Bayes' and Law of Total Probability

For an event space $\{B_1, \ldots, B_m\}$ with $P[B_i] > 0$ for all i,

$$P[B_i|A] = \frac{P[AB_i]}{P[A]} = \frac{P[A|B_i]P[B_i]}{P[A]}$$
$$= \frac{P[A|B_i]P[B_i]}{\sum_{i=1}^{m} P[A|B_i]P[B_i]}$$

We know the a priori probabilities of the B_i(s) and also the probability of an observable event A given B_i. Having seen A (the effect) we want to know the chance that a certain B_i is the cause (that caused A)

Bayes' and Law of Total Probability

- Let A be the event {A world class athlete is seen}
- Consider the event space (World) contains countries of origins. It is W = {China, Jamaica, India, Malaysia, Pakistan, Sri Lanka, UK, US,…}. Let B_1 = China and so on…
- $P[B_i]$ is the probability of the event that the country of origin of a human is B_i
- A priori we could calculate $P[B_i]$ as the ratio of the population of B_i and the population of the world.
- We could also arrive at $P[A|B_i]$, say based on past statistics
- Now say I tell you that I saw an athlete, that is observed A, and I want to know what is probability that the athlete is from India
  - I want to know P[India|A] or $P[B_3|A]$

INNOCENT OR GUILTY

From the book "Randomness"

85% of cabs in your city are red. The rest are blue.

$P[R] = 0.85$

$P[B] = 0.15$

$R \triangleq$ Event that a randomly chosen car is red.

A cab is involved in a hit and run incident.

A witness identifies the cab as blue.

A judge declares that a blue cab is guilty.

$P[\text{Witness identifies blue} \mid B] = 0.8$

$P[\text{Witness identifies red} \mid R] = 0.8$

Did the judge get this right?

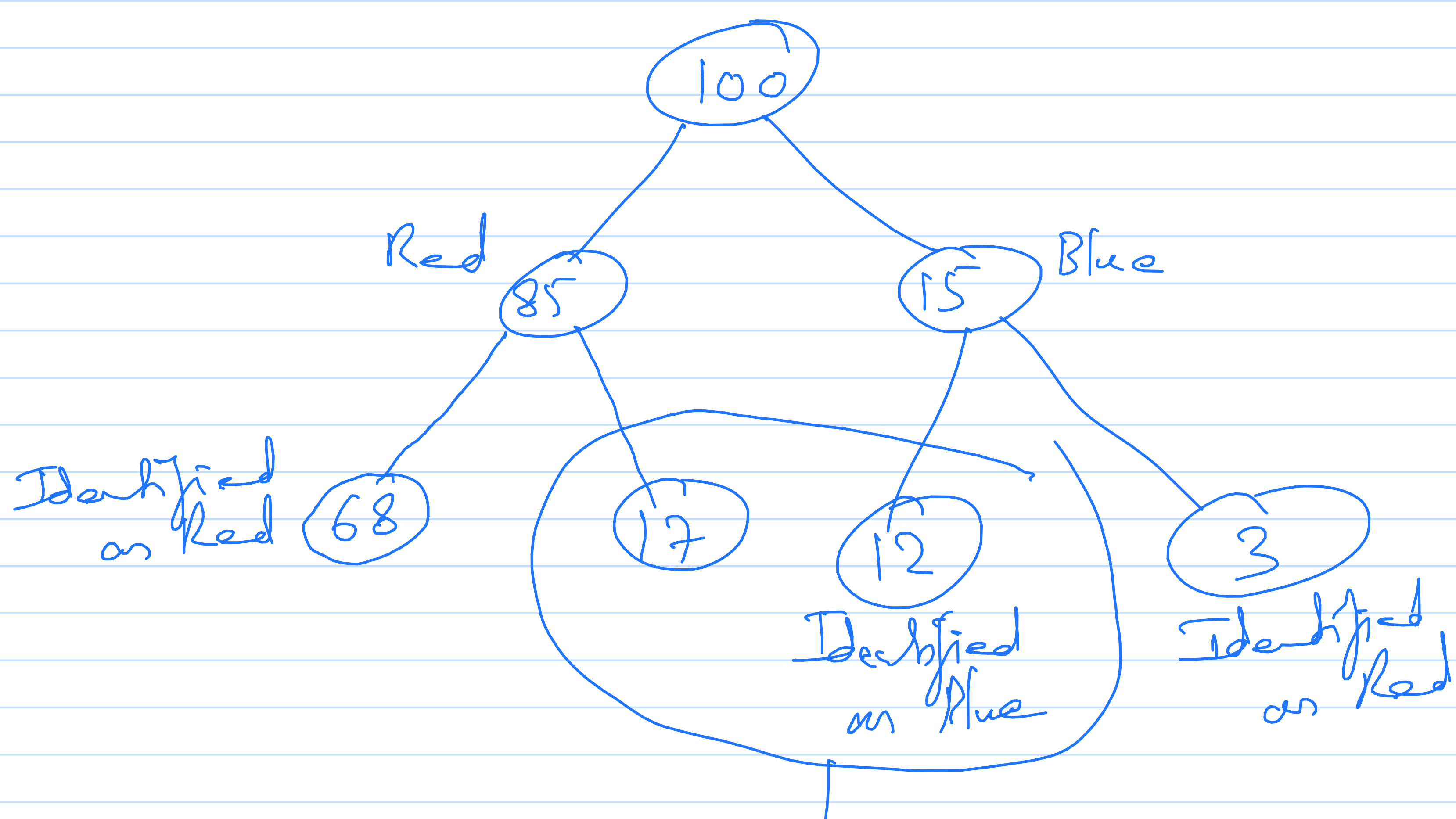
What may make this problem interesting?

