Definition

A rectangular matrix is in *echelon form (or row echelon form)* if it has the following three properties:

1. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
2. All nonzero rows are above any rows of all zeros.

A *leading entry* of a row refers to the leftmost nonzero entry (in a nonzero row).

A *nonzero row* or nonzero column in a matrix means a row or column that contains at least one nonzero entry.

If a matrix is in row echelon form then the leading entry in each nonzero row is called a *pivot*.

If a row echelon form of the augmented matrix of a linear system has a pivot in the augmented column, then it is inconsistent.

Otherwise it's consistent.

If every column other than the augmented column contains a pivot then the system has a unique solution.

If the augmented column doesn't have a pivot **and** one or more columns in the coefficient matrix don't have pivots then the system has infinitely many solutions.

more in detail format statement later

Row Reduction

$R_i \longrightarrow aR_i + bR_j$

not an elementary row op

$R_i \xrightarrow{\text{scaling}} aR_i \xrightarrow{\text{row replacement}} aR_i + bR_j$

Elementary Row Operations

There are three kinds of operations:

1. (Replacement) Replace one row by the sum of itself and a multiple of another row.
2. (Interchange) Interchange two rows.
3. (Scaling) Multiply all entries in a row by a nonzero constant.

Elementary Matrices

Applying an elementary row operation on a matrix is the same multiplying the matrix **on the left** by an *elementary matrix*.

Replacement

The operation $R_i \to R_i + cR_j$ is achieved via left multiplicaton by a matrix of the form

$$E = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & c & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix} \qquad (i < j)$$

or

$$E = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & c & & 1 & \\ & & & & & 1 \end{bmatrix} \qquad (i > j)$$

The matrix has a c as its ij-th entry and otherwise looks like the $m \times m$ identity matrix.

Similarly the operations of row interchange and row scaling can also be achieved via left multiplication by elementary matrices.

Interchange

$$E = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 0 & & 1 & \\ & & & \ddots & & \\ & & 1 & & 0 & \\ & & & & & 1 \end{bmatrix}$$

Scaling

$$E = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & c & & & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}$$

Inverses of Elementary Matrices

Elementary Matrices of all three kinds are invertible.

Replacement: $R_i \to R_i + cR_j$

For $i < j$

$$E = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & c & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}, \quad E^{-1} = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & -c & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}$$

Replacement: $R_i \to R_i + cR_j$

For $i > j$

$$E = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ & & c & & 1 & \\ & & & & & 1 \end{bmatrix}, \quad E^{-1} = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ & & -c & & 1 & \\ & & & & & 1 \end{bmatrix}$$

In terms of row operations, the *reverse* row operation is $R_i \to R_i - cR_j$.

Let's look at our example again.

Interchange

$$E = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 0 & & 1 & \\ & & & \ddots & & \\ & & 1 & & 0 & \\ & & & & & 1 \end{bmatrix} = E^{-1}$$

Not surprisingly, the reverse of the row interchange operation is also itself.

Scaling

$$E = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & c & & & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}, \quad E^{-1} = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & \frac{1}{c} & & & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}$$

$\tilde{A} = EA$

$\tilde{b} = Eb$

The system of equations

$$A\mathbf{x} = \mathbf{b}$$

and

$$EA\mathbf{x} = E\mathbf{b} \qquad \tilde{A}x = \tilde{b}$$

are equivalent systems, having the same set of solutions.

$E^{-1}\tilde{A}x = E^{-1}\tilde{b}$

Reduced Row Echelon Form

RREF

Definition

If a matrix in echelon form satisfies the following additional conditions, then it is in *reduced echelon form (or reduced row echelon form)*:

1. The leading entry in each nonzero row is 1.
2. Each leading 1 is the only nonzero entry in its column.

Example

RREF $\longrightarrow$

$$\begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

EF

Uniqueness of the Reduced Row Echelon Form (RREF)

Theorem

Each matrix is row equivalent to one and only one reduced echelon matrix.

Definition

If a matrix A is row equivalent to an echelon matrix U, we call U *an echelon form (or row echelon form)* of A; if U is in reduced echelon form, we call U *the reduced echelon form* of A.

