Abstract Vector Spaces

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Definition

A *vector space (real vector space)* is a nonempty set V of objects, called *vectors*, on which are defined two operations, called *addition* and *multiplication by scalars (real numbers)*, subject to the ten axioms (or rules) listed below. The axioms must hold for all vectors u, v, and w in V and for all scalars c and d.

1. The sum of u and v, denoted by $u + v$, is in V.
2. $u + v = v + u$
3. $u + (v + w) = (u + v) + w$
4. There is a *zero* vector 0 in V such that $u + 0 = u$.
5. For each u in V , there is a vector $-u$ in V such that $u + (-u) = 0$.

6. The scalar multiple of u by c , denoted by cu, is in V .
7. $c(u+v) = cu + cv$
8. $(c+d)u = cu + du$
9. $c(du) = (cd)u$
10. $1u = u$

$e^{c\log w} = w^c$ Page 3

I had mentioned in the previous lecture that the set $(0, \infty)$ acquires the structure of a vector space via the exponential mapping

$$x \mapsto e^x$$

We may define "addition" and "scalar multiplication" on the set $(0, \infty)$ as follows:

$$w_1 \oplus w_2 = e^{\log w_1 + \log w_2}, \quad \forall w_1, w_2 \in (0, \infty)$$

and

$$\boxed{c \star w = e^{c \log w},} \quad \forall w \in (0, \infty), c \in \mathbb{R}$$

In this particular example, these operations are actually natural,

$$w_1 \oplus w_2 = w_1 w_2, \quad c \star w = w^c$$

Ananya's Question

What if the exponential mapping was replaced with any other bijection?

My apologies to all of you for answering this question incorrectly. It turns out that this technique *does* work if the exponential mapping is in fact replaced by *any* bijection. The resulting operations may not be so natural but they do give rise to a vector space structure.
This vector space structure is presented in the form of a proposition on page 9

$f(x) = e^x$ $\mathbb{R} = (-\infty, \infty)$ $(0, \infty)$

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Proposition

Let $(V, +, .)$ be a real vector space and let W be any other set. If $f : V \to W$ be a bijective mapping, then W acquires the structure of a vector space with vector addition and scalar multiplication defined as follows:

$$w_1 \oplus w_2 := \underbrace{f(f^{-1}(w_1) + f^{-1}(w_2))}, \quad \forall w_1, w_2 \in W$$

and

$$c \star w := f(cf^{-1}(w)), \quad \forall c \in \mathbb{R}, w \in W.$$

$(W, \oplus, \star)$ vector space structure

The proof is a routine verification.

1. Closure under the $\oplus$ operation follows from the fact that W is the codomain of f.

Read the proof of this proposition <u>after</u> proof of uniqueness of additive inverses and additive identity.

2. Commutativity of vector addition in W:

   Let $w_1, w_2 \in W$. Then

$$\begin{aligned} w_1 \oplus w_2 &= f(f^{-1}(w_1) + f^{-1}(w_2)) \\ &= f(f^{-1}(w_2) + f^{-1}(w_1)) \\ &= w_2 \oplus w_1 \end{aligned}$$

3. Associativity of vector addition in W: Let $w_1, w_2, w_3 \in W$. Then

$$\begin{aligned} w_1 \oplus (w_2 \oplus w_3) &= f(f^{-1}(w_1) + f^{-1}(w_2 \oplus w_3)) \\ &= f(f^{-1}(w_1) + f^{-1}(f(f^{-1}(w_2) + f^{-1}(w_3)))) \\ &= f(f^{-1}(w_1) + (f^{-1}(w_2) + f^{-1}(w_3))) \\ &= f((f^{-1}(w_1) + f^{-1}(w_2)) + f^{-1}(w_3)) \\ &= (w_1 \oplus w_2) \oplus w_3 \end{aligned}$$

4 Existence of Additive Identity in W:

Define

$$0_W := f(\mathbf{0}),$$

where $\mathbf{0}$ is a zero vector in V. Let $w \in W$.

$$\begin{aligned} 0_W \oplus w &= f(f^{-1}(0_W) + f^{-1}(w)) \\ &= f(\mathbf{0} + f^{-1}(w)) \\ &= f(f^{-1}(w)) \\ &= w \end{aligned}$$

5 Existence of Additive inverses in W:

Let $w \in W$. Consider $u = f(-f^{-1}(w))$.

