Classification of Linear Systems

A system of linear equations has either

1. no solution, or
2. exactly one solution, or
3. infinitely many solutions.

will prove this formally later on

A system of linear equations is said to be *consistent* if it has either one solution or infinitely many solutions; a system is *inconsistent* if it has no solution.

Row Reduction

$\underline{A}\,\underline{x} = b$

$x = (x_1, \ldots, x_n)$

$[A \; b]$

Elementary Row Operations

There are three kinds of operations:

1. (Replacement) Replace one row by the sum of itself and a multiple of another row.
2. (Interchange) Interchange two rows.
3. (Scaling) Multiply all entries in a row by a nonzero constant.

Definition

A rectangular matrix is in *echelon form (or row echelon form)* if it has the following three properties:

1. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
2. All nonzero rows are above any rows of all zeros.

A *leading entry* of a row refers to the leftmost nonzero entry (in a nonzero row).

A *nonzero row* or nonzero column in a matrix means a row or column that contains at least one nonzero entry.

If a matrix is in row echelon form then the leading entry in each nonzero row is called a *pivot*.

$\rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ (x_1, x_2, x_3)

Let's look at an example in which we try to use this technique to solve a system of equations.

3X

$$\left.\begin{aligned} x_1 - 3x_2 + 4x_3 &= -4 \\ 3x_1 - 7x_2 + 7x_3 &= -8 \\ -4x_1 + 6x_2 - x_3 &= 7 \end{aligned}\right\}$$

Let us first set up the matrix equation $A\mathbf{x} = \mathbf{b}$.

$$A = \begin{bmatrix} 1 & -3 & 4 \\ 3 & -7 & 7 \\ -4 & 6 & -1 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -4 \\ -8 \\ 7 \end{bmatrix}$$

The augmented matrix $[A \quad \mathbf{b}]$ is

$$\begin{bmatrix} 1 & -3 & 4 & -4 \\ 3 & -7 & 7 & -8 \\ -4 & 6 & -1 & 7 \end{bmatrix} \leftarrow$$

$R_1 \quad 1 \quad -3 \quad 4 \quad -4$

$R_2 \quad 3 \quad -7 \quad 7 \quad -8$

$R_2 \to R_2 - 3R_1$:

$$\begin{bmatrix} 1 & -3 & 4 & -4 \\ 0 & 2 & -5 & 4 \\ -4 & 6 & -1 & 7 \end{bmatrix}$$

$x_1 - 3x_2 + 4x_3 = -4$

$2x_2 - 5x_3 = 4$

$-4x_1 + 6x_2 - x_3 = 7$

$R_3 \to R_3 + 4R_1$:

$$\begin{bmatrix} 1 & -3 & 4 & -4 \\ 0 & 2 & -5 & 4 \\ 0 & -6 & 15 & -9 \end{bmatrix}$$

$R_3 \to R_3 + 3R_2$:

$$\left[\begin{array}{ccc|c} 1 & -3 & 4 & -4 \\ 0 & 2 & -5 & 4 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

This corresponds to the equivalent system:

$$
\begin{aligned}
x_1 - 3x_2 + 4x_3 &= -4 \\
2\,x_2 - 5x_3 &= 4 \\
0 &= 3
\end{aligned}
$$

impossible

Clearly this system has no solution, so it is inconsistent.

If a row echelon form of the augmented matrix of a linear system has a pivot in the augmented column, then it is inconsistent.

Otherwise it's consistent.

If every column other than the augmented column contains a pivot then the system has a unique solution.

For example, in the previous lecture we looked at the system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 - 5x_3 &= 10, \end{aligned}$$

having augmented matrix

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix}$$

We used row operations on the augmented matrix to the the echelon form

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & 30 & -30 \end{array}\right]$$

Thus the equivalent system we obtain is

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 30x_3 &= -30 \end{aligned}$$

We solve this by back substitution to obtain

$$\begin{aligned} x_3 &= -1 \\ x_2 &= \cancel{8}\,6 \\ x_1 &= \cancel{8}\;1 \end{aligned}$$

$x_2 = \frac{1}{2}(8 - 8) = 0$

$x_1 = 2x_2 - x_3$

$x_1 = 0$

Why Do Elementary Row Operations Give Us Equivalent Systems?

For interchange and scaling operations, this is something we know from elementary school. But is the answer as obvious for the replacement operation?

Maybe it is. But the answer is obvious if we look at what's going on in terms of matrices.

Elementary Matrices

i,j-th entry

Applying an elementary row operation on a matrix is the same multiplying the matrix **on the left** by an *elementary matrix*.

