Tutorial-2

(1) $y' = x^3 + 2xy^2 + y^3,\ y(0) = 1$ — TAYLORS METHOD

$$y(x_1) = y(x_0) + h f(x_0, y_0) + \frac{h^2}{2!}\left[f_x + f_y f\right]_{(x_0,y_0)} + \frac{h^3}{3!}\left[f_{xx} + 2f f_{xy} + f^2 f_{yy} + f_x f_y + f f_y^2\right]_{(x_0,y_0)}$$

For the Particular Problem, $(x_0, y_0) = (0, 1)$

$$y(x_1) = 1 + h + 5\frac{h^2}{2!} + 29\frac{h^3}{3!}$$

$y(0.2) = 1.33857 \quad (h = 0.2)$

(2) $y'' - xy' - y = 0,\ y(0) = 1;\ y'(0) = 0$ —— TAYLORS METHOD

We have: $y'' = xy' + y$

$$y(x) = y(0) + x \cdot y'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{6} y'''(0) + \frac{x^4}{24} y^{iv}(0) + \frac{x^5}{120} y^{v}(0) + \frac{x^6}{720} y^{vi}(0)$$

Now: $y''(0) = y(0) = 0$

$y'''(x) = x y'' + 2y'$

$y^{iv}(x) = x y''' + 3y''$

$\vdots$

$y'''(0) = 0$

$y^{iv}(0) = 3$

$y^{v}(0) = 0$

$y^{vi}(0) = 5$

$$y(0.1) = 1 + \frac{(0.1)^2}{2} + \frac{(0.1)^4}{24}(3) + \frac{(0.1)^6}{720}(5)$$

$= 1.0050125$

(3). $y' = \dfrac{x^2}{y^2+1}$ $\quad y(0) = 0$ at $x = 0.25, 0.5, 1$ —— PICARDS METHOD.

$$y^{(n)} = y_0 + \int_{x_0}^{x} f(x, y^{(n-1)})\, dx$$

$$y^{(1)} = \int_0^x f(x, y^{(0)})\, dx, \quad y^{(0)} = 0$$

We get $y^{(1)} = \frac{1}{3}x^3$

$$y^{(2)} = \int_0^x \frac{x^2}{\left(\frac{1}{9}\right)x^6 + 1} \cdot dx = \tan^{-1}\left(\frac{x^3}{3}\right) = \frac{1}{3}x^3 - \frac{1}{81}x^9 + \dots$$

Correct to 3 decimal places: $\dfrac{x^9}{81} \leq 0.0005$

$\Rightarrow x \leq 0.7$ //

$\therefore y(0.25) = \frac{1}{3}(0.25)^3 = 0.005$ //.

$y(0.5) = \frac{1}{3}(0.5)^3 = 0.042$ //

$y(1.0) = \frac{1}{3} - \frac{1}{81} = 0.321$ //

(4). $y' = 1 + xy^2 \quad , y(0) = 1$ —— EULERS METHOD.

Eulers method: $y_{n+1} = y_n + h f(x_n, y_n)$, $y' = f(x, y)$

Total solution error: $e_{n+1} = e_n[1 + h f_y(x_n, \xi_n)] + L_{n+1}$, $e_0 = 0$

$$L_{n+1} = \frac{-h^2}{2} y''(\tau_n)$$

Given: $y' = 1 + xy^2 = f(x, y)$

$$y'' = y^2 + 2xy(1 + xy^2) \quad \left(\because y'' = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}\right).$$

For $x = 0.1$, $y_1 = y_0 + h f(x_0, y_0) = 1.1$

Error: $\varepsilon_1 = \frac{-h^2}{2} y''(\xi)$, $0 \le \xi_1 \le 0.1$

At $(0.1, 1.1)$, $\varepsilon_1 = -0.00728$ //.

For $x = 0.2$: $y_2 = y_1 + h f(x_1, y_1) = 1.2121$.

