# Conformal Mapping

In Mathematics, a $\boxed{\text{conformal map}}$ is a function that locally preserves angles but not necessarily lengths.

Formally, let $U$ and $V$ be open subsets of $\mathbb{R}^2$. A function $f: U \longrightarrow V$ is called $\boxed{\text{conformal}}$ or $\boxed{\text{angle-preserving}}$ at a point $u_0 \in U$ if it preserves angles between directed curves through $u_0$, as well as preserving orientation.

**Remark:** Conformal maps preserve both angles and the shapes of infinitesimally small figures, but not necessarily their size or curvature.

**Applications:** Conformal mappings are invaluable to the engineer and physicist as an aid in solving problems in potential theory.

They are a standard method for solving $\boxed{\text{Boundary Value Problems}}$ in two dimensional potential theory and yield rich applications in electrostatics, heat flow, and fluid flow.

The main feature of conformal mappings is that they are angle-preserving (except at some critical points) and allow a $\boxed{\text{geometric approach to complex analysis.}}$

Description: Let us consider a complex function $w = f(z)$ defined in a domain $D$ of the $z$-plane; then to each point in $D$ there corresponds a point in the $w$-plane.

Goal: Our aim is to show that if $f(z)$ is an analytic function, then the mapping given by $w = f(z)$ is a $\boxed{\text{conformal mapping}}$, that is, it preserves angles, except at points where the derivative $f'(z)$ is zero. (Such points are called critical points).

History: Conformality appeared early in the history of construction of maps of the globe. Such maps can be either conformal, that is, give directions correctly, or equiareal, i.e., give areas correctly except for a scale factor.

Connection: Our study of conformality is similar to the approach used in calculus where we study properties of real functions $y=f(x)$ and graph them. Here we study the properties of conformal mappings to get a deeper understanding of the properties of functions.

## Preservation of angles and scale factors:

Let $C$ be a smooth arc, represented by the equation:

$$z = z(t) \,; \quad t \in [a, b]$$

and let $f(z)$ be a function defined at all points $z$ on $C$. The equation

$$w = f[z(t)] \,, \quad t \in [a, b]$$

is a parametric representation of the image $\Gamma$ of $C$ under the transformation $w = f(z)$.

Suppose that $C$ passes through a point $z_0 = z(t_0)$ $(a < t_0 < b)$ at which $f$ is analytic and that $f'(z_0) \neq 0$.

