Complex Analysis and Differential Equations II

Course Code: MA 203

Programme: B.Tech.

Department of Mathematics
Indian Institute of Technology Indore

Contents

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Before mid-semester

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1. **Dr. Bapan Ghosh**:

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1. **Dr. Bapan Ghosh**: First part-complex Analysis

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   Definitions and properties of analytic functions. Cauchy -Riemann equations, harmonic functions. Power series and their properties, Elementary functions, Cauchy's theorem and its applications.

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2. **Dr. Santanu Manna**: Ordinary Differential Equations part
   Review of power series and series solutions of ODE's. Legendre equation and Legendre Polynomials. Regular and singular points, method of Frobenius. Bessel's equation and Bessel's functions. Strum-Liouville problems.

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   Taylor series and Laurent expansion, Residues and Cauchy's residue formula, Evaluation of improper integrals, Conformal mappings, inversion of Laplace transformations

Tentative Evaluation Plan (out of 100 Marks)

1. **Internal evaluation**-Before mid-semester: 15 Marks

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3. **End-semester**: 55 Marks

Complex Analysis

LECTURE 1

July 26, 2023

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1. E. Kreyszig, Advanced Engineering Mathematics, John Wiley & Sons.

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5. S. Ponnusamy, H. Silverman Complex Variable with Applications, Birkhaser.

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6. Like before, we need to extend our number system to obtain a solution.

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2. **Gauss (1831)**: He provided a geometric representation of complex number $z = x + iy$ with the coordinate point (x, y) in the plane. This is what we use today.
3. Gauss was in favor to call such number as "complex number".

Motivations: Real number vs complex number

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3. **Result:** $\mathbb{C}$ cannot be totally ordered in consistence with the usual order on $\mathbb{R}$.
   Explanation

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5. Infinite in complex plane means that we are travelling far away from the origin in any direction.

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3. In this particular discussion we mean complex function by Complex function with complex variable. .

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3. Here, boundedness of a complex function $f(z)$ means $|f(z)|$ is unbounded.

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2. If the first derivative of any complex function $f : \mathbb{C} \to \mathbb{C}$ exists, then all order derivatives must exist on $\mathbb{C}$.

Domain, co-domain and range

We recall the above ideas from real function to discuss the next...

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5. But $e^{(\mathbb{C})} = \mathbb{C} - \{0\}$.

Complex Analysis

LECTURE 2

July 27, 2023

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3. Many theorems (including prime number theorem) in number theory can be proved using complex analysis.

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4. In symbolically, $\mathrm{Re}(z) = x$ and $\mathrm{Im}(z) = y$.
5. The set of all complex numbers is denoted by $\mathbb{C}$.

Operations on complex number

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3. One can easily verify
   1. $z_1 + z_2 = z_2 + z_1$.
   2. $z_1 z_2 = z_2 z_1$.
   3. $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$.

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$$\frac{x_1+iy_1}{x_2+iy_2} = (x_1+iy_1)\left(\frac{x_2-iy_2}{x_2^2+y_2^2}\right) = \frac{(x_1x_2+y_1y_2)+i(x_2y_1-x_1y_2)}{x_2^2+y_2^2}.$$

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1. **Equal complex numbers:** Two complex numbers (x_1, y_1) and (x_2, y_2) are said to be *equal* if both their real parts and imaginary parts are equal.
2. Mathematically,

$$(x_1, y_1) = (x_2, y_2) \text{ if and only if } x_1 = x_2 \text{ and } y_1 = y_2.$$

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4. x-axis is called the real axis.
5. y-axis is called the imaginary axis.

Complex conjugate

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1. The complex conjugate $\overline{z}$ of a given complex number $z = x + iy$ is defined by

$$\overline{z} = x - iy.$$

2. $\overline{z}$ is just the reflection of the point z wrt x-axis.
3. The following properties are easy to prove:
   1. $\mathrm{Re}(z) = \frac{1}{2}(z + \overline{z})$ and $\mathrm{Im}(z) = \frac{1}{2i}(z - \overline{z})$.
   2. $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$.
   3. $\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$, and hence for a real number α, $\overline{\alpha z} = \alpha \overline{z}$.

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3. Addition rule of vectors is also then valid for complex numbers.

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5. We have triangle inequalities:

$$|z_1 + z_2| \leq |z_1| + |z_2|,$$
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6. Conventionally, we fix positive x-axis as the initial vector.

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4. Both values work. Very confusing to fix θ uniquely.

Polar form of complex numbers

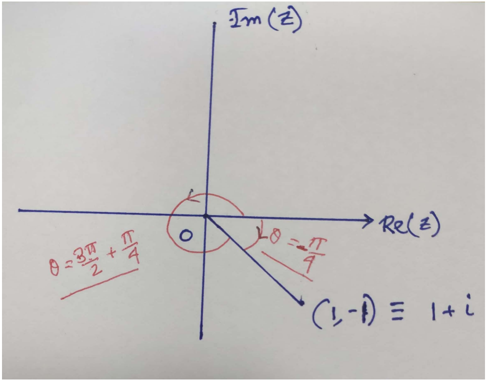

Figure: **3**: Two angles.

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4. $arg(z) = \theta + 2k\pi, k = 0, \pm 1, \pm 2, ...$ (Why $2k\pi$?)

Problems

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Let $z_k = r_k(\cos\theta_k + i\sin\theta_k)$, $k = 1, 2, \ldots, n$. Show that

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   Use method of induction

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2. Generally, De Moiver's formula is invalid for non integer n. **Give an example**.
3. Sometimes, De Moiver's formula is defined with $r = 1$.

Applications of De Moiver's theorem

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2. Plays an important role in finding explicit expressions for the nth roots of unity.

Power of a complex number-Application of De Moiver's theorem

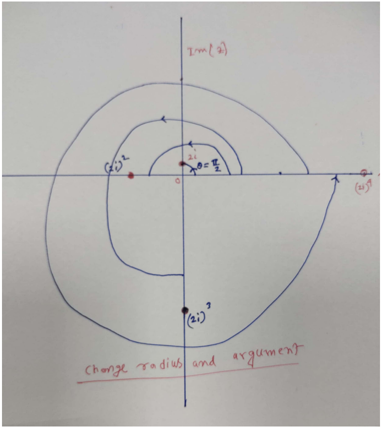

Figure: 4: Geometrical view of power of $2i$

Root finding

Problem: For a given complex number z_0 and $n \in \mathbb{N}$, determine all values of z satisfying $z^n = z_0$.

Check the following example and do by yourself:

Root finding

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7. $z_k = \cos\left(\dfrac{\pi + 2k\pi}{3}\right) + i\sin\left(\dfrac{\pi + 2k\pi}{3}\right)$, $k = 0, 1, 2$.

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7. $z_k = \cos\left(\dfrac{\pi + 2k\pi}{3}\right) + i\sin\left(\dfrac{\pi + 2k\pi}{3}\right)$, $k = 0, 1, 2$.
8. We have only three distinct values of z.

