(1) Heat Equation in One dimension:

$$\partial_t u = \alpha^2 \partial_{xx} u, \quad \text{——— (1)}$$

where, $\alpha^2 = \dfrac{K}{s\rho}$,

where K is the coefficient of conductivity of the material, ρ is the density and s is the specific heat.

In the sequal, we shall discuss finite difference approximations to (1).

Finite-difference Approximations

We divide the (x,t) plane into smaller rectangles by means of the sets of lines,

$$\begin{cases} x = ih, & i = 0, 1, 2, \text{—} \\ y = kl, & K = 0, 1, 2, \text{—} \end{cases}$$

where, $h = \Delta x$, $l = \Delta t$, denoting $u(ih, Kl) = u_i^K$, we have,

$$\partial_t u \approx \frac{u_i^{K+1} - u_i^K}{l} \quad \text{——— (2)}$$

and, $$\partial_{xx} u \approx \frac{1}{h^2}\left(u_{i-1}^K - 2u_i^K + u_{i+1}^K\right) \quad \text{——— (3)}$$

[Recall we have obtained (3) before by Taylor Expansion]

(2) Hence (1) is replaced by the finite difference analogue.

$$\frac{u_i^{K+1} - u_i^K}{l} = \alpha^2 \frac{u_{i-1}^K - 2u_i^K + u_{i+1}^K}{h^2},$$

which simplifies to,

$$u_i^{K+1} = \lambda\, u_{i-1}^K + u_{i+1}^K + (1-2\lambda)\, u_i^K; \quad \text{--- (4)}$$

where

$$\lambda = \frac{\alpha^2 l}{h^2}. \quad \text{--- (5)}$$

In (4), u_i^{K+1} is expressed explicitely in terms of u_{i-1}^K, u_{i+1}^K and u_i^K.

Hence it is called the <u>explicit formula for solution</u> of <u>one-dimensional heat equation</u>.

It can be shown that eq^n (4) approximates (1) only when $0 < \lambda \leq 1/2$, which is called the stability condition for the explicit formulae.

If we set, $\lambda = 1/2$ in (4), we obtain the simple formulae,

$$u_i^{K+1} = \frac{1}{2}\left(u_{i-1}^K + u_{i+1}^K\right) \quad \text{--- (6)}$$

3

which is called Bender-Schmidt recurrence formulae. It is clear that 4 and 6 have limited applications because of the restriction on the values of λ.

A formulae which does not have any restriction on λ is ~~that~~ due to Crank and Nicolson.

In eqⁿ 4 ~~it replaces~~, if we replace $\partial_{xx}u$ by the average of its finite difference approximations on the Kth and $(K+1)$th time levels, we obtain

$$\partial_{xx}u = \frac{1}{2h^2}\left(u_{i-1}^{K} - 2u_i^{K} + u_{i+1}^{K} + u_{i-1}^{K+1} - 2u_i^{K+1} + u_{i+1}^{K+1}\right).$$

Hence, eqⁿ 1 is approximated by,

$$\frac{u_i^{K+1} - u_i^{K}}{l} = \frac{\alpha^2}{2h^2}\left(u_{i-1}^{K} - 2u_i^{K} + u_{i+1}^{K} + u_{i-1}^{K+1} - 2u_i^{K+1} + u_{i+1}^{K+1}\right)$$

which simplifies to,

$$-\lambda u_{i-1}^{K+1} + (2+2\lambda)u_i^{K+1} - \lambda u_{i+1}^{K+1} = \lambda u_{i-1}^{K} + (2-2\lambda)u_i^{K} + \lambda u_{i+1}^{K} \qquad (7)$$

On the left side of (7) we have three unknowns and on the right, all quantities are known.

This is called Crank-Nicolson formula for the one-dimensional heat equation and it is an implicit formula.

It is convergent for all finite values of λ.

If there are N internal mesh points on each time row, then eqn (7) gives N simultaneous eqns for the N unknowns. In a similar way, values of u on all time rows can be calculated.

