ESE MA

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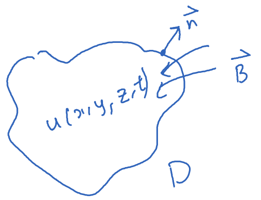
internal energy $u = c_v T$

Δu bet $t = t$ & $t = t + \Delta t$ in a boundary D

$$\int_D \left[u(x,y,z,t+\Delta t) - u(x,y,z,t) \right] dv = \int_t^{t+\Delta t} \int_D q(x,y,z,t,u)\, dv\, dt - \int_t^{t+\Delta t} \int_D \vec{B}(x,y,z,t) \cdot \vec{n}\, dS\, dt$$



q = rate of heat production

$\vec{B} = -K(x,y,z)\,\vec{\nabla} u$

using Gauss Div theorem.

$$\int_D \left(\partial_t u - q - \operatorname{div}(K \vec{\nabla} u) \right) dv = 0$$

$h(x,y,z)$ = cont func^n $\int_\Omega h(x,y,z)\, dv = 0$ for every domain Ω

then $h(x,y,z) = 0$

$$\therefore \quad \partial_t u = q + \operatorname{div}(K \vec{\nabla} u)$$

$q = 0$ $\quad \partial_t u = \operatorname{div}(K \vec{\nabla} u)$

2nd order PDEs

$$A\, d_x^2 u + B\, d_{xy}^2 u + C\, d_y^2 u + D\, d_x u + E\, d_y u + F u = G$$

$A, B, C = f(x,y)$ don't vanish simultaneously

$D, E, F = f(x,y)$ too

$$A d_{xx} u + B d_{xy} u + C d_{yy} u = \Phi(x, y, u, d_x u, d_y u)$$

$$\Delta(x_0, y_0) = \begin{vmatrix} B & 2A \\ 2C & B \end{vmatrix} = B^2 - 4AC$$

$\Delta(x_0, y_0) > 0$ hyperbolic

$\Delta(x_0, y_0) = 0$ parabolic

$\Delta(x_0, y_0) < 0$ elliptic

Change of Co-ordinates

$\xi = \xi(x,y) \qquad \eta = \eta(x,y)$

Jacobian $J = \dfrac{\partial(\xi,\eta)}{\partial(x,y)} = \begin{vmatrix} \partial_x \xi & \partial_y \xi \\ \partial_x \eta & \partial_y \eta \end{vmatrix} \neq 0$

$J \neq 0$ for one-to-one transformation

$u(x,y) = \omega(\xi,\eta)$

forming the new eq^n by substitution

$a\,\partial_{\xi\xi}\omega + b\,\partial_{\xi\eta}\omega + c\,\partial_{\eta\eta}u = \phi(\xi,\eta,\omega,\partial_\xi\omega,\partial_\eta\omega)$

$a = A(\partial_x\xi)^2 + B\,\partial_x\xi\,\partial_y\xi + C(\partial_y\xi)^2$

$b = 2A\,\partial_x\xi\,\partial_x\eta + B(\partial_x\xi\,\partial_y\eta + \partial_y\xi\,\partial_x\eta) + 2C\,\partial_y\xi\,\partial_y\eta$

$c = A(\partial_x\eta)^2 + B\,\partial_x\eta\,\partial_y\eta + C(\partial_y\eta)^2$

$$\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} = \underbrace{\begin{pmatrix} \partial_x\xi & \partial_y\xi \\ \partial_x\eta & \partial_y\eta \end{pmatrix}}_{J} \begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix} \underbrace{\begin{pmatrix} \partial_x\xi & \partial_x\eta \\ \partial_y\xi & \partial_y\eta \end{pmatrix}}_{J^T}$$

$$\begin{vmatrix} a & b/2 \\ b/2 & c \end{vmatrix} = \begin{vmatrix} A & B/2 \\ B/2 & C \end{vmatrix} J^2$$

$$\underbrace{b^2 - 4ac}_{\delta} = \underbrace{(B^2 - 4AC)}_{\Delta} J^2$$

Canonical Form

→ when one/two leading coeff $= 0$

1) $a = c = 0$ hyperbolic family $\quad \partial_{\xi\eta}\omega = \psi(\xi,\eta,\omega,\partial_\xi\omega,\partial_\eta\omega)$
2) $b = 0 \;\; c = -a$ hyperbolic $\quad \partial_{\xi\xi}\omega - \partial_{\eta\eta}\omega = \psi(\,)$
3) $a = 0 \;\; b = 0$ parabolic $\quad \partial_{\eta\eta}\omega = \psi(\,)$
4) $b = 0 \;\; c = a$ elliptic $\quad \partial_{\xi\xi}\omega + \partial_{\eta\eta}\omega = \psi(\,)$