According to the chain rule, if

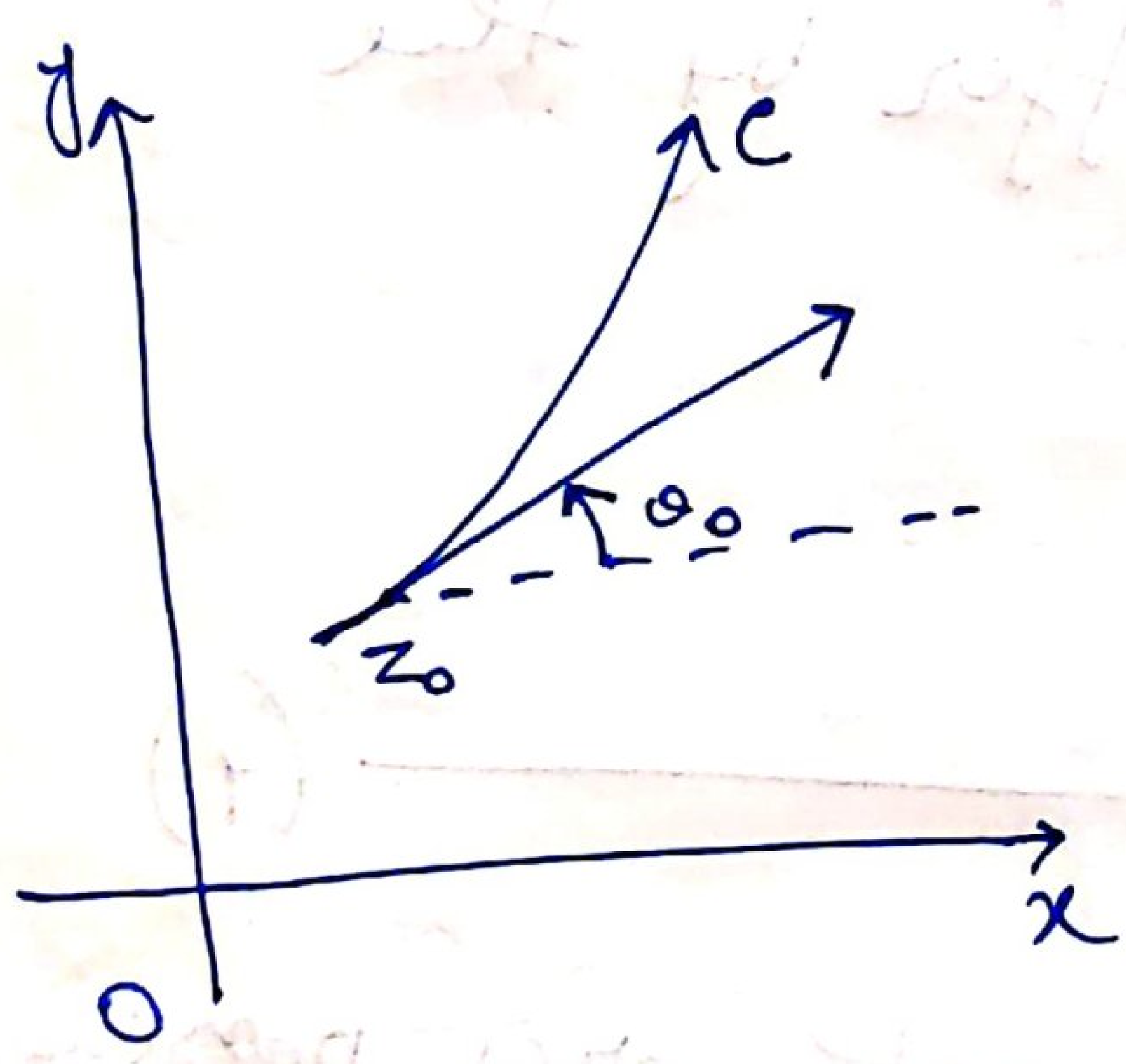
$$w(t) = f[z(t)] \,, \text{ then}$$

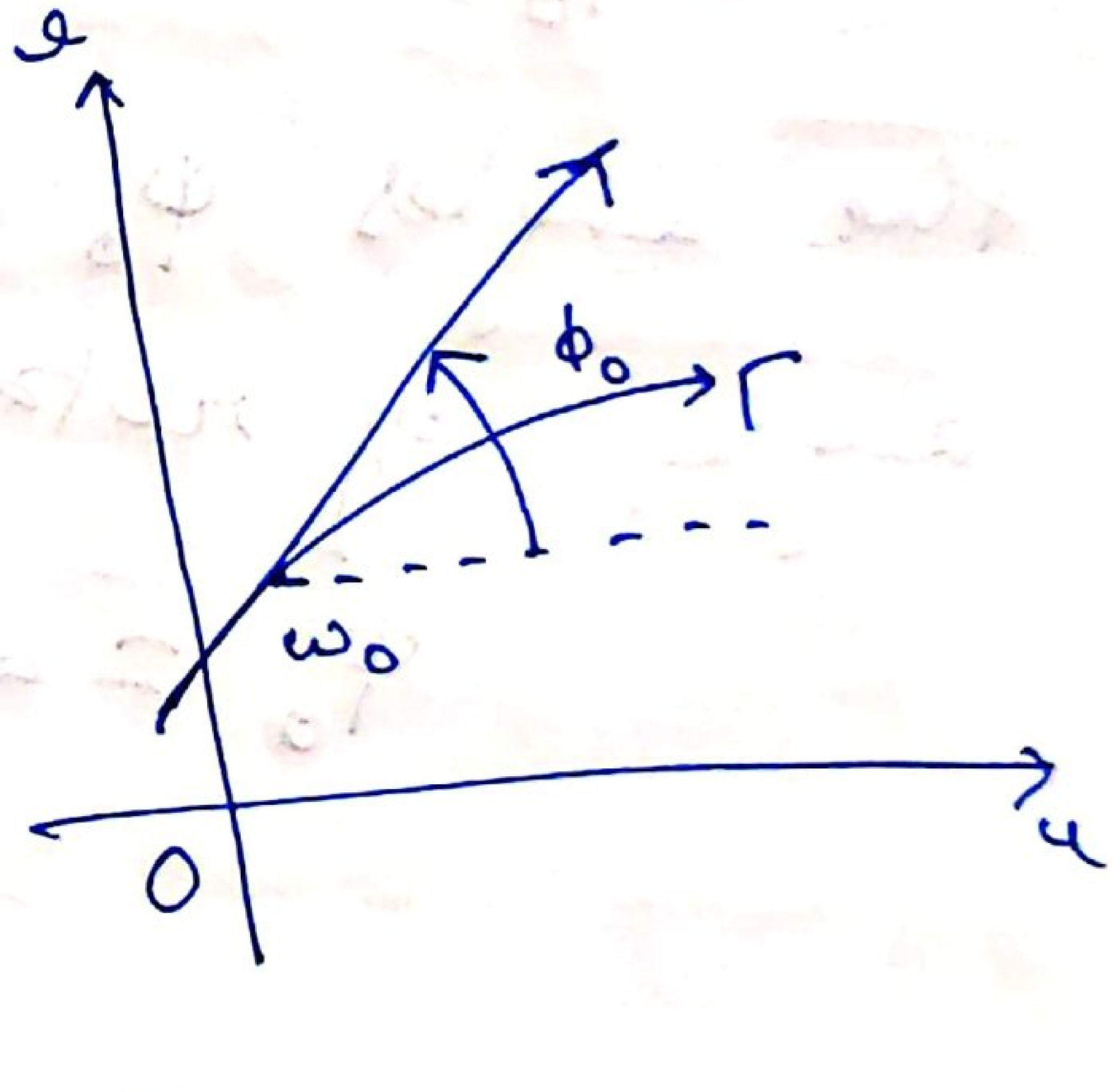
$$w'(t_0) = f'[z(t_0)] \, z'(t_0) \,, \quad (1)$$

and this means that

$$\arg w'(t_0) = \arg f'[z(t_0)] + \arg z'(t_0) \qquad (2)$$

statement (2) is useful in relating the directions of $C$ and $\Gamma$ at the points $z_0$ and $w_0 = f(z_0)$ respectively.




