Indian Institute of Technology Indore
MA203 Complex Analysis and Differential Equations-II
(Autumn Semester 2023)
Tutorial Sheet 3

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1. Find the residues of the following functions at their isolated singular points.

   (a) $\dfrac{1}{z^3 - z^5}$ (b) $\dfrac{z^{2n}}{(1+z)^n}$, $n \in \mathbb{N}$ (c) $\dfrac{z^2+z-1}{z^2(z-1)}$ (d) $\dfrac{\mathrm{e}^z}{z^2(z^2+9)}$

2. Let $f(z) = \dfrac{g(z)}{z-z_0}$, where g is analytic on $|z-z_0| < r$, and $g(z_0) \neq 0$. Then show that
$$\operatorname{Res}(f; z_0) = \lim_{z \to z_0} g(z) = g(z_0).$$

3. Let $f(z) = \dfrac{p(z)}{q(z)}$, where $p(z)$ and $q(z)$ are both analytic at z_0. Further, $p(z_0) \neq 0$, $q(z_0) = 0$, and $q'(z_0) \neq 0$. Then show that
$$\operatorname{Res}(f; z_0) = \frac{p(z_0)}{q'(z_0)}.$$

4. Find $\operatorname{Res}\left(\dfrac{f'(z)}{f(z)}; z_0\right)$ if

   (a) z_0 is a zero of n-th order of the function f,

   (b) z_0 is a pole of n-th order of the function f.

5. Let f be analytic in a simply connected domain D and C be a simple closed curve in the counterclockwise sense. Suppose z_0 is the only zero of f in the region enclosed by C. Show that
$$\oint_C \frac{f'(z)}{f(z)}\,\mathrm{d}z = 2\pi i m$$
where m is the order of zero of f at z_0.

6. Prove that
$$\binom{n}{k} = \frac{1}{2\pi i}\oint_C \frac{(1+z)^n}{z^{k+1}}\,\mathrm{d}z,$$
where $n, k \in \mathbb{N}$, and $n \geq k$.

7. Find the integral $\dfrac{1}{2\pi i}\displaystyle\oint_C \sin\frac{1}{z}\,\mathrm{d}z$, where C is the circle $|z| = r$.

8. Find the residue for $\cot z$ at the point $z = 0$.

9. Find the integral $\dfrac{1}{2\pi i}\displaystyle\oint_C \sin\frac{1}{z}\,\mathrm{d}z$, where C is the circle $|z| = 2$ with anticlockwise orientation.

10. Evaluate the following real integrals.

    (a) $\displaystyle\int_0^{2\pi} \frac{1}{3+2\cos x}\,\mathrm{d}x$ (b) $\displaystyle\int_0^{2\pi} \frac{\sin^2 x}{2+\cos x}\,\mathrm{d}x$ (c) $\displaystyle\int_0^{2\pi} \frac{\cos 2x}{5+4\cos x}\,\mathrm{d}x$

    (d) $\displaystyle\int_0^{\pi} \frac{2}{4+\sin^2 x}\,\mathrm{d}x$ (e) $\displaystyle\int_0^{2\pi} \cos^{2n} x\,\mathrm{d}x$, $n \in \mathbb{N}$.

11. Evaluate the following improper integrals by the residue method.

(a) $\displaystyle\int_0^\infty \frac{1}{1+x^2}\,\mathrm{d}x$ (b) $\displaystyle\int_{-\infty}^\infty \frac{1}{(1+x^2)^3}\,\mathrm{d}x$ (c) $\displaystyle\int_0^\infty \frac{x^2}{(1+x^2)^2}\,\mathrm{d}x$ (d) $\displaystyle\int_0^\infty \frac{\cos sx}{1+x^2}\,\mathrm{d}x$