First Canonical Form

First Canonical Form

we have to select $a=0$ $c=0$ $\quad A(u_i)^2 + B(u_i) + C = 0$

by substituting

$$u_1(x,y) = \frac{d_x \xi}{d_y \xi} = \frac{-B - \sqrt{B^2 - 4AC}}{2A}$$

$$u_2(x,y) = \frac{d_x \eta}{d_y \eta} = \frac{-B + \sqrt{B^2 - 4AC}}{2A}$$

along new coordinete line. $\xi(x,y) = \text{const.}$ $\quad d\xi = 0$

$$d\xi = d_x\xi \, dx + d_y \xi \, dy = 0$$

$$\frac{dy}{dx} = -\frac{d_x \xi}{d_y \xi}$$

Similarly along $\eta(x,y) = \text{const}$ $\quad \frac{dy}{dx} = -\frac{d_x \eta}{d_y \eta}$

Substituting : $\quad A\left(\frac{-dy}{dx}\right)^2 + B\left(\frac{-dy}{dx}\right) + C = 0$

$\hookrightarrow$ characteristic eqn

$$\frac{dy}{dx} = \lambda_1(x,y) = \frac{B - \sqrt{B^2 - 4AC}}{2A}$$

$$\frac{dy}{dx} = \lambda_2(x,y) = \frac{B + \sqrt{B^2 - 4AC}}{2A}$$

$$\xi = y - \frac{B + \sqrt{B^2 - 4AC}}{2A} x = y - \lambda_1 x = c_1$$

$$\eta = y - \frac{B - \sqrt{B^2 - 4AC}}{2A} x = y - \lambda_2 x = c_2$$

$$d_{\xi\eta} \omega = \varphi(\,) \qquad \varphi = \frac{\Phi}{b}$$

$$b = 2A \, d_x\xi \, d_x\eta + B\left(d_x\xi \, d_y\eta + d_y\xi \, d_x\eta\right) + 2C \, d_y\xi \, d_y\eta$$

$$= 4C - \frac{B^2}{A}$$

$$= \frac{-\Delta}{A}$$

Diff eq^n on Laplace & Poisson Eq^n

$d_{xx}u + d_{yy}u = 0$ Laplace $\qquad \Delta u = d_{xx}u + d_{yy}u$

$\qquad\qquad = f(x,y)$ Poisson

taylors formula $\quad f(x) = f(c) + \sum_{k=1}^{n-1} \frac{f^k(c)}{k!}(x-c)^k + \frac{f^{k+1}(\xi)}{(k+1)!}(x-c)^n$

$u(x+h, y) =$ $\qquad u(x, y+k) =$

$u(x-h, y) =$ $\qquad u(x, y-k) =$

add & sub $\qquad$ add & sub $\qquad \boxed{h = k}$

$$d_x u = \frac{1}{2h}\left[u(x+h,y) - u(x-h,y)\right]$$

$$d_{xx} u = \frac{1}{h^2}\left[u(x+h,y) - 2u(x,y) + u(x-h,y)\right]$$

substituting in poisson's

$$u(x+h,y) + u(x-h,y) + u(x,y+h) + u(x,y-h) - 4u(x,y) = h^2 f(x,y)$$

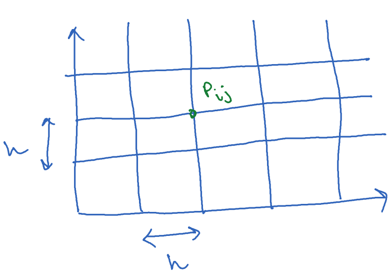
substituting in Laplace's

$$u(x+h,y) + u(x-h,y) + u(x,y+h) + u(x,y-h) - 4u(x,y) = 0$$

h = mesh size $\qquad \left\{\begin{matrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{matrix}\right\} u = h^2 f(x,y)$

$\uparrow$ stencil

Mesh Representation



$P_{ij} = (ih, jh)$ point in mesh.

