Manna Sir MA 203

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Power Series

$$S = \sum_{n=1}^{\infty} C_n (x-a)^n$$

if $\lim_{n\to\infty} S_n$ = finite & unique → maybe convergent

if $\lim_{n\to\infty} S_n = \pm\infty$ divergent surely

Ratio Test ⤵

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = l$$

i) $\sup(l) > 1$ cannot say

ii) $\inf(l) > 1$ divergent

iii) $\sup(l) < 1$ convergent

iv) $\inf(l) < 1$ cannot say

v) $\inf(l) = 1$ cannot say

vi) $\sup(l) = 1$ cannot say

Root test ⤵

$$\lim_{n\to\infty} \left(|a_n|\right)^{1/n} = l$$

Expansions:

i) $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} \dots \infty \qquad x \in (-\infty, \infty)$$

ii) $$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 \dots \infty \qquad x \in (-1, 1)$$

iii) $$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad x \in (-\infty, \infty)$$

iv) $$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad x \in (-\infty, \infty)$$

iv) $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ $\quad x \in (-\infty, \infty)$

v) $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$

Theorems:

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad ROC = R_1$$

if x is in the ROC then

$f(x)$ is diff, cont, integrable, converges uniformly to its limit

Weierstrass M test:

if $f_n(x)$ be a sequence and $|f_n(x)| \le M_n \;\; \forall\, n \ge 1$

and $\sum_{n=1}^{\infty} M_n$ converges then $f_n(x)$ series also converges.

Theorem

if $\sum_{n=1}^{\infty} c_n (x-a)^n$ has ROC R_1 then $\sum_{n=1}^{\infty} n\, c_n (x-a)^{n-1}$ also has the same ROC R_1

Analytic Function

$$f(x) = \sum_{n=1}^{\infty} c_n (x-a)^n$$

$f(x)$ is analytic at $x=a$ if $\lim_{x\to a} f(x)$ is finite and it exists.

Ordinary Point

Consider nth order DE: $\quad y^n(x) + P_{n-1} y^{n-1}(x) + \ldots\ldots P_0\, y(x) = f(x)$

$x = x_0$ is called an ordinary point if all $P_{n-1}(x) \ldots P_0(x)$ and $f(x)$ are analytic at $x = x_0$ ie $f(x)$ and $P_i(x)$ $0 \le i \le n-1$ can be expressed as a power series about $x = x_0$ and having ROC $R > 0$

ie $P_i = \sum^{\infty} c_n (x-x_0)^n$ and $f(x) = \sum^{\infty} \alpha_n (x-x_0)^n$

$n=1$ $\qquad\qquad\qquad\qquad$ $n=1$

Analytic Alternate Definition

$f(x) = \sum_{n=1}^{\infty} c_n (x-a)^n$ is analytic at $x=a$ if its Taylor series exists and converges to $f(x)$

$$f(x) = \sum_{m=0}^{\infty} f^m(0) \frac{(x-a)^m}{m!}$$

Singular Points ($y'' + P_1(x) y' + P_2(x) y = 0$)

$x = x_0$ is called a singular point if $f(x)$ is not analytic at $x = x_0$

ie x_0 is not an ordinary point

i) Irregular Singular Point:

if $(x-x_0) P_1(x)$ and $(x-x_0)^2 P_2(x)$ gives infinity value at $x = x_0$

ii) Regular Singular Point:

if $(x-x_0) P_1(x)$ and $(x-x_0)^2 P_2(x)$ possess derivative of all orders.

Vanishing of all Coefficients.

if a power series $f(x) = \sum_{n=1}^{\infty} c_n (x-a)^n$ has a ROC R_1 and

$f(x) = 0$ $\forall x : |x - x_0| < R_1$ then each coeff of the series must be $= 0$

ie $C_n = 0$ $\forall$ $n \geq 1$

Imp Points

i) always check at the boundary of ROC ie $|x-a| = R$ for the convergence of the Series.

ii) if we have to find cond^n for convergence $\left| \frac{a_{n+1}}{a_n} \right| < 1$ will give cond^n ①

ii) if we have to find cond^n for convergence $\left|\frac{a_{n+1}}{a_n}\right| < 1$ will give cond^n ①

check for $\left|\frac{a_{n+1}}{a_n}\right| = 1$ for cond^n ② as well

iii) We cannot rearrange the terms of a sequence.

