Solution to Question 1 (T2-DE)

Q.1(a) $2x^2 y''(x) + x y'(x) - (x+1) y(x) = 0$

Sol^n: First writing the equation in standard form $\left(y'' + p_1(x) y' + p_0(x) y = 0\right)$, we have

$$y''(x) + \frac{1}{2x} y'(x) - \left(\frac{x+1}{2x^2}\right) y(x) = 0. \quad ; (\text{for } x \neq 0)$$

Here, $p_1(x) = \dfrac{1}{2x}$, $p_0(x) = \dfrac{-(x+1)}{2x^2}$.

Clearly, $x=0$ is a singular point of the D.E. Now,

$$\lim_{x\to 0} x \cdot p_1(x) = \lim_{x\to 0} x \cdot \frac{1}{2x} = \frac{1}{2}$$

$$\lim_{x\to 0} x^2 \cdot p_0(x) = \lim_{x\to 0} x^2 \left[-\frac{(x+1)}{2x^2}\right] = -\frac{1}{2}$$

Both the limits exists and are finite.

Thus, $x=0$ is a regular singular point.

Also notice that both p_0 and p_1 are real analytic everywhere except at 0.

(no other singularities and differentiable infinitely many times, and the Taylor's series around any point converges to the given function)

Hence, we can use the Frobenius method to get a series solution to the DE around $x=0$.

Assume that $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ is the solution for $|x| > 0$, with $a_0 \neq 0$.

where $r \in \mathbb{R}$ and $a_n \in \mathbb{R}$ are to be determined.

Now, we have

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

Here, we assume that the interchange of derivative and sum is allowed and hence term-by-term differentiation

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n x^{n+r-2}$$ ✓ is possible.

Putting in the expressions of y, y', y'' in the given DE, we have

$$2x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n x^{n+r-2} + x \cdot \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} - (x+1) \sum_{n=0}^{\infty} a_n x^{n+r} = 0.$$

$$\therefore \sum_{n=0}^{\infty} 2(n+r)(n+r-1)\, a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r+1} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

Collecting the terms of x^{n+r} together, assuming at each step that such rearrangements and grouping of terms in the series is allowed.

$$\therefore \sum_{n=0}^{\infty} \left[2(n+r)(n+r-1) + (n+r) - 1 \right] a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

$$\therefore \sum_{n=0}^{\infty} (2n+2r+1)(n+r-1)\, a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

shifting the index by 1 to get x^{n+r} instead of x^{n+r+1}.

$$\sum_{n=1}^{\infty} a_{n-1} x^{n+r} = 0$$

$$\therefore (2r+1)(r-1)\, a_0 x^r + \sum_{n=1}^{\infty} (2n+2r+1)(n+r-1)\, a_n x^{n+r} - \sum_{n=1}^{\infty} a_{n-1} x^{n+r} = 0$$

$$\therefore (2r+1)(r-1)\, a_0 x^r + \sum_{n=1}^{\infty} \left[(2n+2r+1)(n+r-1)\, a_n - a_{n-1} \right] x^{n+r} = 0 \quad (1)$$

Notice that eq (1) is true for every $x \in \mathbb{R}$ with $|x| > 0$. Hence, the only way this is true is that the coefficients of powers of x are all zero. This follows from the Linear Algebraic fact that $\{x^{n+r} \mid n \in \mathbb{N}\}$ is a linearly independent set in the space of all infinitely times differentiable functions on $\mathbb{R}$.

That gives us,

$(2r+1)(r-1)\, a_0 = 0$ → this is the indicial equation.

Since $a_0 \neq 0$, either $r = -\frac{1}{2}$ or $r = 1$

Let $r_1 = -\frac{1}{2}$, $r_2 = 1$. Then the roots of the indicial equation are distinct and their difference is NOT an integer $\left(r_1 - r_2 = -\frac{1}{2} - 1 = -\frac{3}{2} \notin \mathbb{Z} \right)$.

Thus, the two linearly independent solutions to the given DE are obtained by substituting $r = r_1$ and $r = r_2$ in the series of y.

Now, the other coefficients are given by the recurrence relation

$$(2n+2r+1)(n+r-1)\,a_n - a_{n-1} = 0, \quad \text{for } \in \mathbb{N}.$$

or, what is the same,

$$a_n = \frac{a_{n-1}}{(2n+2r+1)(n+r-1)}.$$

Notice that since either $r = \frac{-1}{2}$ or $r = 1$, we have $\left(2n+2r+1 = 2n \text{ and } n+r-1 = n-\frac{1}{2}\right)$ or $\left(\begin{matrix}2n+2r+1 = 2n+3 \\ \text{and } n+r-1 = n\end{matrix}\right)$ all of which are non-zero. Hence, such a division is allowed!

From this recurrence relation, we see that $a_0 \in \mathbb{R}$ is a free quantity. We ignore it for the time being, since later in the general solution, the arbitrary constants would absorb a_0 into them!

Particularly, we have

$$a_1 = \frac{a_0}{r(2r+3)}, \quad a_2 = \frac{a_1}{(r+1)(2r+5)} = \frac{a_0}{r(r+1)(2r+3)(2r+5)}$$

$$a_3 = \frac{a_2}{(r+2)(2r+7)} = \frac{a_0}{r(r+1)(r+2)(2r+3)(2r+5)(2r+7)} \text{ and so on}$$

In general, we have

$$\boxed{a_n = \frac{a_0}{r(r+1)\cdots(r+n-1)\,(2r+3)\,(2r+5)\cdots\cdots(2r+2n+1)}} \quad - \text{②}$$

For $r = \frac{-1}{2}$, we have

$$a_n = \frac{a_0}{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}+1\right)\cdots\cdots\left(-\frac{1}{2}+n-1\right)(-1+3)(-1+5)\cdots\cdots(-1+2n+1)}$$

$$= \frac{a_0}{\underbrace{\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\cdots\cdots\left(\frac{2n-3}{2}\right)}_{\text{there are } n\text{- terms with } \frac{1}{2} \text{ as a factor}} \cdot \underbrace{2 \cdot 4 \cdot 6 \cdots\cdots (2n)}_{``2^n \cdot n!"}}$$

$$= \frac{-a_0}{1.1.3.5.\cdots(2n-3).n!}$$

Thus, one of the L.I. solution to the DE is

$$y_{-1/2}(x) = \sum_{n=0}^{\infty} a_n x^{n-1/2}$$

$$\boxed{y_{-1/2}(x) = \frac{a_0}{x^{1/2}}\left[1 - \sum_{n=1}^{\infty} \frac{1}{1.3.5.\cdots(2n-3).n!} \cdot x^n\right]} \quad -(3)$$