Location of roots

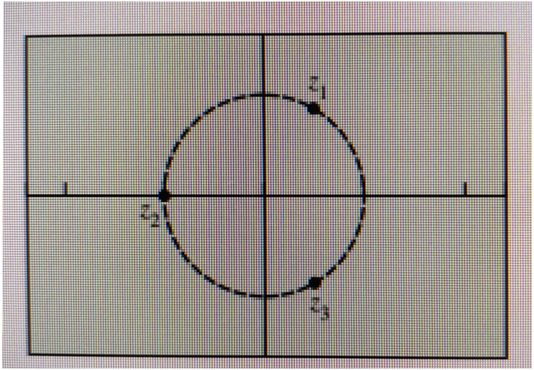

Figure: **5**: Geometrical view of the roots.

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4. I suggest to read (books) about the location of roots (geometrical view) on the complex plane.

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5. $|z - z_0| = 2 \implies \sqrt{(x - x_0)^2 + (y - y_0)^2} = 2$. The subset of points z consists of the points on the circle with center at $z_0 = (x_0, y_0)$ and radius $r = 2$.

Open disk

The collection of z satisfying $|z - z_0| < 2$ represents the subset consisting of the points (red points) **inside** the circle with center at $z_0 = (x_0, y_0)$ and radius $r = 2$.

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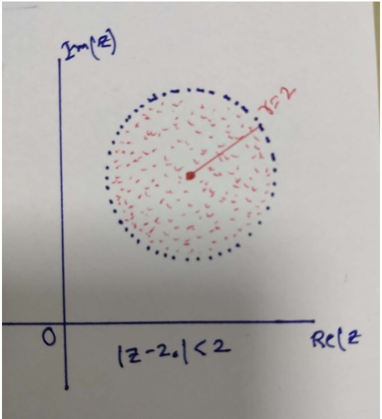

Closed disk

The collection of z satisfying $|z - z_0| \leq 2$ represents the subset consisting of the points inside (red points) and on the circle (blue points) the circle with center at $z_0 = (x_0, y_0)$ and radius $r = 2$.

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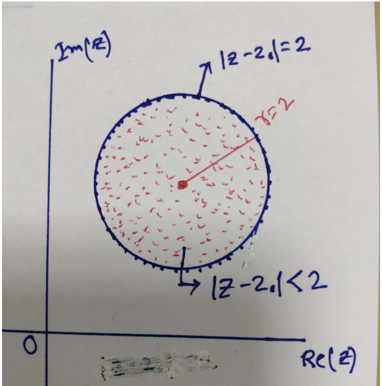

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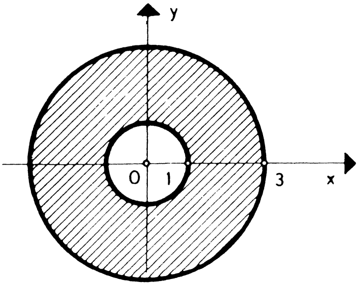

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2. The subset contains all points of $|z - z_0| < 2$ except z_0.
3. It has a hole at z_0.

Complex Analysis

LECTURE 3

July 31, 2023

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3. We can apply these definitions to define limit, continuity and differentiability of functions.

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5. Mathematically, $\hat{N}(z_1; \epsilon) \cap S \neq \emptyset$.

Limit points of a set S

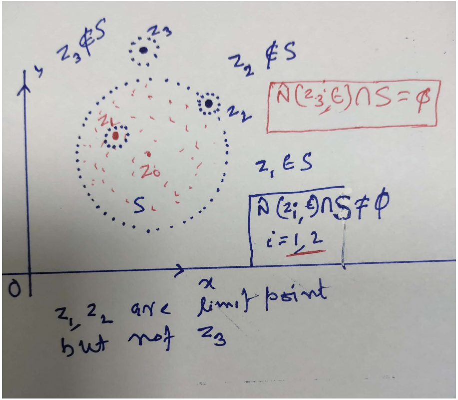

Figure: z_1, z_2 are limit points, but not z_3.

Example: Limit points of a set S

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Limit points of a set S

Problem: Let c_0 be a limit point of S. Then prove that every neighbourhood of c_0 contains infinitely many points of S.

Try yourself.
Ask me if you need help.

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4. In language, neighborhood of z_1 will be completely inside S.

Interior points of a set

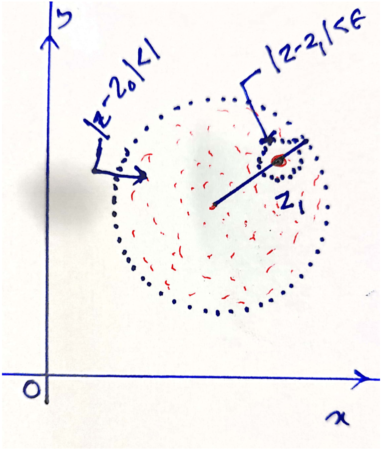

Figure: z_1 is an interior point.

Interior points of a set

Problem: Mathematically prove that all points of $S = \{z : |z - z_0| < 1\}$ are interior points.

Try yourself.

Boundary points of a set S

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4. $S = \{z : y = x^2, x, y \in \mathbb{R}\}$ is an unbounded set.

Connected set

Connected set

A set S be a subset of $\mathbb{C}$ is said to be connected if any two points in S can be joined by a polygonal line (pieces of lines joined end to end) or by any **continuous curve** which lies entirely on S.

Connected set

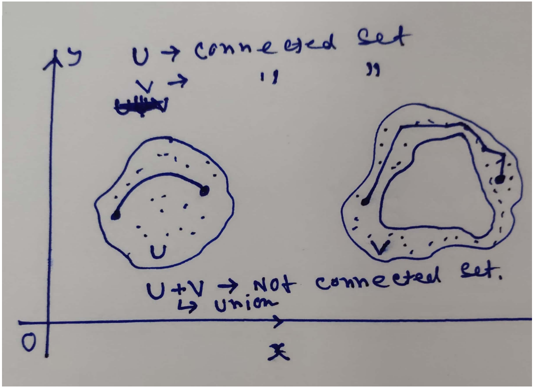

Figure: U, V are connected, but $U \cup V$ is not.

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3. Neighborhood and deleted neighborhood are connected sets.

Domain

An open connected set is called a domain.

Domain

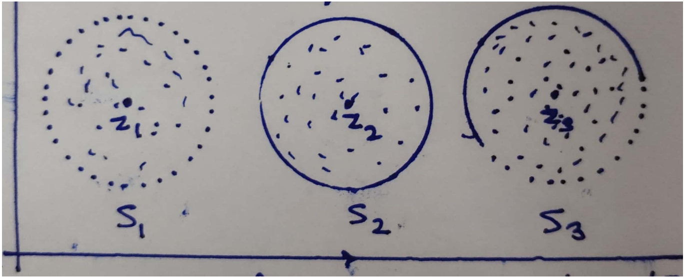

Figure: S_1 is domain, but S_2, S_3 are not.

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Region

A region is a domain together with all, some or none of its boundary points.

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5. A domain is also a region, but the converse is not true.

Complex Analysis

LECTURE 4

August 3, 2023

Complex function

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Complex function

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2. For every point $z = x + iy \in S$, we specify the rule for assigning a corresponding complex number $w = u + iv$.
3. This defines a function of the complex variable z, and the function is denoted by

$$w = f(z).$$

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7. Often we say "domain of a function". Note that "domain of a function" is not necessarily a "domain". Avoid confusion.

