ECE 113- Basic Electronics

Lecture week 9: Phasor and impedance

Dr. Ram Krishna Ghosh, Assistant Professor

Office: B601, Research and Development Block

Email: rkghosh@iiitd.ac.in

INDRAPRASTHA INSTITUTE *of*
INFORMATION TECHNOLOGY
DELHI

SOLUTION OF CIRCUITS CONTAINING DYNAMIC ELEMENTS

Consider the circuit of Fig. 4, which consists of the series connection of a voltage source, a resistor, and a capacitor. Applying KVL around the loop, we obtain the following equation:

$$v_S(t) = v_R(t) + v_C(t)$$

and we get $$\frac{dv_C}{dt} + \frac{1}{RC} v_C = \frac{1}{RC} v_S$$

Let external source be a sinusoidal voltage, described by the expression

$$v_S(t) = V \cos(\omega t)$$

Since the forcing function is a sinusoid, the solution may also be assumed to be of the same form. An expression for $v_C(t)$ is then the following:

$$v_C(t) = A \sin \omega t + B \cos \omega t$$

which is equivalent to $$v_C(t) = C \cos(\omega t + \varphi)$$

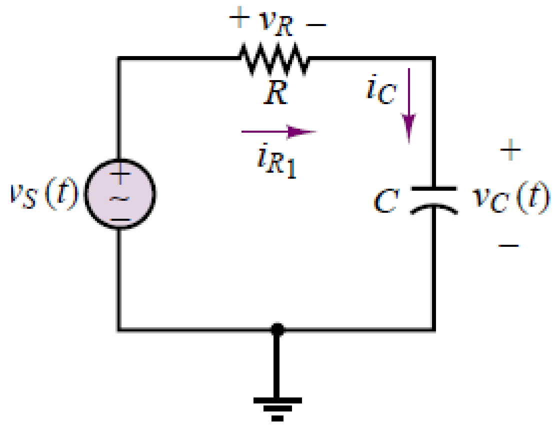


Fig. 4

SOLUTION OF CIRCUITS CONTAINING DYNAMIC ELEMENTS

Substituting this value of in the differential equation for $v_C(t)$ and solving for the coefficients A and B yields the expression

$$A\omega \cos \omega t - B\omega \sin \omega t + 1/(RC)\,(A \sin \omega t + B \cos \omega t) = (1/RC)\, V \cos \omega t$$

A and B are given as following:

$$A = \frac{V\omega RC}{1+\omega^2(RC)^2} \quad \text{and} \quad B = \frac{V}{1+\omega^2(RC)^2}$$

And the solution for $v_C(t)$ may be written as follows

$$v_C(t) = \frac{V\omega RC}{1+\omega^2(RC)^2}\sin \omega t + \frac{V}{1+\omega^2(RC)^2}\cos \omega t$$

This response is plotted in Fig. 5.

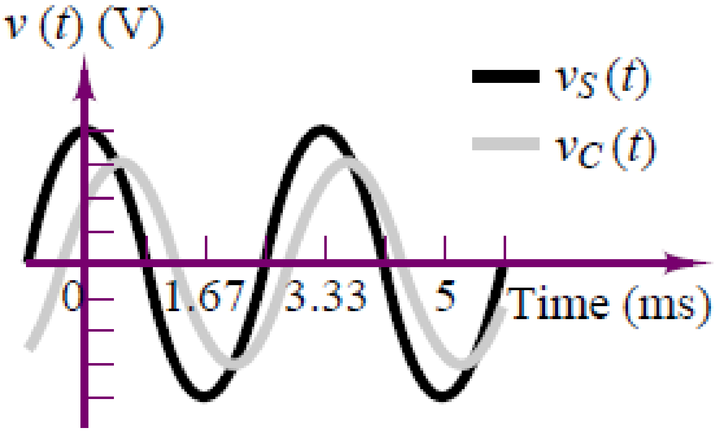


Fig. 5

We shall use an efficient notation to make it possible to represent sinusoidal signals as *complex numbers*, and to eliminate the need for solving differential equations.

