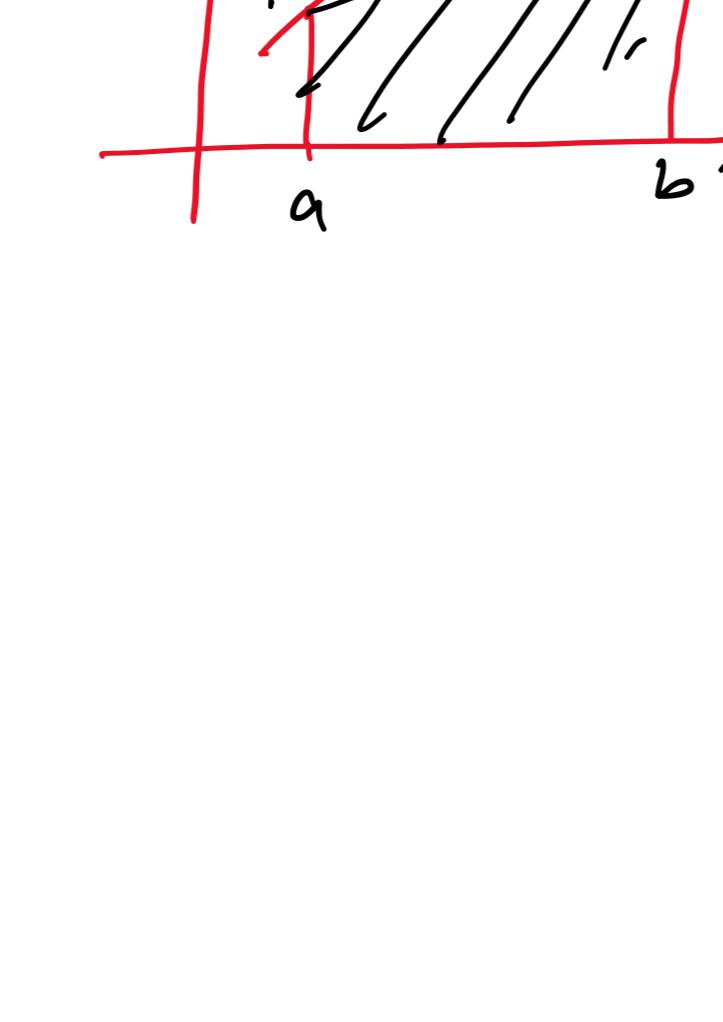


Trapezoidal Rule

$$\begin{aligned} I &= \int_a^b f(x) dx \\ &= \frac{b-a}{2} [f(a) + f(b)] \\ &= (b-a) \left[\frac{f(a) + f(b)}{2} \right] \\ &= \text{Width} \times \text{Average height} \end{aligned}$$



Error in the Trapezoidal rule:

$$I = \int_a^b y_n(x) dx = nh \left[y_0 + \frac{n}{2} \Delta y_0 + \frac{n(n-1)}{12} \Delta^2 y_0 + \dots \right]$$

Where $y_n(x) = y_0 + \frac{x-x_0}{h} \Delta y_0 + \frac{(x-x_0)^2}{2!} \Delta^2 y_0 + \dots = T_0 + T_1 + T_2 + \dots$

In trapezoidal rule, terms from T_2 onwards are neglected. So the truncation error in trapezoidal rule is

