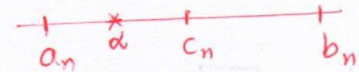
Tutorial Sheet 6

① At each step of the bisection method, the interval in halved.



Therefore

$$b_{n+1} - a_{n+1} = \frac{1}{2}(b_n - a_n), \qquad n = 1, 2, 3, \ldots$$

$$\Rightarrow b_{n+1} - a_{n+1} = \frac{1}{2}(b_n - a_n) = \frac{1}{2^2}(b_{n-1} - a_{n-1}) = \frac{1}{2^3}(b_{n-2} - a_{n-2}) = \ldots = \frac{1}{2^n}(b_1 - a_1)$$

where $a_1 = a$ and $b_1 = b$.

$$\Rightarrow \quad b_{n+1} - a_{n+1} = \frac{1}{2^n}(b_1 - a_1) \quad \Rightarrow \quad b_n - a_n = \frac{1}{2^{n-1}}(b_1 - a_1).$$

The error in the approximation at the n^{th} step is

$|\alpha - c_n| \leq c_n - a_n = b_n - c_n$, where $c_n = \frac{a_n + b_n}{2}$ and α is the actual root.

$$\Rightarrow |\alpha - c_n| \leq b_n - c_n = \frac{1}{2}(b_n - a_n)$$

$$= \frac{1}{2} \cdot \frac{1}{2^{n-1}}(b_1 - a_1) = \frac{1}{2^n}(b_1 - a_1)$$

$$\Rightarrow \quad \boxed{|\alpha - c_n| \leq \frac{b-a}{2^n}}$$

□

(2) Since $2^3 = 8$ and $3^3 = 27$. This implies that

$$2 < \sqrt[3]{25} < 3.$$

Let $\sqrt[3]{25} = x. \Rightarrow x^3 - 25 = 0.$

Let $f(x) \equiv x^3 - 25$. This means that we need to find a root of the $f(x) = 0$ and from the above argument this roots lies inside the interval $[2, 3]$.

number of steps $n = \dfrac{\log_{10}\left(\frac{3-2}{10^{-4}}\right)}{\log_{10} 2} \approx 13.2877$. Hence, 14 steps are needed to achieve the desired accuracy.

n	a_n	b_n	$c_n = (a_n+b_n)/2$	$f(c_n)$
1	2	3	2.5	-9.375
2	2.5	3	2.75	-4.203125
3	2.75	3	2.875	-1.236328125
4	2.875	3	2.9375	0.3474121094
5	2.875	2.9375	2.90625	-0.4529724121
6	2.90625	2.9375	2.921875	-0.05492019653
7	2.921875	2.9375	2.9296875	0.1457095146
8	2.921875	2.9296875	2.92578125	0.04526072741
9	2.921875	2.92578125	2.923828125	-0.004863195121
10	2.923828125	2.92578125	2.924804688	0.02019039821
11	2.923828125	2.924804688	2.924316406	0.00766150991
12	2.923828125	2.924316406	2.924072266	0.001398634529
13	2.923828125	2.924072266	2.923950195	-0.001732411007
14	2.923950195	2.924072266	2.92401123	-0.0001669209171

After 14 steps of the bisection method,

$$\boxed{\sqrt[3]{25} \approx 2.92401123}$$

Let α be a simple root of $f(x)=0$.

3(a) By Taylor's theorem, $(\epsilon_n = \alpha - x_n)$

$$f(\alpha) = f(x_n + \alpha - x_n) = f(x_n + \epsilon_n)$$

$$= f(x_n) + \epsilon_n f'(x_n) + \frac{\epsilon_n^2}{2!} f''(c_n),$$

where c_n is an unknown point between α and x_n.

$\because \alpha$ is the actual root of $f(x)=0$, $f(\alpha)=0$.

Therefore

$$0 = f(x_n) + \epsilon_n f'(x_n) + \frac{\epsilon_n^2}{2} f''(c_n)$$

Since $f'(\alpha) \neq 0$, $f'(x) \neq 0$ in sufficiently small neighbourhood of α.

