MA 203 Complex Analysis and Differential Equations-II

Module-III Partial Differential Equations

Lecture-6 (11 October 2023)

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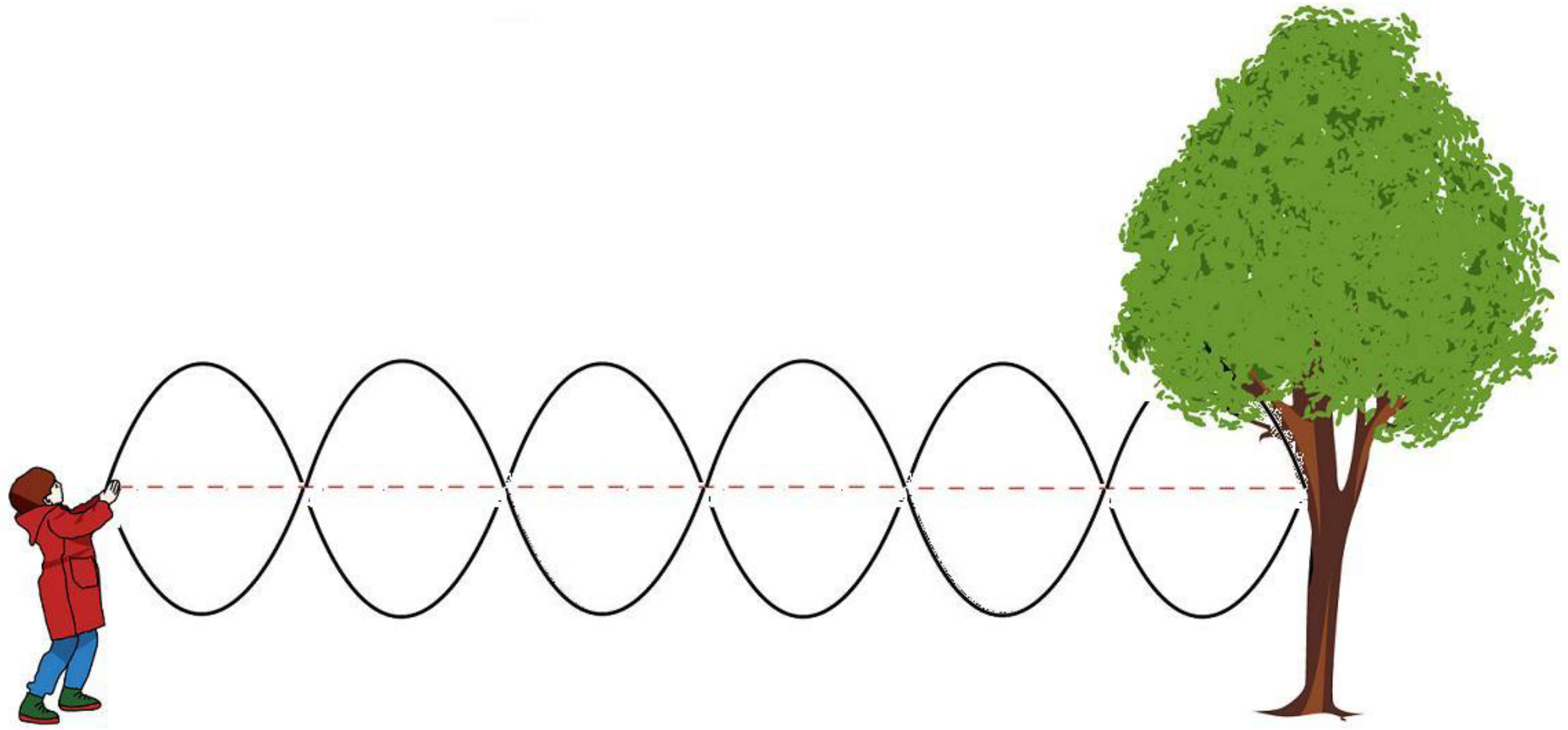
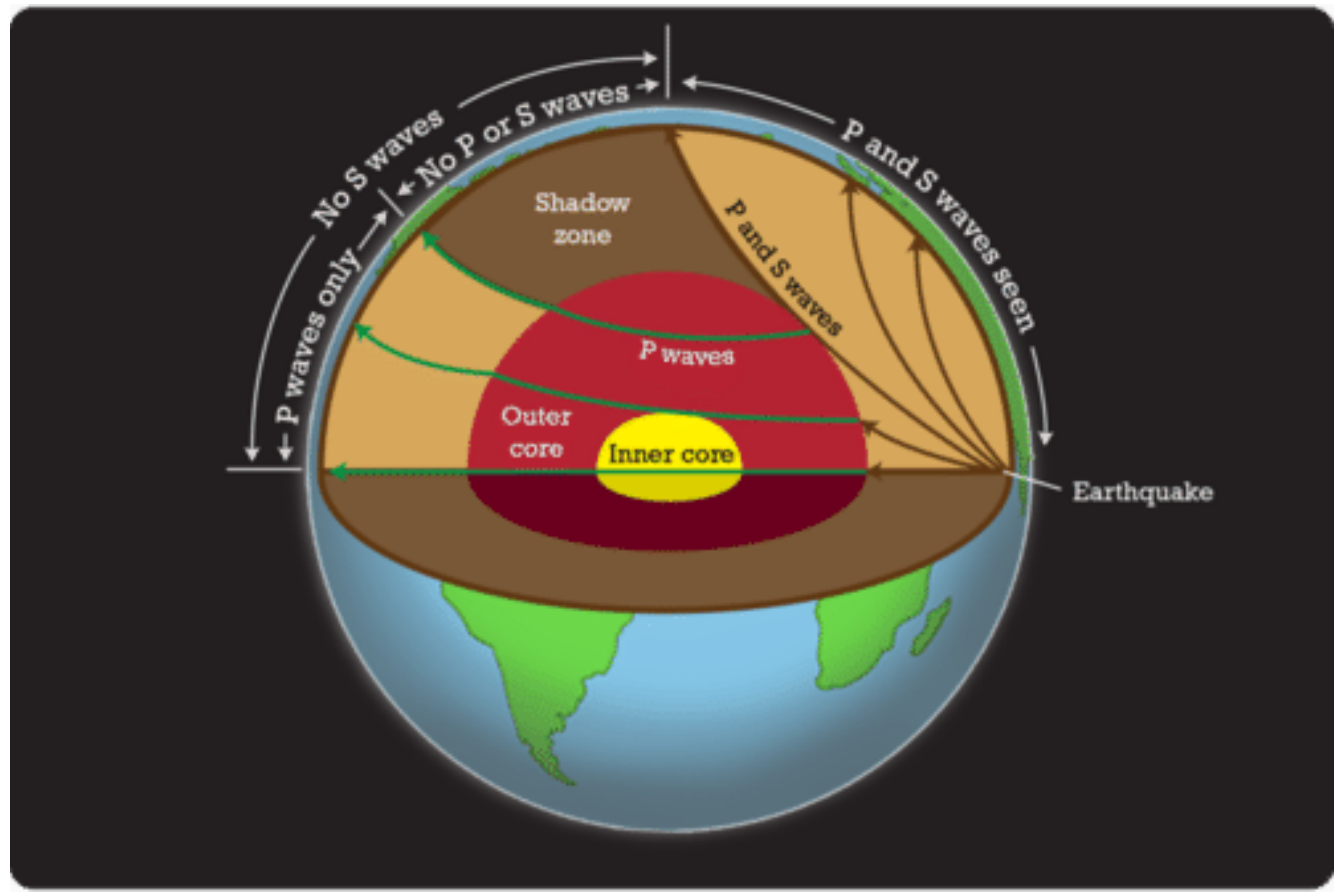
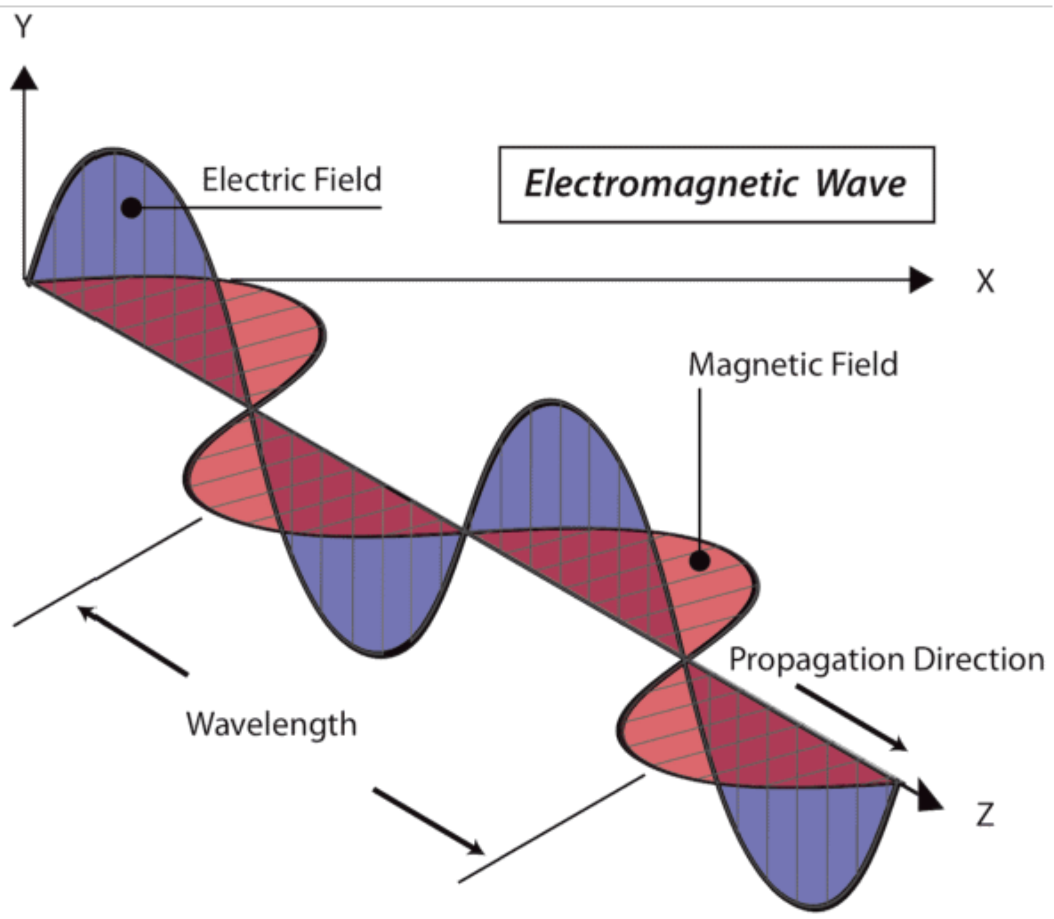
Wave equation

