Evariste Session on Proving Methods

Evariste, the math club of IIITD, is hosting a session on Proofs and Proving Methods.

Timing and Zoom Link

The session is on 10th Feb, Thursday at 6:00pm.

The link for the same is:

https://iiitd-ac-in.zoom.us/j/91848888555?pwd=and3bVpyS1hsSC8yeC9SQjJTeXdOZz09

Meeting ID: 918 4888 8555
Passcode: 565410

Lemma

Let A be an $m \times n$ matrix in reduced echelon form, having k pivot columns, where $1 \leq k \leq m$. Then $\underbrace{\{\mathbf{e}_1, \ldots, \mathbf{e}_k\}}$ is a basis for Col A.

first k columns of I_m

Lemma

Let A be an $m \times n$ matrix and let A' be a matrix obtained by performing a row operation on A. Any linear dependence relation which holds between the columns of A also holds between the corresponding columns of A', and vice versa.

The columns of A are linearly independent if and only if the columns of A' are linearly independent.

$$A' = \underbrace{E_p E_{p-1} \cdots E_1}_{E} A$$

Theorem

Let A be an $m \times n$ matrix. The pivot columns of A form a basis for Col A.

First proof: This is an obvious conclusion of the two preceding lemmas.

Second proof: Let A' be the rref of A. Then there exists an invertible matrix E such that $A' = EA$.

Let $A = [a_1 \cdots a_n]$

Suppose A has k pivot columns. Let $\underbrace{a_{i_1}, a_{i_2}, \ldots, a_{i_k}}$ be the pivot columns of A,

where $1 \le \underline{i_1} < i_2 < \cdots < i_k \le n$

$[\cdots e_1 \cdots]$

$\underline{A'} = EA = [\underline{Ea_1}, \underline{Ea_2}\ Ea_{i_1} \cdots \underline{Ea_n}]$

$\underline{e_1} = Ea_{i_1}, \; e_2 = Ea_{i_2} \cdots e_k = Ea_{i_k}$

Let $b \in \operatorname{col} A$.

Then $\exists$ scalars $c_1, \ldots, c_n$

such that

$$b = c_1 a_1 + c_2 a_2 + \cdots + c_n a_n$$

$$\therefore Eb = \underbrace{c_1 Ea_1 + c_2 Ea_2 + \cdots + c_n Ea_n}$$

$Eb \in \operatorname{col} A'$.

$\therefore$ There exist scalars

$d_1, \ldots, d_k \in \mathbb{R}$ such that

$$Eb = d_1 e_1 + \cdots + d_k e_k.$$

$$\Rightarrow b = d_1 E^{-1} e_1 + \cdots + d_k E^{-1} e_k.$$

$$= d_1 a_{i_1} + \cdots + d_k a_{i_k}$$

$$\in \operatorname{span}\{a_{i_1}, \ldots, a_{i_k}\}$$

$\mathrm{Col}\, A \subset \mathrm{Span}\{a_{i_1}, \ldots, a_{i_k}\}$

Clearly $\{a_{i_1}, \ldots, a_{i_k}\} \subset \mathrm{Col}\, A$

$\Rightarrow \mathrm{Span}\{a_{i_1}, \ldots, a_{i_k}\} \subset \mathrm{Col}\, A$

$\therefore \mathrm{Col}\, A = \mathrm{Span}\{a_{i_1}, \ldots, a_{i_k}\}.$

Next we show $a_{i_1}, \dots, a_{i_k}$ are l.i.

So suppose $\alpha_1 a_{i_1} + \cdots + \alpha_k a_{i_k} = 0$ for some $\alpha_1, \dots, \alpha_k \in \mathbb{R}$.

Then $\alpha_1 E a_{i_1} + \cdots + \alpha_k E a_{i_k} = 0$

$\Rightarrow \alpha_1 e_1 + \cdots + \alpha_1 e_k = 0$.

$$\Rightarrow \quad (\alpha_1, \ldots, \alpha_k, \underbrace{0 \cdots 0}_{m-k \text{ times}}) = 0.$$

$$\Rightarrow \quad \alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$$

$\Rightarrow$ $a_{i_1}, \ldots a_{i_k}$ are l.i.

$\square$

Theorem 4, §1.4 (course textbook)

Theorem

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or they are all false.

- **a** *For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.*
- **b** *Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.*
- **c** *The columns of A span $\mathbb{R}^m$.* $\longrightarrow$ col $A = \mathbb{R}^m$
- **d** *A has a pivot position in every row.*

The equivalence of (a), (b) and (c) is obvious.

