Experiments, Models and Probabilities (Chapter 1 RY)

The fun begins!

Many slides courtesy RY (2nd Ed) instructor material

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Defining Our Universe of Interest

- Our universe of interest has to do with our experiment
- What was it for the coin tossing experiment we performed?
- Set theory provides the formalism we need to capture our universe of interest and its inhabitants

Set Theory

- A set is a collection of objects
- A = {Acushla, Sanjit}
  - A is the set of faculty members teaching probability and statistics in IIITD in 2022
- We will use **capital letters to denote sets**

Set Theory

- Various ways of defining a set

$$A = \{x | x \text{ is an Integer}\}$$

$$B = \{y | y\%2 = 0, \text{ y is an Integer}\}$$

$$B = \{y \text{ is an Integer} | y\%2 = 0\}$$

$$C = \{x^2 | x = 1, 2, 3\}$$

$$D = \{x^2 | x = 1, 2, 3, 4, \ldots\}$$

Set Theory

- Sets can be finite or infinite

- Sets can be finite (and hence also countable) and countably infinite
  - Set of days in a week
  - Set of students in MTH 201
  - Set of integers
  - Set of values that an image pixel may take

  - Set of rational numbers...? Yes/No? Why?

Set Theory

- Sets can be finite or infinite

- Sets can be a continuum
  - Set of all real numbers in the interval (0,1)
  - Set of temperatures
  - Set of heights of humans

Set Theory: Definitions and Notation

$A \subset B$ says that A is a subset of B.
If $A \subset B$, then by definition all members of A are also members of B.

$A \supset B$ implies that A is a superset of B.

Two sets A and B are equal, that is $A = B$, *if and only if* $A \subset B$ and $B \subset A$

Set Theory

- Universal Set S: It contains *everything*
  - All sets are a subset of S

- Null Set ϕ
  - It contains nothing
  - It is a subset of every set

$$\text{For any set } A,\ \phi \subset A$$

Set Theory

- To say that an element *a* belongs to (is a member of) the set A we write

$$a \in A$$

- We can also say

$$\{a\} \subset A$$

Set Operations

Union: $A \cup B$, $A \cup B \cup C$, $\cup_{i=1}^{\infty} A_i$

$\rightarrow x \in A \cup B$ iff $x \in A$ or $x \in B$

if $x \in A \cup B$, then $x \in A$ or $x \in B$

If $x \in A$ or $x \in B$, then $x \in A \cup B$

Set Operations

Union: $A \cup B$, $A \cup B \cup C$, $\cup_{i=1}^{\infty} A_i$

$\rightarrow x \in A \cup B$ iff $x \in A$ or $x \in B$

Intersection: $\cap_{i=1}^{\infty} A_i$

$\rightarrow x \in A \cap B$ iff $x \in A$ and $x \in B$

Complement of A is denoted as A^c

$\rightarrow x \in A^c$ iff $x \notin A$

Difference:

$\rightarrow x \in A - B$ iff $x \in A$ and $x \notin B$

Venn Diagrams

Set Theory

- **Mutually Exclusive Sets**

Sets A_i, $i = 1, 2, \ldots, n$ are mutually exclusive iff $A_i \cap A_j = \phi$ for $i \neq j$

Set Theory

- **Collectively Exhaustive Sets**

Sets A_i, $i = 1, 2, \ldots, n$ are collectively exhaustive iff $\cup_{i=1}^{n} A_i = S$

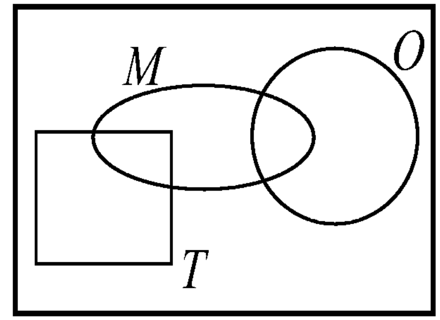
Theorem 1.1

De Morgan's law relates all three basic operations:

$$(A \cup B)^c = A^c \cap B^c.$$

Quiz 1.1 **In-Class Submission**

A pizza at Gerlanda's is either regular (R) or Tuscan (T). In addition, each slice may have mushrooms (M) or onions (O) as described by the Venn diagram at right. For the sets specified below, shade the corresponding region of the Venn diagram.