FORMULA

$$f(x) = \sum_{n=0}^{\infty} C_n x^n$$

$$C_n = \frac{f^n(x)}{n!}$$

FROBENIUS METHOD (use when $\frac{P(x)}{x}$ $\frac{Q(x)}{x^2}$ is not analytic at $x=0$)

$$x^2 y'' + x P(x) y' + Q(x) y = 0$$

if $P(x)$ & $Q(x)$ are analytic throughout ROC

then $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ r = any parametric value

then substitute in DE & find sol^n

$$y(x) = \sum_{n=0}^{\infty} a_n x^{r+n} \qquad y' = \sum_{n=0}^{\infty} (r+n) a_n x^{r+n-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n (r+n)(r+n-1) x^{r+n-2}$$

Let $P(x) = c_0 + c_1 x + c_2 x^2 \dots$

$Q(x) = d_0 + d_1 x + d_2 x^2 \dots$

Substitute in DE

$$x^2 \sum_{n=0}^{\infty} (r+n)(r+n-1) a_n x^{r+n-2} + (c_0 + c_1 x + \dots) x \sum_{n=0}^{\infty} a_n (r+n) x^{r+n-1} + (d_0 + \dots) \sum_{n=0}^{\infty} a_n x^{r+n} = 0$$

equating all coeff of $x^n_r = 0$

$n=0$

equating all coeff of $x^n = 0$

smallest power of x $= x^r$ when $n=0$

$$\left(r(r-1) + c_0 r + d_0\right) a_n = 0 \qquad a_0 \neq 0$$

$$\therefore r^2 + (c_0-1)r + d_0 = 0 \quad \text{indicial equation.}$$

roots $= r_1, r_2$

Case 1 : $r_1 - r_2 \neq$ integer & $r_1 \neq r_2$

$$y(x) = \alpha\, y(x)\Big|_{r=r_1} + \beta\, y(x)\Big|_{r=r_2} \qquad \alpha, \beta = \text{const}$$

Case 2 : $r_1 = r_2$

$$y(x) = \alpha\, y(x)\Big|_{r=r_1} + \beta \frac{\partial y(x)}{\partial r}\Big|_{r=r_1}$$

Case 3: $r_1 - r_2 =$ integer. $r_1 \neq r_2$

if some coeff of $y(x) \to \infty$ when $r = r_1$ then we modify a_0 by $a_0 = b_0(r - r_1)$

we obtain two sol^n $y(x)\Big|_{r=r_1}$ & $\dfrac{\partial y(x)}{\partial r}\Big|_{r=r_1}$

we reject $r = r_2$

Case 4: $r_1 - r_2 =$ int but coeff of $y(x)$ is not $\to \infty$

in this case find a r_1 such that $\sum_{n=0}^{\infty} a_n x^{n+r}$ a_0 & a_1 are constants and every other a_n can be represented in terms of a_0 & a_1. then sol^n is:

$$y(x) = \alpha\, y(x)\Big|_{r=r_1}$$

* use recurrsion to find a_0 a_1 ..... a_n

* if $P(x) = \sin x$ then use its series expansion of $\left(x + \dfrac{x^3}{3!} \cdots\right)$

* if $P(x) = \sin x$ then use its series expansion of $\left(x + \frac{x^3}{3!} \cdots\right)$

then take first few terms as an approximation.

SERIES SOLUTION OF ODE

$$y'' + P(x)\,y' + Q(x)\,y = 0$$

if $P(x)$ & $Q(x)$ are analytic about a point $x = x_0$ then we can write the sol^n of the ODE as:

$$y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$$

TAYLOR SERIES

$$f(x) = \sum_{m=0}^{\infty} f^{m}(x_0)\,\frac{(x-x_0)^m}{m!}$$

Imp Points.

