Spanning Set Theorem, p. 212 of course textbook

Theorem

Let V be a vector space. Let $S = \{v_1, v_2, \ldots, v_p\}$ be a set in V and let $H = Span\{v_1, v_2, \ldots, v_p\}$.

1. *If one of the vectors in S, say v_k, is a linear combination of the remaining vectors in S, then the set formed from S by removing v_k still spans H.*
2. *If $H \neq \{0\}$, some subset of S is a basis for H.*

Theorem

Let V be a finite dimensional vector space. Let $\mathcal{B}_1 = \{b_1, \ldots, b_n\}$ be a basis of V. Let $\mathcal{B}_2 = \{v_1, \ldots, v_n\}$ be any other linearly independent subset of V. Then $\mathcal{B}_2$ is also a basis of V.

Theorem

Let V be a finite dimensional vector space. Any two bases of V must have the same cardinality.

Definition

The cardinality of a basis of a vector space V is called the *dimension* of V.

Theorem

Let V be a finite-dimensional vector space. Any linearly independent subset of V can be extended to a basis for V.

Corollary

Let W be a proper subspace of a vector space V. If W is finite dimensional then

$$\dim W < \dim V$$

Examples

Col $A \subset \mathbb{R}^2$

$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$

Given a spanning set for a vector space, should we try to find a basis by eliminating vectors from the set, or constructing a new set, adding one vector at a time?

Let us first consider this question when we know the dimension of the vector space.

Another basis is $\left\{ \begin{bmatrix} 7 \\ 9 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right\}$

If there are a large number of vectors and the dimension is small, then we're better off if we construct a basis adding one vector at a time.

Basis of Col A : $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 7 \\ 9 \end{bmatrix} \right\}$

Col $A = \mathbb{R}^2$

$$A = \begin{bmatrix} 1 & 7 & 0 & 1 & -2 \\ 2 & 9 & -1 & 3 & \pi \end{bmatrix}$$

Col A $=$

Find Col A and Row A. Span$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 7 \\ 9 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ \pi \end{bmatrix} \right\}$

Basis of Row A : $\left\{ \begin{bmatrix} 1 \\ 7 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 9 \\ -1 \\ 3 \\ \pi \end{bmatrix} \right\}$

Row A $\subset \underline{\underline{\mathbb{R}^5}}$

V

Find a basis for the set of vectors in $\mathbb{R}^2$ on the line $y = 5x$.

$$V = \{ (t, 5t) \mid t \in \mathbb{R} \}$$

$$= \operatorname{Span}\{(2, 10)\}$$

$$\dim V = 2.$$

Find a basis for the set of vectors in $\mathbb{R}^3$ in the plane $x + 2y + z = 0$

$$V = \{ (x, y, z) \in \mathbb{R}^3 \mid x + 2y + z = 0 \}$$

Example of Basis: $\{ (1, 0, -1), (0, 1, -2) \}$

Proposition : Any set of vectors $\{v_1, \ldots, v_n\}$ which contains the zero vector is linearly dependent.

Pf : Wlog, assume $v_1 = 0$.

Put $c_2 = c_3 = \ldots = c_n = 0$

$c_1 = 1$ Then $c_1 v_1 + c_2 v_2 + \ldots + c_n v_n = 0$

$y'' = 5e^{\sqrt{5}x}$

$= 5y$

$\boxed{y = e^{\sqrt{5}x}}$

$y' = \sqrt{5}e^{\sqrt{5}x}$

Find the general solution of the ODE

$$\boxed{y'' - 5y = 0}$$

Second order ODE

Fact from Calculus:

A homogeneous *n*-th order linear ODE has *n* linearly independent solutions.

$y'' = 5e^{-\sqrt{5}x}$

$y = e^{-\sqrt{5}x}$

$y' = -\sqrt{5}e^{-\sqrt{5}x}$

General solution:

$$c_1 e^{\sqrt{5}x} + c_2 e^{-\sqrt{5}x}$$

Gen 2nd order ODE w/ constant coefficients:

$$y'' + ay' + by = 0$$

Approach 1

$$y = e^{mx}$$

$$y' = me^{mx}$$

$$y'' = m^2 e^{mx}$$

$$m^2 \cancel{e^{mx}} + am\cancel{e^{mx}} + b\cancel{e^{mx}} = 0.$$

$\cos mx$

$\sin mx$

repeated:

e^{mx}, xe^{mx}

What about when we don't know the dimension?

Find a basis for the space spanned by the given vectors, $\mathbf{v}_1, \ldots, \mathbf{v}_5$.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ -3 \\ 3 \\ -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -1 \\ 1 \end{bmatrix}$$

First we construct a matrix. Then we row reduce to echelon form, to identify the pivot columns.

