# Pairs of Random Variables

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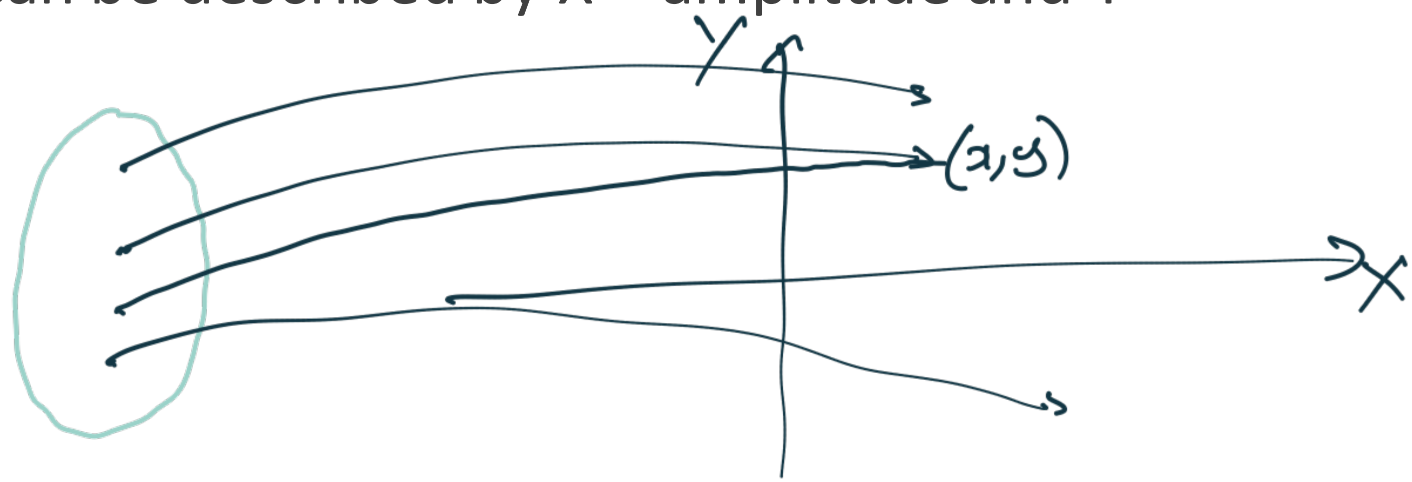
# We started with…

- An Experiment that contains a
  - Procedure, Observations, and Model
- Later we mapped outcomes to numbers
  - One or more outcomes to a point on the line
- We will now map an outcome to a pair of numbers
  - One or more outcomes to a point on a 2-D plane

# We started with…

- The numbers in the pair correspond to RVs X and Y
- For example, a transmitted sinusoid that is received with a random amplitude and random phase
    - Outcomes can be described by X = amplitude and Y = phase



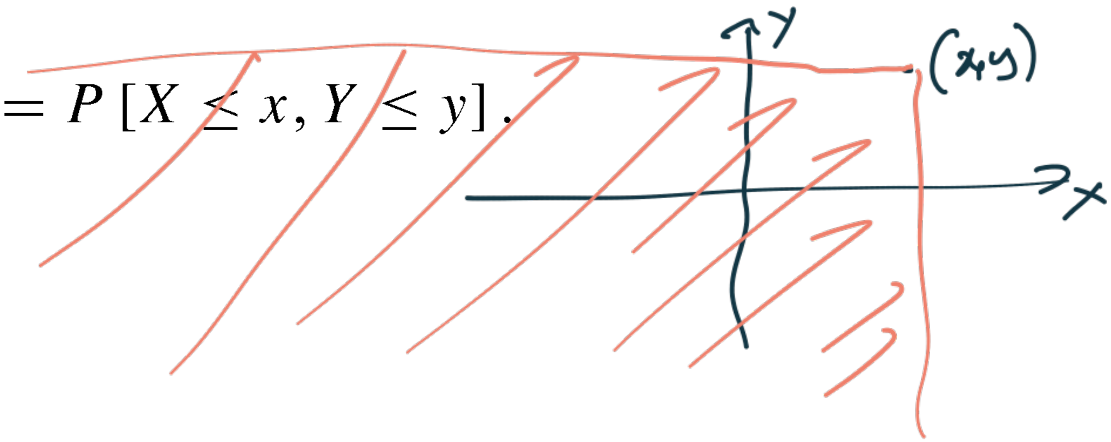
# Pair of RVs

- We defined a CDF for a single RV X
- For the pair of RVs we define a ***joint*** CDF