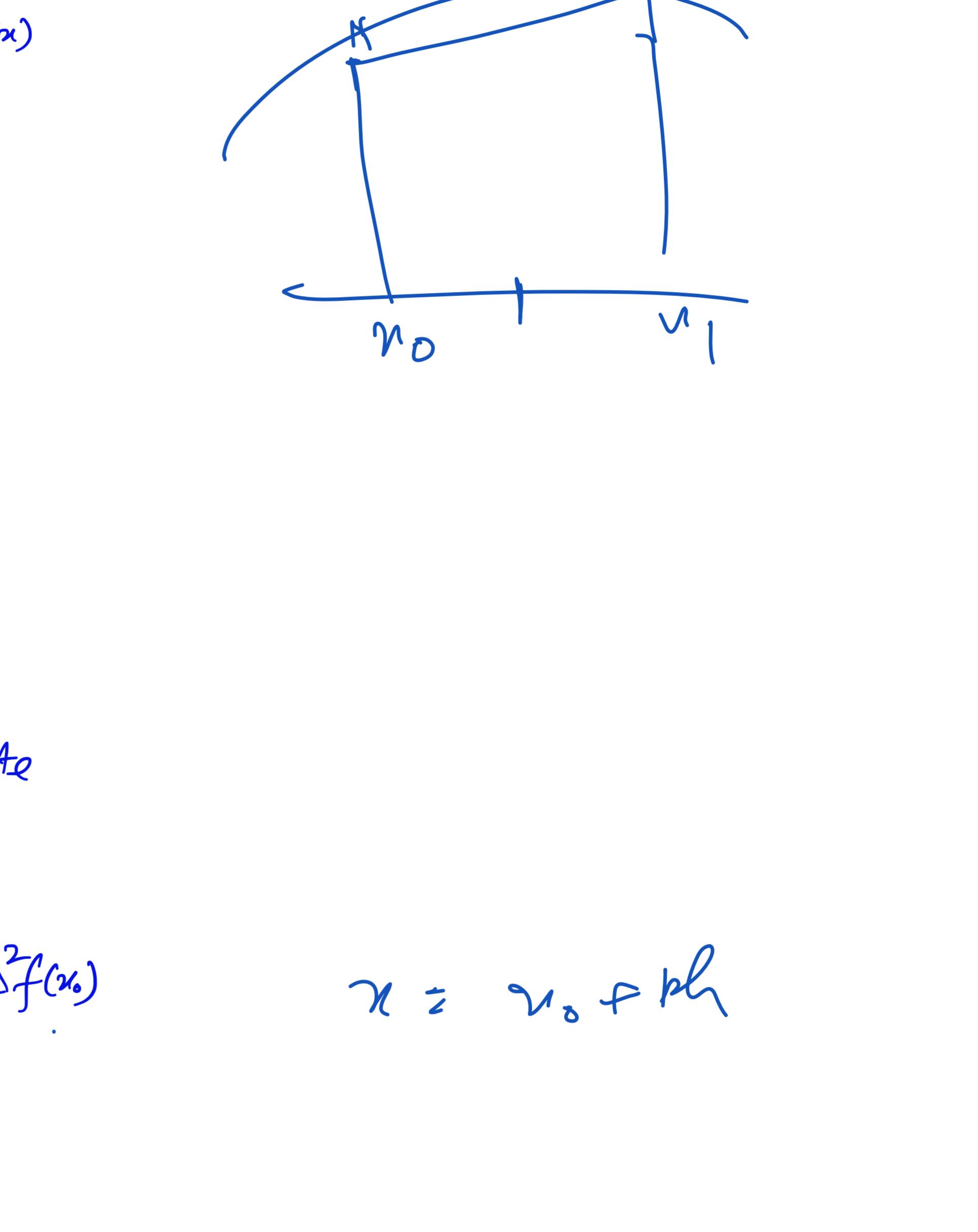
$$\begin{aligned} E_{\text{trap}} &= \int_a^b T_2 dx = \frac{f''(p)}{2} \int_0^h f'(t) h dt \\ &= -f''(p) \frac{h^3}{12} \quad \text{where } x = x_0 + t h, dt = h dp \end{aligned}$$

So, $f''(p) = h^2 f''(x)$

$$\therefore E_{\text{trap}} = -\frac{h^3}{12} f''(x) \quad a \leq x \leq b$$

Composite Trapezoidal Rule:

$[a, b]$ divided in n equal parts of length h .
i.e. $h = \frac{b-a}{n}$, Nodes $x_0, x_0+h, x_0+2h, \dots, x_0+nh$

