Lecture-8

Monday, 12 February 2024 11:59 AM

Linear Boundary Value Problems

Finite Difference Method

$\dfrac{y_{i+1} - y_i}{h}$

Numerical Differentiation

$f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ [Forward difference Formula] OR $y_i'(x_i) = \dfrac{y_i(x_i+h) - y_i(x_i)}{h} + O(h)$

$f'(x) = \lim_{h \to 0} \dfrac{f(x) - f(x-h)}{h}$ [Backward difference Formula] OR $y_i'(x_i) = \dfrac{y_i(x_i) - y_i(x_i-h)}{h} + O(h)$ " $\dfrac{y_i - y_{i-1}}{h}$

$f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x-h)}{2h}$ [Central difference formula] OR $y_i'(x_i) = \dfrac{y_i(x_i+h) - y_i(x_i-h)}{2h} + O(h^2)$

$\dfrac{y_{i+1} - y_{i-1}}{2h}$

The approximat formule

$$f'(x) \approx \frac{f(x+h) - f(x)}{h} = D_h^+ f(x)$$

for sufficiently small $h > 0$

This formula is called forward differ formula

$D_h^- f(x)$

$D_h^0 f(x)$

Theorem

Let $f \in C^2[a,b]$. The mathematical error in F.D.M. is $f'(x) - D_h^+ f(x) = -\frac{h}{2} f''(\xi)$ for some $\xi \in (x, x+h)$

Proof: $f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(\xi)$

$\Rightarrow \dfrac{f(x+h) - f(x)}{h} = f'(x) + \frac{h}{2} f''(\xi)$

$\Rightarrow D_h^+ f(x) = f'(x) + \frac{h}{2} f''(\xi)$

$\Rightarrow f'(x) - D_h^+ f(x) = -\frac{h}{2} f''(\xi)$

[Now How fast the mathematical error converges to 0 as $h \to 0$]

for that $f'(x) - D_h^+ f(x) = -\frac{h}{2} f''(\xi)$ → $|g| = \left|\frac{h}{2}\right| f''(\xi)$

$\Rightarrow \left|\frac{g(h)}{h}\right| = \frac{1}{2} |f''(\xi)|$

expression of Mathematical error

Let $g(h) = f'(x) - D_h^{(+)} f(x)$ then

$$\left|\frac{g(h)}{h}\right| = \frac{1}{2} |f''(\xi)|$$

Let $M > 0$ s.t. $|f''(x)| \le M \quad \forall x \in [a,b]$

(Assume f'' exist & it is a continuous fun & it is restricted to $[a,b]$ So, it is a bounded function)

$\Rightarrow \left|\frac{g(h)}{h}\right| \le \frac{M}{2}$

$\Rightarrow \left|\frac{g(h)}{h}\right| \le \frac{M}{2} \Rightarrow g = O(h)$ as $h \to 0$

If $f(x)$ & $g(x)$ two fun

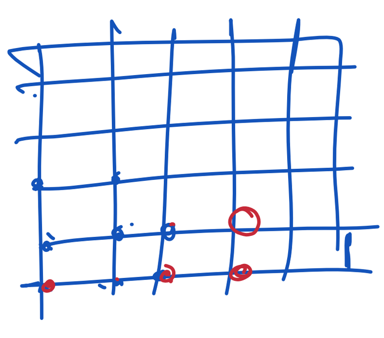
$f(x) = O(g(x))$ as $x \to x_0$

then $|f(x)| \le C |g(x)| \quad \forall x \in (x_0 - h, x_0 + h)$

$\left|\frac{f(x)}{g(x)}\right| \le C$

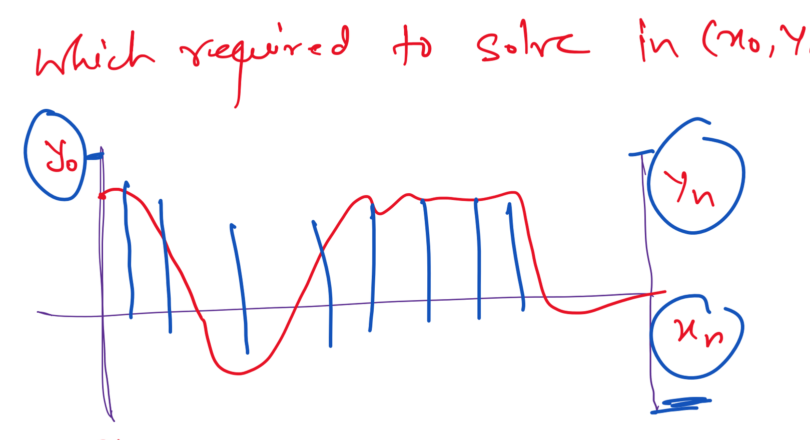
In such situations, we can say that Forward Differen formula $(D_h^{(+)} f(x))$ is of order 1 (Order of accuracy)