On the other hand, if $r=1$, we have

$$a_n = \frac{a_0}{1.2.\cdots.n.5.7.\cdots.(2n+3)} = \frac{a_0}{n!.5.7.\cdots(2n+3)}$$

$\therefore$ Another L.I. solution of the DE is

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$

$$\therefore \boxed{y_1(x) = a_0 x\left[1 + \sum_{n=1}^{\infty} \frac{1}{n!\,5.7.\cdots(2n+3)} x^n\right]} \quad -(4)$$

Now, the general solution is given by

$y(x) = A \cdot y_{-1/2}(x) + B \cdot y_1(x)$, for $A, B \in \mathbb{R}$

$$\therefore \boxed{y(x) = \frac{A}{\sqrt{x}}\left[1 - \sum_{n=1}^{\infty} \frac{x^n}{n!.3.5.\cdots(2n-3)}\right] + B.x.\left[1 + \sum_{n=1}^{\infty} \frac{x^n}{n!.5.7\cdots(2n+3)}\right]} \quad (5)$$

for, $A, B \in \mathbb{R}$.

Q.1 (b) $2x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + (1-x^2)y = 0$

Sol^n: Writing the given DE in the standard form, we have

$$y'' - \frac{1}{2x}y' + \left(\frac{1-x^2}{2x^2}\right)y = 0 \quad , \text{ for } x \neq 0.$$

Here, $p_0(x) = \frac{1-x^2}{2x^2}$, $p_1(x) = \frac{-1}{2x}$.

Clearly, $x=0$ is a singular point of the D.E.

Now,

$$\lim_{x\to 0} x \cdot p_1(x) = \lim_{x\to 0} x \cdot \left(\frac{-1}{2x}\right) = \frac{-1}{2}$$

$$\lim_{x\to 0} x^2 \cdot p_0(x) = \lim_{x\to 0} x^2 \left(\frac{1-x^2}{2x^2}\right) = \frac{1}{2}$$

Both the limits exist and are finite.

Thus, $x=0$ is a regular singular point.

Also, p_0, p_1 are real analytic everywhere except at $x=0$.

Thus, we use the Frobenius method to get a series solution to the given D.E.

Let the series solution be given by

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \quad , \text{ for } |x|>0 \text{ and } a_0 \neq 0, \text{ where } r \in \mathbb{R} \text{ is to be determined.}$$

Now, differentiating the series term-by-term, we get

$$y'(x) = \sum_{n=0}^{\infty} (n+r)\, a_n\, x^{n+r-1}.$$

$$y''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n\, x^{n+r-2}.$$

Since $y(x)$ must satisfy the given D.E., we have

$$2x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n\, x^{n+r-2} - x \cdot \sum_{n=0}^{\infty} (n+r)\, a_n\, x^{n+r-1}$$

$$+ (1-x^2) \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\therefore \sum_{n=0}^{\infty} 2(n+r)(n+r-1)\, a_n x^{n+r} - \sum_{n=0}^{\infty}(n+r)\, a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$- \sum_{n=0}^{\infty} a_n x^{n+r+2} = 0$$

$$\therefore \underbrace{\sum_{n=0}^{\infty}\Big[2(n+r)(n+r-1) - (n+r) + 1\Big] a_n x^{n+r}}_{\text{combining equal powers of } x \text{ under the assumption that such rearrangement of series is allowed!}} - \underbrace{\sum_{n=2}^{\infty} a_{n-2}\, x^{n+r}}_{\text{shifting the index of sum to get } x^{n+r} \text{ instead of } x^{n+r+2}} = 0.$$

$$\therefore \sum_{n=0}^{\infty}\Big[(2n+2r-1)(n+r-1)\Big] a_n x^{n+r} - \sum_{n=2}^{\infty} a_{n-2}\, x^{n+r} = 0$$

$$\therefore (2r-1)(r-1)\, a_0 x^{r} + (2r+1)\cdot r\cdot a_1 x^{r+1} + \sum_{n=2}^{\infty}\Big[(2n+2r-1)(n+r-1)\, a_n - a_{n-2}\Big] x^{n+r} = 0 \quad \text{— (1)}$$

Since equation (1) is true for every $x \neq 0$, we must have all the coefficient zero!

That is,

$(2r-1)(r-1)\, a_0 = 0$

But $a_0 \neq 0$ so that

$(2r-1)(r-1) = 0$ — (this is the indicial equation)

$\therefore \boxed{r = \frac{1}{2} \text{ or } r = 1.}$ — (*)

Thus, we have that the roots of the indicial equation are distinct and the difference of the roots is NOT an integer

$\left(\because r_1 - r_2 = \frac{1}{2} - 1 = \frac{-1}{2} \notin \mathbb{Z}\right)$

Thus, the two linearly independent solutions of the DE are obtained by substituting $r = r_1 \left(= \frac{1}{2}\right)$ and $r = r_2 (=1)$ in the series representation of $y(x)$.

Also, from eq. 1, we obtain

$$(2r+1)\cdot r\cdot a_1 = 0$$

However $r=\frac{1}{2}$ or $r=1$ so that $2r+1 \geq 2 > 0$, and $r \geq \frac{1}{2} > 0$

so that $(2r+1)\cdot r \neq 0$.

Hence $\boxed{a_1 = 0}$.

On the other hand, we have a recurrence relation

$$(2n+2r-1)(n+r-1)\,a_n - a_{n-2} = 0, \quad \forall n \geq 2$$

or, what is the same

$$\boxed{a_n = \frac{a_{n-2}}{(2r+2n-1)(r+n-1)}} \quad \text{— } 2$$

Notice that since $r \geq \frac{1}{2}$ (since $r=\frac{1}{2}$ or $r=1$), we have

$2r+2n-1 \geq 2n \geq 4 > 0$, and $r+n-1 \geq n-\frac{1}{2} \geq 2-\frac{1}{2} > 0$.

Thus $(2r+2n-1)\cdot(n+r-1) \neq 0$ and the division is valid!

From eq. 2, we have

$a_n = 0$, for $n \in \mathbb{N}$ with n odd.