Phasor and Impedance

- A complex numbers can be used to represent sinusoidal signals by using Euler's equation. We have:

$$A\cos(\omega t+\varphi) = \mathrm{Re}\,[Ae^{j(\omega t+\varphi)}]$$

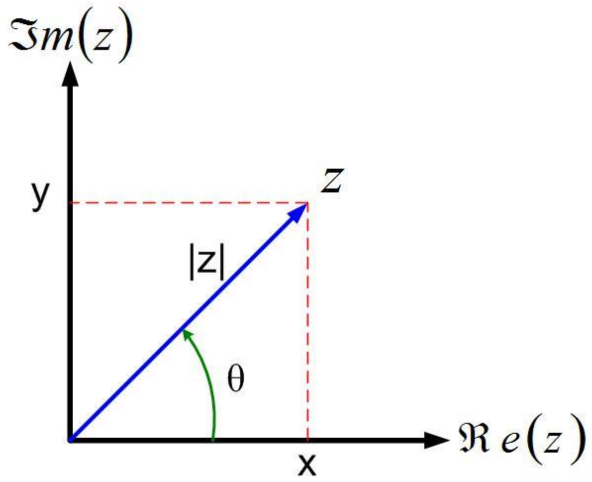
$$e^{j(\omega t)} = \cos\omega t + j\sin\omega t$$

- This equality is easily verified by expanding the right-hand side, as follows:

$$\mathrm{Re}\,[Ae^{j(\omega t+\varphi)}] = \mathrm{Re}\,[A\cos(\omega t+\varphi) + j\,A\sin(\omega t+\varphi)] = A\cos(\omega t+\varphi)$$

- It is possible to express a generalized sinusoid as the real part of a complex vector whose **angle,** is given by ($\omega t + \varphi$) and whose **magnitude,** is equal to the peak amplitude of the sinusoid.
- Similarly, $\mathrm{Im}\,[Ae^{j(\omega t+\varphi)}] = A\sin(\omega t+\varphi)$
- The **complex phasor** corresponding to the sinusoidal signal $A\cos(\omega t+\varphi)$ is therefore defined to be the complex number $Ae^{j\varphi}$:

$$Ae^{j\varphi} = \text{complex phasor notation for } A\cos(\omega t+\varphi) = A\angle\varphi$$