Example:

Use the Bender-Schmidt formulae to solve the heat conduction problem:

$$\partial_t u = \frac{1}{2} \partial_{xx} u$$

with the conditions $u(x,0) = 4x - x^2$

and $u(0,t) = u(4,t) = 0$.

Setting, $h = 1$, we see that $l = 1$ when $\lambda = 1/2$.

Let us now compute the following initial values,

$u(0,0) = 0$, $u(1,0) = 3$,

$u(2,0) = 4$, $u(3,0) = 3$,

and $u(4,0) = 0$.

5

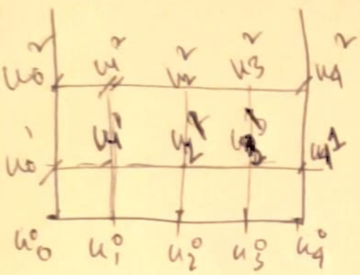
D Equation problem

Further,

$$u(0,t) = u(4,t) = 0.$$

~~For $l=1$ (ie unit time~~

For $l=1$, Bender–Schmidt formula gives,



$$u_1^1 = \frac{1}{2}(0+4) = 2, \quad \left(= \frac{1}{2}(u_0^0 + u_2^0)\right)$$

$$u_2^1 = \frac{1}{2}(3+3) = 3, \quad \left(= \frac{1}{2}(u_1^0 + u_3^0)\right)$$

$$u_3^1 = \frac{1}{2}(4+0) = 2, \quad \left(= \frac{1}{2}(u_2^0 + u_4^0)\right)$$

Similarly, for $l=2$, we obtain,

$$u_1^2 = \frac{1}{2}(u_0^1 + u_2^1) = \frac{1}{2}(0+3) = 1.5$$

$$u_2^2 = \frac{1}{2}(2+2) = 2,$$

$$u_3^2 = \frac{1}{2}(3+0) = 1.5.$$

Continuing in this way, we obtain,

$$u_1^3 = 1, \quad u_2^3 = 1.5, \quad u_3^3 = 1,$$

$$u_1^4 = 0.75, \quad u_2^4 = 1, \quad u_3^4 = 0.75,$$

$$u_1^5 = 0.5, \quad u_2^5 = 0.75, \quad u_3^5 = 0.5,$$

and so on.

(6) Example/ Solve the heat equation problem

$$\partial_t u = \partial_{xx} u \quad \text{—} (8)$$

subject to the conditions $u(x,0) = \sin \pi x$, $0 \le x \le 1$, and $u(0,t) = u(1,t) = 0$. Use Bender–Schmidt's and Crank–Nicolson to compute the value of $u(0.6, 0.04)$ and compare the results with the exact value.

Exact value:

We know sol^n of

$$\begin{cases} \partial_t u = \alpha^2 \partial_{xx} u & \text{in } (0,l)\times(0,t) \\ u(0,t) = 0 = u(l,t) & \text{for } t \ge 0 \\ u(x,0) = f(x) & , 0 < x < l \end{cases}$$

is given as, (recall Problem set II)

$$u(x,t) = \sum_{K=1}^{\infty} \beta_K \sin\left(\frac{K\pi}{l} x\right) e^{-\alpha^2 \pi^2 K^2 t / l^2} \quad \text{—} (*)$$

with $\beta_K = \frac{2}{l} \int_0^l f(x) \sin \frac{K\pi x}{l} \, dx$

Here $u(x,0) = \sin \pi x = \sum_{K=1}^{\infty} \beta_K \sin\left(\frac{K\pi x}{l}\right)$

(ie) $K = 1$

and further for $K=1$, verify that $\beta_K = 1$

and hence exact solution of (8) is

$$u(x,t) = e^{-\pi^2 t} \sin \pi x,$$

so the exact value of $u(0.6, 0.04)$ is

0.6408.

(7)

(a) Bender - Schmidt formulae.

Let, $h = 0.2$, Then, $k = \lambda h^2$

$= \frac{1}{2}(0.04)$

$= 0.02.$

The initial value of u are.