→ if power series about some point $x = x_0$ then

$$y = \sum a_n (x - x_0)^n$$

suppose eq^n is $x y'' \cdots$

$$\Rightarrow x \sum n(n-1)\,a_n (x - x_0)^{n-2}$$

$$\Rightarrow \left[(x - x_0) + x_0\right] \sum n(n-1)\,a_n (x - x_0)^{n-2}$$

$$\Rightarrow \sum n(n-1)\,a_n (x - x_0)^{n-1} + x_0 \sum n(n-1)\,a_n (x - x_0)^{n-2}$$

This is simplified & will be easy to calculate.

Theorem 9 : (Leibniz test) *If (a_n) is decreasing and $a_n \to 0$, then $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges.*

$$a_n = \sum_{k=1}^{2^n} \frac{1}{k}$$
$$= (1) + (1/2) + (1/3 + 1/4) + (1/5 + 1/6 + 1/7 + 1/8) + \cdots$$

$$
\begin{aligned}
a_n &= \sum_{k=1}^{2^n} \frac{1}{k} \\
&= (1) + (1/2) + (1/3 + 1/4) + (1/5 + 1/6 + 1/7 + 1/8) + \cdots \\
&\quad + (1/(2^{n-1}+1) + 1/(2^{n-1}+2) + \cdots + 1/(2^n - 1) + 1/2^n) \\
&\geq \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2}
\end{aligned}
$$

BEZZEL'S EQUATION

$$x^2 y'' + x y' + (x^2 - n^2) y = 0 \qquad x > 0 \quad n \geq 0 \quad n \in \mathbb{R}$$

Since $x = 0$ is a RSP apply Frobenius method $y = \sum_{n=0}^{\infty} C_n x^{n+\rho}$

$\therefore \rho = \pm n$

$$y(x) = C_0 x^n \left[1 - \frac{x^2}{4+4n} + \frac{x^4}{(4+4n)(8n+16)} \cdots \right] = J_n(x)$$

choose $C_0 = \dfrac{1}{2^n \, n!}$

$\Gamma{n+1} = n!$

$\lfloor n = n!$

$$J_n(x) = \frac{1}{n!}\left(\frac{x}{2}\right)^n - \frac{1}{1!\,(n+1)!}\left(\frac{x}{2}\right)^{n+2} + \cdots$$

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r \; \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

Bezzel's equation of first kind.

if $n \neq$ integer

$\text{sol}^n = \alpha J_n(x) + \beta J_{-n}(x)$

if $n =$ integer

$\text{sol}^n = \alpha J_n(x) + \beta Y_n(x)$

$$Y_n(x) = \lim_{r \to n} \frac{\cos(r\pi) J_r(x) - J_{-r}(x)}{\sin(r\pi)}$$

PROPERTIES

1) $\frac{d}{dx}\left[x^n J (x) \right] = x^n J \;\; (x)$

PROPERTIES

1) $\frac{d}{dx}\left[x^n J_n(x) \right] = x^n J_{n-1}(x)$

2) $\frac{d}{dx}\left[x^{-n} J_n(x) \right] = -x^{-n} J_{n+1}(x)$

3) $J_n(x) = \frac{x}{2n}\left[J_{n+1} + J_{n-1} \right]$

4) $J_n'(x) = \frac{1}{2}\left[J_{n-1} - J_{n+1} \right]$

5) $J_n'(x) = \frac{n}{x} J_n - J_{n+1}$

6) $J_{n+1} = \frac{2n}{x} J_n - J_{n-1}$

7) $\int x^{-n} J_{n+1}\, dx = -x^{-n} J_n$

ORTHOGANALITY OF BEZZEL'S FUNCTION

$f(x)$, $g(x)$ are orthogonal on $a \le x \le b$ if

$$\int_a^b f(x)\, g(x)\, dx = 0$$

set of $f_i(x)$ funcⁿ are mutually orthogonal on $x \in [a,b]$ iff:

$$\int_a^b f_i f_j \, dx = 0 \quad \text{for } i \ne j$$
$$> 0 \quad \text{for } i = j$$

BEZZELS

if α_1 α_2 are roots of Bezzels equations.

