MA 203 Complex Analysis and Differential Equations-II

Dr. Debopriya Mukherjee

Lecture-1

September 13, 2023

Contents

- Fourier Series
- Classification of linear second order PDE's in two variables.
- Laplace, wave, and heat equations using separation of variables.
- D' Alembert solution to the wave equations.
- Vibration of a circular membrane.
- Heat equation in the half space.

Questions?

- What is Fourier Series? Series of real numbers or series of functions?

Questions?

- What is Fourier Series? Series of real numbers or series of functions?
- For what kind of functions can we define Fourier Series?

Questions?

- What is Fourier Series? Series of real numbers or series of functions?
- For what kind of functions can we define Fourier Series?
- Is this series convergent? If yes, does it converge to the original function?

Questions?

- What is Fourier Series? Series of real numbers or series of functions?
- For what kind of functions can we define Fourier Series?
- Is this series convergent? If yes, does it converge to the original function?
- Applications of Fourier Series!

History

- Jean-Baptiste Joseph Fourier, a French mathematician and physicist initiated the Fourier series, which eventually developed into Fourier analysis and harmonic analysis, and their applications to problems of heat transfer and vibrations.

History

- Jean-Baptiste Joseph Fourier, a French mathematician and physicist initiated the Fourier series, which eventually developed into Fourier analysis and harmonic analysis, and their applications to problems of heat transfer and vibrations.
- A Fourier series is an expansion of a periodic function into a sum of trigonometric functions.

History

- Jean-Baptiste Joseph Fourier, a French mathematician and physicist initiated the Fourier series, which eventually developed into Fourier analysis and harmonic analysis, and their applications to problems of heat transfer and vibrations.
- A Fourier series is an expansion of a periodic function into a sum of trigonometric functions.
- By expressing a function as a sum of sines and cosines, many problems involving the function are easier to analyze because trigonometric functions are well understood.

History

- Jean-Baptiste Joseph Fourier, a French mathematician and physicist initiated the Fourier series, which eventually developed into Fourier analysis and harmonic analysis, and their applications to problems of heat transfer and vibrations.
- A Fourier series is an expansion of a periodic function into a sum of trigonometric functions.
- By expressing a function as a sum of sines and cosines, many problems involving the function are easier to analyze because trigonometric functions are well understood.
- This has huge applications in solving ODEs and PDEs as the derivatives of trigonometric functions fall into simple patterns.

History

- Jean-Baptiste Joseph Fourier, a French mathematician and physicist initiated the Fourier series, which eventually developed into Fourier analysis and harmonic analysis, and their applications to problems of heat transfer and vibrations.
- A Fourier series is an expansion of a periodic function into a sum of trigonometric functions.
- By expressing a function as a sum of sines and cosines, many problems involving the function are easier to analyze because trigonometric functions are well understood.
- This has huge applications in solving ODEs and PDEs as the derivatives of trigonometric functions fall into simple patterns.

Periodic functions

A function $f : Dom(f) \subset \mathbb{R} \to \mathbb{R}$ is said to be a *periodic function* if there is some positive number p, such that for all $x \in Dom(f)$ we have

$$x + p \in Dom(f) \quad \text{and} \quad f(x+p) = f(x).$$

Periodic functions

A function $f : Dom(f) \subset \mathbb{R} \to \mathbb{R}$ is said to be a *periodic function* if there is some positive number p, such that for all $x \in Dom(f)$ we have

$$x + p \in Dom(f) \quad \text{and} \quad f(x+p) = f(x).$$

Such smallest positive number p is called *period* of the function f.

Periodic functions

A function $f : Dom(f) \subset \mathbb{R} \to \mathbb{R}$ is said to be a *periodic function* if there is some positive number p, such that for all $x \in Dom(f)$ we have

$$x + p \in Dom(f) \quad \text{and} \quad f(x+p) = f(x).$$

Such smallest positive number p is called *period* of the function f.

- The functions $\sin x$, $\cos x$ and $\tan x$ are periodic functions.

