Proposition

Every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation. In other words, there exists a unique $m \times n$ matrix A such that

$$T(\mathbf{x}) = A\mathbf{x}, \quad \forall \mathbf{x} \in \mathbb{R}^n$$

A is called the *standard matrix for the linear transformation* T.

The Change-of-coordinates in $\mathbb{R}^n$

Definition

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be an ordered basis of $\mathbb{R}^n$. The matrix

$$P_{\mathcal{B}} = [\mathbf{b}_1 \quad \ldots \quad \mathbf{b}_n]$$

formed using the basis vectors $\mathbf{b}_1, \ldots, \mathbf{b}_n$ as columns, is called the *change-of-coordinates* matrix from $\mathcal{B}$ to the standard basis in $\mathbb{R}^n$.

The change-of-coordinates matrix takes a coordinate vector with respect to the $\mathcal{B}$ basis and tranforms it to standard coordinates. So if $\mathbf{x}$ is a vector in $\mathbb{R}^n$, then

$$\mathbf{x} = P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}.$$

Proposition

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be an ordered basis of $\mathbb{R}^n$. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the coordinate transformation which sends

$$\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}.$$

The change-of-coordinates matrix $P_{\mathcal{B}}$ is the standard matrix of the inverse T^{-1} of the coordinate transformation.

The standard matrix of the coordinate transformation T is $P_{\mathcal{B}}^{-1}$.

Particular Case: The $\mathcal{B}$-matrix

Definition

Let $T : V \to V$ be a linear transformation from a vector space to itself. Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be an ordered basis for V. There is a unique matrix $[T]_\mathcal{B}$, which we call the $\mathcal{B}$-matrix of T such that

$$[T(v)]_\mathcal{B} = [T]_\mathcal{B}[v]_\mathcal{B}, \quad \forall v \in V.$$

Further, $[T]_\mathcal{B}$ is obtained using the formula

$$[T]_\mathcal{B} = [[T(v_1)]_\mathcal{B} \quad \ldots \quad [T(v_n)]_\mathcal{B}]$$

Proposition

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Let A be the standard matrix of T. Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be any ordered basis of $\mathbb{R}^n$. Let

$$P_{\mathcal{B}} = [\mathbf{b}_1 \quad \cdots \quad \mathbf{b}_n]$$

be the change-of-coordinates matrix from $\mathcal{B}$ to the standard basis in $\mathbb{R}^n$. Then the $\mathcal{B}$-matrix of T is $P_{\mathcal{B}}^{-1} A P_{\mathcal{B}}$.

Since

$$[T]_{\mathcal{B}} = P_{\mathcal{B}}^{-1} A P_{\mathcal{B}}$$

we also obtain the formula

$$\text{Standard matrix of } T = P_{\mathcal{B}} [T]_{\mathcal{B}} P_{\mathcal{B}}^{-1}$$

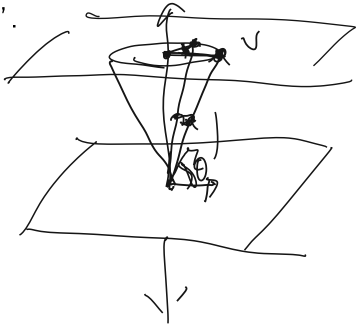
Another Application: 3D Rotations

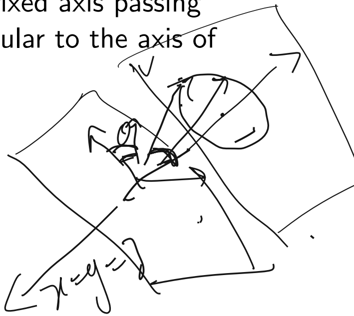
$$T(x, y, z) = ?$$

$$\mathbb{R}^3 \to \mathbb{R}^3$$

$$Tx = Ax$$

We rotate vectors by a fixed angle about a fixed axis passing through the origin, “within planes perpendicular to the axis of rotation”.





Specific example:

Axis of rotation is the line $x = y = z$. Fixed angle is θ.

$\rightarrow v_1 = \{(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})\}$

$T(v_1) = v_1$

$v_2 = \{(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)\}$

$v_3 = \{(-\frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}})\}$

$\beta = \{v_1, v_2, v_3\}$

$$T(v_1) = 1 \cdot v_1 + 0 \cdot v_2 + 0 \cdot v_3$$

$$[T(v_1)]_\beta = (1, 0, 0)$$

Rotate v_2 by an angle of θ.