And.

$$a_2 = \frac{a_0}{(2r+3)(r+1)}, \quad a_4 = \frac{a_2}{(2r+7)(r+3)} = \frac{a_0}{(2r+3)(2r+7)(r+1)(r+3)}.$$

$$a_6 = \frac{a_4}{(2r+11)(r+5)} = \frac{a_0}{(2r+3)(2r+7)(2r+11)(r+1)(r+3)(r+5)}, \text{ and so on.}$$

In general, we have

$$\boxed{a_n = \begin{cases} 0 & ; \ n = 2m-1, \text{ for some } m \in \mathbb{N} \\[2ex] \dfrac{a_0}{(2r+3)(2r+7)\cdots(2r+4m-1)(r+1)(r+3)\cdots(r+2m-1)} & ; \ n = 2m \text{ for some } m \in \mathbb{N}. \end{cases}} \quad \text{— } 3$$

Thus, the series of y becomes

$$\boxed{y(x) = a_0 x^r \left[1 + \sum_{m=1}^{\infty} \frac{x^{2m}}{(2r+3)(2r+7)\cdots(2r+4m-1)(r+1)(r+3)\cdots(r+2m-1)}\right]} \quad (4)$$

Hence, the two linearly independent solutions to the D.E. are

$$y_{1/2}(x) = a_0 x^{1/2}\left[1 + \sum_{m=1}^{\infty} \frac{x^{2m}}{(4)\cdot(8)\cdots(4m)\cdot\left(\frac{3}{2}\right)\left(\frac{7}{2}\right)\cdots\left(\frac{4m-1}{2}\right)}\right]$$

$$= a_0 x^{1/2}\left[1 + \sum_{m=1}^{\infty} \frac{x^{2m}}{4^m \cdot m! \cdot \frac{3\cdot 7 \cdots (4m-1)}{2^m}}\right]$$

$$\therefore \boxed{y_{1/2}(x) = a_0 x^{1/2}\left[1 + \sum_{m=1}^{\infty} \frac{x^{2m}}{2^m \cdot m! \cdot 3\cdot 7 \cdots (4m-1)}\right]} \quad (5)$$

And,

$$y_1(x) = a_0 x \left[1 + \sum_{m=1}^{\infty} \frac{x^{2m}}{5\cdot 7 \cdots (4m+1)(2)\cdot(4)\cdots(2m)}\right]$$

$$\therefore \boxed{y_1(x) = a_0 \cdot x \cdot \left[1 + \sum_{m=1}^{\infty} \frac{x^{2m}}{2^m \cdot m! \cdot 5\cdot 7 \cdots (4m+1)}\right]} \quad (6)$$

Hence, the general solution to the D.E is:

$$y(x) = A \cdot y_{1/2}(x) + B \cdot y_1(x), \quad \text{for } A, B \in \mathbb{R}$$

$$\therefore \boxed{y(x) = A \cdot \sqrt{x} \cdot \left[1 + \sum_{m=1}^{\infty} \frac{x^{2m}}{2^m \cdot m! \cdot 3\cdot 5 \cdots (4m-1)}\right] + B \cdot x \cdot \left[1 + \sum_{m=1}^{\infty} \frac{x^{2m}}{2^m \cdot m! \cdot 5\cdot 7 \cdots (4m+1)}\right]} \quad (*)$$

Q.1(c) $x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + (1+x^2)y = 0$

Solⁿ : Writing the given D.E in the standard form, we have

$$y'' - \frac{1}{x}y' + \frac{(1+x^2)}{x^2}y = 0 \quad , \text{ for } x \neq 0.$$

Clearly, $x=0$ is a singular point of the D.E.

Here, $p_0(x) = \frac{(1+x^2)}{x^2}$, $p_1(x) = -\frac{1}{x}$. Then.

$$\lim_{x\to 0} x \cdot p_1(x) = \lim_{x\to 0} x\left(\frac{-1}{x}\right) = -1$$

$$\lim_{x\to 0} x^2 \cdot p_0(x) = \lim_{x\to 0} x^2\left(\frac{(1+x^2)}{x^2}\right) = 1$$

Both the limits exists and are finite.

Thus, $x=0$ is a regular singular point of the D.E.

We can apply the Frobenius method for obtaining a series solution

Notice that p_0, p_1 are real analytic everywhere except at $x=0$.

Thus, we start with the series

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \quad , \text{ for } |x|>0 \text{ and } a_0 \neq 0 \text{, where } r \in \mathbb{R} \text{ is to be determined.}$$

Differentiating the series term-by-term, we obtain

$$y'(x) = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}.$$

Now, since y is a solution to the given D.E., we have

$$x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} - x \cdot \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + (1+x^2)\sum_{n=0}^{\infty} a_n x^{n+r} = 0.$$

$$\therefore \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} - \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} = 0.$$

$$\therefore \sum_{n=0}^{\infty} \left[(n+r)(n+r-1) - (n+r) + 1\right] a_n x^{n+r} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0$$

$$\therefore \sum_{n=0}^{\infty} (n+r-1)^2 a_n x^{n+r} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0$$

$$\therefore (r-1)^2 a_0 x^r + r^2 a_1 x^{r+1} + \sum_{n=2}^{\infty} \left[(n+r-1)^2 a_n + a_{n-2}\right] x^{n+r} = 0 \quad \text{— (1)}$$

Since eq. (1) is true for all $x \in \mathbb{R}$ with $x \neq 0$, we must have all the coefficients of powers of x to be zero!

Thus, we first have

$$(r-1)^2 a_0 = 0.$$

Since $a_0 \neq 0$, we have

$$(r-1)^2 = 0$$ — (This is the indicial equation)

$$\therefore \boxed{r = 1} \quad \text{— } (*)$$

That is, the roots of the indicial equation are repeated.

One of the linearly independent solutions is then given by substituting r in the series representation of y.

The other linearly independent solution is obtained by differentiating the series of y w.r.t r and then substituting $r = 1$.

That is, the two linearly independent solutions of DE are

$$\boxed{y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}} \quad \text{and} \quad \boxed{y_2(x) = \left.\frac{dy_r}{dr}\right|_{r=1} (x)}$$

To compute the solutions, we must first find the coefficients.

From eq (1), we also have

$$r^2 a_1 = 0$$

But $r = 1 \Rightarrow r^2 = 1 \neq 0$ so that $\boxed{a_1 = 0}$.

Moreover, we get the recurrence relation,

$$(n+r-1)^2 a_n + a_{n-2} = 0, \quad \text{for } n \geq 2$$

or what is the same

$$\boxed{a_n = \frac{-a_{n-2}}{(n+r-1)^2}} \quad \text{— (2)}$$

Notice that since $r=1$, $(n+r-1)^2 = n^2 \geq 4 > 0$ so that the division is allowed. The only time such division is not allowed is when $r = 1-n$ for some $n \in \mathbb{N}$. However $1-n \leq 0$, $\forall n \in \mathbb{N}$, and since taking derivatives is a local problem, $\left.\frac{dy_r}{dr}\right|_{r=1}$ exists after this valid division for $r>0$!