$$P[B \mid \underbrace{\text{Witness identified blue}}_{W_B}] = \frac{P[B, W_B]}{P[W_B]} = \frac{P[W_B \mid B]\, P[B]}{P[(W_B \cap B) \cup (W_B \cap R)]}$$

$\{B, R\}$ is an Event Space.

$$\frac{P[W_B \mid B]\, P[B]}{P[W_B \mid B]\, P[B] + P[W_B \mid R]\, P[R]} = \frac{(0.8)(0.15)}{(0.8)(0.15) + (0.2)(0.85)} = \frac{12}{29}$$

$$P(R \mid \text{Witness identified blue}) = \frac{17}{29}$$



Total identified as Blue

Explain your calculated probabilities

DIAGNOSTIC TESTS.

A disease has a prevalance rate of 1/1000.

A person decides to get tested. What is the probability that the person is infected before the test result is out?

The test has a false +ve rate of 1/1000 & a false −ve rate of 0.

⇓ The corresponding probabilities?

$P[\text{Test is +ve} \mid \text{Person is not infected}] = 10^{-3}$

$P[\text{Test is −ve} \mid \text{Person is infected}] = 0$

The test result is +ve. Is the person infected?

$$P[\text{Person is infected} \mid \text{Test result is +ve}]$$

$$= \frac{P[\text{Test result is +ve} \mid \text{Person is infected}]\, P[\text{Person is infected}]}{P[\text{Test result is +ve} \mid \text{Person is inf}]\, P[\text{Person is inf}] + P[\text{Test result is +ve} \mid \text{Person is not inf}]\, P[\text{Person is not inf}]}$$

SCREAM DETECTION APP

$P[\text{Voice sample is scream}] = p.$

App has false +ve rate > 0 → FPR

$P[\text{App detects scream} \mid \text{Audio isn't scream}]$

& false −ve rate > 0. → FNR

TDR : $P[\text{App detects scream} \mid \text{Sample is a scream}]$

$P[\text{Scream happened} \mid \text{App detected scream}]$

=

$$= \frac{(TDR)\, p}{(TDR)\, p + FPR(1-p)} = \frac{p}{p + \frac{FPR}{TDR}(1-p)}$$

Independence

- **Two Events** A and B are independent *iff* P[AB] = P[A]P[B]
- The occurrence of event B does not change your belief about whether event A has occurred or not
  - Implies P[A|B] = P[?]
- **Definition 1.9**
  - If n>3, the sets A_1, A_2,..., A_n are independent *if*
    - *Every set of n-1 sets taken from* A_1, A_2,..., A_n is independent,
    - $P[A_1 \cap A_2 \cap \ldots \cap A_n] = P[A_1]\, P[A_2] \ldots P[A_n]$

Quiz 1.6

Monitor two consecutive phone calls going through a telephone switching office. Classify each one as a voice call (v) if someone is speaking, or a data call (d) if the call is carrying a modem or fax signal. Your observation is a sequence of two letters (either v or d). For example, two voice calls corresponds to vv. The two calls are independent and the probability that any one of them is a voice call is 0.8. Denote the identity of call i by C_i. If call i is a voice call, then $C_i = v$; otherwise, $C_i = d$. Count the number of voice calls in the two calls you have observed. N_V is the number of voice calls. Consider the three events $N_V = 0$, $N_V = 1$, $N_V = 2$. Determine whether the following pairs of events are independent:

(1) $\{N_V = 2\}$ and $\{N_V \geq 1\}$

(2) $\{N_V \geq 1\}$ and $\{C_1 = v\}$

(3) $\{C_2 = v\}$ and $\{C_1 = d\}$

(4) $\{C_2 = v\}$ and $\{N_V \text{ is even}\}$

Problem 1.6.7

For independent events A and B, prove that

(a) A and B^c are independent.

(b) A^c and B are independent.

(c) A^c and B^c are independent.

Sequential Diagrams

Example 1.25 Problem

Suppose traffic engineers have coordinated the timing of two traffic lights to encourage a run of green lights. In particular, the timing was designed so that with probability 0.8 a driver will find the second light to have the same color as the first. Assuming the first light is equally likely to be red or green, what is the probability $P[G_2]$ that the second light is green? Also, what is $P[W]$, the probability that you wait for at least one light? Lastly, what is $P[G_1|R_2]$, the conditional probability of a green first light given a red second light?