Date:
P. No:

Tutorial-1

2. Second order PDE:

$$A(x,y)\,u_{xx} + B(x,y)\,u_{xy} + C(x,y)\,u_{yy} = \phi(x,y,u,u_x,u_y)$$

Let $\xi = \xi(x,y)$ and $\eta = \eta(x,y)$

$$u(x,y) = \omega(\xi(x,y), \eta(x,y))$$

$$a\,\omega_{\xi\xi} + b\,\omega_{\xi\eta} + c\,\omega_{\eta\eta} = \phi(\xi,\eta,\omega,\omega_\xi,\omega_\eta)$$

$$a = A\,\xi_x^2 + B\,\xi_x\xi_y + C\,\xi_y^2$$
$$b = 2A\,\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\,\xi_y\eta_y$$
$$c = A\,\eta_x^2 + B\,\eta_x\eta_y + C\,\eta_y^2$$

$$b^2 - 4ac = J^2\,[B^2 - 4AC]$$

$$\boxed{J = \frac{\partial(\xi,\eta)}{\partial(x,y)} \neq 0}$$

(i) $b = 0$ and ($a = 0$ or $c = 0$)

Take $a = 0$, $A\,\xi_x^2 + B\,\xi_x\xi_y + C\,\xi_y^2 = 0$

divide by $\xi_y^2 \rightarrow A\left(\frac{\xi_x}{\xi_y}\right)^2 + B\left(\frac{\xi_x}{\xi_y}\right) + C = 0$ — (3)

Along the coordinate line, $\xi = \text{const}$, the total derivative is zero, i.e.

$$d\xi = 0$$
$$\xi_x\,dx + \xi_y\,dy = 0$$
$$\frac{dy}{dx} = -\frac{\xi_x}{\xi_y} \quad \text{— (4)}$$

Date:
P. No:

$$A\left(\frac{dy}{dx}\right)^2 - B\frac{dy}{dx} + C = 0 \qquad -(4)$$

This is characteristic polynomial. $(B^2 - 4AC = 0)$

$$\boxed{\frac{dy}{dx} = \frac{+B}{2A} = \lambda(x,y)} \quad (*)$$

· There is only one real characteristic curve.

$$\boxed{y = f(x) + \xi} \qquad \xi = c_1$$

Now, $b = 0 \rightarrow 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y = 0$

Divide by ξ_y throughout

$$\Rightarrow 2A\left(\frac{\xi_x}{\xi_y}\right)\eta_x + B\left(\frac{\xi_x}{\xi_y}\eta_y + \eta_x\right) + 2C\eta_y = 0$$

$\Rightarrow (B^2 - 4AC)\eta_y = 0$

$\rightarrow$ Since, $B^2 - 4AC = 0$, so we can choose η_y to be arbitrary.

Further, η is arbitrary.

e.g. $\boxed{\eta = x}$

Q. If: For $A, B, C \rightarrow$ constant.

$y = \frac{B}{2A}x + c_1$ | We know that $\xi = c_1$ is only real

$c_1 = y - \frac{B}{2A}x$ | $\left.\begin{array}{l}\xi = y - \frac{B}{2A}x \\ \eta = x\end{array}\right] \rightarrow a = b = 0, \; c = ?$

Date:
P. No:

(2). $\alpha \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$

$A = \alpha,\ B = 0,\ C = 0$

By characteristic eqⁿ for parabolic PDE → $\frac{dt}{dx} = \frac{B}{2A} = 0$

$\therefore \boxed{\begin{array}{l} t = \xi \\ \& \ x = \eta \end{array}}$ Heat eqⁿ is already in canonical form.

$u_t = \omega_\xi \xi_t + \omega_\eta \eta_t = u_\xi$

$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 \omega}{\partial \xi^2}$

$\boxed{k = A = \alpha}$

3. $u = f_1(x-ct) + f_2(x+ct)$

$u(x,0) = f_1(x) + f_2(x) = f(x)$

$\partial_t(u) = -c f_1'(x-ct) + c f_2'(x+ct)$

$\therefore = -c f_1'(x) + c f_2'(x) = g(x)$

~~Also, $-c^2 f_1''(x-ct) + c^2 f_2''(x+ct) = c^2 (f_1'' ...$~~

~~$f''(x-ct) = 0$~~

$u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$

Date:
P. No:

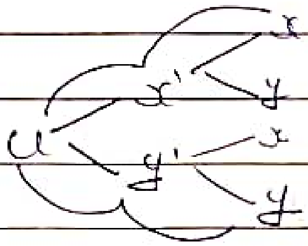
(4). $U_{xx} + U_{yy} = U_{x'x'} + U_{y'y'} = 0$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

α is angle of rotation

$x' = \cos\alpha(x) + \sin\alpha(y)$ —①

$y' = -\sin\alpha(x) + \cos\alpha(y)$ —②



$$U_x = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial y}$$

$$U_x = U_{x'} \cos\alpha - U_{y'} \sin\alpha$$

$$U_{xx} = U_{x'x'} \cos^2\alpha + \sin^2\alpha \, U_{y'y'}$$

$$U_y = \partial u/\partial x' \, \frac{\partial x'}{\partial y} + \partial u/\partial y' \cdot \partial y'/\partial y$$

$$= U_{x'} \sin\alpha + U_{y'} \cos\alpha$$

$$U_{yy} = U_{x'x'} \sin^2\alpha + U_{y'y'} \cos^2\alpha$$

$$U_{xx} + U_{yy} = U_{x'x'} + U_{y'y'}$$