Gaussian(μ,σ)

- The Bell-shaped distribution
- The Normal distribution
- Parameters are the mean μ and standard deviation is σ
  - Variance is σ^2
  - If X is Gaussian we often write X is N[μ, σ^2]

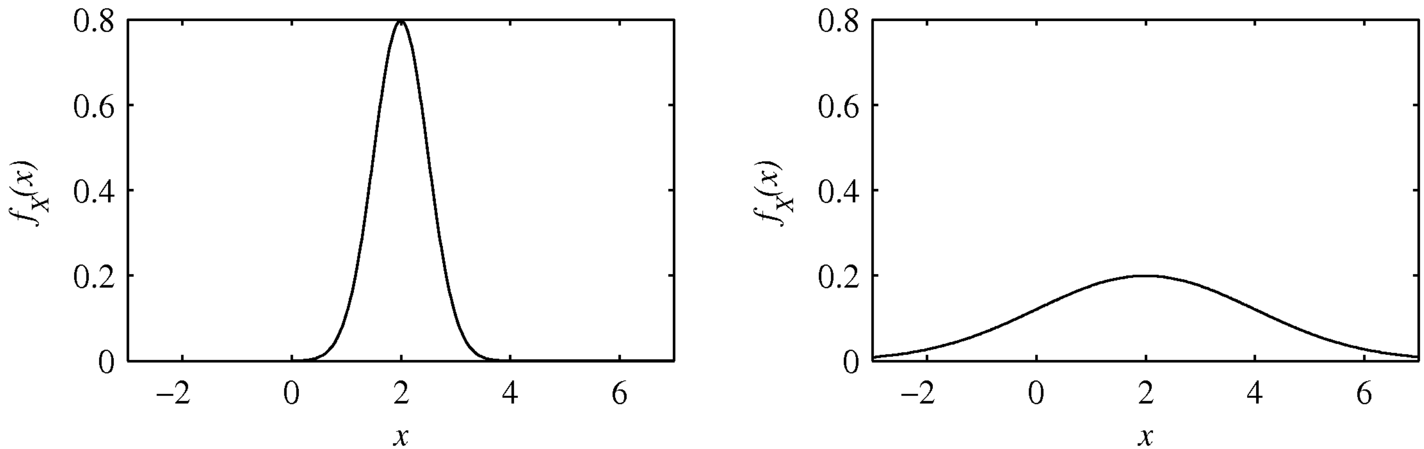
Definition 3.8 *Gaussian Random Variable*

X is a Gaussian (μ, σ) random variable if the PDF of X is

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2},$$

where the parameter μ can be any real number and the parameter $\sigma > 0$.

Figure 3.5



(a) $\mu = 2, \sigma = 1/2$ **(b)** $\mu = 2, \sigma = 2$

Two examples of a Gaussian random variable X with expected value μ and standard deviation σ.

$$Y = aX + b$$

$$dy = a\,dx$$

$$\underbrace{P[Y \le y]} = P\left[X \le \frac{y-b}{a}\right] =$$

$$\underbrace{\int_{-\infty}^{y} f_Y(y)\,dy} \qquad \int_{-\infty}^{\frac{y-b}{a}} \underbrace{f_X(x)}\,dx$$

$$= \int_{-\infty}^{\frac{y-b}{a}} \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\left(\frac{(x-\mu)^2}{2\sigma^2}\right)}\,dx$$

let $x = \frac{y-b}{a}$ $\qquad dx = \frac{dy}{a}$

$$= \int_{-\infty}^{a\left(\frac{y-b}{a}\right)+b} \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\left(\frac{\left(\frac{y-b}{a}-\mu\right)^2}{2\sigma^2}\right)}\,\frac{dy}{a}$$

$$= \int_{-\infty}^{y} \underbrace{\frac{1}{\sqrt{2\pi(a\sigma)^2}}\, e^{-\left(\frac{(y-(a\mu+b))^2}{2(a\sigma)^2}\right)}}\,dy$$

$$\downarrow$$

Gaussian $(a\mu+b,\ a\sigma)$

Linear Transformation of a Gaussian

Theorem 3.13

$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\left(\frac{(x-\mu)^2}{2\sigma^2}\right)}$

If X is Gaussian (μ, σ), $Y = aX + b$ is ______.

