Imagine that you are looking into a mirror. ~~But~~ what you see is actually another world. Suppose you measure your coordinates in your own world with respect to, say the center of your forehead, projected onto the $z = 0$ plane, which is aligned with the mirror.

Suppose you are holding a marble in your left hand. Let's say the coordinates of the marble are $(-2, -3, 5)$. You wish to find the coordinates of the reflection of the marble in the world which is on the other side of the mirror. What do you do? $(-2, -3, -5)$

Suppose the mirror was not the $z = 0$ plane, but some other plane?

The idea that helps us out here is that reflection across a plane (which passes through the origin) is a *linear transformation*.

The special thing about a linear transformation is that if you know what it does to a basis, you know what it does to every other point in space.

So the idea is to choose a basis cleverly and see what happens to that basis, and then define your coordinates with respect to that basis.

But before we actually answer the question about the mirror, we need to understand why it is enough to know what happens to a basis, in order to determine a linear transformation completely.

And more importantly, what is a linear transformation?

A linear transformation is a *mapping* (or a *function*) which *preserves the structure of a vector space.* It *respects linearity* (essentially, it takes "flat" objects to "flat" objects.)

Definition

Let V, W be vector spaces. A function $T : V \to W$ is said to be a *linear transformation* if

(i) $T(v + w) = T(v) + T(w), \quad \forall v, w \in V$

(ii) $T(cv) = cT(v) \quad \forall v \in V, c \in \mathbb{R}$

respects vector addition
and scalar multiplication

Example

If A is an $m \times n$ matrix then the *matrix transformation* $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$T(\mathbf{x}) = A\mathbf{x}$$

is a linear transformation. Clearly,

(i) If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ then
$T(\mathbf{x} + \mathbf{y}) = A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = T(\mathbf{x}) + T(\mathbf{y})$

(ii) If $\mathbf{x} \in \mathbb{R}^n$ and $c \in \mathbb{R}$ then $T(c\mathbf{x}) = A(c\mathbf{x}) = cA\mathbf{x} = cT(\mathbf{x})$.

Special Types of Linear Transformations

Projections

$$(x, y, z) \longmapsto (x, y)$$

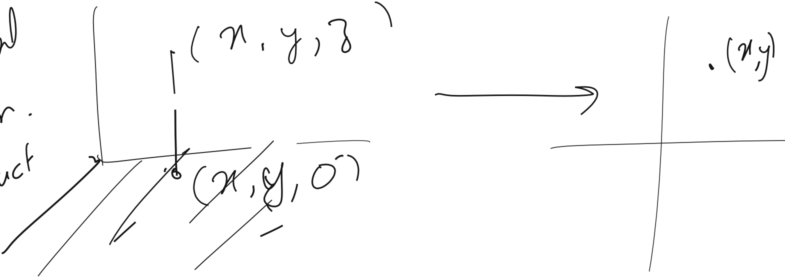
$$\mathbb{R}^3 \longmapsto \mathbb{R}^2.$$

$$(x, y, z) \longmapsto x \qquad \mathbb{R}^3 \to \mathbb{R}$$

Projections, shears, dilations, contractions, rotations, reflections.

orthogonal projection.

