#### **Definition**

A rectangular matrix is in *echelon form (or row echelon form)* if it has the following three properties:

- **I** Each leading entry of a row is in a column to the right of the leading entry of the row above it.
- 2 All nonzero rows are above any rows of all zeros.

A *leading entry* of a row refers to the leftmost nonzero entry (in a nonzero row).

A *nonzero row* or nonzero column in a matrix means a row or column that contains at least one nonzero entry.

**If a matrix is in row echelon form** then the leading entry in each nonzero row is called a *pivot*.

If a row echelon form of the augmented matrix of a linear system has a pivot in the augmented column, then it is inconsistent.

Otherwise it's consistent.

If every column other than the augmented column contains a pivot then the system has a unique solution.

If the augmented column doesn't have a pivot **and** one or more columns in the coefficient matrix don't have pivots then the system has infinitely many solutions.

more in detail

formal statement

Row Reduction

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## **Elementary Row Operations**

There are three kinds of operations:

- (Replacement) Replace one row by the sum of itself and a multiple of another row.
- 2 (Interchange) Interchange two rows.
- (Scaling) Multiply all entries in a row by a nonzero constant.

# **Elementary Matrices**

Applying an elementary row operation on a matrix is the same multiplying the matrix **on the left** by an *elementary matrix*.

## Replacement

The operation  $R_i \rightarrow R_i + cR_j$  is achieved via left multiplication by a matrix of the form

or

The matrix has a c as its ij-th entry and otherwise looks like the  $m \times m$  identity matrix.

Similarly the operations of row interchange and row scaling can also be achieved via left multiplication by elementary matrices.

## Interchange

# Scaling

$$E = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & \\ & & c & & & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & 1 \end{bmatrix}$$

# **Inverses of Elementary Matrices**

Elementary Matrices of all three kinds are invertible.

# **Replacement:** $R_i \rightarrow R_i + cR_j$ For i < j

$$-1 = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & -c & \\ & & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}$$

# **Replacement:** $R_i \rightarrow R_i + cR_j$

For 
$$i > j$$

$$E = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & c & & 1 & \\ & & & & 1 \end{bmatrix}, \quad E^{-1} = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & -c & & 1 & \\ & & & & 1 \end{bmatrix}$$

In terms of row operations, the *reverse* row operation is  $R_i \rightarrow R_i - cR_i$ .

Let's look at our example again.

## Interchange

Not surprisingly, the reverse of the row interchange operation is also itself.

# Scaling

$$E = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & c & & & \\ & & & \ddots & & \\ & & & 1 & \\ & & & 1 \end{bmatrix}, \quad E^{-1} = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & \\ & & \frac{1}{c} & & & \\ & & & \ddots & & \\ & & & 1 & \\ & & & 1 \end{bmatrix}$$

The system of equations
$$A\mathbf{x} = \mathbf{E}\mathbf{A}$$
and
$$EA\mathbf{x} = \mathbf{E}\mathbf{b}$$

$$A\mathbf{x} = \mathbf{b}$$

# Reduced Row Echelon Form

RRET

## **Definition**

If a matrix in echelon form satisfies the following additional conditions, then it is in *reduced echelon form* (or reduced row echelon form):

- $\int$  **I** The leading entry in each nonzero row is 1.
  - **2** Each leading 1 is the only nonzero entry in its column.

## **Example**

# Uniqueness of the Reduced Row Echelon Form (RREF)

#### **Theorem**

Each matrix is row equivalent to one and only one reduced echelon matrix.

#### **Definition**

If a matrix A is row equivalent to an echelon matrix U, we call U an echelon form (or row echelon form) of A; if U is in reduced echelon form, we call U the reduced echelon form of A.

(The proof of the theorem will not be covered in class. Interested students may look at the Proofs document posted in GC.)

Since the reduced echelon form is unique, the leading entries are always in the same positions in any echelon form obtained from a given matrix. These leading entries correspond to leading 1's in the reduced echelon form.

Therefore the pivot positions are the same in any echelon form of a given matrix.

The above fact is intuitively obvious. Students interested in a formal proof should look at the proofs document (we don't really have time to cover too many proofs in detail, as you will see for yourself).

