Review: The Gram-Schmidt Process

Theorem

Given a basis $\{v_1, \ldots, v_p\}$ for an inner product space $(V, \langle ., . \rangle)$, define

$$
\begin{aligned}
w_1 &= v_1 \\
w_2 &= v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 \\
&\ \vdots \\
w_p &= v_p - \frac{\langle v_p, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_p, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2 - \cdots - \frac{\langle v_p, w_{p-1} \rangle}{\langle w_{p-1}, w_{p-1} \rangle} w_{p-1}
\end{aligned}
$$

Then $\{w_1, \ldots, w_p\}$ is an orthogonal basis for V. In addition

$$
\underbrace{\mathrm{Span}\{w_1, \ldots, w_k\}}_{} = \underbrace{\mathrm{Span}\{v_1, \ldots, v_k\}}_{} \qquad \textit{for } 1 \le k \le p
$$

Review

Definition

A matrix A is said to be *orthogonally diagonalizable* if there exists an orthogonal matrix P and a diagonal matrix D such that $A = PDP^T = PDP^{-1}$.

Theorem (The Spectral Theorem for Symmetric Matrices)

An $n \times n$ symmetric matrix A has the following properties:

1. *A has n real eigenvalues, counting multiplicities.*
2. *The dimension of the eigenspace for each eigenvalue λ equals the multiplicity of λ as a root of the characteristic equation.*
3. *The eigenspaces are mutually orthogonal, in the sense that eigenvectors corresponding to different eigenvalues are orthogonal.*
4. *A is orthogonally diagonalizable.*

$$\boxed{U^T U = I}$$

$$\boxed{U : \mathbb{R}^r \to \mathbb{R}^m}$$

Theorem

Let U be an $m \times n$ matrix with orthonormal columns, and let $\mathbf{x}$ *and* $\mathbf{y}$ *be in* $\mathbb{R}^n$. *Then*

1. $\|U\mathbf{x}\| = \|\mathbf{x}\|$
2. $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$
3. $(U\mathbf{x}) \cdot (U\mathbf{y}) = 0$ *if and only if* $\mathbf{x} \cdot \mathbf{y} = 0$

$$② \quad (Ux) \cdot (Uy) = x^T \underbrace{U^T U}_{} y = x^T y = x \cdot y$$

$$① \quad \|Ux\|^2 = (Ux) \cdot (Ux) = x \cdot x = \|x\|^2$$

isometry

$$T(v+w) = T(v) + T(w) \quad \forall v,w \in \mathbb{R}^n$$

$$T(cv) = cT(v) \quad \forall c \in \mathbb{R} \ \forall v \in \mathbb{R}^n$$

Theorem

If $T : \mathbb{R}^n \to \mathbb{R}^n$ preserves the dot product, i.e. if

$$T(\mathbf{x}) \cdot T(\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}, \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n,$$

then T is a linear map.

This justifies the assumption we made earlier that rotations and reflections are linear transformations.

Proof: $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^n$.

Claim: $\{T(e_1), \ldots, T(e_n)\}$ is also a basis of $\mathbb{R}^n$.

For $i \neq j$,

$$T(e_i) \cdot T(e_j) = e_i \cdot e_j = 0.$$

$\therefore \{T(e_1), \ldots, T(e_n)\}$ is an orthogonal set.

For $j = 1, \ldots, n$

$$T(e_j) \cdot T(e_j) = e_j \cdot e_j = 1.$$

$\therefore \{T(e_1), \ldots, T(e_n)\}$ is an orthogonal set of n unit vectors.

$\therefore \{T(e_1), \ldots, T(e_n)\}$ is an orthonormal basis of $\mathbb{R}^n$.

Next let $v, w \in \mathbb{R}^n$.

$$T(v+w) = \underline{\langle T(v+w), T(e_1) \rangle} T(e_1) + \ldots + \langle T(v+w), T(e_n) \rangle T(e_n)$$

$$= \langle v+w, e_1 \rangle T(e_1) + \cdots + \langle v+w, e_n \rangle T(e_n)$$

$$= \underbrace{\langle v, e_1 \rangle T(e_1) + \langle v, e_2 \rangle T(e_2) + \cdots + \langle v, e_n \rangle T(e_n)}$$

$$+ \langle w, e_1 \rangle T(e_1) + \cdots + \langle w, e_n \rangle T(e_n)$$

$$= \underbrace{\langle T(v), T(e_1) \rangle T(e_1) + \cdots + \langle T(v), T(e_n) \rangle T(e_n)}$$

$$+ \langle T(w), T(e_1) \rangle T(e_1) + \cdots + \langle T(w), T(e_n) \rangle T(e_n)$$

$= T(v) + T(w)$.