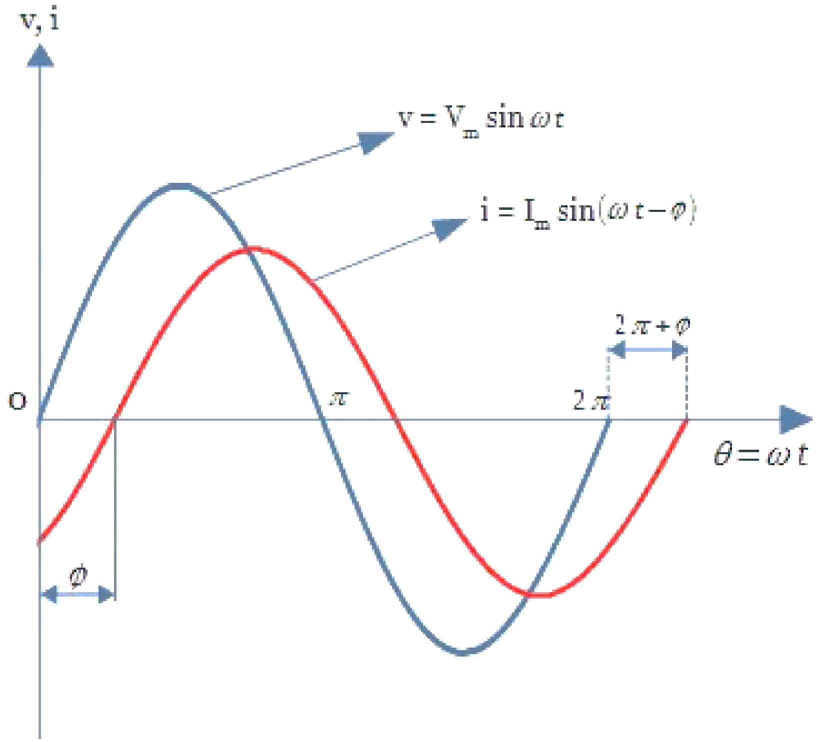


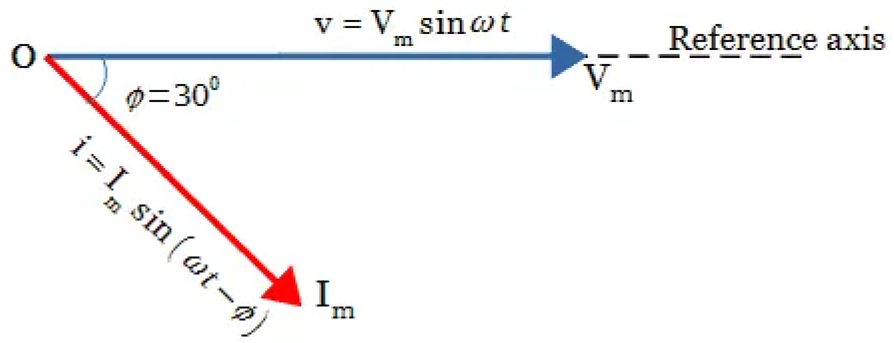
- Any sinusoidal signal may be mathematically represented in one of two ways:
  - a **time-domain form,** $v(t) = A\cos(\omega t+\varphi)$
  - and a **frequency-domain** (or **phasor**) **form,** $\mathbf{V}(j\omega) = Ae^{j\varphi}$
  - $j\omega$ in the notation $\mathbf{V}(j\omega)$, indicating the $e^{j\varphi}$ dependence of the phasor.

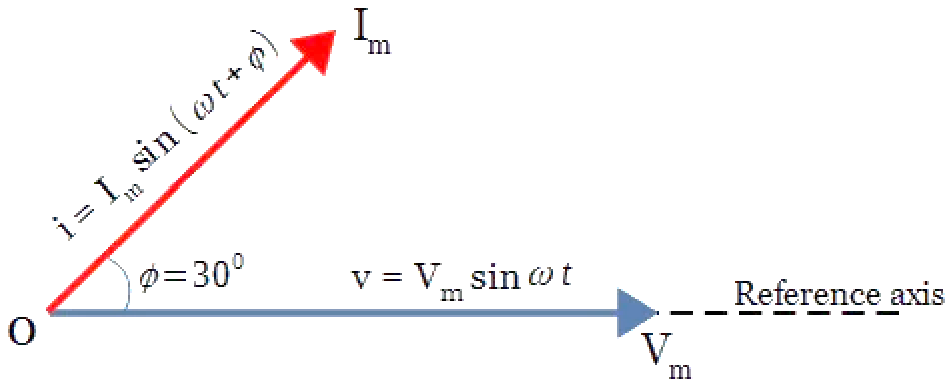
Phasor diagram

Phasor Diagram is a graphical representation of the relation between two or more alternating quantities in terms of magnitude and direction. In other words, it depicts the phase relationship between two or more sinusoidal waveforms having the same frequency.

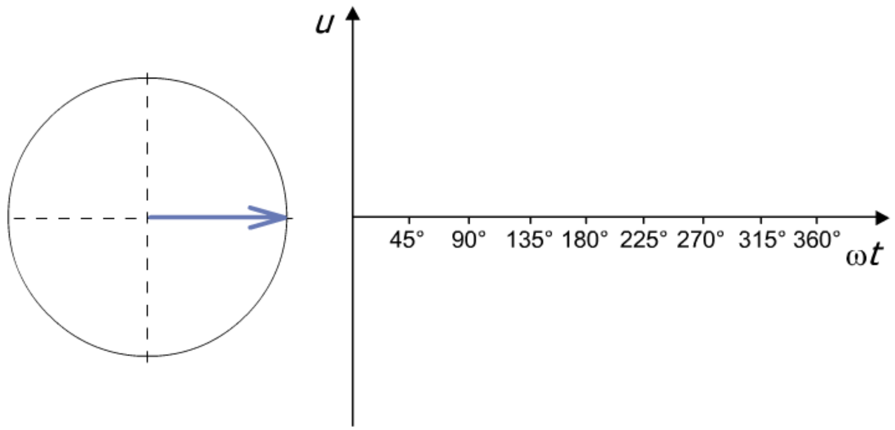
Phasor is a straight line with an arrow at one end, rotating in an anticlockwise direction. The length of the line represents the magnitude of the sinusoidal quantity and the arrow represents the direction.







