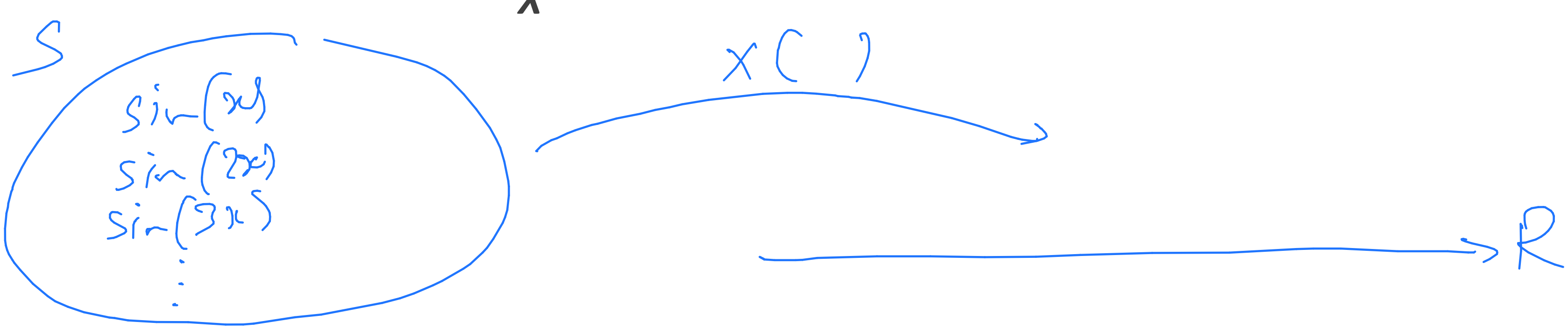
Probability Mass Function

- **Def 2.4** The probability mass function (PMF) of a discrete random variable X is

$$P_X(x) = P[X = x]$$

- Note that the PMF is defined for all *x*, not just the *x* that have a mapping to one or more outcomes
  - **The domain of P_X is the set of real numbers**



Probability Mass Function

- There is nothing sacrosanct about x. Just a convention to use a capital letter for a RV and corresponding small for the value

$$P_X(u) = P[X = u]$$

Fickle Coin

- You toss a fickle coin once. The sample space S={heads, tails, standing}.
- The outcomes heads, tails and standing occur with probabilities 0.7, 0.2 and 0.1 respectively.
- In our model, if tails is observed, then the RV $X = 0$. For the outcome heads, $X = 1$. For the outcome standing $X=2$.