then $J_n(\alpha_1 a) = 0$ $\quad J_n(\alpha_2 a) = 0$

$$\int_0^a x J_n(\alpha_1 x) J_n(\alpha_2 x)\, dx = \begin{cases} 0 & ; \alpha_1 \neq \alpha_2 \\ \dfrac{a^2}{2}\left[J_n'(\alpha_1 a)\right]^2 & ; \alpha_1 = \alpha_2 \end{cases}$$

$$\text{or } \frac{a^2}{2}\left[J_{n+1}(\alpha_1 a)\right]^2$$

Observations:

$$\int_0^1 x J_n^2(x)\, dx = \frac{1}{2} J_{n+1}^2(x)$$

LEGENDRE POLYNOMIALS

$$(1-x^2) y'' - 2x y' + n(n+1) y = 0 \qquad \frac{d}{dx}\left((1-x^2)\frac{dy}{dx}\right) + n(n+1) y = 0$$

put $z = \frac{1}{x}$ $\qquad \dfrac{dy}{dx} = -z^2 \dfrac{dy}{dz}$

$$\frac{d^2y}{dx^2} = \frac{d}{dz}\left(-z^2 \frac{dy}{dz}\right)\frac{dz}{dx}$$

$$(z^4 - z^2)\frac{d^2y}{dz^2} + 2z^3 \frac{dy}{dz} + n(n+1) y = 0$$

$z = 0$ is RSP $\qquad$ Frobenium around $z = 0$

$$y(z) = \sum_{n=0}^{\infty} C_n x^{\rho+n} \qquad C_0 \neq 0 \qquad \rho = -n, n+1$$

$$y(x) = A P_n(x) + B Q_n(x)$$

$$P_n(x) = \sum_{r=0}^{N} \frac{(-1)^r \, (2n-2r)!}{2^r \, r! \, (n-r)! \, (n-2r)!} x^{n-2r}$$

$N = \frac{n}{2}$ if $n =$ even

$= \frac{n-1}{2}$ if $n =$ odd.

$P_0(x) = 1$ $\quad P_1(x) = x$ $\quad P_2(x) = \frac{1 \cdot 3}{2!}\left[x^2 - \frac{2}{2 \cdot 3}\right]$

$P_n(x)$ is a polynomial since its a finite series. and is called Lengdre Polynomial eqn of the first Kind.

Put $C_0 = \frac{(2n)!}{2^n (n!)^2}$

Rodrigue's Formula.

$$P_n(x) = \frac{1}{n! \, 2^n} \frac{d^n}{dx^n} (x^2 - 1)^n$$

Orthogonality

$$\int_{-1}^{1} P_m(x) P_n(x) \, dx = \begin{cases} 0; & m \neq n \\ \frac{2}{2n+1}; & m = n \end{cases}$$

Generating Function of Lengdre Polynomial

$P_n(x)$ is the coefficient of t^n in the expansion of

$(1 - 2xt + t^2)^{-1/2}$ $\quad |x| \leq 1 \quad |t| \leq 1$

$$(1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(x)$$

Generating Function for $J_n(x)$

$$e^{\frac{1}{2}x\left(t - \frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} t^n J_n(x)$$

$$e \quad = \sum_{n=-\infty} \quad t \quad J_n(x)$$

Recurrence Formula for $P_n(x)$

$$(n+1)\, P_{n+1}(x) = (2n+1)\, x\, P_n(x) - n\, P_{n-1}(x)$$

$$n\, P_n(x) = x\, P_n'(x) - P_{n-1}'(x)$$

$$(2n+1)\, P_n(x) = P_{n+1}'(x) - P_{n-1}'(x)$$

$$P_n'(x) = x\, P_{n-1}'(x) + n\, P_{n-1}(x)$$

$$(1-x^2)\, P_n'(x) = n\left[\, P_{n-1}(x) - x\, P_n(x) \right]$$

BEZZELS FUNCTION PROPS:

1) $J_{-n}(x) = (-1)^n J_n(x)$ only for $n \in \mathbb{Z}$

2) $J_0^2 + 2J_1^2 + 2J_2^2 \dots = 1$

3) $\dfrac{d}{dx}\left(J_n^2 + J_{n+1}^2\right) = 2\left(\dfrac{n}{x} J_n^2 - \dfrac{n+1}{x} J_{n+1}^2\right)$

4) $\lim\limits_{n \to \infty} J_n(x) = 0$

LEGENDRE PROPS:

