Sequence: A mapping from the set of positive integers to $\mathbb{C}$ is called a sequence in $\mathbb{C}$, and is denoted by $\{f(n)\}$ or $\{z_n\}$. Examples: $\{\frac{i^n}{n}\}$, $\{i^n\}$, $\{2^n\}$

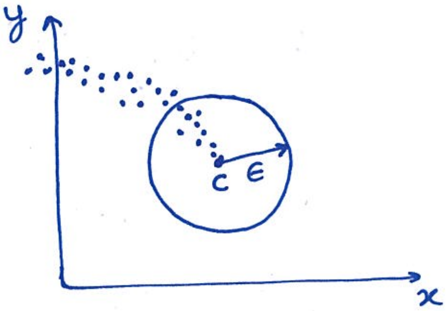
Convergence: A convergent sequence $z_1, z_2, \ldots$ is one that has a limit c, written as (c is unique)

$$\lim_{n\to\infty} z_n = c.$$

By definition of limits this means that for every $\epsilon > 0$, we can find an N such that

$$|z_n - c| < \epsilon \quad \forall \; n > N.$$

Geometrically, all z_n with $n > N$ lie in the open disk of radius ϵ and center c and only finitely many do not lie in that disk.



A divergent sequence is one that does not converge

* A sequence $\{z_n\}$ of complex numbers $z_n = x_n + iy_n$ converges to $c = a + ib$ iff the sequences of real numbers $\{x_n\}$ and $\{y_n\}$ converge to a and b, respectively.

Ex: ① The sequence $\{\frac{i^n}{n}\} = \{i, -\frac{1}{2}, -\frac{i}{3}, \frac{1}{4}, \ldots\}$ is convergent with limit 0.

$$\lim_{n\to\infty} \frac{i^n}{n} = 0.$$

② The sequence $\{i^n\} = \{i, -1, -i, 1, \ldots\}$ is divergent. (Limit does not exist)

Ex. The sequence $\{z_n\}$ with $z_n = x_n + i y_n = 1 - \frac{1}{n^2} + i\left(2 + \frac{4}{n}\right)$

or $\{z_n\} = \left\{6i, \frac{3}{4} + 4i, \frac{8}{9} + \frac{10i}{3}, \dots\right\}$ converges to

$C = 1 + 2i$.

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty}\left[\left(1 - \frac{1}{n^2}\right) + i\left(2 + \frac{4}{n}\right)\right] = 1 + 2i$$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty}\left(1 - \frac{1}{n^2}\right) = 1 = a \qquad c = a + ib.$$

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty}\left(2 + \frac{4}{n}\right) = 2 = b$$

Remark:- Let $\{z_n\}$ and $\{w_n\}$ be two complex sequences converging to A and B, respectively. Then

1. $\lim_{n\to\infty} (z_n \pm w_n) = A \pm B$

2. $\lim_{n\to\infty} (z_n w_n) = AB$

3. $\lim_{n\to\infty} \frac{z_n}{w_n} = \frac{A}{B}$.

Series:- Let $\{z_n\}$ be a sequence of complex numbers. The expression of the form $z_1 + z_2 + z_3 + \dots$ denoted by $\sum_{n=1}^{\infty} z_n$ is called a series.

Partial sums: $S_n = z_1 + z_2 + \dots + z_n$ is called the nth partial sum of the series $\sum_{n=1}^{\infty} z_n$.

Convergence and Divergence of a series:

Let S_n be the n^{th} partial sum of the series $\sum_{n=1}^{\infty} z_n$.

If $\lim_{n\to\infty} S_n = S$ for some S, then we say that the series $\sum_{n=1}^{\infty} z_n$ converges and S is called the sum of the series. If $\{S_n\}$ does not converge, then we say that the series $\sum_{n=1}^{\infty} z_n$ diverges.

Theorem (A necessary condition of convergence)

If $\sum_{n=1}^{\infty} z_n$ converges, then $\lim_{n\to\infty} z_n = 0$.

Proof:- If $\sum_{n=1}^{\infty} z_n$ converges with the sum S, then

$$\left(\text{since } z_n = S_n - S_{n-1}\right)$$

$$\lim_{n\to\infty}(z_n) = \lim_{n\to\infty}(S_n - S_{n-1})$$

$$= \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}$$

$$= S - S = 0.$$

* Note: The condition $\lim_{n\to\infty} z_n = 0$ is necessary for convergence of $\sum_{n=1}^{\infty} z_n$ but not sufficient!

Ex. $\sum_{n=1}^{\infty} \frac{1}{n}$ (Harmonic series) is divergent even though

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

Absolute convergence:-

A series $\sum_{n=1}^{\infty} z_n$ is called absolutely convergent if the series $\sum_{n=1}^{\infty} |z_n|$ is convergent.

Theorem:- If a series is absolutely convergent, then the series is convergent.

* If $\sum_{n=1}^{\infty} z_n$ converges but $\sum_{n=1}^{\infty} |z_n|$ diverges, then the series $\sum_{n=1}^{\infty} z_n$ is called conditionally convergent.

Ex: The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is conditionally convergent.

Real and Imaginary parts:

A series $\sum_{n=1}^{\infty} z_n$, where $z_n = x_n + i y_n$ converges and has the sum $c = a + ib$ if and only if $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ converge and have the sums a and b, respectively.

Ex: ① The series $\sum_{n=1}^{\infty} \frac{(-1)^n + i}{n^2}$ converges as both the real series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge.

② The series $\sum_{n=1}^{\infty} \left[1 + i\frac{(-1)^n}{n}\right]$ does not converge as the series $\sum_{n=1}^{\infty} 1$ does not converge.

Convergence Tests:

1. Comparison Test

If a series $\sum_{n=1}^{\infty} z_n$ is given and we can find a converging series $\sum_{n=1}^{\infty} b_n$ with nonnegative real terms such that

$$|z_n| \leq b_n \quad \text{for } n = 1, 2, 3, \dots$$

then the given series $\sum_{n=1}^{\infty} z_n$ converges, even absolutely

2. Ratio Test:

~~Suppose~~ $\cancel{\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = L}$.

If a series $\sum_{n=1}^{\infty} z_n$ with $z_n \neq 0$ (for $n = 1, 2, 3, \dots$) is such that $\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = L$, then

(a) the series converges absolutely if $L < 1$.

(b) the series diverges if $L > 1$.

(c) the test fails if $L = 1$.

3. Root test:

If a series $\sum_{n=1}^{\infty} z_n$ is such that $\lim_{n\to\infty} \sqrt[n]{|z_n|} = L$, then

(a) the series converges absolutely if $L < 1$.

(b) the series diverges if $L > 1$.

(c) the test fails if $L = 1$.

Power Series

A power series is a series of the form

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

where z is a complex variable, $a_n \in \mathbb{C}$ $(n=0,1,2,\ldots)$ are the <u>coefficients</u> ~~of~~ the series, and $z_0 \in \mathbb{C}$ is called the <u>center</u> of the series.

* If $z_0 = 0$, we obtain as a particular case a power series in powers of z:

$$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots$$

Convergence behaviour of power series

The power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ has a variable z. If we fix this variable at some point, say $z = z_1 \in \mathbb{C}$, then we obtain a complex series $\sum_{n=0}^{\infty} a_n (z_1 - z_0)^n$. This series could be convergent or divergent.

The power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ is said to be convergent or divergent for $z = z_1$ according as the complex series $\sum_{n=0}^{\infty} a_n (z_1 - z_0)^n$ is convergent or divergent.

Ex. ① Convergence for every z

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

is absolutely convergent for every z. By ratio test, $\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| = \lim_{n\to\infty} \left|\frac{z}{n+1}\right| = 0$.

② Convergence only at the center (Useless series)

$$\sum_{n=0}^{\infty} n!\, z^n = 1 + z + 2z^2 + 6z^3 + \cdots$$

converges only at $z=0$ but diverges for all $z \neq 0$. $\lim_{n\to\infty} n!\, z^n \neq 0$ for all $z \neq 0$.

Examples (convergence in a disk).

③ Geometric series

$$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots$$

converges absolutely if $|z|<1$ and diverges if $|z| \geq 1$

* $\lim_{n\to\infty} z^n = \infty \; (\neq 0)$ if $|z| \geq 1$.

* converges with sum $\frac{1}{1-z}$ if $|z|<1$.

$$S_n = 1 + z + z^2 + \cdots + z^n \quad \text{—— ①}$$

$$z S_n = z + z^2 + \cdots + z^n + z^{n+1} \quad \text{—— ②}$$

~~①②~~

① − ② ⇒ $(1-z) S_n = 1 - z^{n+1}$

Now, $1-z \neq 0$ since $z \neq 1$, therefore

$$S_n = \frac{1-z^{n+1}}{1-z} = \frac{1}{1-z} - \frac{z^{n+1}}{1-z}$$

Since $|z|<1$,

$$S = \lim_{n\to\infty} S_n = \frac{1}{1-z} - \cancel{\lim_{n\to\infty} \frac{z^{n+1}}{1-z}}^{\,0}$$

Hence $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ for $|z|<1$.

* The same analysis can be done ~~for~~ by replacing z with $|z|$ to show that $\sum_{n=0}^{\infty} |z|^n$ is also convergent for $|z|<1$.