Subtract row 1 from row 4 : $R_4 \to R_4 - R_1$.

$$\begin{bmatrix} 1 & -2 & 6 & 5 & 0 \\ 0 & 1 & -1 & -3 & 3 \\ 0 & -1 & 2 & 3 & -1 \\ 0 & 3 & -7 & -1 & 1 \end{bmatrix}$$

Add row 2 multiplied by 2 to row 1 : $R_1 \to R_1 + 2R_2$.

$$\begin{bmatrix} 1 & 0 & 4 & -1 & 6 \\ 0 & 1 & -1 & -3 & 3 \\ 0 & -1 & 2 & 3 & -1 \\ 0 & 3 & -7 & -1 & 1 \end{bmatrix}$$

Add row 2 to row 3: $R_3 \to R_3 + R_2$.

$$\begin{bmatrix} 1 & 0 & 4 & -1 & 6 \\ 0 & 1 & -1 & -3 & 3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 3 & -7 & -1 & 1 \end{bmatrix}$$

Subtract row 2 multiplied by 3 from row 4 : $R_4 \to R_4 - 3R_2$.

$$\begin{bmatrix} 1 & 0 & 4 & -1 & 6 \\ 0 & 1 & -1 & -3 & 3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & -4 & 8 & -8 \end{bmatrix}$$

Subtract row 3 multiplied by 4 from row 1 : $R_1 \to R_1 - 4R_3$.

$$\begin{bmatrix} 1 & 0 & 0 & -1 & -2 \\ 0 & 1 & -1 & -3 & 3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & -4 & 8 & -8 \end{bmatrix}$$

Add row 3 to row 2 : $R_2 \to R_2 + R_3$.

$$\begin{bmatrix} 1 & 0 & 0 & -1 & -2 \\ 0 & 1 & 0 & -3 & 5 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & -4 & 8 & -8 \end{bmatrix}$$

Add row 3 multiplied by 4 to row 4 : $R_4 \to R_4 + 4R_3$.

$$\begin{bmatrix} 1 & 0 & 0 & -1 & -2 \\ 0 & 1 & 0 & -3 & 5 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 8 & 0 \end{bmatrix}$$

Finding a Basis for $\operatorname{Nul} A$

Recall that we can find the solution set of the equation $A\mathbf{x} = \mathbf{0}$ using the following method:

Writing a Solution Set (of a consistent system) in Parametric Vector Form

1. Row reduce the augmented matrix to reduced echelon form.
2. Express each basic variable in terms of any free variables appearing in an equation.
3. Write a typical solution $\mathbf{x}$ as a vector whose entries depend on the free variables, if any.
4. Decompose $\mathbf{x}$ into a linear combination of vectors (with numeric entries) using the free variables as parameters.

The vectors in step 4 also give us a basis. Why?

$A : m \times n$

The intuitive answer is that there cannot be any relations between free variables. More formally,

Rank-Nullity Theorem for Matrices

$$\dim \operatorname{Col} A + \dim \operatorname{Nul} A = n$$

We will prove this when we study linear transformations.

$c_1 A^2 + c_2 A + c_3 I = 0.$

Problem

Let V be the vector space of all 2×2 matrices. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \text{where } a \neq 0.$$

Let c_1, c_2 and c_3 be the coefficients of A^2, A and I respectively, in a linear dependence relation. (Here, I denotes the identity matrix.)

Then, if $c_1 = 1$ and $c_3 = det\, A$, then find the value of c_2.

$\boxed{p(A) = 0}$

Cayley Hamilton Thm.

$p(x) = det(A - xI)$

$$A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix}$$

$$\begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix} + \begin{bmatrix} c_2 a & c_2 b \\ c_2 c & c_2 d \end{bmatrix} + \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} = 0$$

$$a^2 + bc + c_2 a + ad - bc = 0.$$

$$c_2 = \frac{bc - ad - (a^2 + bc)}{a}$$

Question

Basis: $\left\{ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right\}$

a $\quad b \quad B$

A $\quad C$

Let

$$V = \left\{ \begin{bmatrix} a & b & c & -c & 0 \\ 0 & -a & b & a & b \\ 0 & 0 & -b & -a & c \end{bmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\}.$$

C

What is the the dimension of V?

Note that V is a subspace of $M_{3\times5}$, the vector space of 3×5 matrices.

$c_1 = 0$

$\boxed{c_1 A + c_2 B + c_3 C} = 0$

Question

Let

$$V = \text{Span}\left\{ \begin{bmatrix} a & b & c & -c & 0 \\ 0 & -a & b & a & b \\ 0 & 0 & -b & -a & c \end{bmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\}.$$

What is the the dimension of V?

Answer : 3

(If W is a vector space, then span $W = W$)

Question

Let $a, b, c \in \mathbb{R}$, where $c \neq 0$.

$$V = \text{Span}\left\{ \begin{bmatrix} a & b & c & -c & 0 \\ 0 & -a & b & a & b \\ 0 & 0 & -b & -a & c \end{bmatrix} \right\}.$$

What is the the dimension of V?

$\dim V = 1$