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Approximating Polynomials. (Lagrange's Form)

$$P(x) = \sum_{i=1}^{k} L(x_i)\, y_i$$

$$\int_{x_0}^{x_K} P(x)\,dx = \int_{x_0}^{x_K} \sum_{i=1}^{k} L(x_i)\, y_i\, dx = \sum_{i=1}^{k} \int_{x_0}^{x_K} L(x_i)\, y_i\, dx$$

* Equally spaced abssicca's $h = x_i - x_{i-1} \quad i = 1, \dots n$

$$P(p) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2}\Delta y_0 \dots \frac{p(p-1)(p-2)\dots(p-k+1)}{k!}(\Delta y_0)^k + R_n(x)$$

$$P(x) = y_0 + (x-x_0)\Delta y_0 + \frac{(x-x_0)(x-x_1)}{2!\,h^2}(\Delta y_0)^2 + \dots \frac{(x-x_0)\dots(x-x_{k-1})}{k!\,h^k}\Delta y_0^k$$

$x = x_0 + ph$

$x_n = x_0 + nh$

$$\int_{x_0}^{x_K} P(x)\,dx = \int_{x_0}^{x_K} f(x)\,dx = h\int_{x_0} P(p)\,dp$$

$$\text{Error } E = \int_{x_0}^{x_K} R(x)\,dx \qquad R(x) = \frac{p(p-1)\dots(p-k)}{(k+1)!} = \frac{(x-x_0)\dots(x-x_k)}{(k+1)!} f^{k+1}(\xi)$$

$a \le \xi \le b$

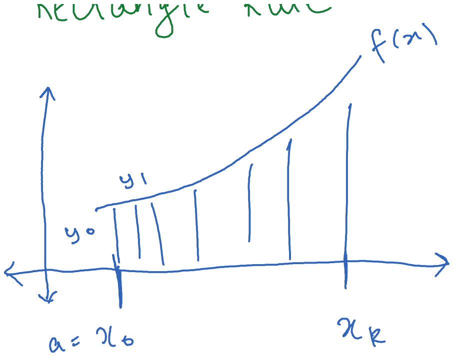
$$\therefore E = \int_{x_0}^{x_K} \frac{(x-x_0)\dots(x-x_K)}{(k+1)!} f^{k+1}(\xi)\,dx$$

$$= \frac{h^{k+2} f^{k+1}(\xi)}{(k+1)!}\int_0^K p(p-1)\dots(p-k)\,dp$$

Rectangle Rule

$$\int_{x_0}^{x_1} f(x)\,dx = \int_{x_0}^{x_1} y_0\,dx = y_0(x_1 - x_0) = h y_0$$

Rectangle Rule



$$\int_{x_0}^{x_1} f(x)\,dx = \int_{x_0}^{x_1} y_0\,dx = y_0(x_1 - x_0) = h y_0$$

$$\therefore \underbrace{\int_{x_0}^{x_k} f(x)\,dx = h\left(y_0 + y_1 + \cdots y_{k-1}\right)}_{\text{Composite formula for } n \text{ integral}}$$

increasing $f(x)$ $\quad \int_a^b f(x)\,dx \geq h\left(y_0 + \cdots y_{n-1}\right)$

decreasing $f(x)$ $\quad \int_a^b f(x)\,dx \leq h\left(y_0 + \cdots y_{n-1}\right)$

* for equally spaced x_i's
$h = x_i - x_{i-1}$

Approximating Integral values

$$I(f) = \int_a^b f(x)\,dx = \int_a^b P(x)\,dx$$

$$I(f) \approx \int_a^b (\text{simpler func}^n)\,dx + R_n$$

Numerical integrator / Quadrature Rule. → Process of approximating an integral

Construct polynomial f for $n+1$ nodes
$x_0 \quad x_1 \cdots x_n$ $\qquad I(f) \approx I(P_n) = \int_a^b \sum f(x_i) L_i\,dx.$