Particularly, from eq ②, we have

$$a_n = 0 \quad \text{, for all odd } n \in \mathbb{N}, \text{ since } a_1 = 0.$$

On the other hand,

$$a_2 = \frac{-a_0}{(r+1)^2}, \quad a_4 = \frac{-a_2}{(r+3)^2} = \frac{+a_0}{(r+1)^2(r+3)^2}$$

$$a_6 = \frac{-a_4}{(r+5)^2} = \frac{-a_0}{(r+1)^2(r+3)^2(r+5)^2}, \text{ and so on.}$$

In general, we have

$$\boxed{a_n = \begin{cases} 0 & ; n = 2m-1 \text{ for some } m \in \mathbb{N}. \\ \dfrac{(-1)^m a_0}{(r+1)^2(r+3)^2 \cdots (r+2m-1)^2} & ; n = 2m. \text{ for some } m \in \mathbb{N}. \end{cases}} \quad \text{—③}$$

That is, we have

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$

$$= a_0 x \left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{2^2 \cdot 4^2 \cdots (2m)^2}\right]$$

$$\therefore \boxed{y_1(x) = a_0 x \left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m} \cdot (m!)^2}\right]} \quad \text{—④}$$

In general, we have

$$y_r(x) = a_0 x^r \left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{(r+1)^2(r+3)^2 \cdots (r+2m-1)^2}\right]$$

Notice that the expression is valid for all $r > 0$. So, we can differentiate y_r for $r > 0$, assuming that term-by-term differentiation is allowed.

Consider the following:

$$\log[y_r(x)] = \log a_0 + r\log x + \log\left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{(r+1)^2 (r+3)^2 \cdots (r+2m-1)^2}\right]$$

Differentiating w.r.t r, we get

$$\frac{1}{y_r(x)} \frac{dy_r(x)}{dr} = \log x + \frac{1}{\left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{(r+1)^2 (r+3)^2 \cdots (r+2m-1)^2}\right]} \cdot \sum_{m=1}^{\infty} \frac{d}{dr}\left[\frac{(-1)^m x^{2m}}{(r+1)^2 (r+3)^2 \cdots (r+2m-1)^2}\right]$$

Notice that $1 + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{(r+1)^2 (r+3)^2 \cdots (r+2m-1)^2} = \frac{y_r(x)}{a_0 \cdot x^r}$

Also, we have

$$\frac{d}{dr}\left[\frac{(-1)^m x^{2m}}{(r+1)^2 (r+3)^2 \cdots (r+2m-1)^2}\right]$$

$$= (-1)^m x^{2m} \sum_{k=1}^{m} \frac{(-2)}{(r+1)^2 \cdots (r+2k-3)^2 \cdot (r+2k-1)^3 \cdot (r+2k+1)^2 \cdots (r+2m-1)^2}$$

$$= \frac{2\cdot(-1)^{m+1} x^{2m}}{(r+1)^2 \cdot (r+3)^2 \cdots (r+2m-1)^2} \cdot \sum_{k=1}^{m} \frac{1}{(r+2k-1)}$$

That is, we have

$$\frac{1}{y_r(x)} \frac{dy_r(x)}{dr} = \log x + \frac{a_0 x^r}{y_r(x)} \cdot \sum_{m=1}^{\infty} \frac{2\cdot(-1)^{m+1} x^{2m}}{(r+1)^2 \cdots (r+2m-1)^2} \left(\sum_{k=1}^{m} \frac{1}{(r+2k-1)}\right)$$

or, what is the same

$$\boxed{\frac{dy_r(x)}{dr} = y_r(x) \cdot \log x + 2a_0 x^r \cdot \sum_{m=1}^{\infty} \frac{(-1)^{m+1} x^{2m}}{(r+1)^2 \cdots (r+2m-1)^2} \left(\sum_{k=1}^{m} \frac{1}{(r+2k-1)}\right)}$$

(5)

Hence, the second linearly independent solution of the D.E is given by

$$\frac{dy_r}{dr}\bigg|_{r=1}(x) = y_1(x)\cdot \log x + 2a_0 x \sum_{m=1}^{\infty} \frac{(-1)^{m+1}\, x^{2m}}{2^2\cdot 4^2 \cdot \ldots \cdot (2m)^2}\cdot\left(\sum_{k=1}^{m}\frac{1}{2k}\right)$$

$$\boxed{\therefore\; y_2(x) = a_0 x\left[\log x\left[1+\sum_{m=1}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m}(m!)^2}\right] + \sum_{m=1}^{\infty}\frac{(-1)^{m+1}x^{2m}}{2^{2m}\cdot(m!)^2}\left(\sum_{k=1}^{m}\frac{1}{k}\right)\right]} \quad \text{— (6)}$$

Hence, the general solution to the DE is

$$y(x) = A\cdot y_1(x) + B\cdot y_2(x), \quad \text{for } A, B \in \mathbb{R}.$$

$$\boxed{\begin{aligned}\therefore\; y_1(x) &= A\cdot x\cdot\left[1+\sum_{m=1}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m}(m!)^2}\right]\\ &+ B\cdot x\cdot\left[\log x\left\{1+\sum_{m=1}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m}(m!)^2}\right\} + \sum_{m=1}^{\infty}\frac{(-1)^{m+1}x^{2m}}{2^{2m}(m!)^2}\left(\sum_{k=1}^{m}\frac{1}{k}\right)\right]\end{aligned}} \quad \text{— (*)}$$

Q.1(d) $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-1)y = 0$

Sol^n: Writing the DE in the standard form, we have

$y'' + \frac{1}{x}y' + \left(\frac{x^2-1}{x^2}\right)y = 0$, for $x \neq 0$.

Clearly, $x=0$ is a singular point of the D.E.

Here, $p_0(x) = \frac{x^2-1}{x^2}$, $p_1(x) = \frac{1}{x}$.

Now,

$\lim_{x\to 0} x \cdot p_1(x) = \lim_{x\to 0} x\cdot\frac{1}{x} = 1$

$\lim_{x\to 0} x^2 \cdot p_0(x) = \lim_{x\to 0} x^2\left(\frac{x^2-1}{x^2}\right) = -1$

Both the limits exists and are finite.

Hence, $x=0$ is a regular singular point.

Also, notice that p_0, p_1 are real analytic every where except at $x=0$.

Thus, we apply the Frobenius method to get a series solution of the D.E around $x=0$.

Let the solution to the given D.E. be

$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$, for $|x|>0$ and $a_0 \neq 0$, where $r \in \mathbb{R}$ is to be determined.

