PROOFS

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NFA TO DFA CONVERSION

If k is the number of states of the NFA, it has 2^k subsets of states. Each subset corresponds to one of the possibilities that the DFA must remember, so the DFA simulating the NFA will have 2^k states. Now we need to figure out which will be the start state and accept states of the DFA, and what will be its transition function. We can discuss this more easily after setting up some formal notation.

PROOF Let $N = (Q, \Sigma, \delta, q_0, F)$ be the NFA recognizing some language A. We construct a DFA $M = (Q', \Sigma, \delta', q_0', F')$ recognizing A. Before doing the full construction, let's first consider the easier case wherein N has no ε arrows. Later we take the ε arrows into account.

1. $Q' = \mathcal{P}(Q)$.
   Every state of M is a set of states of N. Recall that $\mathcal{P}(Q)$ is the set of subsets of Q.
2. For $R \in Q'$ and $a \in \Sigma$ let $\delta'(R, a) = \{q \in Q \mid q \in \delta(r, a) \text{ for some } r \in R\}$.
   If R is a state of M, it is also a set of states of N. When M reads a symbol a in state R, it shows where a takes each state in R. Because each state may go to a set of states, we take the union of all these sets. Another way to write this expression is
   $$\delta'(R, a) = \bigcup_{r \in R} \delta(r, a).$$ [4]
3. $q_0' = \{q_0\}$.
   M starts in the state corresponding to the collection containing just the start state of N.
4. $F' = \{R \in Q' \mid R \text{ contains an accept state of } N\}$.
   The machine M accepts if one of the possible states that N could be in at this point is an accept state.

Consideration of ε

Now we need to consider the ε arrows. To do so we set up an extra bit of notation. For any state R of M we define $E(R)$ to be the collection of states that can be reached from R by going only along ε arrows, including the members of R themselves. Formally, for $R \subseteq Q$ let

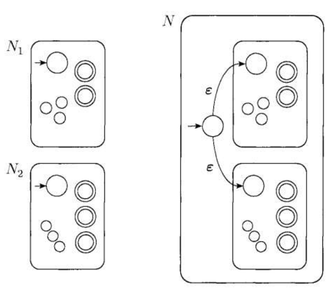
$$E(R) = \{q \mid q \text{ can be reached from } R \text{ by traveling along 0 or more } \varepsilon \text{ arrows}\}.$$

Then we modify the transition function of M to place additional fingers on all states that can be reached by going along ε arrows after every step. Replacing $\delta(r, a)$ by $E(\delta(r, a))$ achieves this effect. Thus

$$\delta'(R, a) = \{q \in Q \mid q \in E(\delta(r, a)) \text{ for some } r \in R\}.$$

Additionally we need to modify the start state of M to move the fingers initially to all possible states that can be reached from the start state of N along the ε arrows. Changing q_0' to be $E(\{q_0\})$ achieves this effect. We have now completed the construction of the DFA M that simulates the NFA N.

The construction of M obviously works correctly. At every step in the computation of M on an input, it clearly enters a state that corresponds to the subset of states that N could be in at that point. Thus our proof is complete.



proof of A∪B using NFA is easy

PROOF

Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize A_1, and
$N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognize A_2.

Construct $N = (Q, \Sigma, \delta, q_0, F)$ to recognize $A_1 \cup A_2$.

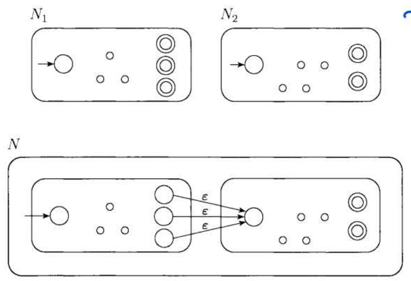
1. $Q = \{q_0\} \cup Q_1 \cup Q_2$.
   The states of N are all the states of N_1 and N_2, with the addition of a new start state q_0.
2. The state q_0 is the start state of N.
3. The accept states $F = F_1 \cup F_2$.
   The accept states of N are all the accept states of N_1 and N_2. That way N accepts if either N_1 accepts or N_2 accepts.
4. Define δ so that for any $q \in Q$ and any $a \in \Sigma_\varepsilon$,

for A∪B

$$\delta(q, a) = \begin{cases} \delta_1(q, a) & q \in Q_1 \\ \delta_2(q, a) & q \in Q_2 \\ \{q_1, q_2\} & q = q_0 \text{ and } a = \varepsilon \\ \emptyset & q = q_0 \text{ and } a \neq \varepsilon. \end{cases}$$

inside machines

at start state



A∘B using NFA

Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize A_1, and
$N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognize A_2.