Newton's FD formula for equally spaced abssisca's

$$I(f) = \int_a^b f(x)\,dx = \int_{x_0}^{x_n} f(x)\,dx = h\int_0^n f(x_0 + ph)\,dp$$

$$\doteq h\int_0^n E^p f(x_0)\,dp = h\int_0^n (1+\Delta)^p \underbrace{f(x_0)}_{y_0}\,dp$$

$$= h\int_0^n \left[1 + p\,\Delta y_0 + \frac{p(p-1)}{2!}\Delta y_0^2 + \cdots\right] dp$$

Quadrature Formula

$$I = \int_a^b f\,dx \approx \sum_{k=0}^{n} w_k f(x_k) = \overbrace{\sum_{k=0}^{n} w_k f(x_k) + R_n(f)}^{\text{form } A}$$

$$\int_a \quad \sum_{k=0} \quad \sum_{k=0}$$

x_k = abscisses

$f(x_k)$ = coordinates

w_k = weights

$$R_n(f) = I - \sum_{k=0}^{n} w_k f(x_k)$$

Integration methods of form A is said to be of order p if $R_n = 0 \ \forall n \le p$

ie $f(x) = 1, x, x^2, \dots x^p$

Error term for x^{p+1} is $E = \int_a^b \underbrace{c(x)}_{\text{weight factor}} x^{p+1} dx - \sum_{k=0}^{n} w_k x_k^{p-1}$

weight factor $c(x) = 1$

$$R_n(f) = \int_a^b f(x)dx - \sum w_k f(x_k)$$

$$= \frac{c}{(p+1)!} f^{p+1}(\xi) \qquad a \le \xi \le b$$

$$\therefore \quad |R_n(f)| \le \frac{|c|}{(p+1)!} \max_{a \le x \le b} |f^{p+1}(x)|$$

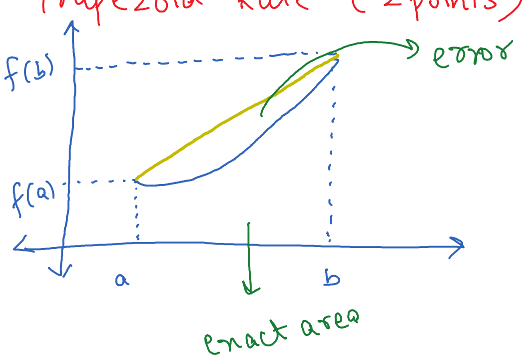
for uniform mesh grids. $a = x_0 \quad b = x_n \quad h = \frac{b-a}{n}$

$$I = \int_a^b f(x)\,dx = \sum w_k f(x_k)$$

$$= w_0 f(x_0) + w_1 f(x_1) \dots$$

Newton Cotes Quadrature formula.

Trapezoid Rule (2 points)



$$f(x) = f(a) + \frac{x-a}{b-a}[f(b) - f(a)]$$

approx func^n

$$I = \int_a^b f(x)\,dx = \frac{b-a}{2}[f(b) + f(a)]$$

This rule gives correct ans for polynomials with deg ≤ 1 $\quad f(x) = 1, x$.

ie $R(f, x) = 0$

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$\therefore$ Order of trapezoid rule is one.

for $f(x) = x^2$ $\quad C = \int_a^b f(x) - \sum' \omega_k f(x_k)$

$$= \int_a^b x^2 dx - \frac{b-a}{2}\left[b^2 + a^2\right]$$

$$= \frac{-1}{6}(b-a)^3$$

$$R_n(f,x) = \frac{c}{2!} f''(\xi) = -\frac{1}{12}(b-a)^3 f''(\xi) \quad a \le \xi \le b$$

$$\therefore \quad |R_n(f,x)| \le \frac{1}{12}|b-a|^3 \max_{a \le x \le b} |f''(x)|$$

