INDIAN INSTITUTE OF TECHNOLOGY INDORE

MA 203: Complex Analysis and Differential Equations-II

Autumn Semester 2023

Tutorial −3 (Differential Equations-II)

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1. Show that
   (a) $J_n(x) = \frac{1}{2}\int_0^{\pi} \cos(n\theta - x\sin\theta)d\theta$, n being an integer
   (b) $J_0^2 + 2J_1^2 + 2J_2^2 + 2J_3^2 + \cdots = 1$
2. Prove the following identities:
   (a) $|J_0(x)| \leq 1$
   (b) $|J_n(x)| \leq 2^{-1/2}$, when $n \geq 1$
3. Express J_5 in terms of $J_0(x)$ and $J_1(x)$
4. Prove that
   (a) $P_n(-1) = (-1)^n$
   (b) $P_n'(-1) = (-1)^{n-1} \times \frac{1}{2}n(n+1)$
   (c) $P_{2m}(0) = (-1)^m \frac{(2m)!}{2^{2m}(m!)^2}$
5. Show that for any function $f(x)$, for which the $n-th$ derivative is continuous,
$$\int_{-1}^{1} f(x)P_n(x)dx = \frac{1}{2^n n!}\int_{-1}^{1}(1-x^2)^n f^n(x)dx$$
6. Show that
$$\int_{-1}^{1} xP_n(x)P_{n-1}(x)dx = \frac{2n}{4n^2-1}$$
7. Show that
$$\int_{-1}^{1}(1-x^2)P_m'(x)P_n'(x)dx = \begin{cases} 0, & \text{when } m \neq n \\ \frac{2n(n+1)}{2n+1} & \text{when } m = n \end{cases}$$
8. Prove that all the roots of $P_n(x)$ are distinct.
9. Express $f(x) = 4x^3 - 2x^2 - 3x + 8$ in terms of the linear combination of the Legendre polynomials.
10. Let $P_n(x)$ be the Legendre polynomial of degree n.
    Prove that $P_n(-x) = (-1)^n P_n(x)$
    Hence, conclude that $P_n(-1) = 1$ if n is even and $P_n(-1) = -1$ if n is odd.

11. Using Rodrigue's formula, show that $P_n(x)$ satisfies

$$\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}P_n(x)\right]+n(n+1)P_n(x)=0$$

12. Express the Lagrange differential equation

$$xy''+(1-x)y'+ny=0$$

$x \neq 0$ in the from of Sturm-Liouville equation.

13. Reduce each of the following differential equations to the Sturm-Lioville equation from indicating the weight function $P(x)$

(a) $(1-x^2)y''-xy'+n^2y=0$

(b) $y''-2xy'+2ny=0$

(c) $xy''+2y'+(x+\lambda)y=0$

14. Determine the eigenvalues and eigenfunctions of the following Sturm-Liouville problems:

(a) $\frac{d}{dx}[x^2y']+\lambda y=0, \quad y(1)=0, \quad y(b)=0, \quad 1<x<b$

(b) $\frac{d}{dx}[xy']+\frac{\lambda}{x}y=0, \quad y(1)=0, \quad y(e^\pi)=0.$

15. Show that the eigenvalues for the BVP

$$y''+\lambda y=0; \quad y(0)=0, \quad y(\pi)+y'(\pi)=0$$

satisfies the equation $\sqrt{\lambda}=-\tan\pi\sqrt{\lambda}$. Prove that the corresponding eigen functions are $\sin(x\sqrt{\lambda_n})$, when λ_n is an eigenvalue.

16. Find all the eigenvalues and eigenfunctions of

$$4(e^{-xy'})'+(1+\lambda)e^{-x}y=0; \quad y(0)=0, \quad y(1)=0.$$