Max Flow / Min Cut

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1 Max Flow

Maximum flow is a fundamental problem in network flow optimization, extensively studied in graph theory and network algorithms. It concerns determining the maximum amount of flow that can be sent from a source node to a sink node in a flow network. A flow network is a directed graph where each edge has a capacity, which represents the maximum amount of flow that can traverse that edge.

The problem can be formally stated as follows: Given a flow network, over a graph $G = (V, E)$, with a source node s and a sink node t, and capacities on each edge, find the maximum flow from s to t.

Capacity

- Represented as $C : E \to \mathbb{R}^+$ where E represents edges.
- Definition: Maximum amount of flow that can be sent through an edge in a flow network.

Flow

Let $G = (V, E)$ be a directed graph, where V is the set of vertices and E is the set of directed edges.

In a graph G, the flow f is defined as $f : V \times V \to \mathbb{R}^+$ such that it follows:

- **Capacity constraint**: $f(u, v) \leq C(u, v)$ for all $(u, v) \in E$
- **Flow Constraint**: For all vertices v except the source s and sink t, the total incoming flow must equal the total outgoing flow:

$$\sum_{u \in V} f(u, v) = \sum_{w \in V} f(v, w)$$

Net Flow

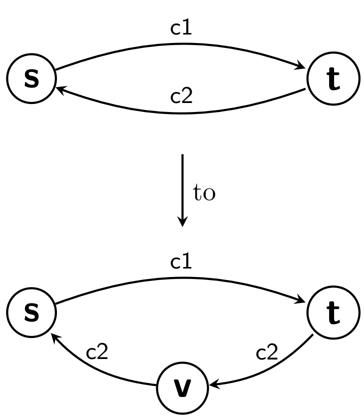
Let $G = (V, E)$ be a flow network with a capacity function c. Let s be the source of the network and t be the sink.

A flow in G is a real-valued function $f : V \times V \to \mathbb{R}$ that satisfies the capacity constraint and flow conservation property. The net flow of the network $|f|$ is defined as:

$$|f| = \sum_{v \in V} f(s, v) - \sum_{v \in V} f(v, s)$$

1.1 Assumption in Maximum Flow Algorithm

- There are no antiparallel edges (edges going in opposite directions with the same vertices): $(u, v) \in E$ implies $(v, u) \notin E$. To model a flow problem with antiparallel edges, the network must be transformed into an equivalent (equivalent in terms of max-flow of the concerned network) one containing no antiparallel edges. To transform the network, choose one of the two antiparallel edges, in this case, (t, s), and split it by adding a new vertex V and replacing edge (t, s) with the pair of edges (t, v) and (v, s). Also, set the capacity of both new edges to the capacity of the original edge (c2). The resulting network satisfies the property that if an edge belongs to the network, the reverse edge does not.



- Self-loops are not present: $(u, u) \notin E$ for all vertices u.
  They aren't considered for the algorithmic purpose as they don't affect the max flow because any non-zero flow through that edge returns to the same vertex eventually.

- For all $u \in V$, u will be in the path from source to sink in the flow network. The conservation law will be violated as none of the vertices except for the source and sink can hold commodities. The incoming flow has to be equal to the outgoing flow. Hence all such vertices which cannot be reached from the source or from which the sink cannot be reached are omitted for algorithmic purposes.

- $|E| \geq |V| - 1$, follows from the previous point,for the graph to be connected.

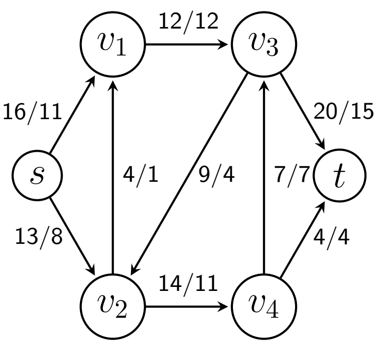


fig 1.

1.2 Objective of the Problem

Let G =(V,E) be a flow network with a capacity function c. Let s be the source of the network and t be the sink. A flow in G is a real-valued function

f : V $\times V \rightarrow R$

that satisfies the Capacity constraint and Flow conservation property.