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Claim:

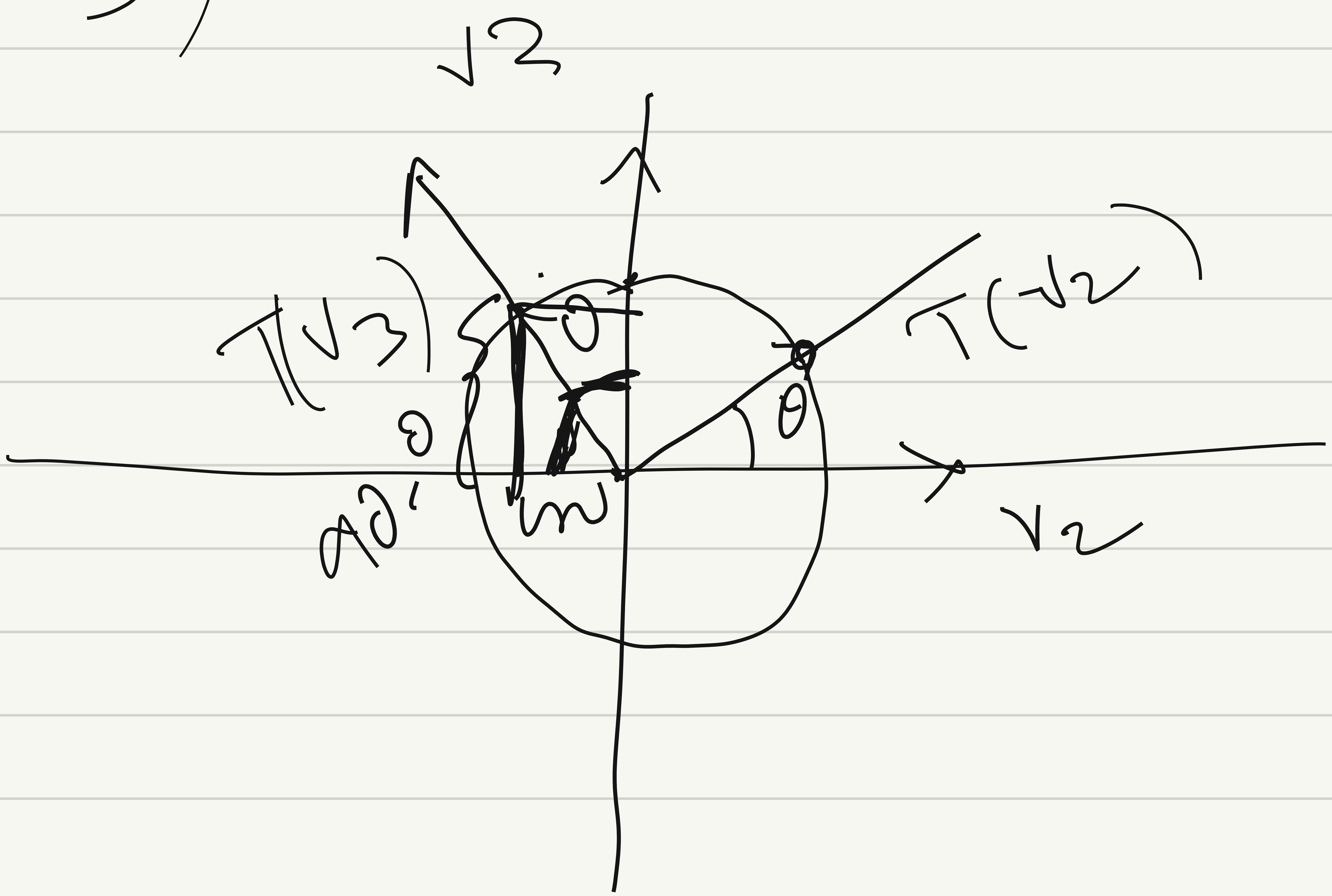
$$T(v_2) = \underbrace{\cos\theta\, v_2 + \sin\theta\, v_3}_{}$$