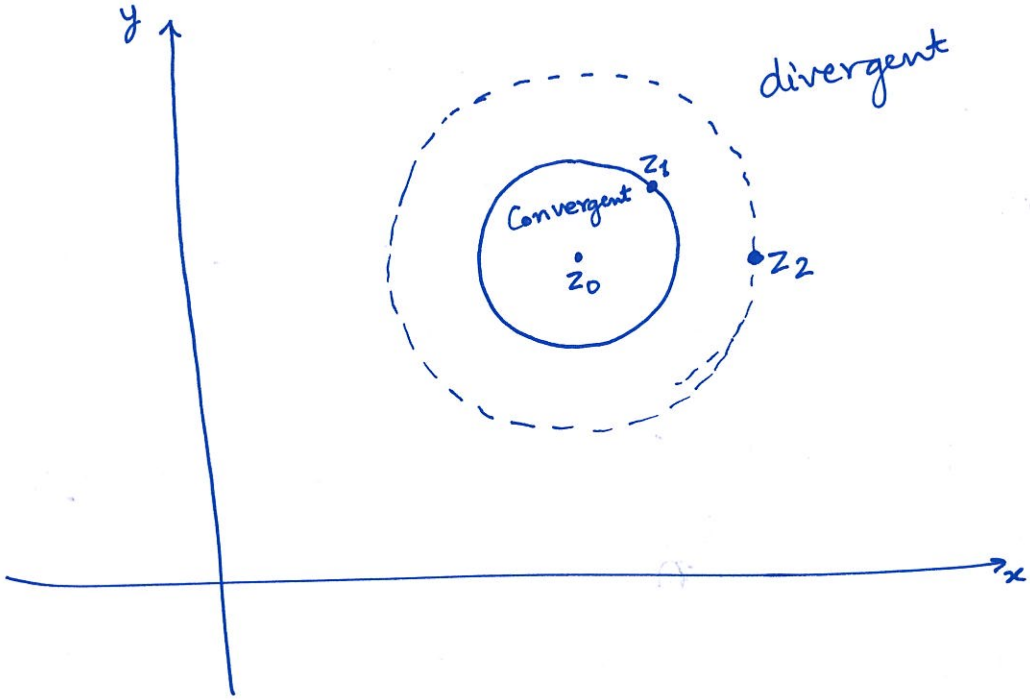
Hence $\sum_{n=0}^{\infty} z^n$ is absolutely convergent for $|z|<1$.

Theorem: (Convergence of a power series)

(a) Every power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges at its center z_0.

(b) If $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges at a point $z = z_1 \neq z_0$, it converges absolutely for every z closer to z_0 than z_1, i.e., for all z such that $|z-z_0| < |z_1 - z_0|$.

(c) If $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ diverges at a point $z = z_2 \neq z_0$, it diverges for every z farther away from z_0 than z_2, i.e., for all z such that $|z-z_0| > |z_2 - z_0|$.



Ex. (Ratio test)

Is the following series convergent or divergent?

$$\sum_{n=0}^{\infty} \frac{(4+3i)^n}{n!}$$

$|4+3i| = 5$

By the ratio test,

$$\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = \lim_{n\to\infty}\left|\frac{4+3i}{n+1}\right| = \lim_{n\to\infty}\frac{5}{n+1} = 0 < 1.$$

Hence the given series is convergent.

Radius of convergence:-

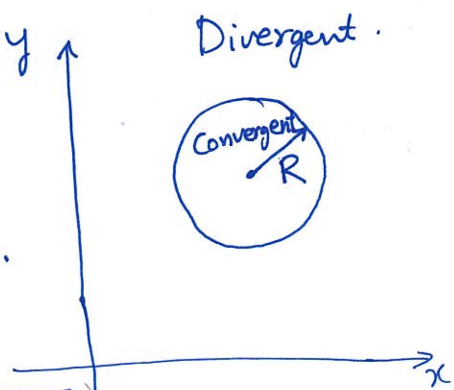
We consider the smallest circle with center z_0 that includes all the points at which the power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges. Let R denote the radius of this circle. The circle

$$|z - z_0| = R$$

is called the circle of convergence and its radius R the radius of convergence of $\sum_{n=0}^{\infty} a_n (z-z_0)^n$.

The above theorem implies convergence for all z for which $|z-z_0| < R$

and the series diverges for all z, for which $|z-z_0| > R$.



* No general statements can be made about convergence of the power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ on the circle of convergence.

* $R=\infty$ if the series converges for all $z \in \mathbb{C}$

* $R=0$ if " " " only at the center z_0.

Ex: Behaviour on the circle of convergence.

On the circle of convergence (radius $R=1$ in all three cases)

(a) $\sum_{n=0}^{\infty} \frac{z^n}{n^2}$ converges everywhere since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

(b) $\sum_{n=1}^{\infty} \frac{z^n}{n}$ converges at -1 (by Leibniz's test) but diverges at 1.

(c) $\sum_{n=1}^{\infty} n! z^n$ diverges everywhere.

Formula for calculating R: (exactly like in reals)

- If $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \mu$, then $R = \frac{1}{\mu}$

- If $\lim_{n \to \infty} |a_n|^{1/n} = \mu$, then $R = \frac{1}{\mu}$,

whenever the above limits exist (with the supposition that division by ∞ (or 0) produces 0 (or ∞)).

Ex: $\sum_{n=1}^{\infty} \frac{z^n}{n}$. Find R.

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{n}{n+1} \right| = 1.$$

$\therefore$ $R = 1/1 = 1$.

$\therefore$ The series $\sum_{n=1}^{\infty} \frac{z^n}{n}$ converges for $|z| < 1$ and diverges for $|z| > 1$.

Ex. Find the radius of convergence R of the following power series

1. $\sum_{n=1}^{\infty} z^n$

2. $\sum_{n=1}^{\infty} n!\, z^n$

3. $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2} z^n$

Solⁿ

1. $\frac{1}{R} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{1}{1}\right| = 1$

$\Rightarrow \boxed{R=1}$

2. $\frac{1}{R} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} |n+1| = \infty$

$\Rightarrow \boxed{R=0}$

3. Here $a_n = \left(1+\frac{1}{n}\right)^{n^2}$. Hence, by the root test,

$$\frac{1}{R} = \lim_{n\to\infty} \left|\left(1+\frac{1}{n}\right)^{n^2}\right|^{1/n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{n} = e$$

$\Rightarrow \boxed{R = 1/e}$

Theorem (Term wise differentiability)

Let $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ be a power series with radius of convergence R, and let it represent a function f(z), i.e.,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots,$$

$$|z-z_0| < R$$

Then the series can be differentiated term-wise in the sense that

$$f'(z) = a_1 + 2a_2(z-z_0) + 3a_3(z-z_0)^2 + \cdots$$

$$f''(z) = 2a_2 + 3 \cdot 2 a_3 (z-z_0) + 4 \cdot 3 a_4 (z-z_0)^2 + \cdots$$

and so on, and the radius of convergence of each resulting series is also R.

Theorem (Term wise integration)

The power series

$$\sum_{n=0}^{\infty} \frac{a_n}{n+1} z^{n+1} = a_0 z + \frac{a_1}{2} z^2 + \frac{a_2}{3} z^3 + \cdots$$

obtained by integrating the series $\sum_{n=0}^{\infty} a_n z^n$ term by term has the same radius of convergence as the original series.

Theorem (Term wise addition or subtraction)
Let r_1, r_2, and r be the radius of convergence of the power series
$$\sum_{n=0}^{\infty} a_n (z-z_0)^n, \quad \sum_{n=0}^{\infty} b_n (z-z_0)^n \text{ and } \sum_{n=0}^{\infty} (a_n \pm b_n)(z-z_0)^n$$
respectively. Then
$$r \geq \min\{r_1, r_2\}.$$

Ex. The series $\sum_{n=0}^{\infty} (1+2^{-n}) z^n$ and the series $\sum_{n=0}^{\infty} (-z^n)$, both have radius of convergence 1, but their sum $\sum_{n=0}^{\infty} 2^{-n} z^n$ has the radius of convergence 2.

Theorem (Multiplication) Consider the Cauchy product $\sum_{n=0}^{\infty} c_n (z-z_0)^n$ of the power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ and $\sum_{n=0}^{\infty} b_n (z-z_0)^n$ where
$$c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0.$$
Then the power series $\sum_{n=0}^{\infty} c_n (z-z_0)^n$ converges absolutely for each z within the circle of convergence of each of the two given series and has the sum $S(z) = f(z)\, g(z)$, where $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ and $g(z) = \sum_{n=0}^{\infty} b_n (z-z_0)^n$.

Taylor Series:-

A power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ with radius of convergence R represents a function analytic in the domain $|z-z_0| < R$.

Theorem:-(Taylor's theorem) Let f be analytic on $D = \{z : |z-z_0| < R\}$. Then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \text{for all } z \in D$$

where for $n = 0, 1, 2, \ldots$

$$a_n = \frac{f^n(z_0)}{n!} = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz \quad \text{—} \circledast$$

with C being any positively oriented simple closed contour enclosing z_0 and lying completely inside D.

* The series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$, where a_n's are given by $\circledast$ is called the Taylor series of f(z) around z_0.
* The Taylor series $\sum_{n=0}^{\infty} a_n z^n$ of a function f(z) around origin is called Maclaurin series.

Theorem (Uniqueness)

Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a power series with radius of convergence R and let it represent a function $f(z)$,

i.e.,
$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots, \quad (*)$$
for $|z-z_0| < R$

Then $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is the Taylor series of $f(z)$ around z_0.

Proof Need to show that
$$a_n = \frac{f^n(z_0)}{n!}, \quad n = 0, 1, 2, \cdots$$

Termwise derivative.

$$(\#) \begin{cases} f'(z) = a_1 + 2a_2(z-z_0) + 3a_3(z-z_0)^2 + \cdots \\ f''(z) = 2a_2 + 3.2\, a_3(z-z_0) + 4.3\, a_4(z-z_0)^2 + \cdots \\ \vdots \qquad\qquad \vdots \end{cases}$$

By putting $z = z_0$ in $(*)$ & $(\#)$ successively, we obtain the desired relations
$$a_n = \frac{f^{(n)}(z_0)}{n!}, \quad n = 0, 1, 2, 3, \cdots \qquad \square$$

* Thus, any infinite power series expansion of an analytic function $f(z)$ must be its Taylor series.

Ex. Find the Taylor series around origin of the following functions.