Then

$$\begin{aligned} w \oplus u &= f(f^{-1}(w) + f^{-1}(u)) \\ &= f(f^{-1}(w) + (-f^{-1}(w))) \\ &= f(\mathbf{0}) \\ &= 0_W \end{aligned}$$

6. Closure with respect to scalar multiplication follows from the fact that W is the codomain of f.
7. First Distributive Law:

   Let $c \in \mathbb{R}, w_1, w_2 \in W$. Then

$$\begin{aligned} c \star (w_1 \oplus w_2) &= f(cf^{-1}(w_1 \oplus w_2)) \\ &= f(c(f^{-1}(w_1) + f^{-1}(w_2))) \\ &= f(cf^{-1}(w_1) + cf^{-1}(w_2)) \end{aligned}$$

$$c \star w_1 \oplus c \star w_2 = f(f^{-1}(c \star w_1) + f^{-1}(c \star w_2))$$
$$= f(cf^{-1}(w_1) + cf^{-1}(w_2))$$

Hence

$$c \star (w_1 \oplus w_2) = c \star w_1 \oplus c \star w_2$$

8. Second Distributive Law:

   Let $c_1, c_2 \in \mathbb{R}, w \in W$. Then

$$(c_1 + c_2) \star w = f((c_1 + c_2)f^{-1}(w))$$
$$= f(c_1 f^{-1}(w) + c_2 f^{-1}(w))$$

$$c_1 \star w \oplus c_2 \star w = f(f^{-1}(c_1 \star w) + f^{-1}(c_2 \star w))$$
$$= f(c_1 f^{-1}(w) + c_2 f^{-1}(w))$$

   Therefore $(c_1 + c_2) \star w = c_1 \star w \oplus c_2 \star w$.

9 Let $c_1, c_2 \in \mathbb{R}, w \in W$.

$$\begin{aligned} c_1 \star (c_2 \star w) &= f(c_1 f^{-1}(c_2 \star w)) \\ &= f(c_1 c_2 f^{-1}(w)) \\ &= c_1 c_2 \star W \end{aligned}$$

10 Let $w \in W$.

$$\begin{aligned} 1 \star w &= f(1.f^{-1}(w)) \\ &= f(f^{-1}(w)) \\ &= w \end{aligned}$$

Proposition

Let V be a vector space. The zero vector in V is unique.

Proof:

Suppose if possible that there are two zero vectors, say $\mathbf{0}$ and z. Then

$$\mathbf{0} + z = \mathbf{0}, \text{ by fourth axiom}$$

$$z + \mathbf{0} = z, \text{ by fourth axiom}$$

$$\mathbf{0} + z = z + \mathbf{0} \text{ by second axiom}$$

Hence $\mathbf{0} = z$.

Proposition

Let V be a vector space. For every u in V there exists a unique $-u$ called the *negative of* u such that $u + (-u) = \mathbf{0}$.

Proof:

Let u be any vector in V. Suppose if possible that there are two vectors, say $-u$ and v, such that $u + (-u) = \mathbf{0}$ and $u + v = \mathbf{0}$ both hold. Then

$$-u = -u + \mathbf{0} = -u + (u + v) = (-u + u) + v = \mathbf{0} + v = v.$$

Hence $-u = v$.

Proposition

Let V be a vector space. Then

$0u = \mathbf{0}$ and $-u = (-1)u$ hold for every u in V, and
$c\mathbf{0} = \mathbf{0}$, holds for every scalar c

Proof:

Let u be any vector in V. Let $v := 0u$. Since
$0u = (0+0)u = 0u + 0u$, it follows that $v = v + v$. Hence

$$\mathbf{0} = v + (-v) = (v+v) + (-v) = v + (v + (-v)) = v + \mathbf{0} = v.$$

Also, $u + (-1)u = 1u + (-1)u = (1 + (-1))u = 0u = \mathbf{0}$.

