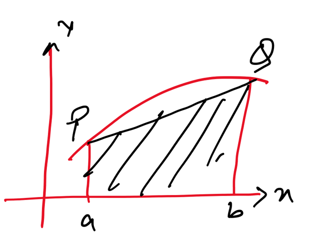
Lecture-3

Wednesday, 31 January 2024 10:38 AM

Trapezoidal Rule



$$I = \int_a^b f(x)\,dx$$

$$= \frac{b-a}{2}\left[f(a)+f(b)\right]$$

$$= (b-a)\left[\frac{f(a)+f(b)}{2}\right]$$

$$= \text{Width} \times \text{Average height}$$

Error in the Trapezoidal rule:

$$I = \int_{x_0}^{x_n} y_n(x)\,dx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \cdots\right]$$

Where $y_n(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \cdots$

$$= T_0 + T_1 + T_2 + \cdots$$

In trapezoidal rule, terms from T_2 onwards are neglected. So the truncation error in trapezoidal rule is

$$E_{trap} = \int_a^b T_2\,dx = \frac{f''(p)}{2}\int_0^1 p(p)\,h\,dp$$

$x = x_0 + ph$
$dx = h\,dp$

$$= -f''(p)\frac{h}{12}$$

So, $f''(p) = h^2 f''(x)$

$$\therefore\ E_{trap} = -\frac{h^3}{12}f''(\xi) \qquad \underline{a \le \xi \le b}$$

$y = f(x) = x^2+1$

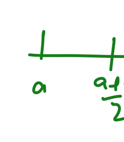
	x_0	x_1	x_2	x_3	x_4
x	1	2	3	4	5
$y = f(x)$	2	5	10	17	26

Composite Trapezoidal Rule:

$[a, b] \xrightarrow{\text{divided}}$ in n equal parts of length h.

ie $h = \frac{b-a}{n}$, Nodes $x_0, x_0+h, x_0+2h, \ldots, x_0+nh$

$$\therefore \int_a^b f(x)\,dx = \underbrace{\frac{h}{2}\left[f(x_0)+f(x_n)+2\{f(x_1)+f(x_2)\cdots+f(x_{n-1})\}\right]}_{\text{Composite formula}}$$



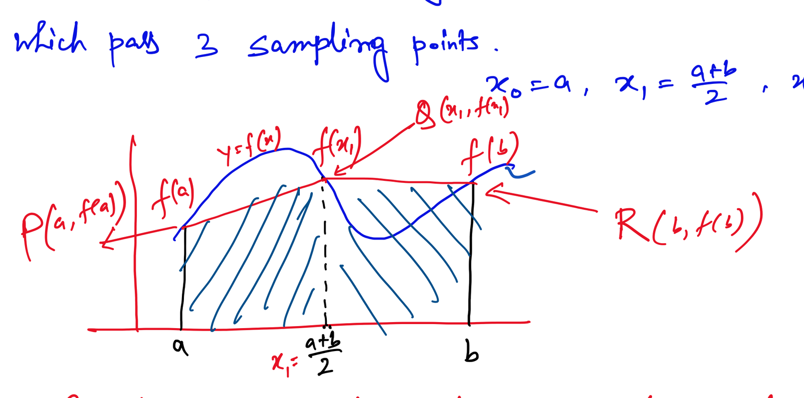
Drawback:

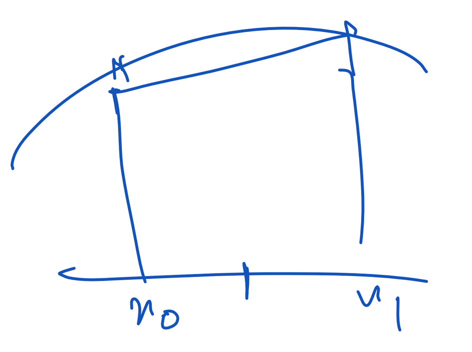
If the length of the interval $[a, b]$ is large, then $(b-a)$ is also large and so the error term will be large. Thus the method becomes meaningless

Simpson's 1/3 rule

$f(x)$ is approximated by a second order polynomial $P_2(x)$ which pass 3 sampling points.