$\therefore u_{i-h,j} + u_{i+h,j} + u_{i,j+h}, u_{i,j-h} = 4u_{i,j}$ Laplace

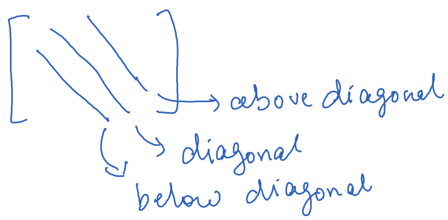
$\qquad = 4u_{i,j} + h^2 u_{i,j}$ Poisson.

steady state of heat flow $d_{xx}u + d_{yy}u = 0$

Apply the stencil formula & obtain stimulltaneous eq^n for all points and then apply any method (Gauss, Seidal, Jacobi)

ADI Method

tridiagonal matrix :



all non zero entries

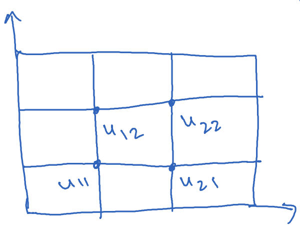
if there are only 3 points in a row or column, then we can apply this ADI method as stencil is $\left\{ 1 \; -4 \; 1 \right\}$

rewrite 5-point formula as:

(a) $u_{i+1,j} + u_{i-1,j} - 4u_{i,j} = -u_{i,j+1} - u_{i,j-1}$

(b) $u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = -u_{i+1,j} - u_{i-1,j}$

initial values $= \left[u_{11}^0 \; u_{21}^0 \; u_{12}^0 \; u_{22}^0 \right]$



using (a) in first itr.

$$\left.\begin{array}{l} u_{12}^1 \quad u_{11}^1 = u_{01} \quad u_{10} \quad u_{12}^0 \\ u_{11}^1 \quad u_{21}^1 \quad u_{31} = u_{20} \quad u_{22}^0 \end{array}\right] \text{row } 1$$

$$\left.\begin{array}{l} u_{21}^1 \quad u_{22}^1 = u_{02} \quad u_{13} \quad u_{11}^0 \\ u_{21}^1 \quad u_{22}^1 = u_{23} \quad u_{21}^0 \end{array}\right] \text{row } 2$$

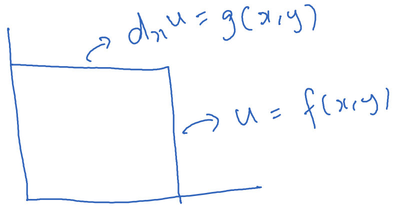
After finding $u_{21}^1 \; u_{22}^1 \; u_{11}^1 \; u_{12}^1$ using (a)

Second itr using (b)

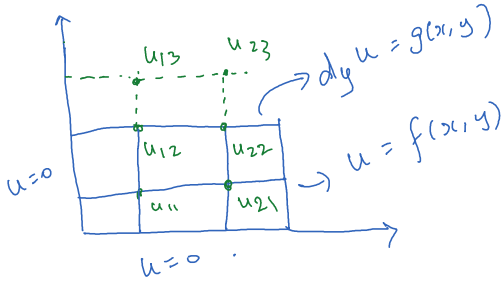
$$\left.\begin{array}{l} u_{11}^2 \quad u_{12}^2 = u_{01} \quad u_{21}^1 \quad u_{10} \\ u_{11}^2 \quad u_{12}^2 = u_{13} \quad u_{02} \quad u_{22}^1 \end{array}\right] \text{Column } 1$$

$$\left.\begin{array}{l} u_{21}^2 \quad u_{22}^2 = u_{11}^1 \quad u_{31}^1 \quad u_{20} \\ u_{21}^2 \quad u_{22}^2 = u_{23} \quad u_{12}^1 \quad u_{32} \end{array}\right] \text{Column } 2$$

Neuman Boundary Problems



in such cases we include points not in grid & approx. them with forward diff formula.



use 5 point formula on P_{11} P_{21}

u_{12} , u_{22} = unknown

as apply 5-point at P_{12} P_{22} as well

but u_{13} u_{23} will come

* Assume Poisson eq^n exists outside grid too

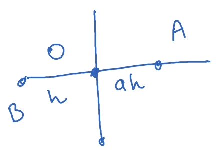
$$\partial_y u_{12} \approx \frac{u_{13} - u_{11}}{2h}$$

find u_{13} & u_{23} through this

$$\partial_y u_{22} \approx \frac{u_{23} - u_{21}}{2h}$$

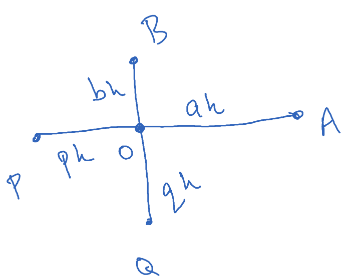
Now you get simultaneous eq^{n}s & solve them.