1) $P_n(-1) = (-1)^n$

2) $P_n(1) = 1$

3) $(1-x^2)\, y'' - 2x\, y' + n(n+1)\, y = 0$

$y = P_n(x)$ is a sol^n

put $x = 1$

$\therefore -2y' + n(n+1)\, y = 0$

$\therefore -2\, P_n'(1) + n(n+1)\, P(1) = 0$ 1

$\therefore P_n'(1) = \underline{n(n+1)}$

$$\therefore P_n'(1) = \frac{n(n+1)}{2}$$

put $x=-1$ $\qquad P_n'(-1) = (-1)^n \frac{n(n+1)}{2}$

4) $P_{2m+1}(0) = 0$

$$P_{2m}(0) = \frac{(2m)!}{2^{2m}(m!)^2}$$

Put $x=0$ in generative func^n and compare.

5) $$x P_n = \frac{n+1}{2n+1} P_{n+1} + \frac{n}{2n+1} P_{n-1}$$

we can multiply both sides by P_{n-1} and integrate $\int_{-1}^{1}$

$$\therefore \int x P_n P_{n-1}\, dx = \frac{2n}{4n^2-1}$$

6) $$\int_{-1}^{1} (1-x^2) P_m' P_n'\, dx = \begin{cases} 0 & ; m \neq n \\ \frac{2n(n+1)}{2n+1} & ; m = n \end{cases}$$

7) All roots of P_n are distinct.

Leibnitz theorem: $(uv)_n = {}^nC_0 u_n v + {}^nC_1 u_{n-1} v + \cdots {}^nC_n u v_n$

$$u_n = \frac{d^n}{dx^n}(u)$$

8) $P_n(-x) = (-1)^n P_n(x)$

in $(1-2xz+z^2)^{-1/2} = \sum z^n P_n(x)$

put $x \to -x$, $z \to -z$ and compare coeff as LHS is same.

Linear Boundary Value Problems

$$p_0(x)\, y'' + p_1(x)\, y' + p_2(x)\, y = r(x)$$

p_0, p_1, p_2, r cont in $[\alpha, \beta]$

$$\left.\begin{aligned} B_1[y] &= a_0 y(\alpha) + a_1 y'(\alpha) + b_0 y(\beta) + b_1 y'(\beta) \\ B_2[y] &= c_0 y(\alpha) + c_1 y'(\alpha) + d_0 y(\beta) + d_1 y'(\beta) \end{aligned}\right\} \text{boundary values}$$

Sturm-Liouville Boundary Value Problem.

$$\frac{d}{dx}\left[P(x) \frac{dy}{dx} \right] + \left[q(x) + \lambda \underbrace{r(x)}_{\text{weight func}^n} \right] y = 0 \quad — ①$$

$A \le x \le B$

p, q, r cont in

$\lambda \in \mathbb{R}$ $\quad p(x) > 0 \quad r(x) > 0$

$$\begin{aligned} a_1\, y(A) + a_2\, y'(A) &= 0 \\ b_1\, y(B) + b_2\, y'(B) &= 0 \end{aligned} \quad — ② \qquad \text{or} \qquad \begin{aligned} y(A) &= y(B) \\ y'(A) &= y'(B) \\ P(A) &= P(B) \end{aligned} \quad — ③$$

A BVP with ① & ② or ① & ③ is considered Sturm Liouville BVP

for some $\lambda \in \mathbb{R}$ if y_λ has non-trivial sol^n ie not $y=0$

Then λ is called E value and y_λ is called E function.

spectrum: set of all Evalues with their E func^n

* to find λ you will take cases:

i) $\lambda = 0$ $\qquad$ ii) $\lambda > 0$ then $\lambda = \mu^2$ $\qquad$ iii) $\lambda < 0$ $\quad \lambda = -\mu^2$

