Legendre Polynomials

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \quad \text{—— (1)}$$

↳ Legendre Differential Equation
by Adrien Marie Legendre (1752 - 1833)

$$y = \sum_{m=0}^{\infty} a_m x^m \quad \text{—— (2)}$$

$$P(x) = -\frac{2x}{1-x^2} \qquad Q = \frac{n(n+1)}{1-x^2}$$

$x=0 \longrightarrow$ O.P.

$$y = a_0\left(1 - \frac{n(n+1)}{\lfloor 2}x^2 + \cdots\right) + a_1\left(x - \frac{(n-1)(n+2)}{\lfloor 3}x^3 + \cdots\right)$$

Putting $x = \frac{1}{z} \Rightarrow z = \frac{1}{x}$

$$\frac{dy}{dx} = -z^2\frac{dy}{dz} \quad \text{—— (3a)}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dz}\left(-z^2\frac{dy}{dz}\right)\frac{dz}{dx}$$

$$= z^2\left(2z\frac{dy}{dz} + z^2\frac{d^2y}{dz^2}\right) \quad \text{—— (3b)}$$

Eq(1) becomes

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

$\Rightarrow$

$\Rightarrow$

$$\Rightarrow (z^4 - z^2)\frac{d^2y}{dz^2} + 2z^3\frac{dy}{dz} + n(n+1)y = 0 \quad \text{—— (4)}$$

$z\,P(z)$

$z^2 Q(z)$ $\qquad z=0$ is an regular singular point of (4)

then the

$$\left.\begin{aligned} y &= \sum_{m=0}^{\infty} c_m z^{\rho+m}, \quad c_0 \neq 0 \\ y' &= \sum (\rho+m)\, c_m z^{\rho+m-1} \\ y'' &= \sum (\rho+m)(\rho+m-1)\, c_m z^{\rho+m-2} \end{aligned}\right\} \quad \text{—— (5)}$$

I.E $= -c_0(\rho+n)(\rho-n-1)$

$$(z^4 - z^2)\sum(\rho+m)(\rho+m-1)z^{\rho+m-2} + 2z^3\sum(\rho+m)c_m z^{\rho+m-1} + n(n+1)\sum c_m z^{\rho+m} = 0 \quad \text{—— (6)}$$

Collecting the coefficient of z^{ρ}, we have

$$C_0 (\rho)(\rho-1) + n(n+1) C_0$$

$$= C_0 \left(-\rho^2+1+n(n+1)\right)$$

$$= -C_0 (\rho+n)(\rho-n-1) \quad \text{—— (7)}$$

Collecting the coeff of $z^{\rho+1}$ and equating to zero

$$-C_1 (\rho+1)(\rho) + n(n+1) C_1 = 0$$

$$\Rightarrow \quad C_1 \left(n(n+1) - (\rho+1)\rho\right) = 0$$

$$\Rightarrow \quad C_1 = 0$$

Coefficient of $z^{\rho+2} = 0$

$$C_0 (\rho)(\rho-1) - C_2 (\rho+2)(\rho+1) + 2C_0(\rho) + n(n+1) C_2 = 0$$

$$\Rightarrow C_0 (\rho)(\rho-1) - C_2 (\rho+2)(\rho+1) + 2C_0(\rho) + n(n+1) C_2 = 0$$

$$\Rightarrow \quad C_2 = \frac{(\rho^2+\rho)}{(\rho+1)(\rho+2) - n(n+1)} C_0 \quad \text{—— (8)}$$

or + (check)

Coff of $z^{\rho+m} = 0$

$$C_m = \frac{(\rho+m-2)(\rho+m-1)}{(\rho+m)(\rho+m-1) - n(n+1)} C_{m-2}$$

$$C_3 = \frac{(\rho+1)(\rho+2)}{(\rho+3)(\rho+2) - n(n+1)} C_1 = 0$$

$$C_1 = C_3 = C_5 = \cdots = 0$$

$$C_4 = \frac{(\rho+2)(\rho+3)}{(\rho+4)(\rho+3) - n(n+1)} C_2$$

$$= \frac{\rho(\rho+1)(\rho+2)(\rho+3)}{\{(\rho+1)(\rho+2) - n(n+1)\}\{(\rho+4)(\rho+3) - n(n+1)\}} C_0$$