Let $\theta_0$ denote a value of $\arg z'(t_0)$ and let $\phi_0$ be a value of $\arg w'(t_0)$. The number $\theta_0$ is the angle of inclination of a directed line tangent to $C$ at $z_0$ and $\phi_0$ is the angle of inclination of a directed line tangent to $\Gamma$ at the point $w_0 = f(z_0)$.

In view of statement (2), there is a value $\psi_0$ of $\arg f'[z(t_0)]$ such that

$$\phi_0 = \psi_0 + \theta_0 \tag{3}$$

Thus, $\phi_0 - \theta_0 = \psi_0$, and we find that the angles $\phi_0$ and $\theta_0$ differ by the angle of rotation

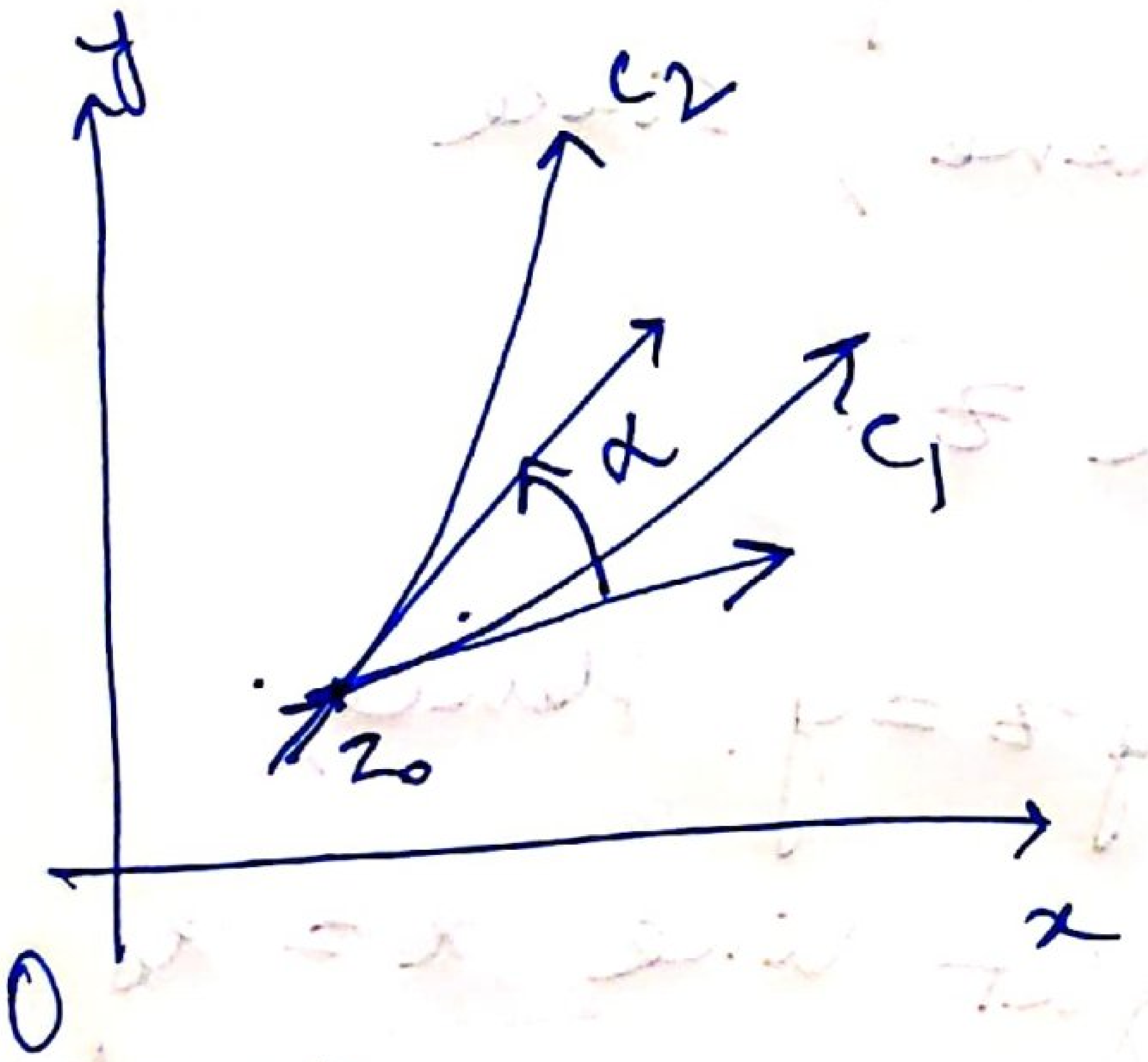
$$\psi_0 = \arg f'(z_0) \tag{4}$$

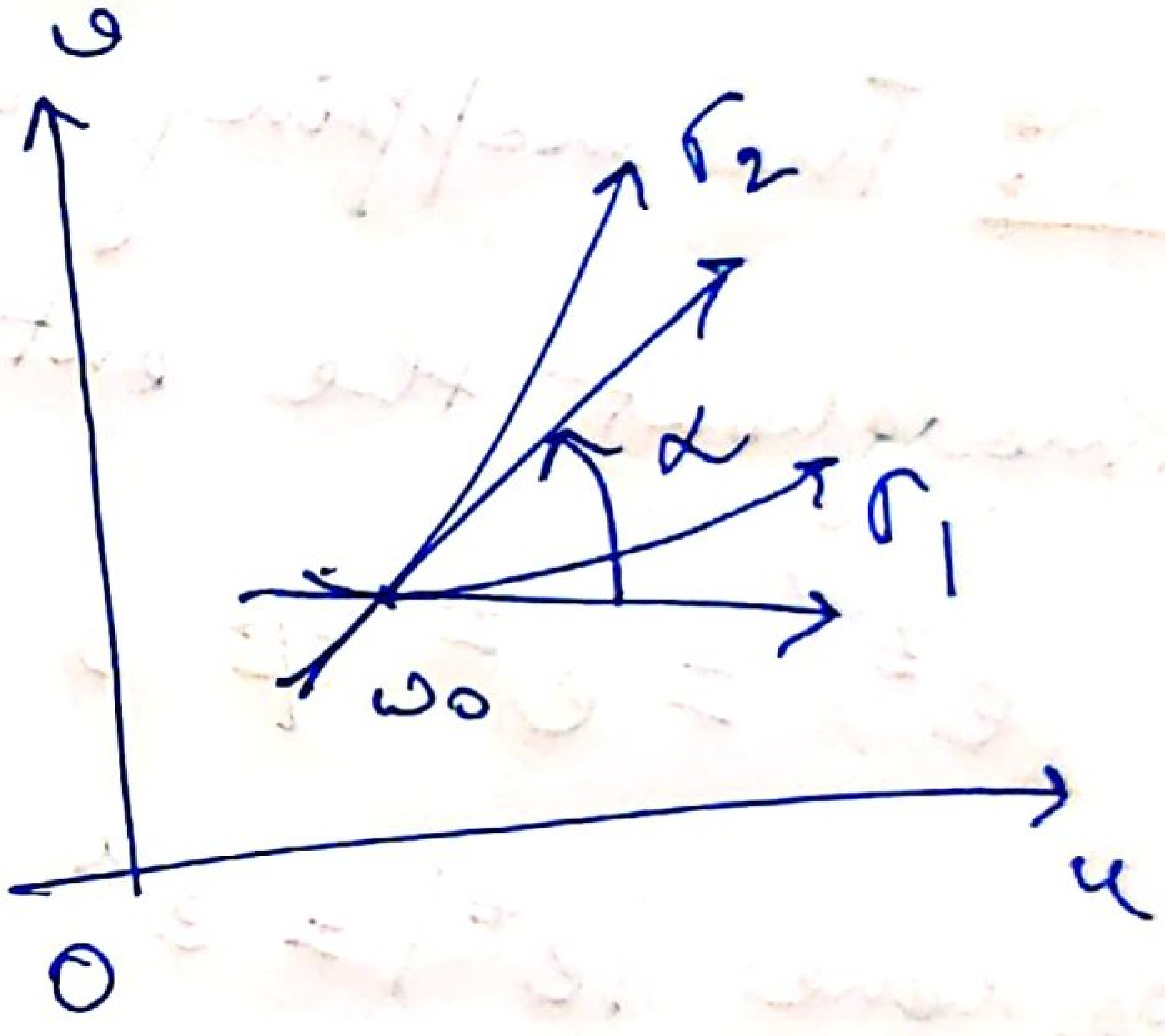
Let $C_1$ and $C_2$ be two smooth arcs passing through $z_0$, and let $\theta_1$ and $\theta_2$ be angles of inclination of directed lines tangent to $C_1$ and $C_2$, respectively at $z_0$. We know,

$$\phi_1 = \psi_0 + \theta_1 \quad \text{and} \quad \phi_2 = \psi_0 + \theta_2$$

are angles of inclination of directed lines tangent to the image curves $\Gamma_1$ and $\Gamma_2$, respectively at the point $w_0 = f(z_0)$.

Thus, $\phi_2 - \phi_1 = \theta_2 - \theta_1$; i.e., the angle $\phi_2 - \phi_1$ from $\Gamma_1$ to $\Gamma_2$ is the same in magnitude and sense as the angle $\theta_2 - \theta_1$ from $C_1$ to $C_2$. Those angles are denoted by $\alpha$.



