MA 203 Complex Analysis and Differential Equations-II

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Lecture-4

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Contents

- Classification of linear second order PDE's in two variables.
- Laplace equation using separation of variables.

Classification of linear second-order PDEs in two variables

- The general second-order linear PDE has the following form:

$$A\, u_{xx} + B\, u_{xy} + C\, u_{yy} + D\, u_x + E\, u_y + F\, u = G, \tag{1}$$

where the coefficients A, B, C, D, F and the free term G are in general functions of the independent variables x and y, but do not depend on the unknown function u.

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- The classification of second-order equations depends on the form of the leading part of the equations consisting of the second-order terms. So, for simplicity of notation, we combine the lower-order terms and rewrite the above equation in the following form

$$A\,u_{xx} + B\,u_{xy} + C\,u_{yy} + I(x, y, u, u_x, u_y) = 0. \tag{2}$$

Classification of linear second-order PDEs in two variables

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- The type of the above equation depends on the sign of the quantity

$$\Delta(x, y) = B^2(x, y) - 4A(x, y)\,C(x, y), \tag{3}$$

  which is called the *discriminant* for (2).

classification of second-order linear PDEs

The classification of second-order linear PDEs is given by the following.

Definition 1

At the point (x_0, y_0), the second-order linear PDE (2) is called

(i) *elliptic*, if $\Delta(x_0, y_0) < 0$

(ii) *parabolic*, if $\Delta(x_0, y_0) = 0$

(iii) *hyperbolic*, if $\Delta(x_0, y_0) > 0$

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Remark 1

1. For each of these categories, equation (2) and its solutions have distinct features.
2. In general, a second order equation may be of one type at a specific point, and of another type at some other point.
3. The terminology is motivated from the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

which—for A, B, C, D, E, F being constants—represents a conic section in the xy-plane and the different types of conic sections arising are determined by $B^2 - 4AC$.

Remark 2

1. The canonical examples of the elliptic, parabolic and hyperbolic PDEs are the two-dimensional Laplace equation, one-dimensional heat equation and one-dimensional wave equations, i.e.,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{(\textbf{Laplace equation})}$$

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{(\textbf{Heat equation})}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{(\textbf{Wave equation})}$$

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Let u denotes the dependent variable in a boundary value problem (BVP).

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Definition 4 (Robin condition)

A condition that prescribes the values of $hu + \partial u/\partial \hat{n}$ at boundary points is known as a *Robin condition*. Here, h is either a constant or a function of the independent variables.

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Definition 5 (Cauchy condition)

If a PDE in u is of second order with respect to one of the independent variables t (time) and if the values of both u and u_t are prescribed at $t = 0$, the boundary condition is known as a *Cauchy-type* condition with respect to t.

Laplace equation

Definition 6 (Two-dimensional Laplace equation)

The two-dimensional Laplace equation is given by

$$\nabla^2 u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{4}$$

Note: the Laplace equation is also referred to as the *potential equation*.

Boundary value problems for the Laplace equation

- The function u, in addition to satisfying the Laplace equation (4) in a bounded region D in the xy-plane, should also satisfy certain boundary conditions on the boundary ∂D of this region. Such problems are referred to as *boundary value problems* (*BVPs*) for the Laplace equation.

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- The function u, in addition to satisfying the Laplace equation (4) in a bounded region D in the xy-plane, should also satisfy certain boundary conditions on the boundary ∂D of this region. Such problems are referred to as *boundary value problems* (*BVPs*) for the Laplace equation.
- We shall consider the following two types of BVPs (or their combinations, in the sense that some boundary conditions in the problem are Dirichlet type while the others are of Neumann type; the problem in this case will neither be called as a Dirichlet problem nor a Neumann problem).

Boundary value problems for the Laplace equation

Definition 7 (Dirichlet problem)

The *Dirichlet problem* for the Laplace equation is to determine the function $\varphi(x, y)$ continuous on D such that $\nabla^2\varphi(x, y) = 0$ within D and $\varphi(x, y) = f(x, y)$ on ∂D, where $f(x, y)$ is a given function.

Definition 8 (Neumann problem)

The *Neumann problem* for the Laplace equation is to determine the function $\varphi(x, y)$ continuous on D such that $\nabla^2\varphi(x, y) = 0$ within D and $\partial\varphi(x, y)/\partial\hat{n} = f(x, y)$ on ∂D, where $f(x, y)$ is a given function, $\hat{n}$ denotes the unit outward normal to the boundary ∂D, and $\partial\varphi/\partial\hat{n}$ denotes the directional derivative or the derivative in the direction of $\hat{n}$.