1. $f(z) = e^z$

2. $f(z) = \frac{1}{1-z}$

3. $f(z) = \sin z$

4. $f(z) = \cos z$

Sol^n ① $f(z) = e^z$ is analytic in whole $\mathbb{C}$. Moreover,

$f^{(n)}(0) = 1$. for $n = 0, 1, 2, 3, \cdots$

Therefore, the Taylor series of $f(z) = e^z$ around $z = 0$ is

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots, \quad z \in \mathbb{C}$$

② $f(z) = \frac{1}{1-z}$ is analytic on $D = \{z : |z| < 1\}$.

$$f'(z) = \frac{1}{(1-z)^2}, \quad f''(z) = \frac{2}{(1-z)^3}, \quad f'''(z) = \frac{2.3}{(1-z)^4}, \cdots$$

$\Rightarrow f^{(n)}(0) = n!$ for $n = 0, 1, 2, 3, \cdots$

Therefore, the Taylor series of $f(z) = \frac{1}{1-z}$ is

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \cdots, \quad |z| < 1$$

③ $f(z) = \sin z$ is analytic over whole $\mathbb{C}$. Moreover,

$f^{(2n)}(0) = 0$, $f^{(2n+1)}(0) = (-1)^n$, $n = 0, 1, 2, \cdots$.

Therefore, the Taylor series of $f(z) = \sin z$ around origin is

$$\sin z = \sum_{n=1}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}, \quad z \in \mathbb{C}.$$

④ $f(z) = \cos z$ is analytic on whole $\mathbb{C}$. Moreover,

$$f^{(2n+1)}(0) = 0, \quad f^{(2n)}(0) = (-1)^n, \quad n = 0, 1, 2, \ldots$$

Therefore, the Taylor series of $f(z) = \cos z$ around origin is

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{((2n)!)}, \quad z \in \mathbb{C}.$$

Problem Find the Taylor series around origin of the following functions.

① $f(z) = \dfrac{1}{(1-z)^2}$

② $f(z) = \dfrac{1}{1+z}$

Sol^n: We know that

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad |z| < 1$$

(Differentiate termwise)

$$\frac{1}{(1-z)^2} = \sum_{n=1}^{\infty} n z^{n-1}, \quad |z| < 1.$$

↑ Taylor series

②
$$\frac{1}{1+z} = \frac{1}{1-(-z)} = \sum_{n=0}^{\infty} (-z)^n, \quad |-z| < 1$$

$$\Rightarrow \quad \frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n, \quad |z| < 1$$

↑ Taylor series.

Prob Find the Taylor series of the function $f(z)=\frac{1}{z}$ in powers of $(z-1)$.

Sol^n

$$\frac{1}{z}=\frac{1}{1+(z-1)}=\sum_{n=0}^{\infty}(-1)^n(z-1)^n, \quad |z-1|<1.$$

Laurent Series:

In Taylor's theorem, we have seen that if a function f is analytic on a domain $D=\{z:|z-z_0|<R\}$, then it can be represented by a series of the form $\sum_{n=0}^{\infty}a_n(z-z_0)^n$ in D.

Is such a series representation possible when f is analytic on D except at point z_0?

The answer is NO as the series $\sum_{n=0}^{\infty}a_n(z-z_0)^n$ is analytic at z_0, but f is not analytic at z_0.

Nevertheless, $f(z)$ in this case may be written as the sum of a series of the form

$$\sum_{n=0}^{\infty}a_n(z-z_0)^n+\sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n}.$$

The following theorem shows that under some condition on f and domain, such a representation is always possible.

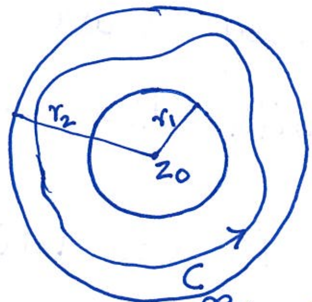
Laurent's Theorem: Let the function f be analytic in an annulus $D = \{z : r_1 < |z - z_0| < r_2\}$. Then at each point $z \in D$, f can be represented by a series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n},$$

where

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz \quad \text{and} \quad b_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{-n+1}}\, dz \qquad \circledast$$

with C being any positively oriented (anticlockwise) simple closed contour enclosing z_0 and lying completely inside D



The series $\displaystyle\sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$, where a_n and b_n are given by $\circledast$ is called the Laurent series of $f(z)$ around z_0. The Laurent series of $f(z)$ can also be written as

$$f(z) = \sum_{-\infty}^{\infty} a_n (z - z_0)^n,$$

where for $n = 0, \pm 1, \pm 2, \ldots,$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz$$

with C being any positively oriented simple closed contour enclosing z_0 and lying completely inside D.

* The series (or finite sum) of negative powers of $(z-z_0)$ is called the principal part of the Laurent series.

* The Laurent series converges and represents $f(z)$ in the open annulus $r_1 < |z-z_0| < r_2$.

Uniqueness:- The Laurent series of a given analytic function $f(z)$ in its annulus of convergence is unique.
However, $f(z)$ may have different Laurent series in two annuli with the same center.

The uniqueness is essential. Therefore to obtain the coefficients in a Laurent series, we generally do not use the integral formulas ⊛; instead we use some other methods to find the Laurent series. If a Laurent series has been found, the uniqueness guarantees that it must be the Laurent series of the given function in the given annulus.

<u>Remark</u>: Let the function $f(z)$ be analytic in the domain $D = \{z : |z - z_0| < r_2\}$, and consider the Laurent series of f. Then, by Cauchy - Goursat theorem[#], for $n = 1, 2, \ldots$, we obtain

$$b_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{-n+1}}\, dz = 0$$

as the integrand $\dfrac{f(z)}{(z - z_0)^{-n+1}}$ is analytic on and inside C.

Thus, in this case, Laurent series of f reduces to its Taylor Series.

<u>Note</u>:- Recall that the expansion

$$\frac{1}{1 - z} = \sum_{n=0}^{\infty} z^n , \qquad |z| < 1$$

is very useful in determining the Laurent series of many functions.

<u>Cauchy - Goursat Theorem</u>:- Let $f(z)$ be a function such that $f(z)$ is analytic on and inside a simple closed contour C. Then

$$\oint_C f(z)\, dz = 0 .$$

Ex. 1 Use of Maclaurin series:

Find the Laurent series of $z^{-5}\sin z$ with center 0.

Sol^n $\sin z$ is analytic for all $z \in \mathbb{C}$ and hence has a Maclaurin series expansion.

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}, \qquad z \in \mathbb{C}$$

$\therefore$ The Laurent series of $z^{-5}\sin z$ is

$$z^{-5}\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n-4} = \frac{1}{z^4} - \frac{1}{6z^2} + \frac{1}{120} - \frac{1}{5040}z^2 + \cdots$$

$$= \frac{1}{z^4} - \frac{1}{3!\,z^2} + \frac{1}{5!} - \frac{1}{7!}z^2 + \frac{1}{9!}z^4 - \cdots$$

$(0 < |z| < \infty)$

Hence the annulus of convergence is the whole complex plane without the origin.

Ex. 2 Substitution:-

Find the Laurent series of $z^2 e^{1/z}$ with center 0.

Sol^n We know that

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad z \in \mathbb{C}$$

Replace z with $1/z$.

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{z}\right)^n \quad \text{for } z \neq 0 \text{ or } |z| > 0$$

$$\Rightarrow z^2 e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!}\,\frac{1}{z^{n-2}} = z^2 + z + \frac{1}{2!} + \frac{1}{3!\,z} + \frac{1}{4!\,z^2} + \cdots$$

$(|z| > 0)$

annulus of convergence is the whole complex plane without the origin.

Example:- Develop $1/(1-z)$ in

(a) nonnegative powers of z

(b) negative powers of z.

Sol^n (a) $\dfrac{1}{1-z} = \displaystyle\sum_{n=0}^{\infty} z^n$ (valid for $|z|<1$)

(b) For $|z|>1$, $\left|\frac{1}{z}\right|<1$

$$\therefore \frac{1}{1-z} = -\frac{1}{z\left(1-\frac{1}{z}\right)} = -\frac{1}{z}\cdot\frac{1}{1-\frac{1}{z}}$$

$$= -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n \quad \text{for } |z|>1$$

$$= -\sum_{n=0}^{\infty}\frac{1}{z^{n+1}} = -\frac{1}{z}-\frac{1}{z^2}-\frac{1}{z^3}-\dots \quad \text{for } |z|>1.$$

Ex. Laurent expansions in different (concentric) annuli:

Find all Laurent series of $1/(z^3-z^4)$ with center 0

problematic points $z=0$ and $z=1$

Sol^n $\dfrac{1}{z^3-z^4} = \dfrac{1}{z^3(1-z)}$

$$\frac{1}{z^3-z^4} = \frac{1}{z^3}\sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} z^{n-3} = \frac{1}{z^3}+\frac{1}{z^2}+\frac{1}{z}+1+z+z^2+z^3+\dots$$

$(0<|z|<1)$

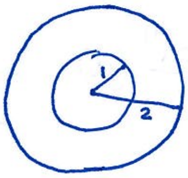
$$\frac{1}{z^3-z^4} = -\frac{1}{z^4}\cdot\frac{1}{1-\frac{1}{z}} = -\frac{1}{z^4}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n \quad \text{for } |z|>1$$

$$= -\sum_{n=0}^{\infty}\frac{1}{z^{n+4}} = -\frac{1}{z^4}-\frac{1}{z^5}-\frac{1}{z^6}-\dots \quad (|z|>1).$$

Example:- Use of partial fractions

Find all Taylor and Laurent series of
$$f(z) = \frac{-2z+3}{z^2-3z+2} \quad \text{with center } 0.$$

Sol^n:- Partial fractions:



$$f(z) = \frac{-2z+3}{(z-2)(z-1)} = -\left(\frac{1}{z-1} + \frac{1}{z-2}\right)$$

$$-\frac{1}{z-1} = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad |z|<1.$$

$$-\frac{1}{z-1} = -\frac{1}{z}\,\frac{1}{1-\frac{1}{z}} = -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \qquad |z|>1.$$

$$-\frac{1}{z-2} = \frac{1}{2}\,\frac{1}{1-\frac{z}{2}} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n, \qquad |z|<2$$

$$-\frac{1}{z-2} = -\frac{1}{z}\,\frac{1}{1-\frac{2}{z}} = -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n, \qquad |z|>2$$

