Extra

Tuesday, 12 September 2023 2:38 AM

Exact Equations

$y' = f(x,y)$

$\Rightarrow \frac{d}{dx}\left(g(x,y)\right) = 0 \hookrightarrow$ Exact

We can extend this concept for 2^{nd} order & higher order.

The concept of exactness which is usually discussed only for first order differential equation can be generalized for higher order differential equations as well.

Def^n: The n^{th} order D.E. $F(x,y,y',\cdots y^n) = 0$ is said to be exact if the function $F(x,y,y',\cdots,y^n)$ is an exact derivative of some differential function of $(n-1)^{th}$ order say $G(x,y,\cdots y^{n-1})$ i.e.,

$$F(x,y,\cdots,y^n) = \frac{d}{dx}\left(G(x,y,\cdots y^{n-1})\right).$$

Here we will discuss 2^{nd} order linear D.E.

Definition 2:

The second-order homogeneous linear differential equation Operator form $L[u]=0$ which is define as

$$L[u] = p_0(x)u''(x) + p_1(x)u'(x) + p_2(x)u(x) = 0 \quad \text{--- (1)}$$

is said to be exact differential equation of 2^{nd} order if and only if, for some $\underbrace{A(x),\ B(x)}_{\text{Differentiable function}} \in C'$

$$L[u] = p_0(x)u'' + p_1(x)u' + p_2(x)u$$
$$= \frac{d}{dx}\left[A(x)u' + B(x)u\right] \quad \text{--- (2)}$$

for all function $u \in C^2$

We say that $L[u]=0$ is an exact D.E. in (1) if and only if $A(x) = p_0$, $p_1 = A' + B$ or $B(x) = p_1 - p_0'$ and $p_2 = B'$.

How one can get this?

From (2) $\frac{d}{dx}[Au' + Bu]$
$= A'u' + Au'' + B'u + Bu'$
$= Au'' + u'(A'+B) + uB'$

$\therefore$ Comparing
$$\left\{\begin{array}{l} p_0 = A(x) \\ p_1 = A' + B \\ p_2 = B' \end{array}\right.$$

Hence (1) is exact if and only if

$$p_2 = B' = (p_1 - A')' = p_1' - (p_0')'$$

$$\Rightarrow p_0'' - p_1' + p_2 = 0$$

Therefore $L(u) = p_0u'' + p_1u' + p_2u = 0$ is exact if $p_0'' - p_1' + p_2 = 0$

Theorem:

The differential equation (1) is exact iff its coefficient functions satisfy

$$p_0'' - p_1' + p_2 = 0$$

and $$p_0(x)u'' + p_1(x)u' + p_2(x)u = \frac{d}{dx}\left[p_0u' + (p_1 - p_0')u\right]$$

Integrating factor:

An integrating factor for the differential equation $L[u]=0$ is a function $v(x)$ such that $vL(u)=0$ is exact.

If an integrating factor v for (1) can be found, then clearly

$$v(x)\left[p_0(x)u'' + p_1(x)u' + p_2(x)u\right] = \frac{d}{dx}\left[A(x)u' + B(x)u\right]$$

Therefore, the solution of the homogen D.E. (1) are given as sol^{n}s of the following linear D.E.

i.e $L(u)=0$

$$\Rightarrow A(x)u' + B(x)u = C \quad \text{--- (3)}$$

$\nwarrow$ arbitary Constant

Also, sol^{n}s of the inhomogeneous differential equation $L[u] = r(x)$ are given by solutions of the following equation

$$A(x)u' + B(x)u = \int v(x)\,r(x)\,dx + C \quad \text{--- (4)}$$

Example 5

Consider the following differential equation

$$x^2y'' + xy' - y = 3x^2, \quad x > 0.$$

We can easily check that it is an exact differential equation as the exactness condition

$$p_0'' - p_1' + p_2 = (x^2)'' - (x)' + (-1) = 2 - 1 - 1 = 0 \text{ follows}$$

Using
$$x^2y'' + xy' - 3$$
$$= \left[x^2y' + (x - 2x)y\right]' = 3x^2$$

We can rewrite $x^2y'' + xy' - y = 3x^2$ as

$$x^2y' - xy = x^3 + C$$

Which is a first order Cauchy-Euler's equation and can be solved to have its general solution as

$$y(x) = x^2 + \frac{C_1}{x} + C_2x$$

Cor A function $v \in C^2$ is an integrating factor for the D.E. (1) if and only if it is a solution of the 2^{nd} O.D.E.