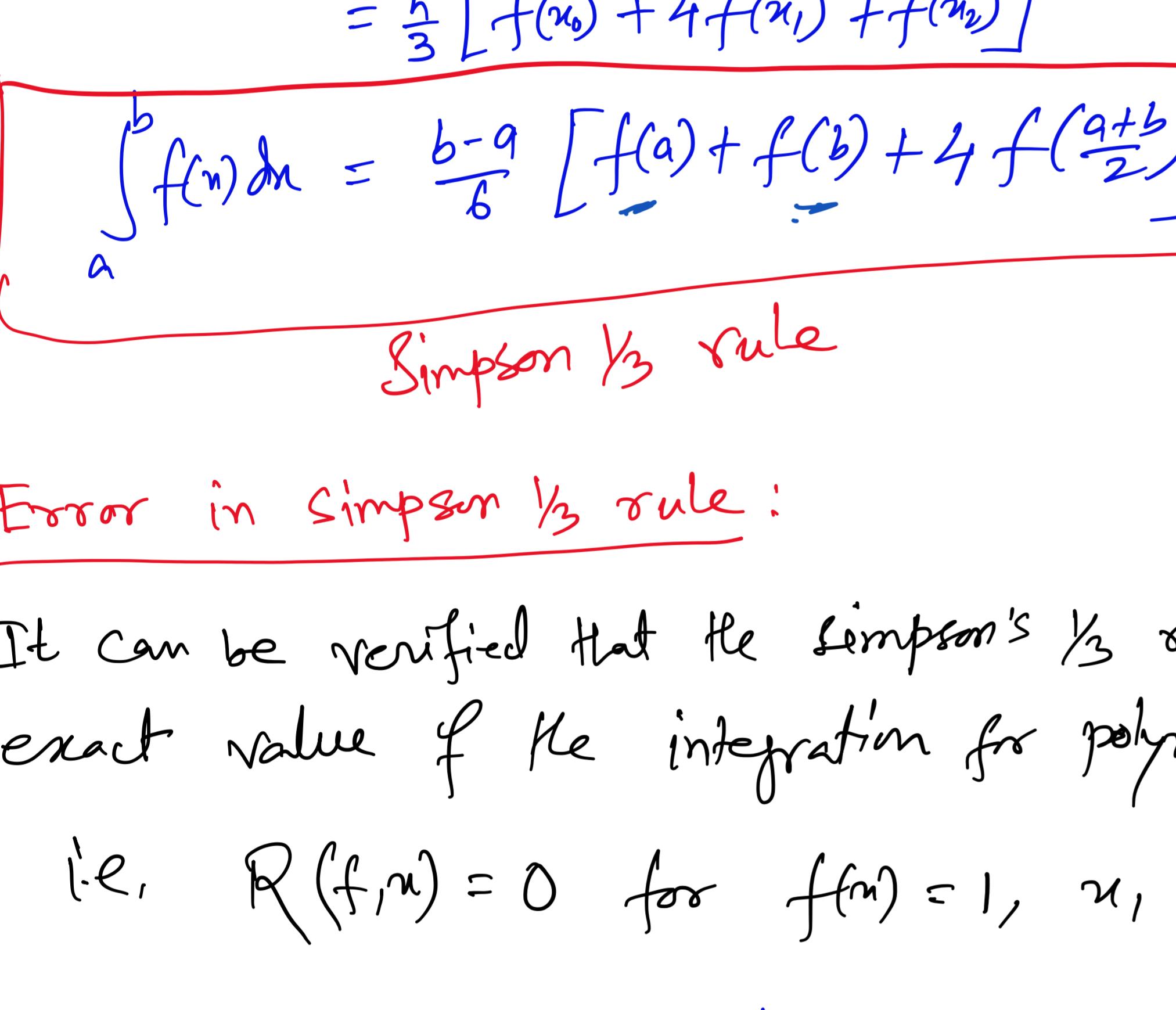


Disadvantage:

If the length of the interval $[a, b]$ is large, then $(b-a)$ is also large and so the error term will be large.
Thus the method becomes meaningless

Simpson's $\frac{1}{3}$ rule

$f(x)$ is approximated by a second order polynomial $P_2(x)$ which pass 3 sampling points.