Solution by the method of separation of variables

- We first solve the Laplace equation (4) without considering the boundary conditions.
- Let $u(x, y) = X(x)\, Y(y)$ be a solution of the Laplace equation (4).

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- Substituting this solution in (4), we obtain

$$X''(x)\,Y(y) + X(x)\,Y''(y) = 0,$$

  where prime denotes the derivative with respect to an independent variable,
- i.e. $X''(x) = \dfrac{\mathrm{d}^2 X}{\mathrm{d}x^2}$ and $Y''(y) = \dfrac{\mathrm{d}^2 Y}{\mathrm{d}y^2}$.

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- i.e. $X''(x) = \dfrac{\mathrm{d}^2 X}{\mathrm{d}x^2}$ and $Y''(y) = \dfrac{\mathrm{d}^2 Y}{\mathrm{d}y^2}$.
- The above equation can be rewritten as

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}.$$

Solution by the method of separation of variables

- The left-hand side of the above equation is a function of x alone whereas the right-hand side is a function of y alone. Therefore, the left- and right-hand sides of the above equation must be constant. Let this constant be k.

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- From this equation, we obtain two *ordinary* differential equations (ODEs), namely

$$\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} - kX = 0 \qquad \text{and} \qquad \frac{\mathrm{d}^2 Y}{\mathrm{d}y^2} + kY = 0. \tag{5}$$

In order to solve these ODEs, there are three possibilities for k. It can be zero, positive or negative. In the following, we shall consider these three cases separately.

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The above equations yield the solution

$$X(x) = c_1 x + c_2 \quad \text{and} \quad Y(y) = c_3 y + c_4,$$

where c_1, c_2, c_3, c_4 are the integration constants and will be determined from the boundary conditions.

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$$\boxed{u(x, y) = (c_1 x + c_2)(c_3 y + c_4)} \tag{6}$$

Case 2: $k > 0$ Let $k = \lambda^2$ and $\lambda > 0$. In this case, eqs. (5) reduce to

$$\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} - \lambda^2 X = 0 \qquad \text{and} \qquad \frac{\mathrm{d}^2 Y}{\mathrm{d}y^2} + \lambda^2 Y = 0.$$

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The above equations yield the solution

$$X(x) = c_5 \mathrm{e}^{\lambda x} + c_6 \mathrm{e}^{-\lambda x} \quad \text{and} \quad Y(y) = c_7 \cos \lambda y + c_8 \sin \lambda y, \tag{7}$$

where c_5, c_6, c_7, c_8 are the integration constants and will be determined from the boundary conditions.

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$$\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} + \lambda^2 X = 0 \qquad \text{and} \qquad \frac{\mathrm{d}^2 Y}{\mathrm{d}y^2} - \lambda^2 Y = 0.$$

The above equations yield the solution

$$X(x) = c_9 \cos \lambda x + c_{10} \sin \lambda x \quad \text{and} \quad Y(y) = c_{11}\mathrm{e}^{\lambda y} + c_{12}\mathrm{e}^{-\lambda y}, \tag{9}$$

where $c_9, c_{10}, c_{11}, c_{12}$ are the integration constants and will be determined from the boundary conditions.

Case 3: $k < 0$ Let $k = -\lambda^2$ and $\lambda > 0$. In this case, eqs. (5) reduce to

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$$\boxed{u(x,y) = (c_9 \cos \lambda x + c_{10} \sin \lambda x)\left(c_{11}\mathrm{e}^{\lambda y} + c_{12}\mathrm{e}^{-\lambda y}\right).} \tag{10}$$

Dirichlet problem for a rectangle

Find the solution of the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{11}$$

satisfying the boundary conditions

$$\begin{aligned} &u(x,0) = 0, \qquad && u(x,b) = 0, \qquad && \text{for} \quad 0 < x < a \\ &u(0,y) = 0, \qquad && u(a,y) = f(y) \qquad && \text{for} \quad 0 < y < b. \end{aligned}$$

Solution

Let $u(x, y) = X(x)\, Y(y)$ be a solution of the Laplace equation (11). Then, as shown above, we obtain two ODEs from the Laplace equation (11), which read

$$\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} - kX = 0 \qquad \text{and} \qquad \frac{\mathrm{d}^2 Y}{\mathrm{d}y^2} + kY = 0. \tag{12}$$

We shall consider the three cases, namely $k = 0$, $k < 0$ and $k > 0$.

