# MA 204 Numerical Methods

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Lecture-3

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# Contents

- Solution of a nonlinear equation, bisection and secant methods, Newton's method, rate of convergence, solution of a system of nonlinear equations.

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- Interpolation by polynomials, divided differences, error of the interpolating polynomial, piecewise linear and cubic spline interpolation.

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- The iterative procedure (given below) is called the **Newton Raphson's method**.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \; n = 0, 1, 2, \cdots .$$

# History

- In numerical analysis, Newton's method, also known as the Newton–Raphson method, named after Isaac Newton and Joseph Raphson, is a root-finding algorithm which produces successively better approximations to the roots (or zeroes) of a real-valued function.

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- The most basic version starts with a real-valued function $f$, its derivative $f$, and an initial guess $x_0$ for a root of $f$.
- This is one of the most powerful methods for solving a root-finding problem. There are many ways of introducing Newton's method.

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- To find a formula for $x_1$, consider the equation of tangent to the graph of $y = f(x)$ at $(x_0, f(x_0))$. It is simply the graph of

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- $(x_1, 0)$ lies on this line:

$$\Longrightarrow \; 0 = f(x_0) + f'(x_0)(x_1 - x_0)$$

$$\Longrightarrow \; x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

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$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \cdots \tag{1}$$

This is **Newton's method** for solving $f(x) = 0$.

# Convergence Analysis

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- This says that the graph of $y = f(x)$ is not tangent to the x-axis when the graph intersects it $x = \alpha$. This case when $f'(\alpha) = 0$ will be discussed later.
- Combining (2) with the continuity of $f'(x)$ implies

$$f'(x) \neq 0 \quad \forall \; x \text{ near } \alpha.$$

# Convergence Analysis

- By Taylor's theorem,

$$\begin{aligned} f(\alpha) &= f(x_n + \alpha - x_n) \\ &= f(x_n) + (\alpha - x_n)f'(x_n) + \frac{(\alpha - x_n)^2}{2} f''(c_n) \qquad (3) \end{aligned}$$

where $c_n$ is an unkown point between $\alpha$ and $x_n$.

## Convergence Analysis

- Note that $f(\alpha) = 0$. Using $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ This implies

$$
\begin{aligned}
& 0 = f(x_n) + (\alpha - x_n) f'(x_n) + \frac{1}{2}(\alpha - x_n)^2 f''(c_n) \\
\Longrightarrow\ & 0 = \frac{f(x_n)}{f'(x_n)} + (\alpha - x_n) + \frac{1}{2}(\alpha - x_n)^2 \frac{f''(c_n)}{f'(x_n)} \\
\Longrightarrow\ & 0 = x_n - x_{n+1} + \alpha - x_n + (\alpha - x_n)^2 \frac{f''(c_n)}{f'(x_n)} \qquad (4)
\end{aligned}
$$

- $\boxed{\alpha - x_{n+1} = (\alpha - x_n)^2 \left[ -\dfrac{f''(c_n)}{2f'(x_n)} \right]}$

# Error Analysis

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- When the initial error is sufficiently small, this shows that the error in the succeeding iterates will decrease very rapidly.

## When to stop iterations?

- Noting that $f(\alpha) = 0$, by Mean Value Theorem

$$f(x_n) = f(x_n) - f(\alpha) = f'(\xi_n)(x_n - \alpha).$$

Thus, error $\varepsilon_n = \alpha - x_n = -\dfrac{f(x_n)}{f'(x_n)}$ provided that $x_n$ is close to $\alpha$ that $f'(x_n) \approx f'(\xi_n)$. This implies $\boxed{\alpha - x_n \approx x_{n+1} - x_n}$.

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- This is the standard error estimation formula for Newton's method and it is usually fairly accurate. However, this formula is not valid if $f'(\alpha) = 0$.

# Multiple Root

- The zero of the function $f$ is said to be of multiplicity m if

  $$f(x) = (x - \alpha)^m g(x)$$

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- A zero of multiplicity 1 is called a simple root or a simple zero.

## Remedy

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- This implies

$$\begin{aligned}\mu(x) &= \frac{(x-\alpha)^m g(x)}{m(x-\alpha)^{m-1}g(x) + (x-\alpha)^m g'(x)} \\ &= (x-\alpha)\frac{g(x)}{mg(x) + (x-\alpha)g'(x)}\end{aligned}$$

also has a zero at $\alpha$. However, $g(\alpha) \neq 0$.

# Newton's method for multiple roots

- Thus, $\mu'(\alpha) = \dfrac{1}{m} \neq 0$, hence $\alpha$ is called a simple zero of $\mu$.
- Newton's method can be applied to $\mu(x)$ to give

$$x_{n+1} = x_n - \frac{f(x_n)f'(x_n)}{[f'(x_n)]^2 - f(x_n)f''(x_n)}$$

- This is called Newton's modified method. This has quadratic convergence regardless of multiplicity of the zeros of f.
- For a simple zero, the original Newton's method requires significantly low computations.

# Newton versus Secant

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- Newton's method or the Secant method is often used to refine an answer obtained by another technique, such as the bisection method, since these methods requires good first approximation but generally give rapid convergence.