Problem 1: Let $P_3(x)$ be the interpolating polynomial for the data $(0,0)$, $(0.5,y)$, $(1,3)$ and $(2,2)$. Find y if the coefficient of x^3 in $P_3(x)$ is 6.

Solution: We have $x_0 = 0,\ x_1 = 0.5,\ x_2 = 1,\ x_3 = 2,$ and
$f(x_0) = 0,\ f(x_1) = y,\ f(x_2) = 3,\ f(x_3) = 2.$

The Lagrange polynomial of order 3, connecting the four points, is given by

$$P_3(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2) + L_3(x)f(x_3),$$

where

$$\begin{aligned}
L_0(x) &= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)},\\
L_1(x) &= \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)},\\
L_2(x) &= \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)},\\
L_3(x) &= \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}.
\end{aligned}$$

Thus, for this problem,

$$\begin{aligned}
L_0(x) &= \frac{(x-0.5)(x-1)(x-2)}{(0-0.5)(0-1)(0-2)} = \frac{x^3 - \frac{7}{2}x^2 + \frac{7}{2}x - 1}{-1} = -x^3 + \frac{7}{2}x^2 - \frac{7}{2}x + 1,\\
L_1(x) &= \frac{(x-0)(x-1)(x-2)}{(0.5-0)(0.5-1)(0.5-2)} = \frac{x^3 - 3x^2 + 2x}{\frac{3}{8}} = \frac{8}{3}x^3 - 8x^2 + \frac{16}{3}x,\\
L_2(x) &= \frac{(x-0)(x-0.5)(x-2)}{(1-0)(1-0.5)(1-2)} = \frac{x^3 - \frac{5}{2}x^2 + x}{-\frac{1}{2}} = -2x^3 + 5x^2 - 2x,\\
L_3(x) &= \frac{(x-0)(x-0.5)(x-1)}{(2-0)(2-0.5)(2-1)} = \frac{x^3 - \frac{3}{2}x^2 + \frac{1}{2}x}{3} = \frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{6}x.
\end{aligned}$$

Thus,

$$\begin{aligned}
P_3(x) &= L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2) + L_3(x)f(x_3)\\
&= L_0(x)\cdot 0 + L_1(x)\cdot y + L_2(x)\cdot 3 + L_3(x)\cdot 2\\
&= \Big(\frac{8}{3}x^3 - 8x^2 + \frac{16}{3}x\Big)y \quad - 6x^3 + 15x^2 - 6x \quad + \frac{2}{3}x^3 - x^2 + \frac{1}{3}x\\
&= \Big(\frac{8}{3}y - 6 + \frac{2}{3}\Big)x^3 + \Big(-8y + 15 - 1\Big)x^2 + \Big(\frac{16}{3}y - 6 + \frac{1}{3}\Big)x\\
&= \Big(\frac{8y-16}{3}\Big)x^3 + \Big(-8y + 14\Big)x^2 + \Big(\frac{16y-17}{3}\Big)x.
\end{aligned}$$

Since we want the coefficient of x^3 to be equal to 6, we need:

$$\frac{8y-16}{3} = 6,$$

or $y = \frac{17}{4} = 4.25.$ ✓

With such y, the polynomial becomes

$$P_3(x) = 6x^3 - 20x^2 + 17x.$$

We can check whether this polynomial interpolates function f, that is, whether we got the correct answer. Note that

$$\begin{aligned} P_3(0) &= 0, \\ P_3(0.5) &= 4.25, \\ P_3(1) &= 3, \\ P_3(2) &= 2. \end{aligned}$$

Problem 2: Let $f(x) = e^x$ for $0 \le x \le 2$. Approximate $f(0.25)$ using linear interpolation with $x_0 = 0$ and $x_1 = 0.5$.

Solution: Linear interpolation is achieved by constructing the Lagrange polynomial P_1 of order 1, connecting the two points. We have:

$$P_1(x) = L_0(x)f(x_0) + L_1(x)f(x_1),$$

where

$$\begin{aligned} L_0(x) &= \frac{x-x_1}{x_0-x_1} = \frac{x-0.5}{-0.5}, \\ L_1(x) &= \frac{x-x_0}{x_1-x_0} = \frac{x}{0.5}. \end{aligned}$$

Thus,

$$P_1(x) = -\frac{x-0.5}{0.5}\cdot e^0 + \frac{x}{0.5}\cdot e^{0.5} = -2x+1+3.2974x = 1.2974x+1.$$

We can check whether this polynomial interpolates function f, that is, whether we got the correct answer. Note that

$$\begin{aligned} P_1(0) &= 1 = e^0, \\ P_1(0.5) &= 1.6487 = e^{0.5}. \end{aligned}$$

Now we can evaluate

$$P_1(0.25) = 1.2974 \cdot 0.25 + 1 = 1.3243, \quad ✓$$

which is an approximation of $f(0.25)$. The true value of f at $x = 0.25$ is $f(0.25) = e^{0.25} = 1.2840$. Thus, we obtained a reasonable approximation.

