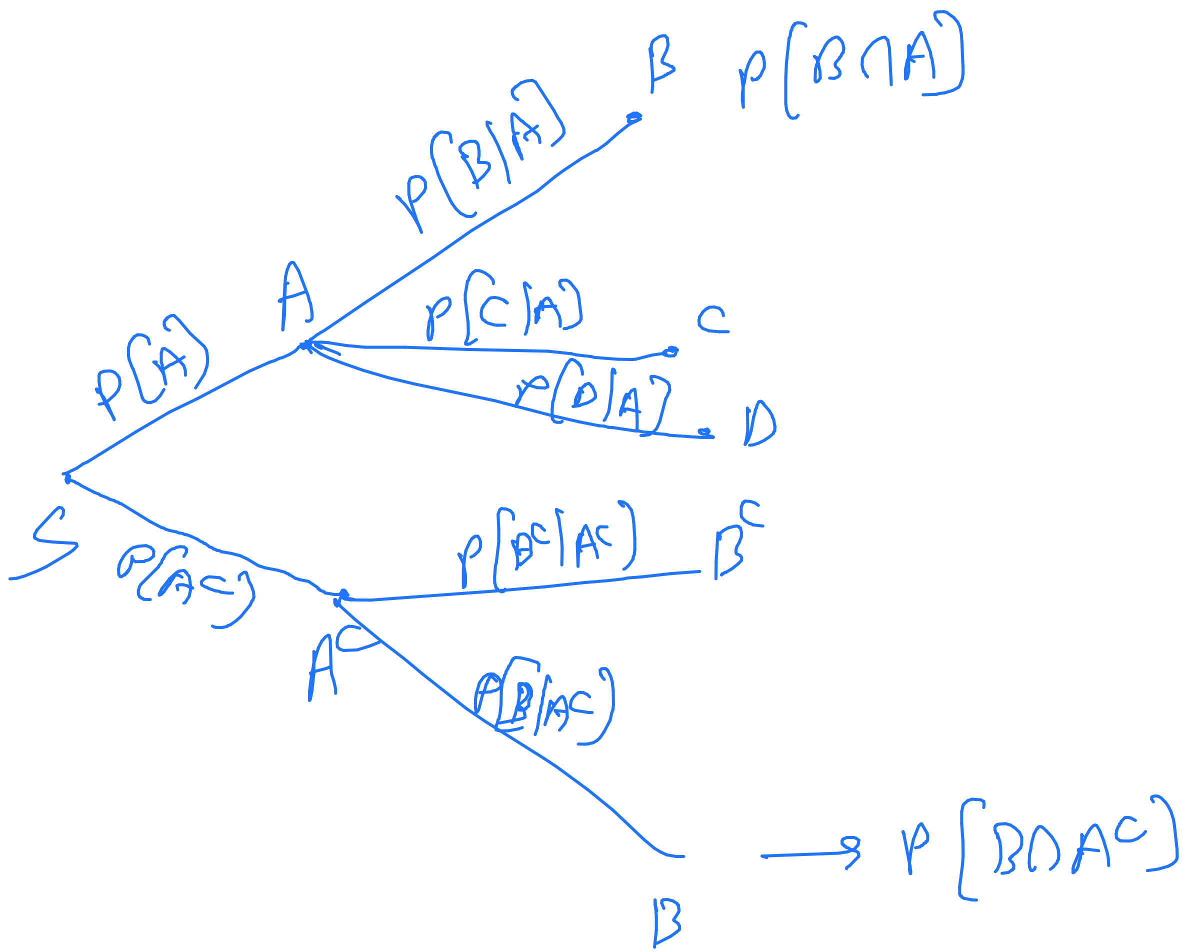
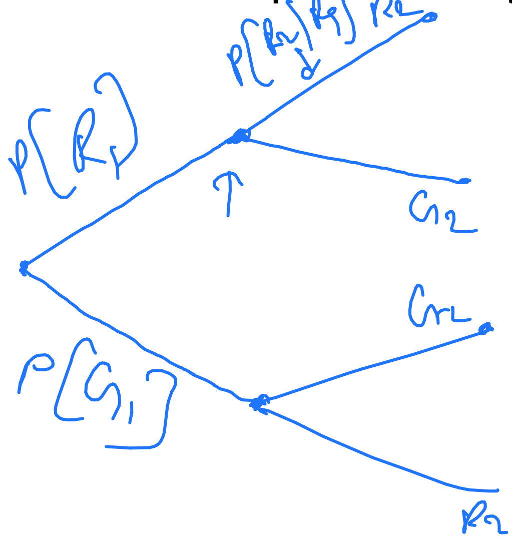
Sequential Diagrams