So condition (1) of definition of linear transformation holds.

Condition : verify for rw.

Definition

Let V be a vector space with an inner product $\langle ., . \rangle$.Let W be a subspace of V. We define the *orthogonal complement* of W in V to be the set

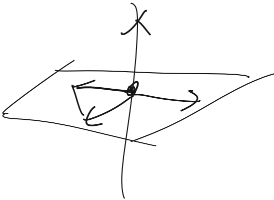
$$W^{\perp} = \{v \in V \mid \langle v, w \rangle = 0, \; \forall w \in W\}.$$

Example:

$V = \mathbb{R}^n$, and $\langle ., , \rangle$ is the standard dot product.

Let A be an $m \times n$ matrix and let

$$W = \operatorname{Nul} A.$$



Then

$$W^{\perp} = \operatorname{Row} A.$$

Orthogonal Decomposition of a Vector Space

Definition

Let V be a vector space. Let U and W be subpaces of V. Then $U + W$ is defined as

$$U + W = \{v \in V \mid v = u + w, \text{ for some } u \in U, w \in W\}.$$

$U + W$ is a subspace of V.

Proof: Since $U = U + \{0\}$,
$U \subset U + W$.

Since $U \neq \emptyset$, $U + W \neq \emptyset$.

Let $v_1, v_2 \in U + W$.

$\Rightarrow$ $v_1 = u_1 + w_1$ for some $u_1 \in U$, $w_1 \in W$

and $v_2 = u_2 + w_2$ for some $u_2 \in U$, $w_2 \in W$.

$\therefore \quad v_1 + v_2 = u_1 + w_1 + u_2 + w_2$

$$= (u_1 + u_2) + (w_1 + w_2)$$

$u_1 + u_2 \in U \quad , \quad w_1 + w_2 \in W$

$\Rightarrow \quad v_1 + v_2 \in U + W .$

$\therefore$ Condition 1 for being a subspace is verified.

Let $c \in \mathbb{R}$, $v \in U + W$.

Then $v = u + w$ for some $u \in U$, $w \in W$.

$\therefore$ $cv = c(u+w) = cu + cw$.

$cu \in U$, $cw \in W \Rightarrow cv \in U + W$.

$\therefore$ 2nd condition verified.

$\therefore$ $U + W$ is a subspace of V.

Proposition

Let V be a vector space with an inner product $\langle ., . \rangle$. Let W be a subspace of V. Then $W^\perp$ is a subspace of V and

$$W \cap W^\perp = \{0\}$$

Further, if V is finite-dimensional then $V = W + W^\perp$ and

$$\dim V = \dim W + \dim W^\perp$$

Pf) Claim: $W^\perp$ is a subspace of V.

Clearly $\langle 0, v \rangle = 0, \ \forall v \in W$

$\therefore \ 0 \in W^\perp \Rightarrow W^\perp \neq \phi$.

Next let $u_1, u_2 \in W^\perp$.

$\Rightarrow \langle u_1, v \rangle = 0, \ \forall v \in W$ — 1 and $\langle u_2, v\rangle = 0, \quad \forall v \in W.$ — ②

From ① & ② we get that

$$\langle u_1 + u_2, v\rangle = \langle u_1, v\rangle + \langle u_2, v\rangle$$
$$= 0, \quad \forall v \in W$$

$\therefore \quad u_1 + u_2 \in W^\perp \Rightarrow W^\perp$ is closed under addition.

Next let $c \in \mathbb{R}$, $u \in W^\perp$.

Then $\langle u, v \rangle = 0$, $\forall v \in W$. — (3).

From (3),

$\langle cu, v \rangle = c \langle u, v \rangle = 0$, $\forall v \in W$

$\Rightarrow cu \in W^\perp$. $\therefore W^\perp$ is closed under scalar mult.

$\therefore$ $W^\perp$ is a subspace of V.

Claim: $W \cap W^\perp = \{0\}$.

Let $v \in W \cap W^\perp$

Since $v \in V$ and $v \in W^\perp$,

$\Rightarrow \langle v, v \rangle = 0$

$\Rightarrow v = 0$. $\Rightarrow W \cap W^\perp \subset \{0\}$.

$\therefore \quad W \cap W^{\perp} = \{0\}.$

Claim: If V is finite dimensional the $V = W + W^{\perp}$ and $\dim V = \dim W + \dim W^{\perp}$.

Let $\dim V = n$.

Since W is a subspace of V, W is finite dimensional.

Let $k = \dim W$.

Let $\beta = \{w_1, \ldots, w_k\}$ be a basis of W.

Extend β to a basis $\{w_1, \ldots, w_k, w_{k+1}, \ldots, w_n\}$ of V.

