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Proposition

→ a statement with value true or false.

Conditional Statements

$p \to q$ if p then q (implificator operator)

i) q is necessary for p

ii) q follows from p

iii) p only if q

iv) q whenever p

v) p is sufficient for q

Properties

$p \to q \equiv \sim q \to \sim p \equiv \sim p \vee q$

Contrapositive

Converse $q \to p$

inverse $\sim p \to \sim q$

$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

Laws

Idenpotent $p \vee p \equiv p$

$p \wedge p \equiv p$

Associative $(p \vee q) \vee r \equiv p \vee (q \vee r)$

Commutative $p \vee q \equiv q \vee p$

Distributive $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

Distributive $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

De Morgan's $\sim(p \vee q) \equiv \sim p \wedge \sim q$

$\sim(p \wedge q) \equiv \sim p \vee \sim q$

Absorption $p \vee (p \wedge q) \equiv p$

$p \wedge (p \vee q) \equiv p$

Formulas

$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$

$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$

$(p \to r) \vee (p \to q) \equiv p \to (q \vee r)$

$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

CNF (Conjunctive Normal Form)

if a statement is represented as conjunction of clauses.

DNF (Disjunctive Normal Form)

If a statement is represented as disjunction of statements.

Logical Implification

$p \to p \vee q$ addition

$p \wedge q \to p$ simplification

$[p \wedge (p \to q)] \to q$ Modus Ponus

$[(p \to q) \wedge \sim q] \to \sim p$ Modus Tollens

$[\sim p \wedge (p \vee q)] \to q$ Disjunctive Syllogism

$[p \to q \wedge q \to r] \to (p \to r)$ Hypothetical Syllogism

$[(p \to q) \wedge (r \to s)] \to [(p \wedge r) \to (q \wedge s)]$

$[(p \leftrightarrow q) \wedge (q \leftrightarrow r)] \to [p \leftrightarrow r]$

Methods

→ first assign propositions to all the statements. (like p, q)

→ then form all the propositional statements (like $p \vee q$)

→ then apply the logical implications to reach at conclusion.

$(p \to q) \wedge q \to p$ fallacy of affirming the conclusion.

$(p \to q) \wedge \sim p \to \sim q$ fallacy of denying the hypothesis.

Resolution Principle.

$C_1 \equiv C_1' \vee L$ $\quad C_2 = C_2' \vee \sim L$

if $C_1 \wedge C_2 \equiv$ true then $(C_1' \vee C_2') \equiv$ true also

↳ resolvent of C_1 C_2

$C_1 \wedge C_2 \to C_1' \vee C_2'$ tautology.

Method

→ Build a resolvent tree

→ compute the resolvent and add it to the tree

→ stop when no more resolverment is possible.

→ the final CNF is our resolution.

Properties

1) $S = \{ c_1 \ c_2 \ c_3 \ - - - \ c_n \}$

if False $\in$ resolvent (S)

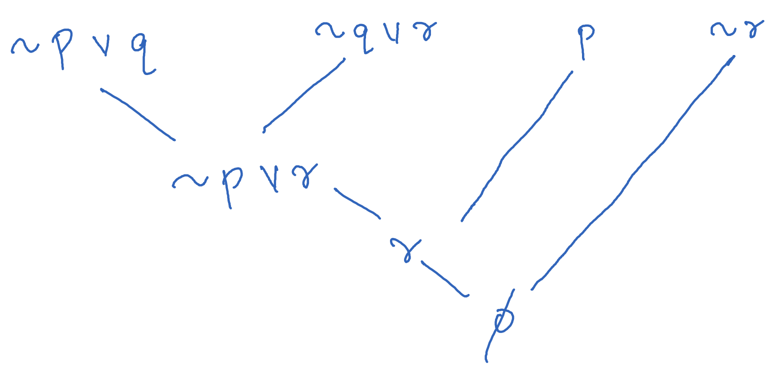
then $c_1 \wedge c_2 \wedge \dots \ c_n \equiv$ False

Proof by Resolution

$S = \{ c_1 , c_2 \ - - - - \ c_n \}$ $\quad$ C_i = clause

$c \in$ resolvent (S) iff $S \cup \{\sim c\}$ is unsatisfiable.

example
$p1 : \ p \to q \ \equiv \ \sim p \vee q$
$p2 : \ q \to r \ \equiv \ \sim q \vee r$
$p3 : \ p \to r \ \equiv \ \sim p \vee r$
$\sim p3 : \ \sim (p \to r) \ \equiv \ \sim r \wedge p$



Predicates.

generalisation / representation of statements. $P(x)$

universe of discourse / domain → set of values of x for which $P(x)$ is defined.

