Review

Proposition

Let V and W be vector spaces. Let $T : V \to W$ be a linear transformation.

T is 1-1 (injective) if and only if $\ker T = \{0\}$.

Review

A 1-1 transformation preserves linear independence.

Proposition

Let V and W be vector spaces. Let $T : V \to W$ be a 1-1 linear transformation.

Then if $\{v_1, \ldots, v_n\}$ is a linearly independent subset of V then $\{T(v_1), \ldots, T(v_n)\}$ is a linearly independent subset of W.

Review

$T : V \longrightarrow \underline{W} \xrightarrow{\text{incl}} W'$

does not

alter the much

picture

Theorem (Rank-Nullity Theorem)

Let V, W be vector spaces. Let $T : \boxed{V} \to W$ be a linear transformation. Then

$$\dim \ker T + \dim \operatorname{range} T = \dim V.$$

If V is infinite dimensional then least one of the summands on left hand side is infinity (i.e. either ker T *or* range T*, or both, must be infinite dimensional).*

Review

$n \times m$

$\mathbb{R}^n \longrightarrow \mathbb{R}^m$

Corollary

Let A be an $m \times n$ matrix. Then

$$\dim \operatorname{col} A + \dim \operatorname{nul} A = n.$$

Definition

Let A be an $m \times n$ matrix. The dimension of col A is called the *rank* of A.

Review

Up to here

This is

the syllabus

Definition

Let A be an $m \times n$ matrix. The dimension of row A is called the *row rank* of A.

Theorem

The row rank of A equals the column rank of A.

Continued

Proposition

Let A be an $m \times n$ matrix. The rows containing pivots in the RREF of A are a basis for the row space of A.

We showed in the previous lecture that the row space of A is the same as the row space of the RREF of A.

Example:

$$(c_1, 0, 0, c_1, 0) + (0, 0, c_2, 2c_2, 0) + (0, 0, 0, 0, c_3)$$

$$= (c_1, 0, 0,$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{row } A \subset \mathbb{R}^5$$

$$\begin{cases} c_1 = 0 \\ c_2 = 0 \\ c_3 = 0 \end{cases}$$

$i_1 = 1$

$i_2 = 3$

$i_3 = 5$

$$\{ \underbrace{(1, 0, 0, 1, 0)}_{a_1^T}, \underbrace{(0, 0, 1, 2, 0)}_{a_2^T}, \underbrace{(0, 0, 0, 0, 1)}_{a_3^T} \}$$

Assertion: Suppose A is an $m \times n$ matrix in reduced echelon form. The the rows of A containing pivots form a basis of the row space of A.

Let $\underbrace{a_1^T, a_2^T, \dots}_{k}, a_m^T$ be the rows of A. A row of A can either contain a pivot or be a row of zeros. Let $a_1^T, a_2^T, \dots, a_k^T$ be those rows of A which contain pivots. Then rest of the rows must be rows of 0s.

$\therefore$ row $A = \operatorname{Span}\{a_1^T, \ldots, a_k^T\}$.

Consider the equation

$$\left.\begin{array}{l} c_1 a_1^T + c_2 a_2^T + \cdots + c_k a_k^T = 0 \\ \text{where } c_1, \ldots, c_k \in \mathbb{R} \end{array}\right\} (*)$$

We wish to show that $(*)$ has only the trivial solution.

Let $(c_1, \ldots, c_k)$ be a solution of $(*)$.

Let the pivot entry in the jth row be called $(A_j^T)_{ij}$.

Then since

$$c_1 a_1^T + \cdots + c_k a_k^T = 0.$$

For any $j = 1, \ldots, k$

$$\left(c_1 a_1^T + \cdots \, c\, a_k^T\right)_{ij} = 0$$

$$\Rightarrow \left(c_j a_j^T\right)_{ij} = 0.$$

$$\Rightarrow c_j = 0.$$

$\Rightarrow$ $(\cancel{*})$ has only the trivial solution. $\square$.