Periodic functions

A function $f : Dom(f) \subset \mathbb{R} \to \mathbb{R}$ is said to be a *periodic function* if there is some positive number p, such that for all $x \in Dom(f)$ we have

$$x + p \in Dom(f) \quad \text{and} \quad f(x + p) = f(x).$$

Such smallest positive number p is called *period* of the function f.

- The functions $\sin x$, $\cos x$ and $\tan x$ are periodic functions.
- On the other hand, the functions x, x^2, x^3, e^x and $\ln x$ are NOT periodic.

Periodic functions

A function $f : Dom(f) \subset \mathbb{R} \to \mathbb{R}$ is said to be a *periodic function* if there is some positive number p, such that for all $x \in Dom(f)$ we have

$$x + p \in Dom(f) \quad \text{and} \quad f(x+p) = f(x).$$

Such smallest positive number p is called *period* of the function f.

- The functions $\sin x$, $\cos x$ and $\tan x$ are periodic functions.
- On the other hand, the functions x, x^2, x^3, e^x and $\ln x$ are NOT periodic.
- If the functions

  $$f : D \subset \mathbb{R} \to \mathbb{R} \quad \text{and} \quad g : D \subset \mathbb{R} \to \mathbb{R}$$

  have period p, then $a\,f + b\,g$, for any constants a and b, also has the period p. [The statement is also true for more than two functions.]

Two distinct functions $f, g : [a, b] \to \mathbb{R}$ are said to be *orthogonal* on this interval if

$$\int_a^b f(x)\, g(x)\, \mathrm{d}x = 0.$$

Theorem 1 (Orthogonality of the trigonometric system)

The trigonometric system

$$\{1, \sin x, \cos x, \sin 2x, \cos 2x, \sin 3x, \cos 3x, \dots, \sin nx, \cos nx, \dots\}$$

is orthogonal on the interval $[-\pi, \pi]$ (hence also on $[0, 2\pi]$ or any other interval of length 2π because of periodicity). In other words, for any integers m and n

$$\int_{-\pi}^{\pi} \cos mx \, \cos nx \, \mathrm{d}x = 0 \qquad (m \neq n) \tag{1a}$$

$$\int_{-\pi}^{\pi} \sin mx \, \sin nx \, \mathrm{d}x = 0 \qquad (m \neq n) \tag{1b}$$

$$\int_{-\pi}^{\pi} \sin mx \, \cos nx \, \mathrm{d}x = 0 \qquad (m \neq n \;\; or \;\; m = n). \tag{1c}$$

Proof.

(1a) For $m \neq n$,

$$\int_{-\pi}^{\pi} \cos mx \, \cos nx \, \mathrm{d}x = \int_{-\pi}^{\pi} \frac{\cos[(m+n)x] + \cos[(m-n)x]}{2} \, \mathrm{d}x$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} \cos[(m+n)x] \, \mathrm{d}x + \frac{1}{2} \int_{-\pi}^{\pi} \cos[(m-n)x] \, \mathrm{d}x$$

$$= \frac{1}{2} \left. \frac{\sin[(m+n)x]}{m+n} \right|_{-\pi}^{\pi} + \frac{1}{2} \left. \frac{\sin[(m-n)x]}{m-n} \right|_{-\pi}^{\pi} = 0.$$

Proof.

(1a) For $m \neq n$,

$$\int_{-\pi}^{\pi} \cos mx \, \cos nx \, \mathrm{d}x = \int_{-\pi}^{\pi} \frac{\cos[(m+n)x] + \cos[(m-n)x]}{2} \, \mathrm{d}x$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} \cos[(m+n)x] \, \mathrm{d}x + \frac{1}{2} \int_{-\pi}^{\pi} \cos[(m-n)x] \, \mathrm{d}x$$

$$= \frac{1}{2} \left. \frac{\sin[(m+n)x]}{m+n} \right|_{-\pi}^{\pi} + \frac{1}{2} \left. \frac{\sin[(m-n)x]}{m-n} \right|_{-\pi}^{\pi} = 0.$$