Therefore, (i) for $|z|<1$,
$$f(z) = \frac{-2z+3}{(z-2)(z-1)} = \sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} \qquad |z|<1$$
$$= \sum_{n=0}^{\infty}\left(1+\frac{1}{2^{n+1}}\right)z^n = \frac{3}{2} + \frac{5}{4}z + \frac{9}{8}z^2 + \cdots \qquad |z|<1.$$

(ii) for $1<|z|<2$,
$$f(z) = \frac{-2z+3}{(z-2)(z-1)} = -\sum_{n=0}^{\infty}\frac{1}{z^{n+1}} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{z^n}{2^n} \qquad 1<|z|<2$$
$$= \left(\frac{1}{2} + \frac{1}{4}z + \frac{1}{8}z^2 + \frac{1}{16}z^3 + \cdots\right) - \left(\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right) \qquad 1<|z|<2.$$

(iii) for $|z|>2$,
$$f(z) = \frac{-2z+3}{(z-2)(z-1)} = -\sum_{n=0}^{\infty}\frac{1}{z^{n+1}} - \sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}} = -\sum_{n=0}^{\infty}\frac{2^n+1}{z^{n+1}} \qquad |z|>2$$
$$= -\frac{2}{z} - \frac{3}{z^2} - \frac{5}{z^3} - \frac{9}{z^4} - \cdots \qquad |z|>2.$$

SINGULARITIES AND ZEROS

Regular point:- If a function f is analytic at a point z_0, then z_0 is called an ORDINARY POINT or a REGULAR POINT of f.

Roughly speaking, a singular point of an analytic function $f(z)$ is a z at which $f(z)$ ceases to be analytic, and a zero is a z at which $f(z)=0$.

* Singularities may be discussed and classified by means of Laurent series.
* Zeros may be discussed by means of Taylor series.

SINGULAR POINT: A function $f(z)$ is said to be SINGULAR or said to have a singularity at a point $z=z_0$ if $f(z)$ is not analytic (perhaps not even defined) at $z=z_0$~~, but every neighbourhood of $z=z_0$ contains points at which $f(z)$ is analytic.~~

We also say that $z=z_0$ is a singular point of $f(z)$.

Ex. ① $f(z)=\frac{1}{z}$, $z=0$ is a singular point as $f(z)$ is not defined at $z=0$.

② $f(z)=|z|^2$, $f(z)=|z|^2$ is differentiable at $z=0$ but it is not analytic at $z=0$ since the function is nowhere differentiab[le] except at the origin. Hence $z=0$ is a singularity of $f(z)=|z|^2$.

ISOLATED SINGULARITY:

A point $z = z_0$ is called an isolated singularity of $f(z)$ if $z = z_0$ has a neighbourhood without further singularities of $f(z)$.

Ex. $\tan z$ has isolated singularities at $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$ etc.

$\tan\left(\frac{1}{z}\right)$ has a non isolated singularity at $z = 0$.

$\tan\left(\frac{1}{z}\right)$ has singularities at $z = 0, \pm\frac{2}{\pi}, \pm\frac{2}{3\pi}, \pm\frac{2}{5\pi}, \pm\frac{2}{(2n+1)\pi}, \ldots$ $(n = 0, 1, 2, \ldots, \infty)$

No matter how small we choose the neighbourhood around $z = 0$, it will contain another singularity.

* Isolated singularities can be classified by Laurent series.

Let $z = z_0$ be an isolated singular point of $f(z)$. Then there exists a $r > 0$ such that f is analytic in the annulus $0 < |z - z_0| < r$. Therefore, by Laurent's theorem, f has a Laurent series representation valid in $0 < |z - z_0| < r$.

Let
$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}, \quad 0 < |z - z_0| < r$$

* The part $\sum_{n=0}^{\infty} a_n (z - z_0)^n$, which is a power series in $(z - z_0)$ represents a function which is analytic at $z = z_0$.
* The other part $\sum_{n=0}^{\infty} \frac{b_n}{(z - z_0)^n}$ is called the PRINCIPAL PART of $f(z)$ at the isolated singularity $z = z_0$.

Depending on the principal part, the following cases may arise.

Case I: The principal part of f has infinitely many terms. In such a case, $z = z_0$ is called an isolated essential singularity of f.

Case II: If all the coefficients $b_1, b_2, b_3, \ldots$ of the principal part are zero, then $z = z_0$ is called a removable singularity of f.

In this case, it is possible to make f analytic in some neighbourhood of z_0 by defining f at z_0.

* A removable singularity is a non-isolated singularity.

Case III: The principal part of f has only finetely many terms, i.e., NOT all of the coefficients $b_1, b_2, b_3, \ldots$ of the principal part of f are zero. In other words, the principal part is of the form

$$\frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots + \frac{b_m}{(z - z_0)^m} \quad (b_m \neq 0).$$

but $b_k = 0 \; \forall \, k > m$

In this case, the singularity of $f(z)$ at $z = z_0$ is called a pole and m is called its order or $z = z_0$ is called a pole of order m.

* Poles of order one are called simple poles.

* Poles " " two " " double poles.

* A pole is an isolated singularity.

Example 1

$$f(z) = \frac{1}{z(z-2)^5} + \frac{3}{(z-2)^2}$$

has a simple pole at $z=0$ and a pole of order 5 at $z=2$.

$$\cancel{f(z) = \frac{1}{z^6}\left(1-\frac{z}{2}\right)}$$

$$f(z) = \frac{1}{-2^5 z}\left(1-\frac{z}{2}\right)^{-5} + \frac{3}{4}\left(1-\frac{z}{2}\right)^{-2}$$

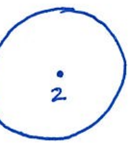
$$\Rightarrow f(z) = -\frac{1}{2^5 z}\left(1 + \frac{5z}{2} + \frac{5(6)}{1.2}\left(\frac{z}{2}\right)^2 + \cdots\right) + \frac{3}{4}\left(1 + z + \frac{2\times 3}{1\times 2}\left(\frac{z}{2}\right)^2 + \cdots\right)$$

$$\left|\frac{z}{2}\right| < 1$$

$$= -\underline{\underline{\frac{1}{2^5 z}}} - \frac{5}{2^6} - \frac{30}{2^6}\frac{z}{2^2} - \cdots$$

$$+\frac{3}{4} + \frac{3}{4}z + \frac{3z^2}{4} + \cdots$$

$z=0$ is a simple pole.



$|z-2|<1$

$$f(z) = \frac{1}{(z-2)^5}\,\frac{1}{2}\left(1+\frac{z-2}{2}\right)^{-1} + \frac{3}{(z-2)^2}$$

$$= \frac{1}{(z-2)^5}\,\frac{1}{2}\left(1 - \frac{z-2}{2} + \frac{1.2}{2!}\left(\frac{z-2}{2}\right)^2 - \left(\frac{z-2}{2}\right)^3 + \cdots\right) + \frac{3}{(z-2)^2}$$

$$= \frac{1}{2}\frac{1}{(z-2)^5} - \frac{1}{4(z-2)^4} + \frac{1}{2^3(z-2)^3} - \frac{1}{2^4}\frac{1}{(z-2)^2} + \frac{1}{2^5} + \frac{1}{2^6}(z-2) + \cdots$$

$$+ \frac{3}{(z-2)^2}$$

$$= \frac{1}{2}\frac{1}{(z-2)^5} - \frac{1}{2^2(z-2)^4} + \frac{1}{2^3(z-2)^3} + \left(3-\frac{1}{16}\right)\frac{1}{(z-2)^2} + \frac{1}{2^5} + \frac{1}{2^6}(z-2) + \cdots$$

$$|z-2| < 2$$

$z=2$ is a pole of order 5.

* It is quite <u>important</u> to consider the Laurent series valid in the <u>immediate neighbourhood</u> of a singular point.

Examples: Isolated singularities

$\sin\frac{1}{z}$ and $e^{1/z}$ have an isolated essential singularity at $z=0$.

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!\,z^n} = 1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \cdots \qquad |z|>0$$

$$\sin\frac{1}{z} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!\,z^{2n+1}} = \frac{1}{z} - \frac{1}{3!\,z^3} + \frac{1}{5!\,z^5} - \cdots \qquad |z|>0$$

(infinitely many terms in the principal parts of f)

*Example (Removable singularity)

$f(z) = \frac{\sin z}{z}$ is analytic everywhere except at $z=0$, where the function is not defined.

Therefore, $f(z)$ has an isolated singularity at $z=0$.

We know that

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}, \quad \forall z \in \mathbb{C}$$

Therefore, for $|z|>0$, we obtain

$$f(z) = \frac{\sin z}{z} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n+1)!} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots$$

(No term in the principal part)

Therefore, $z=0$ is a removable singularity of $\frac{\sin z}{z}$.

We may extend $f(z) = \frac{\sin z}{z}$ to be an entire function by defining $f(z)=1$ at $z=0$.

* Such singularities are of no interest since they can be removed by assigning a suitable value to $f(z_0)$.

The behaviour of an analytic function in a neighbourhood of an essential singularity is entirely different from that in the neighbourhood of a pole.

Behaviour near a pole:

* $f(z) = \frac{1}{z^2}$ has a pole at $z = 0$

$|f(z)| \to \infty$ as $z \to 0$ in any manner.

This illustrates the following theorem.

Theorem: If $f(z)$ is analytic and has a pole at $z = z_0$, then $|f(z)| \to \infty$ as $z \to z_0$ in any manner.

Behaviour near an essential singularity:

$f(z) = e^{1/z}$ has an essential singularity at $z = 0$.

It has no limit for approach along the imaginary axis; it becomes infinite if $z \to 0$ through positive real axis, but it approaches zero if $z \to 0$ through negative real axis.

