Lecture 6
Numerical Methods - 2020

Taylor's Series Method

$\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$

$y(x) = y(x_0 + x - x_0)$
$= y(x_0 + h)$ let $x - x_0 = h$

$y(x_0) = y_0 + h y'(x_0) + \frac{h^2}{2!} y''(x_0) + \frac{h^3}{3!} y'''(x_0) + \cdots$ (1)

$[x_1 = x_0 + h]$

$x_2 = x_1 + h$

$y(x_2) = y(x_1) + h y'(x_1) + \frac{h^2}{2!} y''(x_1) + \frac{h^3}{3!} y'''(x_1) + \cdots$ (2)

where $y'(x_1) = f(x_1, y_1)$
$y'' = f_x + f_y f$
$y''' = f_{xx} + 2 f f_{xy} + f_x f_y + f_y^2 f$

In General

$y(x_{n+1}) = y(x_n) + h y'(x_n) + \frac{h^2}{2!} y''(x_n) + \frac{h^3}{3!} y'''(x_n) + \cdots$
$\cdots + \frac{h^p}{p!} y^{(p)}(x_n) + \text{error}(h)$

Here, ξ is Taylor's remainder.

$\xi = \frac{h^{p+1}}{(p+1)!} y^{(p+1)}(\xi)$ where $x_n \le \xi \le x_{n+1}$

$\Rightarrow \xi = \frac{h^{p+1}}{(p+1)!} y^{(p+1)}(\xi)$

$= \frac{h^{p+1}}{(p+1)!} M$ maximum possible value of $y^{(p+1)}(x)$

So we if we calculate upto $\frac{h^p}{p!} y^{(p)}(x_n)$ then error is $\frac{h^{p+1}}{(p+1)!} y^{(p+1)}(\xi)$

Euler's Method

Taylor's Series method is complicated and requires evaluating the derivatives of function. It is worth that reason

↳ Euler's Method is velocity step method by truncating $y', y'', etc.$

$\therefore y(x_0 + h) = y_0 + h f(x_0, y_0)$
$y_1 = y(x_1) = y_1 + h f(x_0, y_0)$
$y_2 = y(x_2) = y_1 + h f(x_1, y_1)$
$\vdots$
$\text{Error} = \frac{h^2}{2!} y''(x_n) \approx O(h^2)$

Geometry of Euler's Method

Approximate the value of function with tangent line (slope) at (x_n, y_n) & y_{n+1} are found

$M_1 = 3$
$M_2 = 2$
$M_3 = 0.5 = 1$

Z: Euler method error gets on increasing step by step

$\text{slope} = \left(\frac{dy}{dx}\right)_{(x_0, y_0)} = \text{tan}\,\theta = \frac{\Delta y}{\Delta x}$

$\Rightarrow \frac{\Delta y}{\Delta x} = M_1 - M_0$
$= M_1 - M_0$
$= 3 - 0$

$\left(\frac{dy}{dx}\right)_{(x_0, y_0)} = \frac{y_1 - y_0}{h}$

$\Rightarrow f(x_0, y_0) = \frac{y_1 - y_0}{h}$

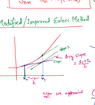
$\Rightarrow y_1 = y_0 + h f(x_0, y_0)$

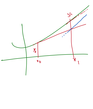
$\Delta y = y_1 - y_0$
$= 3 \times h$
$= 3 \times 0.2$
$= 0.6$

$\Delta x = h$

$\boxed{y_{n+1} = y_n + h f(x_n, y_n)}$

Modified/Improved Euler's Method





Slope are supposed (x_n, y_n) to (x_{n+1}, y_{n+1})

The actual approximate (x_n, y_n) & $(x_{n+1}, y_{n+1} + h f(x_n, y_n))$ → ①

$\therefore y_1 = y_0 + h\left[\frac{f(x_0, y_0) + f(x_1, y_0 + h f(x_0, y_0))}{2}\right]$

If is general

$$\boxed{y_{n+1} = y_n + h\left[\frac{f(x_n, y_n) + f(x_{n+1}, y_n + h f(x_n, y_n))}{2}\right]}$$

next step | present step | slope (average) | x_n & y_n slope | x_{n+1} & y_{n+1} slope (Euler's method)

Ex

Given $y' = \frac{dy}{dx} = xy$, $y(1) = 1$, and $h = 0.1$ approximate $y(1.2)$ using Improved Euler's Method.