## Definition 4.1 Joint Cumulative Distribution Function (CDF)

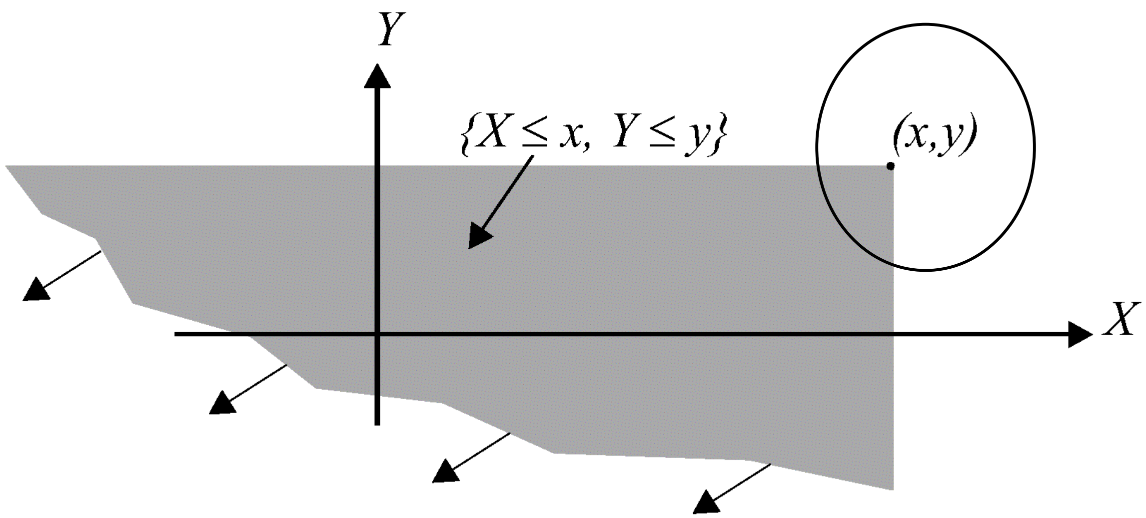
*The* joint cumulative distribution function *of random variables* $X$ *and* $Y$ *is*

$$F_{X,Y}(x, y) = P[X \le x, Y \le y].$$



# Joint CDF

## Figure 4.1




The area of the $(X, Y)$ plane corresponding to the joint cumulative distribution function $F_{X,Y}(x, y)$.

- We are interested in the probability of the intersection of the events {X<=x} and {Y<=y}

# Joint CDF

## Theorem 4.1

For any pair of random variables, $X, Y$,

(a) $\square \leq F_{X,Y}(x, y) \leq \square$

(b) $F_X(x) = \square$,

(c) $F_Y(y) = \square$,

(d) $F_{X,Y}(-\infty, y) = F_{X,Y}(x, -\infty) = \square$

(e) If $x \leq x_1$ and $y \leq y_1$, then $F_{X,Y}(x, y) \; \square \; F_{X,Y}(x_1, y_1)$,

(f) $F_{X,Y}(\infty, \infty) = \square$.

**(g) $F_{X,Y}(\infty, -\infty) = ?$**

# Theorem 4.1

For any pair of random variables, $X, Y$,

(a) $0 \leq F_{X,Y}(x, y) \leq 1$,

(b) $F_X(x) = F_{X,Y}(x, \infty)$,

(c) $F_Y(y) = F_{X,Y}(\infty, y)$,

(d) $F_{X,Y}(-\infty, y) = F_{X,Y}(x, -\infty) = 0$,

(e) If $x \leq x_1$ and $y \leq y_1$, then $F_{X,Y}(x, y) \leq F_{X,Y}(x_1, y_1)$,

(f) $F_{X,Y}(\infty, \infty) = 1$.