It takes on any given value $c = c_0 e^{i\alpha} \neq 0$ in an arbitrary small neighbourhood of $z = 0$.

Picard's theorem: If $f(z)$ is analytic and has an isolated essential singularity at a point z_0, it takes on every value, with at most one exceptional value, in an arbitrarily small neighbourhood of z_0.

Zeros of Analytic Functions

A <u>zero</u> of an analytic function $f(z)$ in a domain D is a point $z = z_0$ in D such that $f(z_0) = 0$.

* A zero has order n if not only f but also the derivatives $f', f'', \dots, f^{(n-1)}$ are all zero at $z = z_0$ but $f^{(n)}(z_0) \neq 0$.

* A first-order zero is called a <u>SIMPLE ZERO.</u>

<u>Ex.</u> ① $f(z) = 1 + z^2$ has simple zeros at $z = \pm i$.

② $f(z) = (1 - z^4)^2$ has zeros at $z = \pm 1, \pm i$.

$$f'(z) = 2(1 - z^4)(-4z^3) = -8z^3(1 - z^4)$$

$$\Rightarrow f'(z) = 0 \quad \text{at } z = \pm 1, \pm i$$

$$f''(z) = -24z^2(1 - z^4) + 32z^6$$

$$\Rightarrow f''(\pm 1) = 32 \neq 0 \quad \text{and} \quad f''(\pm i) = -32 \neq 0$$

$\Rightarrow z = \pm 1, \pm i$ are zeros of order 2 for $f(z) = (1 - z^4)^2$.

③ $f(z) = (z - a)^3$ has a third-order zero at $z = a$.

$$f(a) = 0, \quad f'(z) = 3(z - a)^2 \Rightarrow f'(a) = 0$$

$$f''(z) = 6(z - a) \Rightarrow f''(a) = 0$$

$$f'''(z) = 6 \neq 0.$$

4. $f(z) = e^z$ has no zeros.

5. $f(z) = \sin z$ has simple zeros at $z = n\pi$, where $n = 0, \pm 1, \pm 2, \pm 3, \ldots$.

6. $f(z) = 1 - \cos z$ has ~~simple~~ second-order zeros at $z = 0, \pm 2\pi, \pm 4\pi, \ldots$

teaching order 1

Taylor series at a zero:-

At a n^{th}-order zero $z = z_0$ of $f(z)$, the derivatives $f'(z_0), \ldots, f^{(n-1)}(z_0)$ are zero, by definition. Hence the coefficients $a_0, a_1, \ldots, a_{n-1}$ of the Taylor series are zero, whereas $a_n \neq 0$ so that the series takes the form

$$f(z) = a_n (z - z_0)^n + a_{n+1} (z - z_0)^{n+1} + \cdots \qquad (a_n \neq 0)$$

$$= (z - z_0)^n \left[a_n + a_{n+1} (z - z_0) + \cdots \right]$$

Theorem: A point $z_0 \in \mathbb{C}$ is a zero of $f(z)$ of order n iff f can be expressed in the form

$$f(z) = \psi(z) \, (z - z_0)^n, \qquad |z - z_0| < r$$

for some $r > 0$, where ψ is analytic at z_0, and $\psi(z_0) \neq 0$.

Theorem:- The zeros of an analytic function $f(z)$ $(\neq 0)$ are isolated, i.e., each of them has a neighbourhood that contains no further zeros of $f(z)$.

teaching order 2

Theorem (Poles and Zeros):

Let $f(z)$ be analytic at $z = z_0$ and has a zero of n^{th} order at $z = z_0$. Then $1/f(z)$ has a pole of n^{th} order at $z = z_0$.

* The same holds for $\frac{g(z)}{f(z)}$ if $g(z)$ is analytic at $z = z_0$ and $g(z_0) \neq 0$.

Analytic or Singularity at Infinity (teaching order 8)

If we want to investigate $f(z)$ for large $|z|$, we set $z = 1/w$ and investigate $f(z) = f(1/w) \equiv g(w)$ in a neighbourhood of $w = 0$. We define $f(z)$ to be analytic (or singular) at infinity if $g(w)$ is analytic (or singular) at $w = 0$.

We also define $g(0) = \lim_{w \to 0} g(w)$ if this limit exists.

Furthermore, we say that $f(z)$ has a nth-order zero at infinity if $f(1/w)$ has such a zero at $w=0$. Similarly for poles and essential singularities.

Ex. ① $f(z) = \frac{1}{z^2}$ is analytic at ∞ and has a second-order zero at ∞.

since $g(\omega) = f\left(\frac{1}{\omega}\right) = \omega^2$ is analytic at $\omega = 0$.

And since $g(\omega) = \omega^2$, $g'(\omega) = 2\omega$, $g''(\omega) = 2 \neq 0$; $g(\omega)$ has a second-order zero at $\omega = 0$.

② $f(z) = z^3$ is singular and has a third-order pole at $z = \infty$.

Since $g(\omega) = f\left(\frac{1}{\omega}\right) = \frac{1}{\omega^3}$ is singular at $\omega = 0$.

Also, since $g(\omega) = \frac{1}{\omega^3}$ has a third-order pole at $\omega = 0$.

③ $f(z) = e^z$ has an essential singularity at $z = \infty$

since $g(w) = f\left(\frac{1}{w}\right) = e^{1/w}$ has an " " " $w = 0$.

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teaching order ④

Goes on Page 30

Theorem z_0 is a pole of f of order m iff f can be expressed in the form

$$f(z) = \frac{\psi(z)}{(z - z_0)^m}, \quad 0 < |z - z_0| < r$$

for some $r > 0$, where ψ is analytic at z_0 and $\psi(z_0) \neq 0$.

Proof: Suppose z_0 is a pole of f of order m. Then by Laurent theorem,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{m} \frac{b_n}{(z - z_0)^n}, \quad b_m \neq 0 \quad |z - z_0| > r$$

for some $r > 0$.

$$= \frac{1}{(z - z_0)^m}\left[\sum_{n=0}^{\infty} a_n (z - z_0)^{n+m} + b_1 (z - z_0)^{m-1} + b_2 (z - z_0)^{m-2} + \dots + b_m\right]$$

$$= \frac{\psi(z)}{(z - z_0)^m}$$

where $\psi(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^{n+m} + b_1 (z - z_0)^{m-1} + b_2 (z - z_0)^{m-2} + \dots + b_{m-1}(z - z_0) + b_m$

is analytic at z_0 and $\psi(z_0) = b_m \neq 0$.

Conversely, suppose $f(z) = \dfrac{\psi(z)}{(z-z_0)^m}$

where ψ is analytic at z_0 and $\psi(z_0) \neq 0$.

Since $\psi(z)$ is analytic at z_0, by Taylor's theorem

$$\psi(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

Therefore, it follows that z_0 is an isolated singularity of f. Moreover, in $|z-z_0| < r$

$$f(z) = \frac{1}{(z-z_0)^m} \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

$$= \frac{a_0}{(z-z_0)^m} + \frac{a_1}{(z-z_0)^{m-1}} + \cdots + a_m + \sum_{n=m+1}^{\infty} a_n (z-z_0)^{n-r}$$

Further, $a_0 = \psi(z_0) \neq 0$. Thus z_0 is a pole of f of order m.

<u>Theorem</u> A point $z_0 \in \mathbb{C}$ is a zero of order m iff f can be expressed in the form

$$f(z) = \psi(z)\,(z-z_0)^m \qquad |z-z_0|<r$$

for some $r>0$, where ψ is analyic at z_0 and $\psi(z_0)\neq 0$.

<u>Proof</u> :- Suppose $z_0 \in \mathbb{C}$ is a zero of f of order m.

Since f is analytic at z_0, by Taylor's theorem,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \qquad |z-z_0|<r$$

where $$a_n = \frac{f^{(n)}(z_0)}{n!}.$$

Since z_0 is a zero of f of order m, we have

$$a_n = \frac{f^{(n)}(z_0)}{n!} = 0 \quad \text{for } n=0,1,2,\dots,m-1$$

and $$a_m = \frac{f^{(m)}(z_0)}{m!} \neq 0.$$

Therefore we have,

$$f(z) = \sum_{n=m}^{\infty} a_n (z-z_0)^n \qquad |z-z_0|<r$$

$$= (z-z_0)^m \left(a_m + a_{m+1}(z-z_0) + \dots + \dots\right)$$

$$= (z-z_0)^m \,\psi(z)$$

where $\psi(z) = a_m + a_{m+1}(z-z_0) + a_{m+2}(z-z_0)^2 + \dots$.

Note that $\psi(z)$ is analytic at z_0, and $\psi(z_0) = a_m \neq 0$.

Conversely,

Suppose $f(z) = \psi(z)\,(z-z_0)^m$, where ψ is analytic at z_0 and $\psi(z_0) \neq 0$. We show that $z_0 \in \mathbb{C}$ is a zero of f of order m.

Since ψ is analytic at z_0, we obtain a $r>0$ such that

$$\psi(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \qquad |z-z_0| < r$$

$$\therefore\ f(z) = (z-z_0)^m \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

$$= \sum_{n=0}^{\infty} a_n (z-z_0)^{m+n} \qquad |z-z_0| < r$$

This gives the Taylor series representation of $f(z)$ around z_0. Therefore f is analytic at z_0.

From the uniqueness of Taylor series, we obtain

$$f^{(n)}(z_0) = 0 \quad \text{for } n = 0, 1, 2, \ldots m-1 \text{ and}$$

$$f^{(m)}(z_0) \neq 0.$$

This shows that $z_0 \in \mathbb{C}$ is a zero of f of order m.

teaching order 5

Theorem:- Let z_0 be an isolated singularity of $f(z)$. Then z_0 is a pole of $f(z)$ of order m iff $1/f(z)$ is analytic at z_0 and has a zero of order m.

Proof:- z_0 is a pole of $f(z)$ of order m

$\Leftrightarrow f(z) = \dfrac{\psi(z)}{(z-z_0)^m}$, where ψ is analytic at z_0

and $\psi(z_0) \neq 0$.