Theorem if $p(x)\ q(x)\ r(x)$ are continuous on $[A, B]$

$a_1\ a_2\ b_1\ b_2 \in \mathbb{R}$

$A, B =$ finite $\in \mathbb{R}$

The ST BVP has countably many Evalues with E funcⁿ

Orthogonality

two functions $p(x)$ & $q(x)$ are said to be orthogonal to each other wrt a weight funcⁿ $r(x)$ if:

$$\int_A^B r(x)\, p(x)\, q(x)\, dx = 0$$

$p(x)$ $q(x)$ defined and continuous on $[A, B]$

Let $x(s)$ & $y(s)$ be solⁿs of SL BVP with Evalues λ, μ $(\mu \neq \lambda)$ if $\left[W(x(s), y(s)) \right]_A^B = 0$ then

$$\int_A^B r(s)\, x(s)\, y(s)\, ds = 0$$

EXACT EQUATIONS

$F(x, y, y', y'' \dots y^n) = 0$ is said to be exact if $F(\,)$ is the exact derivative of some $(n-1)^{th}$ order DE $G(x, y, y' \dots y^{n-1})$

$$F(x, y, y', \dots y^n) = \frac{d}{dx}\left(G(x, y, y' \dots y^{n-1}) \right)$$

$L[u]$ operator:

forms 2^{nd} order homogeneous DE $L[u] = 0$

$$L[u] = P_0(x)\, u'' + P_1(x)\, u' + P_2(x)\, u = 0 \quad \text{—} \quad (a)$$

$L[u]$ is exact if

$L[u]$ is exact if

$$L[u] = \frac{d}{dx}\left[A(x)\, u' + B(x)\, u \right]$$

if $p_0'' - p_1' + p_2 = 0$ then ⓐ is exact equation

$A(x) = p_0$ $B(x) = p_1 - p_0'$

Integrating Factor

$v(x)$ is IF for the DE $L[u]=0$ if $v\,L[u]=0$ is exact eqⁿ

$$v(x)\left[p_0 u'' + p_1 u' + p_2 \right] = \frac{d}{dx}\left[A u' + B u \right]$$

$L[u]=0$ $\therefore$ $A u' + B u = c$ $c =$ const.

Solⁿ of inhomogeneous 2^{nd} order DE ie $L[u] = \gamma(x)$

$$A u' + B u = \int \gamma(x)\, v(x)\, dx + C$$

Adjoint Operator: $M[v]$

$v(x)$ is IF of $p_0 u'' + p_1 u' + p_2 = 0$ if $v(x)$ is solⁿ of $M[v]=0$

$$M[v] = \left[p_0 v \right]'' - \left[p_1 v \right]' + p_2 v = 0$$

$$M[v] = p_0 v'' + (2p_0 - p_1) v' + (p_0'' - p_1' + p_2) v = 0$$

$M[v]$ is called the adjoint of $L[u]=0$

Langrange's Identity

$$v\,L[u] - u\,M[v] = \frac{d}{dx}\left(p_0(u'v - uv') - (p_0' - p_1)\, uv \right)$$

$$v\,L[u] - u\,M[v] = \frac{d}{dx}\left(P_0(u'v - uv') - (P_0' - P_1)uv \right)$$

Self Adjoint Equation

Homogeneous Eq^n with $L[u]=0$ $M[v]=0$ $L = M$

eq^{n}s which coincide with their adjoint.

Cond^n: $\boxed{P_0' = P_1}$

a second order DE is adjoint if it is in the form of

$$\frac{d}{dx}\left[P_0(x)\frac{dy}{dx}\right] + P_2(x)\,u = 0 \quad \text{or} \quad \int v\,L[u] = \int u\,L[v]$$

An eq^n can be made self-adjoint by multiplying it by:

$$h(x) = \frac{1}{P_0}\, e^{\int \frac{P_1}{P_0}dx}$$

Integrating Factor to turn to SL BVP

$$y'' + P(x)\,y' + Q(x)\,y = r(x)$$

$$IF = e^{\int P(x)\,dx}.$$

$$\therefore \frac{d}{dx}\left[IF\frac{dy}{dx}\right] + IF\,Q(x)\,y = IF\,r(x)$$

$r(x) = 0$ for SLBVP

Imp Points Tut-2

1) $2\cdot 4\cdot 6 \ldots (2n) = 2^n\, n!$

2) For Frobenius Method :

→ convert eqⁿ to $y'' + P(x)\,y' + Q(x)\,y = 0$

→ check whether $x = a$ is a RSP of $P(x)$ $Q(x)$

ie $\lim_{x \to a} (x-a)\,P(x)$ $\lim_{x \to a} (x-a)^2 Q(x)$ exists and are finite

