Conditioning on an Event

Definition 4.10 Conditional Joint PDF

Given an event B *with* $P[B] > 0,$ *the conditional joint probability density function of* X *and* Y *is*

$$f_{X,Y|B}(x, y) = \begin{cases} \dfrac{f_{X,Y}(x, y)}{P[B]} & (x, y) \in B, \\ 0 & \text{otherwise.} \end{cases}$$

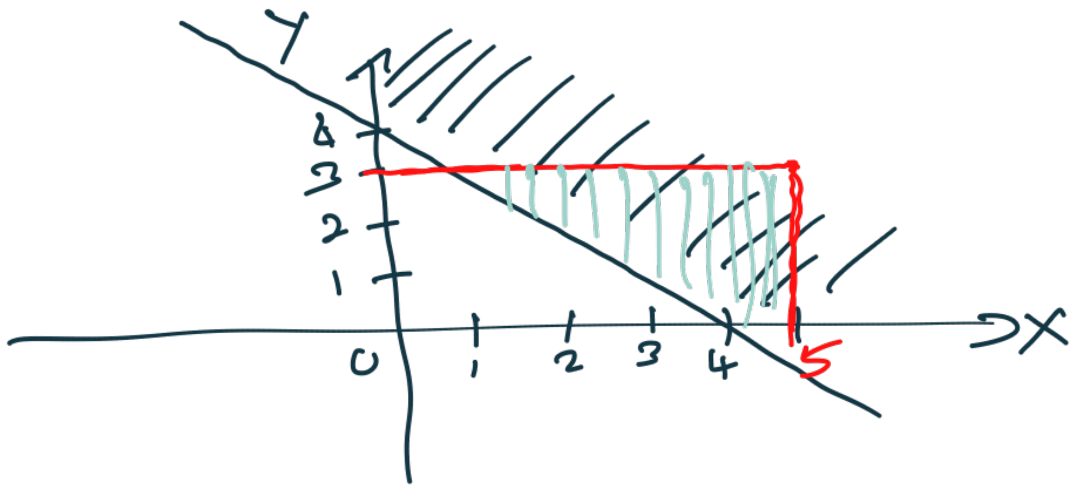
$$P[x \le X < x+dx,\ y \le Y < y+dy \mid B] \simeq f_{X,Y|B}(x,y)\, dx\, dy$$

Example 4.14 Problem

X and Y are random variables with joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 1/15 & 0 \le x \le 5, 0 \le y \le 3, \\ 0 & \text{otherwise.} \end{cases} \tag{4.83}$$

Find the conditional PDF of X and Y given the event $B = \{X + Y \ge 4\}$.



$Y = -X + 4$

Conditioning on an Event

Theorem 4.20 Conditional Expected Value

For random variables X and Y and an event B of nonzero probability, the conditional expected value of $W = g(X, Y)$ given B is

Discrete: $$E[W|B] = \sum_{x \in S_X} \sum_{y \in S_Y} g(x, y) P_{X,Y|B}(x, y)$$

Continuous: $$E[W|B] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{X,Y|B}(x, y)\, dx\, dy.$$

- Nothing special about this

Conditioning on a RV

Definition 4.12 Conditional PMF

For any event $Y = y$ *such that* $P_Y(y) > 0$, *the* conditional PMF *of* X *given* $Y = y$ *is*

$$P_{X|Y}(x|y) = P[X = x|Y = y].$$

$$0 \leq \sum_{y \in S_Y} P\underbrace{[X = x | Y = y]} \leq |S_Y|$$

PMF changes as y changes.

Conditioning on a RV

Theorem 4.22

For random variables X and Y with joint PMF $P_{X,Y}(x, y)$, and x and y such that $P_X(x) > 0$ and $P_Y(y) > 0$,

$$P_{X,Y}(x, y) = P_{X|Y}(x|y)\, P_Y(y) = P_{Y|X}(y|x)\, P_X(x)\,.$$

- Why??

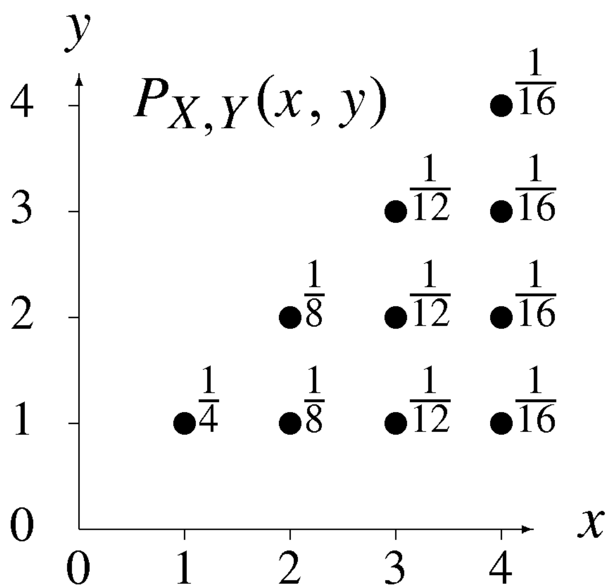
$$P_{X,Y}(x,y) = P[X=x, Y=y] \rightsquigarrow = P[Y=y \mid X=x]\, P[X=x]$$