$$I = \int_{x_0}^{x_n} y_n(x)\,dx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \dots\right]$$

$$y_n(x) = \underbrace{y_0}_{T_0} + \underbrace{p\Delta y_0}_{T_1} + \underbrace{\frac{p(p-1)}{2!}}_{T_2}\Delta^2 y_0 + \dots$$

ignore terms after T_2

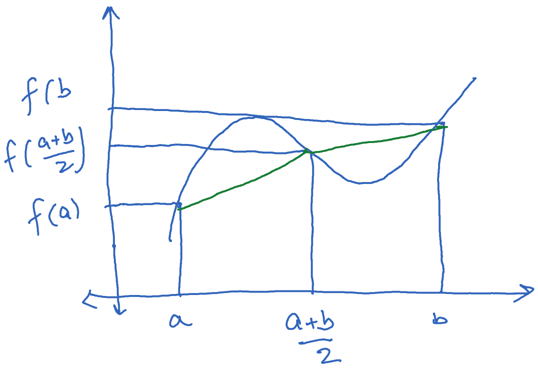
$$E_{Trap} = \int_a^b T_2\,dx = \frac{f''(p)}{2}\int_0^1 P(p)\,h\,dp = -f''(p)\frac{h}{12}$$

$$f''(p) = h^2 f''(x)$$

$$\therefore \quad E_{trap} = \frac{-h^3}{12} f''(\xi) \qquad a \le \xi \le b$$

Simpson's 1/3 Rule

$$f(x) = f(x_0) + \frac{x-x_0}{h}\Delta f(x_0) + \frac{1}{2h^2}(x-x_0)(x-x_1)\Delta^2 f(x_0)$$



$$g(x) = f(x_0) + p\,\Delta f(x_0) + \frac{p(p-1)}{2!}\Delta^2 f(x_0)$$

$$x_0 = a \qquad x_1 = \frac{a+b}{2} \qquad x_2 = b$$

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_2} g(x)\,dx$$

Newton Cotes formula

$$x_1 = x_0 + h$$
$$x_2 = x_0 + 2h$$

$$\Delta f(x_0) = f(x_1) - f(x_0)$$
$$\Delta^2 f(x_0) = f(x_0) - 2f(x_1) + f(x_2)$$

$$\therefore\ I = \int_a^b f(x)\,dx = \frac{b-a}{6}\left[f(a) + f(b) + 4f\left(\frac{a+b}{2}\right)\right]$$

Simpson's 1/3 rule

Error: $R(f,x) = 0$ for $f = 1, x, x^2, x^3$

for $f(x) = x^4$ $\quad R(f,x) = \dfrac{c}{4!} f^4(\xi)$

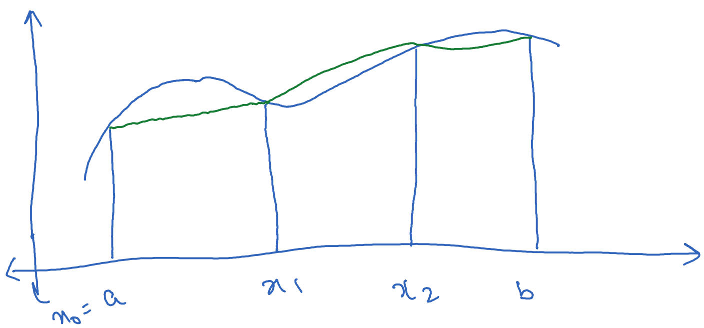
$$\text{where } c = \int_a^b x^4\,dx - \frac{b-a}{6}\left[a^4 + b^4 + 4\left(\frac{a+b}{2}\right)^4\right] = -\frac{(b-a)^5}{120}$$

$$\therefore\ R(f,x) = -\frac{(b-a)^5}{2880} f^4(\xi) = \frac{-h^5}{90} f^4(\xi) \qquad h = \frac{b-a}{2}$$

