$$P[B|A] = \frac{P[A|B]\,P[B]}{P[A]}$$

$$= \frac{P[A|B]\,P[B]}{P[A|B]\,P[B] + P[A|B^c]\,P[B^c]}$$

$$= \frac{P[A|B]\,P[B]}{P[A|B]\,P[B] + P[A|B^c]\,(1 - P[B])}$$

Experiment outcomes S

$$S \xrightarrow{X(\;)} S_X$$

where X is the RV.

$$P_X(x) = P[\underline{X=x}], \quad \forall x \in R$$

Let's say event A occurred.

$$P_{X|A}(x) = P[X=x|A], \quad \forall x \in R$$

$$P[X=x|A] = \frac{P[X=x, A]}{P[A]}$$

Example 2.38 Problem

Let X denote the number of additional years that a randomly chosen 70 year old person will live. If the person has high blood pressure, denoted as event H, then X is a geometric $(p = 0.1)$ random variable. Otherwise, if the person's blood pressure is regular, event R, then X has a geometric $(p = 0.05)$ PMF with parameter. Find the conditional PMFs $P_{X|H}(x)$ and $P_{X|R}(x)$. If 40 percent of all seventy year olds have high blood pressure, what is the PMF of X? $\rightarrow P[H] = 0.4$

$$P_{X|H}(x) = P[X = x \mid H]$$

$$= \begin{cases} (0.9)^{x-1}(0.1) & x = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

$$P_{X|R}(x) = \begin{cases} (0.95)^{x-1}(0.05) & x = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Theorem 2.17

- Consider an Event B. Let B be a subset of the range S_X of RV X.
- What is $P_{X|B}(x)$?

- Start with the definition of conditional probability...

Theorem 2.17

- We have

$$P_{X|B}(x) = \begin{cases} \frac{P_X(x)}{P[B]} & x \in B \\ 0 & \text{otherwise.} \end{cases}$$

Note that this is > $P_X(x)$. Knowledge that B occurred and that {X=x} is in B increases our belief that {X=x} occurred

Example 2.40 Problem

$X \sim P_X(x)$

Suppose X, the time in integer minutes you must wait for a bus, has the uniform PMF

$$P_X(x) = \begin{cases} 1/20 & x = 1, 2, \ldots, 20, \\ 0 & \text{otherwise.} \end{cases} \tag{2.119}$$

Suppose the bus has not arrived by the eighth minute, what is the conditional PMF of your waiting time X?

Event $\{X > 8\} \triangleq A$

$$P_{X|A}(x) = \begin{cases} \dfrac{P[X = x]}{P[A]} & x \in \{X > 8\} = \{9, 10, 11, \ldots, 20\} \\ 0 & \text{otherwise} \end{cases}$$

No Surprises Here!

Theorem 2.18

(a) For any $x \in B$, $P_{X|B}(x) \geq 0$.

(b) $\sum_{x \in B} P_{X|B}(x) = 1$.

(c) For any event $C \subset B$, $P[C|B]$, the conditional probability that X is in the set C, is

$$P\left[C|B\right] = \sum_{x \in C} P_{X|B}\left(x\right).$$

Conditional Expected Value

- Just replace the PMF $P_X(x)$ in the formula for expectation by the conditional PMF $P_{X|B}(x)$
- The same holds for the conditional expected value of a function of a RV too

Definition 2.20 *Conditional Expected Value*

The conditional expected value *of random variable X given condition B is*

$$E[X|B] = \mu_{X|B} = \sum_{x \in B} x P_{X|B}(x) .$$

Expectation of X in Terms of Conditional Expectations Over an Event Space

Theorem 2.19

For a random variable X resulting from an experiment with event space $B_1, \ldots, B_m$,

$$E[X] = \sum_{i=1}^{m} E[X|B_i]\, P[B_i].$$

$$E[X] = \sum_{x \in S_X} x \underbrace{P[X=x]}_{\sum_{i=1}^{m} P[X=x|B_i]\, P[B_i]}$$

$$= \sum_{x \in S_X} x \sum_{i=1}^{m} P[X=x|B_i]\, P[B_i]$$

$$= \sum_{i=1}^{m} \underbrace{\left[\sum_{x \in S_X} x\, P[X=x|B_i]\right]}_{\sum_{x \in B_i} x\, P[X=x|B_i]} P[B_i]$$

$$\cancel{E[X] = \sum_{i=1}^{m} E[X|B_i]}.$$

$$\sum_{x \in B_i} x\, P[X=x|B_i]$$

$\because P[X=x|B_i]=0, \ \forall x \notin B_i$

Conditional Expected Value of Y=g(X)