Construct $N = (Q, \Sigma, \delta, q_1, F_2)$ to recognize $A_1 \circ A_2$.

1. $Q = Q_1 \cup Q_2$.
   The states of N are all the states of N_1 and N_2.
2. The state q_1 is the same as the start state of N_1.
3. The accept states F_2 are the same as the accept states of N_2.
4. Define δ so that for any $q \in Q$ and any $a \in \Sigma_\varepsilon$,

for A∘B

$$\delta(q,a) = \begin{cases} \delta_1(q,a) & q \in Q_1 \text{ and } q \notin F_1 \\ \delta_1(q,a) & q \in F_1 \text{ and } a \neq \varepsilon \\ \delta_1(q,a) \cup \{q_2\} & q \in F_1 \text{ and } a = \varepsilon \\ \delta_2(q,a) & q \in Q_2. \end{cases}$$

imp to consider all possibilities of $\delta(q,a)$

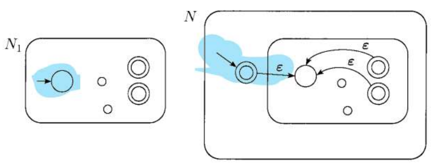


FIGURE 1.50
Construction of N to recognize A^*

PROOF Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize A_1.
Construct $N = (Q, \Sigma, \delta, q_0, F)$ to recognize A_1^*.

1. $Q = \{q_0\} \cup Q_1$.
   The states of N are the states of N_1 plus a new start state.
2. The state q_0 is the new start state.
3. $F = \{q_0\} \cup F_1$.
   The accept states are the old accept states plus the new start state.

4. Define δ so that for any $q \in Q$ and any $a \in \Sigma_\varepsilon$,

$\delta(q,a)$ for A^*

$$\delta(q,a) = \begin{cases} \delta_1(q,a) & q \in Q_1 \text{ and } q \notin F_1 \\ \delta_1(q,a) & q \in F_1 \text{ and } a \neq \varepsilon \\ \delta_1(q,a) \cup \{q_1\} & q \in F_1 \text{ and } a = \varepsilon \\ \{q_1\} & q = q_0 \text{ and } a = \varepsilon \\ \emptyset & q = q_0 \text{ and } a \neq \varepsilon. \end{cases}$$

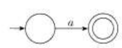
LEMMA 1.55

If a language is described by a regular expression, then it is regular.

PROOF IDEA Say that we have a regular expression R describing some language A. We show how to convert R into an NFA recognizing A. By Corollary 1.40, if an NFA recognizes A then A is regular.

PROOF Let's convert R into an NFA N. We consider the six cases in the formal definition of regular expressions.

1. $R = a$ for some a in Σ. Then $L(R) = \{a\}$, and the following NFA recognizes $L(R)$.



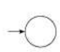
Note that this machine fits the definition of an NFA but not that of a DFA because it has some states with no exiting arrow for each possible input symbol. Of course, we could have presented an equivalent DFA here but an NFA is all we need for now, and it is easier to describe.

Formally, $N = (\{q_1, q_2\}, \Sigma, \delta, q_1, \{q_2\})$, where we describe δ by saying that $\delta(q_1, a) = \{q_2\}$ and that $\delta(r, b) = \emptyset$ for $r \neq q_1$ or $b \neq a$.

2. $R = \varepsilon$. Then $L(R) = \{\varepsilon\}$, and the following NFA recognizes $L(R)$.

Formally, $N = (\{q_1\}, \Sigma, \delta, q_1, \{q_1\})$, where $\delta(r, b) = \emptyset$ for any r and b.

3. $R = \emptyset$. Then $L(R) = \emptyset$, and the following NFA recognizes $L(R)$.



Formally, $N = (\{q\}, \Sigma, \delta, q, \emptyset)$, where $\delta(r, b) = \emptyset$ for any r and b.

4. $R = R_1 \cup R_2$.
5. $R = R_1 \circ R_2$.
6. $R = R_1^*$.

For the last three cases we use the constructions given in the proofs that the class of regular languages is closed under the regular operations. In other words, we construct the NFA for R from the NFAs for R_1 and R_2 (or just R_1 in case 6) and the appropriate closure construction.