Definition

Let V be a vector space. Let S be an infinite subset of V. We say S is a *linearly independent* set if every finite subset of S is linearly independent.

Definition

Let V be a vector space. A set of vectors $\mathcal{B} \subset V$ is said to be a *basis* of V if

(i) $\mathcal{B}$ is linearly independent.

(ii) $\mathcal{B}$ spans V.

Whenever $\mathcal{B}$ is a finite set, we say V is *finite dimensional*.

Example

The columns $\mathbf{e}_1, \ldots, \mathbf{e}_n$ of the $n \times n$ identity matrix I_n form a basis of $\mathbb{R}^n$.

Theorem

Let A be an $m \times n$ matrix. The pivot columns of A form a basis for Col *A.*

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$e_1 \quad e_2 \quad e_3$

$$\begin{array}{cccc} e_1 & e_2 & e_3 & \\ \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$e_1 + 2e_2$

Lemma

Let A be an $m \times n$ matrix in reduced echelon form, having k pivot columns, where $1 \leq k \leq m$. Then $\{\mathbf{e}_1, \ldots, \mathbf{e}_k\}$ is a basis for $\operatorname{Col} A$.

Proof: Clearly $\{e_1, \ldots, e_k\}$ are the pivot columns of A.

$e_1 = (1, 0 \cdots 0)$ $b = (b_1, \ldots, b_m)$ $\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$

$e_2 = (0, 1, 0 \cdots 0)$ $= b_1 e_1 + b_2 e_2 + \cdots + b_m e_m$

Let b be any other column of A which is not a pivot column.

Then $b_j = 0$ for $j = k+1, \ldots m$.

Now $b = b_1 e_1 + \cdots + b_m e_m$

$\Rightarrow b = b_1 e_1 + \cdots + b_k e_k \in \text{span}\{e_1, \ldots, e_k\}$

$\Rightarrow$ all columns of A

are in $\underline{\text{Span}\{e_1, \ldots, e_k\}}$

$\Rightarrow$ $\underline{\text{Col} A \subset \text{Span}\{e_1, \ldots e_k\}}$

Since $\{e_1, \ldots, e_k\}$ are columns of A, $\{e_1, \ldots e_k\} \subset \text{col} A$

$\Rightarrow$ $\text{Span}\{e_1, \ldots, e_k\} \subset \text{col} A$.

$\Rightarrow \quad \operatorname{Col} A = \operatorname{Span}\{e_1, \dots, e_k\}.$

<u>Lemma</u>: If W is a subspace of a vector space V, and $S \subset W$. Then $\operatorname{Span} S \subset W$.

"if": Since W is closed under taking linear combinations,

$$\operatorname{Span} S \subset W.$$

Claim: $e_1, \ldots, e_k$ are l.i.

$c_1 e_1 + \cdots + c_k e_k = 0.$

$\Rightarrow (c_1, c_2, \ldots, c_k, 0 \ldots, 0) = 0$

$\Rightarrow c_1 = \cdots = c_k = 0.$

Lemma

Let A be an $m \times n$ matrix and let A' be a matrix obtained by performing a row operation on A. Any linear dependence relation which holds between the columns of A also holds between the corresponding columns of A', and vice versa.

The columns of A are linearly independent if and only if the columns of A' are linearly independent.

pf) $A' = EA$

where E is an elementary matrix.

Suppose $A = [a_1 \; \cdots \; a_n]$

$A' = [Ea_1 \; \cdots \; Ea_n]$

If $c_1 a_1 + \cdots + c_n a_n = 0$

for $c_1, c_2, \cdots, c_n \in \mathbb{R}$,

then

$$c_1 E a_1 + \cdots + c_n E a_n = 0.$$

$\therefore$ The first statement follows.

Since elementary matrices are invertible,

$$c_1 E a_1 + \cdots + c_n E a_n = 0$$

$$\Rightarrow E^{-1}(c_1 E a_1 + \cdots + c_n E a_n) = 0.$$

$$\Rightarrow c_1 a_1 + \cdots + c_n a_n = 0$$

$\therefore$ the "vice versa" part is true also.

Proposition

Let V be a vector space, and let S be a linearly independent subset of V. Any subset of S is linearly independent.

What is the contrapositive?

What about extending a linearly independent set to a bigger linearly independent set? How would we do this?

Proposition

Let $\{v_1, \ldots, v_n\}$ be a linearly independent set in a vector space V. If $w \notin Span(\{v_1, \ldots, v_n\})$ then the set $\{v_1, \ldots, v_n, w\}$ is linearly independent.

Lemma

Let $S_1 \subset S_2 \subset V$*. Then*

$$\operatorname{Span} S_1 \subset \operatorname{Span} S_2.$$

Lemma

Let W *be a subspace of a vector space* V*. Let* S *be a subset of* W*. Then*

$$\operatorname{Span} S \subset W.$$