Theorem 2.20

The conditional expected value of $Y = g(X)$ given condition B is

$$E[Y|B] = E[g(X)|B] = \sum_{x \in B} g(x) P_{X|B}(x).$$

- Proof: Do as we did for E[Y] in Theorem 2.10.

Word Problem

Quiz 2.9

On the Internet, data is transmitted in packets. In a simple model for World Wide Web traffic, the number of packets N needed to transmit a Web page depends on whether the page has graphic images. If the page has images (event I), then N is uniformly distributed between 1 and 50 packets. If the page is just text (event T), then N is uniform between 1 and 5 packets. Assuming a page has images with probability $1/4$, find the

(1) conditional PMF $P_{N|I}(n)$ $\sim \text{Unif}(1,50)$

(2) conditional PMF $P_{N|T}(n)$ $\sim \text{Unif}(1,5)$

(3) PMF $P_N(n)$ $= P_{N|I}(n)\,P(I) + P_{N|T}(n)\,P[T]$

(4) conditional PMF $P_{N|N\le 10}(n)$ $\rightarrow \dfrac{P[N=n, N\le 10]}{P[N\le 10]} = \begin{cases} \dfrac{P[N=n]}{P[N\le 10]} & n\le 10 \\ 0 & \text{otherwise} \end{cases}$

(5) conditional expected value $E[N|N \le 10]$

(6) conditional variance $\text{Var}[N|N \le 10]$

Problem 2.7.7

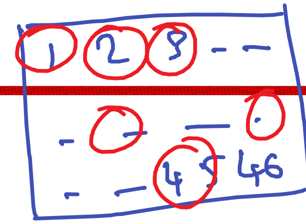
A particular circuit works if all 10 of its component devices work. Each circuit is tested before leaving the factory. Each working circuit can be sold for k dollars, but each nonworking circuit is worthless and must be thrown away. Each circuit can be built with either ordinary devices or ultrareliable devices. An ordinary device has a failure probability of $q = 0.1$ while an ultrareliable device has a failure probability of $q/2$, independent of any other device. However, each ordinary device costs \$1 while an ultrareliable device costs \$3. Should you build your circuit with ordinary devices or ultrareliable devices in order to maximize your expected profit $E[R]$? Keep in mind that your answer will depend on k.

$$E[W_n] = E\left[\frac{n}{K_n+1}\right]$$

$$= E[g(K_n)]$$

$$\neq \frac{n}{E[K_n]+1}$$

Problem 2.7.8



In the New Jersey state lottery, each \$1 ticket has six randomly marked numbers out of $1, \ldots, 46$. A ticket is a winner if the six marked numbers match six numbers drawn at random at the end of a week. For each ticket sold, 50 cents is added to the pot for the winners. If there are k winning tickets, the pot is divided equally among the k winners. Suppose you bought a winning ticket in a week in which $2n$ tickets are sold and the pot is n dollars.

(a) What is the probability q that a random ticket will be a winner? $q = \frac{1}{{}^{46}C_6}$

(b) What is the PMF of K_n, the number of other (besides your own) winning tickets?

$W_n = \frac{n}{K_n + 1}$

(c) What is the expected value of W_n, the prize you collect for your winning ticket?

$S_{K_n} = \{0, 1, 2, \ldots, 2n-1\}$

$P[K_n = 0] = (1-q)^{2n-1}$

$P[K_n = x] = {}^{2n-1}C_x \, q^x (1-q)^{2n-1-x}$

$P[K_n = 1] = {}^{2n-1}C_1 \, q (1-q)^{2n-2}$