Maths I - Linear Algebra (MTH100)

Course Instructor: Acushla Sarswat

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Important Access Codes

Google Classroom Code

7vwn6j5

Online Lectures (Zoom)

Meeting ID: 881 664 1859

Passcode: 9CGn7Z

Contact Information and Office Hours

- Email: acushla@iiitd.ac.in
- Office Hours: Fridays 2-3pm
- Office (via Google Meet): https://meet.google.com/xkg-ncuc-cfk

Email Protocols

Please post all your doubts on Google Classroom before emailing me. If you email me directly I will probably post your query on Classroom and answer it anyway, so you will save time by posting on Classroom.

Please take any **tutorial or textbook related questions** to your TA (the one assigned to your group) before emailing me. You may cc me on the email or forward the email to me or Mr. Soumen Ghosh (Course TF) if it is not answered within three days.

Course Grading Policy

Grade Cutoffs

$A \geq 75$; $B \geq 60$; $C \geq 45$; $D \geq 30$

Intermediate cutoffs will be decided later.

Component Weightage

- Programming Assignment - 10 %
- Midsem - 20 %
- Endsem - 45 %
- Submissions - 25 %

The "Submissions" may be assignments/surprise tests/quizzes announced in advance. Ideally, each Submission will be of **low weightage**, and will not require extra preparation other than regular class attendance.

Course Grading Policy - contd.

Extra Credit Policy

Students can gain **up to 10 %** extra overall marks for:

1. Pointing out mistakes during the lectures (this does not include typos or plus/minus type errors). Please follow this up with an email to me with the subject "Extra Credit Policy" **within two days** to ensure that you receive your due credit.
2. Pointing out mistakes in submission/test/exam questions or solutions or in the lecture slides (not including spelling/punctuation type errors). You will only receive extra credit if you **post publicly on Google Classroom**.

Plagiarism Policy

Students are advised to review the institute plagiarism policy which can be found at:

https://www.iiitd.ac.in/academics/resources/acad-dishonesty.

Course Information - Books

- Course Textbook: *Linear Algebra and its Applications*, David C. Lay, Pearson - 3rd edition)
- Reference: Linear Algebra and Its Applications, Strang
- Reference: Linear Algebra, Schaum's Outline Series, Lipschutz
- Reference: Linear Algebra, Hoffman and Kunze
- Reference: Linear Algebra: A Geometric Approach, Kumaresan
- Reference: Linear Algebra Done Right, Axler
- Reference: Finite-Dimensional Vector Spaces, Halmos

Course Objectives

1. Students are able to compute solutions, forms and metrics related to linear algebra using the applicable results/methods
2. Students are able to test/classify for the given conditions using the given criteria or test
3. Students are able to determine the truth/falsity of statements involving vector spaces and linear transformations and justify or explain the answer using any of the techniques/results covered up to date
4. Students are able to construct proofs for statements involving vector spaces and linear transformations using any of the results covered up to date

Systems of Linear Equations

Definition

A linear equation in the variables $x_1, \ldots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

where b and the coefficients $a_1 \ldots a_n$ are real or complex numbers.

If the coefficients are real we say it's a linear equation with real coefficients and if the coefficients are complex, we say it's a linear equation with complex coefficients.

Examples of Linear Equations

Example 1

$$F = \frac{9}{5}C + 32$$

Example 2

$$x + y + z = 0$$

Examples of equations that are *not* linear

$$x^2 + y^2 = 1, \quad y = \sin x$$

Geometrically, a linear equation represents a line, or a plane or a hyperplane.

Definition

A system of linear equations is a collection of one or more linear equations involving the same variables, say $x_1, \ldots, x_n$. A solution of the system is a list or an *n-tuple* $(s_1, s_2, \ldots, s_n)$ of numbers that makes each equation a true statement when the values $s_1, \ldots s_n$ are substituted for $x_1, \ldots, x_n$, respectively.

Example

$$\begin{aligned} x + y &= 1 \\ x - y &= 1 \end{aligned}$$

This has a unique solution $(1, 0)$.

Definition

The set of all possible solutions is called the *solution set* of the linear system. Two linear systems are called *equivalent* if they have the same solution set.

Example

The system

$$\begin{aligned} y &= 0 \\ x &= 1 \end{aligned}$$

is equivalent to the one on the previous slide.

Classification of Linear Systems

A system of linear equations has either

1. no solution, or
2. exactly one solution, or
3. infinitely many solutions.

A system of linear equations is said to be *consistent* if it has either one solution or infinitely many solutions; a system is *inconsistent* if it has no solution.

