Evariste Session on Proving Methods

Evariste, the math club of IIITD, is hosting a session on Proofs and Proving Methods.

Timing and Zoom Link

The session is on 10th Feb, Thursday at 7:00pm.

The link for the same is:

https://iiitd-ac-in.zoom.us/j/91848888555?pwd=and3bVpyS1hsSC8yeC9SQjJTeXdOZz09

Meeting ID: 918 4888 8555
Passcode: 565410

S l.i. $\Rightarrow$ any subset of S is l.i.

Definition

Let V be a vector space. Let S be an infinite subset of V. We say S is a *linearly independent* set if every finite subset of S is linearly independent.

Proposition

Let V be a vector space, and let S be a linearly independent subset of V. Any subset of S is linearly independent.

What is the contrapositive? Example(s)?

S. l.i. $\Rightarrow$ Every subset of S is l.i.

Contrapositive :

If any one subset of S is l.d. $\Rightarrow$ S is l.d.

If, $\exists$ a l.d. subset of S then S is l.d.

$A \Rightarrow B$ log equiv Contrapositive $\neg B \Rightarrow \neg A$; not B $\neg B$, !B

converse : $B \Rightarrow A$, not equiv to $\neg A \Rightarrow \neg B$.

$V =$ vector space of all real valued functions on $[0,1]\}$.

$$S = \{1, \sin x, \underbrace{\sin^2 x, \cos^2 x}, \cos 3x\}$$

$$c_1 = 1, \quad c_2 = 0, \quad c_3 = -1, \quad c_4 = -1$$

$$c_5 = 0.$$

$$c_1 \cdot 1 + c_2 \sin x + c_3 \sin^2 x + c_4 \cos^2 x$$

$$+ c_5 \cos 3x = 0.$$

$\{v_1, \ldots, v_n\}$ are l.i

$\iff$ the equation

$c_1 v_1 + \cdots + c_n v_n = 0$

has only the trivial solution.

What about extending a linearly independent set to a bigger linearly independent set? How would we do this?

Proposition

Let $\{v_1, \ldots, v_n\}$ be a linearly independent set in a vector space V. If $w \notin Span(\{v_1, \ldots, v_n\})$ then the set $\{v_1, \ldots, v_n, w\}$ is linearly independent.

Pf: Assume

$$c_1 v_1 + \cdots + c_n v_n + \underline{c_{n+1}} w = 0$$

for some scalars $c_1, \ldots, c_{n+1} \in \mathbb{R}$.

Suppose if possible that $\underline{c_{n+1} \neq 0}$.

Then
$$w = -\frac{c_1}{c_{n+1}} v_1 - \frac{c_2}{c_{n+1}} v_2 \cdots - \frac{c_n}{c_{n+1}} v_n$$

$$\Rightarrow w \in \operatorname{Span}\{v_1, \ldots, v_n\}$$

$$\Rightarrow \quad c_{n+1} = 0 .$$

$$\Rightarrow \quad c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$$

Since $v_1, \ldots, v_n$ are linearly independent

$$c_1 = c_2 = \cdots = c_n = 0 .$$

$\square$.

Spanning Set Theorem, p. 212 of course textbook

Theorem

Let V be a vector space. Let $S = \{v_1, v_2, \ldots, v_p\}$ be a set in V and let $H = Span\{v_1, v_2, \ldots, v_p\}$.

1. *If one of the vectors in S, say v_k, is a linear combination of the remaining vectors in S, then the set formed from S by removing v_k still spans H.*
2. *If $H \neq \{0\}$, some subset of S is a basis for H.*

$v_k \in \text{Span}\{v_1, \ldots, v_{k-1}, v_{k+1}, \ldots v_p\}$

Proof of statement 1:

Let $w \in H$.

Then $\exists$ scalars $c_1, \dots, c_p \in \mathbb{R}$ such that

$w = c_1 v_1 + c_2 v_2 + \dots + c_p v_p$.