$$= P_{Y|X}(y|x)\, P_X(x)$$

$$= P[X=x \mid Y=y]\, P[Y=y]$$

$$= P_{X|Y}(x|y)\, P_Y(y)$$

Example 4.17 Problem



Random variables X and Y have the joint PMF $P_{X,Y}(x,y)$, as given in Example 4.13 and repeated in the accompanying graph. Find the conditional PMF of Y given $X = x$ for each $x \in S_X$.

- How many conditional PMF(s) of Y do we have?

VVVVS Problem

- For a given *y*

$$P_{Y|X}(y|1) + P_{Y|X}(y|2) + P_{Y|X}(y|3) + P_{Y|X}(y|4) = ?$$

- For a given *x*

$$P_{Y|X}(1|x) + P_{Y|X}(2|x) + P_{Y|X}(3|x) + P_{Y|X}(4|x) = ?$$

Conditional Expectation

Theorem 4.23 Conditional Expected Value of a Function

X and Y are discrete random variables. For any $y \in S_Y$, the conditional expected value of $g(X, Y)$ given $Y = y$ is

$$E[g(X,Y)|Y=y] = \sum_{x \in S_X} g(x,y) P_{X|Y}(x|y).$$

$P[D,E|F] = P[D|E,F]\,P[E|F]$

$E[g(X,Y)|\underbrace{Y=y}_{A}] = E[g(X,y)|Y=y]$

$\sum_{x \in S_X} \sum_{z \in S_Y} g(x,z)\, P[X=x, Y=z|Y=y] = \sum_{x \in S_X} \sum_{z=y} g(x,z)\, P[X=x|Y=y]$

$= \sum_{x \in S_X} g(x,y)\, P[X=x|Y=y]$

Conditional Expectation

Theorem 4.23 Conditional Expected Value of a Function

X and Y are discrete random variables. For any $y \in S_Y$, the conditional expected value of $g(X, Y)$ given $Y = y$ is

$$E[g(X, Y)|Y = y] = \sum_{x \in S_X} g(x, y) P_{X|Y}(x|y).$$

- The conditional expectation is a function of which variable(s)?

Conditional Expectation

Theorem 4.23 Conditional Expected Value of a Function

X and Y are discrete random variables. For any $y \in S_Y$, the conditional expected value of $g(X, Y)$ given $Y = y$ is

$$E[g(X, Y)|Y = y] = \sum_{x \in S_X} g(x, y) P_{X|Y}(x|y).$$

- E[g(X,Y)] is a function of which variable(s)?

Now For the Continuous Case

Definition 4.13 Conditional PDF

For y such that $f_Y(y) > 0$, the conditional PDF of X given $\{Y = y\}$ is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x, y)}{f_Y(y)}.$$

$$\int_{x \in S_X} f_{X|Y}(x|y)\, dx = \int_{x \in S_X} \frac{f_{X,Y}(x,y)}{f_Y(y)}\, dx$$

Theorem 4.24

$$f_{X,Y}(x, y) = f_{Y|X}(y|x)\, f_X(x) = f_{X|Y}(x|y)\, f_Y(y).$$

Problem

- You are given the conditional pdf $f_{Y|X}(y|x)$ and $f_X(x)$
- How would you find $F_Y(y)$?

$$F_Y(y) = \int_{-\infty}^{y} f_Y(y)\, dy = \int_{-\infty}^{\infty} \int_{-\infty}^{y} f_{X,Y}(x,y)\, dy\, dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{y} f_{Y|X}(y|x)\, f_X(x)\, dy\, dx$$