We apply Gram-Schmidt to obtain and orthogonal basis $\underbrace{\{v_1, \ldots, v_n\}}$ of V such that

$\operatorname{Span}\{v_1, \ldots, v_k\} = \operatorname{Span}\{w_1, \ldots, w_k\} = W$.

$\therefore \{v_1, \ldots, v_k\}$ is a basis of W.

It suffices to show that $\{v_{k+1}, \ldots, v_n\}$ is a basis of $W^\perp$.

Let $v \in W^\perp$.

Then $v = \dfrac{\langle v_1, v \rangle}{\langle v_1, v_1 \rangle} v_1 + \cdots + \dfrac{\langle v_n, v \rangle}{\langle v_n, v_n \rangle} v_n$

Since $v \in W^\perp$, the first k terms vanish.

$$\therefore\ v = \frac{\langle v_{k+1}, v\rangle}{\langle v_{k+1}, v_{k+1}\rangle} v_{k+1} + \cdots + \frac{\langle v_n, v\rangle}{\langle v_n, v_n\rangle} v_n.$$

$$\in \operatorname{Span}\{v_{k+1}, \ldots, v_n\}.$$

$$\therefore\ W^\perp \subset \operatorname{Span}\{v_{k+1}, \ldots, v_n\}.$$

Clearly $v_{k+1}, \dots, v_n \in W^\perp$,
because $\langle v_j, v_i \rangle = 0$, for $j \in \{k+1, \dots, n\}$, $i \in \{1, \dots, k\}$.

$\Rightarrow \operatorname{Span}\{v_{k+1}, \dots, v_n\} \subset W^\perp$

$\therefore W^\perp = \operatorname{Span}\{v_{k+1}, \dots, v_n\}$.

$\dim W^\perp = n - \dim W$. $\square$.

Claim: $V = W + W^{\perp}$.

Let $v \in V$.

$$v = \left(\frac{\langle v_1 , v \rangle}{\langle v_1 , v_1 \rangle} v_1 + \cdots + \frac{\langle v_k , v \rangle v_k}{\langle v_k , v_k \rangle} \right)$$

$$+ \left(\frac{\langle v_{k+1} , v \rangle}{\langle v_{k+1} , v_{k+1} \rangle} v_{k+1} + \cdots + \frac{\langle v_n , v \rangle}{\langle v_n , v_n \rangle} v_n \right)$$

$\in W + W^{\perp}$.

$\therefore \quad V \subset W + W^{\perp}$.

Clearly $W + W^{\perp} \subset V$.

$\therefore \quad V = W + W^{\perp}$.

$\text{Nul}\,A^T = (\text{Row}\,A^T)^\perp = (\text{Col}\,A)^\perp$

Important.

Theorem

Let A be an $m \times n$ matrix. The orthogonal complement of the row space of A is the nullspace of A, and the orthogonal complement of the column space of A is the nullspace of A^T :

$$(\text{Row}\,A)^\perp = \text{Nul}\,A \quad \textit{and} \quad (\text{Col}\,A)^\perp = \text{Nul}\,A^T$$

The k-th row of A is

$(a_{k1}, a_{k2}, \cdots, a_{kn})$.

If $x = (x_1, \dots, x_n) \in \text{Nul} A$.

Then clearly

$$a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n = 0.$$

Since $k \in \{1, \dots, n\}$ is arbitrary

$\therefore$ x is orthogonal to the rows of A.

$\therefore$ $\text{Nul}(A) \subset (\text{Row} A)^{\perp}$.

We know that

$$\dim \operatorname{Nul} A = n - \dim \operatorname{col} A$$

$$= n - \dim \operatorname{row} A$$

$$= \dim (\operatorname{row} A)^{\perp}$$

$$\therefore \operatorname{Nul} A = (\operatorname{row} A)^{\perp}.$$

Since $\quad \mathrm{Col}\,A = \mathrm{Row}\,A^T$,

$$(\mathrm{Col}\,A)^{\perp} = (\mathrm{Row}\,A^T)^{\perp}$$

$$= \mathrm{Nul}\,A^T.$$

Orthogonal Decomposition Theorem

$V = W + W^{\perp}$
$W \cap W^{\perp} = \{0\}$

Theorem

Let V be a finite dimensional vector space with an inner product $\langle ., . \rangle$. Let W be a subspace of V. Then every vector $v \in V$ can be written uniquely in the form

$$v = w + \tilde{w}$$

where $w \in W$ and $\tilde{w} \in W^{\perp}$.

Definition

Let V, W, v, w and $\tilde{w}$ be as in the above theorem. We define the *orthogonal projection of v onto W* to be

$$\operatorname{proj}_W v = w$$

and the *component of v orthogonal to W* to be $\tilde{w} = v - \operatorname{proj}_W v$.