$u_0^0 = 0$, $u_1^0 = \sin(\pi . 0.2) = 0.5878$,

$u_2^0 = \sin(\pi\, 0.4) = 0.9510$,

$u_3^0 = \sin(\pi . 0.6) = 0.9510$,

$u_4^0 = \sin(\pi . 0.8) = 0.5878$

$u_5^0 = \sin(\pi) = 0.$

The Bender - Schmidt formulae gives.

$u_1^1 = \frac{1}{2}(0 + 0.9510) = 0.4755$

$u_2^1 = \frac{1}{2}(0.5878 + 0.9510) = 0.7694$,

$u_3^1 = \frac{1}{2}(0.9510 + 0.5878) = 0.7694$,

$u_4^1 = \frac{1}{2}(0.9510 + 0) = 0.4755$

Similarly calculate,

$u_1^2 = \frac{1}{2}(0.7694) = 0.3847$

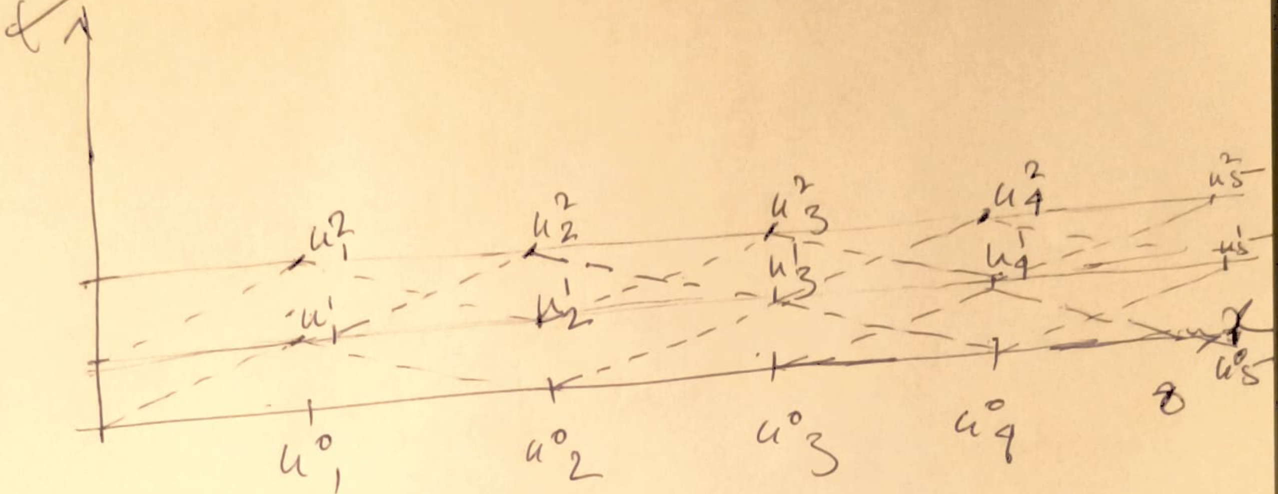
$u_2^2 = 0.62245$, $u_3^2 = 0.62245$

$u_4^2 = 0.3847.$

(8) Therefore,

$$u(0.6, 0.04) = u_3^2 = 0.6224,$$

the error in which is 0.0184.



Crank–Nicolson formula

Let $h = 0.2$ and $l = 0.04$

(ie) $\lambda = 1$.

For, $\lambda = 1$, Crank–Nicolson formula becomes,

$$-u_{i-1}^{K+1} + 4u_i^{K+1} - u_{i+1}^{K+1} = u_{i-1}^{K} + u_{i+1}^{K}. \quad \text{——(8)}$$

Putting, $K=0$ in (i), we obtain,

$$-u_{i-1}^1 + 4u_i^1 - u_{i+1}^1 = u_{i-1}^0 + u_{i+1}^0 \quad \text{——(9)}$$

9

Corresponding to $i = 1, 2, 3,$ and 4 we obtain the four equations,

$$4u'_1 - u'_2 = 0.9510,$$
$$-u'_1 + 4u'_2 - u'_3 = 1.5388,$$
$$-u'_2 + 4u'_3 - u'_4 = 1.5388,$$
$$-u'_3 + 4u'_4 = 0.9510.$$

By symmetry, we have,

$$u'_1 = u'_4 \text{ and } u'_2 = u'_3$$

Solving the above system, we obtain,

$$u'_2 = u'_3 = 0.6460.$$

Hence, $u(0.6, 0.04) \approx 0.6460$, the error in which is 0.0052.