Then, differentiating the series term-by-term, we obtain

$y'(x) = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$

$y''(x) = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$

Since y is a solution to the given D.E., we have

$$x^2\cdot\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + x\cdot\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + (x^2-1)\cdot\sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$\therefore \sum_{n=0}^{\infty}(n+r)(n+r-1)\,a_n\,x^{n+r} + \sum_{n=0}^{\infty}(n+r)\,a_n\,x^{n+r} + \sum_{n=0}^{\infty} a_n\,x^{n+r+2} - \sum_{n=0}^{\infty} a_n\,x^{n+r} = 0.$$

$$\therefore \underbrace{\sum_{n=0}^{\infty}\left[(n+r)(n+r-1) + (n+r) - 1\right] a_n\,x^{n+r}}_{\text{collecting the coefficients of } x^{n+r}\text{, assuming that such a rearrangement of series is allowed!}} + \underbrace{\sum_{n=2}^{\infty} a_{n-2}\,x^{n+r}}_{\text{shifting the index of the sum to get } x^{n+r} \text{ instead of } x^{n+r+2}.} = 0$$

$$\therefore \sum_{n=0}^{\infty}(n+r+1)(n+r-1)\,a_n\,x^{n+r} + \sum_{n=2}^{\infty} a_{n-2}\,x^{n+r} = 0$$

$$\therefore (r+1)(r-1)\,a_0\,x^{r} + (r+2)\cdot r\cdot a_1\cdot x^{r+1} + \sum_{n=2}^{\infty}\left[(n+r+1)(n+r-1)\,a_n + a_{n-2}\right]x^{n+r} = 0. \quad (1)$$

Since equation (1) is true for all $x \in \mathbb{R}$ with $x \neq 0$, we have that the coefficients of the powers of x must be zero!

Thus, we first have

$$(r+1)(r-1)\,a_0 = 0$$

Since $a_0 \neq 0$, we have

$$(r+1)(r-1) = 0 \quad \text{— (this is the indicial equation)}$$

$$\therefore \boxed{r=-1} \text{ or } \boxed{r=1}$$

That is, the roots of the indicial equation are distinct and their difference is an integer

$(r_1 - r_2 = -1-1 = -2 \in \mathbb{Z})$.

From eq (1), we also get

$$(r+2)\cdot r\cdot a_1 = 0.$$

Since $r = \pm 1$, we have $(r+2)\cdot r \neq 0$ so that $\boxed{a_1 = 0}$.

We also get the recurrence relation

$$(n+r+1)(n+r-1)\,a_n + a_{n-2} = 0, \quad \text{for } n \geq 2$$

or what is the same,

$$\boxed{a_n = \frac{-a_{n-2}}{(n+r+1)(n+r-1)}} \quad ; \text{ provided } (n+r+1) \neq 0 \text{ and } n+r-1 \neq 0. \quad (2)$$

Notice that for $r = \pm 1$, we have $n+r+1 \geq n \geq 2 > 0$ so that $n+r+1 \neq 0$ for any $n \geq 2$. However for $r = -1$, we have $n+r-1 = n-2 = 0$ for $n = 2$. Thus, the recurrence relation cannot hold with $r = -1$.

The Frobenius method gives that one of the linearly independent solution to the D.E is given by substituting $r = +1$ in the series representation. The other solution is then given by $\frac{d}{dr}\left[(r+1)\cdot y_r(x)\right]\Big|_{r=-1}$.

That is, we have the two L.I. solutions as

$$\boxed{y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}} \quad \text{and} \quad \boxed{y_2(x) = \left[\frac{d}{dr}(r+1)\cdot y_r(x)\right]_{r=1}}$$

From the recurrence relation (2), we have

$a_n = 0$ for all odd $n \in \mathbb{N}$ $(\because a_1 = 0)$

And,

$$a_2 = \frac{-a_0}{(r+1)(r+3)}, \quad a_4 = \frac{-a_2}{(r+3)(r+5)} = \frac{a_0}{(r+1)(r+3)^2(r+5)}$$

$$a_6 = \frac{-a_4}{(r+5)(r+7)} = \frac{-a_0}{(r+1)(r+3)^2(r+5)^2(r+7)}, \text{ and so on.}$$

In general, we have

$$\boxed{a_n = \begin{cases} 0 & ; \ n = 2m-1, \text{ for some } m \in \mathbb{N} \\ \dfrac{a_0 (-1)^m}{(r+1)(r+3)^2(r+5)^2 \cdots (r+2m-1)^2(r+2m+1)} & ; \ n = 2m \text{ for some } m \in \mathbb{N} \end{cases}} \quad (3)$$

That is, the series representation is

$$y_r(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$\therefore y_r(x) = a_0 x^r \left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{(r+1)(r+3)^2 \cdots (r+2m-1)^2 \cdot (r+2m+1)} \right], \text{ for } r \neq -1, -3, -5, \ldots$$

Thus, one of the L.I. solutions is given by

$$y_1(x) = a_0 x \left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{2 \cdot \underbrace{4^2 \cdot 6^2 \cdots (2m)^2}_{(m-1)\text{ terms containing } 2^2} \cdot 2(m+1)} \right]$$

$$= a_0 x \left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{2^2 \cdot 2^{2(m-1)} \cdot 2^2 \cdot 3^2 \cdots m^2 \cdot (m+1)} \right]$$

$$\therefore \boxed{y_1(x) = a_0 \cdot x \left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m} (m!)^2 \cdot (m+1)} \right]} \quad \text{— (4)}$$

For the other L.I. solutions we have

$$(r+1) \cdot y_r(x) = a_0 x^r \left[(r+1) + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{(r+3)^2 \cdots (r+2m-1)^2 (r+2m+1)} \right]$$

$$\therefore \log\left[(r+1) \cdot y_r(x)\right] = \log a_0 + r \log x + \log \left[(r+1) + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{(r+3)^2 \cdots (r+2m-1)^2 (r+2m+1)} \right]$$

Differentiating w.r.t 'r', we have

$$\frac{1}{(r+1) y_r(x)} \cdot \frac{d}{dr}\left((r+1) y_r(x)\right) = \log x + \frac{1}{\left((r+1) + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{(r+3)^2 \cdots (r+2m-1)^2 \cdot (r+2m+1)} \right)} \times$$

$$\frac{d}{dr} \left[(r+1) + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{(r+3)^2 \cdots (r+2m-1)^2 \cdot (r+2m+1)} \right]$$

Notice that $(r+1) + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{(r+3)^2 \cdots (r+2m-1)^2 (r+2m+1)} = \frac{(r+1) y_r(x)}{a_0 x^r}$