Replacement

The operation $R_i \to R_i + cR_j$ is achieved via left multiplicaton by a matrix of the form

$E = (e_{mn})$

$$E = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & c & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix} \qquad (i < j)$$

$$e_{mn} = \begin{cases} 1, & \text{if } m = n \\ c, & \text{if } m = i,\ n = j \\ 0, & \text{otherwise} \end{cases}$$

or

$$E = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & c & & 1 & \\ & & & & & 1 \end{bmatrix} \qquad (i > j)$$

The matrix has a c as its ij-th entry and otherwise looks like the $m \times m$ identity matrix.

$R_i \to R_i + c R_j$

$i = 3$ $c = -5$

$j = 1$

Let's look at the example we saw earlier from the perspective of elementary matrices.

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 5R_1} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 10 & -10 & 10 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix} \quad //$$

Similarly the operations of row interchange and row scaling can also be achieved via left multiplication by elementary matrices.

$R_i \longleftrightarrow R_j$

Interchange

$$E = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 0 & & 1 & \\ & & & \ddots & & \\ & & 1 & & 0 & \\ & & & & & 1 \end{bmatrix}$$

$E = (e_{mn})$

$$e_{mn} = \begin{cases} 1, & \text{if } m = n,\ m \neq i,\ m \neq j \\ 1, & \text{if } m = i,\ n = j \text{ or if } m = j,\ n = i \\ 0, & \text{otherwise} \end{cases}$$

Let's look at an example.

$$Ax = b$$

$$E^{-1} EAx = E^{-1}Eb$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix} \xrightarrow{R_3 \longleftrightarrow R_1} \begin{bmatrix} 5 & 0 & -5 & 10 \\ 0 & 2 & -8 & 8 \\ 1 & -2 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix}$$

$R_i \to R_i + cR_j$

$i = 1$

$j = 2$

$c = 5$

$$\begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$$

$$\Big\downarrow R_1 \to R_1 + 5R_2$$

$$\begin{bmatrix} 15 & 22 \\ 2 & 3 \end{bmatrix}$$

$R_1 \to R_1 + 5R_2$

$$\begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & 22 \\ 2 & 3 \end{bmatrix}$$

$$R_i \longrightarrow cR_i$$

Scaling

$$E = \begin{bmatrix} 1 & & & & & \\ & \ddots & & 0 & & \\ & & c & & & \\ & 0 & & \ddots & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}$$

ith entry on diagonal

High School Recap - Inverse of a Matrix

Recall that an $n \times n$ matrix A has an *inverse*, which is denoted by A^{-1} if and only if the determinant of A is nonzero.

In fact the inverse of A can be computed by using

Cramer's Rule

If $\det A \neq 0$ then

$$A^{-1} = \frac{1}{\det A} Adj\, A$$

where the *adjoint* of A, written $Adj\, A$, is the $n \times n$ matrix whose ij-th entry is $(-1)^{i+j} \det A_{ji}$, where A_{ij} is the matrix obtained by crossing out the i-th row and j-th column of A. Also recall that

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \ldots \pm a_{n1} \det A_{n1}$$

We will look at the determinants and invertible matrices again, a little later in the course.

Inverses of Elementary Matrices

Elementary Matrices of all three kinds are invertible.

Replacement: $R_i \to R_i + cR_j$

For $i < j$

$R_i \to R_i - cR_j$

$$E = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & c & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}, \quad E^{-1} = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & -c & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}$$

Replacement: $R_i \to R_i + cR_j$

For $i > j$

$$E = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & c & & 1 & \\ & & & & & 1 \end{bmatrix}, \quad E^{-1} = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & -c & & 1 & \\ & & & & & 1 \end{bmatrix}$$

In terms of row operations, the *reverse* row operation is $R_i \to R_i - cR_j$.

Let's look at our example again.

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 5R_1} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 10 & -10 & 10 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 10 & -10 & 10 \end{bmatrix} \xrightarrow{R_3 \to R_3 + 5R_1} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix}$$

Interchange

$$E = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 0 & & 1 & \\ & & & \ddots & & \\ & & 1 & & 0 & \\ & & & & & 1 \end{bmatrix} = E^{-1}$$

Not surprisingly, the reverse of the row interchange operation is also itself.

Scaling

$$E = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & c & & & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}, \quad E^{-1} = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & \frac{1}{c} & & & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}$$

The system of equations

$$A\mathbf{x} = \mathbf{b}$$

and

$$EA\mathbf{x} = E\mathbf{b}$$

are equivalent systems, having the same set of solutions.

$E \leftarrow$ elementary

$EA \neq AE$

in general