(1b) For $m \neq n$,

$$\int_{-\pi}^{\pi} \sin mx \, \sin nx \, \mathrm{d}x = \int_{-\pi}^{\pi} \frac{\cos[(m-n)x] - \cos[(m+n)x]}{2} \, \mathrm{d}x$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} \cos[(m-n)x] \, \mathrm{d}x - \frac{1}{2} \int_{-\pi}^{\pi} \cos[(m+n)x] \, \mathrm{d}x$$

$$= \frac{1}{2} \left. \frac{\sin[(m-n)x]}{m-n} \right|_{-\pi}^{\pi} - \frac{1}{2} \left. \frac{\sin[(m+n)x]}{m+n} \right|_{-\pi}^{\pi} = 0.$$

$\square$

Proof.

1. Case I: $m \neq n$,

$$\int_{-\pi}^{\pi} \sin mx \, \cos nx \, \mathrm{d}x = \int_{-\pi}^{\pi} \frac{\sin[(m+n)x] + \sin[(m-n)x]}{2} \, \mathrm{d}x$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} \sin[(m+n)x] \, \mathrm{d}x + \frac{1}{2} \int_{-\pi}^{\pi} \sin[(m-n)x] \, \mathrm{d}x$$

$$= \frac{1}{2} \left(-\frac{\cos[(m+n)x]}{m+n} \right)_{-\pi}^{\pi} + \frac{1}{2} \left(-\frac{\cos[(m-n)x]}{m-n} \right)_{-\pi}^{\pi}$$

$$= -\frac{\cos[(m+n)\pi] - \cos[-(m+n)\pi]}{2(m+n)}$$

$$-\frac{\cos[(m-n)\pi] - \cos[-(m-n)\pi]}{2(m-n)}$$

$$= 0 \qquad \left[\because \; \cos(-\theta) = \cos\theta \right].$$

Case II: $m = n$,

$$\int_{-\pi}^{\pi} \sin mx \, \cos nx \, \mathrm{d}x = \int_{-\pi}^{\pi} \sin nx \, \cos nx \, \mathrm{d}x = \frac{1}{2} \int_{-\pi}^{\pi} \sin(2nx) \, \mathrm{d}x$$

$$= \frac{1}{2} \left(-\frac{\cos(2nx)}{2n} \right)_{-\pi}^{\pi} = -\frac{\cos(2n\pi) - \cos(-2n\pi)}{4n} = 0$$

Proof.

1. Case I: $m \neq n$,

$$\int_{-\pi}^{\pi} \sin mx \, \cos nx \, \mathrm{d}x = \int_{-\pi}^{\pi} \frac{\sin[(m+n)x] + \sin[(m-n)x]}{2} \, \mathrm{d}x$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \sin[(m+n)x] \, \mathrm{d}x + \frac{1}{2}\int_{-\pi}^{\pi} \sin[(m-n)x] \, \mathrm{d}x$$

$$= \frac{1}{2}\left(-\frac{\cos[(m+n)x]}{m+n}\right)_{-\pi}^{\pi} + \frac{1}{2}\left(-\frac{\cos[(m-n)x]}{m-n}\right)_{-\pi}^{\pi}$$

$$= -\frac{\cos[(m+n)\pi] - \cos[-(m+n)\pi]}{2(m+n)}$$

$$-\frac{\cos[(m-n)\pi] - \cos[-(m-n)\pi]}{2(m-n)}$$

$$= 0 \qquad \left[\because \ \cos(-\theta) = \cos\theta\right].$$

Proof.

1. Case I: $m \neq n$,

$$\int_{-\pi}^{\pi} \sin mx \, \cos nx \, \mathrm{d}x = \int_{-\pi}^{\pi} \frac{\sin\left[(m+n)x\right] + \sin\left[(m-n)x\right]}{2} \, \mathrm{d}x$$
$$= \frac{1}{2}\int_{-\pi}^{\pi} \sin\left[(m+n)x\right] \mathrm{d}x + \frac{1}{2}\int_{-\pi}^{\pi} \sin\left[(m-n)x\right] \mathrm{d}x$$
$$= \frac{1}{2}\left(-\frac{\cos\left[(m+n)x\right]}{m+n}\right)_{-\pi}^{\pi} + \frac{1}{2}\left(-\frac{\cos\left[(m-n)x\right]}{m-n}\right)_{-\pi}^{\pi}$$
$$= -\frac{\cos\left[(m+n)\pi\right] - \cos\left[-(m+n)\pi\right]}{2(m+n)}$$
$$- \frac{\cos\left[(m-n)\pi\right] - \cos\left[-(m-n)\pi\right]}{2(m-n)}$$
$$= 0 \qquad \left[\because \ \cos(-\theta) = \cos\theta\right].$$