Therefore, ~~Assuming that~~ $f'(x_n) \neq 0$ in Newton's method, we have (at for x_n's close to the root)

$$0 = \frac{f(x_n)}{f'(x_n)} + \epsilon_n + \epsilon_n^2 \frac{f''(c_n)}{2f'(x_n)} \quad \text{--- (1)}$$

By Newton's method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \Rightarrow \frac{f(x_n)}{f'(x_n)} = x_n - x_{n+1}$$

$$= (\alpha - x_{n+1}) - (\alpha - x_n)$$

$$= \epsilon_{n+1} - \epsilon_n.$$

Using this (1) becomes

$$0 = \epsilon_{n+1} + \epsilon_n^2 \frac{f''(c_n)}{2f'(x_n)}$$

or $\epsilon_{n+1} = \epsilon_n^2 \left[-\frac{f''(c_n)}{2f'(x_n)} \right]$ or $\frac{|\epsilon_{n+1}|}{|\epsilon_n|^2} = \left| \frac{f''(c_n)}{2f'(x_n)} \right|$

as $n \to \infty$, $x_n \to \alpha$ and hence $c_n \to \alpha$. Therefore,

$$\lim_{n\to\infty} \frac{|\epsilon_{n+1}|}{|\epsilon_n|^2} = \left| \frac{f''(\alpha)}{2f'(\alpha)} \right|.$$

Hence, for Newton's method, the order of convergence $p=2$ and the asymptotic error constant $C = \left| \frac{f''(\alpha)}{2f'(\alpha)} \right|$.

3(b) The secant method gives

$$x_{n+1} = x_n - f(x_n)\,\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}, \quad n = 1, 2, 3, \ldots$$

Let $\epsilon_n = \alpha - x_n$. Hence the secant method gives

$$\alpha - x_{n+1} = \alpha - \epsilon_n - f(\alpha - \epsilon_n)\,\frac{\alpha - \epsilon_n - (\alpha - \epsilon_{n-1})}{f(\alpha - \epsilon_n) - f(\alpha - \epsilon_{n-1})}$$

$$\Rightarrow \quad \epsilon_{n+1} = \epsilon_n - f(\alpha - \epsilon_n)\,\frac{\epsilon_n - \epsilon_{n-1}}{f(\alpha - \epsilon_n) - f(\alpha - \epsilon_{n-1})} \quad \text{— (1)}$$

For any small number ϵ, Taylor's formula gives

$$f(\alpha + \epsilon) = f(\alpha) + \epsilon f'(\alpha) + \frac{\epsilon^2}{2} f''(\alpha) + R_2(\epsilon)$$

where $R_2(\epsilon)$ is the remainder term that vanishes at a faster rate than ϵ^2 as $\epsilon \to 0$.

If α is the root of $f(x) = 0$, then $f(\alpha) = 0$, therefore

$$f(\alpha + \epsilon) \approx \epsilon f'(\alpha) + \frac{\epsilon^2}{2} f''(\alpha) = \epsilon f'(\alpha)(1 + \epsilon M)$$

where $M = \dfrac{f''(\alpha)}{2 f'(\alpha)}$.

$$\therefore \quad f(\alpha - \epsilon_n) \approx -\epsilon_n f'(\alpha)(1 - \epsilon_n M) \quad \text{— (2)}$$

and
$$f(\alpha - \epsilon_n) - f(\alpha - \epsilon_{n-1}) \approx -\epsilon_n f'(\alpha)(1 - \epsilon_n M) + \epsilon_{n-1} f'(\alpha)(1 - \epsilon_{n-1} M)$$

$$= f'(\alpha)\left[-\epsilon_n + \epsilon_{n-1} + (\epsilon_n^2 - \epsilon_{n-1}^2) M\right]$$

$$= -(\epsilon_n - \epsilon_{n-1}) f'(\alpha)\left[1 - (\epsilon_n + \epsilon_{n-1}) M\right] \quad \text{— (3)}$$

From (1), (2) & (3),

$$\epsilon_{n+1} = \epsilon_n - \frac{[-\epsilon_n \cancel{f'(\alpha)}(1 - \epsilon_n M)]\cancel{(\epsilon_n - \epsilon_{n-1})}}{[-\cancel{(\epsilon_n - \epsilon_{n-1})}\cancel{f'(\alpha)}\{1 - (\epsilon_n + \epsilon_{n-1}) M\}]} = \epsilon_n - \frac{\epsilon_n (1 - \epsilon_n M)}{1 - (\epsilon_n + \epsilon_{n-1}) M}$$

$$= \frac{\cancel{\epsilon_n} - (\cancel{\epsilon_n^2} + \epsilon_n \epsilon_{n-1}) M - \cancel{\epsilon_n} + \cancel{\epsilon_n^2 M}}{1 - (\epsilon_n + \epsilon_{n-1}) M} = -\frac{\epsilon_{n-1}\epsilon_n M}{1 - (\epsilon_n + \epsilon_{n-1}) M}$$

$$\approx -\epsilon_{n-1}\epsilon_n M \quad \text{— (4)}$$

For finding the order of convergence,

let $|\epsilon_{n+1}| \approx C\,|\epsilon_n|^P$. as $n \to \infty$. — ⑤ $\left[\text{equivalent to } \lim_{n\to\infty} \frac{|\epsilon_{n+1}|}{|\epsilon_n|^P} = C\right]$

