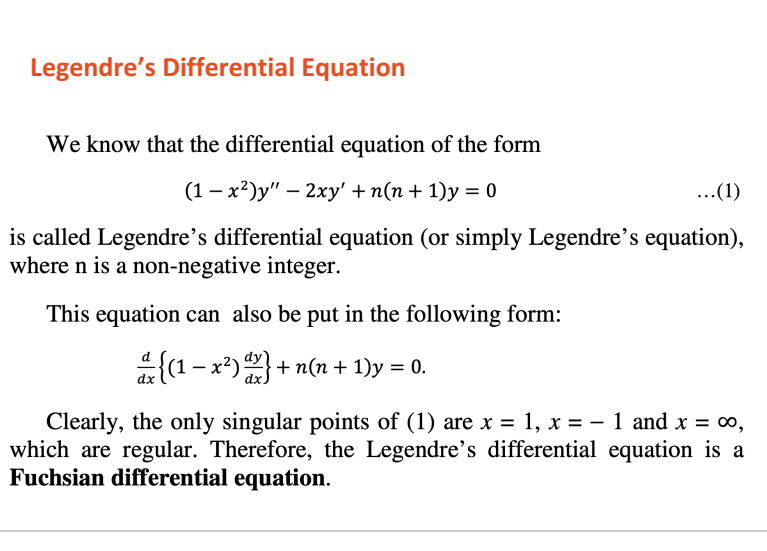
Lecture-9

Monday, 11 September 2023 10:56 PM



Legendre's Differential Equation

We know that the differential equation of the form

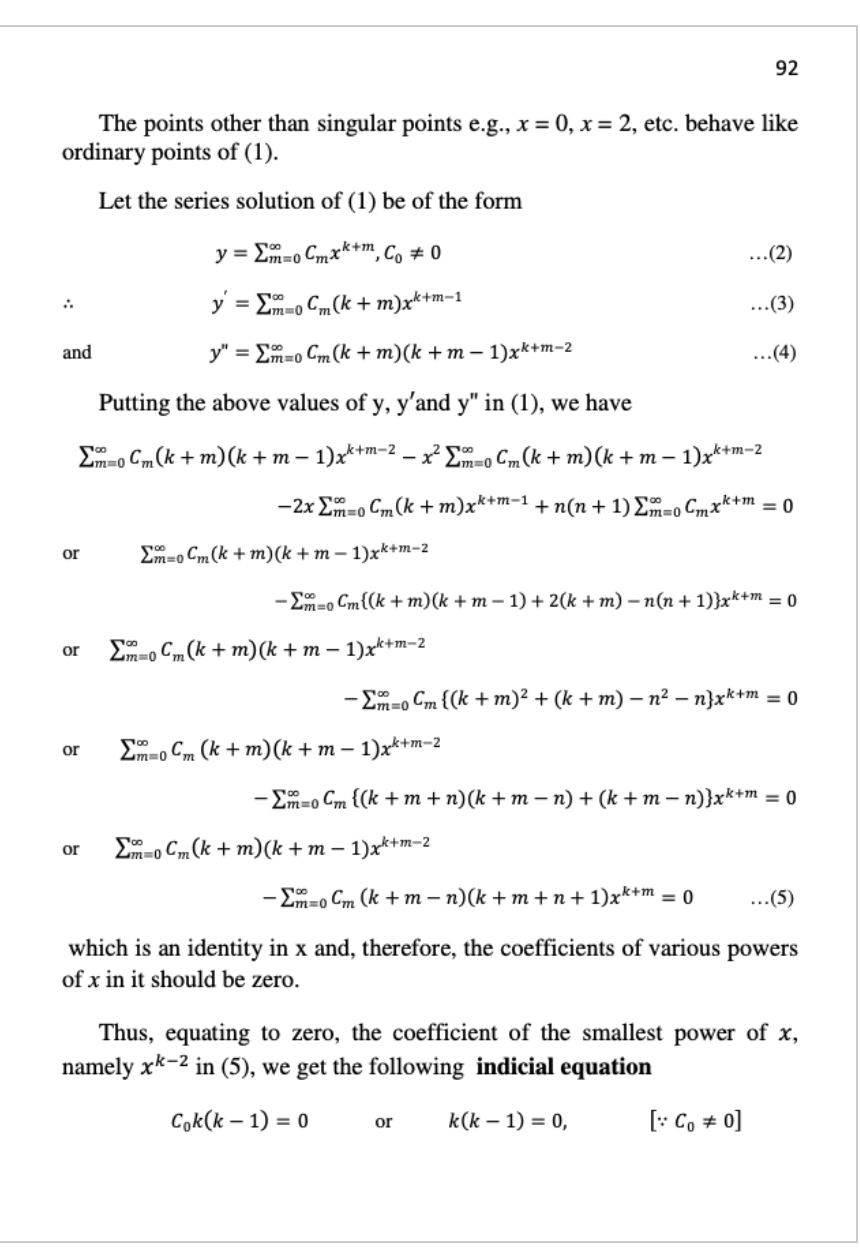
$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \qquad \ldots(1)$$

is called Legendre's differential equation (or simply Legendre's equation), where n is a non-negative integer.

This equation can also be put in the following form:

$$\frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\} + n(n+1)y = 0.$$

Clearly, the only singular points of (1) are $x = 1$, $x = -1$ and $x = \infty$, which are regular. Therefore, the Legendre's differential equation is a **Fuchsian differential equation**.



The points other than singular points e.g., $x = 0$, $x = 2$, etc. behave like ordinary points of (1).