Phasor diagram



AC response of a inductance

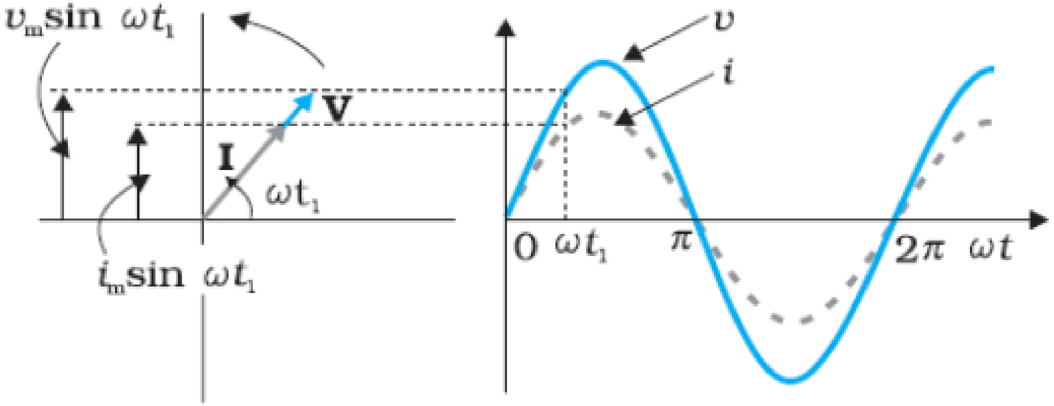
$$v = v_m \sin \omega t$$

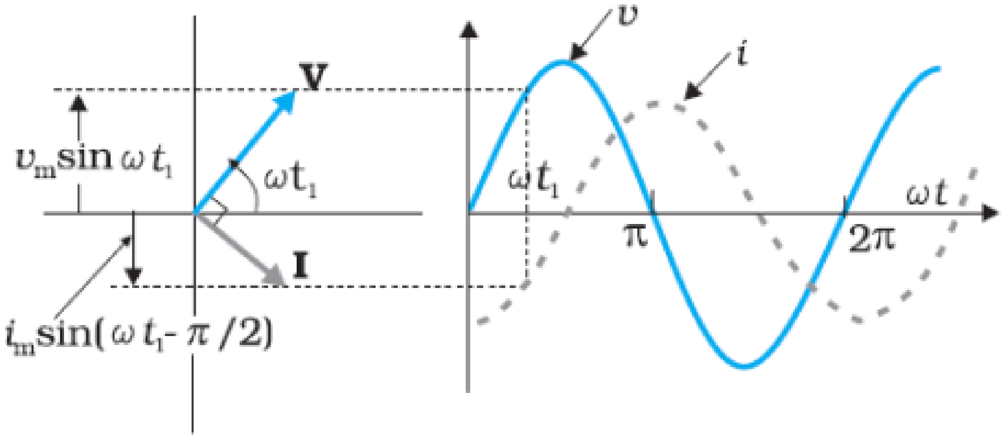
$$i = i_m \sin\left(\omega t - \frac{\pi}{2}\right)$$