10

Iterative methods for the solution of heat equation

Recall that in Crank–Nicolson method, the PDE,

$$\partial_t u = \partial_{xx} u$$

is replaced by the finite-difference equation.

$$(1+\lambda)\, u_{i,j+1} = u_{i,j} + \frac{1}{2}\lambda\left(u_{i-1,j+1} + u_{i+1,j} + u_{i-1,j} - 2u_{i,j}\right) \quad (10)$$

Here simply the notation u_i^K is replaced by u_{ij}

where, $\lambda = \frac{K}{h^2}$.

In eq^n (10), the unknowns are $u_{i,j+1}$, $u_{i-1,j+1}$ and $u_{i+1,j+1}$ and all others are known since they were already computed at the jth step. Hence dropping the j's and setting,

$$c_i = u_{i,j} + \frac{1}{2}\lambda\left(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right). \quad (11)$$

Eq^n (10) can be written as,

$$u_i = \frac{\lambda}{2(1+\lambda)}\left(u_{i-1} + u_{i+1}\right) + \frac{c_i}{1+\lambda} \quad (12)$$

11

From eqⁿ 12, we obtain the iteration formula,

$$u_i^{n+1} = \frac{\lambda}{2(1+\lambda)}\left[u_{i-1}^{n} + u_{i+1}^{n}\right] + \frac{c_i}{1+\lambda}, \quad \text{——— (13)}$$

which expresses the $(n+1)$th iterate in terms of the nth iterate only, and is known as Jacobi's formulae.

It can be seen from Eqⁿ (13) that at time of computing u_i^{n+1}, the latest value of u_{i-1}, namely u_{i-1}^{n+1} is already available. Hence the convergence of Jacobi's iteration formula can be improved by replacing u_{i-1}^{n} in formulae given in (13) by its latest value available, namely by u_{i-1}^{n+1}. Accordingly, we obtain the formulae,

$$u_i^{n+1} = \frac{\lambda}{2(1+\lambda)}\left[u_{i-1}^{n+1} + u_{i+1}^{n}\right] + \frac{c_i}{1+\lambda} \quad \text{——— (14)}$$

which is called the Gauss-Seidel iteration formulae. It can be shown that (14) converges for all finite values of λ and that it converges twice as fast as Jacob's scheme.

Eqⁿ 14 can be written as

~~$u_i^{n+1} = u_i^n + \{ u_i^n +$~~

$$u_i^{n+1} = u_i^n + \left\{ \frac{\lambda}{2(1+\lambda)} \left[u_{i-1}^{n+1} + u_{i+1}^n \right] + \frac{c_i}{1+\lambda} - u_i^n \right\}. \quad \text{—— (15)}$$

from which it is clear ~~that~~ the expression within the curly brackets is the difference between the nth and $(n+1)$th iterates.

Motivated by (15) we introduce the following iterative formulae,

$$u_i^{n+1} = u_i^n + \omega \left\{ \frac{\lambda}{2(1+\lambda)} \left[u_{i-1}^{n+1} + u_{i+1}^n \right] + \frac{c_i}{1+\lambda} - u_i^n \right\} \quad \text{—— (16)}$$

which is called the successive over-relaxation (or SOR) method. ω is called the relaxation factor and it lies, generally between 1 and 2.