Let the series solution of (1) be of the form

$$y = \sum_{m=0}^{\infty} C_m x^{k+m},\ C_0 \neq 0 \qquad \ldots(2)$$

$$\therefore \quad y' = \sum_{m=0}^{\infty} C_m (k+m) x^{k+m-1} \qquad \ldots(3)$$

and

$$y'' = \sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2} \qquad \ldots(4)$$

Putting the above values of y, y′and y″ in (1), we have

$$\sum_{m=0}^{\infty} C_m (k+m)(k+m-1)x^{k+m-2} - x^2 \sum_{m=0}^{\infty} C_m (k+m)(k+m-1)x^{k+m-2}$$
$$-2x\sum_{m=0}^{\infty} C_m (k+m)x^{k+m-1} + n(n+1)\sum_{m=0}^{\infty} C_m x^{k+m} = 0$$

or

$$\sum_{m=0}^{\infty} C_m (k+m)(k+m-1)x^{k+m-2}$$
$$-\sum_{m=0}^{\infty} C_m\{(k+m)(k+m-1) + 2(k+m) - n(n+1)\}x^{k+m} = 0$$

or

$$\sum_{m=0}^{\infty} C_m (k+m)(k+m-1)x^{k+m-2}$$
$$-\sum_{m=0}^{\infty} C_m\{(k+m)^2 + (k+m) - n^2 - n\}x^{k+m} = 0$$

or

$$\sum_{m=0}^{\infty} C_m (k+m)(k+m-1)x^{k+m-2}$$
$$-\sum_{m=0}^{\infty} C_m\{(k+m+n)(k+m-n) + (k+m-n)\}x^{k+m} = 0$$

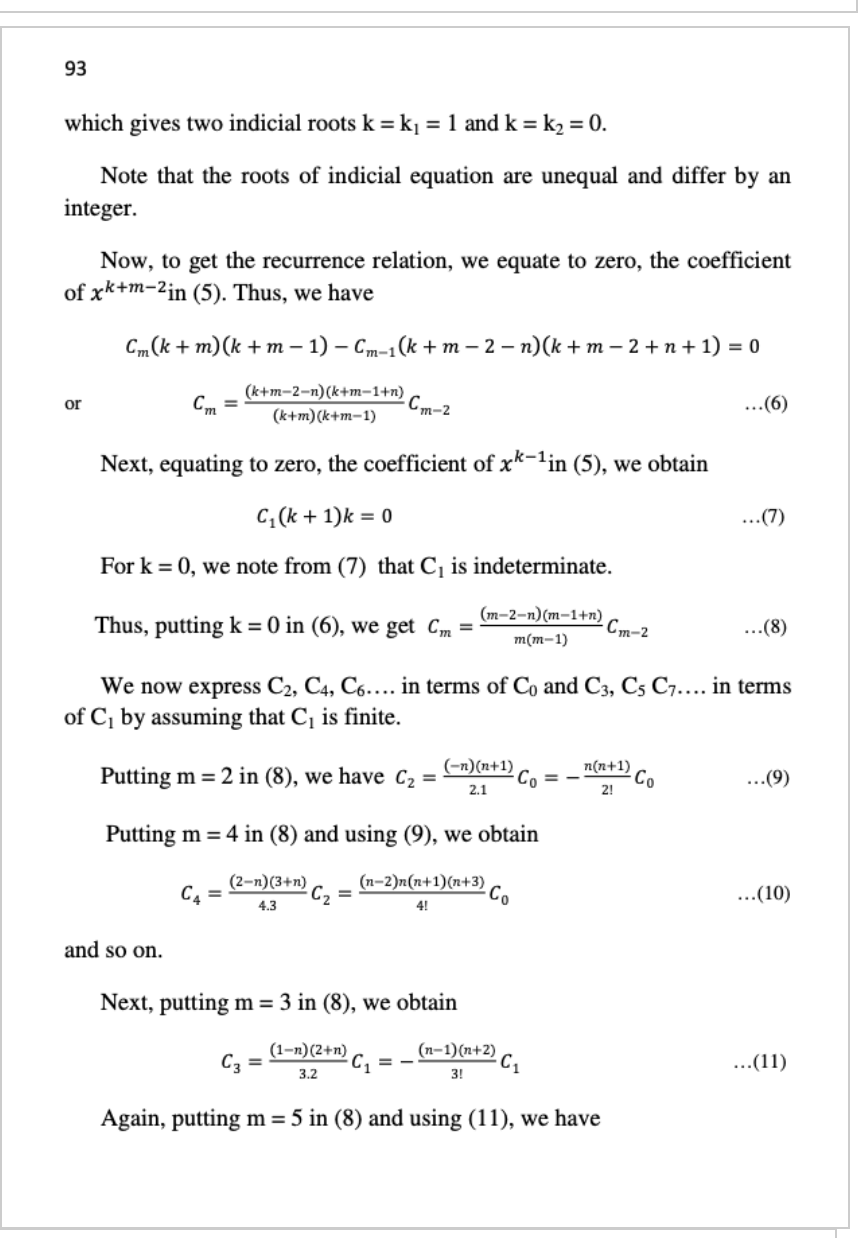
or

$$\sum_{m=0}^{\infty} C_m (k+m)(k+m-1)x^{k+m-2}$$
$$-\sum_{m=0}^{\infty} C_m (k+m-n)(k+m+n+1)x^{k+m} = 0 \qquad \ldots(5)$$

which is an identity in x and, therefore, the coefficients of various powers of *x* in it should be zero.

Thus, equating to zero, the coefficient of the smallest power of *x*, namely x^{k-2} in (5), we get the following **indicial equation**

$$C_0 k(k-1) = 0 \quad \text{or} \quad k(k-1) = 0, \qquad [\because C_0 \neq 0]$$



which gives two indicial roots $k = k_1 = 1$ and $k = k_2 = 0$.

Note that the roots of indicial equation are unequal and differ by an integer.

Now, to get the recurrence relation, we equate to zero, the coefficient of x^{k+m-2}in (5). Thus, we have

$$C_m(k+m)(k+m-1) - C_{m-1}(k+m-2-n)(k+m-2+n+1) = 0$$

or

$$C_m = \frac{(k+m-2-n)(k+m-1+n)}{(k+m)(k+m-1)} C_{m-2} \qquad \ldots(6)$$

Next, equating to zero, the coefficient of x^{k-1}in (5), we obtain

$$C_1(k+1)k = 0 \qquad \ldots(7)$$

For k = 0, we note from (7) that C_1 is indeterminate.