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3. One can also write it in polar form as:

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5. NOTE: For finding limit, it is **NOT** required to define the function at z_0.

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Determine $\lim_{z \to 0} \frac{z}{\overline{z}}$, if it exists.

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3. If we move from the positive of y-axis towards the origin, we obtain $\lim_{z\to 0} \frac{z}{\bar{z}} = \lim_{x=0, y\to 0} \frac{x+iy}{x-iy} = -1$.

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3. If we move from the positive of y-axis towards the origin, we obtain $\lim_{z\to 0} \frac{z}{\bar{z}} = \lim_{x=0, y\to 0} \frac{x+iy}{x-iy} = -1.$
4. If we move along $y = x$ line towards the origin in the first quadrant, $\lim_{z\to 0} \frac{z}{\bar{z}} = \lim_{x\to 0} \frac{x+ix}{x-ix}$, which is a complex number.

Computing limit

Determine $\lim_{z\to 0} \frac{z}{\overline{z}}$, if it exists.

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5. Thus, limit does not exist.

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4. How do we apply the definition if the existence of the limit is not ensured? No answer.
5. Ok, compute a limit along a specific path. Then use the definition. If you are successful, then fine.

Use $\epsilon - \delta$ definition

Prove that $\lim_{z \to i} 2(\frac{z^2 + iz + 2}{z - i}) = 6i$.

Use $\epsilon - \delta$ definition

Show that if the function $f(z) = i\bar{z}/2$ in the open disk $|z| < 1$, then

$$\lim_{z \to 1} f(z) = i/2.$$

Note: We did not define the function for all z in a neighborhood of

$z = 1$. Hence we should consider those z which belong to both the domain $|z| < 1$ and $\hat{N}(1, \delta)$.

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2. Ok, good news. We provide such a facility to find limit of complex functions.
3. Follow the next theorem.

Theorem

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Theorem: Let $f(z) = u(x, y) + iv(x, y)$. Then limit of $f(z)$ exists at $z_0 = x_0 + iy_0$ if and only if limits of $u(x, y)$ and $v(x, y)$ exist at (x_0, y_0).

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Further,

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It is now much easier to verify the existence of limit and computing it.

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1. Recall the usual definition you have studied for real function in single or two-variable.
2. Unlike the case of limit, here, we assume $z_0 \in D \subset \mathbb{C}$.
3. Define a functions $f : D \to \mathbb{C}$.
4. Concept of deleted neighborhood will not work. **WHY?**
5. The function is said to be continuous at z_0 if for every $\epsilon > 0$ (sufficiently small), there exists a $\delta > 0$ (depends on ϵ and the point z_0) such that

$$|f(z) - f(z_0)| < \epsilon \text{ whenever } |z - z_0| < \delta withz \in D.$$

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2. A complex function is said to be continuous in a region R if it is continuous at each point in R

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4. Then we say that f is differentiable at z_0 and the limit denoted by $f'(z_0)$ is called the derivative of f at z_0.

Differentiability

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4. Then the real valued function is differentiable (Why?) everywhere on $\mathbb{R}^2$.

Complex Analysis

LECTURE 5

August 5, 2023

Condition for differentiability

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1. What are the necessary conditions for the differentiability of a complex function.

2. If a complex function is differentiable, how to compute the derivative without using the limit definition?

3. Is there any sufficient condition for the differentiability of a complex function?

Necessary condition for differentiability

Cauchy-Riemann equations (C-R equations):

Let $f(z) = u(x, y) + iv(x, y)$ be differentiable at $z_0 = x_0 + iy_0$. Then the partial derivatives of u and v exist at the point (x_0, y_0),

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$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) = v_y(x_0, y_0) - iu_y(x_0, y_0). \tag{2}$$

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$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) = v_y(x_0, y_0) - iu_y(x_0, y_0).$$

   represents the technique to find derivative without using the limit definition?

3. Note that the violating of the C-R equations implies the non-existence of derivative at the point.

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5. In fact, $f'(0)$ exists (Verify).
6. Now can we claim that the differentiability of the function is due the satisfaction of C-R equations?

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6. If $y = x$ and $x \to 0^+$, then $f'(0) \to \infty$ (complex infinity) –derivative does not exist.
7. Thus, we need alternative conditions for differentiability

Sufficient conditions for differentiability

Theorem:

Sufficient conditions for differentiability

Theorem: Suppose $f = u + iv$ is defined on some neighborhood

$N(z_0; \epsilon)$ of $z_0 = x_0 + iy_0$ such that u_x, u_y, v_x, v_y exist on $N(z_0; \epsilon)$ and are continuous at (x_0, y_0). If u, v satisfies the C-R equations at (x_0, y_0), then f' exists at z_0 and $f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$.

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3. Therefore, u_x, u_y, v_x, v_y are all continuous on $\mathbb{R}^2$, and u, v satisfies the C-R equations $u_x(x,y) = v_y(x,y)$, $u_y(x,y) = -v_x(x,y)$ at each point $(x,y) \in \mathbb{R}^2$.

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4. The C-R equations are satisfied as well.

Application of Sufficient conditions

1. Consider $f(x+iy) = e^{-y}\cos x + ie^{-y}\sin x$.
2. $$u_x(x,y) = -e^{-y}\sin x \qquad u_y(x,y) = -e^{-y}\cos x$$
$$v_y(x,y) = -e^{-y}\sin x \qquad v_x(x,y) = e^{-y}\cos x.$$
3. Therefore, u_x, u_y, v_x, v_y are all continuous on $\mathbb{R}^2$, and u, v satisfies the C-R equations $u_x(x,y) = v_y(x,y)$, $u_y(x,y) = -v_x(x,y)$ at each point $(x,y) \in \mathbb{R}^2$.
4. The C-R equations are satisfied as well.
5. Hence $f'(z) = u_x + iv_x = -e^{-y}\sin x + ie^{-y}\cos x$

Your comment

Your comment

1. Consider the function $f(z)$

$$f(z) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } z \neq 0 \\ 0, & \text{if } z = 0. \end{cases}$$

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2. Check whether the sufficient conditions are satisfied at and in the neighborhood of origin.
3. What is your conclusion on the existence of $f'(z)$ at origin?

Complex Analysis

LECTURE 6

August 7, 2023

Your comment

Your comment

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Our observations

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1. The functions $f(z) = |z|^2$ is differentiable at the origin only.
2. The function $f(z) = |z|^2$ is **NOT** differentiable in any neighborhood of origin.
3. The function $f(x + iy) = x^2 + iy^2$ is differentiable at all points on $y = x$ (HOW?).
4. But the above functions too is **NOT** differentiable in any neighborhood of any point on $y = x$.

Our observations

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3. Therefore, the above functions **are** differentiable in any neighborhood of any point.
4. Thus we obtain a spacial class of functions.
5. **Hence we present the concept** of analytic functions.