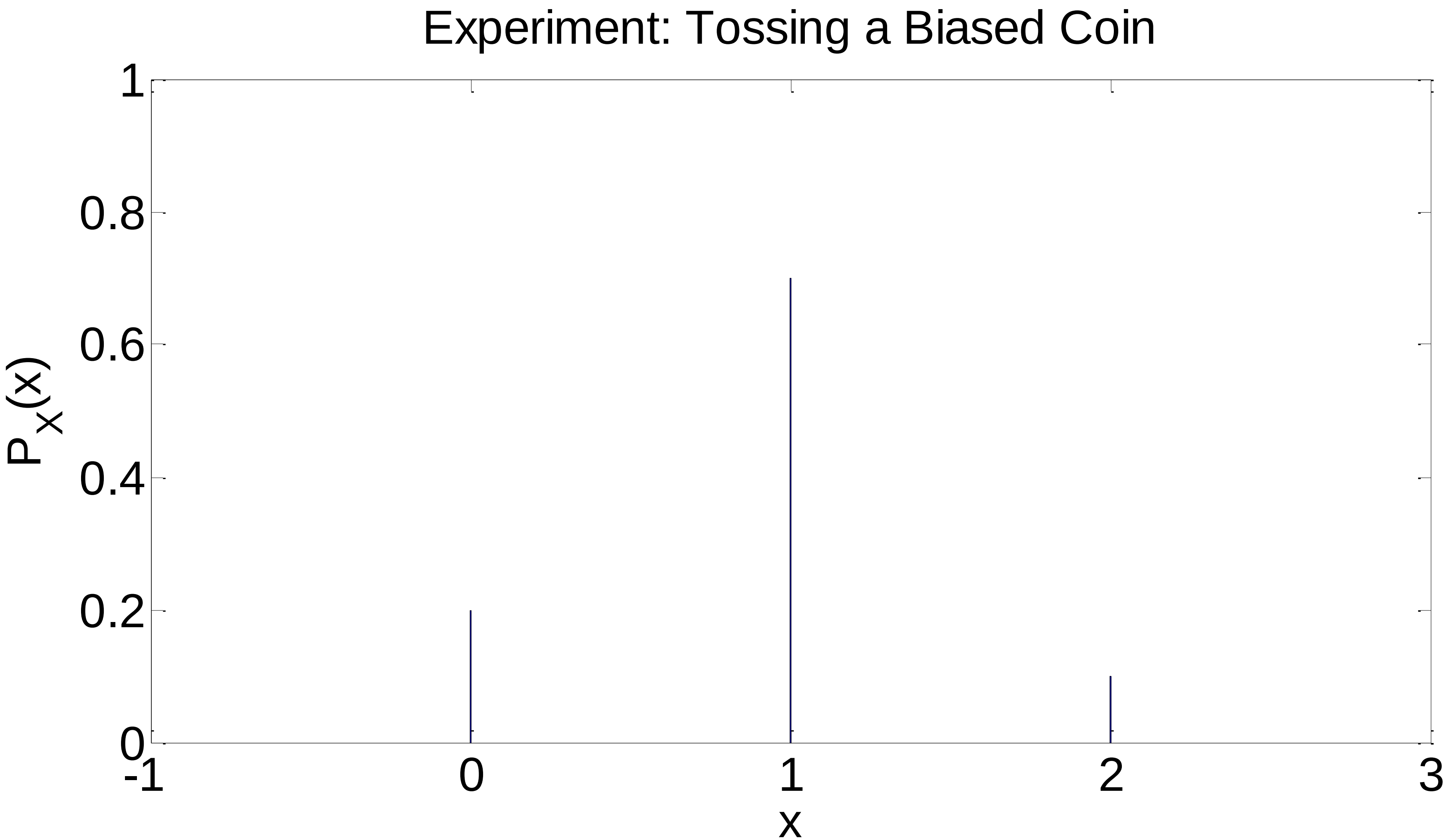
Fickle Coin

- P[X=0] = 0.7, P[X=1] = 0.2, P[X=2] = 0.1

- The PMF is

$$P_X(x) = \begin{cases} 0.2 & x = 0, \\ 0.7 & x = 1, \\ 0.1 & x = 2, \\ 0 & otherwise. \end{cases}$$

PMF: Tossing a Biased Coin…



PMF

- Is $P_X(\pi)$ defined for the experiment? What is its value?

- **The PMF contains all the information about the RV X**

PMF

- Remember that $P_X(x)$ is a probability. Therefore as per Axiom 1:

$$P_X(x) = P[X = x] \geq 0 \quad \forall x$$

- Also

$$\sum_{x \in S_X} P_X(x) = 1$$

PMF

- Remember that $P_X(x)$ is a probability. Therefore:

$$\text{For any } x,\ P_X(x) \geq 0.$$

- Also

$$\Sigma_{x \in S_x} P_X(x) = 1$$

- This is because for every outcome s in the sample space S, there is a number x in the range S_X

PMF Properties

For any event $B \subset S_X$, the probability that X is in set B is $P[B] = \sum_{x \in B} P_X(x)$

This is rather straightforward!