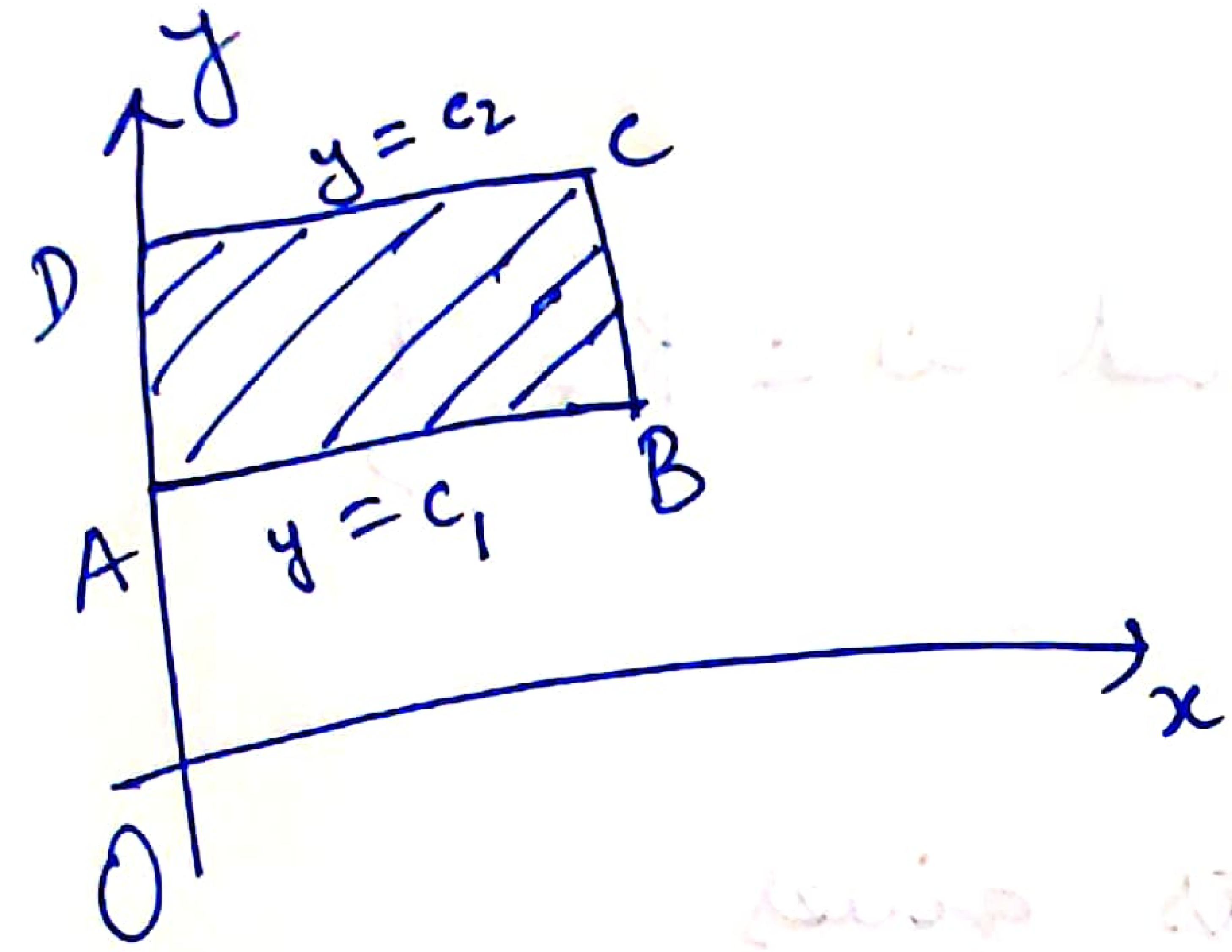

Because of this angle-preserving property, a transformation $w = f(z)$ is said to be **conformal** at a point $z_0$ if $f$ is analytic there and $f'(z_0) \neq 0$.

Question: Prove that the mapping $w = e^z$ is conformal throughout the entire z plane. What is the image of a rectangle under the map $w = e^z$?

Sol^n: The mapping $w = e^z$ is conformal throughout the entire z plane, since

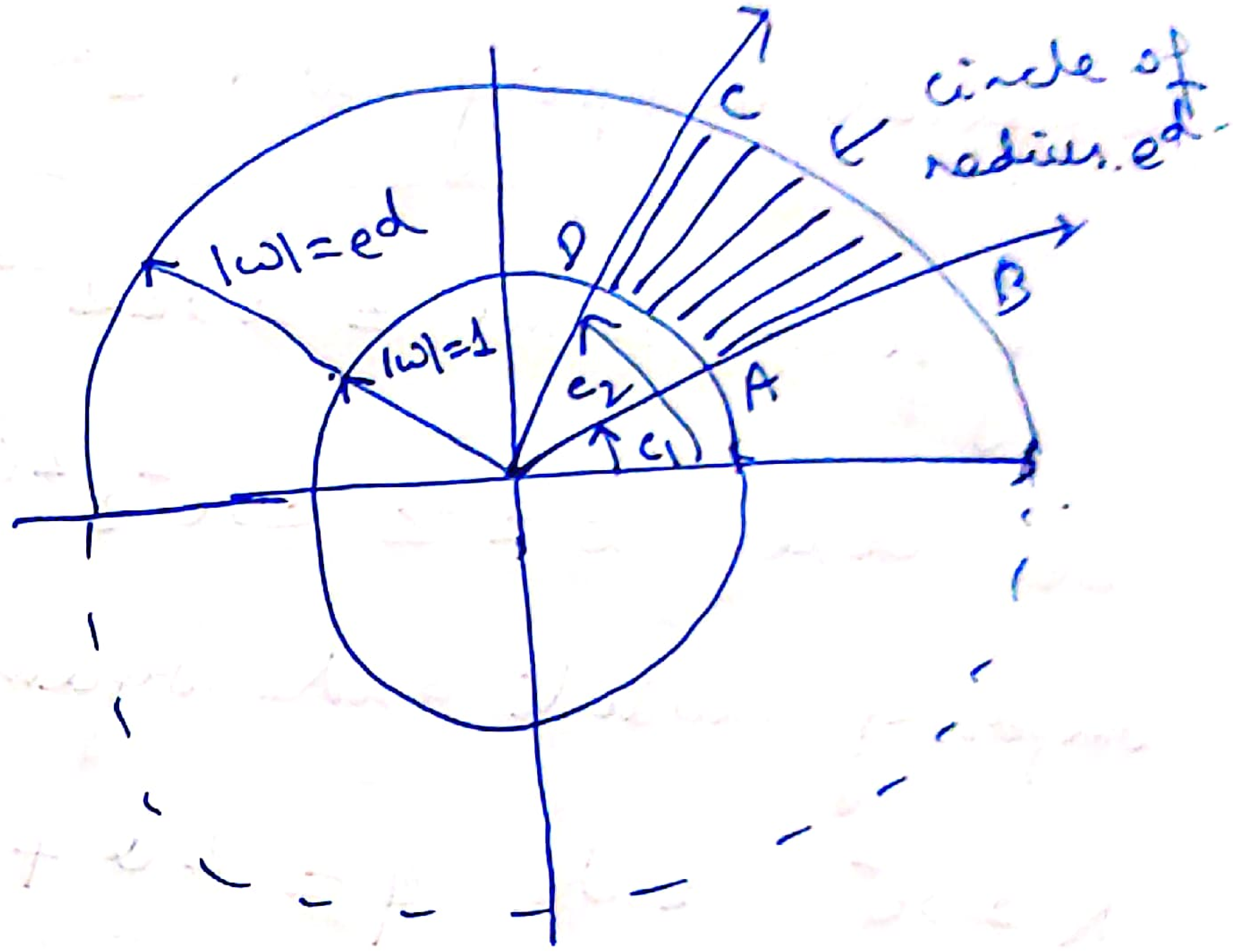
$$(e^z)' = e^z \neq 0 \text{ for each } z.$$

