Event and Event Space

- **Def 1.4 Event Space:** A set of mutually exclusive and collectively exhaustive events is an event space

$S = \{1, 2, \ldots, 6\}$

$E = \{2, 4, 6\}$

$O = \{1, 3, 5\}$

$Z = \{E, O\}$. Given Def 1.4, Z is an event space.

$Z_1 = \{\{1\}, \{2\}, \ldots, \{6\}\}$

Example – Experiment Coin Flips (In Class Exercise Submission)

- Procedure: Flip three coins
- Observation: Sequence of heads (h)/tails (t) that is obtained
- Q1) Give an example outcome
- Q2) What is the Sample space of the experiment?
- How many elements does it contain?
- Let B_i be the event when the sequence contains *i* heads
  - Q3) What range of values can *i* take?
  - Q4) Are the B_i mutually exclusive?
- Q5) Is $B = \{B_0, B_1\}$ an event space?

Why Event Space?

- They maybe easier to handle

- For the above example, sample space contains ? outcomes, while the event space contains just ? events

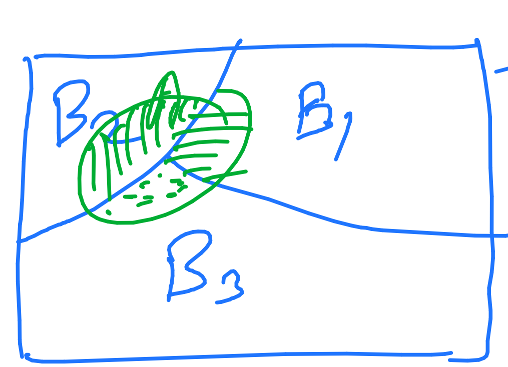
Why Event Space?

- If the sequence length is increased to 50
  - Sample space has 2^{50} outcomes
  - Event Space has just 51 events, and is a much smaller set
- Clearly the event space does not contain all the information of the sample space
  - However, we may not always be interested in all the information

Partitioning An Event into Mutually Exclusive Events

- **Theorem 1.2** $\bigcup_i B_i = S$ $B_i \cap B_j = \Phi \quad \forall i,j \quad i \neq j$

For an event space $B = \{B_1, B_2, \ldots\}$ and any event A in the sample space, let $C_i = A \cap B_i$. For $i \neq j$, the events C_i and C_j are ME and $A = C_1 \cup C_2 \cup \ldots$.



- A very useful theorem!

Experiment Tweet

- Procedure: You send a tweet over amateur radio using Morse code. Your tweet is 140 characters long.
- A character is received incorrectly with probability *p*.
- Your observation is the number of characters that were received correctly.
  - What is your sample space?

$$S = \{0, 1, 2, \ldots, 140\}$$

Experiment Tweet

- Let A_k be the event that at least k characters are received correctly
  - Do the A_k together form an event space?

- Can you think of another set of events that does?
- Express one in terms of the other…

Experiment Tweet

- Can you think of another set of events that does?

- Express one in terms of the other…

Quiz 1.2 In-class exercise

Monitor three consecutive phone calls going through a telephone switching office. Classify each one as a voice call (v) if someone is speaking, or a data call (d) if the call is carrying a modem or fax signal. Your observation is a sequence of three letters (each letter is either v or d). For example, two voice calls followed by one data call corresponds to vvd. Write the elements of the following sets:

(1) $A_1 = \{$first call is a voice call$\}$
(2) $B_1 = \{$first call is a data call$\}$
(3) $A_2 = \{$second call is a voice call$\}$
(4) $B_2 = \{$second call is a data call$\}$
(5) $A_3 = \{$all calls are the same$\}$
(6) $B_3 = \{$voice and data alternate$\}$
(7) $A_4 = \{$one or more voice calls$\}$
(8) $B_4 = \{$two or more data calls$\}$

For each pair of events A_1 and B_1, A_2 and B_2, and so on, identify whether the pair of events is either mutually exclusive or collectively exhaustive or both.

Problem 1.2.2

An integrated circuit factory has three machines X, Y, and Z. Test one integrated circuit produced by each machine. Either a circuit is acceptable (a) or it fails (f). An observation is a sequence of three test results corresponding to the circuits from machines X, Y, and Z, respectively. For example, aaf is the observation that the circuits from X and Y pass the test and the circuit from Z fails the test.

(a) What are the elements of the sample space of this experiment?

(b) What are the elements of the sets

$$Z_F = \{\text{circuit from } Z \text{ fails}\},$$
$$X_A = \{\text{circuit from } X \text{ is acceptable}\}.$$

(c) Are Z_F and X_A mutually exclusive?

(d) Are Z_F and X_A collectively exhaustive?

(e) What are the elements of the sets

$$C = \{\text{more than one circuit acceptable}\},$$
$$D = \{\text{at least two circuits fail}\}.$$

(f) Are C and D mutually exclusive?

(g) Are C and D collectively exhaustive?

Definitions of Probability

- Classical Definition

  Probability of an event A is given by

  $$P[A] = \frac{N_A}{N}$$

  - Basically, number of favorable outcomes divided by the total number of possible outcomes
    - Probability that roll of a die leads to an even number…?
  - How about an improvement to the above?
    - [redacted]
- Relative Frequency Definition

  $$P[A] = \lim_{n\to\infty} \frac{n_A}{n}$$

  - You perform *n* experiments and observe the event A n_A times to come up with the probability above.