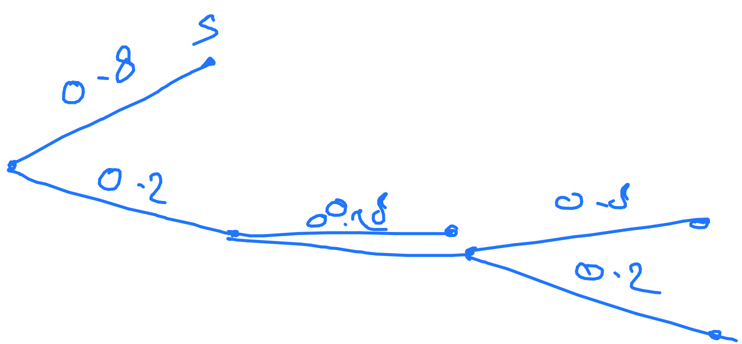
Example 1.25 Problem

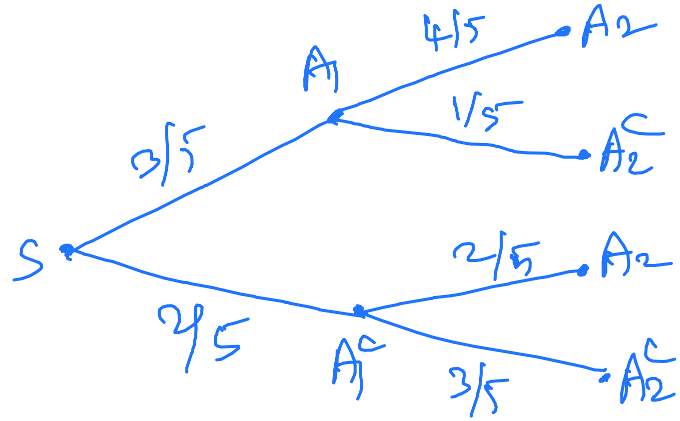
Suppose traffic engineers have coordinated the timing of two traffic lights to encourage a run of green lights. In particular, the timing was designed so that with probability 0.8 a driver will find the second light to have the same color as the first. Assuming the first light is equally likely to be red or green, what is the probability $P[G_2]$ that the second light is green? Also, what is $P[W]$, the probability that you wait for at least one light? Lastly, what is $P[G_1|R_2]$, the conditional probability of a green first light given a red second light?



Quiz 1.7

In a cellular phone system, a mobile phone must be paged to receive a phone call. However, paging attempts don't always succeed because the mobile phone may not receive the paging signal clearly. Consequently, the system will page a phone up to three times before giving up. If a single paging attempt succeeds with probability 0.8, sketch a probability tree for this experiment and find the probability $P[F]$ that the phone is found.





Problem 1.7.6

A machine produces photo detectors in pairs. Tests show that the first photo detector is acceptable with probability $3/5$. When the first photo detector is acceptable, the second photo detector is acceptable with probability $4/5$. If the first photo detector is defective, the second photo detector is acceptable with probability $2/5$.

(a) What is the probability that exactly one photo detector of a pair is acceptable?

(b) What is the probability that both photo detectors in a pair are defective?

Problem 1.7.9

The quality of each pair of photodiodes produced by the machine in Problem 1.7.6 is independent of the quality of every other pair of diodes.

(a) What is the probability of finding no good diodes in a collection of n pairs produced by the machine?