AC response of a resistance

$$v = v_m \sin \omega t$$

$$i = i_m \sin \omega t$$





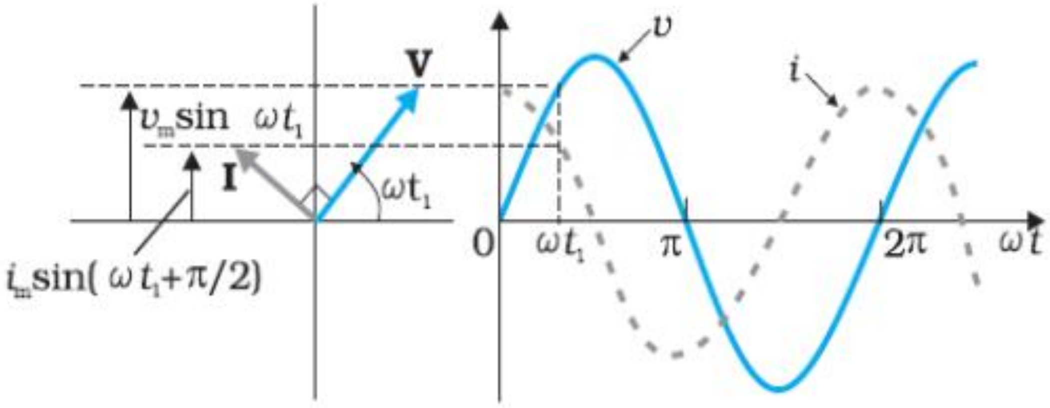
Phasor diagram **voltage versus current graph**

Phasor

AC response of a capacitance

$v = v_m \sin \omega t$

$i = i_m \sin (\omega t + \frac{\pi}{2})$



Phasor diagram

voltage versus current graph

Suppose, we have a complex number (i.e. phasor) $z_1 = |z_1|e^{j\theta} = |z_1|\angle\theta$

And another phasor $z_2 = z_1e^{j\varphi} = |z_1|e^{j(\theta+\varphi)} = |z_1|e^{j\theta}e^{j\varphi}$ i.e. z_2 is leading z_1

If $\varphi = \frac{\pi}{2}$, then $z_2 = jz_1$,

since, $e^{j\frac{\pi}{2}} = cos\frac{\pi}{2} + jsin\frac{\pi}{2} = j$

Clearly, multiplication by *j* means advancement of phase by $\frac{\pi}{2}$ or 90°

If $\varphi = -\frac{\pi}{2}$, then $z_2 = -jz_1 = \frac{1}{j}z_1$ multiplication by -*j* means lagging of phase by $\frac{\pi}{2}$ or 90°

Phasor and Impedance

- Express following voltages as phasors:

$v1(t) = 15 \cos(377t + \pi/4)$ V

$v2(t) = 15 \cos(377t + \pi/12)$ V

The phasors will be following:

$\mathbf{V}_1(j\omega) = 15\angle\pi/4$ V

$\mathbf{V}_2(j\omega) = 15e^{j\,\pi/12} = 15\angle\pi/12$ V

- Converting the phasor voltages from polar to rectangular, we get

$\mathbf{V}1(j\omega) = 10.61 + j10.61$ V

$\mathbf{V}2(j\omega) = 14.49 + j3.88$

If two sources are connected in series, what will be the combined voltage?

v= v1(t) + v2(t), adding in phasor we get,

$\mathbf{V}(j\omega) = 25.10 + j14.49$ V = = $28.98\angle\pi/6$ V

In time domain, we have v(t) = $28.98 \cos(377t + \pi/6)$ V.

Superposition of AC Signals

- Consider the circuit shown in Fig. 6. Find the load current, if I1(t) and I2(t) are given as following:

$$i1(t) = A1\cos(\omega 1t) \;\; and \;\; i2(t) = A2\cos(\omega 2t)$$

- The load current is equal to the sum of the two source currents; that is,

$$iL(t) = i1(t) + i2(t)$$

or, in phasor form,

$$\boldsymbol{IL} = \boldsymbol{I1} + \boldsymbol{I2}$$

- writing **I**1 and **I**2 in a more explicit phasor form as

$$\boldsymbol{I_1} = A_1 e^{j0} \text{ and } \boldsymbol{I_2} = A_2 e^{j0}.$$

- The load current $iL(t) = i1(t) + i2(t) = A1\cos(\omega 1t) + A2\cos(\omega 2t)$

