MA 204 Numerical Methods

Dr. Debopriya Mukherjee

Lecture-1

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Contents

- Solution of a nonlinear equation, bisection and secant methods, Newton's method, rate of convergence, solution of a system of nonlinear equations.

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- Interpolation by polynomials, divided differences, error of the interpolating polynomial, piecewise linear and cubic spline interpolation.

Motivation

Nonlinearity

One of the most frequent problem in engineering and science is to find the root(s) of a non-linear equation

$$f(x) = 0. \tag{1}$$

Here,

- $f : [a, b] \to \mathbb{R}$ is a nonlinear function in x;
- $f \in C^1[a, b]$;
- Roots are **isolated**.

Root of the equation

Definition 1

Given a nonlinear function $f : [a, b] \to \mathbb{R}$, find a value of r for which $f(r) = 0$. Such a solution value for r is called a **root** of the equation

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Approximation to a root

A point $x^\star \in \mathbb{R}$ such that

- $|r - x^\star|$ is **very small**, and
- $f(x^\star)$ is **very close** to 0.

Examples

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$f(x) = x^2 + 4x + 10$

$$f(x) = 0 \implies x^2 + 4x + 10 = 0 \implies (x+2)^2 + 6 = 0.$$

As $(x+2)^2 + 6 \geq 6 \; \forall \; x \in \mathbb{R}$. $\boxed{\text{So, } f(x) = 0 \text{ has no real roots.}}$

Example 4

$f(x) = x^2 + \cos(x) + e^x + \sqrt{x+1}$

The equation $f(x) = 0$ might have real root/roots but the point is that it is very difficult to find the analytic expression of x.

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2. Closed Domain Methods (Bracketing Methods)
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   Advantages: No need to locate the root initially

   Disadvantages: May not converge

For each of the method, we study the following:

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- Geometry of the method (iteration) process
- Computational Scheme
- Application and example

Bisection Method: Basic idea

This method is based on Intermediate Value Theorem.

Theorem

If $f \in C[a, b]$ and K is any number between $f(a)$ and $f(b)$, then there exists $c \in (a, b)$ such that $f(c) = K$.

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Bisection Method

- Suppose that $f(x)$ is continuous on given interval $[a, b]$.
- The function f satisfies the property $f(a)f(b) < 0$ with $f(a) \neq 0$ and $f(b) \neq 0$.
- By Intermediate Value Theorem, there exists a number c such that $f(c) = 0$.

Bisection Method: Description of the method

The Bisection method consists of the following steps:

Step 1: Given an initial interval $[a_0, b_0]$, set $n = 0$.

Step 2: Define $c_n = \frac{(a_n + b_n)}{2}$, the mid-point of interval $[a_n, b_n]$.

Step 3:
- If $f(c_n) = 0$, then $x^\star = c_n$ is the root.
- If $f(c_n) \neq 0$, then either

$$f(a_n)f(c_n) < 0 \quad \text{or} \quad f(a_n)f(c_n) > 0.$$

- If $f(a_n)f(c_n) < 0$, then $a_{n+1} = a_n$, $b_{n+1} = c_n$ and the root $x^\star \in [a_{n+1}, b_{n+1}]$.
- If $f(a_n)f(c_n) > 0$, then $f(b_n)f(c_n) < 0$, this implies $a_{n+1} = c_n$, $b_{n+1} = b_n$ and the root $x^\star \in [a_{n+1}, b_{n+1}]$.

Bisection Method

Step 4: Repeat

Step 5: If the root is not achieved in **Step 3**, then, find the length of new reduced interval $[a_{n+1}, b_{n+1}]$. If the length of the interval $b_{n+1} - a_{n+1}$ is less than a recommended positive number ε, then take the mid-point of this interval $(x^\star = (b_{n+1} + a_{n+1})/2)$ as the approximate root of the equation $f(x) = 0$, otherwise go to Step 2.

Convergence and error in Bisection Method

Let $[a_0, b_0] = [a, b]$ be the initial interval with $f(a)f(b) < 0$. Define the approximate root as $c_n = (a_n + b_n)/2$. Then, there exists a root $x^\star \in [a, b]$ such that

$$|c_n - x^\star| \le (\frac{1}{2})^n (b - a). \tag{2}$$

Moreover, to achieve the accuracy of $|c_n - x^\star| \le \varepsilon$, it is sufficient to take

$$\frac{|b - a|}{2^n} \le \varepsilon \quad \text{i.e.} \quad n \ge \frac{log(|b - a|) - log(\varepsilon)}{log 2}. \tag{3}$$

Error analysis in Bisection Method

$$b_{n+1} - a_{n+1} = \frac{1}{2}(b_n - a_n), \quad n \geq 1. \tag{4}$$

$$\begin{aligned} b_n - a_n &= \frac{1}{2}(b_{n-1} - a_{n-1}) \\ &= \frac{1}{2^2}(b_{n-2} - a_{n-2}) = \frac{1}{2^{n-1}}(b_1 - a_1). \end{aligned} \tag{5}$$

$$\begin{aligned} |c_n - x^\star| \leq c_n - a_n = b_n - c_n &= \frac{1}{2}(b_n - a_n) \\ &= (\frac{1}{2})^n(b_1 - a_1) = (\frac{1}{2})^n(b - a). \end{aligned} \tag{6}$$

Therefore, $|c_n - x^\star| \leq (\frac{1}{2})^n(b - a)$. This implies that the iterates c_n converge to $x^\star$ as $n \to \infty$.

How many iterations?

Our goal is to have $|c_n - x^\star| \leq \varepsilon$. This will be satisfied if

$$
\begin{aligned}
(\frac{1}{2})^n(b-a) \leq \varepsilon &\implies 2^n \geq \frac{b-a}{\varepsilon} \\
&\implies n log_{10} 2 \geq log_{10}(\frac{b-a}{\varepsilon}) \\
&\implies n \geq \frac{log(|b-a|) - log(\varepsilon)}{log 2}.
\end{aligned} \tag{7}
$$

Stop criteria

First select some tolerance $\varepsilon > 0$.

1. Small enough interval i.e., $b_n - a_n \leq \varepsilon$;
2. Small enough difference of consecutive approximations i.e.,

$$|c_{n+1} - c_n| \leq \varepsilon \quad \text{or} \quad \frac{|c_{n+1} - c_n|}{|c_n|} \leq \varepsilon;$$

3. Small enough functional value $|f(c_n)| \leq \varepsilon$;
4. Maximum number of iterations;
5. Any combination of the above.

Pros and Cons

Pros

1. This method is very easy to understand.
2. The sequence forms a cauchy sequence, and always converge to a solution.
3. It is often used as a starter for the more efficient methods.

Cons

1. This method is relatively slow to converge.
2. Choosing a guess close to the root may result in requiring many iterations to converge.