(1) R

(2) $M \cup O$

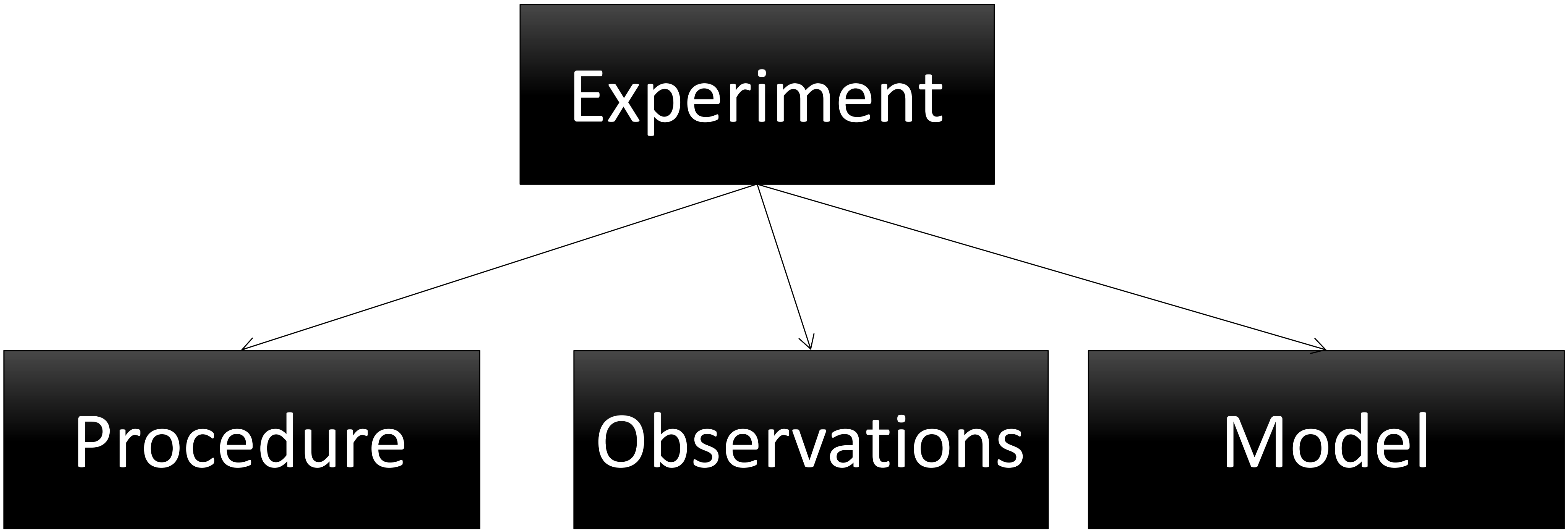
(3) $M \cap O$

(4) $R \cup M$

(5) $R \cap M$

(6) $T^c - M$

The Story That Must Not Be Forgotten



Examples Of Experiments

- Experiment consists of a repeatable procedure
  - Repeats may lead to different observations

- Example Procedures
  - Toss a coin
  - Toss a coin 10 times

- Example Observations
  - Number of heads
  - Sequence of heads and tails
  - Largest number of consecutive heads in the sequence

Outcome

- **Def 1.1:** Outcome
  - An outcome of an experiment is any observation of that experiment

- Outcomes are, by definition, mutually exclusive

Outcome

- Heads, Tails are the possible outcomes of a coin tossing experiment
  - They are mutually exclusive, cannot happen at the same time

Sample Space

- **Def 1.2:** The sample space S of an experiment is the
  - finest-grain,
  - mutually exclusive,
  - collectively exhaustive set of all possible outcomes

- Every element of the sample space is an outcome
  - Each of the outcomes are atomic in nature
    - Cannot be split into more than one outcome
  - The outcomes are also mutually exclusive
  - No outcomes are left out of the sample space

Sample Space

- By default, S is your universe of interest. It is a given.

- *Everything is conditioned on S (super set) having happened*
  - *What's the uncertainty about?*

$\{0\}, \{1\}, \ldots, \{10\}$

$\{0\} \subset S$

$S \supset \{0\}$

- In the ten coin toss experiment $S = \{0,1,2,\ldots,10\}$

Sample Space

- You want to observe the gender sequence obtained after two people have walked past you
- We want to come up with the sample space

- Possible observations are …? Note that we are looking for outcomes

$S = \{MM, FF, FM, MF\}$

Sample Space

- Each one of the above is also an outcome
  - Note MM for your stated observation cannot be split into smaller observations
  - Also, the outcomes are mutually exclusive. Only one can happen at any given time

- S = {MM, FF, FM, MF}

Event and Event Space

- What if I am interested in whether a man was spotted among the first two people?
- No individual outcome is sufficient

Event and Event Space

- **Def 1.3 Event:** An Event is a set of outcomes

- For our example one event of interest, call it E, is
  - E = {MM, MF, FM}

- Note that we say that event E occurs when any of the outcomes MM, MF and FM occur.
  - Think in terms of the definition of union of sets

Event and Event Space

- **Def 1.4 Event Space:** A set of mutually exclusive and collectively exhaustive events is an event space

$S = \{1, 2, \ldots, 6\}$ $E = \{2, 4, 6\}$

$O = \{1, 3, 5\}$

$Z = \{E, O\}$. Given Def 1.4, Z is an event space.

$Z_1 = \{\{1\}, \{2\}, \ldots, \{6\}\}$

Example – Experiment Coin Flips (In Class Exercise Submission)

- Procedure: Flip three coins
- Observation: Sequence of heads (h)/tails (t) that is obtained
- Q1) Give an example outcome
- Q2) What is the Sample space of the experiment?
- How many elements does it contain?
- Let B_i be the event when the sequence contains *i* heads
  - Q3) What range of values can *i* take?
  - Q4) Are the B_i mutually exclusive?
- Q5) Is $B = \{B_0, B_1\}$ an event space?

Why Event Space?

- They maybe easier to handle

- For the above example, sample space contains ? outcomes, while the event space contains just ? events

Why Event Space?

- If the sequence length is increased to 50
  - Sample space has 2^{50} outcomes
  - Event Space has just 51 events, and is a much smaller set
- Clearly the event space does not contain all the information of the sample space
  - However, we may not always be interested in all the information

Partitioning An Event into Mutually Exclusive Events

- **Theorem 1.2**

For an event space $B = \{B_1, B_2, \ldots\}$ and any event A in the sample space, let $C_i = A \cap B_i$. For $i \neq j$, the events C_i and C_j are ME and $A = C_1 \cup C_2 \cup \ldots$.

- A very useful theorem!