$$M[v] = [p_0(x)v]'' - [p_1(x)v]' + p_2(x)v = 0 \quad \text{--- (5)}$$

After Signification

$$[p_0v' + p_0'v]' - p_1v' - p_1'v + p_2v$$
$$\Rightarrow p_0v'' + p_0'v' + p_0'v' + p_0''v - p_1v' - p_1'v + p_2v = 0$$
$$\Rightarrow p_0v'' + (2p_0' - p_1)v' + (p_0'' - p_1' + p_2)v = 0$$

Adjoint of the Linear operator

Definition: The operator M in (5) is called the adjoint of the linear operator L.

The differential equation (5), expanded to the differential equation

$$M[v] = p_0v'' + (2p_0' - p_1)v' + (p_0'' - p_1' + p_2)v = 0 \quad \text{--- (6)}$$

is called the adjoint of the differential equation

$$L[u] = p_0(x)u''(x) + p_1(x)u'(x) + p_2(x)u(x) = 0$$

Lagrange identity:

from eq (1) & eq. (5) we have

$$vL[u] - uM[v] = (\overset{w}{vp_0})u'' - u(\overset{w}{p_0v})'' + (vp_1)u' + u(p_1v)'$$

Since, $wu'' - uw'' = (wu' - uw')'$

$wu'' + \cancel{w'u'} - uw'' - \cancel{u'w'}$
$= wu'' - uw''$

and $(uw)' = uw' + wu'$,

this can be simplified to give the Lagrange identity

$$\boxed{vL[u] - uM[v] = \frac{d}{dx}\left[p_0(u'v - uv') - (p_0' - p_1)uv\right]} \quad \text{--- (7)}$$

Since $M(v) = (vp_0)'' - (vp_1)' + (vp_2) = 0$

$L(u) = p_0u'' + p_1u' + p_2u = 0$

$vL(u) - uM(v) = vp_0''u + p_1u'v + \cancel{p_2vu}$
$- u(vp_0)'' - u(vp_1)'$
$- \cancel{vup_2}$
$= vp_0''u - u(p_0v)'' + p_1u'v$
$- u(vp_1)'$

$= \left[vp_0u' - u(vp_0)'\right]' + (vp_1(x)u)'$

$= \frac{d}{dx}\left[vp_0u' - u(vp_0)' + vp_1u'\right]$

Def (Self-Adjoint)

Homogeneous linear D.E. that coincide with their adjoint are said to be self-adjoint equations.

$Lu=0$ $\quad$ $L=M$.
$Mv=0$

Remark 1

Necessary and sufficient conditions for eq (1) to be self-adjoint is that $2p_0' - p_1 = p_1$

i.e., $p_0' = p_1$.

Moreover, in case of self-adjoint, the last term in (7) vanishes.

$p_0u'' + p_1u' + p_2u = 0$
$p_0u'' + p_0'u' + p_2u = 0$
$\Rightarrow \frac{d}{dx}(p_0u') + p_2u = 0$

Theorem:

The second order linear D.E. (1) is self-adjoint if and only if it has the form

$$\frac{d}{dx}\left[p_0(x)\frac{du}{dx}\right] + p_2(x)u = 0 \quad \text{--- (8)}$$

The D.E. can be made self-adjoint by multiplying through by

$$h(x) = \left[\exp\int\left(\frac{p_1}{p_0}\right)dx\right] \times \frac{1}{p_0} \quad \text{--- (9)}$$

Ex Reduce the following differential equation to Self adjoint form:

$$(1-x^2)u'' - xu' + \lambda u = 0$$