$f(x, y) = xy$ $(x_0, y_0) = (1, 1)$ $x_1 = 1.1$
$(x_1, y_1) = (1.1, 2)$

$y_1 = y_0 + h\left[\frac{f(x_0, y_0) + f(x_1, y_0 + h f(x_0, y_0))}{2}\right]$

$y_1 = 1 + 0.1\left[\frac{(1)(1) + (1.1)(1 + 0.1(1)(1))}{2}\right]$

$y_1 = 1.1105$

$x_2 = 1.1$ $y_2 = 1.1105$, $x_2 = 1.2$

$y_2 = y_1 + h\left[\frac{f(x_1, y_1) + f(x_2, y_1 + h f(x_1, y_1))}{2}\right]$

$y_2 =$

$y_2 = 1.2345268$

$y_2 = 1.2345268 +$

$y_2 = 1.4114949$

$\rightarrow x_2 = 1.2$, $y_2 = 1.2475$

$\rightarrow x_3 = 1.3$, $y_3 = ?$

n	x_n	Euler y_n	Improved y_n	Actual
1	1.1	1.1	1.1105	1.1104
2	1.2	1.221	1.2475	1.2428
3	1.3	1.3431	1.4114	1.4198

Runge-Kutta Method (RK Method)

→ Euler's Simple Method

↳ $y(x_n + h) = y_n + h f(x_n, y_n)$

↳ 1st order RK Method

& error is $O(h^2)$

→ Improved Euler's Method is 2nd order RK Method

$y(x_n) = y(x_n + h) = y(x_n) + h\left[\frac{f(x_n, y_n) + f(x_n + h, y_n + h f(x_n))}{2}\right]$ ↳ ②

$k_2 = f(x_n + h, y_n + k_1)$

We may write this as

$k_1 = h f(x_n, y_n)$
$k_2 = h f(x_n + h, y_n + k_1)$

then ② ⇒

$y(x_n + h) = y(x_n) + h\left[\frac{k_1 + k_2}{2}\right]$

$= y(x_n) + h\left[\frac{k_1 + k_2}{2}\right]$

$\boxed{y(x_{n+1}) = y(x_n) + \frac{1}{2}(k_1 + k_2)}$

↳ This is called 2nd order Runge-Kutta Method.

The error in this method is $O(h^3)$

Third order R.K Method

Taking up to h^3 term of Taylor's Series and on simplification:

$k_1 = h f(x_n, y_n)$
$k_2 = h f\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$
$k_3 = h f(x_n + h, y_n - k_1 + 2k_2)$

Solution is

$y(x_n + h) = y_n + \frac{1}{6}(k_1 + 4k_2 + k_3)$

with error $O(h^4)$

4th order RK Method

Keeping up to h^4 in Taylor's Series and on simplification:

$k_1 = h f(x_n, y_n)$
$k_2 = h f\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$
$k_3 = h f\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$
$k_4 = h f(x_n + h, y_n + k_3)$

& $6th$ of D.F. is

$y(x_n + h) = y(x_n) + \frac{1}{6}\left[k_1 + 2(k_2 + k_3) + k_4\right]$

This formula has error $O(h^5)$.

Since h is small, h^5 is pretty small and this method gives very correct result.

Ex

Solve $\frac{dy}{dx} = 2x + y$, $y(1) = 2$

By 1st, 2nd, 3rd, 4th order RK Method at $x = 1.2$

Ans

$f(x, y) = 2x + y$

$x_0 = 1$, $y_0 = 2$

$y_1(x_1) = y_1(1.2)$

$= y_0 + h f(x_0, y_0)$

$= 2 + 0.2 f(1, 2)$

$= 2 + 0.2 \times 4$

$= 2.8$ → RK-order 1

$k_1 = 0.2 \times 4 = 0.8$

$k_2 = h f(x_0 + h_2, y_0 + k_1)$
$= 0.2 f(1.2, 2.8)$
$= 1.04$

RK-2

$y(1.2) = 2 + \frac{1}{2}(k_1 + k_2)$

$= 2 + \frac{1}{2}(0.8 + 1.04)$

$= 2.92$

RK-3

$k_1 = 0.8$

$k_2 = 0.92$

$k_3 = 1.088$

$y(1.2) = 2.928$

RK-4

$k_1 = 0.8$

$k_2 = 0.92$

$k_3 = 0.932$

$k_4 = 1.0664$