Example of an inconsistent system

$$\begin{aligned} x + y + z &= 0 \\ z &= 0 \\ x + y - z &= 1 \end{aligned}$$

Example of consistent system with infinitely many solutions

$$\begin{aligned} x + y + z &= 0 \\ z &= 0 \\ x + y - z &= 0 \end{aligned}$$

Questions

Is there a way to find out whether a system is consistent or inconsistent without actually solving the system?

Is there a way to find out whether a system has a unique solution without solving it?

The answer is Yes, to both questions.

We can find out a lot about a given system by setting up the equation using a **matrix** and analyzing the properties of the matrix.

Example

Given a linear system of equations

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 - 5x_3 &= 10, \end{aligned}$$

the resulting augmented matrix is

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix}$$

i.e.

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ 5 & 0 & -5, \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 0 \\ 8 \\ 10 \end{bmatrix}$$

Row Reduction

There is a nice algorithm that we can use to answer our questions regarding the system. The idea is to simplify the augmented matrix using a sequence of operations known as *elementary row operations.*

Elementary Row Operations

There are three kinds of operations:

1. (Replacement) Replace one row by the sum of itself and a multiple of another row.
2. (Interchange) Interchange two rows.
3. (Scaling) Multiply all entries in a row by a nonzero constant.

Example

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix}$$

We subtract five times row 1 from row 3. Let's abbreviate this as $R_3 \to R_3 - 5R_1$. This gives us

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 10 & -10 & 10 \end{bmatrix}.$$

Notice that the first column has a 1 as its top entry and 0s below.

Next $R_3 \to R_3 - 5R_2$ gives us

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & 30 & -30 \end{bmatrix}$$

Just by looking at this matrix, we know that the given system is consistent and has a unique solution.

This is because the matrix is in *row echelon form* and every column other than the augmented column has a *pivot*.

Definition

A rectangular matrix is in *echelon form (or row echelon form)* if it has the following three properties:

1. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
2. All nonzero rows are above any rows of all zeros.

A *leading entry* of a row refers to the leftmost nonzero entry (in a nonzero row).

A *nonzero row* or nonzero column in a matrix means a row or column that contains at least one nonzero entry.

If a matrix is in row echelon form then the leading entry in each nonzero row is called a *pivot*.

Before we get into questions of why this technique works, let's examine the relationship between elementary row operations on the augmented matrix and the linear system of equations.

Let's look at the correspondence. The original system was

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 - 5x_3 &= 10, \end{aligned}$$

and the row operations were

$$R_3 \to R_3 - 5R_1$$

followed by

$$R_3 \to R_3 - 5R_2.$$

Thus each elementary row operation on the augmented matrix corresponds to an operation on the system of equations, which produces a system equivalent to the original system.

In the end we get a system of equations which can be solved more easily.

Let's solve the resulting system in the example using **back substitution**.

If a row echelon form of the augmented matrix of a linear system has a pivot in the augmented column, then it is inconsistent.

Otherwise it's consistent.

If every column other than the augmented column contains a pivot then the system has a unique solution.

Let's look at a few practice problems.

Why Do Elementary Row Operations Give Us Equivalent Systems?

For interchange and scaling operations, this is something we know from elementary school. But is the answer as obvious for the replacement operation?

Maybe it is. But it may become more clear if we look at what's going on using matrices.

Elementary Matrices

Applying an elementary row operation on a matrix is the same multiplying the matrix **on the left** by an *elementary matrix*.

Replacement

The operation $R_i \to R_i + cR_j$ is achieved via left multiplicaton by a matrix of the form

$$E = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & c & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix} \qquad (i < j)$$

or

$$E = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & c & & 1 & \\ & & & & & 1 \end{bmatrix} \qquad (i > j)$$

The matrix has a c as its ij-th entry and otherwise looks like the $m \times m$ identity matrix.

Let's look at the example we saw earlier from the perspective of elementary matrices. The operation $R_3 \to R_3 - 5R_1$ was applied to

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix}.$$

Similarly the operations of row interchange and row scaling can also be achieved via left multiplication by elementary matrices.

Interchange

$$E = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 0 & & 1 & \\ & & & \ddots & & \\ & & 1 & & 0 & \\ & & & & & 1 \end{bmatrix}$$

Let's look at an example.

Scaling

$$E = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & c & & & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}$$

Inverse of Elementary Matrix

For a Replacement operation, we replace the constant c by $-c$. What is the corresponding operation?

For the Interchange operation, the inverse of the elementary matrix is itself.

For the Scaling operation, we invert it by replacing c by $\frac{1}{c}$.

Equivalence of Systems

The system

$$A\mathbf{x} = \mathbf{b}$$

becomes

$$EA\mathbf{x} = E\mathbf{b}.$$

If we set $EA = A'$ and $E\mathbf{b} = \mathbf{b}'$ then this new system is $A'\mathbf{x} = \mathbf{b}'$. Both have the same solution set because E is invertible.