**(g) $F_{X,Y}(\infty, -\infty) = 0$.**

## Joint Probability Mass Function

### Definition 4.2 (PMF)

*The* joint probability mass function *of discrete random variables* $X$ *and* $Y$ *is*

$$P_{X,Y}(x, y) = P[X = x, Y = y].$$

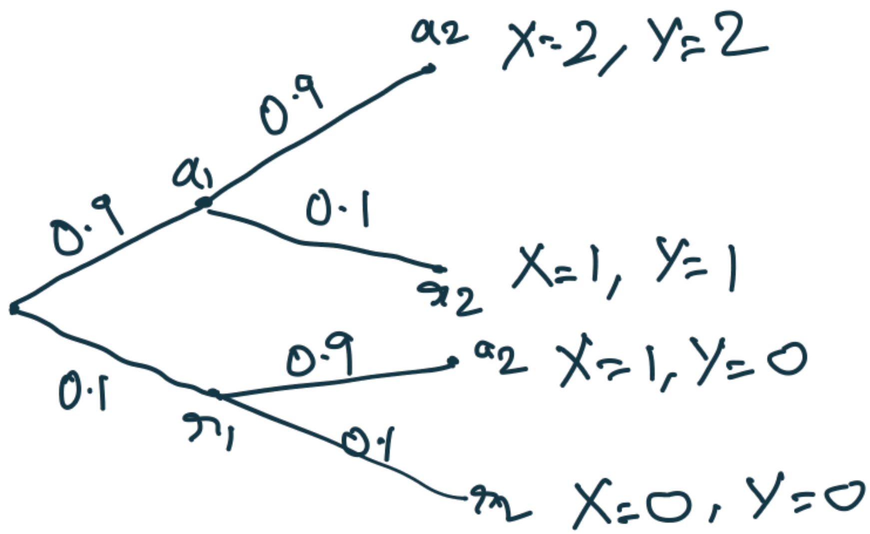
# Example of a Joint PMF

## Example 4.1 Problem

Test two integrated circuits one after the other. On each test, the possible outcomes are $a$ (accept) and $r$ (reject). Assume that all circuits are acceptable with probability 0.9 and that the outcomes of successive tests are independent. Count the number of acceptable circuits $X$ and count the number of successful tests $Y$ before you observe the first reject. (If both tests are successful, let $Y = 2$.) Draw a tree diagram for the experiment and find the joint PMF of $X$ and $Y$.

$S_X = \{0,1,2\}$

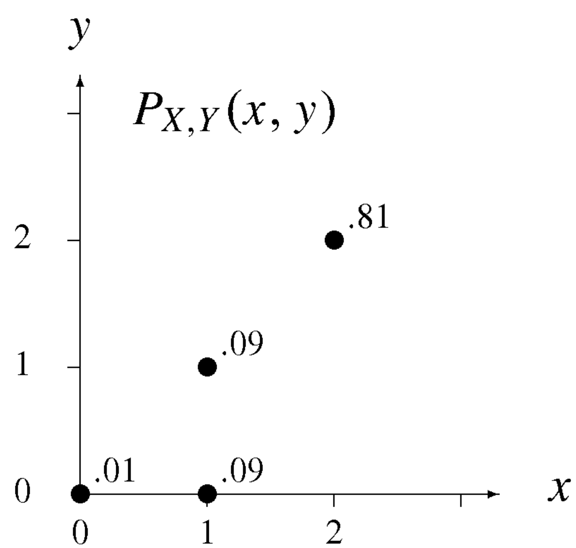
$S_Y = \{0,1,2\}$



# Example of a Joint PMF

## Three possible representations of a Joint PMF

| $P_{X,Y}(x,y)$ | $y=0$ | $y=1$ | $y=2$ |
|---|---|---|---|
| $x=0$ | 0.01 | 0 | 0 |
| $x=1$ | 0.09 | 0.09 | 0 |
| $x=2$ | 0 | 0 | 0.81 |




$$P_{X,Y}(x,y) = \begin{cases} 0.81 & x=2, y=2, \\ 0.09 & x=1, y=1, \\ 0.09 & x=1, y=0, \\ 0.01 & x=0, y=0. \\ 0 & \text{otherwise} \end{cases}$$

# Joint PMF

$$\sum_{x \in S_X} \sum_{y \in S_Y} P_{X,Y}(x, y) = ?$$

# Probability of an Event B that is in the set $S_X \boldsymbol{x} S_Y$

## Theorem 4.2

For discrete random variables $X$ and $Y$ and any set $B$ in the $X, Y$ plane, the probability of the event $\{(X, Y) \in B\}$ is

$$P[B] = \sum_{(x,y)\in B} P_{X,Y}(x, y).$$

# Quiz 4.2

The joint PMF $P_{Q,G}(q,g)$ for random variables $Q$ and $G$ is given in the following table:

| $P_{Q,G}(q,g)$ | $g=0$ | $g=1$ | $g=2$ | $g=3$ |
|---|---|---|---|---|
| $q=0$ | 0.06 | 0.18 | 0.24 | 0.12 |
| $q=1$ | 0.04 | 0.12 | 0.16 | 0.08 |