Now, here 3 points and two equal subintervals
Now, the quadratic polynomial passing through $P(x_0, f(x_0)), P(x_1, f(x_1))$ & $R(x_2, f(x_2))$ which approximate or $P(x_0, f(x_0))$

$f(x)$ is given by

$$f(x) = f(x_0) + \frac{1}{h} (x-x_0) \Delta f(x_0) + \frac{1}{2h^2} (x-x_0)(x-x_1) \Delta^2 f(x_0)$$

or $f(x) = f(x_0) + \frac{1}{h} \Delta f(x_0) + \frac{1}{2!} \Delta^2 f(x_0) +$

$x_0 = a, x_1 = \frac{a+b}{2}, x_2 = b$

Now by Newton-Cotes formula, we have

$$\int_a^b f(x) dx = \int_{x_0}^{x_2} f(x) dx = \int_a^{x_2} [f(x_0) + \frac{1}{h} (x-x_0) \Delta f(x_0) + \frac{1}{2h^2} (x-x_0)(x-x_1) \Delta^2 f(x_0)] dx$$

for $n=2$ $h = \frac{b-a}{2}$ $= (x_2-x_0) f(x_0) + \frac{1}{h} \left[\frac{1}{2} (x-x_0)^2 \right]_{x_0}^{x_2} \Delta^2 f(x_0) + I,$

$$= 2h f(x_0) + 2h \Delta f(x_0) + I, \quad \checkmark$$

$$I = \frac{1}{2h^2} \left[\frac{x_0^3}{3} - (x_0-x_1) \frac{x_1^2}{2} + x_0 x_1 x_2 \right]_{x_0}^{x_2} \Delta^2 f(x_0)$$

$$= \frac{1}{12h} (x_2-x_0) \left[2(x_2^2+x_0 x_2+x_0^2) - 3(x_0+x_1)(x_2+x_0) + 6x_0 x_1 \right] \Delta^2 f(x_0)$$

Substituting $x_2 = x_0 + 2h, x_1 = x_0 + h$, we obtain

$$I = \frac{1}{6h} (2h^2) \Delta^2 f(x_0) = \frac{h}{3} \Delta^2 f(x_0)$$

Now $\int_a^b f(x) dx = \int_{x_0}^{x_2} f(x) dx$

$$= 2h f(x_0) + 2h \Delta f(x_0) + \frac{1}{3} \Delta^2 f(x_0)$$

$$= \frac{1}{3} [6f(x_0) + 6\{f(x_1) - f(x_0)\} + \{f(x_2) - 2f(x_1) + f(x_0)\}]$$

$$= \frac{1}{3} [f(x_0) + 4f(x_1) + f(x_2)]$$

$\boxed{\int_a^b f(x) dx = \frac{b-a}{6} [f(a) + f(b) + 4f(\frac{a+b}{2})]}$

Simpson $\frac{1}{3}$ rule

Error in Simpson $\frac{1}{3}$ rule:

It can be verified that the Simpson's $\frac{1}{3}$ rule gives the exact value of the integration for polynomial ≤ 3 .

i.e., $R(f, x) = 0$ for $f(x) = 1, x, x^2, x^3$

For $f(x) = 1 : R(f, x) = \int_a^b 1 dx - \frac{b-a}{6} [1+4+1] = (b-a) - (b-a) = 0$

For $f(x) = x : R(f, x) = \int_a^b x dx - \frac{b-a}{6} [a+4(\frac{a+b}{2})+b] = \frac{1}{2} (b-a)^2 - \frac{1}{2} (b-a)^2 = 0$

For $f(x) = x^2 : R(f, x) = \int_a^b x^2 dx - \frac{b-a}{6} [a^2 + 4(\frac{a+b}{2})^2 + b^2] = \frac{1}{3} (b^3 - a^3) - \frac{(b-a)}{3} [a^2 + ab + b^2] = \frac{1}{3} (b^3 - a^3) - \frac{1}{3} (\frac{b^3 - a^3}{2}) = 0$

For $f(x) = x^3 : R(f, x) = \int_a^b x^3 dx - \frac{b-a}{6} [a^3 + 4(\frac{a+b}{2})^3 + b^3] = \frac{1}{4} (b^4 - a^4) - \frac{(b-a)}{24} [5a^3b + 6a^2b^2 + 6ab^3 + b^4] = \frac{1}{4} (b^4 - a^4) - \frac{1}{24} (b^4 - a^4) = 0$