Thus, putting k = 0 in (6), we get $C_m = \frac{(m-2-n)(m-1+n)}{m(m-1)} C_{m-2}$ $\qquad \ldots(8)$

We now express C_2, C_4, C_6…. in terms of C_0 and C_3, C_5 C_7…. in terms of C_1 by assuming that C_1 is finite.

Putting m = 2 in (8), we have $C_2 = \frac{(-n)(n+1)}{2.1} C_0 = -\frac{n(n+1)}{2!} C_0$ $\qquad \ldots(9)$

Putting m = 4 in (8) and using (9), we obtain

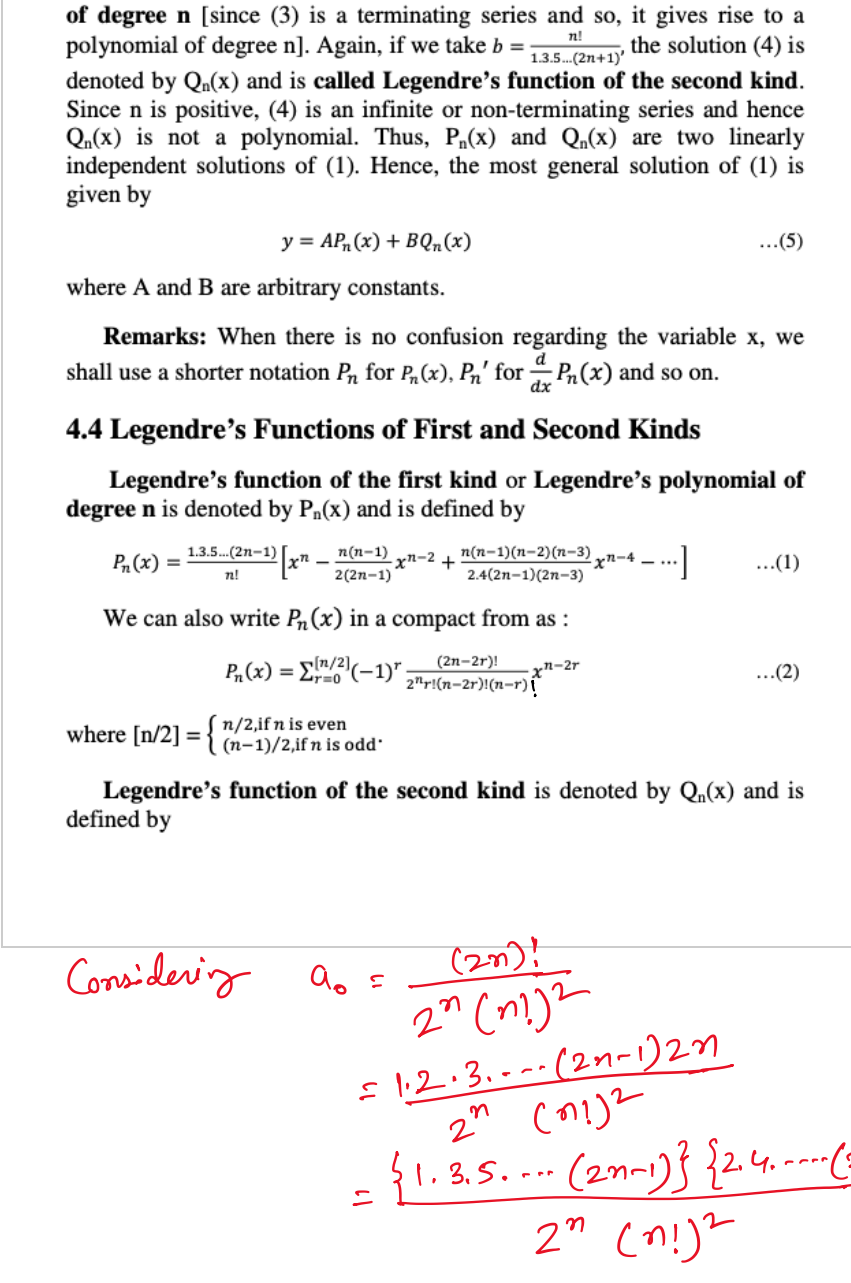
$$C_4 = \frac{(2-n)(3+n)}{4.3} C_2 = \frac{(n-2)n(n+1)(n+3)}{4!} C_0 \qquad \ldots(10)$$

and so on.

Next, putting m = 3 in (8), we obtain

$$C_3 = \frac{(1-n)(2+n)}{3.2} C_1 = -\frac{(n-1)(n+2)}{3!} C_1 \qquad \ldots(11)$$

Again, putting m = 5 in (8) and using (11), we have



$$C_5 = \frac{(3-n)(4+n)}{5.4} C_3 = \frac{(n-3)(n-1)(n+2)(n+4)}{5!} C_1 \qquad \ldots(12)$$

and so on.

Now, the solution (2) can be re-written as:

$$y = x^k(C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + C_5x^5 + \cdots),\ \text{where } k = 0$$

or

$$y = (C_0 + C_2x^2 + C_4x^4 + \cdots) + (C_1x + C_3x^3 + C_5x^5 + \cdots) \qquad \ldots(13)$$

Using the values of C_2, C_3, C_4, C_5,…… in the above equation, we get

$$y = C_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 - \cdots\right]$$
$$+C_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 + \cdots\right] \qquad \ldots(14)$$

which is the required general series solution, C_0 and C_1 being arbitrary constants.