- Linear transformation of a Gaussian gives another Gaussian!
- How do show the above?

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/(2\sigma^2)} \qquad x \in (-\infty, \infty)$$

$$P[Y \leq y] = P[aX + b \leq y]$$

$$= P\left[X \leq \frac{y-b}{a}\right]$$

Linear Transformation of a Gaussian

Theorem 3.13

If X is Gaussian (μ, σ), $Y = aX + b$ is ______.

- If X is not Gaussian, what are E[Y] and E[Y^2]?

Standard Normal Variable and CDF

- **Def 3.9** The standard random variable is Gaussian(0,1) – 0 mean and unit variance

$$P[X \le x] = P[\sigma Z + \mu \le x] = P\left[Z \le \frac{x-\mu}{\sigma}\right]$$

Definition 3.10 ***Standard Normal CDF*** $\rightarrow$ I have:

$P[Z \le z] \ \forall z$

The CDF of the standard normal random variable Z is

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u^2/2}\, du.$$

$Z \sim N(0,1)$ $\rightarrow$ We want $P[X \le x]$

$X \sim N(\mu, \sigma^2)$ $X = \sigma Z + \mu$

Expressing a Gaussian CDF as a N[0,1] CDF

Theorem 3.14

If X is a Gaussian (μ, σ) random variable, the CDF of X is

$$F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right).$$

The probability that X is in the interval $(a, b]$ is

$P[X \le b]$

$- P[X \le a]$

$$\boxed{P[a < X \le b]} = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right).$$

- Show using definition of CDF and substitution of variables in the integral
- All I need are values of ϕ(.)
  - This is important as CDF integral calculations for finite limits can only be done numerically
  - Thankfully, all we now need is a tabulation of ϕ(.)

Standard Normal CDF

ϕ is the CDF of the Standard Gaussian.

- In fact $\phi(-z) = 1 - \phi(z)$
  - Why?
  - Is this true for any Gaussian Distribution?
  - When is it true?
- All I need is $\phi(z)$ for $z >= 0$

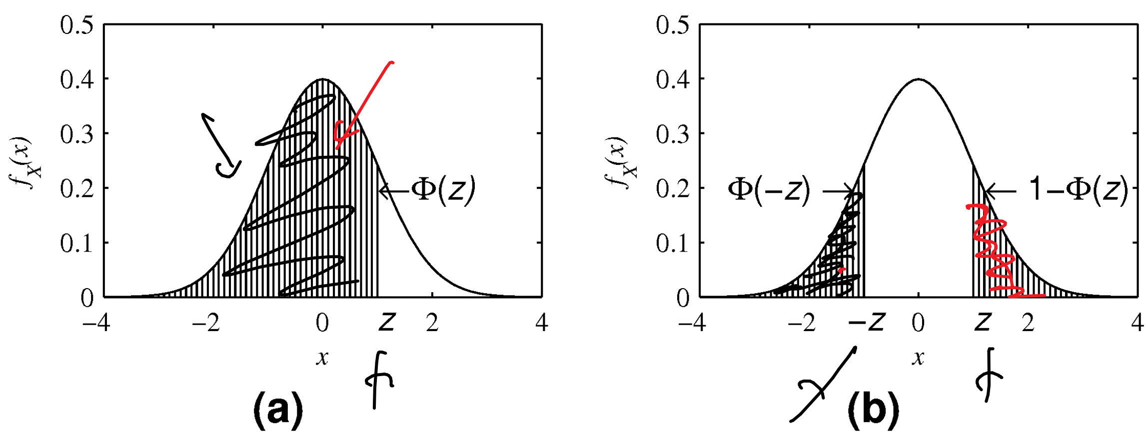
$$\phi(z) = \int_{-\infty}^{z} f_Z(z)\,dz = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\,dz$$

$$= 1 - \int_{z}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\,dz$$

$$= 1 - \int_{-\infty}^{-z} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\,dz$$

We set $z = -z$.

Figure 3.6



Example 3.16 Problem

If X is the Gaussian $(61, 10)$ random variable, what is $P[X \leq 46]$?