Analytic functions

Analytic functions

1. We first provide the definition of analytic functions at a point.

Analytic functions

1. We first provide the definition of analytic functions at a point.
2. Then we define analytic functions on a set.

Analytic functions

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1. **Analytic function at a point**: A function f is said to be analytic at a point z_0 if there exists a neighborhood $N(z_0, \epsilon)$ of z_0, $\epsilon > 0$ (sufficiently small) such that f is differentiable at every point $z \in N(z_0, \epsilon)$.

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2. **Analytic function on a set**: A function f is said to be analytic (or regular or holomorphic) on a set D if it is differentiable at every point of some open set containing D.

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3. If f is **analytic** at all points of a set D, then f is analytic on D.

Necessary and Sufficient Condition for Analyticity

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Necessary and Sufficient Condition for Analyticity

A function $f = u + iv$ is analytic in a **domain** D if and only if u, v satisfies C-R equations in D, and u_x, u_y, v_x, v_y are continuous in D.

Problems

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Apply the theorem to the function on $\mathbb{C}$

1. $f(z) = z$

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1. Check each of the conditions for the above function.

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Apply the theorem to the function on $\mathbb{C}$

1. $f(z) = z$
2. $f(x + iy) = x^2 + iy^2$

1. Check each of the conditions for the above function.

2. What is your conclusion about the analyticity of the function?

Entire functions

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Entire function: A function analytic on the entire complex plane is called an entire function.

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Example

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Example

1. Any polynomial

Entire functions

Entire function: A function analytic on the entire complex plane is called an entire function.

Example

1. Any polynomial
2. $f(x+iy) = e^{-y}\cos x + ie^{-y}\sin x$

Problems

Problem: Suppose f is analytic in a domain D. If $f'(z) = 0$ for all $z \in D$, then f is constant on D.

Contd...

Problems

Problem: Suppose f is analytic in a domain D. If any of $\mathrm{Re} f$, $\mathrm{Im} f$ is constant in D, then f is constant in D.

Problems

Problem: Suppose f is analytic in a domain D such that $|f|$ is constant in D. Then show that f is a constant in D.

Problems

Problem: Let $f = u + iv$ be a non-constant function such that $\overline{f} = u - iv$ be analytic in a domain D. Show that f cannot be analytic in D.

Complex Analysis

LECTURE 7

August 9, 2023

Problems

Problem: Given an **analytic function**

$$w = f(z) = u(x, y) + iv(x, y), z = x + iy,$$

the equations $u(x, y) = \alpha$ and $v(x, y) = \beta$, α and β are constants, define two families of curves in the complex plane. Show that the two families are mutually orthogonal to each other.

The families of curves are called level curves.

Verify the orthogonality

Verify the orthogonality.
Draw the level curves too.

1. $f(z) = z$

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1. $f(z) = z$
2. $f(z) = z^2$
3. $f(x + iy) = x^2 + iy^2$ **(Interesting problem)**

Harmonic functions

Harmonic functions

Definition:

A real-valued function $\phi(x, y)$ of two real variables x and y is said to be harmonic in a given domain D in the xy-plane if ϕ has continuous partial derivatives up to the second order in D

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Definition:
A real-valued function $\phi(x, y)$ of two real variables x and y is said to be harmonic in a given domain D in the xy-plane if ϕ has continuous partial derivatives up to the second order in D AND satisfies the Laplace equation

$$\phi_{xx}(x, y) + \phi_{yy}(x, y) = 0, (x, y) \in D.$$

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1. $\phi(x, y) = x$

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Example

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Example

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2. $\psi(x, y) = y$
3. $\phi(x, y) = x^2 - y^2$

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Example

1. $\phi(x, y) = x$
2. $\psi(x, y) = y$
3. $\phi(x, y) = x^2 - y^2$
4. $\psi(x, y) = 2xy$

Conjugate Harmonic function

Conjugate Harmonic function

Definition:
If two harmonic functions $\phi(x, y), \psi(x, y)$ satisfy C-R equations, namely

$$\phi_x = \psi_y \text{ and } \phi_y = -\psi_x$$

in a domain D, then ψ is called conjugate harmonic function of ϕ.

Important note
Note that harmonic conjugacy is not a symmetric relation, that is, if ψ is conjugate harmonic function of ϕ, then it does not mean that ϕ will be conjugate harmonic function of ψ. This is due to the minus sign in the second Cauchy–Riemann relation.

Examples: Conjugate harmonic function

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4. ψ is a conjugate Harmonic function of ϕ, but ϕ is **NOT** a conjugate Harmonic function of ψ.
5. Verify the statement by C-R equations.

Important observation

1. For $\phi(x,y) = x, \psi(x,y) = y$,
   $f(x+iy) = \phi(x,y) + i\psi(x,y) = z$ is **an analytic function**.

Important observation

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6. Perhaps, special arrangement of $\phi(x, y)$ and $\psi(x, y)$ is constructing an analytic function.

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7. Ok, go to the next theorem.

Next theorem

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2. The theorem provides **conditions for analyticity** of a function.

Next theorem

1. Next theorem give a relationship between analyticity and conjugacy.
2. The theorem provides **conditions for analyticity** of a function.
3. Alternatively, it gives ideas to construct an analytic function from a given harmonic function.

Theorem

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A complex function $f(z) = u(x, y) + iv(x, y)$, $z = x + iy$, is analytic in a domain D if and only if v is a harmonic conjugate of u in D.

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A complex function $f(z) = u(x, y) + iv(x, y)$, $z = x + iy$, is analytic in a domain D if and only if v is a harmonic conjugate of u in D.

Verification

1. Verify for $f(x + iy) = e^{-y}(\cos x + i \sin x)$.
2. Verify for $f(x + iy) = x^2 + iy^2$.

Uniqueness of harmonic conjugate functions

Result

Any two harmonic conjugates v, w of u in a domain D are unique in the sense that they differ by a constant, that is,

$$v(x, y) - w(x, y) = K$$

for all $(x, y) \in D$, where K is a real constant.

Uniqueness of harmonic conjugate functions

Result

Any two harmonic conjugates v, w of u in a domain D are unique in the sense that they differ by a constant, that is,

$$v(x, y) - w(x, y) = K$$

for all $(x, y) \in D$, where K is a real constant.
Proof:

Proof

Construction of harmonic conjugate function

Problem

Find a conjugate harmonic function of $u(x, y) = x^2 - y^2 - y$ in $\mathbb{C}$.

Construction of harmonic conjugate function

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Solution:

Construction of harmonic conjugate function

Construction of harmonic conjugate function

Answer: $v = 2xy + x + K, K \in \mathbb{R}$

Construction of harmonic conjugate function

Answer: $v = 2xy + x + K, K \in \mathbb{R}$
In addition, we can construct the analytic function
$f(x + iy) = x^2 - y^2 - y + i(2xy + x + K) = z^2 + i(z + k)$

Complex Analysis

LECTURE 8

August 10, 2023

Elementary functions

Elementary functions

Elementary functions

Elementary functions

1. Exponential function

Elementary functions

Elementary functions

1. Exponential function
2. Trigonometric functions

Elementary functions

Elementary functions

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2. Trigonometric functions
3. Hyperbolic functions

Elementary functions

Elementary functions

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2. Trigonometric functions
3. Hyperbolic functions
4. Logarithmic function

Complex exponential function

Complex exponential function

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$$e^{i\pi} + 1 = 0.$$

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3. Euler's identity—"The most beautiful equation": When $y = \pi$ in **Euler's formula**, we obtain

$$e^{i\pi} + 1 = 0.$$

4. What would be the form of De-Moiver's theorem in view of Euler's formula?

Properties exponential function

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1. $|e^z| = e^x$, and $\arg e^z = y \pm 2n\pi, \quad (n = 0, 1, 2, \ldots).$

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4. $e^z \neq 0$ for all z.