Error: $\varepsilon_2 = \varepsilon_1(1 + h f_y(x_1, y_1)) - \frac{h^2}{2} y''(\xi)$

$$= \varepsilon_1(1 + h f_y(x_1, y_1)) - \frac{h^2}{2} y''(\xi)\Big|_{(x,y) = (x_2, y_2)}$$

$$= \cancel{0.017}\ 0.00744 - 0.005(1.492 + 0.6272) = -0.008156$$

(5) $y' = 2x + y$; $y(1) = 2$. ——— MODIFIED EULER

Modified Euler:

$$\left.\begin{aligned} y_1^{(n+1)} &= y_0 + \frac{h}{2}\left(f(x_0, y_0) + f(x_1, y_1^{(n)})\right) \\ y_1^{(0)} &= y_0 + h \cdot f(x_0, y_0). \\ y_2^{(n+1)} &= y_1 + \frac{h}{2}\left(f(x_1, y_1) + f(x_2, y_2^{(n)})\right) \\ y_2^{(0)} &= y_1 + h\, f(x_1, y_1) \\ &\vdots \end{aligned}\right\} \text{Iterati process}$$

Exact soln:

$$\boxed{y = \frac{-2}{e}(e - 3e^x + xe)}$$

(6) $y' = y^2 + yx$, $y(1) = 1$ ——— IMPROVED EULER.

~~ME~~: ~~YH~~. Improved Euler:

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1}^{e})\right]$$

$$y_{n+1}^{(e)} = y_n + h\, f(x_n, y_n).$$

From Improved:

$$y(1.2) = 1.564$$

$$y(1.4) = 2.9293$$

$y' = x^2 + y, \quad y(0) = 1$

Modified Euler:

$y_1^{(0)} = 1.05$

$\left.\begin{array}{l} y_1^{(1)} = 1.0513 \\ y_1^{(2)} = 1.0513 \end{array}\right\} y_1 = 1.0513$

$y_2^{(0)} = 1.1040$

$\left.\begin{array}{l} y_2^{(1)} = 1.1055 \\ y_2^{(2)} = 1.1055 \end{array}\right\} y_2 = 1.1055$

First order RK method: $y' = f(x, y)$

$$\left.\begin{array}{l} y_1 = y_0 + h\, f(x_0, y_0) \\ y_2 = y_1 + h\, f(x_1, y_1) \\ \vdots \end{array}\right\} \text{Euler method}$$

Second order RK method:

$$\left.\begin{array}{l} y_1 = y_0 + \frac{k1}{2}(k_1 + k_2) \\ k_1 = h \cdot f(x_0, y_0) \\ k_2 = h\, f(x_1, y_0 + k_1) \end{array}\right\} \text{Improved Euler method.}$$

Third order RK method:

$$y_1 = y_0 + \frac{1}{6}(k_1 + 4k_2 + k_3)$$

$$k_1 = h\, f(x_0, y_0)$$

$$k_2 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = h\, f(x_0 + h, y_0 - k_1 + 2k_2)$$

Fourth order RK method

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = h\, f(x_0, y_0)$$

$$k_2 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = h\, f(x_0 + h, y_0 + k_3)$$

Soln:

$y'' - xy' + y = 0, \quad y(0) = 1, \quad y'(0) = -1 \quad h = 0.3$

let $y' = z \Rightarrow z' = xz - y$

$\therefore \cancel{y' = z} = \cancel{f(x, y, z)}$

$\therefore y' = y = f(x, y, z)$, $y(0) = 1$

$z' = xz - y = g(x, y, z)$, $z(0) = 1$

$k_1 = h\, f(x_0, y_0, z_0)$

$m_1 = h\, g(x_0, y_0, z_0)$

$k_2 = h\, f(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{m_1}{2})$

$m_2 = h\, g(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{m_1}{2})$

$k_3 = h\, f(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{m_2}{2})$

$m_3 = h\, g(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{k_2}{2})$

$k_4 = h\, f(x_0 + h, y_0 + k_3, z_0 + m_3)$

$m_4 = h\, g(x_0 + h, y_0 + k_3, z_0 + m_3)$

$\therefore y(x_1) = y_0 + \frac{1}{6}[k_1 + 2(k_2 + k_3) + k_4)$

$z(x_1) = z_0 + \frac{1}{6}(m_1 + 2(m_2 + m_3) + m_4)$

$k_1 = -1$, $m_1 = -1$

$k_2 = -1.15$, $m_2 = -1.0225$

$k_3 = -1.153375$, $m_3 = -1.0005$

$k_4 = -1.30015$, $m_4 = -0.92992$

$y(0.3) = -0.654655$ //

$z(0.3) = -1.298796$ //

To evaluate $y(0.6)$: Take ~~$y(0.$~~ $x_0 = 0.3$; $y(0.3) = y_1$ $z(0.3) = z_1$ //

(11) Given: $k_1 = h\, f_1(x_n + \alpha h, y_n + \beta k_1)$

$= h\left[f_n + \alpha h \cdot \frac{\partial f}{\partial x} + \beta k_1 \cdot \frac{\partial f}{\partial y} + \cdots\right]$

$= h\left[f_n + \alpha h \cdot \frac{\partial f}{\partial x} + \beta \frac{\partial f}{\partial y} h\left(f_n + \alpha h \frac{\partial f}{\partial x} + \beta k_1 \cdot \frac{\partial f}{\partial y} + \cdots\right)\right]$

$= h f_n + \alpha h^2 \cdot \frac{\partial f}{\partial x} + \beta h^2 f_n \cdot \frac{\partial f}{\partial y} + o(h^3)$

$\therefore y_{n+1} = y_n + w_1\left[h f_n + \alpha h^2 \cdot f_x + h^2 \beta \cdot f_n \cdot f_y + o(h^3)\right]$

Again by Taylor series,

$y_{n+1} = y_n + h f_n + \frac{h^2}{2}\left[f_x + f f_y\right] + \cdots$

By comparision. $w_1 = 1$, $\alpha w_1 = 1/2$, $\beta w_1 = 1/2 \Rightarrow \alpha = \beta = 1/2$

$\therefore y_{n+1} = y_n + h\left[f(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right]$. $k_1 = h f(x_n + \frac{h}{2}, y_n + \frac{k_1}{2})$

Use RK-method of order 4

$2y'' - 5y' - 3y = 45e^{2t}$

$y' = z = f(y, z, t)$

$z' = \frac{5}{2}z + \frac{3}{2}y + \frac{45}{2}e^{2t} = g(y, z, t)$

We get: $k_1 = 1$, $m_1 = 28$

$k_2 = 2.4$, $m_2 = 33.94$

$k_3 = 2.697$ $m_3 = 34.79$

$k_4 = 4.479$, $m_4 = 42.084$

$y(0.1) = \underline{\underline{2.2612}}$ $z(0.1) = \underline{\underline{4.4591}}$

4) Milne Predictor corrector method

$y_4^P = y_0 + \frac{4h}{3}\left[2f(x_1, y_1) - f(x_2, y_2) + 2f(x_3, y_3)\right]$ Error: $\frac{14}{45}h^5 y^{iv}(\xi)$

$y_4^C = y_2 + \frac{h}{3}\left[f(x_2, y_2) + 4f(x_3, y_3) + f(x_4, y_4^P)\right]$ Error: $-\frac{h^5}{90} y_0^{(4)}(\xi)$

$y^P(0.3) = 0.614616$ //

$y^C(0.3) = 0.614776$ //

(15) . Adams Moulton:

$y_4^P = y_3 + \frac{h}{24}\left[55f(x_3, y_3) - 59f(x_2, y_2) + 37f(x_1, y_1) - 9f(x_0, y_0)\right]$

$y_4^C = y_3 + \frac{h}{24}\left[9f(x_4, y_4) + 19f(x_3, y_3) - 5f(x_2, y_2) + f(x_1, y_1)\right]$

Take $x_0 = y_0 = 0$.