(The proof of the theorem will not be covered in class. Interested students may look at the Proofs document posted in GC.)

Since the reduced echelon form is unique, *the leading entries are always in the same positions in any echelon form obtained from a given matrix.* These leading entries correspond to leading 1's in the reduced echelon form.

Therefore the **pivot positions are the same in any echelon form of a given matrix**.

The above fact is intuitively obvious. Students interested in a formal proof should look at the proofs document (we don't really have time to cover too many proofs in detail, as you will see for yourself).

Definition

A *pivot position* in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A. A *pivot column* is a column of A that contains a pivot position.

Example

Row reduce the matrix A below to reduced echelon form, identify the pivot positions, and locate the pivot columns of A.

$$\begin{bmatrix} 0 & -3 & -6 & 4 \\ -1 & -2 & -1 & 3 \\ -2 & -3 & 0 & 3 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} -1 & -2 & -1 & 3 \\ 0 & -3 & -6 & 4 \\ -2 & -3 & 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -2 & -1 & 3 \\ 0 & -3 & -6 & 4 \\ -2 & -3 & 0 & 3 \end{bmatrix}$$

$$\Big\downarrow \quad R_3 \to R_3 - 2R_1$$

$$\begin{bmatrix} -1 & -2 & -1 & 3 \\ 0 & -3 & -6 & 4 \\ 0 & 1 & 2 & -3 \end{bmatrix}$$

$$\Big\downarrow \quad R_2 \leftrightarrow R_3$$

$$\begin{bmatrix} -1 & -2 & -1 & 3 \\ 0 & 1 & 2 & -3 \\ 0 & -3 & -6 & 4 \end{bmatrix}$$

$$R_3 \to R_3 + 3R_2$$

$$\begin{bmatrix} -1 & -2 & -1 & 3 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & -5 \end{bmatrix}$$

$$\Big\downarrow \quad R_1 \to -R_1, \quad R_3 \to \frac{-R_3}{5}$$

$$\begin{bmatrix} 1 & 2 & 1 & -3 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Big\downarrow \quad R_1 \to R_1 - 2R_2$$

$$\begin{bmatrix} 1 & 0 & -3 & +3 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 + 2R_3$
$R_1 \to R_1 - 3R_3$

$$\longrightarrow \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

RREF

$A x = b_1 \qquad A x = b_2 \qquad A x = b_3$

$$\begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The Row Reduction Algorithm

STEP 1

Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.

Example

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

STEP 2

Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Interchange rows 1 and 3.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

STEP 3

Use row replacement operations to create zeros in all positions below the pivot.

$$\longrightarrow \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Subtract row 1 from row 2. $\underline{R_2 \longrightarrow R_2 - R_1}$

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

STEP 4

Cover (or ignore) the row containing the pivot position and cover all rows, if any, above it. Apply steps 1–3 to the submatrix that remains. Repeat the process until there are no more nonzero rows to modify.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Add -3/2 times row 2 to row 3:

$$\longrightarrow \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

STEP 5

Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by a scaling operation.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$R_2 \to R_2 - 2R_3$

$R_1 \to R_1 - 6R_3$

Create 0s in column 5:

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 2 & -4 & 4 & 0 & -14 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 2 & -4 & 4 & 0 & -14 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Scale row 2 by $1/2$:

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow{R_1 \to R_1 + 9R_2}$$

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Create 0s in column 2: $R_1 \to R_1 + 9R_2$

$$\begin{bmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Scale row 1 by $1/3$:

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

The combination of steps 1–4 is called the *forward phase* of the row reduction algorithm. Step 5, which produces the unique reduced echelon form, is called the *backward phase*.

$$\begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Vectors as Ordered Lists or n-tuples

We will temporarily use the word "vector" to refer to an ordered list of numbers.

(Informal note: The last time I taught this course I introduced the abstract definition of a vector very quickly and students were a little lost. So this time I'm postponing it a bit.)

Please review what you were taught about vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ in high school.

Definition

The set of all n-tuples of real numbers is called $\mathbb{R}^n$.

Elements of $\mathbb{R}^n$ are usually **represented** as $n \times 1$ column vectors ($n \times 1$ matrices).