Lecture-7
Sunday, 11 February 2024 6:24 PM

Multi- Step Method

→ Predictor- Corrector Methods.

$\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$, find sol^n at $x = x_0 + h$

→ we will use a pair of formula.

→ 1st formula is called the predictor formula $\left(y_1^{(P)}\right)$

→ 2nd formula is use to correct or refine $y_1^{(P)}$ and denote as $\left(y_1^{(C)}\right)$ ← This is the 1st Correction

→ This corrected formula may be used several times & obtain $y_1^{(C_1)}, y_1^{(C_2)}, \ldots, y_1^{(C_n)}$

Euler's Predictor - Corrector

Euler → $y_1 = y_0 + h f(x_0, y_0)$

Modified Euler → $y_1 = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_0+h, \underbrace{y_0 + h f(x_0,y_0)}_{= y_1})\right]$, $x_1 = x_0 + h$

$\Rightarrow y_1 = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1,y_1)\right]$

Here

$$\boxed{\begin{aligned} y_1^{(P)} &= y_0 + h f(x_0,y_0) \\ y_1^{(C_1)} &= y_0 + \frac{h}{2}\left[f(x_0,y_0) + f\left(x_1, y_1^{(P)}\right)\right] \\ &\vdots \\ y_1^{(C_{n+1})} &= y_0 + \frac{h}{2}\left[f(x_0,y_0) + f\left(x_1, y_1^{(C_n)}\right)\right] \end{aligned}}$$

Error in Euler's Predictor - Corrector Formula

$\frac{dy}{dx} = f \Rightarrow y' = f$

$y_1^{(C_{n+1})} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f\left(x_1, y_1^{(C_n)}\right)\right]$

OR

$y_{m+1} = y_m + \frac{h}{2}\left[y_m' + y_m'(x_m + h)\right] + \text{Error}$, $m = n$

$\Rightarrow y(x_m + h) - y(x_m) = \frac{h}{2}\left[y'(x_m) + y'(x_m + h)\right] + \text{Error}$

$\Rightarrow \cancel{y(x_m)} + \cancel{h y'(x_m)} + \cancel{\frac{h^2}{\lfloor 2} y''(x_m)} + \frac{h^3}{\lfloor 3} y'''(\xi_1) - \cancel{y(x_m)}$

$= \frac{h}{2}\left[\cancel{y'(x_m)} + \cancel{y'(x_m)} + \cancel{h y''(x_m)} + \frac{h^2}{2!} y'''(\xi_2) + \text{Error}\right]$

$\Rightarrow \frac{h^3}{\lfloor 3} y'''(\xi_1) = \frac{h^3}{4} y'''(\xi_2) + \text{Error}$

$\Rightarrow \text{Error} = h^3\left[\frac{1}{6} y'''(\xi_1) - \frac{1}{4} y'''(\xi_2)\right]$

$$\boxed{\begin{aligned} \text{Error} &= -\frac{h^3}{12} f'''(\xi) \\ &\approx O(h^3) \end{aligned}}$$

, $x_m \le \xi \le x_{m+1}$

Milne's Predictor - Corrector Method

For this method we need four prior values (x_0,y_0) , (x_1,y_1) , (x_2,y_2) & (x_3,y_3) & using these data we can find y_4 at $x = x_4$

$\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$

$\Rightarrow \int_{x_0}^{x_4} dy = \int_{x_0}^{x_4} f(x,y)\,dx$ for any $x = x_4$

$\Rightarrow y_4 - y_0 = \int_{x_0}^{x_4} f(x,y)\,dx$

$\Rightarrow y_4 = y_0 + \int_{x_0}^{x_4} f(x,y)\,dx$

Now by interpolation formula :

$f(x,y) = f_0 + n\Delta f_0 + \frac{n(n-1)}{2!}\Delta^2 f_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 f_0 + \cdots$

One may write

$y_4 = y_0 + \int_{x_0}^{x_n}\left(f_0 + n\Delta f_0 + \frac{n(n-1)}{2!}\Delta^2 f_0 + \cdots\right)dx$

Now using $x_n = x_0 + nh$

$y_4 = y_0 + h\int_0^4\left(f_0 + n\Delta f_0 + \frac{n(n-1)}{2!}\Delta^2 f_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 f_0 + \cdots\right)dn$

$y_4 = y_0 + h\left[4f_0 + 8\Delta f_0 + \frac{20}{3}\Delta^2 f_0 + \frac{8}{3}\Delta^3 f_0 + \cdots\right]$