$\Leftrightarrow \dfrac{1}{f(z)} = g(z)(z-z_0)^m$, where $g(z) = \dfrac{1}{\psi(z)}$ is analytic at z_0 and $g(z_0) \neq 0$.

$\Leftrightarrow$ z_0 is a zero of $\dfrac{1}{f(z)}$ of order m.

Remark:- The same holds for $h(z)/f(z)$ if $h(z)$ is analytic at $z = z_0$ and $h(z_0) \neq 0$.

teaching order 6

Theorem (Finding the order of a pole): Let z_0 be a pole of order m. Then for all positive integers k, we have

$$\lim_{z \to z_0} (z-z_0)^k f(z) = \begin{cases} \ell, & k = m \\ 0, & k > m \\ \infty, & k < m \end{cases}$$

teaching order 7

for some $\ell \neq 0$

Ex. $f(z) = \dfrac{1}{z(z-2)^5} + \dfrac{3}{(z-2)^2}$

has singularities at $z=0$ and $z=2$. These singularities are poles.

To find the order of poles:

$$\lim_{z\to 0} z f(z) = \lim_{z\to 0}\left[\frac{1}{(z-2)^5} + \frac{3z}{(z-2)^2}\right] = \frac{1}{(-2)^5} \neq 0$$

So $z=0$ is a pole of order 1 or a simple pole.

$$\lim_{z\to 2} (z-2)^5 f(z) = \lim_{z\to 2}\left[\frac{1}{z} + 3(z-2)^3\right] = \frac{1}{2} \neq 0$$

but $\lim_{z\to 2} (z-2)^k f(z) = \infty$ for $k = 1, 2, 3, 4$.

Hence $z=2$ is a pole of order 5.

Res

Residues and Residue Theorem

Consider the contour integral $\oint_C f(z)\,dz$.

If $f(z)$ has no singularity inside the closed contour C, then the value of the integral is zero by Cauchy-Goursat's theorem.

How do we deal with cases where singularities of f are inside the closed contour C?

In what follows, we illustrate how to apply the method of residues to evaluate such integrals without actually computing the integrals directly.

<u>Def^n</u> (Residue) Let $z = z_0$ be an isolated singularity of $f(z)$. Then there exists a certain deleted neighbourhood $0<|z-z_0|<r$ (for some $r>0$) such that f is analytic everywhere in this deleted neighbourhood and hence has the Laurent series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

valid in this deleted neighbourhood. The coefficient of $\frac{1}{z-z_0}$, that is, b_1 in the Laurent series expansion is called the <u>residue</u> of f at the isolated singularity z_0 and is denoted by $\text{Res}(f; z_0)$ or $\underset{z=z_0}{\text{Res}}\, f(z)$. That is,

$$\operatorname*{Res}_{z=z_0} f(z) = \operatorname{Res}(f; z_0) = b_1 = \frac{1}{2\pi i}\oint_C f(z)\,dz,$$

where C is any simple closed positively oriented contour lying completely inside the deleted neighbourhood.

Ex. ① $\operatorname{Res}\left(\frac{1}{(z-z_0)^k}; z_0\right) = \begin{cases} 1 & \text{if } k=1 \\ 0 & \text{if } k\neq 1 \end{cases}$

② $\operatorname{Res}(e^{1/z}; 0) = 1$ because

$$e^{1/z} = 1 + \boxed{\frac{1}{1!}}\frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \cdots, \quad |z|>0.$$

③ $\operatorname{Res}\left(\frac{1}{(z-1)(z-2)}; 1\right) = \frac{1}{2\pi i}\oint_C \frac{1}{(z-1)(z-2)}\,dz = \frac{1}{2\pi i}\times 2\pi i\,\frac{1}{1-2} = -1$

(by Cauchy integral formula)

The above examples show that sometimes residues can be found in a straightforward manner. But it is not always the case, and we give below some results which are useful in determining residues.

Use: Evaluation of an integral by means of a residue

Evaluate $\oint_C z^{-4}\sin z\,dz$ where C is the counterclockwise oriented unit circle around $z=0$.

$$f(z) = \frac{\sin z}{z^4} = \frac{1}{z^4}\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = \frac{1}{z^3} - \frac{1}{3!\,z} + \frac{z}{5!} - \frac{z^3}{7!} + \cdots, \quad |z|>0$$

$f(z)$ has a pole of third-order at $z=0$ and

$$\operatorname{Res}(f; 0) = -\frac{1}{3!} = -\frac{1}{6}$$

$$\Rightarrow \frac{1}{2\pi i}\oint_C f(z)\,dz = -\frac{1}{6} \Rightarrow \oint_C \frac{\sin z}{z^4}\,dz = -\frac{\pi i}{3}.$$

Theorem (Residue at a simple pole)

Let f be analytic on $0<|z-z_0|<r$ and suppose that f has a simple pole at z_0, then

$$\operatorname{Res}(f; z_0) = \lim_{z\to z_0} (z-z_0) f(z).$$

Proof: $z=z_0$ is a simple pole of $f(z)$, therefore

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0}$$

$$\Rightarrow \lim_{z\to z_0} (z-z_0) f(z) = b_1. \qquad \square$$

Corollary:

① Let $f(z) = \dfrac{g(z)}{z-z_0}$, ~~where~~ where $g(z)$ is analytic on $0<|z-z_0|<r$ and $g(z_0)\neq 0$. Then

$$\operatorname{Res}(f; z_0) = \lim_{z\to z_0} g(z) = g(z_0).$$

Proof $g(z)$ is analytic on $0<|z-z_0|<r$. By Taylor's theorem

$$g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(z_0)}{n!}(z-z_0)^n$$

$$= g(z_0) + \frac{g'(z_0)}{1!}(z-z_0) + \frac{g''(z_0)}{2!}(z-z_0)^2 + \cdots$$

$$\Rightarrow f(z) = \frac{g(z)}{(z-z_0)} = \frac{g(z_0)}{z-z_0} + \frac{g'(z_0)}{1!} + \frac{g''(z_0)}{2!}(z-z_0) + \cdots$$

Not required in proof.

$$\Rightarrow \operatorname{Res}(f; z_0) \cancel{= g(z_0)} = \lim_{z\to z_0} (z-z_0) f(z) = \lim_{z\to z_0} g(z) = g(z_0).$$

(By the above theorem)

② Let $f(z) = \frac{p(z)}{q(z)}$, where $p(z)$ and $q(z)$ are both analytic at z_0. Further, if $p(z_0) \neq 0$, $q(z_0) = 0$ and $q'(z_0) \neq 0$. Then

z_0 is a simple zero of q
$\therefore$ z_0 is a simple pole of f.

$$\operatorname{Res}(f; z_0) = \frac{p(z_0)}{q'(z_0)}.$$

Proof:- ~~Since zeros of an isolate analytic function are isolated~~

Since z_0 is a zero of $q(z)$ and zeros are isolated points, it follows that there exists a deleted neighbourhood $0<|z-z_0|<r$ such that $q(z) \neq 0$ $\forall z \in D = \{z : 0<|z-z_0|<r$

$\therefore$ $f(z) = \frac{p(z)}{q(z)}$ is analytic on $0<|z-z_0|<r$

and hence z_0 is an isolated singularity of $f(z)$.

By the above theorem,

$$\lim_{z \to z_0} (z-z_0) f(z) = \lim_{z \to z_0} \frac{(z-z_0)p(z)}{q(z)}$$

$$= \lim_{z \to z_0} \frac{p(z) + (z-z_0)p'(z)}{q'(z)}$$

$$= \frac{p(z_0)}{q'(z_0)} \neq 0.$$

Hence z_0 is a simple pole of $f(z) = \frac{p(z)}{q(z)}$. Therefore

$$\operatorname{Res}(f; z_0) = \lim_{z \to z_0} (z-z_0) f(z) = \frac{p(z_0)}{q'(z_0)}.$$

□

Theorem (Residue at a pole of order m):

Let f has a pole of order m at z_0. Then

$$R(f;z_0) = \frac{1}{(m-1)!} \lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]$$

Proof:- By the given condn,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_m}{(z-z_0)^m}$$

$$b_m \neq 0, \qquad 0<|z-z_0|<r$$

$$\Rightarrow (z-z_0)^m f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^{n+m} + b_1 (z-z_0)^{m-1} + b_2 (z-z_0)^{m-2} + \cdots + b_m$$

$$\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right] = b_1 (m-1)! + \text{terms containing +ve powers of } (z-z_0)$$

$$\Rightarrow \lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right] = b_1 (m-1)!$$

$$\Rightarrow \text{Res}(f;z_0) = b_1 = \frac{1}{(m-1)!} \lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]. \qquad \square$$

Ex. ① $f(z) = \dfrac{1}{(z-2)(z^2+4)}$. f has simple poles at $2, \pm 2i$.

$$\text{Res}(f;2) = \lim_{z\to 2} (z-2) f(z) = \lim_{z\to 2} \frac{1}{(z^2+4)} = \frac{1}{8}.$$

$$\text{Res}(f;2i) = \lim_{z\to 2i} (z-2i) f(z) = \lim_{z\to 2i} \frac{1}{(z-2)(z+2i)} = \frac{1}{(2i-2)4i}.$$

$$\text{Res}(f;-2i) = \lim_{z\to -2i} (z+2i) f(z) = \lim_{z\to -2i} \frac{1}{(z-2)(z-2i)} = \frac{1}{(2i+2)4i}.$$

② $f(z) = \cot z$ has a simple pole at $z = 0$.

$$\text{Res}(f;0) = \lim_{z \to 0} z \cot z = \lim_{z \to 0} \frac{z}{\sin z} \cos z = 1.$$

③ $f(z) = \dfrac{1}{z^3 - z^4}$ has a pole of order 3 at $z = 0$.

$$\text{Res}(f;0) = \frac{1}{2!} \lim_{z \to 0} \frac{d^2}{dz^2}\left[z^3 f(z)\right]$$

$$= \frac{1}{2!} \lim_{z \to 0} \frac{d^2}{dz^2}\left(\frac{1}{1-z}\right) = \frac{1}{2} \lim_{z \to 0} \frac{2}{(1-z)^3} = 1.$$

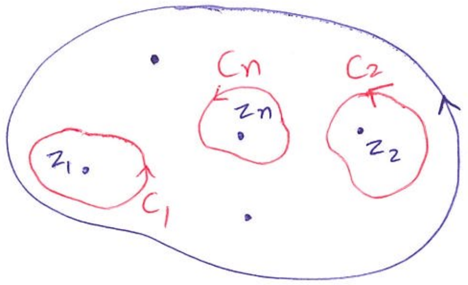
Theorem (Cauchy Residue Theorem)

Let $f(z)$ be analytic inside a simple closed contour C and on C except for finitely many singular points $z_1, z_2, \ldots, z_n$ inside C. Then

$$\oint_C f(z)\, dz = 2\pi i \left[\text{Res}(f;z_1) + \text{Res}(f;z_2) + \cdots + \text{Res}(f;z_n)\right]$$

where the integral is being taken counterclockwise.