4.3. Solution of Legendre's Differential Equation in Descending Powers

Consider Legendre's differential equation of the type

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \qquad \ldots(1)$$

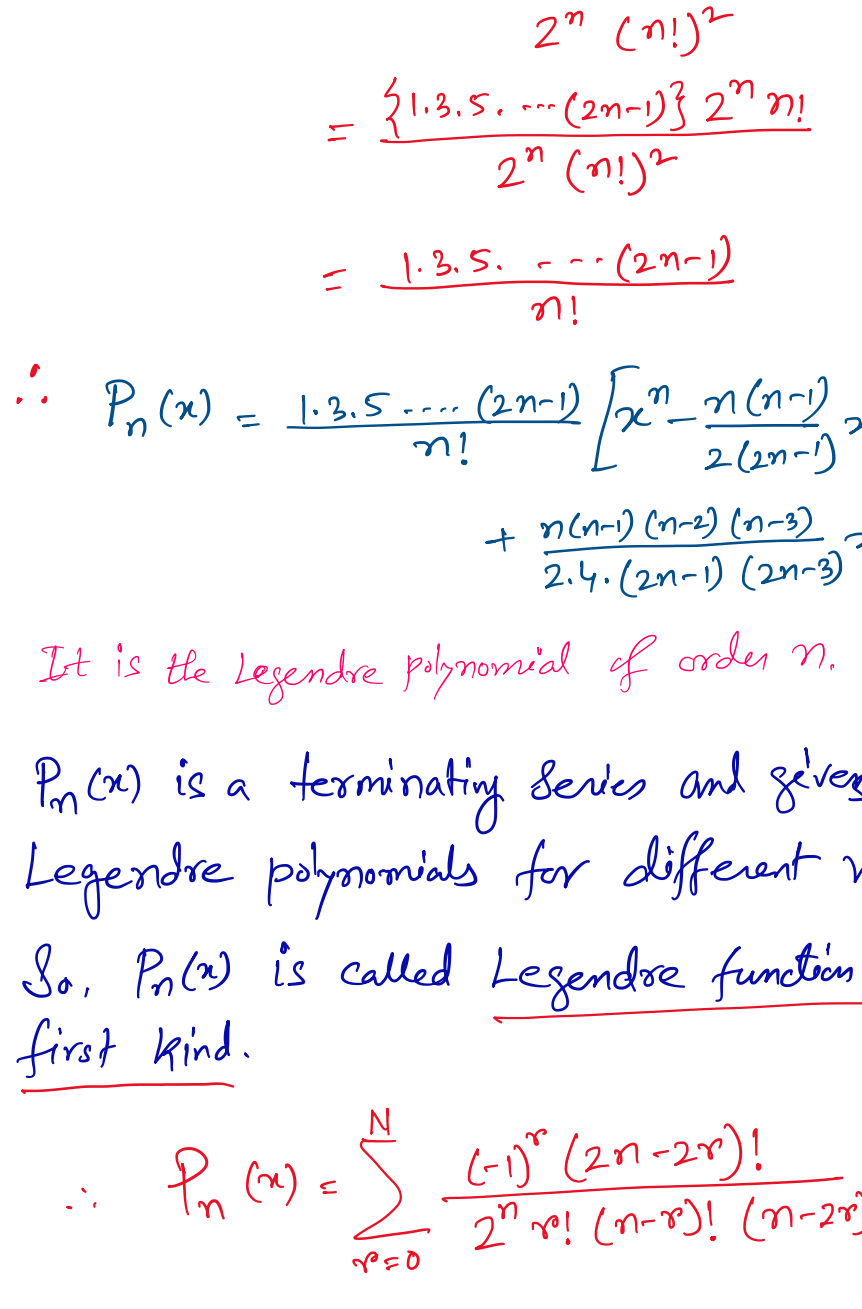
where n is a non-negative integer.

It is possible to obtain the solution of (1) in terms of descending powers of x. Due to its applications to physical problems, this form of solution of Legendre's differential equation is more important.

For such a solution, let us assume that the Legendre's differential equation (1) has a series solution of the form

$$y = \sum_{m=0}^{\infty} C_m x^{k-m},\ C_0 \neq 0 \qquad \ldots(2)$$

Then, by Frobenius method, we can find two linearly independent solutions of (1) in descending powers of x as:



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$$y_1 = a\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}x^{n-4} - \cdots\right] \qquad \ldots(3)$$

and

$$y_2 = b\left[x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)}x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2.4(2n+3)(2n+5)}x^{-n-5} + \cdots\right] \qquad \ldots(4)$$

We found in last lecture

If we take $a = \frac{1.3.5\ldots(2n-1)}{n!}$ the solution (3) is denoted by $P_n(x)$ and is called **Legendre's function of the first kind** or **Legendre's polynomial of degree n** [since (3) is a terminating series and so, it gives rise to a polynomial of degree n]. Again, if we take $b = \frac{n!}{1.3.5\ldots(2n+1)}$, the solution (4) is denoted by $Q_n(x)$ and is **called Legendre's function of the second kind**. Since n is positive, (4) is an infinite or non-terminating series and hence $Q_n(x)$ is not a polynomial. Thus, $P_n(x)$ and $Q_n(x)$ are two linearly independent solutions of (1). Hence, the most general solution of (1) is given by

$$y = AP_n(x) + BQ_n(x) \qquad \ldots(5)$$

where A and B are arbitrary constants.

Remarks: When there is no confusion regarding the variable x, we shall use a shorter notation P_n for $P_n(x)$, P_n' for $\frac{d}{dx}P_n(x)$ and so on.

4.4 Legendre's Functions of First and Second Kinds

Legendre's function of the first kind or **Legendre's polynomial of degree n** is denoted by $P_n(x)$ and is defined by

$$P_n(x) = \frac{1.3.5\ldots(2n-1)}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}x^{n-4} - \cdots\right] \qquad \ldots(1)$$

We can also write $P_n(x)$ in a compact from as :

$$P_n(x) = \sum_{r=0}^{[n/2]} (-1)^r \frac{(2n-2r)!}{2^n r!(n-2r)!(n-r)!} x^{n-2r} \qquad \ldots(2)$$

where $[n/2] = \begin{cases} n/2, \text{if n is even} \\ (n-1)/2, \text{if n is odd} \end{cases}$.