Example:

$$\partial_t u = \partial_{xx} u.$$

subject to the initial condition,

$u = \sin \pi x$ at $t=0$ for $0 \le x \le 1$

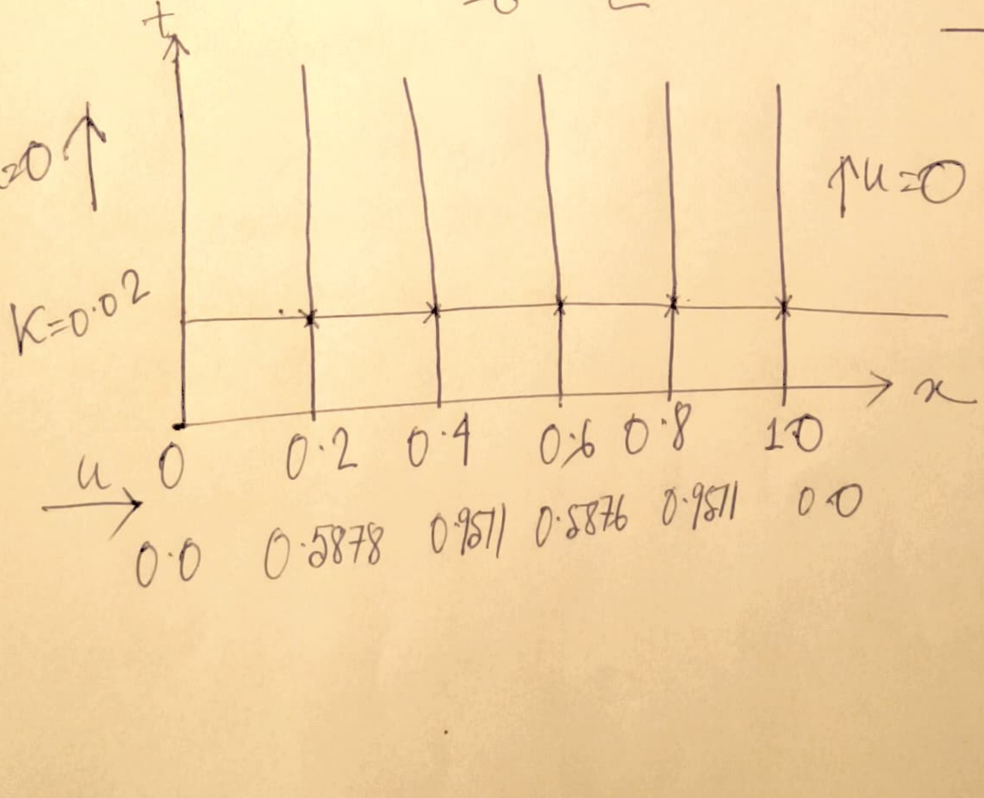
and $u=0$ at $x=0$ and $x=1$, for $t>0$

by the Gauss-Seidel method.

We choose $h = 0.2$ and $K = 0.02$ so that

$\lambda = \frac{K}{h^2} = \frac{1}{2}$. Eqn (14) therefore becomes,

$$u_i^{n+1} = \frac{1}{6}\left[u_{i-1}^{n+1} + u_{i+1}^{n}\right] + \frac{2}{3}c_i \quad \text{—(i)}$$



Wave Equation

The wave equation is defined by the boundary value problem.

$$\partial_t u = c^2 \partial_{xx} u \quad \text{--- (17)}$$

with the boundary conditions,

$$\left.\begin{aligned} u(x,0) &= f(x) \\ \partial_t u(x,0) &= \phi(x) \\ u(0,t) &= \psi_1(t) \\ u(1,t) &= \psi_2(t) \end{aligned}\right\} \text{--- (18)}$$

for, $0 < t \leq T$.

This equation is of hyperbolic type and models the tranverse vibration of a stretched string. As earlier, we use the following difference approximations for the derivatives,

$$\partial_{xx} u = \frac{1}{h^2}\left(u_{i-1}^{k} - 2u_i^{k} + u_{i+1}^{k}\right) + O(h^2) \quad \text{--- (19)}$$

and $$\partial_{tt} u = \frac{1}{l^2}\left(u_i^{k-1} - 2u_i^{k} + u_i^{k+1}\right) + O(l^2) \quad \text{--- (20)}$$

where, $x = ih$, $t = kl$

and $u(x,t) = u(ih, kl) = u_i^k$.