We have $|e^z| = e^x$ and $\operatorname{Arg} z = y$. Hence, $e^z$ maps a vertical straight line $x = x_0$ = const onto the circle $|w| = e^{x_0}$ and a horizontal straight line $y = y_0$ = constant onto the ray $\arg w = y_0$. The rectangle (in Fig (iii)) is mapped onto a region bounded by circles and rays as shown.




AD $\longrightarrow$ $x = 0$

BC $\longrightarrow$ $x = d$




Question: Why do the images of the straight lines $x =$ constant and $y =$ constant under a mapping by an analytic function intersect at right angles? Same question for the curves $|z| =$ constant and $\arg z =$ constant.

$\boxed{\text{Conformal mapping } w = z^2}$

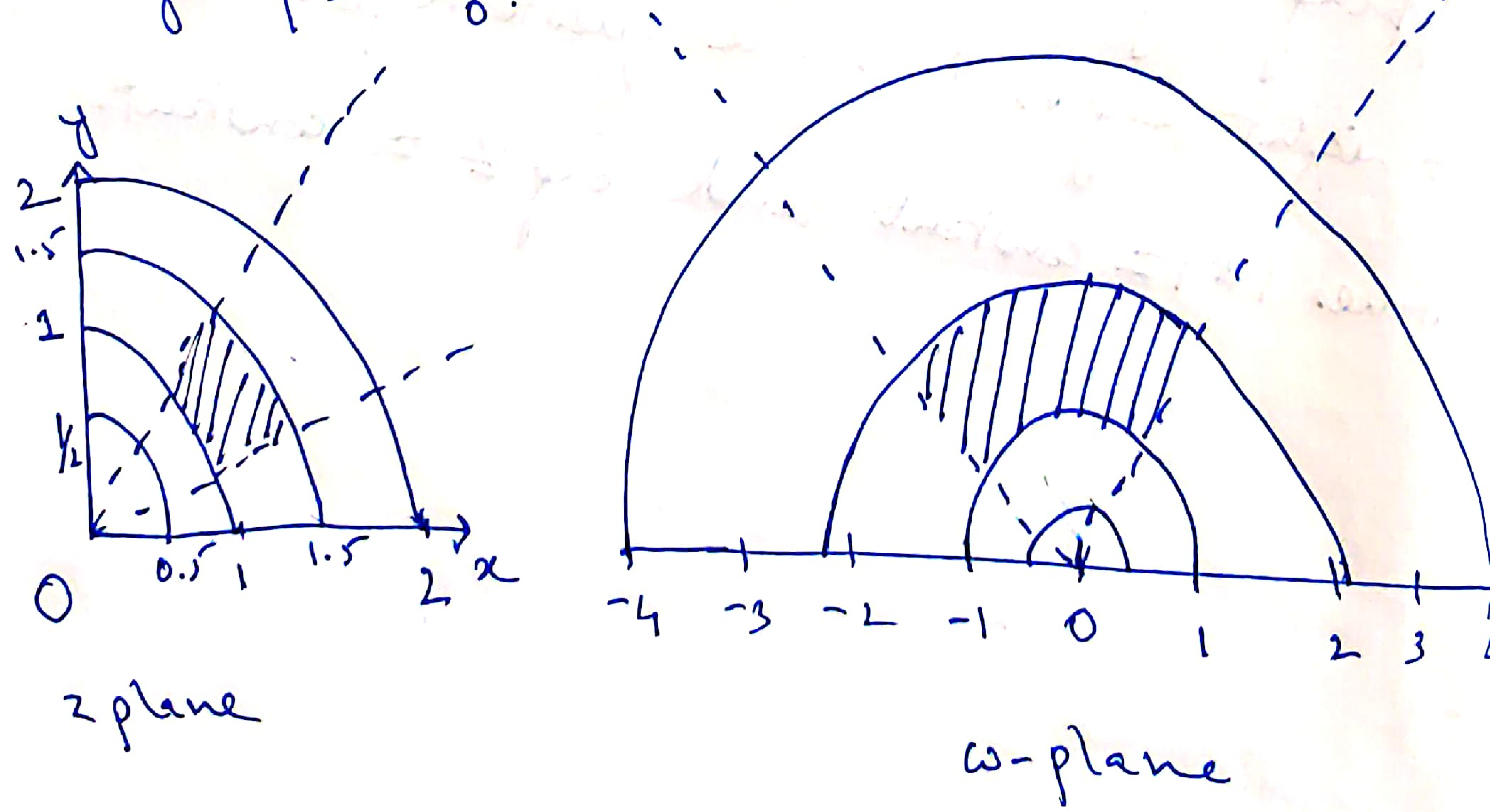
Using polar forms $z = re^{i\theta}$ and $w = Re^{i\phi}$, we have $w = z^2 = r^2 e^{2i\theta}$.