Legendre's function of the second kind is denoted by $Q_n(x)$ and is defined by

Considering $a_0 = \dfrac{(2n)!}{2^n (n!)^2}$

$$= \frac{1.2.3.\cdots(2n-1)2n}{2^n (n!)^2}$$

$$= \frac{\{1.3.5.\cdots(2n-1)\}\{2.4.\cdots(2n)\}}{2^n (n!)^2}$$

$$= \frac{\{1.3.5.\cdots(2n-1)\}2^n n!}{2^n (n!)^2}$$

$$= \frac{1.3.5.\cdots(2n-1)}{n!}$$

$$\therefore P_n(x) = \frac{1.3.5\cdots(2n-1)}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)}x^{n-4} - \cdots\right]$$

It is the Legendre polynomial of order n.

$P_n(x)$ is a terminating series and gives Legendre polynomials for different values of n. So, $P_n(x)$ is called Legendre function of the first kind.

$$\therefore P_n(x) = \sum_{r=0}^{N} \frac{(-1)^r (2n-2r)!}{2^n r!(n-r)!(n-2r)!} x^{n-2r}$$

Where $N = \frac{n}{2}$ if n is even

& $N = \frac{n-1}{2}$ if n is odd

Now Considering y_2 and fixed the value of a_0 as

$$a_0 = \frac{n!}{1.3.5,\cdots(2n-1)}$$

$$Q_n(x) = \frac{n!}{1.3.5.\cdots(2n-1)} x^{-(n+1)}\left[1 + \frac{(n+1)(n+2)}{1.2.(2n+3)} + \cdots\right]$$

$Q_n(x)$ is called Legendre function of the 2nd Kind.

$P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1.3}{\underline{|2}}\left(x^2 - \frac{2}{2.3}\right)$

$= \frac{1.3}{\underline{|2}}\left(\frac{3x^2-1}{3}\right)$

$P_2(x) = \frac{3x^2-1}{2}$

$P_2(1) = 1$

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$

$P_3(1) = 1$

$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

$P_4(1) = 1$

Q. Express $f(x) = x^4 + 3x^3 - x^2 + 5x - 2$ in terms of Legendre polynomials.

Solⁿ $P_0(x) = \frac{1}{0!}x^0 = 1$ $\qquad P_1(x) = \frac{1}{1!}x^1 = x$

$P_2(x) = \frac{1}{2}(3x^2 - 1)$

$P_3(x) = \frac{1.3.5}{3!}\left[x^3 - \frac{3.2}{2.5}x^1\right] = \frac{1}{2}(5x^3 - 3x)$

$P_4(x) = \frac{1.3.5.7}{4!}\left[x^4 - \frac{4.3}{2.7}x^2 + \frac{4.3.2.1}{2.4.7.5}x^0\right] = \frac{1}{8}(35x^4 - 30x^2 + 3)$

$\Rightarrow 8P_4(x) = 35x^4 - 30x^2 + 3$

$\Rightarrow 35x^4 = 8P_4(x) + 30x^2 - 3$

$\Rightarrow x^4 = \frac{8}{35}P_4(x) + \frac{6}{7}x^2 - \frac{3}{35}$

$\because x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}x$, $\quad x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}$

$f(x) = x^4 + 3x^3 - x^2 + 5x - 2$

$= \left[\frac{8}{35}P_4(x) + \frac{6}{7}x^2 - \frac{3}{35}\right] + 3\left[\frac{2}{5}P_3(x) + \frac{3}{5}x\right]$

$- \left[\frac{2}{3}P_2(x) + \frac{1}{3}\right] + 5x - 2$

$= \frac{8}{35}P_4(x) + \frac{6}{5}P_3(x) - \frac{2}{21}P_2(x) + \frac{34}{5}x - \frac{224}{105}$

$= \frac{8}{35}P_4(x) + \frac{6}{5}P_3(x) - \frac{2}{21}P_2(x) + \frac{34}{5}P_1(x) - \frac{224}{105}P_0(x)$

Ex Express $2 - 3x + 4x^2$ in terms of Legendre Polynomials

Here $2 - 3x + 4x^2$

$= 2P_0(x) - 3P_1(x) + \frac{4}{3}\left[2P_2(x) + 1\right]$

$= 2P_0(x) - 3P_1(x) + \frac{8}{3}P_2(x) + \frac{4}{3}P_0(x)$

$= \left(\frac{10}{3}\right)P_0(x) - 3P_1(x) + \frac{8}{3}P_2(x)$

Rodrigue's formula for Legendre Polynomial

$$P_n(x) = \frac{1}{n!\,2^n}\frac{d^n}{dx^n}(x^2-1)^n$$

Proof

Let $v = (x^2-1)^n$

$\frac{dv}{dx} = 2nx(x^2-1)^{n-1}$ —— ①

$\Rightarrow (x^2-1)\frac{dv}{dx} = 2nx(x^2-1)^n = 2nxv$

$\Rightarrow (1-x^2)\frac{dv}{dx} + 2nxv = 0$ —— ②

Differentiating w.r.t. x

$(1-x^2)\frac{d^2v}{dx^2} - 2x\frac{dv}{dx} + 2nx\frac{dv}{dx} + 2nv = 0$

$\Rightarrow (1-x^2)\frac{d^2v}{dx^2} + 2(n-1)x\frac{dv}{dx} + 2nv = 0$ —— ③

Diff n times w.r.t x using Leibnitz's theorem.

$\Rightarrow \left\{(1-x^2)\frac{d^{n+2}v}{dx^{n+2}} + {}^nC_1(-2x)\frac{d^{n+1}v}{dx^{n+1}} + \frac{n(n+1)(-2)}{2}\frac{d^nv}{dx^n}\right\}$

$+2(n-1)\left\{x\frac{d^{n+1}v}{dx^{n+1}} + n.1.\frac{d^nv}{dx^n}\right\} + 2n\frac{d^nv}{dx^n} = 0$

$\Rightarrow (1-x^2)\frac{d^{n+2}v}{dx^{n+2}} + \left\{-2nx + 2(n-1)x\right\}\frac{d^{n+1}v}{dx^{n+1}}$

$+\left\{-n(n+1) + 2n(n-1) + 2n\right\}\frac{d^nv}{dx^n} = 0$

$\Rightarrow (1-x^2)\frac{d^{n+2}v}{dx^{n+2}} - 2x\frac{d^{n+1}v}{dx^{n+1}} + n(n+1)\frac{d^nv}{dx^n} = 0$

Let $v_n = \frac{d^nv}{dx^n}$

then the above equation becomes

$(1-x^2)\frac{d^2v_n}{dx^2} - 2x\frac{dv_n}{dx} + n(n+1)v_n = 0$ —— ④

v_n satisfies the Legendre's equation.