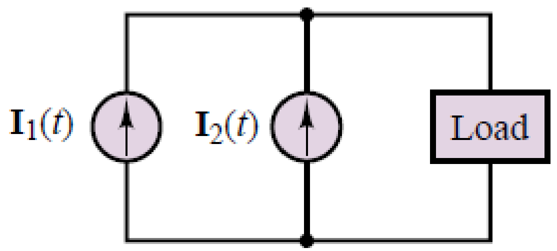


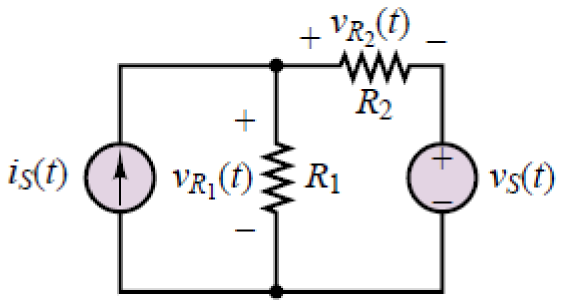
Fig. 6

Example of AC Superposition

- Compute the voltages $v_{R1}(t)$ and $v_{R2}(t)$ in the circuit of Figure 7. Given

  $i_S(t) = 0.5\cos(2\pi 100t)$ A

  $v_S(t) = 20\cos(2\pi 1{,}000t)$ V



$R_1 = 150\ \Omega$, $R_2 = 50\ \Omega$

Figure 7

- Since the two sources are at different frequencies, we must compute a separate solution for each. Consider the current source first, with the voltage source set to zero (short circuit) as shown in Figure 8. The circuit thus obtained is a simple current divider. Writing the source current in phasor notation:

$$I_S(j\omega) = 0.5e^{j0} = 0.5\angle 0\ \text{A} \qquad \omega = 2\pi 100\angle\text{rad/s}$$

$$\mathbf{V}_{R1}(\mathbf{I}_S) = \mathbf{I}_S\frac{R_2}{R_1 + R_2}R_1 = 0.5\ \angle 0\left(\frac{50}{150+50}\right)150 = 18.75\ \angle 0\ \text{V}$$

$$\omega = 2\pi 100\ \text{rad/s}$$

$$\mathbf{V}_{R2}(\mathbf{I}_S) = \mathbf{I}_S\frac{R_1}{R_1 + R_2}R_2 = 0.5\ \angle 0\left(\frac{150}{150+50}\right)50 = 18.75\ \angle 0\ \text{V}$$

$$\omega = 2\pi 100\ \text{rad/s}$$

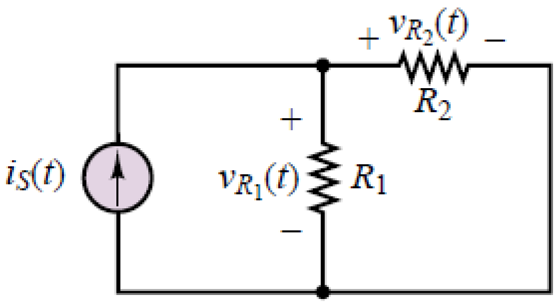


Figure 8

Example of AC Superposition (cont.)

- Next, we consider the voltage source, with the current source set to zero (open circuit), as shown in Figure 9. We first write the source voltage in phasor notation: *is (t)* = 0.5 cos*(2π100t)* A

$$\mathbf{V}_S(j\omega) = 20e^{j0} = 20\angle 0 \text{ V} \qquad \omega = 2\pi 1{,}000 \text{ rad/s}$$

$$\mathbf{V}_{R1}(\mathbf{V}_S) = \mathbf{V}_S \frac{R_1}{R_1 + R_2} = 20\angle 0 \left(\frac{150}{150+50} \right) = 15 \angle 0 \text{ V}$$

$$\omega = 2\pi 1{,}000 \text{ rad/s}$$

$$\mathbf{V}_{R2}(\mathbf{V}_S) = -\mathbf{V}_S \frac{R_2}{R_1 + R_2} = -20\angle 0 \left(\frac{50}{150+50} \right) = -5\angle 0 = 5\angle \pi \text{ V}$$

$$\omega = 2\pi 1{,}000 \text{ rad/s}$$

$$\mathbf{V}_{R1} = \mathbf{V}_{R1}(\mathbf{I}_S) + \mathbf{V}_{R1}(\mathbf{V}_S)$$

$$v_{R1}(t) = 18.75 \ \cos(2\pi 100t) + 15 \ \cos(2\pi 1{,}000t) \text{ V}$$

$$\mathbf{V}_{R2} = \mathbf{V}_{R2}(\mathbf{I}_S) + \mathbf{V}_{R2}(\mathbf{V}_S)$$

$$v_{R2}(t) = 18.75 \ \cos(2\pi 100t) + 5 \ \cos(2\pi 1{,}000t + \pi) \text{ V.}$$