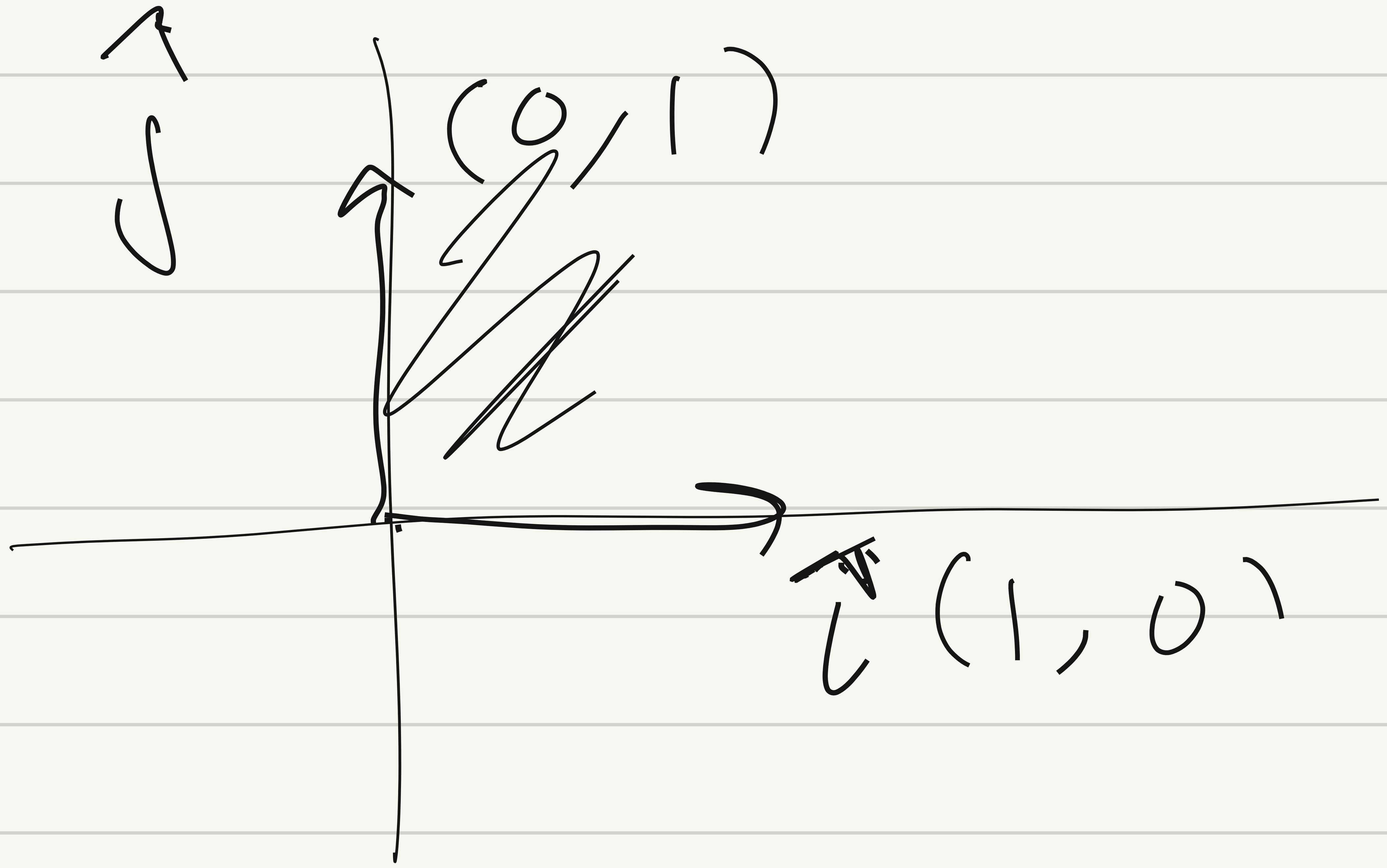
inner product

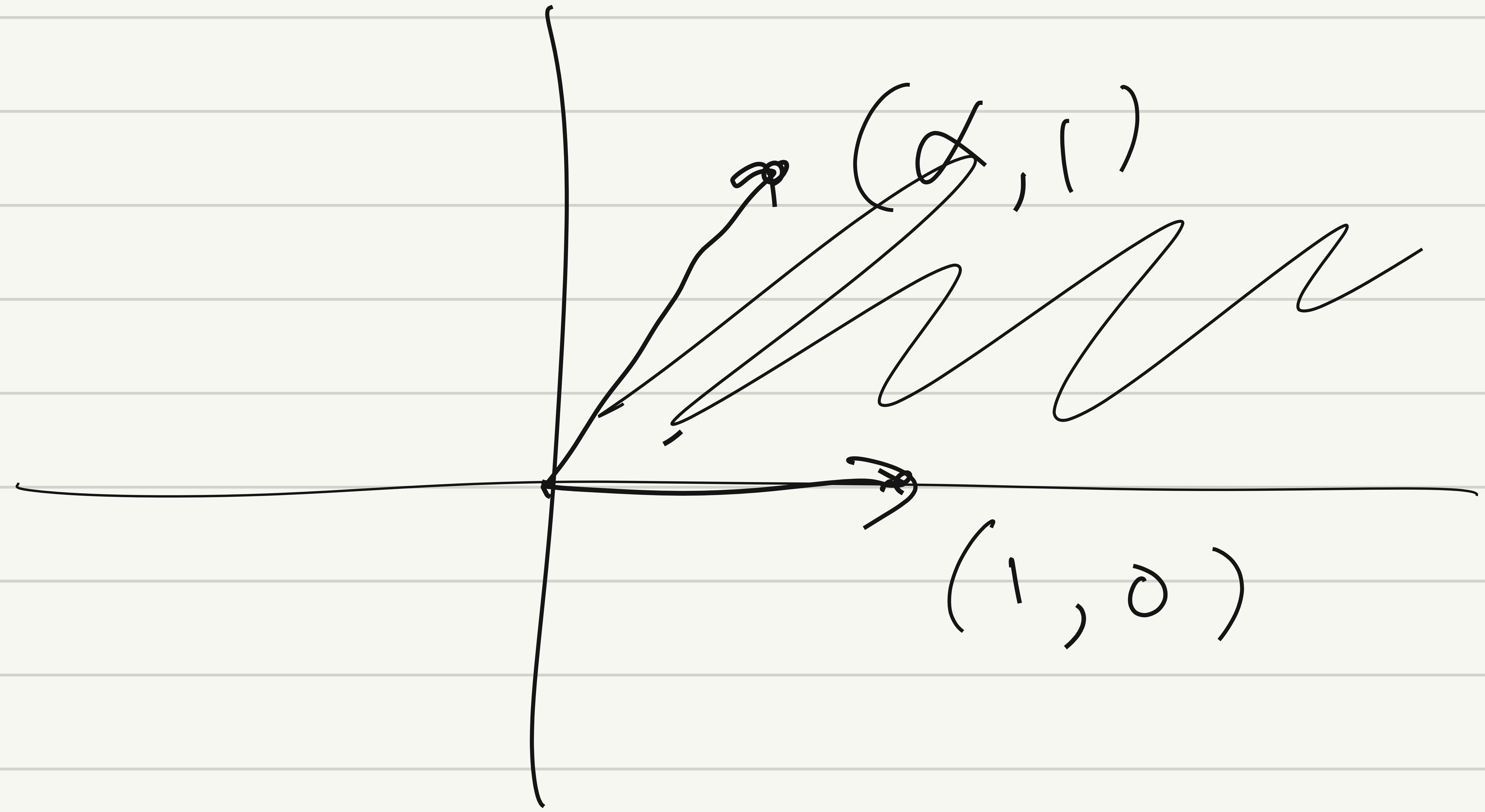


$$T(x, y, z) = (x, y)$$

$$T : \mathbb{R}^3 \to \mathbb{R}^2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$





$$\begin{bmatrix} 1 & 0 \\ \alpha & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + \alpha y \\ y \end{bmatrix} \quad \alpha > 0$$

$$\begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{bmatrix} \alpha \\ 1 \end{bmatrix}$$

