- **g.** *There is an $n \times n$ matrix D such that $AD = I$.*
- **h.** *A^T is an invertible matrix.*

Proof of Theorem (parts listed)

We already know that (a) is equivalent to (b), and that (a) is equivalent to (h).

We will show that

1. (b) $\iff$ (c)
2. (a) $\implies$ (d)
3. (d) $\implies$ (c)
4. (a) $\implies$ (e)
5. (e) $\implies$ (c) (equivalently, not (c) $\implies$ not (e))
6. (e) holds if and only if (g) holds.
7. (h) holds if and only if (f) holds.

$\boxed{8}$ (a) $\iff$ (h). $\longrightarrow$ we already know.

(b) $\implies$ (c) is obvious.

(c) $\implies$ (b): If there are n pivot positions, they must be on the diagonal.

(a) $\implies$ (d): Multiply both sides by A^{-1}.

(d) $\implies$ (c): By the Existence and Uniqueness theorem, a consistent system has a unique solution if and only if there are no free variables.

(a) $\implies$ (e): Multiply both sides by A^{-1}.

not (c) $\implies$ not (e): If A does not have n pivot positions, then the RREF of A must have at least one row which does not contain a pivot. This can only happen if it is a row of zeros.

Let A' be the RREF of A. Suppose the i-th row of A' consists of zeros. Then the equation

$$A'\mathbf{x} = \mathbf{e}_i$$

has no solution.

(e) $\implies$ (g): The columns of D are solutions of

$$A\mathbf{x} = \mathbf{e}_i, \quad i = 1, \ldots, n.$$

(g) $\implies$ (e): Conversely, if

$$\mathbf{b} = b_1\mathbf{e}_1 + \ldots + b_n\mathbf{e}_n$$

and if $D = [\mathbf{d}_1 \quad \ldots \quad \mathbf{d}_n]$, then

$$\begin{aligned} A(b_1\mathbf{d}_1 + \ldots + b_n\mathbf{d}_n) &= b_1A\mathbf{d}_1 + \ldots + b_nA\mathbf{d}_n \\ &= b_1\mathbf{e}_1 + \ldots + b_n\mathbf{e}_n \\ &= \mathbf{b}. \end{aligned}$$

$(e) \Rightarrow (g)$:

Let d_i be any one solution of the equation

$$Ax = e_i$$

where $I_n = [e_1 \cdots e_n]$.

i.e. $Ad_i = e_i \quad \forall\, i = 1, \ldots, n$.

$D = [d_1 \cdots d_n]$

$$AD = A[d_1 \; \cdots \; d_n]$$

$$= [Ad_1 \; \cdots \; Ad_n]$$

$$= [e_1 \; \cdots \; e_n]$$

$$= I. \qquad \square.$$

$(g) \Rightarrow (e)$: Suppose a matrix D exists such that $AD = I$.

Let $D = [d_1 \; \cdots \; d_n]$, where $d_i \in \mathbb{R}^n$, for $i = 1, \ldots, n$.

$e_1 = (1, 0, \ldots, 0)$

$\Rightarrow \quad A[d_1 \; \cdots \; d_n] = [e_1 \; \cdots \; e_n]$

$\Rightarrow \quad [Ad_1 \; \cdots \; Ad_n] = [e_1 \; \cdots \; e_n]$

$\Rightarrow \quad Ad_i = e_i, \quad \forall i = 1, \ldots, n.$

We are given that $b \in \mathbb{R}^n$.

Let $b = (b_1, \ldots, b_n)$.

Then $b = b_1 e_1 + b_2 e_2 + \cdots + b_n e_n$.

$$\Rightarrow b = b_1 A d_1 + b_2 A d_2 + \cdots + b_n A d_n$$

$$= (AD) b$$

$$= A (Db)$$

Let $y = Db$.

Ab

$A = [a_1 \cdots a_n]$

$b = (b_1, \ldots, b_n)$

$Ab = b_1 a_1 + \cdots + b_n a_n$

$AD = [Ad_1 \cdots Ad_n]$

$Ay = b. \Rightarrow Ax = b$ has a solution.

Alternatively, we could proceed as follows:

$$AD = I$$

$$ADb = b.$$

$$A(Db) = b.$$

$\therefore$ $y = Db$ is a solution of $Ax = b$.

$$A^T D = I \implies (A^T D)^T = I^T$$
$$\Downarrow$$
$$D^T A = I.$$

(h) $\implies$ (f): By what we have just shown, there exists a matrix D such that $A^T D = I$. Hence $D^T A = I$.