Proof:



By Cauchy-Goursat's theorem for multiply-connected domains,

$$\oint_C f(z)\, dz = \oint_{C_1} f(z)\, dz + \oint_{C_2} f(z)\, dz + \cdots + \oint_{C_n} f(z)\, dz$$

$$= 2\pi i\, \text{Res}(f;z_1) + 2\pi i\, \text{Res}(f;z_2) + \cdots + 2\pi i\, \text{Res}(f;z_n)$$

$$= 2\pi i \sum_{n=1}^{\infty} \text{Res}(f;z_n).$$

$\square$

Ex. Evaluate $\oint_C \frac{4-3z}{z^2-z}\,dz$ around any counterclockwise simple closed contour C such that

(i) 0 and 1 are inside C.

(ii) 0 is inside but 1 is outside C.

(iii) 0 is outside but 1 is inside C.

(iv) 0 and 1 are outside C.

Soln: $f(z) = \frac{4-3z}{z^2-z}$ has singularities at $z=0$ and $z=1$.

Moreover, both 0 and 1 are simple poles of $f(z)$. Therefore,

$$\text{Res}(f;0) = \lim_{z\to 0} z\cdot\left(\frac{4-3z}{z^2-z}\right) = \lim_{z\to 0}\frac{4-3z}{z-1} = -4.$$

$$\text{Res}(f;1) = \lim_{z\to 1}(z-1)\left(\frac{4-3z}{z^2-z}\right) = \lim_{z\to 1}\frac{4-3z}{z} = 1.$$

(i) $$\oint_C \frac{4-3z}{z^2-z}\,dz = 2\pi i\left[\text{Res}(f;0) + \text{Res}(f;1)\right]$$
$$= 2\pi i\,(-4+1) = -6\pi i.$$

(ii) $$\oint_C \frac{4-3z}{z^2-z}\,dz = 2\pi i\ \text{Res}(f;0) = -8\pi i.$$

(iii) $$\oint_C \frac{4-3z}{z^2-z}\,dz = 2\pi i\ \text{Res}(f;1) = 2\pi i$$

(iv) $$\oint_C \frac{4-3z}{z^2-z}\,dz = 0$$ (by Cauchy-Goursat's theorem)

Ex. Evaluate $\oint_C z e^{\pi/z} dz$, where C is the ellipse $9x^2+y^2=9$ (oriented counterclockwise).

Sol^n $z=0$ is a singularity of $f(z)=z e^{\pi/z}$.

$$f(z) = z e^{\pi/z} = z \sum_{n=0}^{\infty} \frac{\pi^n}{n!\, z^n} = \sum_{n=0}^{\infty} \frac{\pi^n}{n!\, z^{n-1}}, \qquad |z|>0.$$

It follows that 0 is ~~the~~ essential Singularity of $f(z)=z e^{\pi/z}$ and $\text{Res}(f;0) = b_1 = \frac{\pi^2}{2!}$. Therefore,

$$\oint_C z e^{\pi/z} dz = 2\pi i \times \text{Res}(f;0) = \pi^3 i.$$

Ex. Integrate $f(z) = \frac{1}{z^3-z^4}$ clockwise around the circle $C: |z| = \frac{1}{2}$.

Sol^n $z=0$ and 1 are the only singularities of $f(z)=1/(z^3-z^4)$

Also $z=0$ is a pole of order 3 because

$$\lim_{z\to 0} z^3 f(z) = \lim_{z\to 0}\frac{1}{1-z} = 1$$

but $\lim_{z\to 0} z f(z) = \lim_{z\to 0} z^2 f(z) = \infty$.

Also the singularity $z=1$ is outside of given C, so it i[s] of no interest.

$$\text{Res}(f;0) = \frac{1}{2!}\lim_{z\to 0}\frac{d^2}{dz^2}\left(z^3 f(z)\right) = \frac{1}{2}\lim_{z\to 0}\frac{d^2}{dz^2}\frac{1}{1-z} = \frac{1}{2}\lim_{z\to 0}\frac{2}{(1-z)^3} = 1.$$

$$\therefore \int_C f(z)\, dz = \underbrace{-2\pi i \times \text{Res}(f;0) = -2\pi i}_{\text{clockwise } C}.$$

* Note that had we used wrong Laurent's, expansion

$$\frac{1}{z^3-z^4} = -\frac{1}{z^4} - \frac{1}{z^5} - \frac{1}{z^6} - \cdots, \qquad |z|>1$$

we would have obtained wrong answer 0.

Evaluation of Real Integrals by Residue Methods

We consider a real integral involving trigonometric functions of the form

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$$

where $f(x,y)$ is a rational function defined inside the unit circle $|z|=1$, $z=x+iy$.

The real integral involving $\sin\theta$ and $\cos\theta$ can be converted into a contour integral around the unit circle by the following substitutions.

$$z = e^{i\theta}$$

$$\Rightarrow dz = i e^{\theta}\, d\theta = i z\, d\theta$$

$$\Rightarrow d\theta = \frac{dz}{iz}$$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{1}{2}\left(z+\frac{1}{z}\right)$$

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{1}{2i}\left(z-\frac{1}{z}\right).$$

Let C be the unit circle $|z|=1$ traversed in the anticlockwise direction. The real integral can then be transformed as

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta = \oint_C \frac{1}{iz} f\left(\frac{z+\frac{1}{z}}{2}, \frac{z-\frac{1}{z}}{2i}\right) dz$$

$$= 2\pi i \left[\text{Sum of residues of } \frac{1}{iz} f\left(\frac{z+\frac{1}{z}}{2}, \frac{z-\frac{1}{z}}{2i}\right) \text{inside } C: |z|=1\right]$$

Ex. Show that $\displaystyle\int_0^{2\pi} \frac{1}{\sqrt{2}-\cos\theta}\, d\theta = 2\pi.$

Sol^n:- Let $z = e^{i\theta}$.

$$\Rightarrow \quad dz = i e^{i\theta} d\theta = iz\, d\theta$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z+\frac{1}{z}}{2}.$$

Let C be the unit circle $|z|=1$ traversed in the anticlockwise direction. Then we have

$$\int_0^{2\pi} \frac{1}{\sqrt{2}-\cos\theta}\, d\theta = \oint_C \frac{1}{iz\left(\sqrt{2}-\frac{1}{2}\left(z+\frac{1}{z}\right)\right)} dz$$

$$= \oint_C \frac{1}{i\cancel{z}\left(-\frac{1}{2}\right)\frac{(z^2-2\sqrt{2}z+1)}{\cancel{z}}} dz$$

$$= -\frac{2}{i} \oint_C \frac{1}{(z-\sqrt{2}-1)(z-\sqrt{2}+1)} dz$$

$$= -\frac{2}{i} \times 2\pi i \,\text{Res}\left[\frac{1}{z^2-2\sqrt{2}z+1}; \sqrt{2}-1\right]$$

$$= -4\pi \times \lim_{z\to\sqrt{2}-1} \cancel{(z-\sqrt{2}+1)} \frac{1}{(z-\sqrt{2}-1)\cancel{(z-\sqrt{2}+1)}}$$

$$= -4\pi \times \frac{1}{\sqrt{2}-1-\sqrt{2}-1} = -4\pi \times \frac{1}{(-2)} = 2\pi.$$

$$z = \frac{2\sqrt{2} \pm \sqrt{8-4\times1\times1}}{2} = \sqrt{2} \pm 1$$

$$z = \begin{cases} \sqrt{2}+1 > 1 \\ \sqrt{2}-1 < 1 \end{cases}$$

Improper Integrals of Rational Functions

We now consider real integrals of the type

$$\int_{-\infty}^{\infty} f(x)\, dx .$$

Such an integral for which the interval of integration is not finite, is called an Improper integral, and it has the meaning

$$\int_{-\infty}^{\infty} f(x)\, dx = \lim_{a\to-\infty}\int_a^0 f(x)\, dx + \lim_{b\to\infty}\int_0^b f(x)\, dx .$$

If both limits exist, we may couple the two independent passages to $-\infty$ and ∞, and write

$$\int_{-\infty}^{\infty} f(x)\, dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)\, dx .$$

The expression $\lim_{R\to\infty}\int_{-R}^{R} f(x)\, dx$ is called the Cauchy principal value of the integral; it may exist even if the limits $\lim_{a\to-\infty}\int_a^0 f(x)\, dx$ and/or $\lim_{b\to\infty}\int_0^b f(x)\, dx$ do not exist. For instance,