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3. $e^{z_1+z_2} = e^{z_1}e^{z_2}$.
4. $e^z \neq 0$ for all z.
5. e^z is periodic with the fundamental period $2\pi i$ (Ohh! purely imaginary period),

$$\text{i.e.,} e^{z+2k\pi i} = e^z, z \in \mathbb{C}, k \in \mathbb{Z}$$

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Properties exponential function

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$$\text{i.e.,} e^{z+2k\pi i} = e^z, z \in \mathbb{C}, k \in \mathbb{Z}$$

6. e^z is NOT injective.
7. If $H = \{x + iy : -\pi < y \leq \pi\}$. Then e^z is bijective from H to $\mathbb{C} \setminus \{0\}$.

Trigonometric function

Trigonometric function

1. Euler formula yields

$$e^{iy} = \cos y + i \sin y \quad \text{and} \quad e^{-iy} = \cos y - i \sin y.$$

Trigonometric function

1. Euler formula yields

$$e^{iy} = \cos y + i \sin y \quad \text{and} \quad e^{-iy} = \cos y - i \sin y.$$

2. Solving the above equations, we obtain

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3. Here sine and cosine functions are expressed in term of complex exponential functions.

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3. Now using sine and cosine we can define $\tan z, \sec z, \mathrm{cosec}\ z$ as in the real case.

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3. Real and imaginary parts of $\sin z$ and $\cos z$ are:

$$\begin{aligned}\sin z &= \sin x \cosh y + i \cos x \sinh y\\ \cos z &= \cos x \cosh y - i \sin x \sinh y.\end{aligned}$$

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3. $\sin z = 0$ iff $z = k\pi$, and $\cos z = 0$ iff $z = k\pi + \frac{\pi}{2}$, where $k \in \mathbb{Z}$.

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2. $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$—punctured complex plane.
3. $H = \{x + iy : -\pi < y \leq \pi\}$- a strip in the complex plane (visualize it)

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5. Certainly, $\text{Log} : \mathbb{C}^* \to H$ is well defined i.e., single-valued function.

Example

1. When $z = 1 + i$,

$$\log\,(1+i) = \ln\sqrt{2} + i arg(z) = \ln\sqrt{2} + i\left(\frac{\pi}{4} + 2k\pi\right), k \in \mathbb{Z}.$$

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Complex Analysis

LECTURE 9

August 16, 2023

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3. For $z \notin H$, Log $\mathrm{e}^z \neq z$.
4. For real $x > 0$, Log $x = \ln x$.
5. Log z is not continuous on the negative real axis $\mathbb{R}^- = \{z = x + iy : x < 0, y = 0\}$.

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2. $\dfrac{\mathrm{d}}{\mathrm{d}z}\text{Log } z = \dfrac{1}{z}$.
3. The identity Log $(z_1 z_2) =$ Log $z_1 +$ Log z_2 is true iff Arg $z_1 +$ Arg $z_2 \in (-\pi, \pi]$.

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defined as

$$\gamma(t) = x(t) + iy(t),$$

with $x, y : [a, b] \to \mathbb{R}$ being continuous.

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2. Line segment:
   The function $\gamma : [0, 1] \to \mathbb{C}$ given by

   $$\gamma(t) = tz_1 + (1 - t)z_0$$

   gives the line segment joining z_0 and z_1.

Types of curves

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3. Alternatively, a smooth curve does not have any **sharp point or corner**.

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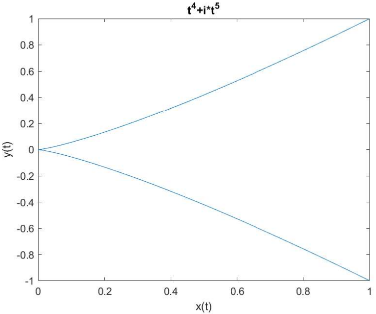

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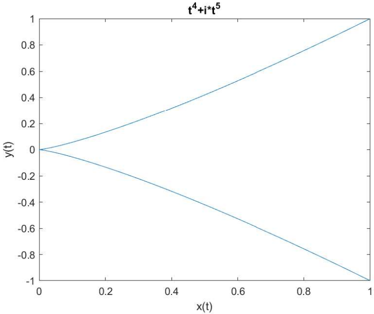

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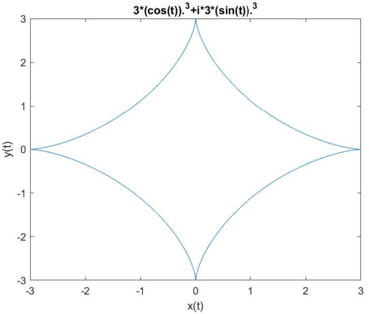

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2. In this case, the curve may have property $\gamma(t_1) = \gamma(t_2)$ for some $t_1 \neq t_2, t_1, t_2 \in (a, b)$ too.

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3. **Examples**: A circle is simple, the curve of the shape U are examples of simple contour, but the curve of the shapes 8 and g are not simple

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Curve with Reverse Orientation

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2. Thus for a contour C, the contour with the negative orientation $-C$ make sense.

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4. Easy to calculate.

Important results

Result: Let $f : [a, b] \to \mathbb{C}$ be a piecewise continuous function. Then

$$\left| \int_a^b f(t)\, dt \right| \leq \int_a^b |f(t)|\, dt.$$

Proof:

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Proof:

1. Let $\phi = \text{Arg}\ \left(\int_a^b f(t)\, dt \right)$
2. $$\begin{aligned} \left| \int_a^b f(t)\, dt \right| &= \mathrm{e}^{-i\phi} \int_a^b f(t)\, dt \qquad (\because |z| = \mathrm{e}^{-i\text{Arg } z} z) \\ &= \int_a^b \mathrm{e}^{-i\phi} f(t)\, dt \end{aligned}$$

Proof

$$=\text{Re } \int_a^b e^{-i\phi} f(t)\, dt \ \left(\because \int_a^b e^{-i\phi} f(t)\, dt = \left|\int_a^b f(t)\, dt\right| \text{ is real }\right)$$

$$= \int_a^b \text{Re } \left(e^{-i\phi} f(t)\right)\, dt$$

[Note that this is Riemann integral of the real function Re $\left(e^{-i\phi} f(t)\right)$]

$$\leq \int_a^b \left|e^{-i\phi} f(t)\right|\, dt$$

[Using the property of Riemann integral with the fact that

$$\text{Re } \left(e^{-i\phi} f(t)\right) \leq \left|e^{-i\phi} f(t)\right|]$$

$$= \int_a^b |f(t)|\, dt.$$

Remark

ϕ is NOT defined if $\left|\int_a^b f(t)\,dt\right| = 0$. Then there is nothing to prove.