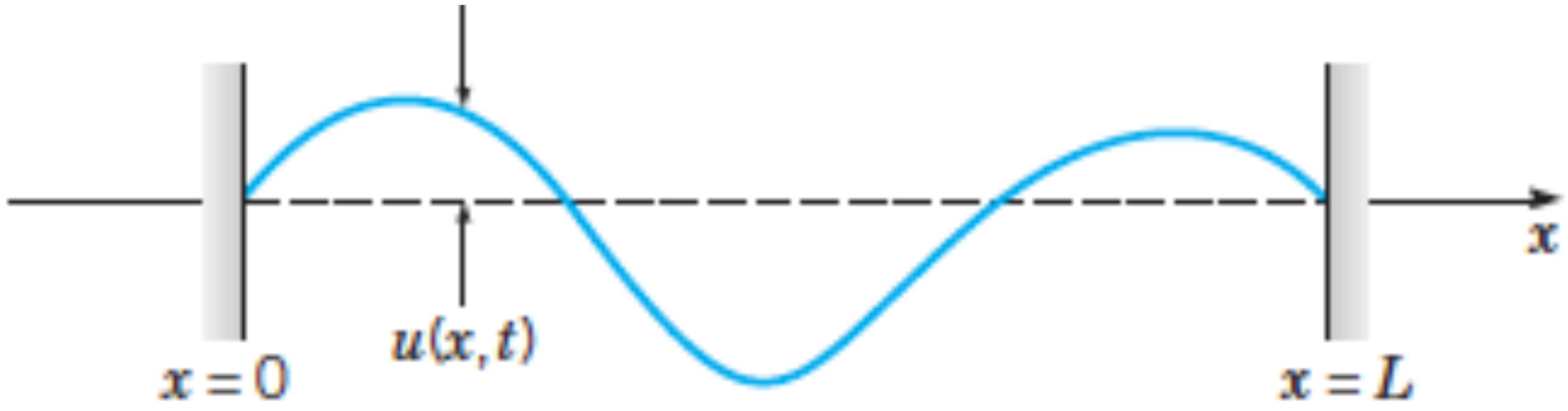
- arises almost in any mathematical analysis of phenomena involving the propagation of waves in a continuous medium