Problem 3: For a function f, the forward divided differences are given by

$$\begin{array}{lllll} x_0 & = & 0.0 & f[x_0] & & \\ & & & & f[x_0, x_1] & \\ x_1 & = & 0.4 & f[x_1] & & f[x_0, x_1, x_2] = \frac{50}{7} \\ & & & & f[x_1, x_2] = 10 & \\ x_2 & = & 0.7 & f[x_2] = 6 & & \end{array}$$

Determine the missing entries.

Solution: This problem is on Newton's divided differences.

The zeroth divided difference of f with respect to x_i is

$$f[x_i] = f(x_i).$$

The first divided difference of f with respect to x_i and x_{i+1} is

$$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}.$$

The second divided difference is

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}.$$

① Thus, first we find $f[x_1]$:

$$\begin{aligned} f[x_1, x_2] &= \frac{f[x_2] - f[x_1]}{x_2 - x_1}, \\ 10 &= \frac{6 - f[x_1]}{0.7 - 0.4}, \\ f[x_1] &= \mathbf{3}. \quad \checkmark \end{aligned}$$

② We find $f[x_0, x_1]$:

$$\begin{aligned} f[x_0, x_1, x_2] &= \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}, \\ \frac{50}{7} &= \frac{10 - f[x_0, x_1]}{0.7 - 0.0}, \\ f[x_0, x_1] &= \mathbf{5}. \quad \checkmark \end{aligned}$$

③ We now find $f[x_0]$:

$$\begin{aligned} f[x_0, x_1] &= \frac{f[x_1] - f[x_0]}{x_1 - x_0}, \\ 5 &= \frac{3 - f[x_0]}{0.4 - 0.0}, \\ f[x_0] &= \mathbf{1}. \quad \checkmark \end{aligned}$$

Note that steps ① and ② could be interchanged. However, step ③ could only be done last.

Problem 4: Let $i_0, i_1, \ldots, i_n$ be a rearrangement of the integers $0, 1, \ldots, n$. Show that $f[x_{i_0}, x_{i_1}, \ldots, x_{i_n}] \;=\; f[x_0, x_1, \ldots, x_n]$.

Solution: Let P_c and P_d be two polynomials, such that P_c interpolates f at $x_0, x_1, \ldots, x_n$ and P_d interpolates f at $x_{i_0}, x_{i_1}, \ldots, x_{i_n}$:

$$\begin{aligned} P_c &= c_0 + c_1(x - x_0) \;+\; \ldots \;+\; c_n(x - x_0)(x - x_1)\cdots(x - x_{n-1}), \\ P_d &= d_0 + d_1(x - x_{i_0}) \;+\; \ldots \;+\; d_n(x - x_{i_0})(x - x_{i_1})\cdots(x - x_{i_{n-1}}), \end{aligned}$$

We can rewrite the polynomials above as

$$\begin{aligned} P_c &= c_n x^n \;+\; \text{lower order terms}, \\ P_d &= d_n x^n \;+\; \text{lower order terms}. \end{aligned}$$

Since P_c and P_d were defined to be in the form of Newton's polynomials, we know that c_n and d_n are nth divided differences, $c_n = f[x_0, x_1, \ldots, x_n]$ and $d_n = f[x_{i_0}, x_{i_1}, \ldots, x_{i_n}]$:

$$\begin{aligned} P_c &= f[x_0, x_1, \ldots, x_n]x^n \;+\; \text{lower order terms}, \\ P_d &= f[x_{i_0}, x_{i_1}, \ldots, x_{i_n}]x^n \;+\; \text{lower order terms}. \end{aligned}$$

We also know that the polynomial interpolating the same nodes is unique, that is, $P_c = P_d$. Thus,

$$f[x_0, x_1, \ldots, x_n] \;=\; f[x_{i_0}, x_{i_1}, \ldots, x_{i_n}]. \quad \checkmark$$

Problem 5: Give explicit formulas for $f[a]$, $f[a,b]$, $f[a,b,c]$ in terms of $f(a)$, $f(b)$, $f(c)$.
Optional: Give an explicit formula for $f[x, x+h, x+2h, \ldots, x+nh]$.