Hence, the solution may be some multiple of v_n

i.e., $P_n(x) = c\,v_n$ —— ⑤

$\left[\frac{d^2y}{dx^2} - m^2y = 0,\quad y = e^{mx} \text{ is a sol}^n \quad \frac{d^2}{dx^2}\left(\frac{dy}{dx}\right) - m^2\left(\frac{dy}{dx}\right) = 0 \quad y = \frac{e^{mx}}{m},\ \frac{dy}{dx} = e^{mx}\right]$

⑤ $\Rightarrow P_n(x) = c\frac{d^n}{dx^n}(x^2-1)^n$

To find c, collect the coefficients of x^n from both the sides

$\frac{\underline{|2n}\,x^n}{2^n(\underline{|n})^2} = c.2n(2n-1)(2n-2)\cdots(n+1)x^n$

$= \frac{c.\{2n(2n-1)\cdots(n+1)\}\{n(n-1)\cdots2.1\}x^n}{\underline{|n}}$

$\Rightarrow \frac{\underline{|2n}}{2^n(\underline{|n})^2}x^n = \frac{c\,\underline{|2n}}{\underline{|n}}x^n$

$\Rightarrow \boxed{c = \frac{1}{2^n\underline{|n}}}$

Hence, the Rodrigue's formula is

$$\boxed{P_n(x) = \frac{1}{2^n\underline{|n}}\frac{d^n}{dx^n}(x^2-1)^n}$$

Orthogonality of Legendre Polynomials

We shall prove that

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = \begin{cases} 0, & m \neq n \\ \frac{2}{2n+1}, & m = n \end{cases}$$

$(1-x^2)u'' - 2xu' + m(m+1)u = 0 \times v$

$(1-x^2)v'' - 2xv' + n(n+1)v = 0 \times u$ & subtract

int -1 to 1, we get

$m \neq n$, $\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0$ (orthogonality of L.P.)

when m=n → Rodrigue's f

Generating function of $P_n(x)$

$P_n(x)$ is the coefficient of t^n in the expansion of $(1-2xt+t^2)^{-1/2}$ in ascending powers of t,

i.e., $$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(x), \quad |x| \leq 1,\ |t| \leq 1$$

Now, $(1-2xt+t^2)^{-1/2} = \left[1 - t(2x-t)\right]^{-1/2}$

if $|t(2x-t)| < 1$, then we can expand.

$(n+1)^{th}$ term is

$= \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\cdots\left(-\frac{1}{2}-n+1\right)(-1)^n t^n(2x-t)^n}{\underline{|n}}$

$= \frac{1.3.5.\cdots(2n-1)\,t^n(2x-t)^n}{2^n\underline{|n}}$

Now, $(1-2x+t^2)^{-1/2} = \left[1 - t(2x-t)\right]^{-1/2}$

$= 1 + t(2x-t) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{\underline{|2}}t^2(2x-t)^2 + \cdots$

$\cdots + \frac{1.3.5.\cdots(2n-5)}{2^{n-2}\underline{|n-2}}t^{n-2}(2x-t)^{n-2}$

$+ \frac{1.3.5.\cdots(2n-3)}{2^{n-1}\underline{|n-1}}t^{n-1}(2x-t)^{n-1}$

$+ \frac{1.3.5.\cdots(2n-1)\,t^n(2x-t)^n}{2^n\underline{|n}}$ —— ①

Now, $t^{n-2}(2x-t)^{n-2} = t^{n-2}\left[(2x)^{n-2} - (n-2)t(2x)^{n-3} + \frac{(n-2)(n-3)}{\underline{|2}}t^2(2x)^{n-4}\right]$

$t^{n-1}(2x-t)^{n-1} = t^{n-1}\left[(2x)^{n-1} - (n-1)t(2x)^{n-2} + \cdots\right]$

$\Rightarrow t^n(2x-t)^n = t^n\left[(2x)^n - n(t)(2x)^{n-1} + \cdots\right]$

Putting the values in ① and collecting the coefficients as follows.

Coefficient of $t = x = P_1(x)$

Coefficient of $t^2 = -\frac{1}{2} + \frac{3}{2}x^2 = \frac{1}{2}(3x^2-1) = P_2(x)$

⋮

Coefficient of $t^n =$

$\frac{1.3.5.\cdots(2n-1).2^nx^n}{2^n\underline{|n}} + \frac{1.3.5.\cdots(2n-3)\{-(n-1)\}}{2^{n-1}\underline{|n-1}}(2x)$

$+ \frac{1.3.5.\cdots(2n-5)}{2^{n-2}\underline{|n-2}}\frac{(n-2)(n-3)}{\underline{|2}}(2x)^{n-4} + \cdots$

$= \frac{1.3.5.\cdots(2n-1)}{\underline{|n}}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)}x^{n-4} + \cdots\right]$

$= P_n(x)$

Ex Express $x^3 + 2x^2 - x - 3$ in terms of Legendre polynomial.