$x_0 = a,\ x_1 = \frac{a+b}{2},\ x_2 = b$





So here 3 points and two equal subinterval

Now, the quadratic polynomial passing through $P(a, f(a))$, $Q(x_1, f(x_1))$ & $R(b, f(b))$ which approximate or $P(x_0, f(x_0))$
$f(x)$ is given by

$$f(x) = f(x_0) + \frac{1}{h}(x-x_0)\Delta f(x_0) + \frac{1}{2h^2}(x-x_0)(x-x_1)\Delta^2 f(x_0)$$

$x = x_0 + ph$

OR $f(x) = f(x_0) + p\Delta f(x_0) + \frac{p(p-1)}{2!}\Delta^2 f(x_0)$

$x_0 = a,\quad x_1 = \frac{a+b}{2},\quad x_2 = b$

Now by Newton-Cotes formula, we have

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_2} f(x)\,dx = \int_{x_0}^{x_2}\left[f(x_0) + \frac{1}{h}(x-x_0)\Delta f(x_0) + \frac{1}{2h^2}(x-x_0)(x-x_1)\Delta^2 f(x_0)\right]dx$$

For $n=2$, $h = \frac{b-a}{2}$

$$= (x_2-x_0)f(x_0) + \frac{1}{h}\left[\frac{1}{2}(x-x_0)^2\right]_{x_0}^{x_2}\Delta f(x_0) + I_1$$

$$= 2h f(x_0) + 2h\Delta f(x_0) + I_1$$

$$I_1 = \frac{1}{2h^2}\left[\frac{x^3}{3} - (x_0-x_1)\frac{x^2}{2} + x_0x_1x\right]_{x_0}^{x_2}\Delta^2 f(x_0)$$

$$= \frac{1}{12h^2}(x_2-x_0)\left[2(x_2^2+x_0x_2+x_0^2) - 3(x_0+x_1)(x_2+x_0) + 6x_0x_1\right]\Delta^2 f(x_0)$$

Substituting $x_2 = x_0+2h$, $x_1 = x_0+h$, we obtain

$$I_1 = \frac{1}{6h}(2h^2)\Delta^2 f(x_0) = \frac{h}{3}\Delta^2 f(x_0)$$

Now $$\int_a^b f(x)\,dx = \int_{x_0}^{x_2} f(x)\,dx$$

$$= 2hf(x_0) + 2h\Delta f(x_0) + \frac{h}{3}\Delta^2 f(x_0)$$

$$= \frac{h}{3}\left[6f(x_0) + 6\{f(x_1)-f(x_0)\} + \{f(x_0)-2f(x_1)+f(x_2)\}\right]$$

$$= \frac{h}{3}\left[f(x_0)+4f(x_1)+f(x_2)\right]$$

$$\boxed{\int_a^b f(x)\,dx = \frac{b-a}{6}\left[f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right]}$$

Simpson 1/3 rule

Error in Simpson 1/3 rule:

It can be verified that the simpson's 1/3 rule gives the exact value of the integration for polynomial ≤ 3.

i.e, $R(f, x) = 0$ for $f(x) = 1, x, x^2, x^3$

For $f(x) = 1$: $R(f,x) = \int_a^b 1\,dx - \frac{b-a}{6}[1+4+1]$

$$= (b-a)-(b-a) = 0$$

For $f(x) = x$: $R(f,x) = \int_a^b x\,dx - \frac{(b-a)}{6}\left[a+4\left(\frac{a+b}{2}\right)+b\right]$

$$= \frac{1}{2}(b^2-a^2) - \frac{1}{2}(b^2-a^2) = 0$$

For $f(x) = x^2$: $R(f,x) = \int_a^b x^2\,dx - \frac{(b-a)}{6}\left[a^2+4\left(\frac{a+b}{2}\right)^2+b^2\right]$

