Introduction to Algorithms

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IIIT, Delhi
Summer Semester,
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About Myself

Name: Subhabrata Samajder

Research Interests:

- Lattice based cryptography
- Statistical aspects of symmetric key cryptanalysis
- Broadcast Encryption
- Blockchain
- e-Voting
- Random graphs

About Myself

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Course Webpage: On Google classroom

- Class Code: s34dfee

Academic Integrity Policy

- Anyone caught cheating or copying will be penalised.
- Plagiarism cases will be *dealt strictly*.
- Take this opportunity to stay away from plagiarism forever.

Grading plan: Tentative Grading Components

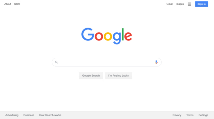
Components	Number	Weightage
MidSem Theory	1	25%
MidSem Lab	1	10%
EndSem Theory	1	25%
EndSem Lab	1	10%
Lab	11 to 13	20%
Quiz and/or Homework	≥ 4	10%

Introduction to Algorithms

Algorithms: In Our Daily Lives

Cooking

Traffic Lights



Google Search

Sorting Vinyl Records

Work Commute

Online Shopping

Algorithms: Rubik's Cube

Solve:

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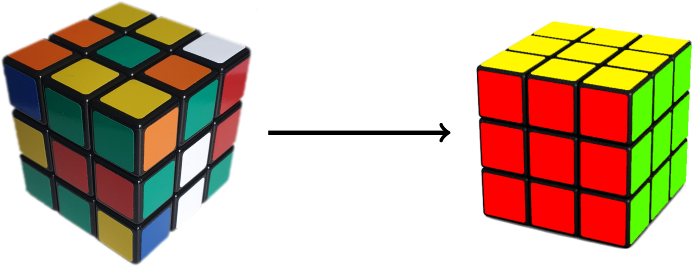

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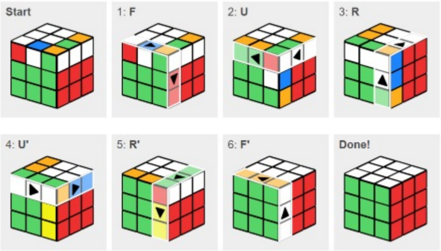

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Elementary Operations

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The elementary operations that we will consider will be at a higher level and include arithmetic and logical operations.

Finiteness

Algorithms: Finiteness $\Rightarrow$ an algorithm must stop.

- **Example:** Compute $(a + b) * (c + d)$

$$t_1 = a + b; \quad t_2 = c + d; \quad t_3 = t_1 * t_2.$$

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Computational Method: A procedure that has all of the characteristics of an algorithm except that it possibly *lacks finiteness*.

- **Example:** $\mathrm{while}(1)\{\}$

Sequence

$$t_1 = a + b; \quad t_2 = c + d; \quad t_3 = t_1 * t_2.$$

- We emphasise on the sequential nature of the procedure.

- Any permutation of this sequence *does not* give the same desired output.
  - **Example:** $t_3 = t_1 * t_2; \quad t_2 = c + d; \quad t_1 = a + b$, is not the same as the algorithm above.

Finite Sequence

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Note: Sometimes different orderings of the operations may give rise to the *same* result.

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- **Single computing unit (a processor):** *Sequential execution*.
- **Two computing unit:** Can be executed simultaneously!

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We would only be concentrating on sequential algorithms!!

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  Example: Searching Problem

  - **I/P:** A list L of integer values and another value v.
  - **Question:** Does $v \in L$?
  - **O/P:** '*yes*' if $v \in L$; else it returns '*no*'.

Note: *Decision problems appear rather simple but much of the sophistication of the area of algorithms can be discovered by studying such algorithms*!!

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- Here, resources $\equiv$ the time of execution and the space required by the algorithm.

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- **Other resources:** For example, power consumption is important for battery operated devices.

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- **Size of inputs:** A *function* from the *set of all possible inputs to* $\mathbb{Z}^+$.

- Fixing a positive integer n fixes the set of all inputs of size n and this is a typically a *finite set*.

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  - Additions: 2
  - Multiplications: 1
  - Time: $2\times$Cost of Additions $+ 1\times$Cost of Multiplication

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Example:

- *Search Problem:* $|L|$.
- *Arithmetic Problem:* $\max\{\log_2 a, \log_2 b, \log_2 c, \log_2 d\}$.
  - Additions: 2
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Note:

- # steps can vary across two different inputs of size n.
- $\therefore$ given n, one *cannot define a unique $t(n)$* such that the algorithm requires *exactly $t(n)$ steps* on any input of size n.

Two Ways to Tackle this Problem

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2. **Average-case time complexity:** Considers the average case behaviour of the algorithm.
   - For each n, the set of all inputs of size n is assumed to be finite.
   - Define a *uniform distribution* on this set.
   - Then the time function $T(n)$ becomes a *random variable*.
   - *Average-case time complexity* $= E[T(n)]$ (function of n).

Runtime Function of an Algorithm (Cont.)

- We Will mostly focus on the worst-case time complexity.

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- Analogously, one can also formulate the worst-case and average-case *space* required by an algorithm.

Arithmetic Problem

f(a, b, c, d):

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\begin{aligned}
t_1 &= a + b \\
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  - Depends on the size of integers a, b, c and d.
  - The sizes of a, b, c and d can vary.
  - Assume that $n = \max\{\lceil \log_2 a \rceil, \lceil \log_2 b \rceil, \lceil \log_2 c \rceil, \lceil \log_2 d \rceil\}$.
  - Adding two n-bit integers take time proportional to n.
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  - Adding two n-bit integers take time proportional to n.
  - Multiplying two n-bit integers take time proportional to $n^{\log_2 3}$.
- **Size of input:** n.
- **Time complexity:** proportional to $n^{\log_2 3}$.

Books Consulted

1. Chapter 2 of *A Course on Cooperative Game Theory* by Satya R. Chakravarty, Palash Sarkar and Manipushpak Mitra.

2. *Introduction to Algorithms: A Creative Approach* by Udi Manber.

Thank You for your kind attention!