IRREGULAR BOUNDARY



$$u_A = u_0 + ah\,\partial_x u_0 + \frac{1}{2}(ah)^2 \partial_{xx} u_0 \ldots$$

$$u_B = u_0 - h\,\partial_x u_0 + \frac{h^2}{2} \partial_{xx} u_0 \ldots$$



$$\Delta u_0 \approx \frac{2}{h^2}\left[\frac{u_A}{a(a+p)} + \frac{u_B}{b(b+q)} + \frac{u_P}{p(p+a)} + \frac{u_Q}{q(q+b)} - \frac{ap+bq}{apbq}u_0\right]$$

Heat Equation:

$$\partial_t u = \alpha^2 \partial_{xx} u$$

define (x,y) in plane: $x = ih$ $h = \Delta x$
$y = kl$ $l = \Delta t$

$y = kl \qquad t = -$

$u(x,y) = u(ih, kl) = u_i^k$

$$d_t u = \frac{u_i^{k+1} - u_i^k}{l} \qquad d_{xx} u = \frac{1}{h^2}\left[u_{i+1}^k - 2u_i^k + u_{i+1}^k\right]$$

substitute :

$$u_i^{k+1} = \lambda u_{i-1}^k + \lambda u_{i+1}^k + (1-2\lambda) u_i^k$$ Explicit formula

$$0 \le \lambda \le \frac{1}{2}$$ Region of Stability $\lambda = \frac{\alpha^2 l}{h^2}$

$$u_i^{k+1} = \frac{1}{2}\left[u_{i-1}^k + u_{i+1}^k\right]$$ Bender Schmidt formula ($\lambda = \frac{1}{2}$ in explicit)

$$-\lambda u_{i-1}^{k+1} + (2+2\lambda) u_i^{k+1} - \lambda u_{i+1}^{k+1} = \lambda u_{i-1}^k + (2-2\lambda) u_i^k + \lambda u_{i+1}^k$$ Crank Nicolson formula (no restriction on λ)

$\hookrightarrow$ $d_{xx} u$ as avg of forward diff on k th & $(k+1)$ th levels

$$d_{xx} u = \frac{1}{2h^2}\left[u_{i-1}^k - 2u_i^k + u_{i+1}^k + u_{i-1}^{k+1} - 2u_{i-1}^{k+1} + u_{i-1}^{k+1}\right]$$

for $\lambda = 1$, Crank Nicolson formula becomes:

$$-u_{i-1}^{k+1} + 4u_i^{k+1} - u_{i+1}^{k+1} = u_{i-1}^k + u_{i+1}^k$$

Iterative formula for Crank Nicolson / heat Equation.

$u_{i,j}$ used now $\qquad d_t u = d_{xx} u \qquad \alpha^2 = 1 \qquad \lambda = \frac{k}{h^2}$

$$(1+\lambda) u_{i,j+1} = u_{i,j} + \frac{\lambda}{2}\left[u_{i+1,j+1} + u_{i+1,j} + u_{i-1,j} - 2u_{i,j}\right]$$

$$c_i = u_{i,j} + \frac{\lambda}{2}\left[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right]$$

$$\therefore \; u_i = \frac{\lambda}{2(1+\lambda)}\left[u_{i-1} + u_{i+1}\right] + \frac{c_i}{(1+\lambda)}$$

$$u_i^n = \frac{\lambda}{2(1+\lambda)}\left[u_{i-1}^{n} + u_{i+1}^n\right] + \frac{c_i}{(1+\lambda)}$$ Jacobi formula] keep $\lambda = 1$

$$u_i = \frac{\lambda}{2(1+\lambda)}\left[u_{i-1} + u_{i+1}\right] + \frac{c_i}{(1+\lambda)}$$ Jacobi formula

$$u_i^n = \frac{\lambda}{2(1+\lambda)}\left[u_{i-1}^{n+1} + u_{i+1}^n\right] + \frac{c_i}{(1+\lambda)}$$ Seidal formula.

Keep $\lambda = \frac{1}{2}$

$$u_i^{n+1} = u_i^n + \omega\left[\frac{\lambda}{2(1+\lambda)}\left(u_{i-1}^{n+1} + u_{i+1}^n\right) + \frac{c_i}{(1+\lambda)} - u_i^n\right]$$ Successive over Relaxation formula.

ω → relaxation factor $1 \le \omega \le 2$

Wave Equation

$$d_{tt}u = c^2 d_{xx}u$$

$$u(x,0) = f(x)$$
$$\partial_t u(x,0) = \phi(x)$$
$$u(0,t) = \varphi_1(t)$$
$$u(1,t) = \varphi_2(t)$$

$$u_i^{k+1} = -u_i^{k-1} + \alpha^2\left(u_{i-1}^k + u_{i+1}^k\right) + 2(1-\alpha^2)u_i^k$$

$\alpha < 1$ for stability

$$\alpha = \frac{cl}{h}$$