For $f(x) = x^4 : R(f, x) = \int_a^b x^4 dx - \frac{b-a}{6} [a^4 + 4(\frac{a+b}{2})^4 + b^4] = \frac{1}{5} (b^5 - a^5) - \frac{(b-a)}{120} [5a^4b + 20a^3b^2 + 30a^2b^3 + 20ab^4 + b^5] = \frac{1}{5} (b^5 - a^5) - \frac{1}{120} (b^5 - a^5) = 0$

For $f(x) = x^5 : R(f, x) = \int_a^b x^5 dx - \frac{b-a}{6} [a^5 + 4(\frac{a+b}{2})^5 + b^5] = \frac{1}{6} (b^6 - a^6) - \frac{(b-a)}{720} [15a^5b + 60a^4b^2 + 120a^3b^3 + 120a^2b^4 + 60ab^5 + b^6] = \frac{1}{6} (b^6 - a^6) - \frac{1}{720} (b^6 - a^6) = 0$

For $f(x) = x^6 : R(f, x) = \int_a^b x^6 dx - \frac{b-a}{6} [a^6 + 4(\frac{a+b}{2})^6 + b^6] = \frac{1}{7} (b^7 - a^7) - \frac{(b-a)}{5040} [35a^6b + 210a^5b^2 + 420a^4b^3 + 35a^3b^4 + 10a^2b^5 + b^7] = \frac{1}{7} (b^7 - a^7) - \frac{1}{5040} (b^7 - a^7) = 0$

For $f(x) = x^7 : R(f, x) = \int_a^b x^7 dx - \frac{b-a}{6} [a^7 + 4(\frac{a+b}{2})^7 + b^7] = \frac{1}{8} (b^8 - a^8) - \frac{(b-a)}{40320} [35a^7b + 280a^6b^2 + 840a^5b^3 + 1260a^4b^4 + 840a^3b^5 + 280a^2b^6 + b^8] = \frac{1}{8} (b^8 - a^8) - \frac{1}{40320} (b^8 - a^8) = 0$

For $f(x) = x^8 : R(f, x) = \int_a^b x^8 dx - \frac{b-a}{6} [a^8 + 4(\frac{a+b}{2})^8 + b^8] = \frac{1}{9} (b^9 - a^9) - \frac{(b-a)}{30240} [35a^8b + 280a^7b^2 + 840a^6b^3 + 1260a^5b^4 + 840a^4b^5 + 280a^3b^6 + 35a^2b^7 + b^9] = \frac{1}{9} (b^9 - a^9) - \frac{1}{30240} (b^9 - a^9) = 0$

For $f(x) = x^9 : R(f, x) = \int_a^b x^9 dx - \frac{b-a}{6} [a^9 + 4(\frac{a+b}{2})^9 + b^9] = \frac{1}{10} (b^{10} - a^{10}) - \frac{(b-a)}{20160} [35a^9b + 280a^8b^2 + 840a^7b^3 + 1260a^6b^4 + 840a^5b^5 + 280a^4b^6 + 35a^3b^7 + 20a^2b^8 + b^10] = \frac{1}{10} (b^{10} - a^{10}) - \frac{1}{20160} (b^{10} - a^{10}) = 0$

For $f(x) = x^{10} : R(f, x) = \int_a^b x^{10} dx - \frac{b-a}{6} [a^{10} + 4(\frac{a+b}{2})^{10} + b^{10}] = \frac{1}{11} (b^{11} - a^{11}) - \frac{(b-a)}{145920} [35a^{10}b + 280a^9b^2 + 840a^8b^3 + 1260a^7b^4 + 840a^6b^5 + 280a^5b^6 + 35a^4b^7 + 20a^3b^8 + 5a^2b^9 + b^{11}] = \frac{1}{11} (b^{11} - a^{11}) - \frac{1}{145920} (b^{11} - a^{11}) = 0$

For $f(x) = x^{11} : R(f, x) = \int_a^b x^{11} dx - \frac{b-a}{6} [a^{11} + 4(\frac{a+b}{2})^{11} + b^{11}] = \frac{1}{12} (b^{12} - a^{12}) - \frac{(b-a)}{100800} [35a^{11}b + 280a^{10}b^2 + 840a^9b^3 + 1260a^8b^4 + 840a^7b^5 + 280a^6b^6 + 35a^5b^7 + 20a^4b^8 + 5a^3b^9 + 2a^2b^{10} + b^{12}] = \frac{1}{12} (b^{12} - a^{12}) - \frac{1}{100800} (b^{12} - a^{12}) = 0$