Wave equation



$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad t > 0, \quad 0 < x < L$$

- Governs the vertical displacement $u(x, t)$ in the string
- Assumptions:
  - damping effects, such as air resistance, are neglected
  - amplitude of the motion is not too large
- $c^2 = T/\rho$; T: tension in the string; ρ: mass per unit length of the string material
- c is velocity; units of c: length/time

Wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad t > 0, \quad 0 < x < L$$

- Solution by separation of variables
- Assumption: the solution is $u(x,t) = X(x)\,T(t)$
- Substitute this assumption in the heat equation
- we get $X(x)\,T''(t) = c^2 X''(x)\,T(t) \qquad \Longrightarrow \qquad \dfrac{X''(x)}{X(x)} = \dfrac{1}{c^2}\dfrac{T''(t)}{T(t)} = k$
- we obtain two ordinary differential equations

$$\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} - kX = 0 \qquad \text{and} \qquad \frac{\mathrm{d}^2 T}{\mathrm{d}t^2} - k\,c^2 T = 0$$

- k can be zero, positive or negative.

Solution of the heat equation

- Case 1: $k = 0$

$$\text{Solution: } u(x, t) = (c_1 x + c_2)(c_3 t + c_4)$$

- Case 2: $k > 0$

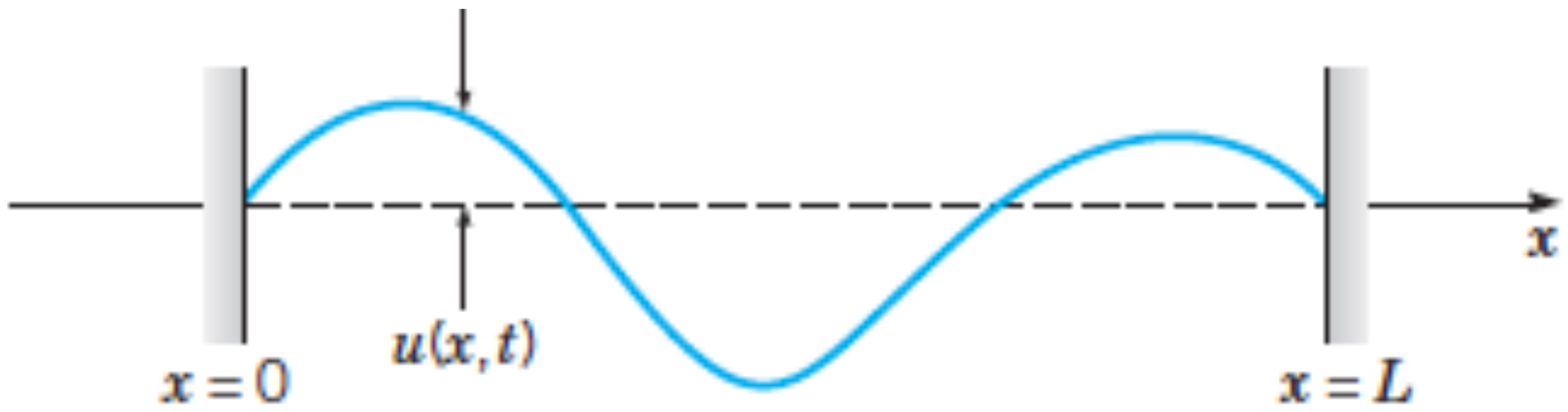
$$\text{Solution: } u(x, t) = \left(c_5 \mathrm{e}^{\lambda x} + c_6 \mathrm{e}^{-\lambda x}\right)\left(c_7 \mathrm{e}^{\lambda c t} + c_8 \mathrm{e}^{-\lambda c t}\right)$$

