MA 204 Numerical Methods

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Lecture-4

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Contents

- Solution of a nonlinear equation, bisection and secant methods, Newton's method, rate of convergence.

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- Interpolation by polynomials, divided differences, error of the interpolating polynomial, piecewise linear and cubic spline interpolation.

Polynomial Interpolation

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Given, $(n+1)$ points, say (x_i, y_i) where $i = 0, 1, 2, \cdots, n$ with distinct x_i, not necessarily sorted, we want to find a polynomial of degree n,

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such that it interpolates these points, i.e.,

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Note: The total number of data points is 1 larger than the degree of the polynomial.

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Answer:

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- On the other hand: look for an interpolating polynomial.

Importance and Limitations

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The interpolating polynomial happens to be

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The relative error is given by

$$E_r(p_2(0.75) = \frac{f(0.75) - p_2(0.75)}{f(0.75)} \approx 0.6129283.$$

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- Find the values between the points for discrete data set;
- To approximate a (probably complicated) function by a polynomial;
- Then, it is easier to do computations such as derivative, integration etc.

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i. $x = 0, y = 1 \quad : P_2(0) = a_0 = 1$

ii. $x = 1, y = 0 \quad : P_2(1) = a_2 + a_1 + a_0 = 0$

iii. $x = \frac{2}{3}, y = 0.5 \quad : P_2(\frac{2}{3}) = (\frac{4}{9})a_2 + (\frac{2}{3})a_1 + a_0 = 0.5$

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We have 3 linear equations and 3 unknowns (a_2, a_1, a_0).

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In matrix-vector form

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ \frac{4}{9} & \frac{2}{3} & 1 \end{pmatrix} \begin{pmatrix} a_2 \\ a_1 \\ a_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0.5 \end{pmatrix}$$

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Easy to solve:

$$a_2 = -\frac{3}{4}, \;\; a_1 = -\frac{1}{4}, \;\; a_0 = 1.$$

Then,

$$P_2(x) = -\frac{3}{4}x^2 - \frac{1}{4}x + 1.$$

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$$\begin{aligned}
P_n(x_0) = y_0 \quad &: \quad x_0^n a_n + x_0^{n-1} a_{n-1} + \cdots + x_0 a_1 + a_0 = y_0 \\
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Putting this in matrix-vector form

$$\begin{pmatrix} x_0^n & x_0^{n-1} & \cdots x_0 & 1 \\ x_1^n & x_1^{n-1} & \cdots x_1 & 1 \\ \vdots & \vdots & \vdots & \\ x_n^n & x_n^{n-1} & \cdots x_n & 1 \end{pmatrix} \begin{pmatrix} a_n \\ a_{n-1} \\ \vdots \\ a_0 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix}$$

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If x_i's are distinct, then $\mathbf{X}$ is invertible, therefore $\vec{a}$ has a unique solution.

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Theorem 1

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In other words,

Given $n+1$ distinct points $x_0, x_1, \cdots, x_n$ and $n+1$ ordinates $y_0, \cdots, y_n$, there is a polynomial $p(x)$ of degree $\leq n$ that interpolates y_i at x_i, $i = 0, 1, \cdots, n$. This polynomial $p(x)$ is unique among the set of all polynomials of degree at most n.

Proofs

Recall the Vandermonde matrix **X** is given by

$$V_n(x) = \det \begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ \vdots & & & & \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^n \\ 1 & x & x^2 & \cdots & x^n \end{pmatrix} \tag{1}$$

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- One can show that $V_n(x)$ is a polynomial of degree n, and that its roots are $x_0, \cdots, x_{n-1}$. We can obtain the formula

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- Expand the last row of $V_n(x)$ by minors to show that $V_n(x)$ is a polynomial of degree n and to find the coefficient of the term x^n.
- One can show that

$$\det(X) = V_n(x_n) = \prod (x_i - x_j)$$

Bad news: $\mathbf{X}$ has a very large condition number for large n, therefore, not effective to solve if n is large.

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- Lagrange polynomials
- Newton's divided differences

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Locally supported in discrete sense. The cardinal functions $l_i(x)$ can be written as

$$\begin{aligned} l_i(x) &= \prod_{j=0, j\neq i}^{n} \left(\frac{x - x_j}{x_i - x_j} \right) \\ &= \frac{x - x_0}{x_i - x_0} \frac{x - x_1}{x_i - x_1} \cdots \frac{x - x_{i+1}}{x_i - x_{i+1}} \cdots \frac{x - x_n}{x_i - x_n}. \end{aligned}$$

Lagrange Interpolation

Verify:

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Lagrange form of the interpolation polynomial

Lagrange form of the interpolation polynomial can be simply expressed as

$$P_n(x) = \sum_{i=0}^{n} l_i(x) y_i.$$

It is easy to check the interpolating property:

$$P_n(x_j) = \sum_{i=0}^{n} l_i(x) y_i = y_j, \quad \text{for every } j.$$

Example

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Answer. The data set corresponds to

$$x_0 = 0,\ x_1 = \frac{2}{3},\ x_2 = 1,\ y_0 = 1\ y_1 = 0.5,\ y_2 = 0.$$

We first compute the cardinal functions

$$l_0(x) = \frac{3}{2}(x - \frac{2}{3})(x - 1)$$
$$l_1(x) = -\frac{9}{2}x(x - 1)$$
$$l_2(x) = 3x(x - \frac{2}{3}).$$

Thus,

$$P_2(x) = \frac{3}{2}(x - \frac{2}{3})(x - 1) - \frac{9}{2}x(x - 1)(0.5) + 0.$$

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- Elegant formula
- Slow to compute, each $l_i(x)$ is different,
- Not flexible: if one changes a points x_j, or add on an additional point x_{n+1}, one must re-compute all l_i's.

Newton's Divided Differences

Given $(n+1)$ data set, we will describe an algorithm in a recursive form.

Main idea: Given $P_k(x)$ that interpolates $k+1$ data points $\{x_i, y_i\},\ i = 0, 1, 2, \cdots, k$, compute $P_{k+1}(x)$ that interpolates one extra point, $\{x_{k+1}, y_{k+1}\}$, by using P_k and adding an extra term.

- For $n = 0$, we set $P_0(x) = y_0$. Then, $P_0(x) = y_0$.
- For $n = 1$, we set

$$P_1(x) = P_0(x) + a_1(x - x_0)$$

where a_1 is to be determined.

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  where a_1 is to be determined.
  Then, find a_1 by the interpolation property $y_1 = P_1(x_1)$, we have

  $$\begin{aligned} y_1 &= P_0(x_1) + a_1(x_1 - x_0) \\ &= y_0 + a_1(x_1 - x_0). \end{aligned}$$

  This gives us

  $$a_1 = \frac{y_1 - y_0}{x_1 - x_0}.$$

For $n = 2$: we set

$$P_2(x) = P_1(x) + a_2(x - x_0)(x - x_1).$$

Then,

$$P_2(x_0) = P_1(x_0) = y_0, P_2(x_1) = P_1(x_1) = y_1.$$

Determine a_2 by the interpolating property $y_2 = P_2(x_2)$.

$$y_2 = P_1(x_2) + a_2(x_2 - x_0)(x_2 - x_1),$$

$$a_2 = \frac{\dfrac{y_2 - y_1}{x_2 - x_1} - \dfrac{y_1 - y_0}{x_1 - x_0}}{x_2 - x_0}.$$

Newton's form for the interpolation polynomial:

$$P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots$$
$$+ a_n(x - x_0)(x - x_1)\cdots(x - x_{n-1}).$$