New Chapter

$A \in M_{n \times n}(\mathbb{R})$. square matrix

Definition

An *eigenvector* of an $n \times n$ matrix A is a nonzero vector $\mathbf{x} \in \mathbb{R}^n$ (or in $\mathbb{C}^n$) such that $A\mathbf{x} = \lambda\mathbf{x}$ for some $\lambda \in \mathbb{R}$ (or $\lambda \in \mathbb{C}$). A number λ is called an *eigenvalue* of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an $\mathbf{x}$ is called an *eigenvector corresponding to* λ.

Let's look at an example when $n = 2$.

$A\underline{x} = \lambda \underline{x}$

number

Let $a \in \mathbb{R}$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ -a \end{bmatrix} = \boxed{0}\begin{bmatrix} a \\ -a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\lambda_1 = 0$ $\qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a \\ -a \end{bmatrix}$ is an eigenvector.

$b \in \mathbb{R}$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} b \\ b \end{bmatrix} = 2\begin{bmatrix} b \\ b \end{bmatrix} = \begin{bmatrix} 2b \\ 2b \end{bmatrix}$$

$\lambda_2 = 2$ $\qquad \begin{bmatrix} b \\ b \end{bmatrix}$ is an eigenvector

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

The equation

$Ax = \lambda x$ is the

same as

$$\boxed{(A - \lambda I)x = 0}$$

$$\boxed{Ax = \lambda x}$$

$$\Updownarrow$$

$$Ax - \lambda x = 0$$

$$\Updownarrow$$

$$(A - \lambda I)x = 0$$

$$\underline{B} = A - \lambda I \qquad \boxed{Bx = 0}$$

$$= \begin{bmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix}$$

$$\det B = (1-\lambda)^2 + 1$$

$$= \lambda^2 - 2\lambda + 2$$

$$\boxed{\begin{matrix} 1+i \\ 1-i \end{matrix}}$$

$p(B) = 0$ $\qquad$ $\boxed{Bx = 0}$

$B^2 - 2B + 2I = 0.$

λ_1, λ_2 are roots of $p(x)$.

$\lambda_1 + \lambda_2 = 2.$

The eigenvalues of A are

$\lambda_1 = 1+i$, $\underline{\lambda_2 = 1-i}$.

First system is

$$Ax = (1+i)x.$$

$$\iff (A-(1+i)I)x = 0$$

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$B_1 = A - (1+i)I = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}$$

$$B_1 x = 0$$

$$\Longleftrightarrow \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \xrightarrow{R_2 \to R_2 - iR_1} \begin{bmatrix} -i & -1 \\ 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 \to +iR_1} \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$$

$x_1 - ix_2 = 0$

$x_1 = ix_2$

$$\begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_2 \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} ix_2 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} i-1 \\ i+1 \end{bmatrix}$$

$$(1+i) \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} i-1 \\ i+1 \end{bmatrix} \checkmark$$

Find the other eigenspace yourself.

$(A-\lambda I)x = 0$ $\boxed{Bx = 0}$ has a solution $\Longleftrightarrow$ B is not invertible

$Ax = \lambda \cancel{I}$

Theorem

λ is an eigenvalue of A if and only if $\det(A - \lambda I) = 0$*.*

Proof.

$\boxed{\det(A-\lambda I) = 0}$

λ is an eigenvalue of A

$\Longleftrightarrow$ the equation $(A - \lambda I)\cancel{\mathbf{x}} = \mathbf{0}$ has a nontrivial solution, where $x \in \mathbb{R}^n$ or $x \in \mathbb{C}^n$.

$\Longleftrightarrow$ $A - \lambda I$ is not invertible

$\Longleftrightarrow$ $\det(A - \lambda I) = 0$.

□

$(A - \lambda I)x = 0$.

$$p(\lambda) = \det(A - \lambda I)$$
$$= 0.$$

Definition

$p(\lambda) = \det(A - \lambda I)$ is called the *characteristic polynomial* of A.
The equation

$$\det(A - \lambda I) = 0$$

is called the *characteristic equation* of A.