Case 1: $k = 0$ In this case, we obtain the solution of the Laplace equation (11)

$$u(x, y) = (c_1 x + c_2)(c_3 y + c_4). \tag{13}$$

We now compute the integration constants c_1, c_2, c_3, c_4 using the given boundary conditions.

$$u(x, 0) = 0 \quad \Longrightarrow \quad (c_1 x + c_2) c_4 = 0 \quad \Longrightarrow \quad \boxed{c_4 = 0}$$

since $c_1 x + c_2$ cannot be zero for all $x \in (0, a)$.

$$u(0, y) = 0 \quad \implies \quad c_2(c_3 y + c_4) = 0 \quad \implies \quad \boxed{c_2 = 0}$$

since $c_3 y + c_4$ cannot be zero for all $y \in (0, b)$. Therefore, with these two boundary conditions the solution of the Laplace equation (11) in this case reduces to

$$u(x, y) = c_1 c_3 xy. \tag{14}$$

Now, let us apply the remaining boundary conditions.

$$u(x, b) = 0 \quad \implies \quad c_1 c_3 xb = 0 \quad \implies \quad c_1 c_3 x = 0$$

since $b \neq 0$. With this, the solution (14) further reduces to

$$u(x, y) = 0,$$

which cannot be the general solution of the Laplace equation with the given boundary conditions. Therefore, $k = 0$ is not possible.

Case 2: $k < 0$

- Let $k = -\lambda^2$ and $\lambda > 0$. In this case, we obtain the solution of the Laplace equation (11)

$$u(x, y) = (c_9 \cos \lambda x + c_{10} \sin \lambda x)\big(c_{11} e^{\lambda y} + c_{12} e^{-\lambda y}\big). \tag{15}$$

We now compute the integration constants $c_9, c_{10}, c_{11}, c_{12}$ using the given boundary conditions.

$$u(x, 0) = 0 \quad \Longrightarrow \quad (c_9 \cos \lambda x + c_{10} \sin \lambda x)(c_{11} + c_{12}) = 0 \quad \Longrightarrow \quad \boxed{c_{12} = -c_{11}}$$

since $c_9 \cos \lambda x + c_{10} \sin \lambda x$ cannot be zero for all $x \in (0, a)$.

Case 2: $k < 0$

$$u(0, y) = 0 \quad \Longrightarrow \quad c_9\left(c_{11}\mathrm{e}^{\lambda y} + c_{12}\mathrm{e}^{-\lambda y}\right) = 0 \quad \Longrightarrow \quad \boxed{c_9 = 0}$$

since $c_{11}\mathrm{e}^{\lambda y} + c_{12}\mathrm{e}^{-\lambda y}$ cannot be zero for all $y \in (0, b)$. Therefore, with these two boundary conditions the solution of the Laplace equation (11) in this case reduces to

$$u(x, y) = c_{10}c_{11} \sin \lambda x\left(\mathrm{e}^{\lambda y} - \mathrm{e}^{-\lambda y}\right). \tag{16}$$

Now, let us apply the remaining boundary conditions.

$$u(x, b) = 0 \quad \Longrightarrow \quad c_{10}c_{11} \sin \lambda x\left(\mathrm{e}^{\lambda b} - \mathrm{e}^{-\lambda b}\right) = 0 \quad \Longrightarrow \quad c_{10}c_{11} \sin \lambda x = 0$$

since $\mathrm{e}^{\lambda b} - \mathrm{e}^{-\lambda b} \neq 0$ (as $\lambda, b \neq 0$). With this, the solution (16) further reduces to

$$u(x, y) = 0,$$

which cannot be the general solution of the Laplace equation with the given boundary conditions. Therefore, $k < 0$ is also not possible.

Case 3: $k > 0$

Let $k = \lambda^2$ and $\lambda > 0$. In this case, we obtain the solution of the Laplace equation (11)

$$u(x, y) = \left(c_5 e^{\lambda x} + c_6 e^{-\lambda x}\right)(c_7 \cos \lambda y + c_8 \sin \lambda y). \tag{17}$$

We now compute the integration constants c_5, c_6, c_7, c_8 using the given boundary conditions.

$$u(x, 0) = 0 \quad \Longrightarrow \quad \left(c_5 e^{\lambda x} + c_6 e^{-\lambda x}\right) c_7 = 0 \quad \Longrightarrow \quad \boxed{c_7 = 0}$$

since $c_5 e^{\lambda x} + c_6 e^{-\lambda x}$ cannot be zero for all $x \in (0, a)$.

$$u(0, y) = 0 \quad \Longrightarrow \quad (c_5 + c_6)(c_7 \cos \lambda y + c_8 \sin \lambda y) = 0 \quad \Longrightarrow \quad \boxed{c_6 = -c_5}$$

since $c_7 \cos \lambda y + c_8 \sin \lambda y$ cannot be zero for all $y \in (0, b)$.