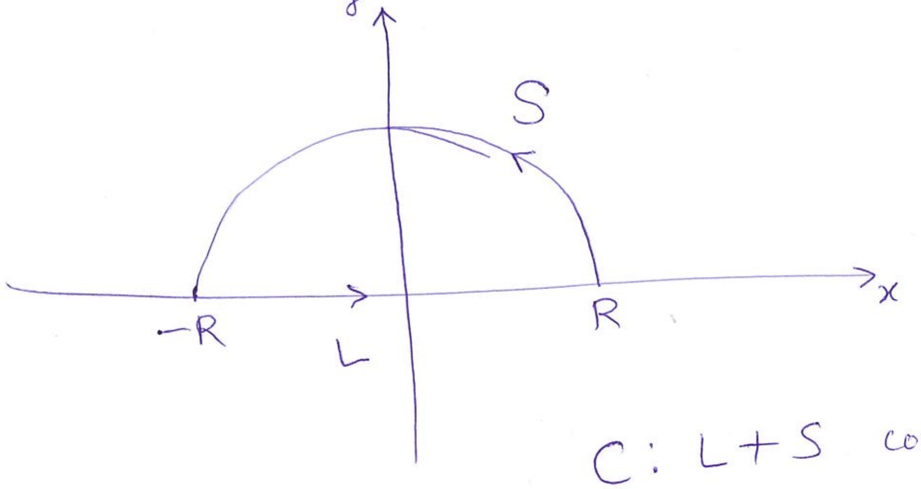
$$\lim_{R\to\infty}\int_{-R}^{R} x\, dx = \lim_{R\to\infty}\left(\frac{R^2}{2} - \frac{R^2}{2}\right) = 0 \quad \text{but} \quad \lim_{b\to\infty}\int_0^b x\, dx = \infty .$$

We assume that $f(x)$ in $\int_{-\infty}^{\infty} f(x)\, dx$ is a real function whose denominator is different from zero for all real x and is of degree at least two higher than the degree of the numerator. Then the limits $\lim_{a\to-\infty}\int_a^0 f(x)\, dx$ and $\lim_{b\to\infty}\int_0^b f(x)\, dx$ exist and we may start with $\lim_{R\to\infty}\int_{-R}^{R} f(x)\, dx$.

We consider the contour integral

$$\oint_C f(z)\,dz$$

around a path C as shown below



C: $L+S$ counterclockwise

Since $f(x)$ is rational, $f(z)$ has finitely many poles in the upper-half plane, and if we choose R large enough, the C encloses all these poles. By residue theorem, we then obtain

$$\oint_C f(z)\,dz = \int_S f(z)\,dz + \int_{-R}^{R} f(z)\,dz = 2\pi i \sum \operatorname{Res} f(z)$$

where the sum consists of all the residues of $f(z)$ at the points in the upper-half plane at which $f(z)$ has a pole. From this, we have

$$\int_{-R}^{R} f(x)\,dx = 2\pi i \sum \operatorname{Res} f(z) - \int_S f(z)\,dz.$$

We prove that if $R \to \infty$, the value of the integral $\int_S f(z)\, dz \to 0$.

If we set $z = Re^{i\theta}$, then S is represented by $R=$ constant, and as z ranges along S, the variable θ ranges from 0 to π. Since by assumption, the degree of the denominator of $f(z)$ is at least two units higher than the degree of the numerator, we have

$$|f(z)| < \frac{k}{|z|^2} \qquad (|z| = R > R_0)$$

for sufficiently large constants k and R_0. By ML-inequality,

$$\left| \int_S f(z)\, dz \right| < \frac{k}{R^2} \times \pi R = \frac{k\pi}{R} \qquad (R > R_0)$$

Hence, as $R \to \infty$, $\int_S f(z)\, dz \to 0$. Consequently,

$$\boxed{\int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \sum \text{Res } f(z)}$$

where the sum is over all the residues of $f(z)$ corresponding to the poles of $f(z)$ in the upper-half plane.

Ex. 1 Evaluate $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx$. by the residue method.

Soln

Here $f(x) = \frac{1}{1+x^2}$.

Note that $f(x)$ satisfies both conditions, namely,

(i) its denominator is nonzero for all real x,

(ii) the degree of the denominator is at least two higher than the degree of the numerator.

Moreover, $f(z) = \frac{1}{1+z^2}$ has poles at $z = \pm i$ and only the pole $z = i$ lies in the upper-half plane.

~~A~~lso, $z = i$ is a simple pole and hence

$$\text{Res}\left(\frac{1}{1+z^2}; i\right) = \lim_{z \to i} (z-i) \underset{\frac{1}{1+z^2}}{\underset{\parallel}{f(z)}} = \lim_{z \to i} \frac{1}{z+i} = \frac{1}{2i}.$$

Therefore $\int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \;\text{Res}(f; i)$

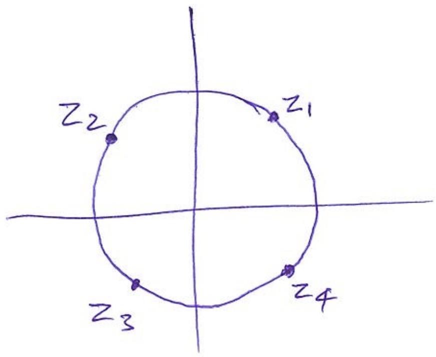
$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx = 2\pi i \times \text{Res}\left(\frac{1}{1+z^2}; i\right)$$

$$= 2\pi i \times \frac{1}{2i} = \pi.$$

Ex. $$\int_0^\infty \frac{1}{1+x^4} = \frac{\pi}{2\sqrt{2}}$$

Solⁿ $f(z) = \dfrac{1}{1+z^4}$ Conditions on f are satisfied for this $f(z)$.

$f(z) = \dfrac{1}{1+z^4}$ has four simple poles at $z_1 = e^{\pi i/4}$, $z_2 = e^{3\pi i/4}$, $z_3 = e^{-3\pi i/4}$, $z_4 = e^{-\pi i/4}$



The poles z_1 & z_2 lie in the upper-half plane.

$$\text{Res}\left(\frac{1}{1+z^4};\, e^{\pi i/4}\right) = \left[\frac{1}{4z^3}\right]_{z=z_1}$$

(Res f(z) z_0 = $\dfrac{p(z_0)}{q'(z_0)}$ formula for $f(z) = \dfrac{p(z)}{q(z)}$)

$$= \frac{1}{4e^{3\pi i/4}} = -\frac{1}{4}e^{\pi i/4}$$

$$\text{Res}\left(\frac{1}{1+z^4};\, e^{3\pi i/4}\right) = \left[\frac{1}{4z^3}\right]_{z=z_2}$$

$$= \frac{1}{4e^{9\pi i/4}} = \frac{1}{4}e^{-\pi i/4}$$

$$\therefore \int_{-\infty}^{\infty} \frac{1}{1+x^4}\,dx = 2\pi i \sum \text{Res}(f(z)) = 2\pi i\left(-\frac{1}{4}e^{\pi i/4} + \frac{1}{4}e^{-\pi i/4}\right)$$

$$= \frac{2\pi i}{4}\left(-\cancel{\frac{1}{\sqrt{2}}} - i\frac{1}{\sqrt{2}} + \cancel{\frac{1}{\sqrt{2}}} - i\frac{1}{\sqrt{2}}\right) = \frac{\pi i}{2} \times \left(-\frac{2i}{\sqrt{2}}\right) = \frac{\pi}{\sqrt{2}}.$$

$$\therefore \int_0^\infty \frac{1}{1+x^4}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{1+x^4}\,dx = \frac{\pi}{2\sqrt{2}}.$$

Integrals of the form $\int_{-\infty}^{\infty} f(x)\cos sx\,dx$ and $\int_{-\infty}^{\infty} f(x)\sin sx\,dx$

~~Let~~ $f(x)$ is a rational function satisfying the conditions as before that its denominator is nonzero for all real x and its denominator has degree at least 2 higher than the degree of its numerator.

Let $\alpha_1, \alpha_2, ---, \alpha_n$ be the complete list of poles of $f(z)$ in the upper-half plane. Then (as before)

$$\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f(z);\alpha_k).$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\cos sx\,dx + i\int_{-\infty}^{\infty} f(x)\sin sx\,dx$$

$$= -2\pi \sum_{k=1}^{n} \operatorname{Im}\left(\operatorname{Res}(f(z);\alpha_k)\right) + i.2\pi \sum_{k=1}^{n} \operatorname{Re}\left(\operatorname{Res}(f(z);\alpha_k)\right)$$

Comparing the real and imaginary parts on both sides

$$\int_{-\infty}^{\infty} f(x)\cos sx\,dx = -2\pi \sum_{k=1}^{n} \operatorname{Im}\left(\operatorname{Res}(f(z);\alpha_k)\right)$$

$$\int_{-\infty}^{\infty} f(x)\sin sx\,dx = 2\pi \sum_{k=1}^{n} \operatorname{Re}\left(\operatorname{Res}(f(z);\alpha_k)\right).$$

Ex. Show that $\displaystyle\int_{-\infty}^{\infty} \frac{\cos sx}{k^2+x^2}\,dx = \frac{\pi}{k} e^{-ks}$ and

$$\int_{-\infty}^{\infty} \frac{\sin sx}{k^2+x^2}\,dx = 0 \qquad (s>0,\ k>0)$$

Solⁿ $\displaystyle f(z) = \frac{e^{isz}}{k^2+z^2}$ has only one pole $z = ik$ in the upper-half plane.

$$\mathrm{Res}(f(z);\, ik) = \lim_{z\to ik} (z-ik)\frac{e^{isz}}{k^2+z^2} = \lim_{z\to ik} \frac{e^{isz}}{z+ik}$$

$$= \frac{e^{-ks}}{2ik}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{isx}}{k^2+x^2}\,dx = 2\pi i\ \mathrm{Res}(f(z);\, ik)$$
$$= \cancel{2}\pi \cancel{i} \times \frac{e^{-ks}}{\cancel{2i}k} = \frac{\pi}{k} e^{-ks}.$$

Comparing the real and imaginary parts on both sides,

$$\int_{-\infty}^{\infty} \frac{\cos sx}{k^2+x^2}\,dx = \frac{\pi}{k} e^{-ks}$$

and $$\int_{-\infty}^{\infty} \frac{\sin sx}{k^2+x^2}\,dx = 0.$$