We also know that

$v_k \in \text{Span}\{v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_p\}$

$\therefore \exists$ scalars $d_1, \ldots, d_{k-1}$, $d_{k+1}, \ldots, d_p$ such that

$$v_k = d_1 v_1 + \cdots + d_{k-1} v_{k-1} + d_{k+1} v_{k+1} + \cdots + d_p v_p$$

$$w = c_1 v_1 + \cdots + c_k v_k + \cdots + c_p v_p$$

$$= c_1 \underline{v_1} + c_2 v_2 + \ldots + c_{k-1} v_{k-1} + \underline{c_k (d_1 v_1} + d_2 v_2 \ldots + d_{k-1} v_{k-1} + d_{k+1} v_{k+1} + \cdots d_p v_p)$$

$$+ \cdots + c_p v_p$$

$$= (c_1 + c_k d_1) v_1 + (c_2 + c_k d_2) v_2$$

$$+ \cdots + (c_{k-1} + c_k d_{k-1}) v_{k-1}$$

$$+ (c_{k+1} + c_k d_{k+1}) v_{k+1}$$

$$+ \cdots + (c_p + c_k d_p) v_p$$

$$\in \operatorname{Span}\{v_1, \ldots, v_{k-1}, v_{k+1}, \ldots, v_p\} \,\square.$$

The following lemmas, which have already been proved earlier, are useful in the proof of the above theorem.

Lemma

Let $S_1 \subset S_2 \subset V$. Then

$$\mathrm{Span}\, S_1 \subset \mathrm{Span}\, S_2.$$

Lemma

Let W be a subspace of a vector space V. Let S be a subset of W. Then

$$\mathrm{Span}\, S \subset W.$$

Proof of Statement 2

$A \setminus B = \{x \in A \mid x \notin B\}$ $V = \mathbb{R}^2$

$S = \{(1,0), (0,1), (-1,0), (0,-1)\}$

$S_1 = \{(0,1), (-1,0), (0,-1)\}$

$S_2 = \{(-1,0), (0,-1)\}$

Will the following idea work? Why or why not?

does NOT work

Let $S' = \{v \in S \mid v \notin \mathrm{Span}(S \setminus \{v\})\}$?

set minus

Two other ideas:

1. Remove vectors from S one by one until you get a linearly independent set. This is the approach used in the Sheldon Axler text, as well as in your course textbook.
2. Pick the biggest linearly independent subset of S that you can find. This may not be unique, but works.

$\{(1,0), (0,1)\}$, $\{(1,0), (0,-1)\}$

$\{(-1,0), (0,-1)\}$, $\{(0,1), (-1,0)\}$

Let S' be a linearly independent subset of S having maximum cardinality, say

$$S' = \{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\}$$

where $1 \leq i_1 < i_2 < \cdots < i_k \leq p$.

<u>Claim</u>: S' is a basis of H!

Let $v \in \text{span}\{v_1, \dots, v_n\} = H$.

Suppose if possible $v \notin \text{Span}\{v_{i_1}, \dots, v_{i_k}\}$

By Proposition proved earlier,

$\{v_{i_1}, \dots, v_{i_k}, v\}$ is linearly independent

$\rightarrow\leftarrow$. ← contradiction symbol

Hence $v \in \text{Span}\{v_{i_1}, \dots, v_{i_k}\}$.

$$\Rightarrow \operatorname{Span}\{v_{i_1}, \ldots, v_{i_k}\}$$

$$= H.$$

$\therefore \{v_{i_1}, \ldots, v_{i_k}\}$ is a basis of H.

almost

Same as Statement 1 of Spanning Set Theorem but stated differently

Proposition

Let $S = \{v_1, \ldots, v_n\}$ be ~~an ordered~~ a set of vectors in V, let $w \in V$ be any vector, and let $S' = \{w, v_1, \ldots, v_n\}$. Then $\operatorname{Span} S = \operatorname{Span} S'$ if and only if $w \in \operatorname{Span} S$.

Proof: Exercise

Aim: To establish that every basis has the same cardinality => dimension well-defined.

Theorem

Let V be a finite dimensional vector space. Let $\mathcal{B}_1 = \{b_1, \ldots, b_n\}$ be a basis of V. Let $\mathcal{B}_2 = \{v_1, \ldots, v_n\}$ be any other linearly independent subset of V. Then $\mathcal{B}_2$ is also a basis of V.