a	b	n	Closed Newton-Cotes Formula	h	Truncation Error
x_0	x_1	1	$h \cdot \dfrac{f(x_0) + f(x_1)}{2}$	$\dfrac{(b-a)}{1}$	$-\frac{1}{12} h^3 f''(\xi)$
x_0	x_2	2	$\frac{1}{3} \cdot h \cdot [f(x_0) + 4f(x_1) + f(x_2)]$	$\dfrac{(b-a)}{2}$	$-\frac{1}{90} h^5 f^{(iv)}(\xi)$
x_0	x_3	3	$\frac{3}{8} \cdot h \cdot [f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)]$	$\dfrac{(b-a)}{3}$	$-\frac{3}{80} h^5 f^{(iv)}(\xi)$
x_0	x_4	4	$\frac{2}{45} \cdot h \cdot [7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)]$	$\dfrac{(b-a)}{4}$	$-\frac{8}{945} h^7 f^{(vi)}(\xi)$

$$\frac{\Delta^n y_0}{h^n} = \frac{d^n y}{dx^n}$$

Simpson's 3/8 Rule



$$f(x) = f(x_0) + \frac{x - x_0}{h}\Delta f(x_0) + \frac{(x-x_0)(x-x_1)}{2h^2}\Delta^2 f(x_0) + \frac{(x-x_0)(x-x_1)(x-x_2)}{6h^2}\Delta^3 f(x_0)$$

$$I = \int_a^b f(x)\,dx = \int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}\Big[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\Big]$$

$$h = \frac{b-a}{3}$$

$$R(f,x) = \frac{-3h^5}{80} f^4(\xi)$$

1. $\Delta^2 y_0 = \Delta(\Delta y_0) = \Delta(y_1 - y_0) = \Delta y_1 - \Delta y_0$
$= (y_2 - y_1) - (y_1 - y_0)$
$= y_2 - 2y_1 + y_0$

2. $\nabla^2 y_n = \nabla(\nabla y_n) = \nabla(y_n - y_{n-1})$
$= \nabla y_n - \nabla y_{n-1}$
$= (y_n - y_{n-1}) - (y_{n-1} - y_{n-2})$
$= y_n - 2y_{n-1} + y_{n-2}$

3. $E^2 y_0 = E(E y_0) = E y_1 = y_2$

4. $\delta^2 y_x = \delta\left[y(x+\frac{h}{2}) - y(x-\frac{h}{2})\right] = \delta y_{x+\frac{h}{2}} - \delta y_{x-\frac{h}{2}}$
$= (y_{x+h} - y_x) - (y_x - y_{x-h})$
$= y_{x+h} - 2y_x + y_{x-h}$

$\Delta y_0 = y_1 - y_0 \Rightarrow \Delta y_r = y_{r+1} - y_r$

$\nabla y_n = y_n - y_{n-1} \Rightarrow \nabla y_1 = y_1 - y_0$

$E y_0 = y_1$ $\quad E y_r = y_{r+1}$ etc

$\delta(f(x)) = f(x+\frac{h}{2}) - f(x-\frac{h}{2})$
$\delta = [E^{1/2} - E^{-1/2}]$

So, $\Delta f(x_r) = f(x_r + h) - f(x_r)$

Relation
$E = 1 + \Delta$
$\Delta = E - 1$
$E = (1 - \nabla)^{-1}$
$\delta = E^{1/2} - E^{-1/2}$
$\Delta\nabla = \Delta - \nabla$
& $\Delta\nabla = \Delta - \nabla = \delta^2$

if we reach x from x_0 through p steps
then we must reach y from y_0 through p steps

$$y = E^p y_0 = (1+\Delta)^p y_0 = \Big[1 + p\Delta + p(p-1)\,\Delta^2 + p(p-1)(p-2)\,\Delta^3 \Big]u.$$

$$y = E^p y_0 = (1+\Delta)^p y_0 = \left[1 + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \frac{p(p-1)(p-2)}{3!}\Delta^3 \dots\right] y_0$$

