MA 203 Complex Analysis and Differential Equations-II

Module-III Partial Differential Equations

Lecture-5 (10 October 2023)

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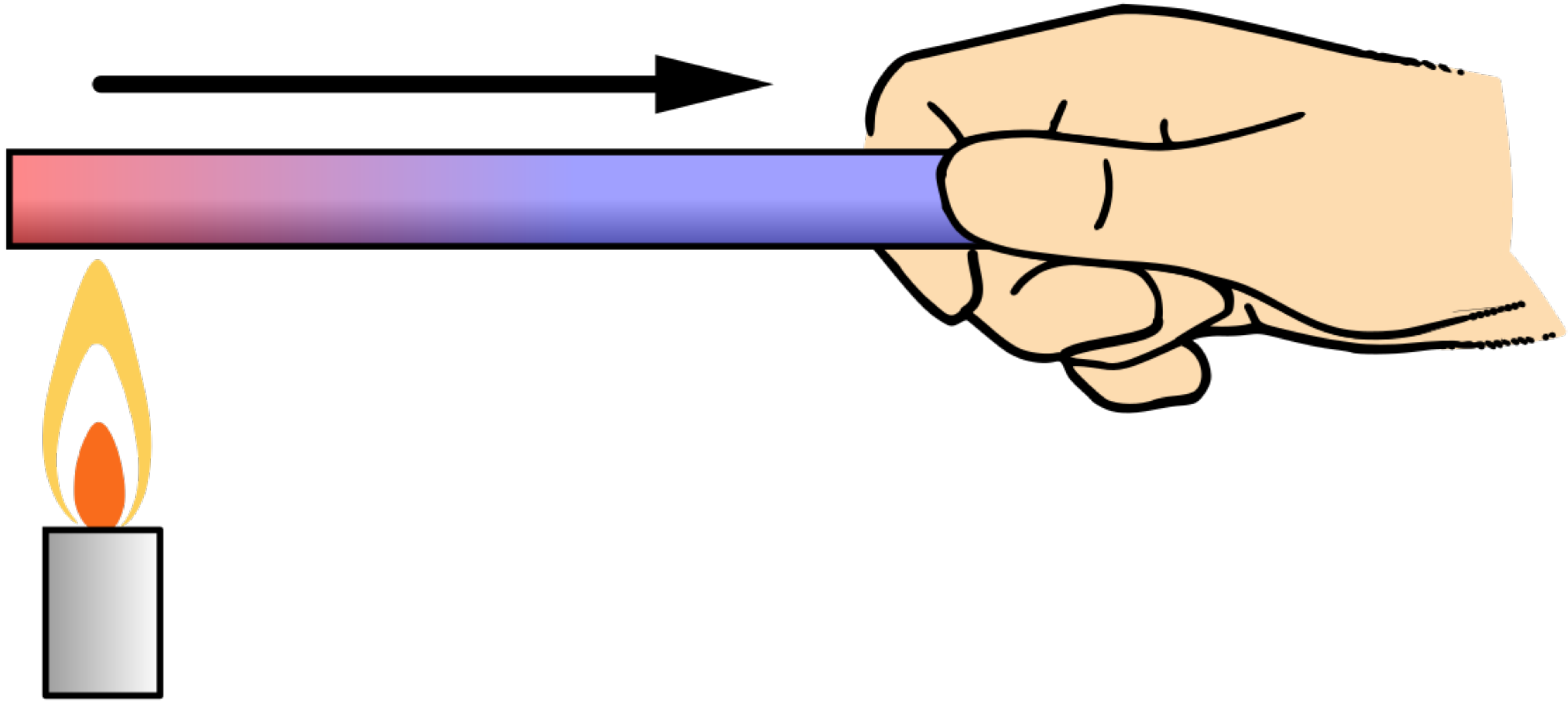
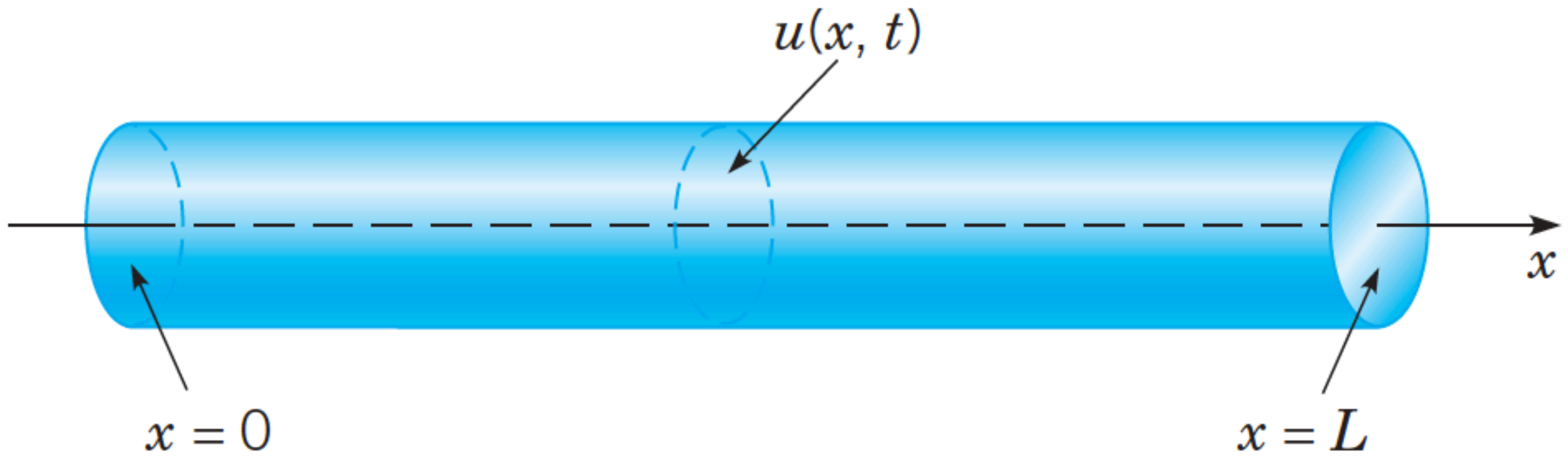
Heat equation



