06/12/24

Santanu sir

1) $$\int_0^h f(x)\,dx = h\{a f(0) + b f(\tfrac{h}{3}) + c f(h)\}$$

Choose: $f(x) = 1, x, x^2$

for $f(x) = 1$, $$\int_0^h f(x)\,dx = \int_0^h 1\,dx = h$$

$$= h\{a f(0) + b f(\tfrac{h}{3}) + c f(h)\}$$

$$= h\{a+b+c\}$$

$\therefore$ $h = h\{a+b+c\}$

$$\boxed{a+b+c=1} \quad -(1)$$

for $f(x) = x$: $$\int_0^h f(x)\,dx = \int_0^h x\,dx = \frac{h^2}{2}$$

RHS: $$h\{\cancel{a}^{0} + \frac{bh}{3} + ch\}$$

$\therefore$ $$\frac{h^2}{2} = h(\cancel{a}^{0} + \frac{bh}{3} + ch)$$

$$\frac{1}{2} = \frac{b}{3} + c \quad -(2)$$

for $f(x) = x^2$: $$\int_0^h f(x)\,dx = \int_0^h x^2\,dx = \frac{h^3}{3}$$

$$h\left(a(0) + b\cdot\frac{h^2}{9} + ch^2\right) = \frac{h^3}{3}$$

$$\frac{b}{9} + c = \frac{1}{3} \quad -(3)$$

Solving (1), (2), (3) $\quad a = 0;\ b = 3/4;\ c = 1/4$

Truncation Error:

$$TE = \frac{c}{(k+1)!} f^{(k+1)}(\xi), \quad 0 \le \xi \le h$$

Choose: $f(x) = x^3$

$$TE = \frac{c}{3!} f^{3}(\xi) \qquad 0 < \xi < h$$

$$C = \int_0^h x^3 dx - h\left\{ \frac{bh^3}{27} + c\frac{h^3}{} \right\}$$

$$= -\frac{h^4}{36}$$

$$\therefore \; TE = -\frac{h^4}{216} f^3(\xi) = O(h^4)$$

2) $$I = \int_1^2 \frac{2x\,dx}{1+x^4}$$

Gauss legendre formulas:

$$\boxed{\begin{array}{ll} \text{1 point:} & \int_{-1}^{1} f(x)\,dx = 2f(0) \\ \text{2nd point:} & \int_{-1}^{1} f(x)\,dx = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) \\ \text{3 point:} & \int_{-1}^{1} f(x)\,dx = \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right) \end{array}}$$

first transform: $[1,2] \to [-1,1]$

Choose: $x(t) :\equiv x = at + b$ $\left(\begin{array}{l} x(-1) = 1 \\ x(1) = 2 \end{array} \right)$

$$\left.\begin{array}{l} x(-1) = 1 \Rightarrow 1 = -a + b \\ x(1) = 2 \Rightarrow 2 = a + b \end{array}\right\} \Rightarrow b = 3/2 \;;\; a = 1$$

$$x = \frac{t}{2} + \frac{3}{2} = \frac{t+3}{2} \Rightarrow dx = \frac{dt}{2}$$

$$I = \int_{-1}^{1} \frac{8(t+3)\,dt}{[16 + (t+3)^4]} = \int_{-1}^{1} f(t)\,dt \qquad f(t) = \frac{8(t+3)}{16 + (t+3)^4}$$

using 1^{st} point rule: $I = 2f(0) = 0.4946$

" 2^{nd} " " : $I = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$

$= 0.3842 + 0.1592 = 0.5434$

" 3^{rd} " " : $I = \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)$

$= \frac{5}{9} \times 0.4393 + \frac{8}{9}(0.2474) + \frac{5}{9}(0$

$I = 0.5408$

---

3) 2 point formula:

$$\int_0^\infty e^{-x} f(x)\,dx = \frac{(2+\sqrt{2})}{4} f(2-\sqrt{2}) + \frac{(2-\sqrt{2})}{4} f(2+\sqrt{2})$$