$0 \cdot v_1$

$T(v_2) = c_1 v_1 + c_2 v_2 + c_3 v_3$

$[T(v_2)]_\beta = (c_1, c_2, c_3)$

$$T(v_3) = -\sin\theta\, v_2 + \cos\theta\, v_3$$



$1 = \|v_2\| = \|v_3\|$

$1 = \|T(v_2)\|$

$1 = \|T(v_3)\|$

$$[T]_\beta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

standard matrix of T

$$A = P_\beta [T]_\beta P_\beta^{-1}$$

$$(x, y, z) = x\hat{i} + y\hat{j} + z\hat{k}$$

$$T(x\hat{i} + y\hat{j} + z\hat{k}) = ?,$$

$$A = \underbrace{\begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -1/\sqrt{2} & \frac{1}{\sqrt{2}} & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & \frac{2}{\sqrt{6}} \end{bmatrix}}_{\text{Find out for yourself.}}$$

$$T(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= A(x\hat{i} + y\hat{j} + z\hat{k})$$

Matrix of a Linear Transformation

Proposition (M)

Let V, W be vector spaces. Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be an ordered basis for V and $\mathcal{C} = \{w_1, \ldots, w_m\}$ be an ordered basis for W. Let $T : V \to W$ be a linear transformation. There exists a unique $m \times n$ matrix A such that

$$[T(v)]_{\mathcal{C}} = A[v]_{\mathcal{B}}, \quad \text{for every } v \in V.$$

Further, we have

$$A = [[T(v_1)]_{\mathcal{C}} \quad \ldots \quad [T(v_n)]_{\mathcal{C}}]$$

This unique matrix A is called is called the *matrix of T* with respect to the bases $\mathcal{B}$ and $\mathcal{C}$, and is denoted by $[T]_{\mathcal{B},\mathcal{C}}$.

Proposition

Let U, V and W be vector spaces. Let $T : U \to V$ and $S : V \to W$ be linear transformations. Then the composite $S \circ T : U \to W$ is also a linear transformation.

Let us look at some examples of how to find this matrix, when working with matrices and/or polynomials.

$V = \mathbb{P}_n \longrightarrow$ all polynomials of degree $\leq n$.

$W = \mathbb{P}_{n-1} \longrightarrow$ all polynomials of degree $\leq n-1$

$T: V \to W$

is defined by

$$T(f(t)) = f'(t) = \frac{df}{dt}$$

$f, g \in \mathbb{P}_n$

$$(i) \quad T(f+g) = \frac{d}{dt}(f+g) = \frac{df}{dt} + \frac{dg}{dt}$$

$$= Tf + Tg.$$

$c \in \mathbb{R}, f \in \mathbb{P}_n$

$$(ii) \quad T(cf) = \frac{d}{dt}(cf) = c\frac{df}{dt} = cT(f)$$

Fix a basis for $\mathbb{P}_n$.

$$\beta = \underbrace{\{\overset{v_1}{1}, \overset{v_2}{t}, t^2, t^3, t^4 \ldots, \overset{v_{n+1}}{t^n}\}}_{n+1 \text{ vectors.}}$$

Fix a basis for $\mathbb{P}_{n-1}$.

$$\gamma = \{\overset{w_1}{1}, t, \ldots, \overset{w_n}{t^{n-1}}\}$$

$$[T]_{\beta,e} = \begin{bmatrix} [T(v_1)]_e & \cdots & [T(v_{n+1})]_e \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & 0 & \cdots & \vdots \\ \vdots & \vdots & 0 & 3 & 0 & & \vdots \\ \vdots & \vdots & \vdots & \vdots & 4 & & \vdots \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \cdots & n \end{bmatrix}$$

$T(v_{n+1})$

$= nt^{n-1}$

$T(v_1) = 0 \quad T(v_2) = 1, \quad T(v_2) = 2t, \quad T(v_3) = 3t^2$

Submission 5 :

$\cos^2 x - \sin^2 x = \cos 2x$

$\in \operatorname{Span}\{\sin^2 x, \cos^2 x\}$

$\sin^2 x + \cos^2 x = 1$

$1 \in \operatorname{Span}\{\sin^2 x, \cos^2 x\}$

$$W = \operatorname{Span}\{\sin^2 x, \cos^2 x, 1, \cos 2x\}$$

$$= \underline{\operatorname{Span}\{\sin^2 x, \cos^2 x\}}$$

$$\dim W = 2.$$

$$\| \operatorname{Span}\{1, \sin^2 x\}$$