(f) $\implies$ (h): If $CA = I$ then $A^T C^T = I$. So by the equivalence of parts (a) and (g), which we have already shown, A^T is invertible.

(h) $\Rightarrow$ (f): A^T is invertible

Definition

If $\mathbf{v}_1, \ldots, \mathbf{v}_p \in \mathbb{R}^n$, then the set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ is denoted by *Span*$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ and is called the subset of $\mathbb{R}^n$ *spanned (or generated)* by $\mathbf{v}_1, \ldots, \mathbf{v}_p$. That is, *Span*$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is the collection of all vectors that can be written in the form

$$\underline{c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p}$$

with $c_1, \ldots, c_p$ scalars.

If we look at this in $\mathbb{R}^2$, linear combinations with positive weights can be thought of as the region enclosed between the rays going out from the origin in the directions of $\mathbf{v}_1$ and $\mathbf{v}_2$.

$c_1 \geq 0; \; c_2 \geq 0$
$c_1 \geq 0$
$c_2 < 0$
v_1
v_2
0
Here, v_1 & v_2 are in 1st quadrant
$c_1 v_1 + c_2 v_2$
$c_1 \leq 0$
$c_2 \geq 0$
$c_1 \leq 0$
$c_2 \leq 0$

$$V = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \qquad X = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$y \in \mathrm{Span}\{v_1, v_2, v_3\} \iff c_1 v_1 + c_2 v_2 + c_3 v_3 = y$ has a solution.

$\boxed{VX = y}$ is consistent

Example

For what value(s) of h will $\mathbf{y}$ be in Span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ if

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

$$\text{and} \quad \mathbf{y} = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$$

$h = 5$

$$\begin{bmatrix} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & h \end{bmatrix}$$

We row reduce the matrix

$$\begin{bmatrix} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & h \end{bmatrix}$$

Add row 1 to row 2 : $R_2 \to R_2 + R_1$.

$$\begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ -2 & -7 & 0 & h \end{bmatrix}$$

Add row 1 multiplied by 2 to row 3 : $R_3 \to R_3 + 2R_1$.

$$\begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & -6 & h-8 \end{bmatrix}$$

~~Subtract row 2 multiplied by 5 from row 1 : $R_1 \to R_1 - 5R_2$.~~

$$\begin{bmatrix} 1 & \cancel{0}\,5 & \cancel{7}\,-3 & -\cancel{1}\,4 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & -6 & h-8 \end{bmatrix}$$

Subtract row 2 multiplied by 3 from row 3 : $R_3 \to R_3 - 3R_2$.

$$\begin{bmatrix} 1 & 0 & 7 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & h-5 \end{bmatrix}$$

Augmented column contains a pivot $\iff h-5 \neq 0 \iff h \neq 5$

Definition

An indexed set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} \in \mathbb{R}^n$ is said to be *linearly independent* if the vector equation

$$x_1\mathbf{v}_1 + \ldots + x_p\mathbf{v}_p = 0$$

has only the trivial solution. The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is said to be *linearly dependent* if there exist weights $c_1, \ldots, c_p$, not all zero, such that

$$c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p = 0$$

linear dependence relation.

Example: Are vectors v_1, v_2, and v_3 on the previous slide independent or dependent?

Does the equation

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$$

have non trivial solutions?

Does the matrix $[v_1\ v_2\ v_3]$ have a non-pivot column?

$$\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} 1 & 5 & -3 \\ -1 & 4 & 1 \\ -2 & 7 & 0 \end{bmatrix}$$

$$\xleftrightarrow{\text{row reduction}} \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

not a pivot column.

$\{v_1, v_2, v_3\}$ are linearly dependent.

$$x_1 + 7x_3 = 0$$

$$x_2 - 2x_3 = 0.$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -7x_3 \\ 2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -7 \\ 2 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \qquad X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

$$V = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}.$$

$Vc = 0$ has a solution.

→ row reducing matrix (product of all elementary matrices used in row reduction)

$EVc = 0$ has a solution.

∴ x and c are the same. Any multiple of $(-7, 2, 1)$ gives me a linear dependence relation.

$$-7v_1 + 2v_2 + v_3 = 0$$

$$-7\begin{bmatrix}1\\-1\\-2\end{bmatrix}+2\begin{bmatrix}5\\-4\\-7\end{bmatrix}+\begin{bmatrix}-3\\1\\0\end{bmatrix}$$

$$=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

l.d. relation.