Indian Institute of Technology Indore
MA203 Complex Analysis and Differential Equations-II
(Autumn Semester 2023)
Tutorial Sheet 2

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1. Find the Laurent series of the following functions around the given points.

   (a) $f(z) = \dfrac{1}{z^2 - 3z + 2}$, around $z = 0$.

   (b) $f(z) = \dfrac{1}{z^2 - 3z + 2}$, around $z = 3/2$.

2. Expand the function $e^{\frac{1}{1-z}}$ in a Laurent series around the point $z = 1$. Also determine the domain within which the expansion holds.

3. Expand the function $e^{z+\frac{1}{z}}$ in a Laurent series in the domain $0 < |z| < \infty$.

4. Find Laurent series of the function $f(z) = \dfrac{1}{z^3 - z^4}$ in the regions

   (a) $\{z \in \mathbb{C} : 0 < |z| < 1\}$,

   (b) $\{z \in \mathbb{C} : |z| > 1\}$.

5. Find Laurent series of the function $f(z) = \dfrac{1}{1-z}$ in the domains $|z| > 1$ and $|z| < 1$.

6. Find the Laurent series of the function $f(z) = \dfrac{1}{z(1-z)^2}$ on the sets

   (a) $D_1 := \{z : 0 < |z - 1| < 1\}$,

   (b) $D_2 := \{z : |z - 1| > 1\}$.

7. Can the given functions be expanded in a Laurent series around the given points? Why or why not?

   (a) $\cos \dfrac{1}{z}$, $z = 0$.

   (b) $\sec \dfrac{1}{z-1}$, $z = 1$.

   (c) $z^2 \operatorname{cosec} \dfrac{1}{z}$, $z = 0$.

   (d) $\operatorname{Log} z$, $z = 0$.

8. Find the Laurent series of $f(z) = z^3 - 3z^2 + 3z - 1 + \dfrac{1}{z-2}$ around $z = 2$.

9. Discuss the singularities of the following functions:

   (a) $\dfrac{1}{z - z^3}$ (b) $\dfrac{z^4}{1 + z^4}$ (c) $\dfrac{z^5}{(1-z)^2}$ (d) $\dfrac{e^z}{1 + z^2}$ (e) e^{-1/z^2}

   (f) $\dfrac{\cos z}{z^2}$ (g) $\dfrac{\sin z}{z^2}$ (h) $\dfrac{\sin z}{z}$ (i) $\tan z$ (j) $\sin \dfrac{1}{z}$.

10. Discuss the singularities of the functions $\dfrac{1}{\sin z}$ and $\dfrac{1}{\sin \frac{1}{z}}$ at $z = 0$.

11. Which of the following singularities are removable/pole?

    (a) $\dfrac{\sin z}{z^2 - \pi^2}$, $z = \pi$

    (b) $\dfrac{\sin z}{(z - \pi)^2}$, $z = \pi$

12. Find the zeros and their orders for the following functions:

    (a) $z^2(\mathrm{e}^{z^2} - 1)$ (b) $z^2 + 9$ (c) $\dfrac{z^2 + 9}{z^4}$ (d) $z \sin z$.

13. Prove the following theorems.

    Theorem 1 *A point $z_0 \in \mathbb{C}$ is a zero of f of order m if and only if f can be expressed in the form*

    $$f(z) = \psi(z)(z - z_0)^m, \qquad z \in D = \{z : |z - z_0| < r\},$$

    for some $r > 0$, where ψ is analytic at z_0 and $\psi(z_0) \neq 0$.

    Theorem 2 *A point $z_0 \in \mathbb{C}$ is a pole of f of order m if and only if f can be expressed in the form*

    $$f(z) = \frac{\psi(z)}{(z - z_0)^m}, \qquad z \in \hat{D} = \{z : 0 < |z - z_0| < r\},$$

    for some $r > 0$, where ψ is analytic at z_0 and $\psi(z_0) \neq 0$.

14. Let z_0 be a pole of $f(z)$ of order m. Then show that

    $$\lim_{z \to z_0} (z - z_0)^k f(z) = \begin{cases} l, & k = m, \\ 0, & k > m, \\ \infty, & k < m. \end{cases}$$

    for some $l \neq 0$.