Solⁿ $P_3(x) = \frac{1}{2}(5x^3 - 3x)$

$P_2(x) = \frac{1}{2}(3x^2 - 1)$

$P_1(x) = x$

$P_0(x) = 1$

$\Rightarrow 5x^3 - 3x = 2P_3(x)$

$\Rightarrow 5x^3 = 2P_3(x) + 3P_1(x)$

$\Rightarrow x^3 = \frac{2P_3(x) + 3P_1(x)}{5}$

$3x^2 - 1 = 2P_2(x)$

or $3x^2 = 2P_2(x) + P_0(x)$

$\therefore x^2 = \frac{2P_2(x) + P_0(x)}{3}$

$= \frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)$

$x^3 + 2x^2 - x - 3 = \frac{2P_3(x) + 3P_1(x)}{5} + \frac{4}{3}P_2(x) + \frac{2}{3}P_0(x)$

$- P_1(x) - 3P_0(x)$

$= \frac{2}{5}P_3(x) - \frac{2}{5}P_1(x) + \frac{4}{3}P_2(x) - \frac{7}{3}P_0(x)$

Generating Function for $J_n(x)$

$$e^{\frac{1}{2}x\left(t - \frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} t^n J_n(x)$$

Bessel function of various order can be derived as coefficients of different power of t

Recurrence Formula for $P_n(x)$

① $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$

② $nP_n(x) = xP_n'(x) - P_{n-1}'(x)$

③ $(2n+1)P_n(x) = P_{n+1}'(x) - P_{n-1}'(x)$

④ $P_n'(x) = xP_{n-1}'(x) + nP_{n-1}(x)$

⑤ $(1-x^2)P_n'(x) = n\left[P_{n-1}(x) - xP_n(x)\right]$

Linear Boundary value Problems:

Consider the second-order linear differential Equation

$P_0(x)y'' + P_1(x)y' + P_2(x)y = r(x)$ —— ①

in the interval $I = [\alpha, \beta]$, where we consider that the function $P_0(x)$, $P_1(x)$, $P_2(x)$ & $r(x)$ are continuous in I.

Together with the differential equation (1), we shall consider the boundary conditions of the form

$$\left.\begin{aligned} B_1[y] &= a_0y(\alpha) + a_1y'(\alpha) + b_0y(\beta) + b_1y'(\beta) = A \\ B_2[y] &= c_0y(\alpha) + c_2y'(\alpha) + d_0y(\beta) + d_1y'(\beta) = B \end{aligned}\right] \text{—②}$$

$\overset{\alpha}{\vdash}\!\!\!-\!\!\!\overset{\beta}{\dashv}$

where a_i, b_i, c_i, d_i, $i = 0, 1$ and A, B are given Constants

Sturm-Liouville boundary value Problem

Consider a D.E of the form

$$\frac{d}{dx}\left\{p(x)\frac{dy}{dx}\right\} + \left[q(x) + \lambda r(x)\right]y = 0, \quad A \leq x \leq B \quad \text{—①}$$

where p, q, r are real-valued Continuous functions on [A, B] and λ is a real parameter, here $p(x) > 0$, $r(x) > 0$.

Let us consider two sets of boundary Condition as.

$$\left.\begin{aligned} a_1y(A) + a_2y'(A) &= 0 \\ b_1y(B) + b_2y'(B) &= 0 \end{aligned}\right] \text{—② Separated bdⁿ Condⁿ}$$

where $a_1^2 + a_2^2 \neq 0$ & $b_1^2 + b_2^2 \neq 0$

and atleast one of a_1 & a_2 and at least one of b_1 & b_2 are non-zero

OR

$$\left.\begin{aligned} y(A) &= y(B) \\ y'(A) &= y'(B) \\ p(A) &= p(B) \end{aligned}\right] \text{—③ periodic bdⁿ Condⁿ.}$$

Defⁿ A BVP consisting of equations ① with ② or ① with ③ is called regular Sturm-Liouville BVP.

Defⁿ Suppose for a value of λ ∈ ℝ, y_λ is a non-trivial solution of ① with ② or ① with ③. Then λ is called an "eigenvalue" and y_λ is Called an "eigenfunction" of the S-L Problem.

The set of all eigenvalues of a regular problem is called its spectrum.

Theorem:

All eigenvalue of Strum Liouville problem are real.

Ex Find the eigenvalues and eigenfunctions of the BVP

$y'' + \lambda y = 0, \quad 0 \leq x \leq \pi$ —— ④

with $y(0) = 0$

& $y(\pi) = 0$

This is a self-adjoint form bdⁿ Condⁿ

Separated bdⁿ Condⁿ

Solⁿ

Case I: When $\lambda = 0$ then $y(x) = 0$

$y'' = 0$, $y(x) = Ax + B$, $\quad y(0) = 0$, $B = 0$; $y(\pi) = 0$ $A = 0$

$\therefore y(x) = 0$ is the Solution,

so we doesn't have any non-trival solⁿ in this case.

Hence, $\lambda = 0$ is not an eigenvalue of ④.

Case II

When $\lambda < 0$, then we can write $\lambda = -\mu^2$ for $\mu \in \mathbb{R}$.

Here $y'' - \mu^2y = 0 \Rightarrow y(x) = Ae^{\mu x} + Be^{-\mu x}$

$y(0) = 0$

$y(\pi) = 0 \Rightarrow A + B = 0$

$Ae^{\pi x} + Be^{-\pi t} = 0$

$\begin{bmatrix} 1 & 1 \\ e^{\pi x} & e^{-\pi x} \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Phi \neq 0 \rightarrow$ so, it is invertible

Solving ④, we get $y(x) = 0$ is the Solution, so we doesn't have any non-trivial Solution in this case.

Hence $\lambda < 0$ is not an eigenvalue of ④.

Case III

When $\lambda > 0$, then we can write $\lambda = \mu^2$ for some $\mu \in \mathbb{R}$.

Then, general solution of ④ can be written as

$y(x) = C_0\cos\mu x + C_1\sin\mu x$

$y'' + \mu^2y = 0 \Rightarrow$

$y(0) = 0 = y(\pi) \Rightarrow 0 = C_0$

$\Rightarrow 0 = C_1\sin\mu\pi$

$\mu\pi = n\pi$, $n = 1, 2, \cdots$

Now $y(0) = 0 \Rightarrow C_0 = 0$ & $y(\pi) = 0$ implies that $C_1\sin\mu\pi = 0$. For non-trivial solⁿ of ④, we must have $\mu\pi = n\pi$, where $n \in \mathbb{Z}$ i.e. $\mu_n = n$ for each $n \in \mathbb{Z}$

and in this Case, $\lambda_n = n^2$ is the eigenvalue and $\phi_n(x) = \sin nx$ is the corresponding eigenfunction for each $n \in \mathbb{Z}$.