$k_1 = 0.2$, $k_2 = 0.202$, $k_3 = 0.20204$, $k_4 = 0.20816$ $\therefore$ $\boxed{y(0.2) = 0.2048}$

$k_1 = 0.2082$ $k_2 = 0.2188$ $k_3 = 0.2195$ $k_4 = 0.2356$ $y(0.4) = 0.4269$

|||ly $y(0.6) = 0.7209$

Using this: $y^P(0.8) = 1.08242$ //

$y^C(0.8) = 1.082449$ //

(19) $y'' = xy$: $y(0) + y'(0) = 1$, $y_2 = 1$

(i) Finite difference approx:

$$y_i' = \frac{y_{i+1} - y_{i-1}}{2h}$$

$$y_i'' = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}$$

Now: $y'' = xy$

$\Rightarrow$ $y_i'' = x_i y_i \Rightarrow \frac{1}{h^2}(y_{i-1} - 2y_i + y_{i+1}) - x_i y_i = 0$ — (1)

For $h = 0.5$; $4(y_{i-1} - 2y_i + y_{i+1}) - x_i y_i = 0$

Also $y_0 + y_0' = 1$

$\Rightarrow y_0 + \left[\frac{y_1 - y_{-1}}{2h}\right] = 1$

$\Rightarrow \boxed{y_0 + y_1 - y_{-1} = 1}$

$i=0$ in (1) $\Rightarrow$ $y_{-1} - 2y_0 + y_1 = 0$

$i=1$ in (1) $\Rightarrow$ $4y_0 - 8.5y_1 = -4$

We get: $\underline{y_0 = -1}$: $\underline{y_1 = 0}$

$$\boxed{\begin{array}{l} y(0) = -1 \\ y(0.5) = 0 \end{array}}$$

(ii) $xy'' + xy' - 2y = 2(x+1)$ $y(0) = 1$, $y'(1) = 0$

We get eq:

$i=1$: $7y_2 - 16y_1 + 5y_0 = 16/3$ | $i=3$: $10.5y_4 - 20y_3 + 7.5y_2 = 4$

$i=2$: $7y_3 - 14y_2 + 5y_1 = 10/3$

~~Y.S.~~ Boundary con: $y_0 = 0$, $y_3' = 0 \Rightarrow y_4 = y_2$

We get: $y_1 = -0.8407$

$y_2 = -1.16027$

$y_3 = -1.2442$

11. 9 $y' = f(x,y) = \lambda y$, $\quad k_1 = hf(x_n, y_n) = h\lambda y_n$

$k_2 = hf(x_n + \alpha h, y_n + \beta k_1) = h\lambda(y_n + \beta k_1) = h\lambda(y_n + \beta \cdot \lambda h y_n)$

$= h\lambda y_n(1 + \beta\lambda h)$

$\therefore y_{n+1} = y_n + \left(1 - \frac{1}{2\alpha}\right)\lambda h y_n + \frac{1}{2\beta}\lambda h(1 + \beta\lambda h) y_n$

$= \left[1 + \lambda h\left(1 - \frac{1}{2\alpha}\right) + \frac{\lambda h}{2\beta}(1 + \beta\lambda h)\right] y_n$

$= \left[1 + \lambda h\left(1 - \frac{1}{2\beta}\right) + \frac{\lambda h}{2\beta}(1 + \lambda h\beta)\right] y_n$ [Choosing $\alpha = \beta$]

$= \left(1 + \lambda h + \frac{\lambda^2 h^2}{2}\right) y_n$, the sequence will converge

$\therefore E(\lambda h) = 1 + \lambda h + \frac{\lambda^2 h^2}{2}$, for stability $\left|1 + \lambda h + \frac{\lambda^2 h^2}{2}\right| < 1$

$\Rightarrow \lambda h \in (-2, 0)$

if $h = \frac{1}{4}$, $\lambda = 3$, the method is not stable

$h = \frac{1}{2}$, $\lambda = -2$, the method is stable.

$y(0.6)$ ...