μ_X $\quad$ σ_X

$$P[X - \mu_X \leq 46 - \mu_X]$$

$$= P\left[\frac{X - \mu_X}{\sigma_X} \leq \frac{46 - \mu_X}{\sigma_X}\right]$$

$$P\left[Z \leq \frac{46 - \mu_X}{\sigma_X}\right] = \Phi_Z\left(\frac{46 - \mu_X}{\sigma_X}\right)$$

Example 3.17 Problem

If X is a Gaussian random variable with $\mu = 61$ and $\sigma = 10$, what is $P[51 < X \leq 71]$?

$$P\left[\frac{51-\mu}{\sigma} < \frac{X-\mu}{\sigma} \leq \frac{71-\mu}{\sigma}\right]$$

$$= P\left[Z \leq \frac{71-\mu}{\sigma}\right] - P\left[Z \leq \frac{51-\mu}{\sigma}\right]$$

Standard Normal CCDF

Definition 3.11

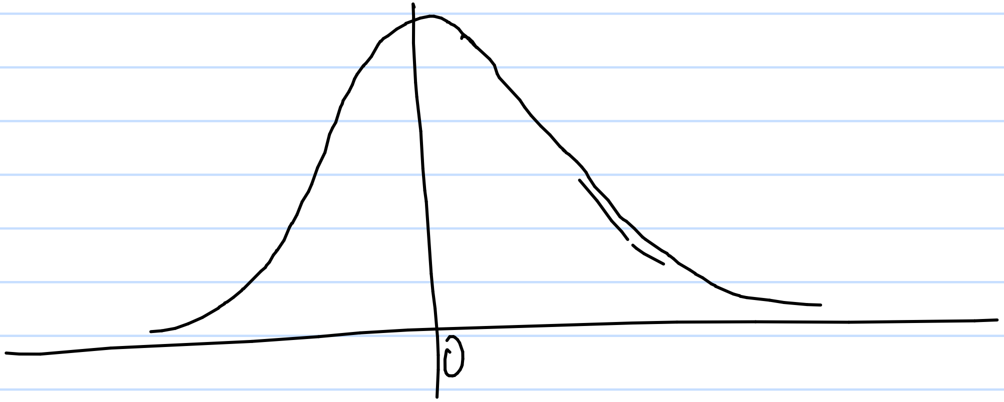
The standard normal complementary CDF *is*

$$Q(z) = P[Z > z] = \frac{1}{\sqrt{2\pi}} \int_z^\infty e^{-u^2/2}\, du = 1 - \Phi(z).$$

Q-func

$$\Phi(3) = 0.9987,\ \ \Phi(4) = 0.9999768$$

$$Q(3) = 1.35 \times 10^{-3}, Q(4) = 3.17 \times 10^{-5}$$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$$P[Y_{20} > 1000] = P\left[\frac{Y_{20} - E[Y_{20}]}{\sigma_{Y_{20}}} > \frac{1000 - E[Y_{20}]}{\sigma_{Y_{20}}}\right]$$

Problem 3.5.6

A professor pays 25 cents for each blackboard error made in lecture to the student who points out the error. In a career of n years filled with blackboard errors, the total amount in dollars paid can be approximated by a Gaussian random variable Y_n with expected value $40n$ and variance $100n$. What is the probability that Y_{20} exceeds 1000? How many years n must the professor teach in order that $\underbrace{P[Y_n > 1000]} > 0.99$?

Problem 3.5.7

Suppose that out of 100 million men in the United States, 23,000 are at least 7 feet tall. Suppose that the heights of U.S. men are independent Gaussian random variables with a expected value of $5'10''$. Let N equal the number of men who are at least $7'6''$ tall.

(a) Calculate σ_X, the standard deviation of the height of men in the United States.

(b) In terms of the $\Phi(\cdot)$ function, what is the probability that a randomly chosen man is at least 8 feet tall?

(c) What is the probability that there is no man alive in the U.S. today that is at least $7'6''$ tall?

(d) What is $E[N]$?

- Six Sigma Event?
    - Use MATLAB’s qfunc

Quiz 3.5

X is the Gaussian $(0, 1)$ random variable and Y is the Gaussian $(0, 2)$ random variable.

(1) Sketch the PDFs $f_X(x)$ and $f_Y(y)$ on the same axes.

(2) What is $P[-1 < X \leq 1]$?

(3) What is $P[-1 < Y \leq 1]$?

(4) What is $P[X > 3.5]$?

(5) What is $P[Y > 3.5]$?