Lecture 6
Friday 6 October 2023

Bessel's differential equation & Bessel's function

Bessel's equation of order n, n being a non-negative (real) constant

$$x^2 y'' + x y' + (x^2 - n^2) y = 0 \qquad (1) \qquad n \geq 0$$

Called Bessel diff equation after the name of German mathematician & astronomer Friedrich Wilhelm Bessel (1784-1846)

This equation arises in the problem finding solution of wave equation in regions possessing circular symmetry.

Use $x = 0$ as a regular singular point of (1)

Be equal to Frobenius method (method by series)

Use $y = \sum_{m=0}^{\infty} c_m x^{k+m}$ $\quad c_0 \neq 0 \quad$ assume

$$y'(x) = \sum_{m=0}^{\infty} c_m (k+m) x^{k+m-1}$$

$$y''(x) = \sum_{m=0}^{\infty} c_m (k+m)(k+m-1) x^{k+m-2} \qquad (2)$$

Putting (2) in (1), we get

$$x^2 \sum_{m=0}^{\infty} c_m (k+m)(k+m-1) x^{k+m-2} + x \sum_{m=0}^{\infty} c_m (k+m) x^{k+m-1} + (x^2 - n^2) \sum_{m=0}^{\infty} c_m x^{k+m} = 0 \qquad (3)$$

Collect the coeff of like powers of x (Ex. x^k)

$$c_0 (k)(k-1) + c_0 k + c_0 (-n^2) = 0 \qquad (4)$$

Equate to zero the coefficient of x^{k+1} we get

$$c_1 (k+1)k + c_1 (k+1) - n^2 c_1 = 0$$

$$c_1 [(k+1)^2 - n^2] = 0$$

$$c_1 = 0$$

Equate to zero the coefficient of x^{k+m}

$$c_m (k+m)(k+m-1) + c_m (k+m) - n^2 c_m + c_{m-2} = 0$$

$$\Rightarrow c_m [(k+m)^2 - n^2] = -c_{m-2}$$

$$\text{or} \quad c_m = -\frac{c_{m-2}}{(k+m)^2 - n^2}$$

putting $m = 2$ in (5)

$$c_2 = -\frac{c_0}{(k+2)^2 - n^2} \qquad (6)$$

Putting $m = 4$

$$c_4 = -\frac{c_2}{(k+4)^2 - n^2} = \frac{c_0}{[(k+2)^2 - n^2][(k+4)^2 - n^2]}$$

Now we have series in

$$y = c_0 x^k \left[1 - \frac{x^2}{(k+2)^2 - n^2} + \frac{x^4}{[(k+2)^2 - n^2][(k+4)^2 - n^2]} - \cdots\right] \qquad (7)$$

To justify the equation (7), (5) must be equal to zero, $c_0 \neq 0$ $\Rightarrow$ is the indicial equation

and $c_1 = 0$, hence

$$c_1 = 0 \quad \begin{cases} \text{if } k = n \\ \text{if } k = -n \end{cases}$$

Putting $k = n$ in (5)

$$y_1 = c_0 x^n \left[1 - \frac{x^2}{2^2(n+1)} + \frac{x^4}{2^4(n+1)(n+2)} - \cdots\right]$$

$$= c_0 2^n \left[\frac{x}{2}\right]^n \left[1 - \frac{\left(\frac{x}{2}\right)^2}{1!(n+1)} + \frac{\left(\frac{x}{2}\right)^4}{2!(n+1)(n+2)} - \cdots\right]$$

$$= \frac{c_0}{\Gamma(n+1)} \left[\frac{x}{2}\right]^n \left[1 - \frac{\left(\frac{x}{2}\right)^2}{1!(n+1)} + \frac{\left(\frac{x}{2}\right)^4}{2!(n+1)(n+2)} - \cdots\right]$$

$$\boxed{y_1 = J_n(x)}$$

$$\text{where } J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(n+m+1)} \left(\frac{x}{2}\right)^{n+2m}$$

$$\left[\frac{x}{2}\right]^n \left[1 - \frac{\left(\frac{x}{2}\right)^2}{1!(n+1)} + \frac{\left(\frac{x}{2}\right)^4}{2!(n+1)(n+2)} - \cdots\right]$$

$$\