Therefore, with these two boundary conditions the solution of the Laplace equation (11) in this case reduces to

$$u(x, y) = c_5 c_8 \big(\mathrm{e}^{\lambda x} - \mathrm{e}^{-\lambda x}\big) \sin \lambda y = A\big(\mathrm{e}^{\lambda x} - \mathrm{e}^{-\lambda x}\big) \sin \lambda y, \tag{18}$$

where $A = c_5 c_8$ is another constant. Now, let us apply the remaining boundary conditions.

$$u(x, b) = 0 \quad \Longrightarrow \quad A\big(\mathrm{e}^{\lambda x} - \mathrm{e}^{-\lambda x}\big) \sin \lambda b = 0 \quad \Longrightarrow \quad \boxed{\sin \lambda b = 0}$$

since $\mathrm{e}^{\lambda x} - \mathrm{e}^{-\lambda x}$ cannot be zero for all $x \in (0, a)$ and $A \neq 0$ for a nontrivial solution. This yields

$$\sin \lambda b = \sin n\pi \quad \Longrightarrow \quad \lambda = \frac{n\pi}{b}, \qquad n = 1, 2, 3, \ldots$$

Note that we have not taken $n = 0$ because $n = 0$ will yield $\lambda = 0$ whereas $\lambda \neq 0$ in this case. With this, the solution (18) becomes

$$u(x, y) = A\left(\mathrm{e}^{\frac{n\pi x}{b}} - \mathrm{e}^{-\frac{n\pi x}{b}}\right) \sin \frac{n\pi y}{b}, \qquad n = 1, 2, 3, \ldots$$

Neglecting the constant of proportionality A, we conclude that functions

$$u_n(x, y) = \left(\mathrm{e}^{\frac{n\pi x}{b}} - \mathrm{e}^{-\frac{n\pi x}{b}}\right) \sin \frac{n\pi y}{b}, \qquad n = 1, 2, 3, \ldots \tag{19}$$

satisfy the Laplace equation (11) and the above three boundary conditions. The functions in (19) are referred to as the *eigenfunctions* for the given problem.

Therefore, the most general solution is given by

$$u(x, y) = \sum_{n=1}^{\infty} A_n \left(\mathrm{e}^{\frac{n\pi x}{b}} - \mathrm{e}^{-\frac{n\pi x}{b}} \right) \sin \frac{n\pi y}{b}.$$

Now we shall use the final given boundary condition.

$$u(a, y) = f(y) \quad \Longrightarrow \quad \sum_{n=1}^{\infty} A_n \left(\mathrm{e}^{\frac{n\pi a}{b}} - \mathrm{e}^{-\frac{n\pi a}{b}} \right) \sin \frac{n\pi y}{b} = f(y),$$

which is a Fourier sine series for $f(y)$, $0 < y < b$. Therefore, the Fourier coefficients in the series are given by

$$A_n \left(\mathrm{e}^{\frac{n\pi a}{b}} - \mathrm{e}^{-\frac{n\pi a}{b}} \right) = \frac{2}{b} \int_0^b f(y) \sin \frac{n\pi y}{b} \, \mathrm{d}y,$$

which gives

$$A_n = \frac{2}{b} \frac{1}{\mathrm{e}^{\frac{n\pi a}{b}} - \mathrm{e}^{-\frac{n\pi a}{b}}} \int_0^b f(y) \sin \frac{n\pi y}{b} \, \mathrm{d}y.$$

Therefore, the solution of the given problem is

$$\boxed{u(x,y) = \frac{2}{b}\sum_{n=1}^{\infty} \frac{\mathrm{e}^{\frac{n\pi x}{b}} - \mathrm{e}^{-\frac{n\pi x}{b}}}{\mathrm{e}^{\frac{n\pi a}{b}} - \mathrm{e}^{-\frac{n\pi a}{b}}} \left(\int_0^b f(y) \sin \frac{n\pi y}{b} \, \mathrm{d}y \right) \sin \frac{n\pi y}{b}}$$

or

$$\boxed{u(x,y) = \frac{2}{b}\sum_{n=1}^{\infty} \frac{\sinh \frac{n\pi x}{b}}{\sinh \frac{n\pi a}{b}} \sin \frac{n\pi y}{b} \int_0^b f(y) \sin \frac{n\pi y}{b} \, \mathrm{d}y}$$