→ check whether $P(x)$ and $Q(x)$ are analytic at all other points

→ then assume $y = \sum_{n=0}^{\infty} a_n (x-a)^{n+\rho}$ $a_0 \neq 0$

3) When we have to take $\left.\frac{\partial y}{\partial r}\right|_{r=r_1}$

suppose $y_r = a_0 x^r \left[1 + \sum_{m=1}^{\infty} \underbrace{\frac{(-1)^m x^{2m}}{(r+1)^2 (r+3)^2 + \cdots (r+2m-1)^2}}_{P} \right]$

take log

$$\log y = \log a_0 + \log x^r + \log [1 + P]$$

diff wrt r

$$\frac{1}{y_r(x)} \frac{dy_r}{dx} = \log x + \frac{1}{1+P} \frac{d}{dr}(1+P)$$

$$1+P = \frac{y_r(x)}{a_0 x^r}$$

$$\frac{1}{y_r(x)} \frac{d\,y_r(x)}{dx} = \log x + \frac{a_0 x^r}{y_r(x)} \frac{d}{dr}[1+P]$$

BEZZELS

1) $\Gamma(z) = \int_0^{\infty} x^{z-1} e^{-x}\, dx$

2) $\Gamma(z) = \Gamma(z-1)\ (z-1)$

3) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

4) $J_n(x) = \dfrac{x^n}{2^n \Gamma(n+1)} \left[1 - \dfrac{x^2}{2(2n+2)} + \dfrac{x^4}{2\cdot 4\cdot (2n+2)(2n+4)} \cdots\cdots \right]$

Wallis Formula.

1) $$\int_0^{\pi/2} \cos^m x\, dx = \int_0^{\pi/2} \sin^m x\, dx = I(m)$$

$$I(m) = \begin{cases} \frac{1}{2}\frac{3}{4}\frac{5}{6}\cdots\frac{m-1}{m}\cdot\frac{\pi}{2} & m = \text{even} \\ \frac{2}{1}\frac{4}{3}\frac{6}{5}\cdots\frac{m}{m-1} & m = \text{odd} \end{cases}$$

BEZZEL'S PROPERTIES

1) $$\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin(n\pi)}$$

2) Since J_n and J_{-n} both satisfy $y'' + \frac{1}{x}y' + \left(1 - \frac{n^2}{x^2}\right)y = 0$

$$J_n'' + \frac{1}{x}J_n' + \left(1 - \frac{n^2}{x^2}\right)J_n = 0 \quad \times J_{-n}$$

$$- \quad J_{-n}'' + \frac{1}{x}J_{-n}'' + \left(1 - \frac{n^2}{x^2}\right)J_{-n} = 0 \quad \times J_n$$

$$\underbrace{J_n'' J_{-n} - J_{-n}'' J_n}_{V'} + \frac{1}{x}\left(\underbrace{J_n' J_{-n} - J_{-n}' J_n}_{V}\right) = 0$$

you would get $J_n' J_{-n} - J_{-n}' J_n = \frac{c}{x}$ Now compare coeff of x^{-1} to find out c.

3) $$J_n(x) = \frac{1}{n!}\left(\frac{x}{2}\right)^n - \frac{1}{(n+1)!}\left(\frac{x}{2}\right)^{n+2} + \frac{1}{2!\,(n+2)!}\left(\frac{x}{2}\right)^{n+4} \cdots\cdots$$

Integration Props:

1) $$\int_0^{\pi} \cos mx \cos nx \, dx = \int_0^{\pi} \sin mx \sin nx \, dx = \frac{\pi}{2} \quad m=n$$
$$= 0 \quad m \neq n.$$

2) $J_n(0)=0$ but $J_0(0)=1$

3) $$\int_{-1}^{1} f(x) P_n(x) = \frac{1}{2^n n!} \int_{-1}^{1} (1-x^2)^n f^n(x) \, dx.$$

4) if a func^n has a repitition in root for $x=a$

then $f(a)=0$ $f'(a)=0$

Imp Points:

1) Try to use generative func^n or recurrence relation to solve the question. If it doesn't, then try acutual formula.