Important results

Result: For $\alpha \in \mathbb{R}$, show that

$$\int_a^b e^{i\alpha t}\, dt = \frac{e^{i\alpha b} - e^{i\alpha a}}{i\alpha}.$$

Complex Analysis

LECTURE 10

August 17, 2023

Important results

Result: Let $f : [a, b] \to \mathbb{C}$ be a continuous function. Then it is NOT necessary that there must exist $c \in (a, b)$ such that

$$\int_a^b f(t)\,dt = f(c)(b-a).$$

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Example:

1. Set $f(t) = e^{it}, a = 0, b = 2\pi$.
2. Then $\int_a^b f(t)\,dt = 0$
3. But $f(t)$ is zero nowhere.

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5. Such an integration is called a contour integration.

Contour Integral

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4. We define the integration using **Riemann sum**.

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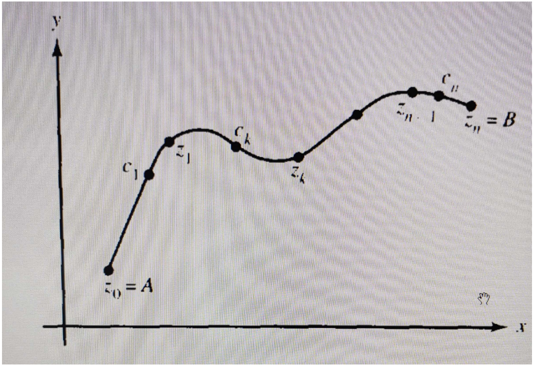

Figure:

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5. Then the Riemann sun for the partition P_n is

$$S(P_n) = \sum_{k=1}^{n} f(c_k)\Delta z_k$$

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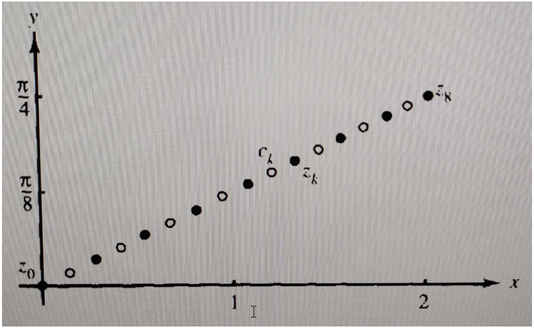
Example-Riemann sum

Use the Riemann sum to find an approximation $S(P_8)$ to the contour integral $\int_C e^z\, dz$ where C is the line joining the starting point $A = 0$ and terminal point $B = 2 + i\pi/4$.

Example-Riemann sum



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5. Later we can calculate the analytical value
$$-1 + \frac{e^2}{\sqrt{2}} + i\frac{e^2}{\sqrt{2}} \approx 4.22485 + 4.22485i$$

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3. How to convert?

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5. We solve many problems using the final expression.

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3. Note the RHS expression could be remembered from the expansion of $\int_C (u + iv)(dx + idy)$

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6. Ok, wait a bit. We'll see the beauty of the Cartesian form soon.

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4. $\int_C (f(z)+g(z))\,dz = \int_C f(z)\,dz + \int_C g(z)\,dz.$

Example

Evaluate

$$I = \int_C z^2 \, dz,$$

where C is along x-axis from 0 to 1 and then along the line parallel to y-axis from 1 to $1 + 2i$.

Example-contd

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4. $$\int_{C_2} z^2\, dz = \int_0^1 (1+2it)^2 2i\, dt = -4 - \frac{2}{3} i.$$

Example-contd

$$\int_C z^2 \, dz = \int_{C_1} z^2 \, dz + \int_{C_2} z^2 \, dz = -\frac{11+2i}{3}.$$

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3. What is its role in integration?

Integration using Indefinite Integral

Theorem: Let f be a continuous function defined on a domain D and there exists a function F defined on D such that $F' = f$. Let $z_1, z_2 \in D$. Then for any contour C lying in D starting from z_1, and ending at z_2,

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$$\int_C f(z)\,dz = F(z_2) - F(z_1).$$

Proof:

1. Suppose that that C is given by a map $\gamma : [a, b] \to \mathbb{C}$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t} F(\gamma(t)) = F'(\gamma(t))\gamma'(t).$$

Proof-Contd.

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$$
\begin{aligned}
\int_C f(z)\,dz &= \int_a^b f(\gamma(t))\gamma'(t)\,dt \\
&= \int_a^b F'(\gamma(t))\gamma'(t)\,dt \\
&= \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}F(\gamma(t))\,dt \\
&= F(\gamma(t))|_a^b = F(\gamma(b)) - F(\gamma(a)) \\
&= F(z_2) - F(z_1).
\end{aligned}
$$

Proof-Contd.

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1. Geometrically, it means that the integration does not depend on the path.

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1. Geometrically, it means that the integration does not depend on the path.
2. The integration depends on the end points only.

Application of the theorem

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Evaluate

$$I = \int_C z^2 \, dz,$$

where C is along x-axis from 0 to 1 and then along the line parallel to y-axis from 1 to $1+2i$.

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4. Hence, we CAN apply the result.

Example-contd

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$$\int_C z^2 \, dz = \left[\frac{z^3}{3}\right]_0^{1+2i} = -\frac{11+2i}{3}.$$

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3. Good news. We do not calculate line integration now.

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Example

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Can you apply the theorem to evaluate $\int_C |z|^2\, dz$, where C is an arc of unit circle $|z| = 1$ traversed in the clockwise direction with initial point -1 and final point i?

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3. Obviously, C lies in D.
4. Therefore,

$$\int_C \frac{1}{z}\,dz = \Big[\mathrm{Log}\ z\Big]_{-i}^{i} = \mathrm{Log}\ (i) - \mathrm{Log}\ (-i) = i\frac{\pi}{2} - i\frac{-\pi}{2} = i\pi.$$

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$$\begin{aligned} L(\gamma) &= \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt \\ &= \int_a^b |\gamma'(t)|\, dt \end{aligned}$$

ML formula or ML inequality

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If L is the arc length of a contour $C : \gamma(t)$, $a \leq t \leq b$ and M is a positive number such that $|f(z)| \leq M$ for all $z \in C = \gamma$ ($f(z)$ is a piece-wise continuous function), then

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$$\left| \int_C f(z)\, dz \right| \leq ML.$$

Proof

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$$\begin{aligned}
\left|\int_C f(z)\,dz\right| &= \left|\int_a^b f(\gamma(t))\gamma'(t)\,dt\right| \\
&\leq \int_a^b |f(\gamma(t))|\,|\gamma'(t)|\;dt \\
&\leq \int_a^b M\,|\gamma'(t)|\;dt \\
&= M\int_a^b |\gamma'(t)|\;dt \\
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1. It give some estimate of an upper bound of the modulus of the integration.
2. We can prove some important theorems using this ML-inequality.

Important Integral Theorems

We present three important theorems

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1. Cauchy's Integral Theorem

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1. Cauchy's Integral Theorem
2. The Cauchy-Goursat Theorem
3. Cauchy's Integral Formula

Cauchy's Integral Theorem: by Cauchy in 1846

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Let $f(z)$ be a function such that

1. $f(z)$ is analytic on and inside a simple closed contour C, and
2. $f'(z)$ be continuous on and inside C.