Comparing moduli and arguments gives

$R = r^2$ and $\phi = 2\theta + 2n\pi$; $n \in \mathbb{Z}$.

Choose $n = 0$. Then, $\boxed{\phi = 2\theta}$

Hence, circles $r = r_0$ are mapped onto circles $R = r_0^2$ and rays $\theta = \theta_0$ onto rays $\phi = 2\theta_0$.




Mapping $w = z^2$, Lines $|z| = \text{constant}$, $\arg z = \text{constant}$ and their images in the $w$-plane

Fig I

In cartesian coordinates, we have

$$z = x + iy, \quad \text{and}$$

$$u = \mathrm{Re}(z^2) = x^2 - y^2, \quad v = \mathrm{Im}(z^2) = 2xy$$

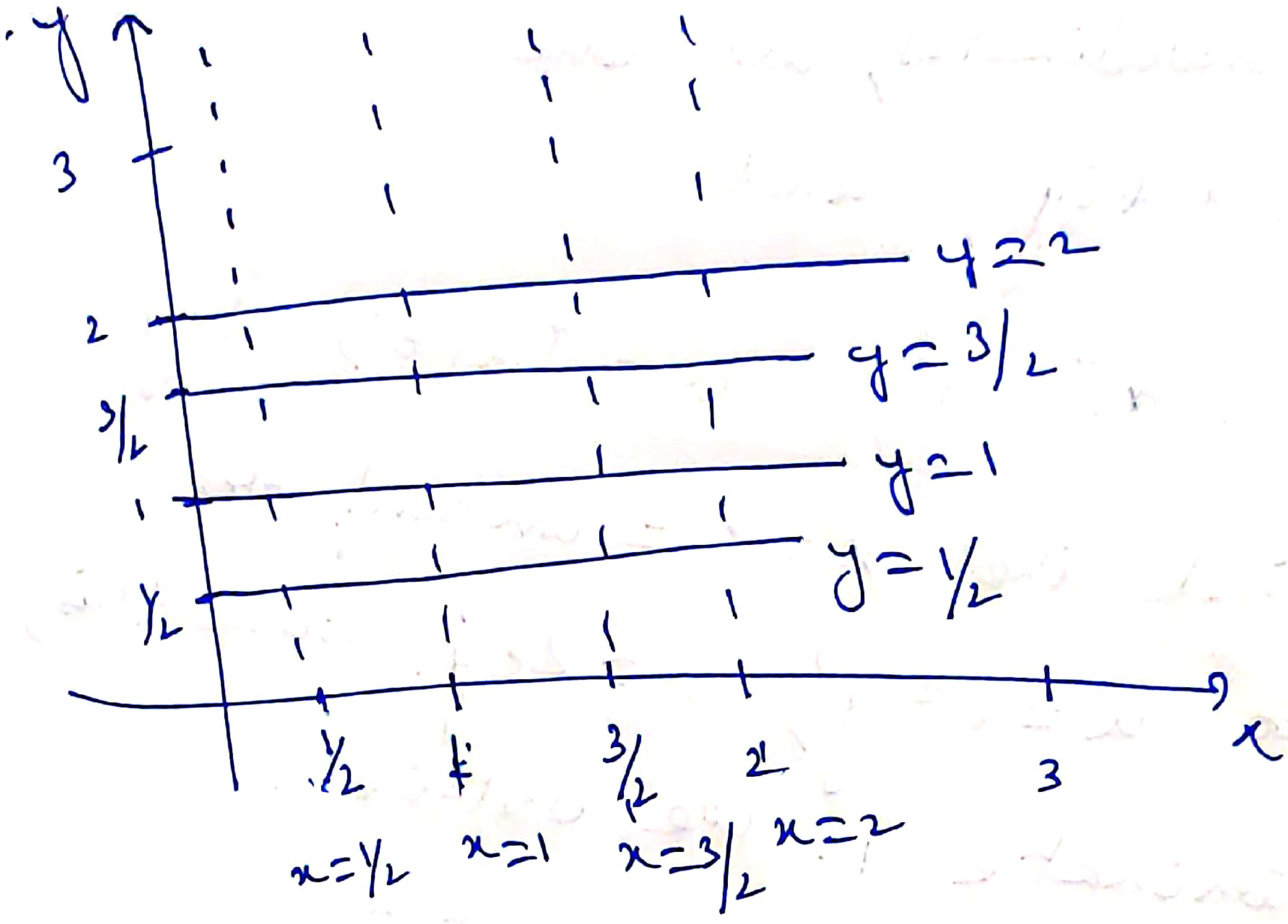
Hence, vertical lines $x = c =$ constant are mapped onto $u = c^2 - y^2$, $v = 2cy$. From this, we can eliminate $y$. We obtain