- Case 3: $k < 0$

$$\text{Solution: } u(x, t) = (c_9 \cos \lambda x + c_{10} \sin \lambda x)(c_{11} \cos \lambda c t + c_{12} \sin \lambda c t)$$

- Constants $c_1, c_2, \ldots, c_{12}$ are determined from the initial and boundary conditions.

Initial and boundary conditions



$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad t > 0, \quad 0 < x < L$$

- The ends are assumed to remain fixed. Boundary conditions

$$u(0,t) = 0, \quad u(L,t) = 0 \quad \text{for} \quad t \geq 0$$

- Initial conditions
  - initial position of the string

$$u(x,0) = f(x) \quad \text{for} \quad 0 \leq x \leq L$$

  - initial velocity of the string

$$u_t(x,0) = g(x) \quad \text{for} \quad 0 \leq x \leq L$$

- For consistency, $f(0) = g(0) = 0$ and $f(L) = g(L) = 0$

Example 1 (Elastic string with a nonzero initial displacement)

Find the solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad t > 0, \quad 0 < x < L$$

satisfying the following boundary and initial conditions

$$u(0,t) = 0, \quad u(L,t) = 0 \quad \text{for} \quad t \geq 0$$

$$u(x,0) = f(x), \quad u_t(x,0) = 0 \quad \text{for} \quad 0 \leq x \leq L.$$

Example 2 (Elastic string with nonzero initial velocity)

Find the solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad t > 0, \quad 0 < x < L$$

satisfying the following boundary and initial conditions

$$u(0,t) = 0, \quad u(L,t) = 0 \quad \text{for} \quad t \geq 0$$

$$u(x,0) = 0, \quad u_t(x,0) = g(x) \quad \text{for} \quad 0 \leq x \leq L.$$

Example 3 (Elastic string with general initial conditions)

Find the solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad t > 0, \quad 0 < x < L$$

satisfying the following boundary and initial conditions

$$u(0,t) = 0, \quad u(L,t) = 0 \quad \text{for} \quad t \geq 0$$

$$u(x,0) = f(x), \quad u_t(x,0) = g(x) \quad \text{for} \quad 0 \leq x \leq L.$$

Recap: Fourier Series of even and odd functions

Theorem 4: (i) The Fourier series of an *even* function $f(x)$ of period $2L$ is a **Fourier cosine series**

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right), \tag{11}$$

with the coefficients

$$a_0 = \frac{1}{L}\int_0^L f(x)\,\mathrm{d}x, \tag{12a}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)\mathrm{d}x, \qquad n = 1, 2, 3, \ldots \tag{12b}$$

(ii) The Fourier series of an *odd* function $f(x)$ of period $2L$ is a **Fourier sine series**

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right), \tag{13}$$

with the coefficients

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)\mathrm{d}x, \qquad n = 1, 2, 3, \ldots \tag{14}$$