Gauss Forward Formula

$$y = y_0 + G_1 \Delta y_0 + G_2 \Delta^2 y_{-1} + G_3 \Delta^3 y_{-1} + \dots$$

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_1 + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-1}$$

$$\Delta^2 y_{-1} = \Delta^2 E^{-1} y_0 = \Delta^2 (1+\Delta)^{-1} y_0$$

$$= \Delta^2 (1 - \Delta + \Delta^2 - \Delta^3 + \Delta^4 - \dots) y_0$$

$$= \Delta^2 y_0 - \Delta^3 y_0 - \Delta^5 y_0 \dots$$

For N.F :

$$y_p = y_0 + {}^pC_1 \Delta y_0 + {}^pC_2 \Delta^2 y_0 + {}^pC_3 \Delta^3 y_0 + \dots$$

N.B. : $y_p = y_0 + {}^pC_1 \Delta y_{-1} + {}^{(p+1)}C_2 \Delta^2 y_{-2} + \dots$

G.F. $y_p = y_0 + {}^pC_1 \Delta y_0 + {}^pC_2 \Delta^2 y_{-1} + {}^{(p+1)}C_3 \Delta^3 y_{-1} + \dots$

G.B. $y_p = y_0 + {}^pC_1 \Delta y_{-1} + {}^{(p+1)}C_2 \Delta^2 y_{-1} + {}^{(p+1)}C_3 \Delta^3 y_{-2} + \dots$

Stirling: $y_p = y_0\left(\frac{1+1}{2}\right) + \frac{\Delta y_1 + \Delta y_0}{2}\,{}^pC_1 + \Delta^2 y_{-1}\left({}^{p+1}C_3 + {}^pC_2\right)$

$$+ {}^{p+1}C_3\left(\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2}\right) + \dots$$

Bessel $y_p = \frac{y_0 + y_1}{2} + \Delta y_0 \left(\frac{{}^pC_1 + {}^{p+1}C_1}{2}\right) + {}^pC_2\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right)$

$$+ \Delta^3 y_{-1}\left({}^{p+1}C_3 + {}^pC_3\right) + \dots$$

Gaussian Quadrature.

$$\int_a^b f(x)\,dx = \sum_{k=1}^{n} c_k f(x_k)$$

exact for polynomial with degree $\leq 2n-1$

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

sol^n is $P_n(x)$

$$y(x) = P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left((x^2-1)^n\right)$$

$$\int_{-1}^{1} P_m(x)\, P_n(x)\, dx = 0 \qquad n \neq m$$
$$= \frac{2}{2n+1} \qquad n = m$$

$$\int_{-1}^{1} P_n(x)\, x^m\, dx = 0 \qquad m < n$$

$$\int_{-1}^{1} f(x)\,dx = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$ open type formula

exact for poly with deg $\leq 2(2)-1 = 3$

$$\int_a^b f(x)\,dx \longrightarrow \int_{-1}^{1} f(x)\,dx \qquad X = \frac{b-a}{2}x + \frac{b+a}{2}$$

Gauss Legendre Formulas

1 point: $\int_{-1}^{1} f(x)\,dx = 2f(0)$

2 point $\int_{-1}^{1} f(x)\,dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$

3 point $\int_{-1}^{1} f(x)\,dx = \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)$

2 point: $\int_0^{\infty} e^{-x} f(x)\,dx = \frac{2+\sqrt{2}}{4} f(2-\sqrt{2}) + \frac{2-\sqrt{2}}{4} f(2+\sqrt{2})$

Picards Method of Successive Approximation.

$$\frac{dy}{dx} = f(x,y) \qquad y(x_0) = y_0$$

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\, dx$$

$\left| y_{k+1}(x) - y_k(x) \right| \leq \varepsilon$ then we conclude that it converged.