Then

$$\int_C f(z)\, dz = 0.$$

Proof

Review of Green's theorem

1. Let C be a piece-wise, simple closed curve enclosing a region R.

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Then

$$\int_C (Mdx + Ndy) = \iint_R (N_x - M_y)dxdy,$$

where the line integral is taken in **counterclockwise sense**.

Proof-Contd.

1. We know (alternative approach can be found in many books)

$$\int_C f(z)\,dz = \int_C (udx - vdy) + i\int_C (vdx + udy)$$

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$$\int_C f(z)\,dz = \iint_D (-v_x - u_y)dxdy + i\iint_D (u_x - v_y)dxdy$$

3. Because of analyticity, CR equations are satisfied on D.
4. Hence

$$\int_C f(z)\,dz = 0 + i0 = 0.$$

Cauchy-Goursat theorem

Let $f(z)$ be a function such that $f(z)$ is analytic on and inside a simple closed contour C. Then

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5. Cauchy-Goursat theorem is a stronger version of Cauchy's integral theorem.
6. We use this theorem to solve problems.

Alternative form of Cauchy-Goursat theorem

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Let $f(z)$ be a function analytic throughout a simply connected domain D and C be a simple closed contour lying completely inside D. Then

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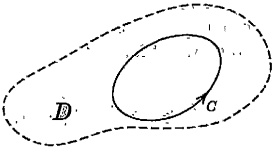

Figure:

Examples- Application of Cauchy-Goursat theorem

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Evaluate the following integrals where C denotes the circle of unit radius with center at zero.

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3. $\displaystyle\int_C \frac{\mathrm{e}^{iz^2}}{z^2+4}\,dz.$

Examples- Application of Cauchy-Goursat theorem

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1. Can you apply Cauchy-Goursat's theorem to evaluate $\int_C \operatorname{cosec}^2 z$, where C denotes the circle of unit radius with center at zero.

Examples- Application of Cauchy-Goursat theorem

1. Can you apply Cauchy-Goursat's theorem to evaluate $\int_C \operatorname{cosec}^2 z$, where C denotes the circle of unit radius with center at zero.
2. Determine the value of the integration.

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3. Note that $f(z)$ is analytic at $\mathbb{C} \setminus \{n\pi : n \in \mathbb{Z}\}$.

4. Consider the annulus $D = \{z : 0 < |z| < \pi\}$. This is our modified domain.

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2. The contour C lies entirely inside D.
3. Also $F'(z) = f(z)$ on D, where $F(z) = -\cot z$.
4. Since C is a closed contour with starting and end points z_1 and z_2. Then $z_1 = z_2$.
5. Hence $\int_C \mathrm{cosec}^2 z = F(z_2) - F(z_1) = 0$.

Complex Analysis

LECTURE 11

August 22, 2023

Deformation of simple closed counter

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3. Visualize the statement.
4. What is the advantage/application of such deformation?

Trouble with ugly contour

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5. Thus we have **NO trouble**.

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Trouble with ugly contour

1. For simplicity, we consider the function $f(z) = 1/z$ and the contour encloses a domain including origin.
2. Thus, we CANNOT apply Cauchy-Goursat theorem.
3. Thus, we need to calculate the integration **DIRECTLY** along the curve.
4. So parametric representation of the ugly contour is required.
5. This makes our life terrible.

.

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3. Follow the **next theorem**.

Theorem on deformation contour

Theorem on deformation contour

Theorem

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Let C_1 and C_2 be two simple closed **positively** oriented contours such that C_2 lies interior to C_1. If $f(z)$ is analytic in a domain D that contains both the contours and the region between them, then

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

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1. Both the contours have positive orientation.
2. It is not necessary that the function needs to be analytic inside the domain enclosed by C_2.
3. In practical situations (examples), $f(z)$ is NOT analytic inside the domain enclosed by C_2.

Proof-outline

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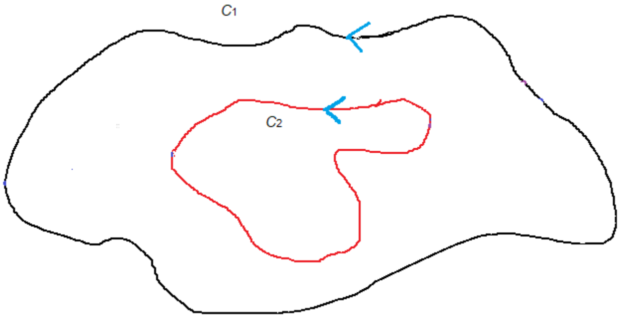

Figure:

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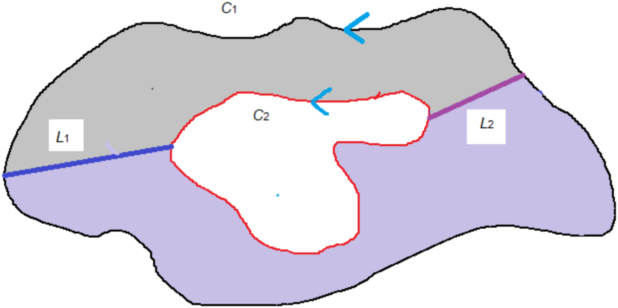

Figure: $f(z)$ is analytic on the curve and shaded region. It may not be analytic on the white domain enclosed by C_2.

Proof-outline

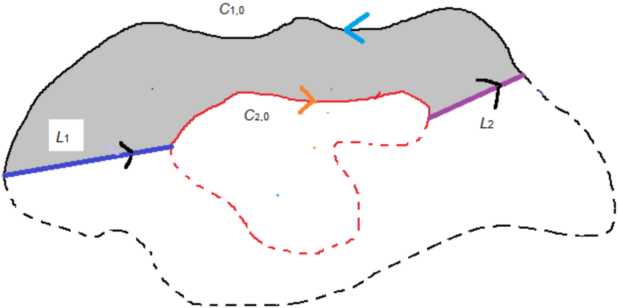

Figure: Integration along $C_{1,0} + L_1 + C_{2,0} + L_2$ is 0 by Cauchy-Goursat theorem.

Proof-outline

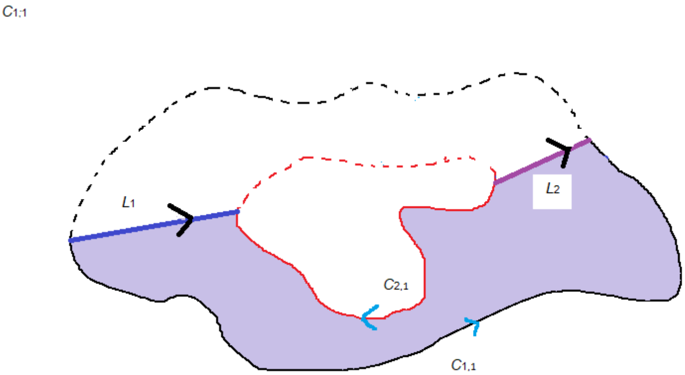

Figure: Integration along $C_{1,1} - L_2 + C_{2,1} - L_1$ is 0 by Cauchy-Goursat theorem.