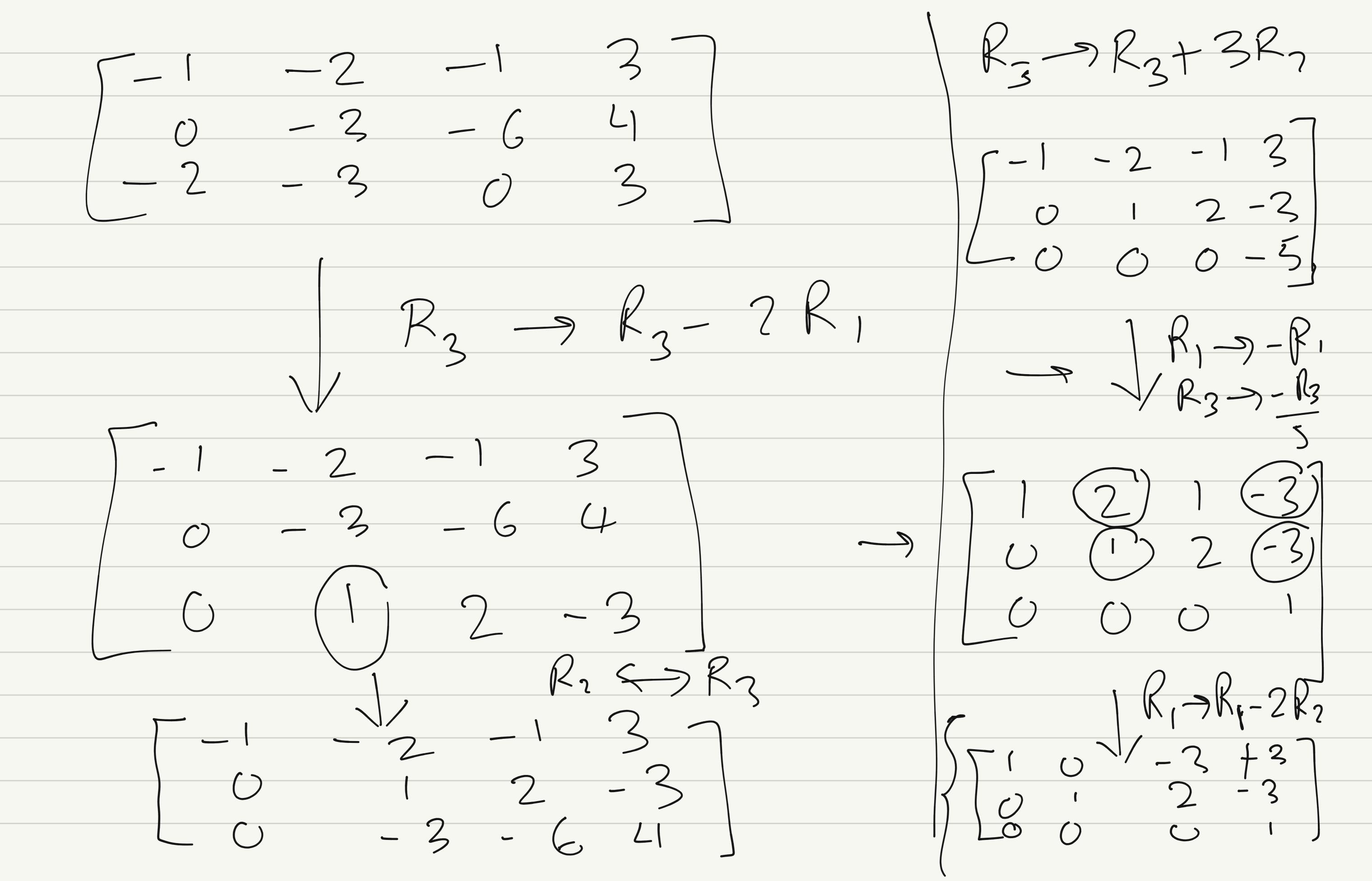
## **Definition**

A pivot position in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A. A pivot column is a column of A that contains a pivot position.

## Example

Row reduce the matrix A below to reduced echelon form, identify the pivot positions, and locate the pivot columns of A.

$$\begin{bmatrix} 0 & -3 & -6 & 4 \\ -1 & -2 & -1 & 3 \\ -2 & -3 & 0 & 3 \end{bmatrix} \xrightarrow{R_1 + R_2} \begin{bmatrix} -1 & -2 & -1 & 3 \\ 0 & -3 & -6 & 4 \\ -2 & -3 & 0 & 3 \end{bmatrix}$$



$$R_{2} \rightarrow R_{2} + \frac{2}{3} + \frac{2}{3} = \frac{2}{3} + \frac{2}{3} = \frac{2}{3}$$

$$Ax = b_1$$

$$Ax = b_2$$

$$Ax = b_3$$

$$\begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# The Row Reduction Algorithm

#### STEP 1

Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.

## **Example**

Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Interchange rows 1 and 3.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Use row replacement operations to create zeros in all positions below the pivot.

Cover (or ignore) the row containing the pivot position and cover all rows, if any, above it. Apply steps 1–3 to the submatrix that remains. Repeat the process until there are no more nonzero rows to modify.

$$\begin{bmatrix}
3 & 9 & 12 & 9 & 6 & 15 \\
0 & 2 & -4 & 4 & 2 & -6 \\
0 & 3 & -6 & 6 & 4 & -5
\end{bmatrix}$$

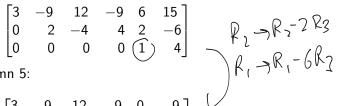
Add -3/2 times row 2 to row 3:

Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by a scaling operation.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Create 0s in column 5:

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 2 & -4 & 4 & 0 & -14 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$



$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 2 & -4 & 4 & 0 & -14 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Scale row 2 by 1/2:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$
1/2:
$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

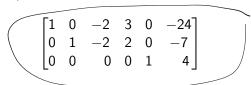
$$\begin{matrix} \mathcal{R}, \longrightarrow \mathcal{R}_1 + \mathcal{I} \mathcal{R}_2 \\ \longrightarrow \end{matrix}$$