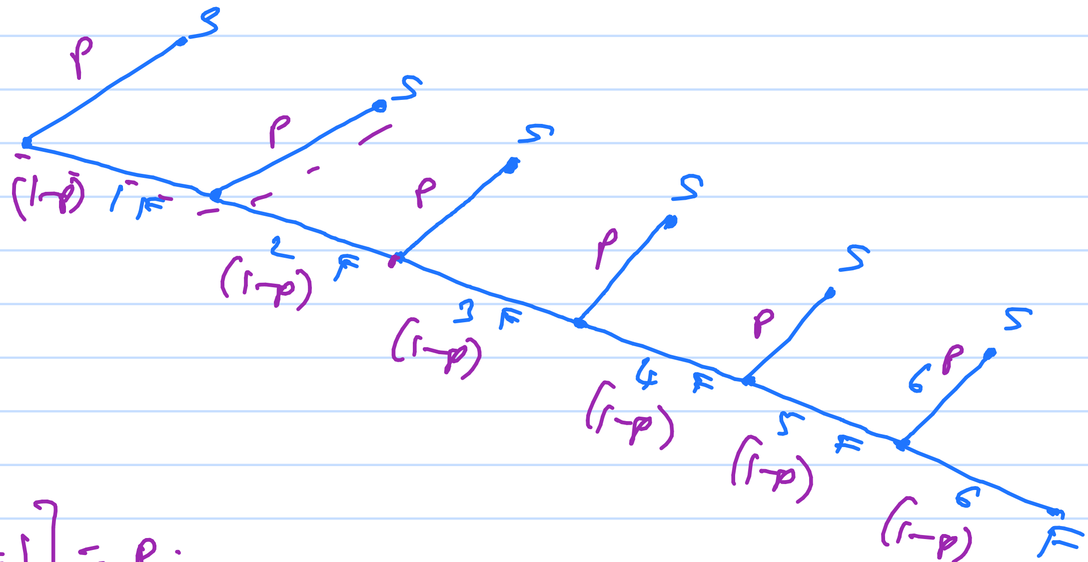
Problem 2.2.1

The random variable N has PMF

$$P_N(n) = \begin{cases} c(1/2)^n & n = 0, 1, 2, \\ 0 & \text{otherwise.} \end{cases}$$

(a) What is the value of the constant c?

(b) What is $P[N \leq 1]$?



$$P[K=1] = p.$$

$$P[K=2] = (1-p)\,p$$

$$\vdots$$

$$P[K=5] = (1-p)^4\,p$$

$$P[K=6] = P[(K=6 \cap F) \cup (K=6 \cap S)]$$

$$= P[K=6 \cap F] + P[K=6 \cap S]$$

$$= (1-p)^6 + (1-p)^5\,p.$$

Problem 2.2.9

When someone presses "SEND" on a cellular phone, the phone attempts to set up a call by transmitting a "SETUP" message to a nearby base station. The phone waits for a response and if none arrives within 0.5 seconds it tries again. If it doesn't get a response after $n = 6$ tries the phone stops transmitting messages and generates a busy signal.

(a) Draw a tree diagram that describes the call setup procedure.

(b) If all transmissions are independent and the probability is p that a "SETUP" message will get through, what is the PMF of K, the number of messages transmitted in a call attempt?

(c) What is the probability that the phone will generate a busy signal?

(d) As manager of a cellular phone system, you want the probability of a busy signal to be less than 0.02 If $p = 0.9$, what is the minimum value of n necessary to achieve your goal?

$(1-p)^n < 0.02$

Families of Random Variables

- We added the RV and its PMF to the model of an experiment

- In the real world many different experiments may be modeled by the same family of random variables!