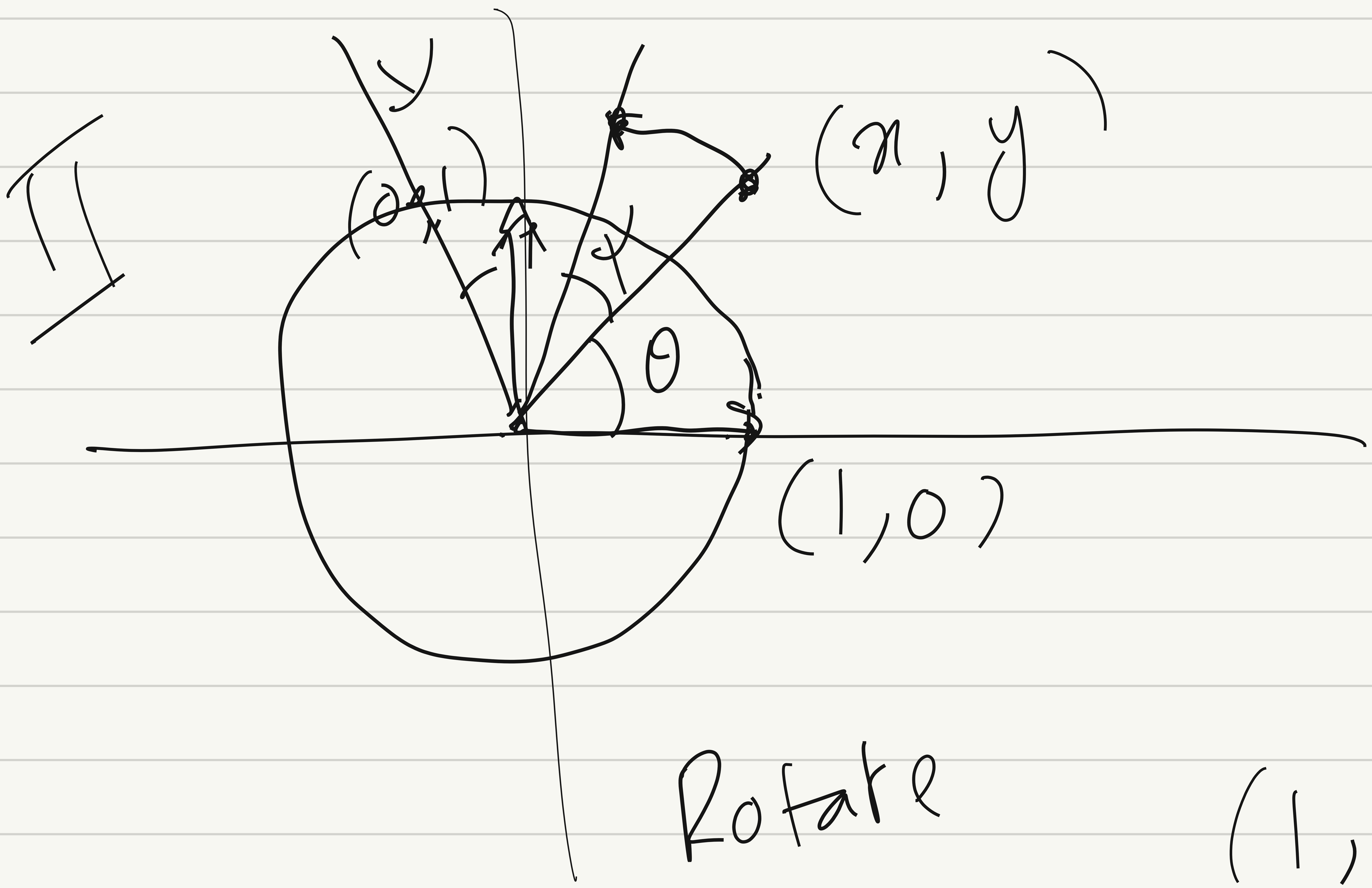
Dilation $\alpha > 1$

$$(x, y, z) \longmapsto (\alpha x, \alpha y, \alpha z)$$

$$\begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix} = \alpha I.$$

Contraction: $\alpha < 1$

Rotations : $\mathbb{R}^2$.



Rotate by ψ

$x(1,0) + y(0,1)$.

$(1,0) \longmapsto (\cos\psi, \sin\psi)$

$(0,1) \longmapsto (-\sin\psi, \cos\psi)$

$$(x, y) = \underline{x}(1,0) + y(0,1)$$

$$T(x,y) = xT(1,0) + yT(0,1)$$

$$= (x\cos\psi, x\sin\psi) + y(-\sin\psi, \cos\psi)$$

$$= (x\cos\psi - y\sin\psi, x\sin\psi + y\cos\psi)$$

$$\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = T(x, y).$$

matrix of the transformation