$y_4 = y_0 + \frac{4h}{3}\left[2f(x_1,y_1) - f(x_2,y_2) + 2f(x_3,y_3)\right]$

[After substituting values of Δf_0, $\Delta^2 f_0$, $\Delta^3 f_0$]

$$\boxed{\begin{aligned} &\text{with error} = \frac{14}{45} h^5 y^{iv}(\xi) \\ &y_4^{(P)} = y_0 + \frac{4h}{3}\left[2f_1 - f_2 + 2f_3\right] \end{aligned}}$$

Called Milne's predictor formula

For Corrector formula :

$\int_{y_0}^{y_2} dy = \int_{x_0}^{x_2} f(x,y)\,dx$

$\Rightarrow y_2 = y_0 + h\int_0^2\left[f_0 + n\Delta f_0 + \frac{n(n-1)}{2!}\Delta^2 f_0 + \cdots\right)dn$

$= y_0 + h\left[2f_0 + 2\Delta f_0 + \frac{1}{3}\Delta^2 f_0\right] - \frac{h}{90}\Delta^4 f_0 + \cdots$

$\Rightarrow y_2 = y_0 + \frac{h}{3}\left[f_0 + 4f_1 + f_2\right] - \frac{h}{90}\Delta^4 f_0$

Which may be written as

$y_4 = y_2 + \frac{h}{3}\left[f_2 + 4f_3 + f_4\right]$

with error $= -\frac{h}{90}\Delta^4 f_0 = -\frac{h^5}{90} y_0^{(4)}(\xi)$

where $x_{m-1} < \xi < x_{m+1}$

The predictor formula is

$y_4^{(P)} = y_0 + \frac{4h}{3}\left[2f(x_1,y_1) - f(x_2,y_2) + 2f(x_3,y_3)\right]$

and corrector formula is

$y_4^{(C)} = y_2 + \frac{h}{3}\left[f(x_2,y_2) + 4f(x_3,y_3) + f(x_4,y_4)\right]$

NOTE :

If four points are there then directly → $y_4^{(P)}$ & $y_4^{(C)}$

If only one prior value/pts (x_0,y_0) is given then we need to generate other three values using one of the following method

① Euler ② Modified Euler

③ Euler Predictor - Corrector formula

④ RK Method.

1st order Simultaneous Differential equations

$\frac{dy}{dx} = f_1(x,y,z)$, $\frac{dz}{dx} = f_2(x,y,z)$

with initial conditions $y(x_0) = y_0$, $z(x_0) = z_0$

h

x_0 x_1

$x_1 = x_0 + h$

We want $y(x_0 + h)$ and $z(x_0 + h)$

$k_1 = h f(x_0, y_0)$

k_2

k_3

k_4

$\begin{cases} k_1 = h f_1(x_0, y_0, z_0) \\ m_1 = h f_2(x_0, y_0, z_0) \end{cases}$

$\begin{cases} k_2 = h f_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{m_1}{2}\right) \\ m_2 = h f_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{m_1}{2}\right) \end{cases}$

$k_3 = h f_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{m_2}{2}\right)$

$m_3 = h f_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{m_2}{2}\right)$

$k_4 = h f_1\left(x_0 + h, y_0 + k_3, z_0 + m_3\right)$

$m_4 = h f_2\left(x_0 + h, y_0 + k_3, z_0 + m_3\right)$

$y(x_0 + h) = y_0 + \frac{1}{6}\left[k_1 + 2(k_2 + k_3) + k_4\right]$

$z(x_0 + h) = z_0 + \frac{1}{6}\left[m_1 + 2(m_2 + m_3) + m_4\right]$

Accuracy of the method $O(h^5)$

$\frac{dy}{dx} = x + y + z = f_1(x,y,z)$

$\frac{dz}{dx} = 2x + 3y + 4z = f_2(x,y,z)$

with $y(0) = 1$, $z(0) = 1$ at $x = 0.2$

$x_0 = 0$, $h = 0.2$

$k_1 = 0.4$ | $m_1 = 1.4$

$k_2 = 0.6$ | $m_2 = 2.12$

$k_3 = 0.69$ | $m_3 = 2.468$

$k_4 = 1.052$ | $m_4 = 3.8296$

$y(x_1) = y(x_0 + 0.2) = 1.672$

$z(x_1) = z(x_0 + 0.2) = 3.401$ 3.6

$\frac{dy}{dx} = 3x + 4y + 1$, $y(0) = 1$, find y at $x = 0.8$

$x = 0$

by Milne's Method.

	y_0	y_1	y_2	y_3
x	0	02	0.4	0.6
y	1	2	3.92	7.496
f(x,y)				

$y_4^{(P)}(0.8) = 18.85$

$y_4^{(C)}(0.8) = 19.186$

$y(0.2) = y_0 + h f(x_0, y_0) =$

$y(0.4) =$

$y(0.6) =$