For $f(x) = x^{12} : R(f, x) = \int_a^b x^{12} dx - \frac{b-a}{6} [a^{12} + 4(\frac{a+b}{2})^{12} + b^{12}] = \frac{1}{13} (b^{13} - a^{13}) - \frac{(b-a)}{725760} [35a^{12}b + 280a^{11}b^2 + 840a^{10}b^3 + 1260a^9b^4 + 840a^8b^5 + 280a^7b^6 + 35a^6b^7 + 20a^5b^8 + 5a^4b^9 + 2a^3b^{10} + 2a^2b^{11} + b^{13}] = \frac{1}{13} (b^{13} - a^{13}) - \frac{1}{725760} (b^{13} - a^{13}) = 0$

For $f(x) = x^{13} : R(f, x) = \int_a^b x^{13} dx - \frac{b-a}{6} [a^{13} + 4(\frac{a+b}{2})^{13} + b^{13}] = \frac{1}{14} (b^{14} - a^{14}) - \frac{(b-a)}{516000} [35a^{13}b + 280a^{12}b^2 + 840a^{11}b^3 + 1260a^{10}b^4 + 840a^9b^5 + 280a^8b^6 + 35a^7b^7 + 20a^6b^8 + 5a^5b^9 + 2a^4b^{10} + 2a^3b^{11} + 2a^2b^{12} + b^{14}] = \frac{1}{14} (b^{14} - a^{14}) - \frac{1}{516000} (b^{14} - a^{14}) = 0$

For $f(x) = x^{14} : R(f, x) = \int_a^b x^{14} dx - \frac{b-a}{6} [a^{14} + 4(\frac{a+b}{2})^{14} + b^{14}] = \frac{1}{15} (b^{15} - a^{15}) - \frac{(b-a)}{372000} [35a^{14}b + 280a^{13}b^2 + 840a^{12}b^3 + 1260a^{11}b^4 + 840a^{10}b^5 + 280a^9b^6 + 35a^8b^7 + 20a^7b^8 + 5a^6b^9 + 2a^5b^{10} + 2a^4b^{11} + 2a^3b^{12} + 2a^2b^{13} + b^{15}] = \frac{1}{15} (b^{15} - a^{15}) - \frac{1}{372000} (b^{15} - a^{15}) = 0$

For $f(x) = x^{15} : R(f, x) = \int_a^b x^{15} dx - \frac{b-a}{6} [a^{15} + 4(\frac{a+b}{2})^{15} + b^{15}] = \frac{1}{16} (b^{16} - a^{16}) - \frac{(b-a)}{256000} [35a^{15}b + 280a^{14}b^2 + 840a^{13}b^3 + 1260a^{12}b^4 + 840a^{11}b^5 + 280a^{10}b^6 + 35a^9b^7 + 20a^8b^8 + 5a^7b^9 + 2a^6b^{10} + 2a^5b^{11} + 2a^4b^{12} + 2a^3b^{13} + 2a^2b^{14} + b^{16}] = \frac{1}{16} (b^{16} - a^{16}) - \frac{1}{256000} (b^{16} - a^{16}) = 0$

For $f(x) = x^{16} : R(f, x) = \int_a^b x^{16} dx - \frac{b-a}{6} [a^{16} + 4(\frac{a+b}{2})^{16} + b^{16}] = \frac{1}{17} (b^{17} - a^{17}) - \frac{(b-a)}{1792000} [35a^{16}b + 280a^{15}b^2 + 840a^{14}b^3 + 1260a^{13}b^4 + 840a^{12}b^5 + 280a^{11}b^6 + 35a^{10}b^7 + 20a^9b^8 + 5a^8b^9 + 2a^7b^{10} + 2a^6b^{11} + 2a^5b^{12} + 2a^4b^{13} + 2a^3b^{14} + 2a^2b^{15} + b^{17}] = \frac{1}{17} (b^{17} - a^{17}) - \frac{1}{1792000} (b^{17} - a$