Case II: $m = n$,

$$\int_{-\pi}^{\pi} \sin mx \, \cos nx \, \mathrm{d}x = \int_{-\pi}^{\pi} \sin nx \, \cos nx \, \mathrm{d}x = \frac{1}{2}\int_{-\pi}^{\pi} \sin(2nx) \, \mathrm{d}x$$
$$= \frac{1}{2}\left(-\frac{\cos(2nx)}{2n}\right)_{-\pi}^{\pi} = -\frac{\cos(2n\pi) - \cos(-2n\pi)}{4n} = 0$$

Let $f : Dom(f) \subset \mathbb{R} \to \mathbb{R}$ be a periodic function with period 2π. The *Fourier series* representation of f is given by

$$\mathcal{S}_f(x) := \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx). \tag{2}$$

The coefficients $a_0, a_1, a_2, a_3, \ldots, b_1, b_2, b_3, \ldots$ are referred to as the *Fourier coefficients* of f and are given by

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\cos nx\,\mathrm{d}x, \qquad n = 0, 1, 2, 3, \ldots \tag{3a}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\sin nx\,\mathrm{d}x, \qquad n = 1, 2, 3, \ldots \tag{3b}$$

Remark 1

A Fourier series for $f(x)$ does NOT always converge to $f(x)$; the sum of the series at some specific point $x = x_0$ may differ from the value $f(x_0)$ of the function at $x = x_0$.

Definition 2

A function $f : [a, b] \to \mathbb{R}$ is said to be piecewise smooth (or sectionally smooth) if this interval can be divided into a finite number of subintervals such that

1. f has a continuous derivative f' in the interior of each of these subintervals,
2. and both $f(x)$ and $f'(x)$ approach finite limits as x approaches either endpoint of each of these subintervals from its interior.

In other words, we may say that f is piecewise smooth on $[a, b]$ if both f and f' are piecewise continuous on $[a, b]$.

Theorem 3 (Representation by a Fourier series)

Let $f : [-\pi, \pi] \to \mathbb{R}$ *be periodic with period* 2π *and be piecewise smooth in the interval* $[-\pi, \pi]$*. Then, the Fourier series of* f*, i.e.,* $\mathcal{S}_f(x)$ *converges at every point* x *to the value*

$$\frac{f(x+) + f(x-)}{2} \tag{4}$$

where $f(x+)$ *is the right hand limit of* f *at* x *and* $f(x-)$ *is the left hand limit of* f *at* x*. In particular, if* f *is also continuous at* x*, the value (4) reduces to* $f(x)$ *and* $\mathcal{S}_f(x) = f(x)$*.*

Find the Fourier series of the periodic function $f(x)$ defined by

$$f(x) = \begin{cases} -k & \text{if} \quad -\pi \le x < 0 \\ k & \text{if} \quad 0 \le x < \pi \end{cases} \quad \text{and} \quad f(x + 2\pi) = f(x).$$

Hence show that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}.$$

Proof.

Let the Fourier series representation of f be given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

where the *Fourier coefficients* are given by

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\cos nx \,\mathrm{d}x, \qquad n = 0, 1, 2, 3, \ldots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\sin nx \,\mathrm{d}x, \qquad n = 1, 2, 3, \ldots$$

□

Proof.

Let us compute the Fourier coefficients as follows.

