Vectors as Ordered Lists or n-tuples

We will **temporarily** use the word "vector" to refer to an ordered list of numbers.

Definition

The set of all n-tuples of real numbers is called $\mathbb{R}^n$.

Elements of $\mathbb{R}^n$ are usually **represented** as $n \times 1$ column vectors ($n \times 1$ matrices).

The vector whose entries are all zero is called the **zero vector** and is denoted by $\mathbf{0}$.

Equality of vectors in $\mathbb{R}^n$ and the operations of scalar multiplication and vector addition in $\mathbb{R}^n$ are defined entry by entry just as in $\mathbb{R}^2$.

Algebraic Properties of $\mathbb{R}^n$

For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^{\mathbf{n}}$ and all scalars c and d,

- $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
- $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$.
- $\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$ (where $-\mathbf{u}$ denotes $(-1)\mathbf{u}$)
- $c(\mathbf{u} + \mathbf{v}) = \mathbf{cu} + \mathbf{cv}$
- $(c + d)\mathbf{u} = \mathbf{cu} + \mathbf{du}$
- $c(d\mathbf{u}) = (\mathbf{cd})\mathbf{u}$
- $1\mathbf{u} = \mathbf{u}$

Definition

Given vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots \mathbf{v_p} \in \mathbb{R}^{\mathbf{n}}$ and given scalars $c_1, c_2, \ldots, c_p$, the vector $\mathbf{y}$ defined by

$$\mathbf{y} = c_1\mathbf{v_1} + \ldots + c_p\mathbf{v_p}$$

is called a *linear combination* of $\mathbf{v_1}, \ldots, \mathbf{v_p}$ with *weights* or *coefficients* $c_1, \ldots, c_p$.

This is well defined because of associativity of vector addition.

Definition

If $\mathbf{a_1}, \ldots, \mathbf{a_n}$ are the columns of A then

$$A\mathbf{x} := [\mathbf{a}_1 \quad \mathbf{a}_2 \ldots \quad \mathbf{a}_n] \begin{bmatrix} x_1 \\ . \\ . \\ . \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + \ldots + x_n\mathbf{a}_n$$

The above definition agrees with the definition of $A\mathbf{x}$, when $\mathbf{x}$ is viewed as a column matrix, as the $k, 1$-th entry of the column matrix $A\mathbf{x}$ is the same as the k-th entry of $A\mathbf{x}$, viewed as a vector in $\mathbb{R}^m$.

Theorem

If A is an $m \times n$ matrix, with columns $\mathbf{a_1}, \ldots, \mathbf{a_n}$, and if $\mathbf{b}$ is in $\mathbb{R}^m$, the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + \ldots + x_n\mathbf{a}_n = \mathbf{b}$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$[\mathbf{a_1} \quad \mathbf{a_2} \ldots \quad \mathbf{a_n} \quad \mathbf{b}]$$

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Homogeneous Linear Systems

Definition

A system of linear equations is said to be *homogeneous* if it can be written in the form

$$A\mathbf{x} = \mathbf{0},$$

where A is an $m \times n$ matrix.

Fact

The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the equation has at least one free variable.

Parametric Vector Form

Whenever the solution set of a linear system is described explicitly in terms of a linear combination of vectors with variable and/or fixed weights, we say that the solution is in *parametric vector form*.

For a homogeneous system, the weights are all variable.

For the non-homogeneous system

$$A\mathbf{x} = \mathbf{b}, \quad \mathbf{b} \neq \mathbf{0}$$

we first check if the system is consistent.

Writing a Solution Set (of a consistent system) in Parametric Vector Form

1. Row reduce the augmented matrix to reduced echelon form.
2. Express each basic variable in terms of any free variables appearing in an equation.
3. Write a typical solution $\mathbf{x}$ as a vector whose entries depend on the free variables, if any.
4. Decompose $\mathbf{x}$ into a linear combination of vectors (with numeric entries) using the free variables as parameters.

$$x_1 - 3x_2 - 5x_3 = 0$$
$$x_2 + x_3 = 3$$

Example

Find the general solution of the linear system whose augmented matrix is

$$\begin{bmatrix} 1 & -3 & -5 & 0 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

Let us reduce this to RREF.

$$\xrightarrow{R_1 \to R_1 + 3R_2} \begin{bmatrix} 1 & 0 & -2 & 9 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

Let us write the system of equations.