A predicate becomes a proposition when it is assigned a value.

Quantifiers:

1) Universal Quantifier $\forall x\, P(x)$

$\forall x\, P(x) \equiv P(x_1) \wedge P(x_2) \cdots \wedge P(x_n)$ Domain $= \{x_1, x_2 \cdots x_n\}$

2) Existential Quantifier

$\exists x\, P(x) \equiv P(x_1) \vee P(x_2) \vee \cdots P(x_n)$ Domain $= \{x_1, x_2 \cdots x_n\}$

3) Uniqueness Quantifier.

$\exists!\, x\, P(x) \iff \exists x \left[P(x) \wedge y : (P(y) \rightarrow y = x)\right]$

Imp Points

1) Every student $\forall x\, [S(x) \rightarrow P(x)]$
2) Some students $\exists x\, [S(x) \wedge P(x)]$

De Morgans Law

1) $\sim [\forall x : P(x)] \equiv \exists x : \sim P(x)$
2) $\sim [\exists x : P(x)] \equiv \forall x : \sim P(x)$

Rules Of Inference

1) Universal Instantiation

$\forall x\, P(x) \Rightarrow$ $P(c)$ is true for any arbitary element c in the universe of discourse

2) Universal Generalisation

$P(c)$ is true for any arbitary element c in domain $\Rightarrow \forall x\, P(x)$

3) Existential Generalisation

3) Existential Generalisation

$\exists x\, P(x) \Rightarrow P(c)$ is true for some arbitary element c in U

4) Existential Instantiation

$P(c)$ is true for some c in U $\Rightarrow$ $\exists x\, P(x)$

Method

→ First define domain U

→ assign predicates to statements

→ make CNFs of statements

→ Apply rules of inference to convert to propositions.

→ Apply rules of inference for propositions to reach to conclusion.

Methods of Proving

1) Direct Proof

$$[p \wedge (p \rightarrow q)] \rightarrow q$$

Show that conclusion is true assuming principle hypothesis is true.

2) Indirect Proof

A) Proof by Contrapositivity

$$p \rightarrow q \equiv \sim q \rightarrow \sim p$$

use when direct proving technique is not working

B) Vaccuous Proof

$p \rightarrow q$ if p is a false statement irrespective of q.

c) Proof by Contradiction

if prove $p \to q$

show $[(p \wedge \sim q) \to F]$ is tautology

assume $\sim q$ to be true.

Proof By Contradiction

→ to show that p is true by contradiction.

→ assume that a statement r is true. and $\sim p$ is true.

→ using p and r arrive at $\sim r$ is true.

→ $[\sim p \to (r \wedge \sim r)]$ ie $[\sim p \to F]$

$\Rightarrow \Leftarrow$

Proof Strategies:

i) $(P_1 \wedge P_2 \wedge \dots P_n) \to Q$ proof by example.

ie $(\sim Q \to \sim P_1) \vee (\sim Q \to \sim P_2) \dots \vee (\sim Q \to \sim P_n)$ is true.

2) $(P_1 \vee P_2 \vee \dots P_n) \to Q$ proof by cases

ie $(P_1 \to Q) \wedge (P_2 \to Q) \dots \wedge (P_n \to Q)$ is true

We can use Without Loss Of Generalisation (WLOG) when our cases are not given a specific value but are variable.

Non Constructive Proof.

generate such a case, such that without knowing the true value of the case we can arrive at a conclusion generate such a case, such that without knowing the true value of the case we can arrive at a conclusion

like $(\sqrt{2})^{\sqrt{2}}$ → irrational then $((\sqrt{2})^{\sqrt{2}})^{\sqrt{2}}$ is rational
→ rational.

Uniqueness Proof.

P1: there exists a sample x that satisfies the property

P2: there exists no other sample other than x which satisfies the property.

Backwards Reasoning

→ To prove that q is true.

→ device a statement p such that $p \to q$ is true.

* basically reverse engineering

Proof by Mathematical Induction.

