MA 204 Numerical Methods

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Lecture-6

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Contents

- Solution of a nonlinear equation, bisection and secant methods, Newton's method, rate of convergence.

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- Interpolation by polynomials, divided differences, error of the interpolating polynomial, piecewise linear and cubic spline interpolation.

Errors in Polynomial Interpolation

Given a function $f(x)$ on $x \in [a, b]$, and a set of distinct points $x_i \in [a, b]$, $i = 0, 1, \cdots, n$. Let $P_n(x) \in \mathcal{P}_n$ s.t.

$$P_n(x_i) = f(x_i), \quad i = 0, 1, 2, \cdots, n.$$

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Error function: $e(x) = f(x) - P_n(x)$, $x \in [a, b]$.

Theorem 1

There exists some value $\xi \in (a, b)$, such that

$$e(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) \prod_{i=0}^{n} (x - x_i), \quad \textit{for all} \quad x \in [a, b]. \tag{1}$$

Proof. If $f \in \mathcal{P}_n$, then, by Uniqueness Theorem of polynomial interpolation, we must have $f(x) = P_n(x)$. Then, $e(x) = 0$ and the proof is trivial.

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it holds

$$W(x_i) = 0, \quad W(x) = x^{n+1} + \cdots, \quad W^{(n+1)} = (n+1)!$$

Fix an x such that $a \leq x \leq b$ and $x \neq x_i$ for any i. We define a constant

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We find all the zeros for $\varphi(y)$. We see that $x_i's$ are zeros since

$$\varphi(x_i) = f(x_i) - P_n(x_i) - cW(x_i) = 0.$$

Also, x is a zero because

$$\varphi(x) = f(x) - P_n(x) - cW(x) = 0.$$

Here goes our deduction:

$\varphi(x)$ has atleast $\quad (n+2) \quad$ zeros on $[a,b]$.
$\varphi'(x)$ has atleast $\quad (n+1) \quad$ zeros on $[a,b]$.
$\varphi''(x)$ has atleast $\quad n \quad$ zeros on $[a,b]$.
$\ldots$
$\varphi^{(n+1)}(x)$ has atleast $\quad 1 \quad$ zero on $[a,b]$.
$\boxed{\text{Call it } \xi \quad s.t. \quad \varphi^{(n+1)}(\xi)=0.}$ So, we have

$$\varphi^{(n+1)}(\xi) = f^{(n+1)}(\xi) - 0 - cW^{(n+1)}(\xi) = 0.$$

Recall $W^{n+1} = (n+1)!$, we have, for every y,

$$f^{(n+1)}(\xi) = cW^{(n+1)}(\xi) = \frac{f(y)-P_n(y)}{W(y)}(n+1)!$$

$$e(x) = f(x) - P_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) \prod_{i=0}^{n} (x - x_i),$$

for some $\xi \in [a, b]$.

Disadvantages of polynomial interpolation $P_n(x)$

- $n-$ times differentiable; We do not need such high smoothness;
- big error in certain intervals (esp. near the ends);
- no convergence result;
- heavy to compute for large n.