Definitions of Probability

- Axiomatic Definition
  - A few axioms and a few definitions is all you need ☺
  - In this course we will use the Axiomatic definition
  - Once in a while we will go back to the Relative Frequency Definition

Axioms of Probability

- We started with an Experiment
- Experiments involved a procedure, observations and a model
- We started with a model by mapping observations to outcomes, sample space, events, and event space
- Now we want to associate a probability (a number, a measure) with each event in S

Let $P[A]$ denote the probability of event A

Three Axioms of Probability

A1: For any event A, $P[A] \geq 0$

A2: $P[S] = 1$

A3: For any countable collection $A_1, A_2, \ldots$ of ME events $P[A_1 \cup A_2 \cup \ldots] = P[A_1] + P[A_2] + \ldots$

- We assume that the above are true.
- All of the rest follows from these axioms
- This is also called the Axiomatic approach and is a fairly recent approach (starting early 1900(s))

Use A1, A2, A3 to show that

$$P[A_1 \cup A_2 \cup \dots \cup A_n] = P[A_1] + P[A_2] + \dots + P[A_n]$$

when $A_1, A_2, \dots, A_n$ are mutually exclusive.

$B_i = A_i$, $i = 1, 2, \dots, n$

$B_j = \Phi$, $j = n+1, \dots$

$$P[A_1 \cup A_2 \cup \dots \cup A_n] = P\left[\bigcup_{j=1}^{\infty} B_j\right]$$

$$= P[B_1] + P[B_2] + \dots + P[B_n] + P[\Phi] + P[\Phi] + \dots = P[A_1] + \dots + P[A_n] + P[\Phi] + P[\Phi] + \dots$$

$$\Phi = \underbrace{(\Phi \cap B_1)}_{\triangleq C_1} \cup \underbrace{(\Phi \cap B_2)}_{\triangleq C_2} \dots$$

let $\Phi \cap B_i = C_i$

$\Phi = C_1 \cup C_2 \cup C_3 \dots$ $\qquad C_i \cap C_j = \Phi \quad i \neq j$

$$P[\Phi] = P[(\Phi \cap B_1) \cup (\Phi \cap B_2) \cup \dots]$$

$$= \sum_{i=1}^{\infty} P[\Phi \cap B_i] = \sum_{i=1}^{\infty} P[\Phi]$$

Theorem 1.4

If the events A_i, $i = 1, 2, \ldots, m$ are ME, then $P[A_1 \cup A_2 \cup \ldots \cup A_m] = \sum_{i=1}^{m} P[A_i]$

We will use just the three axioms!

Let $B_1, B_2, \ldots$ be mutually exclusive sets. Axiom $A3$ applies to the sets B_i.

Let $B_i = A_i$ for $1 \leq i \leq m$ and for $i > m$, let $B_i = \phi$

Proof of Theorem 1.4

Therefore, we have $\cup_{i=1}^{m} A_i = \cup_{i=1}^{\infty} B_i$

Also $P[\cup_{i=1}^{m} A_i] = P[\cup_{i=1}^{\infty} B_i]$

$$= \sum_{i=1}^{m} P[A_i] + \sum_{i=m+1}^{\infty} P[\phi]$$

Show that $P[\phi] = 0$ and we are done!

Using Theorem 1.2 from basic set theory, write $\phi = (\phi \cap B_1) \cup (\phi \cap B_2) \cup \ldots$ and... we are done (How?)

Theorem 1.5

The probability of an event $B = \{s_1, s_2, \ldots, s_m\}$ is the sum of the probabilities of the outcomes contained in the event:

$$P[B] = \sum_{i=1}^{m} P[\{s_i\}].$$

Theorem 1.6

For an experiment with $S = \{s_1, s_2, \ldots, s_n\}$ in which each outcome s_i is equally likely

$P[s_i] = 1/n,\ 1 \leq i \leq n$

Proof: Simple. How?

Example – In Class Exercise

Score T is an integer between 0 and 100 corresponding to the outcomes $s_0, \ldots, s_{100}$. A score of 90 to 100 is an A and below 60 is a failing grade of F. Given that all scores between 51 and 100 are equally likely and a score of 50 or less never occurs,

find $P[\{s_{79}\}]$, $P[T \geq 90]$, $P[student\ passes]$.

Theorem 1.8

For any event A, and event space $\{B_1, B_2, \ldots, B_m\}$,

- Think Event Space…
- How are the B(s) related?
- Can I express A in terms of the event space?
- How about the P[A]?

$$P[A] = \sum_{i=1}^{m} P[A \cap B_i]$$

Quiz 1.4

Monitor a phone call. Classify the call as a voice call (V) if someone is speaking, or a data call (D) if the call is carrying a modem or fax signal. Classify the call as long (L) if the call lasts for more than three minutes; otherwise classify the call as brief (B). Based on data collected by the telephone company, we use the following probability model: $P[V] = 0.7$, $P[L] = 0.6$, $P[VL] = 0.35$. Find the following probabilities:

(1) $P[DL]$

(2) $P[D \cup L]$

(3) $P[VB]$

(4) $P[V \cup L]$

(5) $P[V \cup D]$

(6) $P[LB]$

$L = (L \cap D) \cup (L \cap V)$

$\because \{D, V\}$ is an event space

$P[L] = P[(L \cap D) \cup (L \cap V)]$

$= P[L \cap D] + P[L \cap V]$

$= P[LD] + P[LV]$