We show the equivalence of (c) and (d).

Claim

Idea: The columns of A span $\mathbb{R}^m$ if and only if the columns of the RREF of A span $\mathbb{R}^m$.

Proof of claim:

Let A' be the RREF of A.

Then $\exists$ an invertible matrix E such that $A' = EA$.

First we show that $\operatorname{col} A = \mathbb{R}^m$

$\Rightarrow \operatorname{col} A' = \mathbb{R}^m$

Let $A = [a_1 \; - \; - \; - \; a_n]$

Let $b \in \mathbb{R}^m \Rightarrow E^{-1} b \in \mathbb{R}^m = \text{col} A$

$\therefore \exists \, c_1, \ldots, c_n \in \mathbb{R}$ such that

Then $E^{-1} b = c_1 a_1 + \cdots + c_n a_n$

$\therefore \quad b = c_1 E a_1 + \cdots + c_n E a_n$

$\Rightarrow b \in \text{col} A' \Rightarrow \mathbb{R}^m \subset \text{col} A'$

$\Rightarrow \underline{\text{col} A' = \mathbb{R}^m}.$

Conversely, let $b \in \mathbb{R}^m$

Since $\operatorname{col} A' = \mathbb{R}^m$,

$\exists$ scalars $d_1, \ldots, d_n$ such that

$\Rightarrow Eb = d_1 Ea_1 + \ldots + d_n Ea_n$

$\in \operatorname{col} A' = \mathbb{R}^m$.

$$\Rightarrow \quad b = d_1 a_1 + \cdots + d_n a_n$$

$$\Rightarrow b \in \text{col } A. \Rightarrow \mathbb{R}^m \subset \text{col} A$$

$$\Rightarrow \text{col } A = \mathbb{R}^m.$$

$(c) \Rightarrow (d)$: $\text{col } A = \mathbb{R}^m$

$$\Rightarrow \text{col } A' = \mathbb{R}^m$$

$\Rightarrow$ The pivot columns of A' span $\mathbb{R}^m$.

$\Rightarrow \mathbb{R}^m = \text{span}\{e_1, \dots, e_k\}$

where $e_1, \dots, e_k$ are the pivot columns of A'.

$\Rightarrow k = m \Rightarrow$ There is a pivot in every row

(d) $\Rightarrow$ (c): Let k be the number of pivot of A. If there is a pivot in every row,

then $k = m$

$\Rightarrow \{e_1, \dots, e_m\}$ is a basis of $\operatorname{col} A'$

$\Rightarrow \operatorname{col} A' = \mathbb{R}^m$

$\Rightarrow \operatorname{col} A = \mathbb{R}^m$.

Invertible Matrix Theorem (more parts)

Let A be an $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a. A is an invertible matrix.

b. A is row equivalent to the n×n identity matrix.

c. A has n pivot positions.

d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

e. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

f. There is an $n \times n$ matrix C such that $CA = I$.

g. There is an $n \times n$ matrix D such that $AD = I$.

h. A^T is an invertible matrix.

i. The columns of A form a linearly independent set.

j. The columns of A span $\mathbb{R}^n$.

Proof - continued from earlier

Conditions (d) and (i) are equivalent.

Conditions (e) and (j) are equivalent.

Definition

Let V be a vector space. Let S be an infinite subset of V. We say S is a *linearly independent* set if every finite subset of S is linearly independent.

Proposition

Let V be a vector space, and let S be a linearly independent subset of V. Any subset of S is linearly independent.

What is the contrapositive?

Proof: Let $S = \{v_1, \ldots, v_p\}$ be a linearly independent subset of V.

Let $\{v_{i_1}, \ldots, v_{i_k}\}$ be any subset of S, where $k \leq p$ and $1 \leq i_1 < i_2 < \cdots < i_k \leq p$.

Eg $\{v_1, v_2, v_3\}$ $c_1 = 0$ $\qquad c_2 v_2 + c_3 v_3 = 0$

$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$

Suppose

$$c_{i_1} v_{i_1} + c_{i_2} v_{i_2} + \cdots + c_{i_k} v_{i_p} = 0$$

for some scalars $c_{i_1}, \ldots, c_{i_k} \in \mathbb{R}$.

put $c_j = 0$, $\forall j \in \{1, \ldots, p\} \setminus \{i_1, \ldots, i_k\}$

$$\Rightarrow c_1 v_1 + c_2 v_2 + \cdots + c_p v_p = 0$$

$$\Rightarrow \quad c_1 = c_2 = \cdots = c_p = 0$$

$\square$.