Hence ④ implies that

$$C\,|\epsilon_n|^P \approx |\epsilon_{n-1}|\,|\epsilon_n|\,|M|$$

$$\Rightarrow \quad |\epsilon_n|^{P-1} \approx \frac{|M|}{C}\,|\epsilon_{n-1}|$$

$$\Rightarrow \quad |\epsilon_n| \approx \left(\frac{|M|}{C}\right)^{\frac{1}{P-1}} |\epsilon_{n-1}|^{\frac{1}{P-1}} \quad — ⑥$$

Comparing ⑤ & ⑥,

$$P = \frac{1}{P-1} \quad \Rightarrow \quad P^2 - P - 1 = 0 \Rightarrow P = \frac{1 \pm \sqrt{5}}{2}$$

but since $P > 0$, $\boxed{P = \frac{1+\sqrt{5}}{2} \approx 1.618}$

and $$C = \left(\frac{|M|}{C}\right)^{\frac{1}{P-1}} \quad \Rightarrow \quad C^{P-1} = \frac{|M|}{C} \Rightarrow C^P = |M|$$

$$\Rightarrow C = |M|^{1/P} \quad \Rightarrow C = \left|\frac{f''(\alpha)}{2f'(\alpha)}\right|^{\frac{2}{1+\sqrt{5}}} \Rightarrow \boxed{C = \left|\frac{f''(\alpha)}{2f'(\alpha)}\right|^{\frac{\sqrt{5}-1}{2}}}$$

4 (a)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, 3, \ldots$$

(a) $f(x) \equiv \ln(x-1) + \cos(x-1) = 0$, $x_0 = 1.65$

See the table below. The solution of the given equation accurate to within 10^{-5} is 1.397748.

n	x_n	$f(x_n)$	$x_n - x_{n-1}$
0	1.65	0.3653009	
1	1.258582	-0.3857896	-0.3914182
2	1.365403	-0.07277412	0.1068214
3	1.395989	-0.00375424	0.03058537
4	1.397743	-0.00001120807	0.001754641
5	1.397748	$-1.005722 \times 10^{-10}$	5.269807×10^{-6}

(b) $f(x) \equiv e^x + 2^{-x} + 2\cos x - 6$, $x_0 = 1.5$

See the table below. The solution of the given equation accurate to within 10^{-5} is 1.829384.

n	x_n	$f(x_n)$	$x_n - x_{n-1}$
0	1.5	-1.023283	
1	1.95649	0.5797014	0.4564897
2	1.841533	0.05034095	-0.1149567
3	1.829506	0.0005021213	-0.01202705
4	1.829384	5.151614×10^{-8}	-0.0001223987
5	1.829384	8.881784×10^{-16}	-1.256032×10^{-8}

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$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, 3, \ldots$$

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5(a) $$x_{n+1} = x_n - f(x_n)\,\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}, \quad n = 1, 2, 3, \ldots$$

$$f(x) \equiv \ln(x-1) + \cos(x-1)$$

$$x_0 = 1.65$$

$$x_1 = 1.26$$

n	x_n	$f(x_n)$	$x_n - x_{n-1}$
0	1.65	0.3653009	
1	1.26	-0.3806837	-0.39
2	1.459021	0.1178275	0.1990211
3	1.411981	0.0295507	-0.04704037
4	1.396234	-0.003229541	-0.01574679
5	1.397785	0.00007828671	0.001551389
6	1.397749	2.018452×10^{-7}	-0.00003671689
7	1.397748	$-1.265221 \times 10^{-11}$	-9.491118×10^{-8}