$Ax = \lambda x$

Definition

If λ is an eigenvalue of A then the nullspace of $A - \lambda I$ is called the *eigenspace* corresponding to λ. When $\lambda \in \mathbb{R}$, the set

$$\{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \lambda\mathbf{x}\}$$

is called the *real λ-eigenspace*, and the set

$$\{\mathbf{x} \in \mathbb{C}^n \mid A\mathbf{x} = \lambda\mathbf{x}\}$$

is called the *complex λ-eigenspace*.

Example

$$A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$$

Find a basis for the eigenspace corresponding to the eigenvalue 2.

We solve $\underline{\det(A-\lambda I)=0}$.

$$\det(A-\lambda I) = \begin{vmatrix} 4-\lambda & -1 & 6 \\ 2 & 1-\lambda & 6 \\ 2 & -1 & 8-\lambda \end{vmatrix}$$

Check computations yourself.

Find eigenvalues : DIY.

$$B = A - 2I = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix}$$

Solve : $Bx = 0$.

$$\begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 2 & -1 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to \frac{1}{2} R_1$

$$\begin{bmatrix} 1 & -\frac{1}{2} & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 \quad \underbrace{x_2 \quad x_3}_{\text{free}}$$

$$x_1 = \frac{1}{2} x_2 - 3x_3$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

The 2-eigenspace is

$$\mathrm{Span}\left\{ \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

A and B are similar

if $\exists$ an invertible matrix P

such that $\boxed{PAP^{-1} = B}$.

Theorem

Similar matrices have the same characteristic polynomial and therefore the same eigenvalues, **with the same multiplicities**.

Claim: $\underline{\det(A - \lambda I) = \det(B - \lambda I)}$

[multiplicity means multiplicity as a root]

$$\det(B-\lambda I)$$

$$=\det(PAP^{-1}-\lambda I)$$

$$=\det(PAP^{-1}-\lambda PIP^{-1})$$

$$=\det(P(A-\lambda I)P^{-1})$$

$$=\cancel{\det(P)}\det(A-\lambda I)\cancel{\det(P^{-1})}$$

$$\det(AB)=\det A\det B$$

$$I=II=PP^{-1}$$

$$\det(P^{-1})=\frac{1}{\det P}$$

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 2 & 7 \\ 0 & 0 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

Theorem

The eigenvalues of a triangular matrix are the entries on its main diagonal.

Pf: $A = (a_{ij})$

If A is a triangular matrix

Then $(A - \lambda I)$ is also triangular.

$\therefore\ p(\lambda) = \det(A - \lambda I) = (a_{11} - \lambda)(a_{22} - \lambda)$

$\cdots (a_{nn} - \lambda)$

roots of $p(\lambda)$ are $a_{11}, a_{22}, \ldots, a_{nn}$.

Diagonalization

$[T]_{\mathcal{B}} = P_{\mathcal{B}}^{-1} A P$ $\quad \mathcal{B} = \{b_1, \dots, b_n\}$

$x \mapsto Ax$ $\quad P_{\mathcal{B}} = [b_1 \cdots b_n]$

T

Definition

A square matrix A is said to be *diagonalizable* if A is similar to a diagonal matrix, that is, if $A = PDP^{-1}$ for some invertible matrix P and some diagonal matrix D.

Theorem

Let A be an $n \times n$ matrix. Then A is diagonalizable if and only if there exists a basis of $\mathbb{R}^n$ consisting of eigenvectors of A.

(In other words, an $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors. This statement is called the Diagonalization Theorem in your course textbook.)

Example

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ 2x_2 \\ 3x_3 \end{pmatrix}$$

Recall

Proposition

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Let A be the standard matrix of T. Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be any ordered basis of $\mathbb{R}^n$. Let

$$P_{\mathcal{B}} = [\mathbf{b}_1 \quad \cdots \quad \mathbf{b}_n]$$

be the change-of-coordinates matrix from $\mathcal{B}$ to the standard basis in $\mathbb{R}^n$. Then the $\mathcal{B}$-matrix of T is $P_{\mathcal{B}}^{-1} A P_{\mathcal{B}}$.

Example

Diagonalize the following matrix, if possible.

$$A = \begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & -1 \\ 1 & -1 & 5 \end{bmatrix}.$$