INDIAN INSTITUTE OF TECHNOLOGY INDORE

MA 203 Complex Analysis and Differential Equations-II

Autumn Semester

Tutorial – 8 (Complex Analysis)

1. For what simple closed contour C will it follow from Cauchy-Goursat theorem that

$$\text{(a)} \int_C \frac{1}{z}\,dz = 0 \quad \text{(b)} \int_C \frac{\cos z}{z^6 - z^2}\,dz = 0 \quad \text{(c)} \int_C \frac{e^{\frac{1}{z}}}{z^2+9}\,dz = 0.$$

2. Evaluate the following integrals. It would useful to draw the contours/domain in understanding the theory in a better way.

   (a) $\displaystyle\int_C \frac{1}{z-3i}\,dz$, C the circle $|z| = \pi$, counter clockwise — Ans: $2\pi i$

   (b) $\displaystyle\int_C \frac{\text{Log }(z-1)}{z-6}\,dz$, C the circle $|z-6| = 4$, counter clockwise — Ans: $2\pi i\text{Log } 5$

   (c) $\displaystyle\int_C \frac{z^2}{z-1}\,dz$, C the circle $|z| = 2$, counter clockwise — Ans: $2\pi i$

   (d) $\displaystyle\int_C \frac{z^2 \sin z}{(z-\pi)^3}\,dz$, C the circle $|z| = 2\pi$, counter clockwise — Ans: $-4\pi^2 i$

   (e) $\displaystyle\int_C \frac{e^z}{z}\,dz$, C consists of $|z| = 2$ (counterclockwise) and $|z| = 1$ (clockwise) — Ans:0

   (f) $\displaystyle\int_C \frac{1}{z^2+1}\,dz$, C: (a) $|z+i| = 1$, (b) $|z-i| = 1$ counter clockwise — Ans: $-\pi,\ \pi$

   (g) $\displaystyle\int_C \frac{2z^3+z^2+4}{z^4+4z^2}\,dz$, C the circle $|z-2| = 4$ clockwise — Ans: $-4\pi i$

   (h) $\displaystyle\int_C (z-z_1)^{-1}(z-z_2)^{-1}\,dz$ for a simple closed path C enclosing z_1 and z_2. — Ans:0

3. Evaluate $\displaystyle\int_C \frac{e^z}{z^2(z-1)}\,dz$, without using partial fractions, where C is the circle $|z| = 2$, traversed in the counter clockwise direction — Ans: $2\pi i(e-2)$

4. Give an example of a function $f(z)$ for which $\displaystyle\int_{|z|=r} f(z)\,dz = 0$ for any $r > 0$, even thought $f(z)$ is not analytic everywhere.

5. Evaluate the following integrals:

   (a) $\displaystyle\int_0^{2\pi} e^{e^{i\theta}}\,d\theta$ — Ans: 2π

   (b) $\displaystyle\int_0^{2\pi} e^{(e^{i\theta}-i\theta)}\,d\theta$ — Ans: 2π

6. Is it possible to have an analytic function F in a domain D such that $F'(z) = |z|^2$ for all $z \in D$? Give reason for your answer.

7. Let $D = \{z \in \mathbb{C} : |z| < 1\}$ and f be an analytic function defined on D. Suppose $a, b \in D$ and $C : \gamma(t) = a + t(b-a)$, $t \in [0,1]$ is the straight line joining a and b.

   (a) Prove that $\displaystyle\frac{f(b)-f(a)}{b-a} = \int_0^1 f'(\gamma(t))\,dt$

(b) If Re $f'(z) > 0$ for all $z \in D$ then prove that f is injective.

8. Let D be a simply connected domain and $f : D \to \mathbb{C}$ be an analytic function. Then prove that

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta$$

for every $r > 0$ such that $N(z_0; r)$ is contained in D.

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