5(b) $$f(x) \equiv e^{-x} + 2^{-x} + 2\cos x - 6$$

$$x_0 = 1.5$$

$$x_1 = 1.96$$

n	x_n	$f(x_n)$	$x_n - x_{n-1}$
0	1.5	-1.023283	
1	1.96	0.597452	0.46
2	1.79043	-0.1546295	-0.1695699
3	1.825294	-0.01671603	0.03486393
4	1.82952	0.0005584875	0.004225738
5	1.829383	-1.917857×10^{-6}	-0.0001366187
6	1.829384	$-2.188854 \times 10^{-10}$	4.675458×10^{-7}

⑥ $f(x) = e^x - x - 1$ $\quad f'(x) = e^x - 1$ $\quad f''(x) = e^x$

$f(0) = 1 - 0 - 1 = 0$ $\quad f'(0) = 1 - 1 = 0$ $\quad f''(0) \neq 0$

$\Rightarrow$ $f(x)$ has a zero of multiplicity 2 at $x = 0$.

With Newton's method: slow convergence to the root. $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$.

n	x_n	$f(x_n)$	$x_n - x_{n-1}$
0	1	0.7182818	
1	0.5819767	0.2075957	-0.4180233
2	0.319055	0.05677201	-0.2629217
3	0.1679962	0.01493591	-0.1510589
4	0.08634887	0.003837726	-0.0816473
5	0.0437957	0.000973187	-0.04255317
6	0.02205769	0.0002450693	-0.02173802
7	0.01106939	0.00006149235	-0.0109883
8	0.005544905	0.00001540144	-0.005524483
9	0.002775014	3.853917*10^(-6)	-0.00276989
10	0.001388149	9.639248*10^(-7)	-0.001386866
11	0.0006942351	2.410369*10^(-7)	-0.0006939139
12	0.0003471577	6.026621*10^(-8)	-0.0003470774
13	0.0001735889	1.506742*10^(-8)	-0.0001735688
14	0.00008679696	3.766965*10^(-9)	-0.00008679194
15	0.00004339911	9.417549*10^(-10)	-0.00004339785
16	0.00002169971	2.354406*10^(-10)	-0.0000216994
17	0.00001084989	5.886025*10^(-11)	-0.00001084982
18	5.424953*10^(-6)	1.471512*10^(-11)	-5.424935*10^(-6)

With modified Newton's method: fast convergence to the root.

$$x_{n+1} = x_n - \frac{f(x_n)\,f'(x_n)}{[f'(x_n)]^2 - f(x_n)\,f''(x_n)}.$$

n	x_n	$f(x_n)$	$x_n - x_{n-1}$
0	1	0.7182818	
1	-0.2342106	0.02540578	-1.234211
2	-0.00845828	0.00003567061	0.2257523
3	-0.00001189018	7.06879*10^(-11)	0.00844639
4	-4.176666*10^(-11)	0	0.00001189014
5	-4.176666*10^(-11)	0	0

⑦ See the mathematica notebook.

7. $\underline{\text{Sol}^n}$: For convenience we write a instead of 1.37. Then $1/a$ is the root of the equation

$$f(x)=0 \quad \text{where } f(x) = a - \frac{1}{x}.$$

Now, $f'(x) = \frac{1}{x^2}$. Thus, Newton's method yields the iteration:

$$x_{n+1} = x_n - \frac{a - 1/x_n}{1/x_n^2} = x_n(2 - ax_n).$$

Note that the expression $x_n(2-ax_n)$ can be evaluated on our defective calculator.

choose x_0 reasonably close to $1/1.37$.

The choose $x_0 = 1$ would work out fine, but we will start off a little closer, maybe noting that 1.37 is about $4/3$, so its reciprocal is about $3/4$.

choose x_0 = 0.75.

We get $x_1 = x_0(2 - 1.37x_0) = 0.729375$.

Thus, $x_2 = 0.729926589$, and

$$x_3 = 0.729927007.$$

It turns out that $x_4 = x_3$ upto 9 decimal places. Thus,

$\frac{1}{1.37}$ is equal to 0.72992701 to 8 decimal places.

!!

(5) $f(x) \equiv x^3 + 4x^2 - 10$

$f(1) = -5$, $f(2) = 14$

Since $f(1) \cdot f(2) < 0$, by the intermediate value theorem, the given equation has a solution in $[1, 2]$.