1) Regular Induction

to prove $\forall n \; P(n)$

base case $P(b)$ is true for a specific b

inductive hypothesis $P(k)$ is true for all $k \geq b$

inductive step $P(k+1)$ is true from $P(k)$

$$\begin{array}{l} \therefore \; P(k) \to P(k+1) \quad \forall k \geq b \\ \therefore \; P(b) \\ \hline \forall n \; P(n) \quad n \geq b. \end{array}$$

$$\frac{P(b)}{\therefore \forall n\ P(n)} \quad n \geq b.$$

b) Strong Induction

base case: $P(b)$ true for a specific b

inductive hypothesis: $P(b) \wedge P(b+1) \ldots\ P(k)$ is true for all $k \geq b$

inductive step $P(k+1)$ is true.

$$\therefore \frac{P(b) \qquad \forall k\ P(b) \wedge P(b+1) \ldots P(k)}{\forall n\ P(n)}$$

Fundamental theorem of Algebra:

$\forall n \in \mathbb{Z}^{+} \qquad n = 2^a 3^b 5^c \ldots$

any positive integer n can be represented as product of powers of prime no.

SETS

1) Equality of sets $A = B$

iff $A \subseteq B$ and $B \subseteq A$

or $\forall x\ (x \in A \leftrightarrow x \in B)$ is tautology

2) Cardinallity of set = number of elements in set A $\qquad n(A) = |A|$

3) Power set $S(A)$
set of all subsets of A $\qquad |S(A)| = 2^{|A|}$

4) Subset of A $\quad A \subseteq B$

4) Subset of A $\quad A \subseteq B$

$\forall x \, (x \in A \rightarrow x \in B)$

cartesian product $\quad A \times B = \{ (a,b) : (a \in A) \wedge (b \in B) \}$

Difference of Sets

$$A - B = \{ x : x \in A \wedge x \in B \}$$

Symmetric Difference of Sets.

$$A \Delta B = (A - B) \cup (B - A)$$

RELATIONS

relation R defined on set A & B

$R \subseteq A \times B \qquad |A| = n \qquad |B| = m \qquad |R| \leq n * m$

R is defined on set A & B if $a \in A$ and $b \in B$ and $(a,b) \in R$

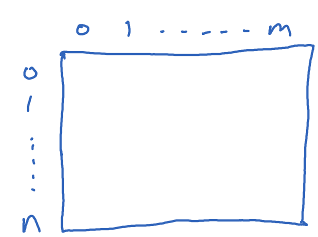
representation aRb

Matrix Representation

$|A| = m \quad |B| = n \quad$ matrix $M = m \times n$

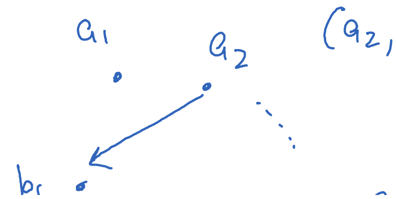
relation R defined on set A & B $\quad R \subseteq A \times B$

if $(a_i, b_j) \in R$ then $M_{i,j} = 1$ else $M_{i,j} = 0$



Binary Relations / Graph Relations.

$A = \{ a_1 \; a_2 \; \cdots \; a_m \} \qquad B = \{ b_1, b_2 \; \cdots \; b_n \}$



$(a_2, b_1) \in R$

$A = \{a_1\ a_2\ \cdots\ a_m\}$ $B = \{b_1, b_2 \ldots b_n\}$

if $(a_i, b_j) \in R$ connect vertex a_i & b_j

Vertex $V = \{a_1\ a_2 \cdots a_m, b_1, b_2 \cdots b_n\}$

Types of Relations

1) Reflexive relations

relation R defined from set A to itself

$\forall a \in A [(a,a) \in R]$ is true

diagonal elements $=1$ in matrix

graphs should have self loops.

$\forall a [a \in A \rightarrow (a,a) \in R]$

ϕ is a reflexive relation.

if $|A| = n$ then 2^{n^2-n} reflexive relations possible.

2) Irreflexive Relation

$\forall a (a \in A \rightarrow (a,a) \notin R)$

ϕ is a irreflexive relation.

3) Symmetric Relation.

R is defined from set A to set B

$\forall a \in A, \forall b \in B [(a,b) \in R \rightarrow (b,a) \in R]$

Matrix must be symmetric
graph should have loops.

4) Asymmetric Relation.

$R \subseteq A \times B$

$\forall a \in A, \forall b \in B : \{ (a,b) \in R \rightarrow (b,a) \notin R \}$

diagonal elements $= 0$ & no $M_{i,j} = M_{j,i}$

5) Anti Symmetric Relation.

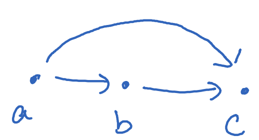
$\forall a \in A, \forall b \in B : \{ (a,b) \in R \wedge (b,a) \in R \longrightarrow b = a \}$

if $a \neq b$ then :	$(a,b) \in R$	$(b,a) \in R$
	1	0
	0	1
	0	0

+ $\emptyset$ can satisfy reflexive, symmetric, asymmetric, antisymmetric relations.