$$x_1 - 2x_3 = 9$$
$$x_2 + x_3 = 3$$

Rewrite expressing basic variables in terms of free variables (if any).

$$x_1 = 9 + 2x_3$$
$$x_2 = 3 - x_3$$
$$x_3 \text{ free}$$

In Parametric Vector Form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{x} = \begin{bmatrix} 9 \\ 3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

The solution set of $Ax = b$

$= \{ x \in \mathbb{R}^n \mid Ax = b \} = S$

Theorem

Suppose the equation

$$A\mathbf{x} = \mathbf{b}$$

is consistent for some given $\mathbf{b}$, *and let* $\mathbf{p}$ *be a solution (i.e.* $A\mathbf{p} = \mathbf{b}$*). Then the solution set of* $A\mathbf{x} = \mathbf{b}$ *is the set of all vectors of the form*

$$\mathbf{w} = \mathbf{p} + \mathbf{v_h},$$

where $\mathbf{v_h}$ *is any solution of the homogeneous equation*

$$A\mathbf{x} = \mathbf{0}.$$

$T = \{ w \mid w = p + v_h, \text{ where } v_h \in \mathbb{R}^n \text{ and } Av_h = 0 \}$

all of

1. Make sure your symbols are well-defined when you write a proof.

Proof

Suppose $\mathbf{w}$ is any solution of the equation $A\mathbf{x} = \mathbf{b}$. This means

$$A\mathbf{w} = \mathbf{b}. \tag{1}$$

Also, we're given that

$$A\mathbf{p} = \mathbf{b}. \tag{2}$$

If we subtract equation (2) from (1) we get

$$A(\mathbf{w} - \mathbf{p}) = \mathbf{0}.$$

Hence $\mathbf{w} - \mathbf{p}$ is a solution of the equation $\underline{A\mathbf{x} = \mathbf{0}}$.

Put $\mathbf{v_h} = \mathbf{w} - \mathbf{p}$. Then

$$\underline{\mathbf{w} = \mathbf{p} + \mathbf{v_h}}. \quad = p + (w - p)$$

Next we show that any vector of the form $p + v_h$ is a solution of $Ax = b$,

~~Conversely, suppose~~ where $\mathbf{v_h}$ is any solution of the homogeneous system $A\mathbf{x} = \mathbf{0}$. This means

$$A\mathbf{v_h} = \mathbf{0}. \tag{3}$$

Put $\mathbf{w} = \mathbf{p} + \mathbf{v_h}$. Then using equations (2) and (3) we get

$$A\mathbf{w} = \mathbf{Ap} + \mathbf{Av_h} = \mathbf{b}.$$

$b + 0$

which means $\mathbf{w}$ is a solution of $A\mathbf{x} = \mathbf{b}$.

$$Aw = A(p + v_h) = Ap + Av_h$$

$$= Ap + 0 = b.$$

To show that two sets S and T are equal, we have to show $S \subset T$ as well as $T \subset S$.

subset relation

Matrices - Review

Theorem

Let A, B*, and* C *be* $m \times n$ *matrices, and let* r *and* s *be scalars.*

- **a.** $A + B = B + A$ *(Commutativity of addition)*
- **b.** $A + (B + C) = (A + B) + C$ *(Associativity of addition)*
- **c.** $A + 0 = A$ *(Additive Identity)*
- **d.** $r(A + B) = rA + rB$ *(Distributive Law)*
- **e.** $(r + s)A = rA + sA$ *(Distributive Law)*
- **f.** $r(sA) = (rs)A$

Matrix Multiplication

Let $A = (a_{ij})$ be an $m \times n$ matrix and $B = (b_{ij})$ be an $n \times p$ matrix. We define the matrix product $C := AB$ as the $m \times p$ matrix whose entries are

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}. \quad \text{(Row-Column Rule)}$$

Theorem

Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined.

- **a.** $A(BC) = (AB)C$ *(associative law of multiplication)*
- **b.** $A(B + C) = AB + AC$ *(left distributive law)*
- **c.** $(B + C)A = BA + CA$ *(right distributive law)*
- **d.** $r(AB) = (rA)B = A(rB)$ *for any scalar r*
- **e.** $I_m A = A = A I_n$ *(identity for matrix multiplication)*

Let's verify the associative law.

Powers of a Matrix

$$A^k = \underbrace{A \dots A}_{k}$$

Transpose of a Matrix

Let A and B denote matrices whose sizes are appropriate for the following sums and products.