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$$\int_{C_{1,0}+L_1+C_{2,0}+L_2} = 0$$

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3. Adding together
$$\int_{C_{1,0}+C_{1,1}+C_{2,0}+C_{2,1}} = 0$$

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4. Note that we have applied Cauchy-Goursat theorem TWICE.

Example

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4. Direct integration yield $2\pi i$ for any r.

ALTERNATIVE Proof-outline

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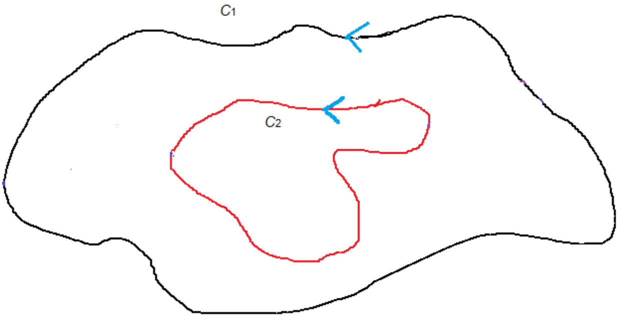
We can prove the theorem by applying cauchy-Goursat theorem only ONE time.



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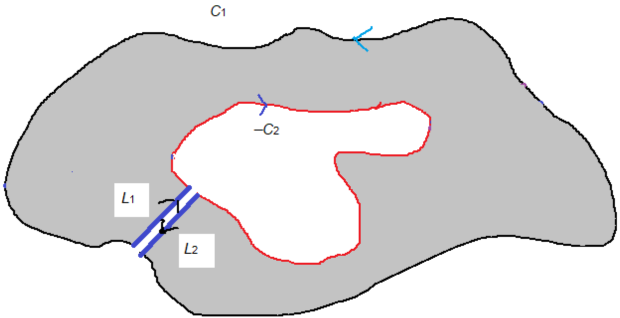

Figure: One one simply connected domain (shaded portion). Integration along $C_1 + L_1 - C_2 + L_2$ is 0 by Cauchy-Gaursat theorem.

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Cauchy-Goursat's theorem for multiply connected domain

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Theorem

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Let C be a positively oriented simple closed contour and C_k, $k = 1, 2, \ldots n$ denote a finite number of positively oriented simple closed contours all lying wholly within C, but each C_k lies in the exterior of every other whose interior have no points in common. If a function f is analytic throughout the closed region D consisting of all points within and on C except for the points interior to each C_k.

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$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \cdots + \int_{C_n} f(z)\,dz.$$

Proof:- for doubly connected domain

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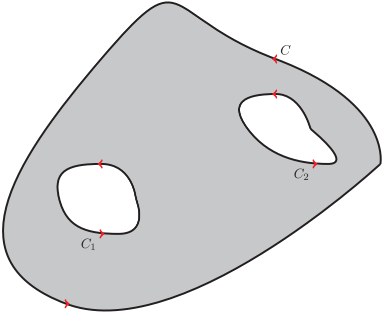

Figure: Think about the proof with for doubly connected domain

Proof:- for doubly connected domain

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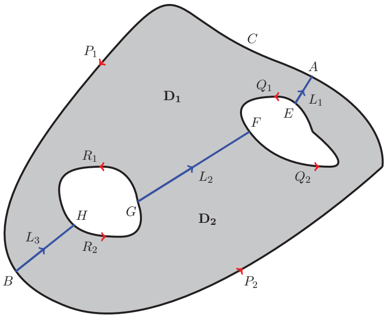

Figure: Apply the concepts from earlier the theorem

Example with multiply connected domain

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4. Then by Cauchy-Goursat theorem for doubly connected domain

$$\int_C \frac{1}{z^2+1} dz = \int_{C_1} \frac{1}{z^2+1} dz + \int_{C_2} \frac{1}{z^2+1} dz$$

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5. Finally, $\displaystyle\int_{C} \frac{1}{z^2+1}dz = \frac{1}{2i}[2\pi i - 0] + \frac{1}{2i}[0 - 2\pi i] = 0.$

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6. But we can find an easy way follows from the next theorem.

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Let the function $f(z)$ be analytic on and inside a positively oriented simple closed contour C and z_0 be any point inside C.

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Let the function $f(z)$ be analytic on and inside a positively oriented simple closed contour C and z_0 be any point inside C.
Then

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz. \tag{3}$$

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3. We apply Cauchy's integral formula
4. Solution:

$$\int_C \frac{\cos(e^z)}{z} dz = 2\pi i \cos(e^z)|_{z=0} = 2\pi i \cos(1).$$

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4. Then by Cauchy-Goursat theorem for doubly connected domain

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3. Finally, $\displaystyle\int_{C} \frac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-2)(z-3)}dz = -4\pi i.$

Derivative of contour integrals

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Statement

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If f is analytic on a simply connected domain D then f has derivatives of all orders in D (which are then analytic in D) and for any $z_0 \in D$, one has

$$f^n(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} \, dz, \tag{4}$$

where C is a simple closed contour (oriented counterclockwise) around z_0 in D.

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2. Solution:

$$\int_C \frac{e^{z^2}}{z^2} dz = \frac{2\pi i}{1!} \frac{d}{dz}(e^{z^2})|_{z=0} = 0.$$

Converse of Cauchy-Goursat's Theorem

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1. Answer is NO.
2. Consider the domain $D := \{z : |z| < 1\}$
3. $f(z) = \begin{cases} z & \text{if } z \in D \setminus \{\frac{1}{2}\} \\ 1 & \text{if } z = \frac{1}{2} \end{cases}.$

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4. For every closed contour, one can see:
$$\int_C f(z)\, dz = 0.$$
5. But, note that $f(z)$ is NOT analytic in D.

Morera's Theorem

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2. Why partially?
3. Because: Instead of the function $f(z)$, we have imposed continuity property on $f(z)$ on the domain.

Consequences of Cauchy's Integral Formula

We discuss about three consequences:

1. Cauchy Estimates

Consequences of Cauchy's Integral Formula

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2. Liouville's Theorem

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3. Fundamental Theorem of Algebra

Cauchy Estimates

Let $f(z)$ be analytic on and inside the circle $C : |z - z_0| = r$. If $|f(z)| \leq M$ for all $z \in C$, then for all $n \geq 0$,

$$|f^{(n)}(z_0)| \leq \frac{n!M}{r^n}.$$

Proof

Liouville's Theorem

Liouville's Theorem: Bounded entire function must be a constant function.

Proof

Fundamental Theorem of Algebra

Before we present the Fundamental Theorem of Algebra we prove the following result.

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Result: $|P(z)| \to \infty$ as $|z| \to \infty$ for every polynomial $P(z)$ of degree $n \geq 1$.

Proof

Proof

Fundamental Theorem of Algebra

Fundamental Theorem of Algebra: Every polynomial $P(z)$ of degree $n \geq 1$ has a root in $\mathbb{C}$.

Proof

Fundamental Theorem of Algebra

Corollary: Every polynomial $P(z)$ of degree $n \geq 1$ has exactly n (not necessarily distinct) roots in $\mathbb{C}$.

Proof

Thank You