(b) How many pairs of diodes must the machine produce to reach a probability of 0.99 that there will be at least one acceptable diode?

$$P[A_1^c \cap A_2^c]$$

$$P[(A_{1,1}^c \cap A_{2,1}^c) \cap (A_{1,2}^c \cap A_{2,2}^c) \cdots \cap (A_{1,n}^c \cap A_{2,n}^c)]$$

$$= (P[(A_{1,1}^c \cap A_{2,1}^c)]) \cdots (P[A_{1,n}^c \cap A_{2,n}^c])$$

Section 1.8

Counting Methods

Definition 1.10 Fundamental Principle of Counting

If subexperiment A has n possible outcomes, and subexperiment B has k possible outcomes, then there are nk possible outcomes when you perform both subexperiments.

Example 1.29 Problem

Shuffle a deck and observe each card starting from the top. The outcome of the experiment is an ordered sequence of the 52 cards of the deck. How many possible outcomes are there?

Example 1.30 Problem

Shuffle the deck and choose three cards in order. How many outcomes are there?

Theorem 1.12

The number of k-permutations of n distinguishable objects is

$$(n)_k = n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}.$$

Theorem 1.13

The number of ways to choose k objects out of n distinguishable objects is

$$\binom{n}{k} = \frac{(n)_k}{k!} = \frac{n!}{k!(n-k)!}.$$

Definition 1.11 n choose k

For an integer $n \geq 0$, we define

$$\binom{n}{k} = \begin{cases} \dfrac{n!}{k!(n-k)!} & k = 0, 1, \ldots, n, \\ 0 & \textit{otherwise.} \end{cases}$$

Example 1.33 Problem

A laptop computer has PCMCIA expansion card slots A and B. Each slot can be filled with either a modem card (m), a SCSI interface (i), or a GPS card (g). From the set $\{m, i, g\}$ of possible cards, what is the set of possible ways to fill the two slots when we sample with replacement? In other words, how many ways can we fill the two card slots when we allow both slots to hold the same type of card?

Theorem 1.14

Given m distinguishable objects, there are m^n ways to choose with replacement an ordered sample of n objects.

Theorem 1.15

For n repetitions of a subexperiment with sample space $S = \{s_0, \ldots, s_{m-1}\}$, there are m^n possible observation sequences.

Example 1.38 Problem

For five subexperiments with sample space $S = \{0, 1\}$, how many observation sequences are there in which 0 appears $n_0 = 2$ times and 1 appears $n_1 = 3$ times?

Theorem 1.16

The number of observation sequences for n subexperiments with sample space $S = \{0, 1\}$ with 0 appearing n_0 times and 1 appearing $n_1 = n - n_0$ times is $\binom{n}{n_1}$.

Theorem 1.17

For n repetitions of a subexperiment with sample space $S = \{s_0, \ldots, s_{m-1}\}$, the number of length $n = n_0 + \cdots + n_{m-1}$ observation sequences with s_i appearing n_i times is

$$\binom{n}{n_0, \ldots, n_{m-1}} = \frac{n!}{n_0! n_1! \cdots n_{m-1}!}.$$

$^{n}C_{n_0} \quad {}^{n-n_0}C_{n_1} \quad {}^{n-n_1-n_0}C_{n_2} \quad - \; - \; - \; - \quad {}^{n-(n_1+n_2+\cdots+n_{m-2})}C_{n_{m-1}}$

Definition 1.12 Multinomial Coefficient

For an integer $n \geq 0$, *we define*

$$\binom{n}{n_0, \ldots, n_{m-1}} = \begin{cases} \dfrac{n!}{n_0! n_1! \cdots n_{m-1}!} & n_0 + \cdots + n_{m-1} = n; \\ & n_i \in \{0, 1, \ldots, n\}, i = 0, 1, \ldots, m-1, \\ 0 & \textit{otherwise.} \end{cases}$$

Quiz 1.8

Consider a binary code with 4 bits (0 or 1) in each code word. An example of a code word is 0110.

(1) How many different code words are there?

(2) How many code words have exactly two zeroes?

(3) How many code words begin with a zero?

(4) In a constant-ratio binary code, each code word has N bits. In every word, M of the N bits are 1 and the other $N - M$ bits are 0. How many different code words are in the code with $N = 8$ and $M = 3$?

Problem 1.8.7

An instant lottery ticket consists of a collection of boxes covered with gray wax. For a subset of the boxes, the gray wax hides a special mark. If a player scratches off the correct number of the marked boxes (and no boxes without the mark), then that ticket is a winner. Design an instant lottery game in which a player scratches five boxes and the probability that a ticket is a winner is approximately 0.01.