Now, FDM for linear boundary value Problem



$$f_1(x)\, y'' + f_2(x)\, y' + f_3(x)\, y = r(x)$$

which required to solve in (x_0, y_0) under $y(x_0) = y_0$ & $y(x_n) = y_n$



$y'' = \dfrac{d^2y}{dx^2}$

$y' = \dfrac{dy}{dx} = f(x,y)$

For 2nd order ODE, boundary condition may be any one of the form

1. $y(x_0) = y_0$ & $y(x_n) = y_n$ → Dirichlet's Condition
2. $y'(x_0) = y_0$ & $y'(x_n) = y_n$ → Neumann Condition
3. $y'(x_0) = y_0$ & $y(x_n) = y_n$ → (One side Dirichlet's condition)
4. $y(x_0) = y_0$ & $y'(x_n) = y_n$ →
5. $y(x_0) = y_0$ & $y(x_n) + y'(x_n) = b$ → Mixed Condition
6. $y(x_0) + y'(x_0) = a$ & $y'(x_n) = y_n$ → Mixed Condition

Dirichlet Problem by FDM

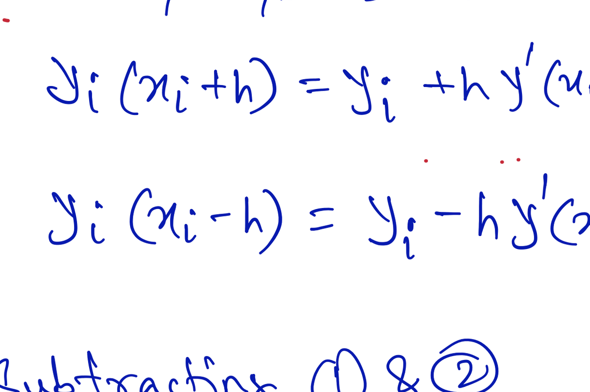
$x_i = x_0 + ih$

Method consists of dividing (x_0, x_n) into n subinterval $x_1, x_2, \dots x_{n-1}$ s.t. $x_i = x_0 + ih$

Let D.E.

$$f_1(x)\, y'' + f_2(x)\, y' + f_3(x)\, y = r(x)$$

with $y(x_0) = y_0$ & $y(x_n) = y_n$



Let $y(x_i) = y_i$

Now, by Taylor Series expansion we have

$$y_i(x_i + h) = y_i + h y'(x_i) + \frac{h^2}{2!} y''(x_i) + \frac{h^3}{3!} y'''(x_i) + \cdots \quad (1)$$

$$\& \quad y_i(x_i - h) = y_i - h y'(x_i) + \frac{h^2}{2!} y''(x_i) - \frac{h^3}{3!} y'''(x_i) + \cdots \quad (2)$$

(1) − (2)

$y_i(x_i + h) - y_i(x_i - h) = 2h\, y'(x_i) + \frac{h^3}{3} y'''(x_i) + \cancel{O(h^3)}$

$\Rightarrow \boxed{\dfrac{y_{i+1} - y_{i-1}}{2h} = y'(x_i)} + O(h^2)$

Subtracting (1) & (2)

$$y_i'(x_i) = \frac{y_i(x_i + h) - y_i(x_i - h)}{2h} + O(h^2)$$

$$= \frac{y_{i+1} - y_{i-1}}{2h}$$

Adding (1) & (2)

$$y_i(x_i + h) + y_i(x_i - h) = 2y_i + 2\frac{h^2}{2!} y''(x_i) + O(h^4)$$

$$\Rightarrow y_{i+1} - 2y_i + y_{i-1} = h^2 y''(x_i) + O(h^4)$$

$O(h^2)$

$$\Rightarrow \boxed{y''(x_i) = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + O(h^2)}$$

$f_1(x)\, y'' + f_2(x)\, y' + f_3(x)\, y = r(x)$

Now,

$$f_1(x_i) \left[\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right] + f_2(x_i) \left[\frac{y_{i+1} - y_{i-1}}{2h}\right] + f_3(x_i)\, y_i = r(x_i)$$

Which on arranging.

$$\left[2f_1(x_i) - h f_2(x_i)\right] y_{i-1} + \left[2h^2 f_3(x_i) - 4 f_1(x_i)\right] y_i + \left[2f_1(x_i) + h f_2(x_2)\right] y_{i+1} = 2h^2 r(x_i)$$

??

$i = 1, 2, \dots (n-1)$