$$\int_0^\infty \frac{e^{-x}}{1+x^2}\,dx \Rightarrow f(x) = \frac{1}{1+x^2}$$

Ans: 0.64706

---

4) 3 point formule

Ans: $\int_0^\infty e^{-x} f(x)\,dx = 0.65101$

---

4) $I = \int_0^1 \frac{dx}{1+x}$

Divide $[0,1] \rightarrow [0, 1/2] \cup [1/2, 1]$

$$I = \int_0^{1/2} \frac{dx}{1+x} + \int_{1/2}^1 \frac{dx}{1+x}$$

$I = I_1 + I_2$

5) $f(x) = \dfrac{5}{3x^2-2}, \quad x > 1$

a)

x	2	2.25	2.5	2.75	3
$f(x)$	0.5	0.38	0.30	0.24	0.2

$h = 0.25$

$$f(2.5) = \frac{5}{3(2.5)^2-2} = 0.298 = 0.30$$

1374) b) Trapezium rule:

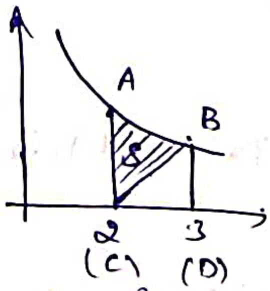
$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(x_0) + 2\left(f(x_1) + f(x_2) + \dots + f(x_{n-1})\right) + f(x_n)\right]$$

$$h = \frac{b-a}{n}$$

$$\int_2^3 \frac{5}{3x^2-2}\,dx = \frac{0.25}{2}\left[0.5 + 0.2 + 2\{0.38 + 0.30 + 0.24\}\right]$$

$$= 0.3175$$

c)



Area of shaded region = Area of under curve AB − Area of ΔBCD

$$= 0.3175 - \frac{1}{2}(1 \times 0.2)$$

$$= 0.2175 \text{ unit}^2$$

6) $\int_a^b f(x)\,dx = \frac{h}{3}\left[f(x_0) + f(x_n) + 4\{f(x_1) + f(x_3) + \dots + f(x_{n-1})\} \right.$

$2\{f(x_2) + f(x_4) + \dots + f(x_n$

$a = 0 ; b = 2$

Choosing 4 intervals: $n=4$

$h = \frac{b-a}{n} = \frac{2-0}{4} = 1/2 = 0.5$

$f(x_0) = f(0) = 0.2$

$f(x_1) = f(0.5) = 13.575$

$f(x_2) = f(1) = 30.2$

$f(x_3) = f(1.5) = 54.575$

$f(x_4) = f(2) = 94.2$

$I = \frac{0.5}{3}\left[0.2 + 94.2 + 4(13.575 + 54.575) + 2(30.2)\right]$

$I = 71.2333$

$\int_0^2 f(x)\,dx = \int_0^2 (0.2 + 25x + 3x^2 + 2x^4)\,dx = 71.2$

$|\varepsilon_t| = \left| \frac{71.2 - 71.2333}{71.2} \right| = 0.047$

---

7) $\int_0^2 (e^{x^2} - 1)\,dx$

Trapezoidal rule:

$\int_a^b f(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2\{y_1 + y_2 + y_3 + \dots + y_n\right.$

Simpson 1/3 formula:

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[y_0 + y_n + 4(y_1+y_3+\cdots) + 2(y_2+y_4+\cdots) \right]$$