Section 1.9

Independent Trials

Example 1.39 Problem

What is the probability $P[S_{2,3}]$ of two failures and three successes in five independent trials with success probability p.

${}^5C_3 \; {}^3C_3$ mutually exclusive ways.

Any way occurs with prob $(1-p)(1-p)ppp$

$= (1-p)^2 p^3$.

$\therefore P[S_{2,3}] = {}^5C_3 (1-p)^2 p^3$.

Theorem 1.18

The probability of n_0 failures and n_1 successes in $n = n_0 + n_1$ independent trials is

$$P\left[S_{n_0,n_1}\right] = \binom{n}{n_1}(1-p)^{n-n_1}p^{n_1} = \binom{n}{n_0}(1-p)^{n_0}p^{n-n_0}.$$

Example 1.41 Problem

To communicate one bit of information reliably, cellular phones transmit the same binary symbol five times. Thus the information "zero" is transmitted as 00000 and "one" is 11111. The receiver detects the correct information if three or more binary symbols are received correctly. What is the information error probability $P[E]$, if the binary symbol error probability is $q = 0.1$?

$$P[E] = P[E \mid \text{Info } 0]\, P[\text{Info } 0] + P[E \mid \text{Info } 1]\, P[\text{Info } 1]$$

$$= 1 - \big(P[\text{No bit flipped}] + P[\text{Exactly 1 flipped}] + P[\text{Exactly 2 flipped}] \big)$$

Theorem 1.19

A subexperiment has sample space $S = \{s_0, \ldots, s_{m-1}\}$ with $P[s_i] = p_i$. For $n = n_0 + \cdots + n_{m-1}$ independent trials, the probability of n_i occurences of s_i, $i = 0, 1, \ldots, m-1$, is

$$P\left[S_{n_0,\ldots,n_{m-1}}\right] = \binom{n}{n_0, \ldots, n_{m-1}} p_0^{n_0} \cdots p_{m-1}^{n_{m-1}}.$$

Quiz 1.9

Data packets containing 100 bits are transmitted over a communication link. A transmitted bit is received in error (either a 0 sent is mistaken for a 1, or a 1 sent is mistaken for a 0) with probability $\epsilon = 0.01$, independent of the correctness of any other bit. The packet has been coded in such a way that if three or fewer bits are received in error, then those bits can be corrected. If more than three bits are received in error, then the packet is decoded with errors.

(1) Let $S_{k,100-k}$ denote the event that a received packet has k bits in error and $100 - k$ correctly decoded bits. What is $P[S_{k,100-k}]$ for $k = 0, 1, 2, 3$?

(2) Let C denote the event that a packet is decoded correctly. What is $P[C]$?

$P[\text{Group 1 kicker kicks a field goal}] = 1/2$

$P[\text{Group 2 \_\_\_\_\_\_\_\_}] = \frac{1}{3}$

kicker from G_1

kicker from G_i

$$K = (K \cap G_1) \cup (K \cap G_2)$$

$$P[K] = P[K \cap G_1] + P[K \cap G_2]$$

$$= \underbrace{P[K \mid G_1]}\, P[G_1] + P[K \mid G_2]\, P[G_2]$$

$$= (1/2)\, P[G_1] + \frac{1}{3} P[G_2]$$

$$= \left(\frac{1}{2}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)\left(2/3\right) =$$

Problem 1.9.5

There is a collection of field goal kickers, which can be divided into two groups 1 and 2. Group i has $3i$ kickers. On any kick, a kicker from group i will kick a field goal with probability $1/(i+1)$, independent of the outcome of any other kicks by that kicker or any other kicker.

(a) A kicker is selected at random from among all the kickers and attempts one field goal. Let K be the event that a field goal is kicked. Find $P[K]$.

(b) Two kickers are selected at random. For $j = 1, 2$, let K_j be the event that kicker j kicks a field goal. Find $P[K_1 \cap K_2]$. Are K_1 and K_2 independent events?

(c) A kicker is selected at random and attempts 10 field goals. Let M be the number of misses. Find $P[M = 5]$.