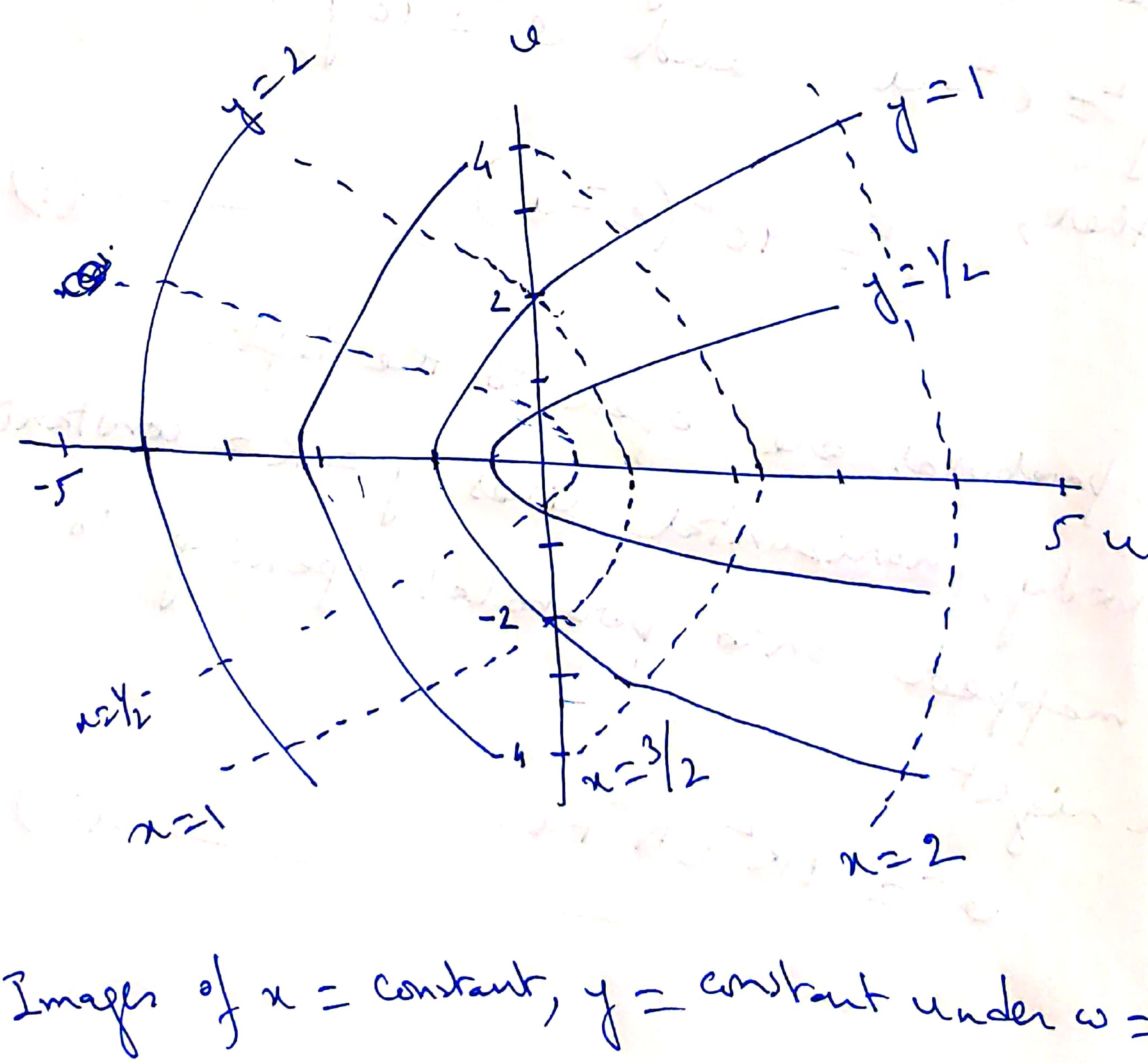
$$y^2 = c^2 - u \quad \text{and} \quad v^2 = 4c^2y^2.$$

Together, $v^2 = 4c^2(c^2 - u)$ ——— (i)

These parabolas are open to the left.
Similarly, horizontal lines $y = k =$ constant are mapped onto parabolas opening to the right.

$v^2 = 4k^2(k^2 + u)$ ——— (ii)




Images of $x$ = constant, $y$ = constant under $\omega = z^2$

$w = z^2$ has a critical point at $z = 0$ where $f'(z) = 2z = 0$, so that conformality fails at $z = 0$.

Question 1: Find all points at which the mapping $f(z) = z^2 + \frac{1}{z^2}$ is not conformal.

solution: $\pm 1, \pm i, 0$.

Question 2: Conformality of $w = z^n$, $n \geq 2$.