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \frac{1}{\pi}\left(\int_{-\pi}^{0} f(x)\,\mathrm{d}x + \int_{0}^{\pi} f(x)\,\mathrm{d}x\right) \\
&= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\,\mathrm{d}x + \int_{0}^{\pi} k\,\mathrm{d}x\right] = \frac{1}{\pi}\left(-k\int_{-\pi}^{0}\mathrm{d}x + k\int_{0}^{\pi}\mathrm{d}x\right) \\
&= \frac{1}{\pi}\big(-k\times\pi + k\times\pi\big) = 0.
\end{aligned}
$$

$\square$

Proof.

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\cos nx\,\mathrm{d}x = \frac{1}{\pi}\left(\int_{-\pi}^{0} f(x)\,\cos nx\,\mathrm{d}x + \int_{0}^{\pi} f(x)\,\cos nx\,\mathrm{d}x\right) \\
&= \frac{1}{\pi}\left[\int_{-\pi}^{0} (-k)\cos nx\,\mathrm{d}x + \int_{0}^{\pi} k\,\cos nx\,\mathrm{d}x\right] \\
&= \frac{1}{\pi}\left(-k\int_{-\pi}^{0} \cos nx\,\mathrm{d}x + k\int_{0}^{\pi} \cos nx\,\mathrm{d}x\right) \\
&= \frac{1}{\pi}\left[-k\left(\frac{\sin nx}{n}\right)_{-\pi}^{0} + k\left(\frac{\sin nx}{n}\right)_{0}^{\pi}\right] = 0 \qquad \text{for all} \quad n = 1, 2, 3, \ldots
\end{aligned}
$$

□

Proof.

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\sin nx\,\mathrm{d}x = \frac{1}{\pi}\left(\int_{-\pi}^{0} f(x)\,\sin nx\,\mathrm{d}x + \int_{0}^{\pi} f(x)\,\sin nx\,\mathrm{d}x\right)\\
&= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\sin nx\,\mathrm{d}x + \int_{0}^{\pi} k\,\sin nx\,\mathrm{d}x\right]\\
&= \frac{1}{\pi}\left(-k\int_{-\pi}^{0}\sin nx\,\mathrm{d}x + k\int_{0}^{\pi}\sin nx\,\mathrm{d}x\right)\\
&= \frac{1}{\pi}\left[-k\left(-\frac{\cos nx}{n}\right)_{-\pi}^{0} + k\left(-\frac{\cos nx}{n}\right)_{0}^{\pi}\right]\\
&= \frac{k}{n\pi}[\cos 0 - \cos(-n\pi)] - \frac{k}{n\pi}[\cos n\pi - \cos 0] = \frac{2k}{n\pi}(1-\cos n\pi) \quad \text{for} \quad n = 1, 2, 3,
\end{aligned}
$$

□

Proof.

Noting that $\cos n\pi = (-1)^n$,

$$b_n = \frac{2k}{n\pi}[1-(-1)^n] \qquad \text{for} \quad n = 1, 2, 3, \dots$$

Thus we have,

$$b_1 = \frac{4k}{\pi}, \quad b_2 = 0, \quad b_3 = \frac{4k}{3\pi}, \quad b_4 = 0, \quad b_5 = \frac{4k}{5\pi}, \quad \dots$$

Therefore, the Fourier series of given f is

$$\boxed{\frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{1}{7}\sin 7x + \dots\right)} \qquad (\#)$$

□

Proof.

Note that the function $f(x)$ is discontinuous at the points $x = n\pi$ for all integers n. Nevertheless, at all other points than these, the function $f(x)$ is continuous and its left- and right-hand derivatives exist. Hence, the Fourier series (#) converges to the given $f(x)$ for all $x \neq n\pi$, where n is an integer. Therefore, at $x = \pi/2$,

$$f\left(\frac{\pi}{2}\right) = \frac{4k}{\pi}\left[\sin\left(\frac{\pi}{2}\right) + \frac{1}{3}\sin\left(\frac{3\pi}{2}\right) + \frac{1}{5}\sin\left(\frac{5\pi}{2}\right) + \frac{1}{7}\sin\left(\frac{7\pi}{2}\right) + \dots\right]$$

$$\implies \quad k = \frac{4k}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right)$$

$$\implies \quad \boxed{1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}} \tag{$*$}$$

$\square$