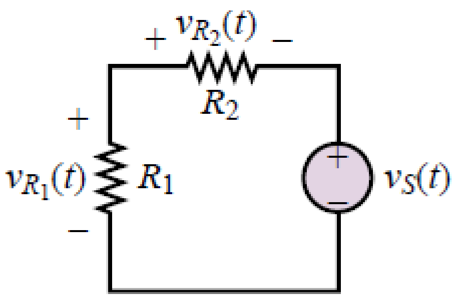
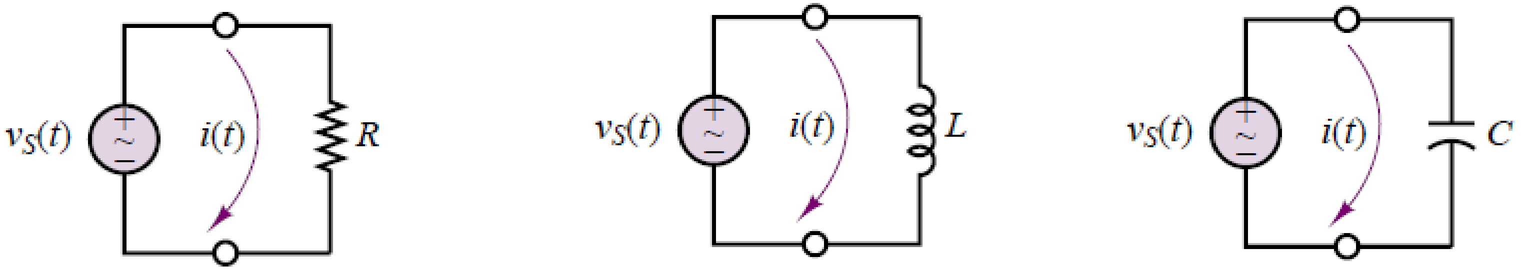


Figure 9

Impedance

- We will consider the *i-v* relationship of the three ideal circuit elements (R, L and C) in light of the new phasor notation.
- The result will be a new formulation in which resistors, capacitors, and inductors will be described in the same notation.
- A direct consequence of this result will be that the circuit theorems discussed earlier will be extended to AC circuits.
- In the context of AC circuits, any one of the three ideal circuit elements defined so far will be described by a parameter called **impedance,** which may be viewed as a *complex resistance*.
- The impedance concept is equivalent to stating that capacitors and inductors act as *frequency-dependent resistors*, that is, as resistors whose resistance is a function of the frequency of the sinusoidal excitation.
- Figure 10 shows the conventional circuit representation.

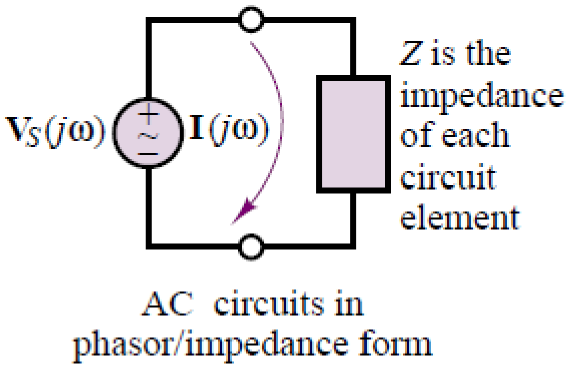


Conventional form Representation

Figure 10

Impedance

- The phasor-impedance form representation in shown in Figure 11 in which the representation explicitly shows phasor voltages and currents and treats the circuit element as a generalized "impedance."
- R, L and C circuit elements will be represented as one such impedance element.
- Let the source voltage in the circuit of Figure 11 be defined by
  - $v_S(t) = A\cos \omega t$ or $\mathbf{V}_S(j\omega) = Ae^{j0} = A\angle 0$
- Then the current $i(t)$ is defined by the i-v relationship for each circuit element. Let us examine the frequency-dependent properties of the resistor, inductor, and capacitor, one at a time.