$z_0 = z(0.3)$

(ii)

$\therefore K_1 = hf(x_n + \alpha h, y_n + \beta K_1)$

$= h\left[f(x_n, y_n) + \alpha h \frac{\partial f}{\partial x} + \beta K_1 \frac{\partial f}{\partial y} + \cdots \right]$

$= hf_n + \alpha h^2 \frac{\partial f}{\partial x} + h\beta \frac{\partial f}{\partial y} K_1 + \cdots$

$\therefore K_1 = hf_n + \alpha h^2 \frac{\partial f}{\partial x} + h\beta \frac{\partial f}{\partial y}\left(hf_n + \alpha^2 h^2 \frac{\partial f}{\partial x} + h\beta \frac{\partial f}{\partial y} K_1 + \cdots \right) + \cdots$

$\therefore y_{n+1} = y_n + w_1 \left\{ hf_n + \alpha h^2 \frac{\partial f}{\partial x} + h^2 \beta f_n \frac{\partial f}{\partial y} + O(h^3) \right\}$

(Taylor series)

Again, $y(x_{n+1}) = y(x_n) + hf(x_n, y_n) + \frac{h^2}{2}\left(\frac{\partial f}{\partial x} + f \frac{\partial f}{\partial y} \right) + \cdots$

$\Rightarrow \quad w_1 = 1, \quad \alpha w_1 = 1/2, \quad \beta w_1 = 1/2 \quad \Rightarrow \alpha = \beta = 1/2$

$\therefore \quad y_{n+1} = y_n + hf\left(x_n + \frac{h}{2}, y_n + \frac{K_1}{2}\right), \quad K_1 = hf\left(x_n + \frac{h}{2}, y_n + \frac{K_1}{2}\right)$

Q.12

Solve $y' = -2xy^2$, $y(0) = 1$ with $h = 0.3$ using 2nd order implicit Runge-Kutta method.

$$y_{n+1} = y_n + k_1$$

$$k_1 = h f\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$f(x,y) = -2xy^2, \quad x_0 = 0, \; y_0 = 1$$

$$k_1 = h\left[-2\left(x_n + \frac{h}{2}\right)\left(y_n + \frac{k_1}{2}\right)^2\right]$$

$$= -h(2x_n + h)\left(y_n + \frac{k_1}{2}\right)^2$$ which is an implicit equation for k_1 and one may use any iterative method.

define $F(k_1) = k_1 + h(2x_n + h)\left(y_n + \frac{k_1}{2}\right)^2$

$$= k_1 + 0.3(2x_n + 0.3)\left(y_n + \frac{k_1}{2}\right)^2$$

Let us propose to use Newton-Raphson method

$$k_1^{(l+1)} = k_1^{(l)} - \frac{F(k_1^{(l)})}{F'(k_1^{(l)})}, \quad l = 0, 1, 2 \ldots$$

assume $k_1^{(0)} = h f(x_0, y_0) = -h 2 x_0^2 y_0^2$

$$= -2(0.3)(0) = 0$$

$$F(k_1) = k_1 + 0.3(2x_n + 0.3)(y_n + \frac{k_1}{2})^2$$

$$F'(k_1) = 1 + 0.3(2x_n + 0.3)\,2(y_n + \frac{k_1}{2})\frac{1}{2}$$

$$= 1 + 0.3(2x_n + 0.3)(y_n + \frac{k_1}{2})$$

$$F'(k_1^{(0)}) = 1 + 0.3(2(0) + 0.3)(1 + 0) = 1.09$$

$$F(k_1^{(0)}) = 0 + 0.3(2(0) + 0.3)(1 + 0)^2 = 0.09$$

$$F(k_1^{(0)}) = 0.09 \; ; \quad F'(k_1^{(0)}) = 1.09$$

$$\therefore \; k_1^{(1)} = k_1^{(0)} - \frac{F(k_1^{(0)})}{F'(k_1^{(0)})} = 0 - \frac{0.09}{1.09}.$$

$$= -0.0825688$$

$$F(k_1^{(1)}) = 0.00015151$$

$$\Rightarrow k_1^{(2)} = -0.0826994$$

$$F'(k_1^{(1)}) = 1.08628$$

one proceeds until $|k_1^{(l+1)} - k_1^{(l)}| < \epsilon$ (preassigned)

$$y(0.3) \approx y_1 = 1 + (-0.08269) = \cancel{0.8}\ 0.9173006.$$