(4.12)

Calculate the following probabilities:

(1) $P[Q=0]$

(2) $P[Q=G]$ $= P[Q=0, G=0] + P[Q=1, G=1]$

(3) $P[G>1]$ $= P[Q=0, G>1] + P[Q=1, G>1] = P[G=2] + P[G=3]$

(4) $P[G>Q]$

$\{G>Q\} = \{(0,1), (0,2), (0,3), (1,2), (1,3)\}$

# Marginal PMF

- For discrete RVs X and Y with **joint** PMF $P_{X,Y}(x,y)$, $P_X(x)$ and $P_Y(y)$ are defined as the **marginal** PMFs of X and Y respectively

$$P_X(x) = \sum_{y \in S_Y} P_{X,Y}(x,y)$$

Suppose we have the joint $P_{X,Y,Z}(x,y,z) = P[X=x, Y=y, Z=z]$

$$P_X(x) = \sum_{z \in S_Z} \sum_{y \in S_Y} P_{X,Y,Z}(x,y,z)$$

# Quiz 4.3

The probability mass function $P_{H,B}(h,b)$ for the two random variables $H$ and $B$ is given in the following table. Find the marginal PMFs $P_H(h)$ and $P_B(b)$.

| $P_{H,B}(h,b)$ | $b=0$ | $b=2$ | $b=4$ |
|---|---|---|---|
| $h=-1$ | 0 | 0.4 | 0.2 |
| $h=0$ | 0.1 | 0 | 0.1 |
| $h=1$ | 0.1 | 0.1 | 0 |

(4.20)

# Marginal PMF

## Theorem 4.3

For discrete random variables $X$ and $Y$ with joint PMF $P_{X,Y}(x, y)$,

$$P_X(x) = \sum_{y \in S_Y} P_{X,Y}(x, y), \qquad P_Y(y) = \sum_{x \in S_X} P_{X,Y}(x, y).$$

# Marginal PMF

## Theorem 4.3

For discrete random variables $X$ and $Y$ with joint PMF $P_{X,Y}(x, y)$,

$$P_X(x) = \sum_{y \in S_Y} P_{X,Y}(x, y), \qquad P_Y(y) = \sum_{x \in S_X} P_{X,Y}(x, y).$$

- We looked at the joint CDF and the joint PMF. What do you expect next?

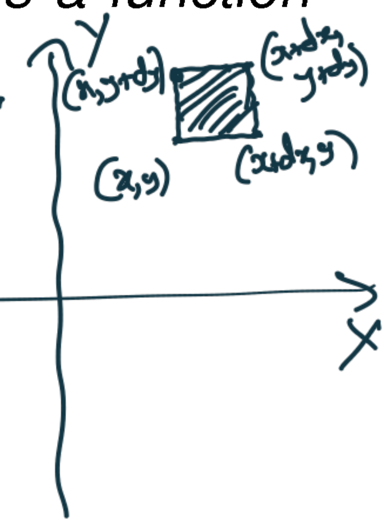
# Joint PDF

## Joint Probability Density

## Definition 4.3 Function (PDF)

*The joint PDF of the continuous random variables* $X$ *and* $Y$ *is a function* $f_{X,Y}(x, y)$ *with the property*

$$F_{X,Y}(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(u, v)\, dv\, du.$$




$$P[\, x \le X < x+dx,\ y \le Y < y+dy \,]$$

$$\approx f_{X,Y}(x,y)\, dx\, dy.$$


$$P[\, x < X \le x+dx \,] \approx f_X(x)\, dx$$

# Joint PDF

## Theorem 4.4

$$f_{X,Y}(x, y) = \frac{\partial^2 F_{X,Y}(x, y)}{\partial x \, \partial y}$$

$$z = g(x,y) = x^3 + 3x^2y^2 + y^3 + y^4 + 10$$

$$\frac{\partial^2 g(x,y)}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y} g(x,y)\right]$$

$$= \frac{\partial}{\partial x}\left[6x^2y + 3y^2 + 4y^3\right]$$

$$= 12xy + 0 + 0$$

$$= 12xy$$

# Properties of a Joint PDF

## Theorem 4.6

A joint PDF $f_{X,Y}(x, y)$ has the following properties corresponding to first and second axioms of probability (see Section 1.3):

$P[X \leq \infty, Y \leq \infty] = 1$

$\Rightarrow \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx\, dy = 1$