Next, let c be any scalar. Let $w := c\mathbf{0}$. Since $c(\mathbf{0}+\mathbf{0}) = c\mathbf{0}$ it follows that $w = w + w$. Hence $w = \mathbf{0}$.

$$\boxed{V = v + v}$$

$$V - v = v$$

$$V + (-v) = v + v + (-v)$$

$$\cancel{-2 + 2} + x = 3 + (-2)$$

Alternative "intuitive" definition: A subspace W of a vector space V is a subset which is also a vector space, with the same addition & scalar multip. and

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Definition

A *subspace* of a vector space V is a subset H of V that has two properties:

a. H is closed under vector addition. That is, for each u and v in H, the sum $u + v$ is in H.

b. H is closed under multiplication by scalars. That is, for each u in H and each scalar c, the vector cu is in H.

Definition

Let $v_1, \ldots, v_p$ be distinct elements of a vector space V, and let $c_1, \ldots, c_p$ be scalars. The vector

$$c_1 v_1 + \ldots + c_p v_p$$

is called a *linear combination* of the vectors $v_1, \ldots, v_p$.

$(f \in V$ means $f : [0,1] \longrightarrow \mathbb{R})$

V = vectors of a continuous real valued functions defined on the $[0,1]$ interval.

$v_1 = \sin x$

$v_2 = \cos x$

$2\sin x + 3\cos x$

2 subspace of vector space of all real valued function defined on $[0,1]$

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$$\text{Span}\, S = \{ v \in V \mid v = c_1 v_1 + \cdots + c_n v_n \text{ for some } v_1, \ldots, v_n \in S,\ c_1, \ldots, c_n \in \mathbb{R} \}$$

Definition

Let S be a nonempty subset of a vector space V. The set of all elements of V that can be expressed as linear combinations of elements of S is called the *span* of S, and is denoted by $Span\, S$. If S is the empty set, we define $Span\, S$ to be the singleton set $\{\mathbf{0}\}$.

Proposition

Let $S \subset V$. Then $Span\, S$ is a subspace of V.

If $S = \phi$, then

$\operatorname{Span} S = \{0\}$.

$\{0\}$ is a subspace of V

because (i) $0 + 0 = 0$

and (ii) $c0 = 0$, $\forall c \in \mathbb{R}$.

Next we consider the case when S is a finite set.

Let $S = \{v_1, \dots, v_p\}$ for some $p \in \mathbb{N}$.

- Closure under vector addition:

Let $u, v \in \operatorname{Span} S$.

$\Rightarrow \exists$ scalars $c_1, \ldots, c_p \in \mathbb{R}$

such that

$$u = c_1 v_1 + \cdots + c_p v_p$$

and $\exists$ scalars $d_1, \ldots, d_p \in \mathbb{R}$

such that

$$v = d_1 v_1 + \cdots + d_p v_p$$

$$u + v = c_1 v_1 + \cdots + c_p v_p$$
$$+ d_1 v_1 + \cdots + d_p v_p$$

$$= \underbrace{(\underline{c_1 + d_1}) v_1 + \cdots + (\underline{c_p + d_p}) v_p}_{\in \operatorname{Span} S.}$$

$\therefore$ Span S is closed under vector addition.

Next let $c \in \mathbb{R}$, $u \in \operatorname{Span} S$.

$\exists\ c_1, \ldots, c_p \in \mathbb{R}$ such that

$u = c_1 v_1 + \cdots + c_p v_p$.

$$\Rightarrow cu = c(c_1 v_1 + \cdots c_p v_p)$$

$$= cc_1 v_1 + \cdots + cc_p v_p$$

$$\in \operatorname{Span} S$$

$\Rightarrow$ Span S is closed under scalar multiplication.

$\Rightarrow$ Span S is a subspace of V.