Simpson 3/8 rule:

$$\int_a^b f(x)\,dx = \frac{3h}{8}\left[y_0 + y_n + 3(y_1+y_2+y_4+y_5+\cdots) + 2(y_3+y_6+\cdots) \right]$$

$$h = \frac{b-a}{n} = \frac{2-0}{6} = \frac{1}{3}$$

Trapezoidal rule: $I = 16.398$

Simpson 1/3: $I = 14.692$

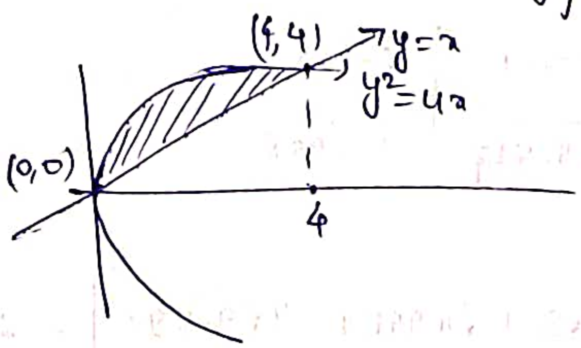
Simpson 3/8: $I = 14.883$

multiple of 2 | multiple of 3 } Interval

---

8) $\rho = \sqrt{x^2+y^2}$

Mass of Plane Lamina $= \iint \sqrt{x^2+y^2}\,dx\,dy$



(Bounded area ⇒ shaded region)

$$I = \int_{x=0}^{4} \int_{y=x}^{2\sqrt{x}} \sqrt{x^2+y^2}\,dx\,dy$$

let $F(x) = \int_{x}^{2\sqrt{x}} \sqrt{x^2+y^2}\, dy$

$$I = \int_{x=0}^{4} F(x)\, dx$$

Dividing $[0,4]$ into four subintervals.

$$I = \frac{1}{3}\left[F(0) + 4\left(F(1)+F(3)\right) + F(4) + 2F(2)\right]$$

$F(0) = 0$

$$F(1) = \int_{1}^{2} \sqrt{1+y^2}\, dy$$

Taking $h = 0.5$

y	1	1.5	2
$\sqrt{1+y^2}$	$\sqrt{2}$	$\sqrt{3.25}$	$\sqrt{5}$

$$F(1) = \frac{0.5}{3}\left[\sqrt{2} + \sqrt{5} + 4\sqrt{3.25}\right] = 1.81$$

$$F(2) = \int_{2}^{2\sqrt{2}} \sqrt{4+y^2}\, dy$$

Taking $h = 0.414$

y	2	2.414	2.828
$\sqrt{1+y^2}$	$\sqrt{8}$	$\sqrt{9.827}$	$\sqrt{11.998}$

$$F(2) = \frac{0.414}{3}\left[\sqrt{8} + \sqrt{11.998} + 4\sqrt{9.827}\right] = 2.59$$

$$F(3) = \int_{3}^{3.464} \sqrt{9+y^2}\, dy$$

Taking $h = 0.232$

y	3	3.232	3.464
$\sqrt{1+y^2}$	$\sqrt{18}$	$\sqrt{19.446}$	$\sqrt{\ }$

$f(3) = 9.047$

$$F(4) = \int_{0}^{4} \sqrt{16+y^2} = 0$$

$$I = \frac{1}{3}\left[0 + 4\,(1.81 + 2.047) + 2, 2.594\right] = 6.875$$

---

10) $$\int_0^1 \int_0^2 \frac{2xy}{(1+x^2)(1+y^2)}\, dx\, dy = I$$

let $$I_1 = \int_0^2 \frac{2xy}{(1+x^2)(1+y^2)}\, dx$$

$h = 0.25$

x	0	0.25	0.5	0.75
$\frac{xy}{(1+x^2)(1+y^2)}$	0	$\frac{0.25y}{(1.0625)(1+y^2)}$	$\frac{0.5y}{1.25(1+y^2)}$	$\frac{0.75y}{1.5625(1+y^2)}$

1	1.25	1.50	1.75	2
$\frac{y}{2(1+y^2)}$	$\frac{1.25y}{2.5625(1+y^2)}$	$\frac{1.5y}{3.25(1+y^2)}$	$\frac{1.75y}{4.0625(1+y^2)}$	$\frac{2y}{5(1+y^2)}$