Moreover,

$$\frac{d}{dr} \left[(r+1) + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{(r+3)^2 \cdots (r+2m-1)^2 \cdot (r+2m+1)} \right]$$

$$= 1 + \sum_{m=1}^{\infty} (-1)^m x^{2m} \cdot \left(\sum_{k=1}^{m-2} \frac{(-2)}{(r+3)^2 \cdots (r+2k-1)^2 \cdot (r+2k+1)^3 \cdot (r+2k+3)^2 \cdots (r+2m-1)^2 \cdot (r+2m+1)} + \frac{(-1)}{(r+3)^2 \cdots (r+2m+1)^2} \right)$$

$$= 1 + \sum_{m=1}^{\infty} \frac{(-1)^{m+1} x^{2m}}{(r+3)^2 \cdots (r+2m-1)^2 \cdot (r+2m+1)} \left(\sum_{k=1}^{m-2} \frac{2}{r+2k+1} + \frac{1}{r+2m+1} \right)$$

Thus, we have

$$\frac{1}{(r+1)\,y_r(x)} \frac{d}{dr}\left((r+1)\,y_r(x)\right) = \log x + \frac{a_0 x^r}{(r+1)\,y_r(x)} \cdot \left[1 + \sum_{m=1}^{\infty} \frac{(-1)^{m+1} x^{2m}}{(r+3)^2 \cdots (r+2m-1)^2 (r+2m+1)} \times \left(\sum_{k=1}^{m-2} \frac{2}{r+2k+1} + \frac{1}{r+2m+1} \right) \right]$$

$$\therefore \frac{d}{dr}\left((r+1)\,y_r(x)\right) = (r+1)\,y_r(x) \log x + a_0 x^r \cdot \left[1 + \sum_{m=1}^{\infty} \frac{(-1)^{m+1} x^{2m}}{(r+3)^2 \cdots (r+2m-1)^2 (r+2m+1)} \times \left(\sum_{k=1}^{m-2} \frac{2}{r+2k+1} + \frac{1}{r+2m+1} \right) \right]$$

$$= \log x \cdot a_0 x^r \cdot \left[r+1 + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{(r+3)^2 \cdots (r+2m-1)^2 \cdot (r+2m+1)} \right]$$

$$+ a_0 x^r \left[1 + \sum_{m=1}^{\infty} \frac{(-1)^{m+1} x^{2m}}{(r+3)^2 \cdots (r+2m-1)^2 (r+2m+1)} \cdot x \left(\sum_{k=1}^{m-2} \frac{2}{r+2k+1} + \frac{1}{r+2m+1} \right) \right]$$

Thus, we have

$$\left.\frac{d}{dr}\left((r+1)\,y_r(x)\right)\right|_{r=-1} = \frac{a_0 \log x}{x} \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{2^2 \cdot 4^2 \cdots (2m-2)^2 \cdot (2m)} + \frac{a_0}{x} \cdot \left[1 + \sum_{m=1}^{\infty} \frac{(-1)^{m+1} x^{2m}}{2^2 \cdots (2m-2)^2 \cdot (2m)} \cdot \left(\sum_{k=1}^{m-2} \frac{2}{2k} + \frac{1}{2m} \right) \right]$$

$$\boxed{\therefore y_2(x) = a_0 \left[\log x \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m-1}}{2^{2m-2} \cdot \left((m-1)!\right)^2 \cdot m} + \frac{1}{x} \left\{ 1 + \sum_{m=1}^{\infty} \frac{(-1)^{m+1} x^{2m}}{2^{2m-2} \left((m-1)!\right)^2 \cdot m} \times \left(\sum_{k=1}^{m-2} \frac{1}{k} + \frac{1}{2m} \right) \right\} \right]}$$

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Thus, the general solution to the DE is

$y(x) = A\cdot y_1(x) + B\cdot y_2(x)$, for $A, B \in \mathbb{R}$.

$$\boxed{\begin{aligned}\therefore\ y(x) &= A\cdot x\cdot\left[1+\sum_{m=1}^{\infty}\frac{x^{2m}}{2^{2m}\,(m!)^2(m+1)}\right]\\ &+B\cdot\left[\log x\cdot\sum_{m=1}^{\infty}\frac{x^{2m-1}}{2^{2m-2}\cdot(m-1)!\cdot m}+\frac{1}{x}\left\{1-\sum_{m=1}^{\infty}\frac{x^{2m}}{2^{2m-2}(m-1)!\cdot m}\left(\sum_{k=1}^{m-2}\frac{1}{k}+\frac{1}{2m}\right)\right\}\right]\end{aligned}} \tag{*}$$

Remarks :-

① In all the solutions, we have assumed term-by-term manipulation to be valid. This does NOT cause any issue with Frobenius method.

② The series solution is defined only for $x \neq 0$.
In case $\lim_{x\to 0} y(x)$ exist, we may extend it to $x=0$ by continuity.

#2) We shall use the following solutions of Bessel's equation —

$$\boxed{x^2y'' + xy' + (x^2 - n^2)y = 0}$$

Result: I: $y = A J_n(x) + B J_{-n}(x)$, where, n is not an integer.

Result: II: $y = A J_n(x) + B Y_n(x)$, where, n is an integer.

Here $J_n(x)$ is Bessel's fn. of first kind.

& $Y_n(x) = \lim_{p \to n} \dfrac{J_p(x)\cos(p\pi) - J_{-p}(x)}{\sin(p\pi)}$, $n \in \mathbb{Z}$.

$Y_n(x)$ is Bessel's fn. of second kind of order n.

(i) Given, $x^2y'' + xy' + (x^2 - 5^2)y = 0$, which is Bessel's eqn. of order 5, which is an integer.

So, General soln $\boxed{y = A J_5(x) + B Y_5(x)}$; A, B arbitrary constants.

(ii) Rewriting the given eqn. as —

$$x^2y'' + xy' + \{x^2 - (2/5)^2\}y = 0.$$

As $n = 2/5 \notin \mathbb{Z}$, soln.: $\boxed{y = A J_{2/5}(x) + B J_{-2/5}(x)}$

(iii) Rewriting the given eqn. as —

$$\boxed{z^2y'' + zy' + (z^2 - 0^2)y = 0}$$

As $n = 0 \in \mathbb{Z}$, soln. $\boxed{y = A J_0(z) + B Y_0(z)}$

Remark: The solution provided here should be followed if the question is for 1 mark. If the same question is asked for more marks, you need to derive the complete solution!