Number of iterations needed for achieving the accuracy of 10^{-5}:

$$n \geq \frac{\log_{10}\left(\frac{b-a}{\epsilon}\right)}{\log_{10} 2} = \frac{\log_{10}\left(\frac{2-1}{10^{-5}}\right)}{\log_{10} 2} \approx 16.6096$$

$\Rightarrow n = 17$ iterations are required.

n	a_n	b_n	$c_n = \frac{a_n+b_n}{2}$	$f(c_n)$
1	1	2	1.5	2.375
2	1	1.5	1.25	-1.796875
3	1.25	1.5	1.375	0.162109375
4	1.25	1.375	1.3125	-0.8483886719
5	1.3125	1.375	1.34375	-0.350982666
6	1.34375	1.375	1.359375	-0.09640884399
7	1.359375	1.375	1.3671875	0.03235578537
8	1.359375	1.3671875	1.36328125	-0.03214997053
9	1.36328125	1.3671875	1.365234375	0.00007202476263
10	1.36328125	1.365234375	1.364257813	-0.01604669075
11	1.364257813	1.365234375	1.364746094	-0.007989262813
12	1.364746094	1.365234375	1.364990234	-0.003959101523
13	1.364990234	1.365234375	1.365112305	-0.00194365901
14	1.365112305	1.365234375	1.36517334	-0.0009358472819
15	1.36517334	1.365234375	1.365203857	-0.0004319187993
16	1.365203857	1.365234375	1.365219116	-0.0001799489032
17	1.365219116	1.365234375	1.365226746	-0.00005396254153

After 17 iterations, the solution of the given equation is 1.365226746.

⑥ (a) $f(x) = e^x - x^2 + 3x - 2$, , $a = 0$, $b = 1$, $\epsilon = 10^{-4}$

$$n \geq \frac{\log_{10}\left(\frac{b-a}{\epsilon}\right)}{\log_{10} 2} = \frac{4}{\log_{10} 2} = 13.2877$$

$\Rightarrow$ number of iterations needed for achieving accuracy of 10^{-4} is 14.

n	a_n	b_n	$c_n = (a_n+b_n)/2$	$f(c_n)$
1	0	1	0.5	0.8987212707
2	0	0.5	0.25	-0.02847458331
3	0.25	0.5	0.375	0.4393664146
4	0.25	0.375	0.3125	0.2066816912
5	0.25	0.3125	0.28125	0.08943319623
6	0.25	0.28125	0.265625	0.03056423414
7	0.25	0.265625	0.2578125	0.00106636769
8	0.25	0.2578125	0.25390625	-0.01369868371
9	0.25390625	0.2578125	0.255859375	-0.006314806791
10	0.255859375	0.2578125	0.2568359375	-0.002623882347
11	0.2568359375	0.2578125	0.2573242188	-0.0007786731029
12	0.2573242188	0.2578125	0.2575683594	0.0001438683406
13	0.2573242188	0.2575683594	0.2574462891	-0.0003173971182
14	0.2574462891	0.2575683594	0.2575073242	-0.00008676307322

After 14 iterations, the solution of the given equation is 0.2575073242.

(b) $f(x) = x + 1 - 2\sin(\pi x)$, $a = 0.5$, $b = 1$, $\epsilon = 10^{-4}$

$$n \geq \frac{\log_{10}\left(\frac{b-a}{\epsilon}\right)}{\log_{10} 2} = \frac{\log_{10}(0.5 \times 10^{4})}{\log_{10} 2} = 12.2877$$

$\Rightarrow$ number of iterations needed for achieving accuracy of 10^{-4} is 13.

n	a_n	b_n	$c_n = (a_n+b_n)/2$	$f(c_n)$
1	0.5	1	0.75	0.3357864376
2	0.5	0.75	0.625	-0.222759065
3	0.625	0.75	0.6875	0.02456077539
4	0.625	0.6875	0.65625	-0.1075925287
5	0.65625	0.6875	0.671875	-0.04358222
6	0.671875	0.6875	0.6796875	-0.0100196305
7	0.6796875	0.6875	0.68359375	0.00714433889
8	0.6796875	0.68359375	0.681640625	-0.001469329874
9	0.681640625	0.68359375	0.6826171875	0.002829599108
10	0.681640625	0.6826171875	0.6821289063	0.0006781563091
11	0.681640625	0.6821289063	0.6818847656	-0.0003960816036
12	0.6818847656	0.6821289063	0.6820068359	0.0001409136779
13	0.6818847656	0.6820068359	0.6819458008	-0.0001276148853

After 13 iterations, the solution of the given equation is 0.6819458008.