Lecture-5A

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$\frac{dy}{dx} = x+y \quad , \; y(1) = -1 \qquad$ (IVP)

$\frac{d^2y}{dx^2} - x\frac{dy}{dx} + x^2 y = \cos x \quad , \; y(0)=0$ & $y(3)=6 \longrightarrow$ (IVP)

$y^{n}(x) = F(x, y, y^{(1)}(x), y^{(2)}(x), \ldots, y^{(n-1)}(x))$

$y(x_0) = y_0 \; , \; y'(x_0) = y_0' \; , \; y^{(2)}(x_0) = y_0'' \ldots$

Semianalytical :

Example $\quad \frac{dy}{dx} = 1+y^2 \quad , \; y(0) = 0 \longrightarrow$ (IVP) $\quad$ (*)

Sol $\quad \frac{dy}{dx} = (1+y^2) \qquad y(0) = 0$

Integrating (*) from x_0 to x

$$\int_{x_0}^{x} dy = \int_{x_0}^{x} (1+y^2)\,dx$$

$$\Rightarrow y(x) - y(x_0) = \int_{x_0}^{x} (1+y^2)\,dx$$

$$\Rightarrow y(x) = y(x_0) + \int_{x_0}^{x} (1+y^2)\,dx \quad \text{——} \quad (**)$$

In order to solve (**), we suggest an iteration method

Let $y^{(0)}(x) = y(x_0) = y_0$

$$y(x) = y(x_0) + \int_0^x (1+y_0^2)\,dx$$

$$= 0 + \int_0^x (1+0)\,dx = x$$

$y_1(x) = x$

Similarly $\quad y_2(x) = y(x_0) + \int_0^x (1+y_1^2)\,dx$

$$= 0 + \int_0^x (1+x^2)\,dx = x + \frac{x^3}{3}$$

at $x = 0.2$

$= 0.0123$

$$y_3(x) = y(x_0) + \int_0^x (1+y_2^2)\,dx$$

$$= 0 + \int_0^x \left[1 + \left(x + \frac{x^3}{3}\right)^2\right] dx$$

$$= \int_0^x \left[1 + x^2 + \frac{x^6}{9} + \frac{2}{3}x^4\right] dx$$

$$= x + \frac{x^3}{3} + \frac{x^7}{7 \cdot 9} + \frac{2}{3}\frac{x^5}{5}$$

at $x = 0.2$

$= 0.0210$

0.0232

→ Picard's Method of successive approximation

$$|y_{(k+1)}(x) - y_{(k)}(x)| \le \varepsilon \quad \text{(preassigned)}$$

then we conclude that the method Convergence

eg

$$|y_{(5)}(0.3) - y_{(4)}(0.3)| \le 0.324 \quad \text{(say)}$$

But

$$|y_{(8)}(0.3) - y_{(7)}(0.3)| \le 0.00359$$

Single-step Method

$y' = \frac{dy}{dx} = f(x,y) \quad , \; y(x_0) = y_0$

$y_n, y_n', h \rightarrow$ [input] $\rightarrow$ [y_{n+1}] output

Process: $\quad y(x_{n+1}) = F(x_n, y_n, y_n', h)$

↳ processes is depending only one past value

Ex $\quad y_{n+1} = y_n + x_n h$

Ex $\quad y_{n+1} = 2y_n - 3y_n'$

Taylor Series Method

Consider $\quad \frac{dy}{dx} = f(x,y) \quad \text{——} \quad (1)$

$y(x_0) = y_0 \quad , \; x \in [x_0, b]$

Assumption:

The eq (1) has a unique solⁿ $y(x)$ on $[x_0, b]$ and $y(x)$ has continuous partial derivatives of order say $(p+1)$ on $[x_0, b]$, $p \geq 1$.

Solution $y(x)$ of (1) can be expanded in a Taylor's Series about any point say $x = x_0$

$$y(x) = y(x_0 + x - x_0)$$

$$= y(x_0) + (x-x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \cdots$$

$$\cdots + \frac{(x-x_0)^p}{p!}y^{(p)}(x_0) + \frac{(x-x_0)^{p+1}}{(p+1)!}y^{(p+1)}(\xi_n) \quad \text{——} \quad (2)$$

$$\boxed{x_0 < \xi_n < x}$$

$$R_n = \frac{(x-x_0)^{p+1}}{(p+1)!}y^{(p+1)}(\xi_n)$$

If we generalized

$$y(x_{n+1}) = y(x_n) + h y'(x_n) + \frac{h^2}{2!}y''(x_n) + \cdots + \frac{h^p}{p!}y^{(p)}(x_n) + R_n$$

Remark:

The higher order derivatives are to be computed from the IVP.

$y' = f(x,y)$

assumption: $f(x,y)$ is differentiable as many times as we require.

$y'(x_n) = f(x_n, y_n)$

$y'(x) = f(x,y)$

$$y'' = \frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx} = \frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}$$

$$y''(x_n) = \left[\frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}\right]_{(x_n, y_n)}$$

$$y(x_{n+1}) = y(x_n) + h f(x_n, y_n) + \frac{h^2}{2}\left[\left(\frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}\right)_{x_n, y_n}\right]$$

$$+ \cdots + \frac{h^p}{p!}f^{(p-1)}(x_n, y_n) + \varepsilon_{p+1} \quad \text{——} \quad (\times)$$

Residual/Error

Note

$$\varepsilon_{p+1} = \frac{h^{p+1}}{(p+1)!}y^{(p+1)}(\xi_n)$$

$$= \frac{h^{n+1}}{(n+1)!}f^{n}(\xi, y(\xi)) \quad , \; n = p$$

$x - x_0 = h$

$$\Rightarrow y(x) = y(x_0) + h y'(x_0) + \frac{h^2}{2!}y''(x_0) \; \cdots +$$

$$y'(x_0) = \frac{dy(x_0)}{dx} = f(x_0, y_0)$$

$$y''(x_0) = f'(x_0, y_0)$$

$$y'''(x_0) = f''(x_0, y_0)$$

$y' = f$

$y'' = f_x + f_y f$

$$y''' = f_{xx} + 2f f_{xy} + f^2 f_{yy} + f_x f_y + f f_y^2 \Big]_{(x_0, y_0)}$$

$f_x = \frac{\partial f}{\partial x}$

$f_y = \frac{\partial f}{\partial y}$

$f_{xy} = \frac{\partial^2 f}{\partial x \partial y}$

$$y(x_1) = y(x_0 + h)$$

$$y(x_1) = y_0 + h f(x_0, y_0) + \frac{h^2}{2}\left[f_x(x_0,y_0) + f_y(x_0,y_0) f(x_0,y_0)\right]$$

$$+ \frac{h^3}{3!}\left[\quad\right]$$

$h = x_1 - x_0$

$\Rightarrow x_1 = x_0 + h$

$\Rightarrow x_2 = x_1 + h$ or $= x_0 + 2h$

$$y(x_2) = y(x_1) + h y'(x_1) + \frac{h^2}{2!}y''(x_1) + \cdots$$