Bessel's eq^n: $x^2y'' + xy' + (x^2 - n^2)y = 0$

Sol^n → $AJ_n(x) + BJ_{-n}(x)$ $\quad n \notin \mathbb{Z}$

or $y = AJ_n(x) + BY_n(x)$ $\quad n \in \mathbb{Z}$

$$J_n(x) = \frac{x^n}{2^n \Gamma(n+1)} \left[1 - \frac{x^2}{2.2(n+1)} + \frac{x^4}{2.4.2^2(n+1)(n+2)} - \dots \right]$$

$$\boxed{\begin{array}{l} \Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx \\ \Gamma(z) = \Gamma(z-1).(z-1) \end{array}}$$

Q3

a) $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$

$$\frac{x^{-1/2}}{2^{-1/2}\Gamma(1/2)} \left[1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \dots \right] \qquad \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

$$= \sqrt{\frac{2}{\pi x}} \cos x \qquad \left[\because \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \dots \right]$$

b) $J_{3/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x \right)$

Using $J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$

Replace n by $1/2$.

$$J_{-1/2}(x) + J_{3/2}(x) = \frac{2}{2x} J_{1/2}(x)$$

$$\Rightarrow J_{3/2}(x) = -J_{-1/2}(x) + \frac{1}{x} J_{1/2}(x)$$

$$= -\sqrt{\frac{2}{\pi x}} \cos x + \frac{1}{x} \sqrt{\frac{2}{\pi x}} \sin x$$

$$\boxed{\begin{aligned} J_{1/2}(x) &= \frac{x^{1/2}}{2^{1/2}\Gamma(3/2)} \left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} \dots \right] \\ &= \sqrt{\frac{x}{2}} \cdot \frac{1}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} \cdot \frac{1}{x} \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \right] \end{aligned}}$$

$$= +\sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x \right)$$

c) $J_2'(x) = \left(1 - \frac{4}{x^2}\right) J_1(x) + \frac{2}{x} J_0(x)$

Use $\boxed{J_n'(x) = J_{n-1}(x) - \frac{n}{x} J_n(x)}$

put $n=2$

$$J_2'(x) = J_1(x) - \frac{2}{x} J_2(x) \quad \text{— (1)}$$

Since $\boxed{J_{n+1}(x) + J_{n-1}(x) = \frac{2n}{x} J_n(x)}$

$n=1$

$$J_2(x) + J_0(x) = \frac{2}{x} J_1(x) \quad \text{— (2)}$$

Substitute J_2 from (2) in (1)

$$J_2'(x) = J_1(x) - \frac{2}{x}\left[\frac{2}{x} J_1(x) - J_0(x)\right]$$

d) $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x \qquad J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$

$$\cancel{J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)}$$

$$\left[J_{1/2}(x)\right]^2 + \left[J_{-1/2}(x)\right]^2 = \frac{2}{\pi x} (\sin^2 x + \cos^2 x) = \frac{2}{\pi x}$$

e) $\lim_{z \to 0} \frac{J_n(z)}{z^n} = \frac{1}{2^n \Gamma(n+1)^n} \quad ; \quad n > -1$

$$J_n(z) = \frac{z^n}{2^n \Gamma(n+1)} \left[1 - \frac{z^2}{4(n+1)} + \frac{z^4}{4 \cdot 8 \cdot (n+1)(n+2)} - \cdots\right]$$

$$\lim_{z \to 0} \frac{J_n(z)}{z^n} = \frac{1}{2^n \Gamma(n+1)}$$

f) $\int_0^1 \frac{u J_0(xu)}{(1-u^2)^{1/2}} du = \frac{\sin x}{x}$

we have $J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots$

$$\therefore \int_0^1 \frac{u\,J(\alpha u)}{(1-u^2)^{1/2}}\,du = \int_0^1 \frac{u}{(1-u^2)^{1/2}}\left(1-\frac{x^2}{4}u^2+\frac{x^4}{4.16}u^4 \dots\right)du$$

put $u = \sin\theta$ $\quad du = \cos\theta\, d\theta$

$$\int_0^{\pi/2} \frac{\sin\theta}{\cos\theta}\left(1-\frac{x^2}{4}\sin^2\theta+\frac{x^4}{64}\sin^4\theta - \dots\right)\cos\theta\, d\theta$$

$$= \int_0^{\pi/2} \sin\theta\, d\theta - \frac{x^2}{4}\int_0^{\pi/2}\sin^3\theta\, d\theta + \frac{x^4}{64}\int_0^{\pi/2}\sin^5\theta\, d\theta$$

Wallis formula $\int_0^{\pi/2}\cos^n x\, dx = \int_0^{\pi/2}\sin^n x\, dx$

n even $\quad \int_0^{\pi/2}\sin^n x\, dx = \frac{1.3.5\dots(n-1)}{2.4.6\dots n}\cdot\frac{\pi}{2}$

odd $\quad \int_0^{\pi/2}\sin^n x\, dx = \frac{2.4.6\dots(n-1)}{1.3.5\dots n}$

$$= -\cos\theta\Big|_0^{\pi/2} - \frac{x^2}{4}\times\frac{2}{3} + \frac{x^4}{64}\times\frac{2.4}{3.5}\dots$$

$$= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots = \frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right) = \frac{\sin x}{x}$$

g) $J_n J_{-n}' - J_n' J_{-n} = -\frac{2\sin \pi n}{\pi x}$

we know J_n, J_{-n} are sol^{n}s of Bessel eq $y'' + \frac{1}{x}y' + \left(1-\frac{n^2}{x^2}\right)y = 0$

i.e. $J_n'' + \left(\frac{1}{x}\right)J_n' + \left(1-\frac{n^2}{x^2}\right)J_n = 0 \quad$ — (1)

$J_{-n}'' + \frac{1}{x}J_{-n}' + \left(1-\frac{n^2}{x^2}\right)J_{-n} = 0 \quad$ — (2)

$J_{-n}\times(1) - J_n\times(2)$

$J_n''J_{-n} - J_{-n}''J_n + \frac{1}{x}\left(J_n'J_{-n} - J_{-n}'J_n\right) = 0 \quad$ — (3)

Let $J_n'J_{-n} - J_{-n}'J_n = v \quad \underset{\text{diff}}{\Rightarrow} \quad J_n''J_{-n} + \cancel{J_n'J_{-n}'} - \left(J_{-n}''J_n + \cancel{J_{-n}'J_n'}\right) = v'$

$$J_n'' J_{-n} - J_{-n}'' J_n = v'$$

③ $\Rightarrow$ $v' + \frac{1}{x} v = 0$ $\quad \frac{dv}{dx} = -\frac{v}{x}$ $\quad \Rightarrow \frac{dv}{v} = -\frac{dx}{x}$