2) $P_0(x)=1$

$P_1(x)=x$

$P_2(x)=\frac{1}{2}(3x^2-1)$

$P_3(x)=\frac{1}{2}(5x^3-3x)$

$P_4(x)=\frac{1}{8}(35x^4-30x^2+3)$

Imp Approach Methods

Imp Approach Methods

1) $\displaystyle\int_{-1}^{1} \underbrace{(1-x^2)\,P_n'}_{\text{I}}\,\underbrace{P_m'}_{\text{II}}\,dx$

$$= \underbrace{(1-x^2)\,P_n'\,P_m\Big|_{-1}^{1}}_{\to 0} - \int_{-1}^{1}\left(-2xP_n' + (1-x^2)P_n''\right)P_m\,dx$$

$$(1-x^2)P_n'' - 2xP_n' + n(n+1)P_n = 0$$

$$= + \int_{-1}^{1} n(n+1)\,P_n\,P_m$$

$$= 0 \quad \text{if } n \neq m$$

$$\frac{n(n+1)}{2n+1} \quad \text{if } n = m$$

2) $\displaystyle\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}P_n(x)\right] + n(n+1)P_n = 0$ prove P_n satisfies

$$P_n = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

Let $v = (x^2-1)^n$

$$v_1 = 2xn(x^2-1)^{n-1}$$

$$(x^2-1)\,v_1 = 2xn\,v$$

$$(x^2-1)\,v_2 + 2x\,v_1 = 2n\,v_1 x + 2nv$$

$$(x^2-1)\,v_2 + 2x(1-n)\,v_1 - 2nv = 0$$

Diff wrt x n times and apply Leibnitz theorem

$$(1-x^2)\,v_{n+2} - 2x\,v_{n+1} + n(n+1)\,v_n = 0$$

$$\therefore \frac{d}{dx}\left((1-x^2) y_{n+1} \right) + n(n+1) v_n = 0$$

$$\frac{d}{dx}\left[(1-x^2) \frac{d}{dx} v_n \right] + n(n+1) v_n = 0$$

$$y_n = \frac{d^n}{dx^n} (x^2-1)^n$$

$$\frac{v_n}{2^n n!} = P_n$$

$\therefore$ Proved

3) $$I = \int_{-1}^{1} f(x) P_n(x) dx \qquad P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$$

$$= \int_{-1}^{1} \underbrace{f(x)}_{\mathrm{I}} \underbrace{\frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n dx}_{\mathrm{II}}$$

$$I = \frac{1}{2^n n!}\left[f(x) \frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n \Big|_{-1}^{1} - \int_{-1}^{1} f'(x) \frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n dx \right]$$

$$\frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n = \frac{d^{n-1}}{dx^{n-1}} (x-1)^n (x+1)^n$$

$$= (x-1)^n (n-1)! (x+1) + n(x-1)^{n-1} (n-2)! (x+1)^2 \cdots$$

$$= 0 \text{ at } x = \pm 1$$

$$\therefore I = \frac{1}{2^n n!}\left[-\int_{-1}^{1} f'(x) \frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n dx \right]$$

Basically $\frac{d^n}{dx^n} \to \frac{d^{n-1}}{dx^{n-1}}$ and $f(x) \to f'(x)$

$$\therefore I = \frac{1}{2^n n!} (-1)^n \int_{-1}^{1} (x^2-1)^n f^n(x)\, dx \quad \text{also.}$$

4) $$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$$

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\, \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$n = -1/2$

$$J_{-1/2}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\, \Gamma(r+\frac{1}{2})} \left(\frac{x}{2}\right)^{2r-\frac{1}{2}}$$

$$= \frac{1}{0!\, \sqrt{\pi}} \left(\frac{x}{2}\right)^{-1/2} - \frac{1}{1!\, \frac{1}{2}\sqrt{\pi}} \left(\frac{x}{2}\right)^{\frac{3}{2}} \cdots$$

$$= \frac{1}{\sqrt{\pi}} \left(\frac{x}{2}\right)^{-\frac{1}{2}} \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots \right]$$

$$= \sqrt{\frac{2}{\pi x}} \cos x$$