Theorem

Assume that

ⓐ A, B are finite real numbers;

ⓑ The functions p(x), q(x) & r(x) are real valued Continuous function on [A, B]; and

ⓒ a_1, a_2, b_1, b_2 are real numbers.

Then, the Sturm-Liouville problem ① with ② or ① with ③ has Countably many eigenvalues with no finite limit point.

Corresponding to each eigenvalue, there exists an eigenfunction.

Orthogonal:

Two distinct functions x and y, defined and continuous on [A, B] are said to be orthogonal w.r.t. a Continuous Weight function r(x) if

$$\int_A^B r(s)\,x(s)\,y(s)\,ds = 0 \quad \text{——⑤}$$

Theorem:

Let all the assumptions ⓐ and ⓑ of above theorem hold.

For the parameters, λ, μ (λ ≠ μ),

let x and y be the corresponding solutions of ① such that $\left[pW(x,y)\right]_A^B = 0$, where W(x,y) is the Wronskian of x and y. Then

$$\int_A^B r(s)\,x(s)\,y(s)\,ds = 0$$

Ex Find the eigenvalues and the Corresponding eigenfunctions of $y'' + \lambda y = 0$ with $y(0) = 0$, $y'(L) = 0$

Solⁿ Given $y'' + \lambda y = 0$ —— ①

bdⁿ Condⁿ $y(0) = 0$ & $y'(L) = 0$ —— ②

Case I: Let $\lambda = 0$ then solⁿ of ① is

$y(x) = Ax + B$ —— ③

from ③ $\quad y'(x) = A$ —— ④

$y(0) = 0 \Rightarrow B = 0$.

$y'(L) = 0 \Rightarrow A = 0$ $\quad \therefore y(x) = 0$, Since $y(x) \neq 0$.

So there is no eigen function corresponding to $\lambda = 0$

Case II: Let $\lambda = -\mu^2$ where $\mu \neq 0$ (i.e λ is -ve)

$\therefore y'' - \mu^2y = 0$

$\Rightarrow y(x) = Ae^{\mu x} + Be^{-\mu x}$ —— ⑧

Now $y'(x) = A\mu e^{\mu x} - B\mu e^{-\mu x}$

$\therefore A + B = 0$ & $A\mu e^{\mu L} - B\mu e^{-\mu L} = 0$

Solving above eq, we find $A = B = 0$.

Since $y(x) \neq 0$. So, there is no eigen function Corresponding to $\lambda = -\mu^2$

Case III: Let $\lambda = \mu^2$ where $\mu \neq 0$

then $y'' + \mu^2y = 0$

$\Rightarrow y(x) = A\cos\mu x + B\sin\mu x$

$\Rightarrow y'(x) = -A\mu\sin\mu x + B\mu\cos\mu x$

Using bdⁿ Condⁿ $\quad 0 = A$ & $0 = B\mu\cos\mu L$

$\Rightarrow A = 0$ & $B\cos\mu L = 0$ as $\mu \neq 0$

$B\cos\mu L = 0$

If $B = 0$, then with $A = 0$ reduces to $y(x) = 0$, which is not an eigen function.

So, $B \neq 0$ for the existence of eigen functions.

∴ $\cos\mu L = 0 \Rightarrow \mu L = (2n-1)\frac{\pi}{2}$ $\quad n = 1, 2, 3, \cdots$

$\mu = (2n-1)\frac{\pi}{2L}$

using $A = 0$ and above relation reduces to

$y(x) = B\sin\left\{(2n-1)\frac{\pi x}{2L}\right\}$ $\quad n = 1, 2, 3, \cdots$

and then $\lambda = \mu^2 = (2n-1)^2\frac{\pi^2}{4L^2}$, $\quad n = 1, 2, 3, \cdots$

So, required eigenfunctions $y_n(x)$ with Corresponding eigenvalue λ_n are $y_n(x) = B_n\sin\left\{(2n-1)\frac{\pi x}{2L}\right\}$

& $\lambda_n = \frac{(2n-1)^2\pi^2}{4L^2}$, $\quad n = 1, 2, 3, \cdots$ $\quad$ [$B_n = 1$ may take]

Ex Verify that the system $y'' + \lambda y = 0$, $y(0) = 0$, $y'(l) = 0$ is a Strum-Liouville system. Find the eigen values and eigen functions of the system.

Prove that the eigen functions are orthogonal and determine the Corresponding orthogonal function.

Solⁿ Given $y'' + \lambda y = 0$, & $y(0) = 0$ $y'(l) = 0$ —— ⓐ

S-L System is

$\frac{d}{dx}\left\{p(x)\frac{dy}{dx}\right\} + \left[q(x) + \lambda r(x)\right]y = 0$ —— ①

Bdˢ $a_1y(a) + a_2y'(a) = 0$ —— ②

$b_1y(b) + b_2y'(b) = 0$ —— ③

Comparing with ⓐ and (1-3)

$p = 1$, $q = 0$, $r = 1$ $\qquad a = 0$, $a_1 = 1$, $a_2 = 0$

$b = l$, $b_1 = 0$, $b_2 = 1$

∴ The given System is a Strum-Liouville system as the values and Conditions are satisfied.

Solⁿ of ⓐ $\quad y = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$ —— ④

$y(0) = 0 \Rightarrow A = 0$ $\qquad y(x) = B\sin\sqrt{\lambda}x$

$y' = B\sqrt{\lambda}\cos\sqrt{\lambda}l$

$\Rightarrow \lambda = (2n-1)^2\frac{\pi^2}{4l^2}$

$\therefore y_n = B_n\sin\frac{(2n-1)}{2l}\pi x$ —— E. F.

Now $\int_0^l B_m\sin\frac{(2m-1)\pi x}{2l}\,B_n\sin\frac{(2n-1)}{2l}\pi x\,dx$

$\times\frac{2}{2}$

$= B_mB_n\int_0^l 2\sin\frac{(2m-1)}{2l}\pi x\,\sin\frac{(2n-1)}{2l}\pi x\,dx$

$= 0$