$\log v + \log x = \log c$ $\quad$ or $\quad v = c/x$

$$\Rightarrow J_n' J_{-n} - J_{-n}' J_n = c/x$$

Substitute

$$\begin{cases} J_n(x) = \frac{1}{2^n \Gamma(n+1)} \left[x^n - \frac{x^{n+2}}{4(n+1)} + \frac{x^{n+4}}{4 \cdot 8 \cdot (n+1)(n+2)} - \cdots \right] \\ J_{-n}(x) = \frac{1}{2^{-n} \Gamma(-n+1)} \left[x^{-n} - \frac{x^{2-n}}{4(1-n)} + \frac{x^{4-n}}{4 \cdot 8 (1-n)(2-n)} - \cdots \right] \end{cases}$$

$$\frac{1}{2^n \Gamma(n+1)} \left[n x^{n-1} - \frac{(n+2) x^{n+1}}{4(n+1)} + \frac{(n+4) x^{n+3}}{4 \cdot 8 (n+1)(n+2)} - \cdots \right]$$

$$\times \frac{1}{2^{-n} \Gamma(-n+1)} \left[x^{-n} - \frac{x^{2-n}}{4(1-n)} \cdots \right] - \frac{1}{2^{-n} \Gamma(-n+1)} \left[-n x^{-n-1} \right.$$

$$\left. - \frac{(2-n) x^{1-n}}{4(1-n)} + \frac{(4-n) x^{3-n}}{4 \cdot 8 \cdot (1-n)(2-n)} \right] \left[J_n(x) \right] = \frac{c}{x}$$

Compare coeff of $\frac{1}{x}$ on both sides

$$\frac{n}{2^n \Gamma(n+1) \cdot 2^{-n} \Gamma(-n+1)} + \frac{n}{2^{-n} \Gamma(-n+1)\, 2^n \Gamma(n+1)} = c$$

or $\quad c = \frac{2n}{n \Gamma(n) \Gamma(1-n)} = \frac{2}{\pi / \sin n\pi} = \frac{2 \sin n\pi}{\pi}$

$$\boxed{\because \Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}}$$

$$\Rightarrow J_n J_{-n}' - J_n' J_{-n} = -\frac{2 \sin n\pi}{\pi x} \qquad \text{—} (*)$$

Now $\div$ (*) by J_n^2

$$\frac{J_n J_{-n}' - J_n' J_{-n}}{J_n^2} = -\frac{2 \sin n\pi}{\pi x J_n^2}$$

or $\quad \frac{d}{dx}\left(\frac{J_{-n}}{J_n} \right) = -\frac{2 \sin n\pi}{\pi x J_n^2}$

Prove that J_n (x) is the coefficient of z^n in the expansion of $e^{\frac{x}{2}\left(z-\frac{1}{z}\right)}$

Proof. We know that $e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots$

$$e^{\frac{xz}{2}} = 1 + \left(\frac{xz}{2}\right) + \frac{1}{2!}\left(\frac{x}{2z}\right)^2 - \frac{1}{3!}\left(\frac{x}{2z}\right)^3 + \cdots \qquad \ldots(1)$$

$$e^{\frac{x}{2z}} = 1 - \left(\frac{x}{2z}\right) + \frac{1}{2!}\left(\frac{x}{2z}\right)^2 - \frac{1}{3!}\left(\frac{x}{2z}\right)^3 + \cdots \qquad \ldots(2)$$

On multiplying (1) and (2), we get

$$e^{\frac{x}{2}\left(z\frac{1}{z}\right)} = \left[1 + \left(\frac{xz}{2}\right) + \frac{1}{2!}\left(\frac{xz}{2}\right)^2 + \frac{1}{3!}\left(\frac{xz}{2}\right)^3 + \cdots\right] \times \left[1 - \frac{x}{2z} + \frac{1}{2!}\left(\frac{x}{2z}\right)^2 - \frac{1}{3!}\left(\frac{x}{2z}\right)^3 + \cdots\right] \ldots(3)$$

The coefficient of z^n in the product of (3), we get

$$= \frac{1}{n!}\left(\frac{x}{2}\right)^n - \frac{1}{(n+1)!}\left(\frac{x}{2}\right)^{n+2} + \frac{1}{2!(n+2)!}\left(\frac{x}{2}\right)^{n+4} - \cdots = J_n(x)$$

Similarly, coefficient of z^{-n} in the product of (3) = $J_{-n}(x)$

$$\therefore \quad e^{\frac{x}{2}\left(z\frac{1}{z}\right)} = J_0 + z J_1 + z^2 J_2 + z^3 J_3 + \ldots + z^{-1} J_{-1} + z^{-2} J_{-2} + z^{-3} J_{-3} + \ldots$$

$$e^{\frac{x}{2}\left(z\frac{1}{z}\right)} = \sum_{n=-\infty}^{\infty} z^n J_n(x)$$

For this reason $e^{\frac{x}{2}\left(z\frac{1}{z}\right)}$ is known as the generating function of Bessel's functions.

Example 1. *Show that Bessel's Function $J_n(x)$ is an even function when n is even and is odd function when n is odd.*

Solution. We know that

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\sqrt{n+r+1}} \left(\frac{x}{2}\right)^{n+2r} \qquad ...(1)$$

Replacing x by – x in (1), we get

$$J_n(-x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\sqrt{n+r+1}} \left(\frac{-x}{2}\right)^{n+2r} \qquad ...(2)$$

Case I. If *n* is even, then n + 2r is even $\Rightarrow \left(\frac{-x}{2}\right)^{n+2r} = \left(\frac{x}{2}\right)^{n+2r}$

Thus (2), becomes

$$J_n(-x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\sqrt{n+r+1}} \left(\frac{x}{2}\right)^{n+2r}$$

$$= J_n(x) \qquad \begin{bmatrix} For\ even\ function \\ f(-x) = f(x) \end{bmatrix}$$

Hence, J_n (x) is even function

Case II. If *n* is odd, then *n* + 2r is odd $\Rightarrow \left(\frac{-x}{2}\right)^{n+2r} = -\left(\frac{x}{2}\right)^{n+2r}$

Thus (2). Becomes

$$J_n(-x) = -\sum_{r=0}^{\infty} \frac{(-1)^r}{r!\sqrt{n+r+1}} \left(\frac{x}{2}\right)^{n+2r}$$

$$= -J_n(x) \qquad \begin{bmatrix} For\ odd\ function \\ f(-x) = -f(x) \end{bmatrix}$$

Proved.

Hence, J_n (x) os odd function.