Families of Random Variables

- Experiment 1: Select a random human being and note the gender. Your observation is either male/female

- Experiment 2: Toss a fair coin. Your observation is either heads/tails

Experiments and Models

- Both experiments can be modeled by the RV X with PMF

$$P_X(x) = \begin{cases} 0.5 & x = 0 \\ 0.5 & x = 1 \\ 0 & \text{otherwise} \end{cases}$$

- The values x=0, x=1 could also correspond to an odd number and even number respectively, on the roll of a die

Experiments and Models

- The fact that the same models may work for very different experiments is also a good reason to map outcomes to numbers
- Different experiments may have different sample spaces
- They may have the same range space
- They may have the same probabilistic model (PMF for the discrete RV)

Bernoulli(p) RV

- The simplest of the many we will see
- Very useful to model reality
- Sample space of your experiment has two outcomes
  - A circuit either passes or fails a test (S = {pass, fail})
  - A bit is either received correctly or in error (S = {correct, err})
  - A coin toss results in either heads or tails (S = {heads, tails})

Bernoulli(p) RV

- Sample space of your experiment has two outcomes
  - A circuit either passes or fails a test (S = {pass, fail})
  - A bit is either received correctly or in error (S = {correct, err})
  - A coin toss results in either heads or tails (S = {heads, tails})

- Your model required just a single parameter p
  - p is a probability and can be
    - the probability that a circuit fails a test,
    - probability that a bit is received in error,
    - probability that a coin toss leads to heads and so on…(add your examples)

Bernoulli(p) RV

- **Def 2.5:** X is a Bernoulli(p) RV if the PMF of X has the form

$S_X = \{0, 1\}$

$$P_X(x) = \begin{cases} 1-p & x = 0, \\ p & x = 1, \\ 0 & \text{otherwise.} \end{cases}$$

where $0 < p < 1$.

Geometric RV

- Your perform trials of an experiment till a trial results in a desired outcome/ event

- You want to count the number of trials you must do, including the one that gave you the desired outcome

$$S_X = \{1, 2, 3, \ldots\}$$

$$P(X=3) = (1-p)^2 p$$

$$P[X=1] = p$$

$P[X=2] = P$[First Bern trial resulted in a 0 $\cap$ Second Bern trial in a 1] $= (1-p)p$ [Since we assumed indep...

Geometric RV

- Your perform trials of an experiment till a trial results in a desired outcome/ event
- You want to count the number of trials you must do till you get the desired outcome
- Any trial can either result in your event/outcome of interest or its complement

Geometric RV

- Any trial can either result in your event/outcome of interest or its complement

- A trial is modeled by a ………………… RV?

- The parameter of the RV is …………………

Geometric RV

- Your perform trials of an experiment till a trial results in a desired outcome/ event

- You want to count the number X of trials you must do till you get the desired outcome

- Suppose all trials are independent of each other

  - We have a random __ number of ____________ trials

Geometric RV

- Your perform trials of an experiment till a trial results in a desired outcome/ event
- You want to count the number of trials you must do till you get the desired outcome
- Suppose all trials are independent of each other
- Each trial is modeled by a single parameter ….., the probability of success (your event of interest)

Geometric RV

- The Geometric RV counts the number of trials you must do till success (the desired outcome/event occurs)

- Range space?

Geometric RV

- The Geometric RV counts the number of trials you must do till success (the desired outcome/event occurs)

- PMF?

Geometric RV

- Your experiment involves looking at a sequence of received bits (1 or 0), starting with the first bit of the sequence
- Your observation is the number x, where the x^{th} bit is the very first bit that was received in error
  - Bit 1, Bit 2, …, Bit x-1 are all received correctly
  - Bit x is in error
- Each bit is in error with probability p independent of any other bit that was transmitted

Geometric RV

- Let X be the RV corresponding to outcomes of the experiment
- P[X = x] = P[Event that Bit 1 is correct ∩ Event that Bit 2 is correct ∩ …. ∩ Event that Bit x-1 is correct ∩ Event that Bit x is incorrect]
- $P[X = x] = (1-p)^{(x-1)} p$, for $x = 1,2,\ldots$
- It is 0 for all other x.
- Note that **each subexperiment is a Bernoulli trial**