Further, $\partial_t u(x,t)$ is approximated by,

$$\partial_t u(x,t) = \frac{u_i^{K+1} - u_i^{K-1}}{2l} + O(l^2) \quad \text{--- (21)}$$

Substituting from eq's (19) and (20) in (17),; we obtain,

$$\frac{1}{l^2}\left(u_i^{K-1} - 2u_i^K + u_i^{K+1}\right) = \frac{c^2}{h^2}\left(u_{i-1}^K - 2u_i^K + u_{i+1}^K\right).$$

Setting, $\alpha = \frac{cl}{h}$ in the above and rearranging the terms, we have

$$u_i^{K+1} = -u_i^{K-1} + \alpha^2\left(u_{i-1}^K + u_{i+1}^K\right) + 2(1-\alpha^2)u_i^K \quad \text{--- (22)}$$

Eq^n (22) shows ~~that~~ the function values at the Kth and $(K-1)$th time levels are required to determine ~~those~~ at the $(K+1)$th time level. Such difference schemes are called three level difference schemes compared to the two level schemes derived in the parabolic case.

Formula (22) holds good if $\alpha < 1$, which is the condition for stability.

Example

Solve the eq^n $\partial_{tt} u = \partial_{xx} u$ subject to the following conditions

$$u(0,t) = 0, \quad u(1,t) = 0, \quad t > 0.$$

and

$$\partial_t u(x,0) = 0, \quad u(x,0) = \sin^3(\pi x),$$

$$0 \le x \le 1.$$

This problem has an exact solution given by,

$$u(x,t) = \frac{3}{4} \sin \pi x \cos \pi t - \frac{1}{4} \sin 3\pi x \sin 3\pi t.$$

Let $h = 0.25$ and $l = 0.2$.

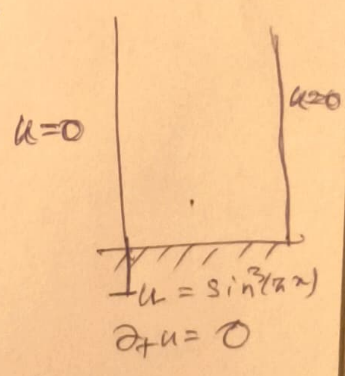
Then, $\alpha = 0.8 < 1$.

The given conditions are.

$$u_0^K = 0, \quad u_4^K = 0$$

$$u_i^0 = \sin^3(\pi i h).$$

$$i = 1, 2, 3, 4$$

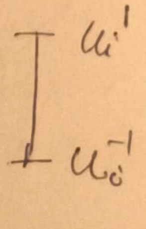


Also by divided difference,

$$\frac{\partial u(x,0)}{\partial t} = 0$$

$$\Rightarrow u_i^1 - u_i^{-1} = 0$$

$$\Rightarrow u_i^{-1} = u_i^1.$$



With $\alpha = 0.8$, the explicit formula becomes,

$$u_i^{K+1} = -u_i^{K-1} + 0.64\left(u_{i-1}^K + u_{i+1}^K\right) + 2(0.36)u_i^K.$$

Setting, $K=0$, the above gives:

$$u_i^1 = -u_i^{-1} + 0.64\left(u_{i-1}^0 + u_{i+1}^0\right) + 0.72u_i^0$$

$$\Rightarrow u_i^1 = 0.32\left(u_{i-1}^0 + u_{i+1}^0\right) + 0.36u_i^0$$

since, $u_i^{-1} = u_i^1$.

Therefore,

$$u_1^1 = 0.32\left(u_0^0 + u_2^0\right) + 0.36u_1^0.$$

$$= 0.32(0 + 1) + 0.36u_1^0$$

$$\left[\text{Since, } u_2^0 = \sin^3\left(\pi \cdot \tfrac{1}{2}\right) = 1\right]$$

$\cancel{= 0.32(}$

$$= 0.32 + 0.36(\underbrace{0.3537}_{\sin^3\left(\frac{\pi}{4}\right)})$$

$$= 0.4473 \quad (\text{error} = 0.0365)$$

Similarly, $u_2^1 = 0.5867$ (error $= 0.0571$)

and $u_3^1 = 0.4473$ (error $= 0.0365$)

The computations can be continued for, $K = 1, 2, \ldots$