phasor-impedance form

Figure 11

Impedance

The Resistance

If a voltage $v_S(t) = A \cos \omega t$ is applied across a resistance R resulting in a current $i(t)$, then $i(t)$ is given as follow:

$$i(t) = \frac{v_S(t)}{R} = \frac{A}{R}\cos(\omega t)$$

In phasor form we have,

$$\mathbf{V}_S(j\omega) = A\angle 0$$

$$\mathbf{I}(j\omega) = \frac{A}{R}\angle 0$$

Finally, the *impedance* of the resistor is defined as the ratio of the phasor voltage across the resistor to the phasor current flowing through it, and the symbol Z_R

$$Z_R(j\omega) = \frac{\mathbf{V}_S(j\omega)}{\mathbf{I}(j\omega)} = R \qquad \text{Impedance of a resistor}$$

Impedance

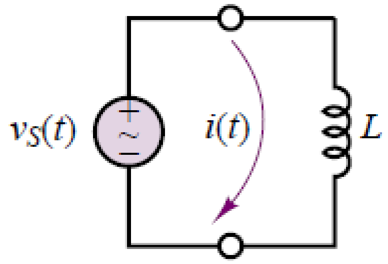
The Inductance

If a voltage $v_S(t) = A \cos \omega t$ is applied across the inductor L, then $i_L(t)$ is given as

$$i_L(t) = i(t) = \frac{1}{L}\int v_S(t')dt' = \frac{1}{L}\int A\cos(\omega t')dt' = \frac{A}{\omega L}\sin \omega t$$

We find dependence on the radian frequency of the source on the expression for the inductor current.

Further, the inductor current is shifted in phase (by 90°) with respect to the voltage. This fact can be seen by writing the inductor voltage and current in time-domain form:



$$v_S(t) = v_L(t) = A\cos \omega t \quad and \quad i(t) = i_L(t) = \frac{A}{\omega L}\cos(\omega t - \pi/2)$$

Using phasor notation,

$$v_S(j\omega) = A\angle 0^0 \qquad \text{and} \quad I(j\omega) = \frac{A}{\omega L}\angle -{}^{\pi}/_{2}$$

Finally, the *impedance* of the inductor is given

$$Z_L(j\omega) = \frac{v_S(j\omega)}{I(j\omega)} = \omega L\angle {}^{\pi}/_{2} = j\omega L \qquad \text{Impedance of Inductor}$$

The inductor appears to behave like a *complex frequency-dependent resistor,* and that the magnitude of this complex resistor, ωL, is proportional to the signal frequency, ω.

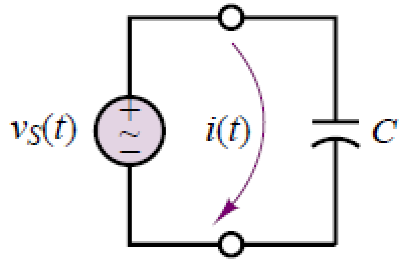
Impedance

The Capacitor

If a voltage $v_S(t) = A\cos\omega t$ is applied across the capacitor C, the capacitor voltage $v_c(t) = v_S(t)$ then $i_C(t)$ is given as

$$i_C(t) = C\frac{dv_C}{dt} = \mathrm{C}\,\frac{d}{dt}(A\cos\omega t) = -C\,(A\omega\sin\omega t) = \omega C\,A\cos(\omega + {}^{\pi}/_{2})$$

We find dependence on the radian frequency of the source on the expression for the capacitor current.



In phasor form,

$$v_S(j\omega) = A\,\angle 0^0 \qquad \text{and} \quad I(j\omega) = \omega C\,A\angle{}^{\pi}/_{2}$$

Finally, the *impedance* of the capacitor $Z_C(j\omega)$is defined as

$$Z_C(j\omega) = \frac{v_S(j\omega)}{I(j\omega)} = \frac{1}{\omega C}\angle -{}^{\pi}/_{2} = \frac{-j}{\omega C} = \frac{1}{j\omega C} \qquad \text{Impedance of a capacitor}$$

The impedance of a capacitor is also a frequency-dependent complex quantity, with the impedance of the capacitor varying as an inverse function of frequency;