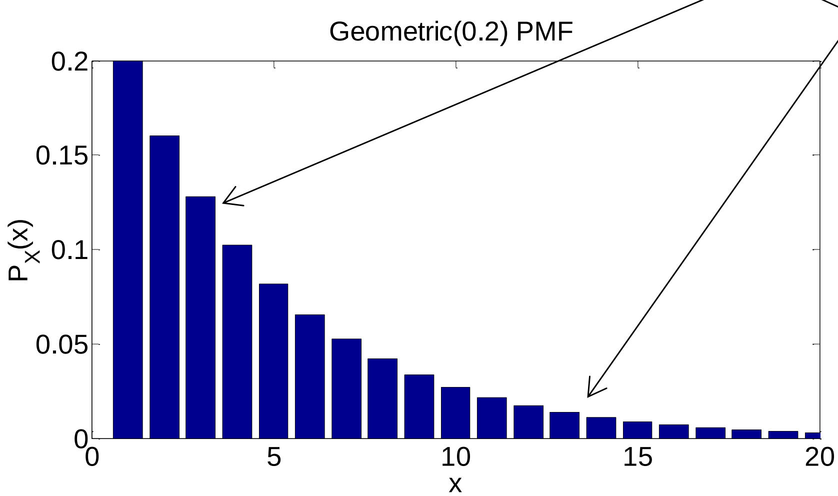
Geometric RV

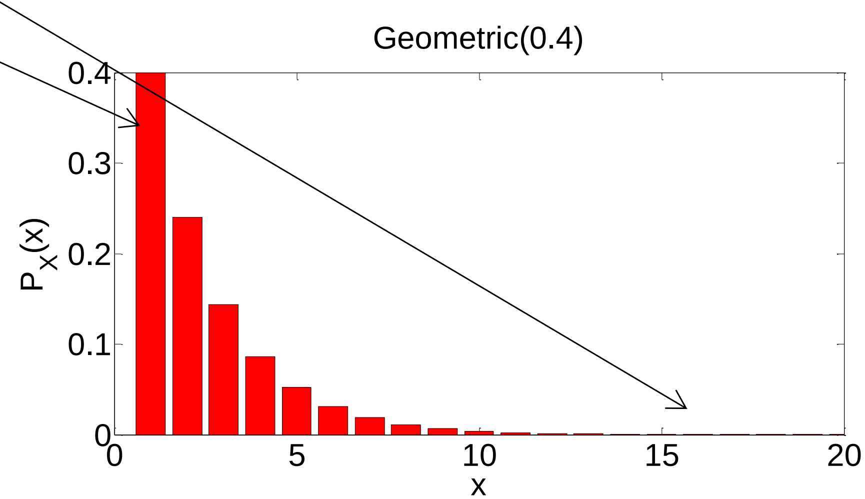
- **Def 2.6** X is a geometric(p) random variable if the PMF of X has the form

$$P_X(x) = \begin{cases} p(1-p)^{x-1} & x = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

Plot the PMF(s)! It will take a minute.
Play around with the parameter *p*

Compare





Binomial(n,p) RV

- Here the number of trials is fixed to n
- The Binomial RV counts the number of successful trials amongst the total of n trials
- Each trial is modelled by the RV ___________(___)

Binomial(n,p) RV

- Here the number of trials is fixed to n

- The Binomial RV counts the number of successful trials amongst the total of n trials

- Range space?

$X \sim$ Binomial(n,p) RV

- Here the number of trials is fixed to n
- The Binomial RV counts the number of successful trials amongst the total of n trials

$$S_X = \{0, 1, 2, \ldots, n\}$$

- PMF? Simply calculate probabilities for all values in the range space...

$$P[X=x] = \begin{cases} {}^{n}C_{k}(1-p)^{n-k}p^{k} & k = 0, 1, 2, \ldots, n \\ 0 & \text{otherwise} \end{cases}$$

$$P[X=k] = P[k \text{ successes in } n \text{ indep and id Bern trials}]$$