From indicial equation (7)

$$\rho = -n, \; n+1$$

$$y = C_0 z^{\rho} \left[1 + \frac{\rho(\rho+1)}{(\rho+2)(\rho+1) - n(n+1)} z^2 + \frac{\rho(\rho+1)(\rho+2)(\rho+3)}{[(\rho+1)(\rho+2) - n(n+1)][(\rho+3)(\rho+4) - n(n+1)]} z^4 + \cdots\cdots \right] \quad \text{—— (A)}$$

Putting $\rho = -n$ in (A)

$$y_1 = c_0 z^{-n}\left[1 - \frac{n(n-1)}{2(2n-1)} z^2 + \frac{n(n-1)(n-2)(n-3)z^4}{2\cdot 4\cdot (2n-1)\ (2n-3)} - \cdots\right]$$

$$y_1 = C_0 x^n\left[1 - \frac{n(n-1)}{2(2n-1)} x^{-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4\cdot (2n-1)(2n-3)} x^{-4} - \cdots\right] \quad \text{— (I)}$$

Substituting $\rho = (n+1)$ in (A)

$$y_2 = C_0 z^{n+1}\left[1 + \frac{(n+1)(n+2)}{1\cdot 2\cdot (2n+3)} z^2 + \cdots\right]$$

$x = \frac{1}{z}$

$$y_2 = C_0 x^{-(n+1)}\left[1 - \frac{(n+1)(n+2)}{1\cdot 2\,(2n+3)} x^{-2} + \frac{(n+1)(n+2)(n+3)(n+4)}{2\cdot 4\cdot\ (2n+3)\ (2n+5)} x^{-4} - \cdots\right]$$

The general sol^n is

$$y = Ay_1 + By_2 = A\,P_n(x) + B\,Q_n(x)$$

If we consider in (I) as

$$C_0 = \frac{\lfloor 2n}{2^n (\lfloor n)^2}$$

$$= \frac{1.2.3.\ \cdots (2n-1)(2n)}{2^n (\lfloor n)^2}$$

$$= \frac{\{1.3.5.\cdots(2n-1)\}\{2.4.6.\cdots 2n\}}{2^n (\lfloor n)^2}$$

$$= \frac{\{1.3.5\cdots(2n-1)\}\,\cancel{2^n}\,\cancel{\lfloor n}}{\cancel{2^n}(\lfloor n)^{\cancel{2}}}$$

$$2.4.6\cdots 2n$$
$$= 2.(2.2).(2.3)\cdots($$
$$= 2^n(1.2.3.\cdots)$$
$$= 2^n \lfloor n$$

$$= \frac{1.3.5.\ \cdots (2n-1)}{\lfloor n}$$

From (I), y_1 can be express as

$$P_n(x) = \frac{1.3.5.\cdots(2n-1)}{\lfloor n}\left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)\ (2n-3)} x^{n-4} - \cdots\right]$$

This is a Legendre polynomial of order n.

If $C_0 = \frac{\lfloor n}{1.3.5.\cdots(2n+1)}$

If $C_0 = \dfrac{\cdots}{1\cdot 3\cdot 5\cdots(2n+1)}$

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \dfrac{1\cdot 3}{\lfloor 2}\left(x^2 - \dfrac{2}{2\cdot 3}\right)$

$$\boxed{P_n(x) = \sum_{r=0}^{N} \frac{(-1)^r\,(2n-2r)!}{2^n\, r!\,(n-r)!\,(n-2r)!}\, x^{n-2r}}$$

where $N = \dfrac{n}{2}$ if n is even

$N = \dfrac{n-1}{2}$ if n is odd.