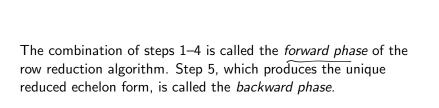
$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

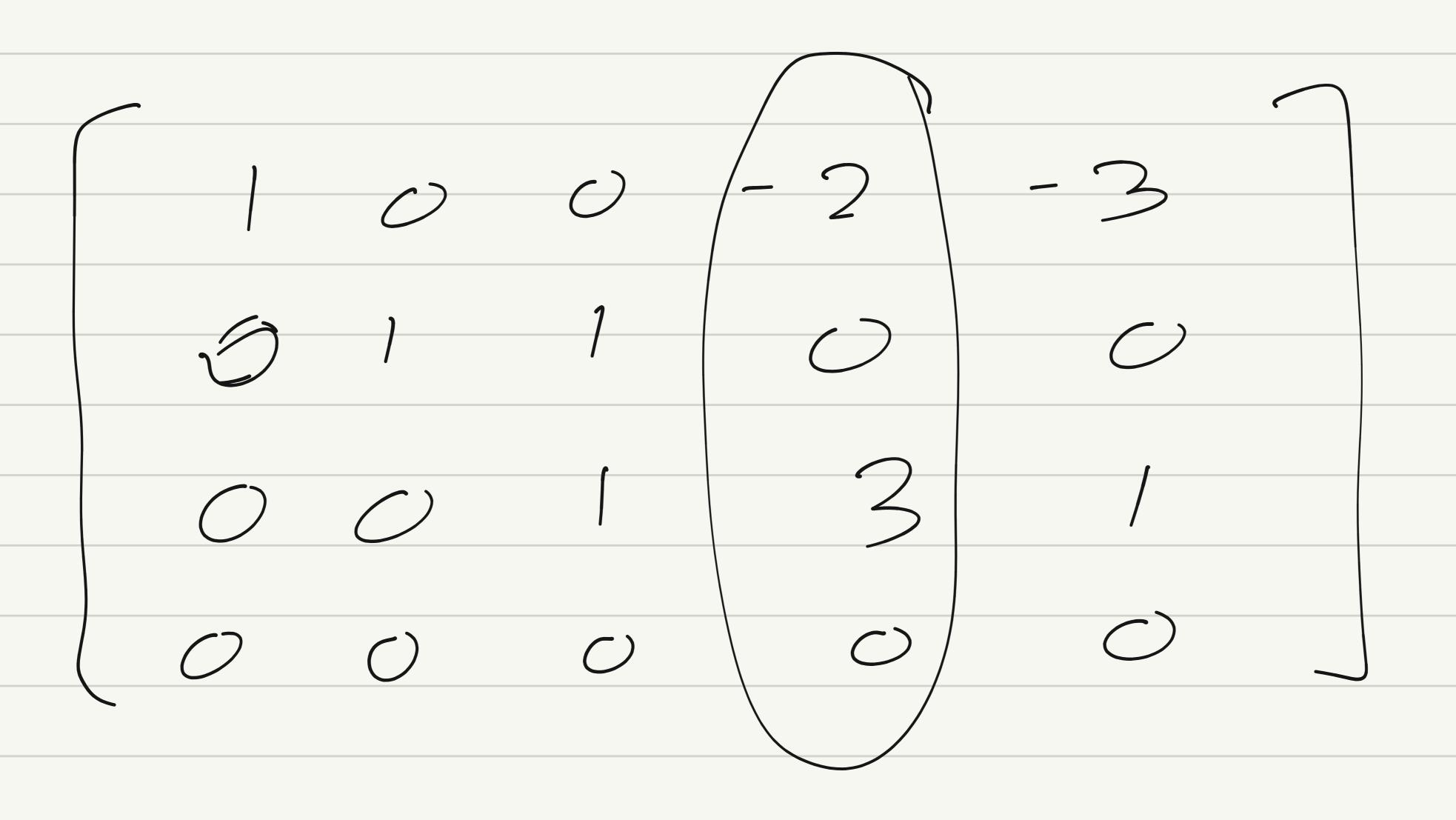
$$\begin{bmatrix}
3 & 0 & -6 & 9 & 0 & -72 \\
0 & 1 & -2 & 2 & 0 & -7 \\
0 & 0 & 0 & 0 & 1 & 4
\end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Scale row 1 by 1/3:







# **Vectors as Ordered Lists or** *n***-tuples**

We will temporarily use the word "vector" to refer to an ordered list of numbers.

(Informal note: The last time I taught this course I introduced the abstract definition of a vector very quickly and students were a little lost. So this time I'm postponing it a bit.)

Please review what you were taught about vectors in  $\mathbb{R}^2$  and  $\mathbb{R}^3$  in high school.

#### **Definition**

The set of all *n*-tuples of real numbers is called  $\mathbb{R}^n$ .

Elements of  $\mathbb{R}^n$  are usually **represented** as  $n \times 1$  column vectors  $(n \times 1 \text{ matrices})$ .