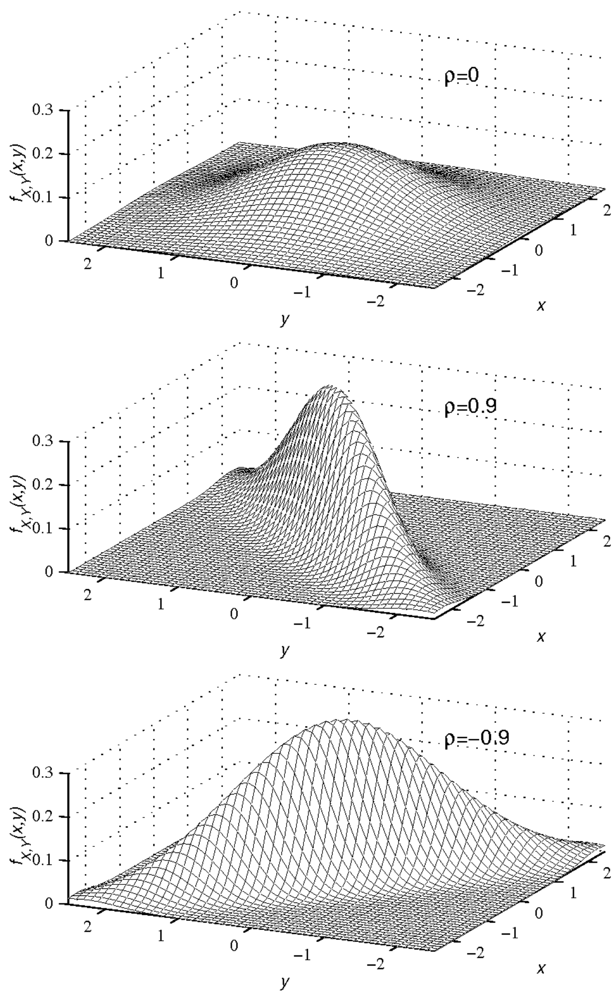
Definition 4.14 Conditional Expected Value of a Function

For continuous random variables X *and* Y *and any* y *such that* $f_Y(y) > 0$*, the* conditional expected value *of* $g(X, Y)$ *given* $Y = y$ *is*

$$E[g(X, Y)|Y = y] = \int_{-\infty}^{\infty} g(x, y) f_{X|Y}(x|y)\, dx.$$

Examples of a Joint Gaussian PDF

Figure 4.5



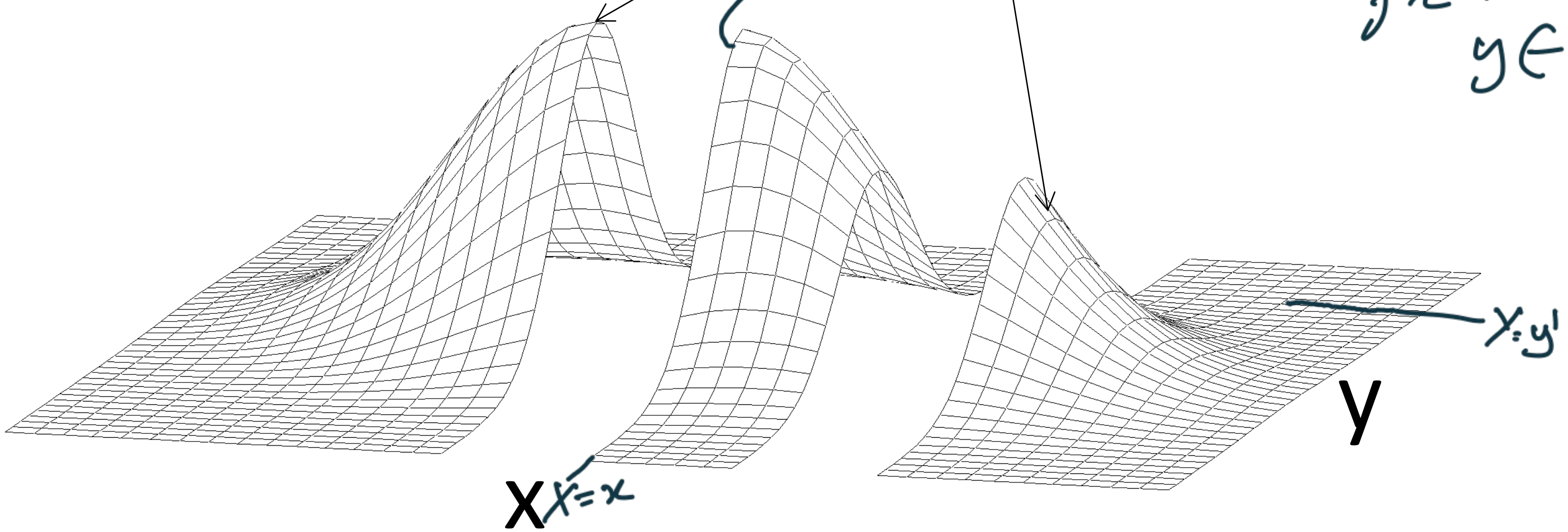
The Joint Gaussian PDF $f_{X,Y}(x, y)$ for $\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 1$, and three values of ρ.

The Conditional Density

Figure 4.6

Each bell shaped curve corresponds to $f_{Y|X}(y|x)$ scaled by $f_X(x)$

$f_{X,Y}(x,y)$ for $X=x$, $y \in S_Y$



Example Joint Gaussian Density