$$= \frac{1}{3}(b^3-a^3) - \frac{(b-a)}{3}\left[a^2+ab+b^2\right]$$

$$= \frac{1}{3}(b^3-a^3) - \frac{1}{3}(b^3-a^3)$$

$$= 0$$

For $f(x) = x^3$: $R(f,x) = \int_a^b x^3\,dx - \frac{(b-a)}{6}\left[a^3+4\left(\frac{a+b}{2}\right)^3+b^3\right]$

$$= \frac{1}{4}(b^4-a^4) - \frac{(b-a)}{6}\left[a^3+a^2b+ab^2+b^3\right]$$

$$= \frac{1}{4}(b^4-a^4) - \frac{1}{4}(b^4-a^4) = 0$$

For $f(x) = x^4$:

$$C = \int_a^b x^4\,dx - \frac{(b-a)}{6}\left[a^4+4\left(\frac{a+b}{2}\right)^4+b^4\right]$$

$$= \frac{1}{5}(b^5-a^5) - \frac{(b-a)}{24}\left[5a^4+4a^3b+6a^2b^2+4ab^3+5b^4\right]$$

$$= \frac{1}{120}\left[24(b^5-a^5) - 5(b-a)(5a^4+4a^3b+6a^2b^2+4ab^3+5b^4)\right]$$

$$= -\frac{(b-a)}{120}\left[b^4-4ab^3+6a^2b^2-4a^3b+a^4\right]$$

$$C = -\frac{(b-a)^5}{120}$$

Using the definition of error it is given by

$$R(f,x) = \frac{C}{4!}f^{(4)}(\xi)$$

$$= -\frac{(b-a)^5}{2880}f^{(4)}(\xi), \qquad a \le \xi \le b$$

$$= -\frac{h^5}{90}f^{(4)}(\xi). \qquad \text{Since } h = \frac{b-a}{2}$$

The bond of the error is given by

$$|R(f,x)| \le \frac{(b-a)^5}{2880}\max_{a\le x\le b}\left|f^{(4)}(x)\right|$$

Since, Simpson's 1/3 rule gives exact result for polynomials of degree ≤ 3, the method is of order 3

Substituting in (*)

i.e. it takes upto $\Delta^2 y_0$

& Error at $\Delta^3 y_0$

or for $n=2$ in *

$$h\left(\frac{n^4-4n^3+4n^2}{24}\right)\Delta^3 y_0$$

but at $n=2$ this term is give zero

So the error will be next term

$$h\left(\frac{6n^5-45n^4+110n^3-90n^2}{720}\right)\Delta^4 y_0,\ n=2$$

$$\Rightarrow -\frac{h}{90}\cdot\Delta^4 y_0$$

$$\frac{d^4 y}{dx^4} = \frac{\Delta^4 y_0}{h^4} \Rightarrow \Delta^4 y_0 = h^4\frac{d^4 f}{dx^4}$$

for $y = f$

$$\text{Error} = -\frac{h^5}{90}f^{iv}(x)$$

$$|\text{Error}| < \left|\frac{h^5}{90}\max_{x\in[a,b]} f^{iv}(x)\right|$$

$$\int_a^b f(x)\,dx = \frac{h}{3}$$

a	b	n	Closed Newton-Cotes Formula	h	Truncation Error
x_0	x_1	1	$h\cdot\frac{f(x_0)+f(x_1)}{2}$	$\frac{(b-a)}{1}$	$-\frac{1}{12}h^3 f''(\xi)$
x_0	x_2	2	$\frac{1}{3}\cdot h\cdot[f(x_0)+4f(x_1)+f(x_2)]$	$\frac{(b-a)}{2}$	$-\frac{1}{90}h^5 f^{(iv)}(\xi)$
x_0	x_3	3	$\frac{3}{8}\cdot h\cdot[f(x_0)+3f(x_1)+3f(x_2)+f(x_3)]$	$\frac{(b-a)}{3}$	$-\frac{3}{80}h^5 f^{(iv)}(\xi)$
x_0	x_4	4	$\frac{2}{45}\cdot h\cdot[7f(x_0)+32f(x_1)+12f(x_2)+32f(x_3)+7f(x_4)]$	$\frac{(b-a)}{4}$	$-\frac{8}{945}h^7 f^{(vi)}(\xi)$