Let A be an $n \times n$ matrix (where $n \geq 2$). A_{ij} denotes the $n-1 \times n-1$ submatrix formed by deleting the i-th row and j-th column of A, for $1 \leq i, j \leq n$.

Definition

For $n \geq 2$, the determinant of an $n \times n$ matrix $A = (a_{ij})$ is the sum of n terms of the form $\pm a_{1j} \det A_{1j}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of A. In symbols,

$$\begin{aligned}\det A &= a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n} \\ &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}\end{aligned}$$

The determinant of a 1×1 matrix is the single entry of that matrix.

Definition

Let $A = (a_{ij})$ be an $n \times n$ matrix (where $n \geq 2$). The *(i, j)-cofactor of A* is the number C_{ij} given by

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

same formula as previous page $\leftarrow$ $\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$

Theorem

The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column. The expansion across the i-th row using the cofactors is

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

The cofactor expansion down the j-th column is

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}.$$

strictly out of syllabus

A little group theory (namely the facts that S_n is generated by *transpositions*), and that cosets corresponding to transpositions partition S_n, and some induction can be used to establish that

permutations of the set $\{1, \ldots, n\}$

$$\det A = \sum_{\pi \in S_n} \operatorname{sign}(\pi) \prod_{j=1}^{n} a_{j,\pi(j)}$$

(12)

$a_{12}a_{21}a_{33}\cdots a_{nn}$

$$\begin{bmatrix} a & 0 & 0 \\ b & d & 0 \\ c & e & f \end{bmatrix} \qquad \begin{bmatrix} a & d & e \\ 0 & b & f \\ 0 & 0 & c \end{bmatrix} \leftarrow \text{upper triangular}$$

lower triangular

Theorem

If A is a triangular matrix, then det A is the product of the entries on the main diagonal of A.

Idea: Use definition and induction - lower triangular case.

For the upper triangular case, expand along first column instead.

Proof: A lower triangular. For $n = 2$,

$$\det \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - 0 \cdot a_{21}$$
$$= a_{11}a_{22}.$$

Assume that the result

holds true for $n = k \geq 2$.

We show that the result holds true when $n = k+1$.

$$\det A = \sum_{j=1}^{k+1} (-1)^{1+j} a_{1j} \det A_{1j} = a_{11} \det A_{11}$$

$\det A_{11} = a_{22} a_{33} \cdots a_{k+1,k+1}$, by induction hypothesis.

$$\therefore \quad \det A = a_{11} a_{22} \cdot \cdots \cdot a_{k+1,k+1}$$

$\square$.

$$
\begin{bmatrix}
a_{11} & 0 & 0 & 0 & 0 & 0 \\
a_{21} & a_{22} & 0 & & \cdots & \\
\vdots & & \ddots & & & \\
\vdots & & & & & \\
a_{n1} & & & & & a_{nn}
\end{bmatrix}
$$

Upper triangular case - similar.

$$
\begin{bmatrix}
a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\
0 & a_{22} & & & \\
0 & 0 & \ddots & & \\
0 & 0 & 0 & \cdots & a_{nn}
\end{bmatrix}
$$

$R_i \to cR_i$

i,i-th entry

$$\begin{bmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & c & & \\ & & & \ddots & \\ 0 & & & & 1 \end{bmatrix}$$

Proposition

Let E be an $n \times n$ elementary matrix. Then

1. $\det E = c$, when E corresponds to scaling a row by a nonzero scalar c
2. $\det E = 1$, when E corresponds to a row replacement operation
3. $\det E = -1$, when E corresponds to a row interchange operation

In case of row replacement or row scaling, E is triangular ...

We show that $\det E = -1$, when E corresponds to a row interchange, using induction on n.

Base case .. what is n?

E corresponds to replacing R_i and R_j.

$$E = \left(e_{kl} \right)$$

$e_{kl} = \begin{cases} 1, & \text{if } k = l \\ 0, & \text{otherwise} \end{cases}$

Identity

$$e_{kl} = \begin{cases} 1, & \text{if } k \neq i,\ l \neq j \text{ and } k = l \\ & \text{or if } k = i \text{ and } l = j \\ & \text{or if } k = j \text{ and } l = i \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\det E = e_{11} \det E_{11} - e_{12} \det E_{12}$$

$$+ \cdots \pm e_{1n} \det E_{1n}$$

$$=$$

Back to earlier proof:

Base case: $n = 2$.

only possibility: interchange R_1 and R_2.

$$\det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1.$$

Induction hypothesis: Assume that the result holds for $2 \leq n < k$. We show that the hypothesis holds for $n = k$.

Assume that the induction hypothesis is true for $2 \leq k < n$. We show that it holds when $n = k$.

Main idea: Expand along a row which is not involved in the row interchange.

If the interchange is $R_i \leftrightarrow R_j$

then we expand along row l

where $l \neq i$ and $l \neq j$.

$$\det E = (-1)^{l+1} e_{l1} \det E_{l1} + \cdots + (-1)^{l+n} e_{ln} \det E_{ln}.$$

$$= (-1)^{l+l} e_{ll} \det E_{ll} = \det E_{ll}$$

The matrix E_{11} also an elementary matrix which corresponds to a row interchange.

$\therefore$ By induction hypothesis.

$\det E_{11} = -1.$

$\therefore \quad \det E = -1.$

Suppose E corresponds to $R_i \leftrightarrow R_j$. We expand the determinant along row l, where $l \neq i, l \neq j$. Let $E = (a_{ij})$. Then

$$\det E = \det E_{ll}$$

Since E_{ll} is an $k-1 \times k-1$ elementary matrix corresponding to a row interchange operation, it follows by the induction hypothesis that

$$\det E_{ll} = -1.$$

Therefore $\det E = -1$.

Proposition

Let A be an $n \times n$ matrix having two identical rows. Then the determinant of A is zero.

Idea(s) behind proof:

First consider the case where the identical rows are adjacent. Expand.

Next consider the non-adjacent case. Use induction.

Consider the case where R_i, R_{i+1} are identical.

Expanding along the ith row, we get

(1) $$\det A = (-1)^{i+1} \det A_{i1} + \cdots + (-1)^{i+n} \det A_{in}$$

along i+1th row,

(2) $$\det A = (-1)^{i+2} \det A_{i+1,1} + \cdots + (-1)^{i+1+n} \det A_{i+1,n}$$

Clearly $A_{ij} = A_{i+1,j}$.

$\therefore$ ① & ② $\Rightarrow$

$\det A = -\det A$

$\Rightarrow \det A = 0$.