6) Transitive Relations.

if $a, b, c \in A \{ (a,b) \in R \wedge (b,c) \in R \rightarrow (a,c) \in R \}$

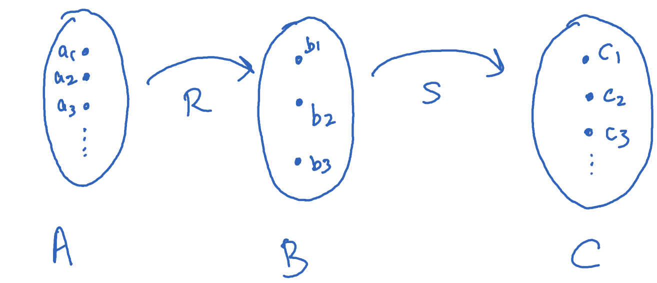


graph

Operations on Relations.

1) intersection $\cap$
2) union $\cup$
3) Difference $-$
4) ZOR $\oplus$ $A \oplus B = (A - B) \cup (B - A)$

$S \circ R$ relations



$R \subseteq A \times B$

$S \subseteq B \times C$

$R^m = R^{m-1} \circ R$

$S \circ R = \{ (a_i, c_k) : \exists\, b_j \in B \wedge (a_i, b_j) \in R \wedge (b_j, c_k) \in S \}$

$\forall m \ (a_i, a_j) \in R^m$ interpretation: there exists a path of length m from a_i to a_j in the graph

$a_i \to a' \to a'' \to a''' \to a_j$

Closure of a Relation.

1) Reflexive closure: MINIMAL superset such that all (a_i, a_i) are present in that relation.

Reflexive closure of $R = R \cup \{ (a_1, a_1), (a_2, a_2) \dots, (a_n, a_n) \}$

2) Symmetric Closure: MINIMAL superset

$R \cup \underbrace{\{ (b_j, a_i) : (a_i, a_j) \in R \}}_{R^{-1}}$ $\quad \therefore R \cup R^{-1} =$ symmetric closure.

3) Transitive closure: MINIMAL superset. is made using recursion.

Properties.

i) a relation R is transitive iff $R^n \subseteq R$

Properties:

1) a relation R is transitive iff $R^n \subseteq R$

$\Rightarrow$ $R^n \subseteq R$

$(a,b) \in R$ $(b,c) \in R$ $R^2 = R \circ R$ $\therefore$ $(a,c) \in R^2$

But $R^2 \subseteq R$ $\therefore$ $(a,c) \in R$

$\therefore$ $(a,b) \in R \wedge (b,c) \in R \rightarrow (a,c) \in R$

$\Leftarrow$ R is transitive

Base case: $n = 1$ true

Hypo: $R^n \subseteq R$ for all $1 \leq i \leq n$

step: $R^{n+1} \subseteq R$

suppose $(a,c) \in R^{n+1}$ $\therefore$ $\exists x$ st $(a,x) \in R^n$ $(x,c) \in R$

$R^n \subseteq R$ $\therefore$ $(a,x) \in R \wedge (x,c) \in R$

$(a,x) \in R \wedge (x,c) \in R \longrightarrow (a,c) \in R$

TRANSITIVE CLOSURE

Connectivity relation $R^* = R \cup R^2 \cup R^3 \dots R^{|R|}$

$(a_i, a_j) \in R^*$ iff there exists a path of any length b/w a_i & a_j

R^* is the transitive closure of R.

$S \circ R = M_R \odot M_S$ — boolean product

M_R = relation matrix of $R \subseteq A \times B$

M_S = relation matrix of $S \subseteq B \times C$

Let S = transitive set then $S^n \subseteq S$

$S^* = S \cup S^2 \cup S^3 \dots S^n$ $S^* \subseteq S$

if S is transitive then $R \subseteq S$, $R^* \subseteq S$

if $(a,b) \in R^n \Rightarrow (a,b) \in S$

as $R \subseteq S$ $\quad R^n \subseteq S^n$ $\quad R^* \subseteq S^*$

Now if $(a,b) \in R^*$ then $(a,b) \in S^*$

since $S^* \subseteq S$ $\quad (a,b) \in S$

$\therefore$ if $(a,b) \in R^*$ then $(a,b) \in S$ $\quad$ Hence R^* is the smallest superset.