- **a.** $(A^T)^T = A$
- **b.** $(A+B)^T = A^T + B^T$
- **c.** $(rA)^T = rA^T$ for any scalar r
- **d.** $(AB)^T = B^T A^T$

A New Perspective

$B = [b_1 \; . \; . \; . \; b_p]$

This matrix product has the property that if $\mathbf{b_1}, \ldots, \mathbf{b_p}$ are the columns of B then

$$AB = [A\mathbf{b}_1 \quad A\mathbf{b}_2 \quad \ldots \quad A\mathbf{b_p}]$$

Each column of AB is a linear combination of the columns of A using weights from the corresponding column of B.

3×2

$$A = \begin{array}{cc} a_1 & a_2 \\ \end{array}$$
$$A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 3 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 5 & 1 & 3 \\ 1 & 2 & 0 \end{bmatrix} \qquad 2 \times 3$$
$$\begin{array}{ccc} b_1 & b_2 & b_3 \end{array}$$

$$AB = \begin{bmatrix} 6 & 3 & 3 \\ 10 & 2 & 6 \\ 16 & 5 & 9 \end{bmatrix} = \begin{bmatrix} Ab_1 & Ab_2 & Ab_3 \end{bmatrix}$$

Row col rule:

$$AB = \begin{bmatrix} 6 & 3 & 3 \\ 10 & 2 & 6 \\ 16 & 5 & 9 \end{bmatrix}$$

$$Ab_1 = 5 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Proof

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

Let $A = (a_{ij})$ be an $m \times n$ matrix and let $B = (b_{ij})$ be an $n \times p$ matrix whose columns are $\mathbf{b_1}, \ldots, \mathbf{b_p}$.

Let $k \in \{1, \ldots, p\}$. Then for $1 \leq l \leq m$, the l-th entry of the k-th column of AB is simply the l, k-th entry of the matrix product AB, which is

$$\sum_{j=1}^{n} a_{lj} b_{jk}$$

If we denote the columns of A by $\mathbf{a}_1, \ldots, \mathbf{a}_n$, then

$$b_k = \begin{bmatrix} b_{1k} \\ b_{2k} \\ \vdots \\ b_{nk} \end{bmatrix}$$

$$A\mathbf{b}_k = b_{1k}\mathbf{a}_1 + \ldots + b_{nk}\mathbf{a}_n = \sum_{j=1}^{n} b_{jk}\mathbf{a}_j$$

$$b_{jk}\, a_j = \begin{bmatrix} b_{jk} a_{1j} \\ b_{jk} a_{2j} \\ b_{jk} a_{lj} \\ \vdots \\ b_{jk} a_{mj} \end{bmatrix}$$

l-th entry

Thus the l-th entry of the column vector $A\mathbf{b}_k$ is

$$\sum_{j=1}^{n} b_{jk} a_{lj}. \quad = \quad \sum_{j=1}^{n} a_{lj} b_{jk}$$

Therefore $A\mathbf{b}_k$ is the k-th column of AB.

Similarly, each row of AB is a linear combination of the rows of B using weights from the corresponding row of A. In other words

$$row_i(AB) = row_i(A)B.$$

Proof.

$$\begin{aligned} row_i(AB) &= (col_i((AB)^T))^T \\ &= (col_i(B^T A^T))^T \\ &= (B^T col_i(A^T))^T \\ &= row_i(A)B. \end{aligned}$$

□

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 3 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 5 & 1 & 3 \\ 1 & 2 & 0 \end{bmatrix}$$

$$\text{row}_1(AB) = 1\begin{bmatrix} 5 & 1 & 3 \end{bmatrix} + 1\begin{bmatrix} 1 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 3 & 3 \end{bmatrix}$$

$$\text{row}_2(AB) = 2\begin{bmatrix} 5 & 1 & 3 \end{bmatrix} + 0\begin{bmatrix} 1 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 2 & 6 \end{bmatrix}$$

$$\text{row}_3(AB) = 3\begin{bmatrix} 5 & 1 & 3 \end{bmatrix} + 1\begin{bmatrix} 1 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 5 & 9 \end{bmatrix}$$

$$AB = \begin{bmatrix} 6 & 3 & 3 \\ 10 & 2 & 6 \\ 16 & 5 & 9 \end{bmatrix}$$

Inconsistent system with free variable.

$$x_1 + x_2 = 1$$

$$x_1 + x_2 = 2$$

not pivot column

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$5x_4 = 0$$