Heat equation



- Also known as the heat conduction equation or the diffusion equation
- Governs the variation of the temperature
- Assumptions:
  - sides of the bar are perfectly insulated
  - the bar is very thin
- Heat equation $$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \qquad t > 0, \quad 0 < x < L$$
- $\alpha^2 = \kappa/(\rho s)$: thermal diffusivity; units: (length)^2/time
- κ: thermal conductivity; ρ: density; s: specific heat

Heat equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

- Solution by separation of variables
- Assumption: the solution is $u(x,t) = X(x)\,T(t)$
- Substitute this assumption in the heat equation
- we get $X(x)\,T'(t) = \alpha^2 X''(x)\,T(t) \quad \Longrightarrow \quad \dfrac{X''(x)}{X(x)} = \dfrac{1}{\alpha^2}\dfrac{T'(t)}{T(t)} = k$
- we obtain two ordinary differential equations

$$\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} - kX = 0 \qquad \text{and} \qquad \frac{\mathrm{d}T}{\mathrm{d}t} = k\alpha^2 T$$

- k can be zero, positive or negative.

Solution of the heat equation

- Case 1: $k = 0$

  The solution is $u(x,t) = A_1 x + B_1$

- Case 2: $k > 0$

  The solution is $u(x,t) = \left(A_2 \mathrm{e}^{\lambda x} + B_2 \mathrm{e}^{-\lambda x}\right)\mathrm{e}^{\lambda^2 \alpha^2 t}$

- Case 3: $k < 0$

  The solution is $u(x,t) = (A_3 \cos \lambda x + B_3 \sin \lambda x)\mathrm{e}^{-\lambda^2 \alpha^2 t}$

- Constants $A_1, B_1, A_2, B_2, A_3, B_3$ are determined from the boundary conditions.

Example 1 (Homogeneous boundary conditions)

Find the solution of the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \qquad t > 0, \quad 0 < x < L$$

satisfying the following boundary conditions and initial condition

$$u(0,t) = 0, \quad u(L,t) = 0, \quad \text{for} \quad t > 0$$

$$u(x,0) = f(x) \quad \text{for} \quad 0 \leq x \leq L.$$

Example 2 (Nonhomogeneous boundary conditions)

Find the solution of the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \qquad t > 0, \quad 0 < x < L$$

satisfying the following boundary conditions and initial condition

$$u(0,t) = T_1, \quad u(L,t) = T_2, \quad \text{for} \quad t > 0$$

$$u(x,0) = f(x) \quad \text{for} \quad 0 \le x \le L.$$

Example 3 (Bar with insulated ends)

Find the solution of the heat equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \qquad t > 0, \quad 0 < x < L$$

satisfying the following boundary conditions and initial condition

$$u_x(0,t) = 0, \quad u_x(L,t) = 0, \quad \text{for} \quad t > 0$$

$$u(x,0) = f(x) \quad \text{for} \quad 0 \le x \le L.$$

Recap: Fourier Series of even and odd functions

Theorem 4: (i) The Fourier series of an *even* function $f(x)$ of period $2L$ is a **Fourier cosine series**

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right), \tag{11}$$

with the coefficients

$$a_0 = \frac{1}{L}\int_0^L f(x)\,\mathrm{d}x, \tag{12a}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)\mathrm{d}x, \qquad n = 1, 2, 3, \ldots \tag{12b}$$

(ii) The Fourier series of an *odd* function $f(x)$ of period $2L$ is a **Fourier sine series**

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right), \tag{13}$$

with the coefficients

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)\mathrm{d}x, \qquad n = 1, 2, 3, \ldots \tag{14}$$