(Simpson 1/3)

$\therefore \quad I_1 = \frac{0.25}{3}\left[\text{———} \right]$

$$I_1 = 1.507 \frac{y}{1+y^2}$$

$$I = \int_0^1 I_1\, dy \Rightarrow I = \int_0^1 \frac{1.507\, y}{1+y^2}\, dy$$

$\underline{k = 0.25}$

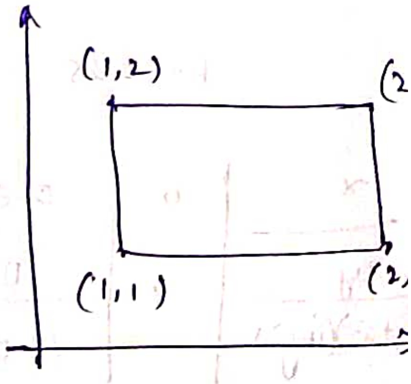
y	0	0.25	0.5	0.75	1
$\frac{y}{1+y^2}$	0	0.235	0.4	0.48	0.5

$$I = 1.507 \times \frac{0.25}{3}\left(0 + 0.5 + 2 \times 0.4 + 4(0.48 + 0.235)\right)$$

$$I = 0.5224$$

---

11) $\iint_D \frac{1}{(x^2+y^2)}\, dx\, dy$



$$I = \int_{y=1}^{2} \int_{x=1}^{2} \frac{1}{(x^2+y^2)}\, dx\, dy$$

let $I_1 = \int_{x=1}^{2} \frac{1}{x^2+y^2}\, dx$

$$\underline{I = 0.2311}$$

9)
$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right]$$

$$x_i = x_0 + ih \qquad i = 1,2,3$$

Choose $f(x) = 1$

$$\int_{x_0}^{x_3} f(x)\,dx = \int_{x_0}^{x_3} dx = x_3 - x_0 = x_0 + 3h - x_0 = 3h$$

Now, $\frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right]$

$$= \frac{3h}{8}\left[1+3+3+1\right] = \frac{3h}{8}[8] = \underline{3h}$$

$f(x) = x, x^2, x^3$

Now choose: $f(x) = x^4$

$$TE = \frac{c}{4!} f^{(iv)}(\eta) \qquad 0 < \eta < h$$

$$c = \int_{x_0}^{x_3} x^4\,dx - \frac{3h}{8}\left[x_0^4 + 3x_1^4 + 3x_2^4 + x_3^4\right]$$

$$= \frac{1}{5}\left[x_3^5 - x_0^5\right] - \frac{3h}{8}\left[\quad " \quad\right]$$

(using $x_i = x_0 + ih$)

$$= \frac{1}{5}\left[(x_0+3h)^5 - x_0^5\right] - \frac{3h}{8}\left[x_0^4 + 3(x_0+h)^4 + 3(x_0+2h)^4 + (x_0+3h)^4\right]$$

$$c = -\frac{9}{10}h^5$$

$$TE = -\frac{9h^5}{10 \times 24} f^{iv}(\eta) = -\frac{3}{80}h^5 f^{iv}(\eta).$$

$I = \int_0^1 \frac{dx}{1+x}$ $\quad x_0 = 1$, $x_1 = 1/3$, $x_2 = 2/3$, $x_3 = 1$, $h = 1/3$

$$I = \frac{3}{8}\left(\frac{1}{3}\right)\left[f(0) + 3f(1/3) + 3f(2/3) + f(1)\right]$$

$$= \frac{1}{8}\left[1 + \frac{9}{4} + \frac{9}{3} + \frac{1}{2}\right] = 0.69375$$

$$I = \int_0^1 \frac{dx}{1+x} = \ln(1+x)\Big|_0^1 = \ln 2$$

Error : $\quad E = \left|\ln 2 - 0.69375\right|$