Given the network flow of graph G, the objective is to find the flow f such that $|f|$ is maximum. Here, $|f|$ is defined as:

$$|f| \; = \sum_{v \in V} f(s, v) - \sum_{v \in V} f(v, s)$$

where:

- s is the source vertex of the graph.
- V is the set of all vertices in the flow network.

Mathematically, this means we need to maximize the commodities which can be transferred from the source to the sink.

1.3 Residual Network

Given a network flow graph $G = (V, E)$ and flow f,the residual network graph G_f consists of of edges whose capacities represent how the flow can change on edges of G. Intuitively, an edge of the flow network can admit an amount of additional flow equal to the edge's capacity minus the flow on that edge.If that value is positive, that edge goes into G_f with a "residual capacity" of $c_f(u,v)$=$c(u,v)$-$f(u,v)$ where:

- $f(u,v)$ represents the flow from node u to node v.
- $c(u,v)$ is the capacity constraint of the edge u,v.

Those edges u,v of G whose flow equal their capacity have $c_f(u,v)$=0 and they do not belong to G_f. In order to represent a possible decrease in the positive flow $f(u,v)$ on an edge in G the residual network G_f contains an edge $(v,u$ with a residual capacity of $c_f(u,v) = f(u,v)$, that is, an edge that can admit flow in the opposite direction to (u,v), at most cancelling out the flow on (u,v). These reverse edges in the residual network allow an algorithm to send back flow it has already sent along an edge. Sending flow back along an edge is equivalent to decreasing the flow on the edge.

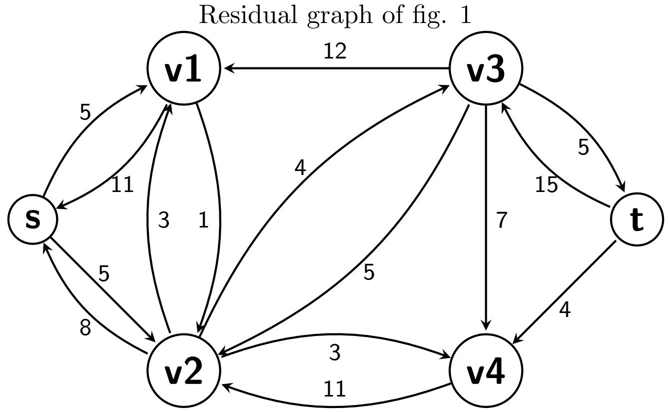
More formally for a flow network $G = (V,E)$ with source s,sink t,and a flow f,consider a pair of vertices $(u,v) \in V$. We define the residual Capacity $c_f(u,v)$ defined by:

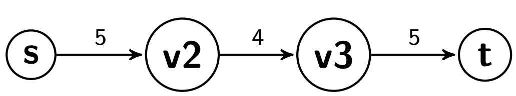
The function $c_f(u,v)$ is defined as follows:

$$c_f(u,v) = \begin{cases} c(u,v) - f(u,v) & \text{if } (u,v) \in E, \\ f(v,u) & \text{if } (v,u) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, given a flow network $G = (V,E)$ and a flow f the residual network induced by f is $G_f = (V, E_f)$, where
E_f={(u,v) $\in V \times V : c_f(u,v) \geq 0$}

Residual graph of fig. 1





1.4 Augmenting Flows in Residual Networks

In the residual network, flow guides how to add more flow to the original network. If f represents the flow in network G, and f' is the flow in the corresponding residual network G_f, then we define $f \uparrow f'$, the augmentation of flow f by f', to be a function from E to $\mathbb{R}$, defined by

$$f(u,v) = \begin{cases} f(u,v) + f'(u,v) - f'(v,u) & \text{if } (u,v) \in E \\ 0 & \text{otherwise} \end{cases}$$

The intuition behind this definition follows the definition of the residual network. The flow on (u,v) increases by $f'(u,v)$, but decreases by $f'(v,u)$ because pushing flow on the reverse edge in the residual network signifies decreasing the flow in the original network.