Solution:
Similar to problem 3, we can write the Newton's divided difference formulas as:

$$\begin{aligned}
f[a] &= f(a), \quad \checkmark \\
f[a,b] &= \frac{f[b]-f[a]}{b-a} \\
&= \frac{f(b)-f(a)}{b-a}, \quad \checkmark \\
f[a,b,c] &= \frac{f[b,c]-f[a,b]}{c-a} \\
&= \frac{\frac{f(c)-f(b)}{c-b} - \frac{f(b)-f(a)}{b-a}}{c-a} \\
&= \frac{(f(c)-f(b))(b-a)-(f(b)-f(a))(c-b)}{(c-b)(b-a)(c-a)}. \quad \checkmark
\end{aligned}$$

Optional: Note that

$$\begin{aligned}
f[x] &= f(x), \\
f[x,x+h] &= \frac{f(x+h)-f(x)}{h}, \\
f[x,x+h,x+2h] &= \frac{f[x+h,x+2h]-f[x,x+h]}{2h} \\
&= \frac{\frac{f(x+2h)-f(x+h)}{h} - \frac{f(x+h)-f(x)}{h}}{2h} \\
&= \frac{f(x+2h)-2f(x+h)+f(x)}{2h^2}, \\
f[x,x+h,x+2h,x+3h] &= \frac{f[x+h,x+2h,x+3h]-f[x,x+h,x+2h]}{3h} \\
&= \frac{\frac{f(x+3h)-2f(x+2h)+f(x+h)}{2h^2} - \frac{f(x+2h)-2f(x+h)+f(x)}{2h^2}}{3h} \\
&= \frac{f(x+3h)-3f(x+2h)+3f(x+h)-f(x)}{6h^3},
\end{aligned}$$

By observation, we have

$$f[x,x+h,x+2h,\ldots,x+nh] = \frac{f(x+nh)-nf(x+(n-1)h)+\ldots \pm nf(x+h) \mp f(x)}{n!\, h^n}.$$

The exact signs of $\pm$ and $\mp$ depend on whether n is even or odd.

Let $f(x) = 3^x$ for every $x \in \mathbb{R}$.

(a) Use Lagrange interpolation to find a polynomial $p(x)$ of degree at most two that agrees with this function at the points $x_0 = 0$, $x_1 = 1$, and $x_2 = 2$. (Do not simplify!)

(b) Find a bound on $|f(x) - p(x)|$ for each $x \in [0, 2]$.

Solution (a). The Lagrange interpolation polynomials for the points $x_0 = 0$, $x_1 = 1$, and $x_2 = 2$ are

$$\begin{aligned}
\ell_0(x) &= \frac{(x-1)(x-2)}{(-1)(-2)} = \frac{(x-1)(x-2)}{2}, \\
\ell_1(x) &= \frac{(x-0)(x-2)}{(1)(-1)} = x(2-x), \\
\ell_2(x) &= \frac{(x-0)(x-1)}{(2)(1)} = \frac{x(x-1)}{2}.
\end{aligned}$$

The Lagrange interpolation formula therefore yields

$$\begin{aligned}
p(x) &= f(x_0)\ell_0(x) + f(x_1)\ell_1(x) + f(x_2)\ell_2(x) \\
&= f(0)\frac{(x-1)(x-2)}{2} + f(1)x(2-x) + f(2)\frac{x(x-1)}{2} \\
&= 3^0\frac{(x-1)(x-2)}{2} + 3x(2-x) + 3^2\frac{x(x-1)}{2} \\
&= \frac{(x-1)(x-2)}{2} + 3x(2-x) + 9\frac{x(x-1)}{2}.
\end{aligned}$$

□

Solution (b). Because f is thrice differentiable over $\mathbb{R}$, the Lagrange Interpolation Remainder Theorem states that for every $x \in [0, 2]$ there exists some $z_x \in (0, 2)$

$$f(x) - p(x) = \frac{1}{3!}f'''(z_x)x(x-1)(x-2).$$

Because

$$\begin{aligned}
f'(x) &= \log(3)3^x, \\
f''(x) &= (\log(3))^2 3^x,
\end{aligned}$$

(1) Let $f(x) = 3^x$ for every $x \in \mathbb{R}$.

(a) Use Lagrange interpolation to find a polynomial $p(x)$ of degree at most two that agrees with this function at the points $x_0 = 0$, $x_1 = 1$, and $x_2 = 2$. (Do not simplify!)

(b) Find a bound on $|f(x) - p(x)|$ for each $x \in [0, 2]$.

Solution (a). The Lagrange interpolation polynomials for the points $x_0 = 0$, $x_1 = 1$, and $x_2 = 2$ are

$$\ell_0(x) = \frac{(x-1)(x-2)}{(-1)(-2)} = \frac{(x-1)(x-2)}{2},$$
$$\ell_1(x) = \frac{(x-0)(x-2)}{(1)(-1)} = x(2-x),$$
$$\ell_2(x) = \frac{(x-0)(x-1)}{(2)(1)} = \frac{x(x-1)}{2}.$$

The Lagrange interpolation formula therefore yields

$$\begin{aligned} p(x) &= f(x_0)\ell_0(x) + f(x_1)\ell_1(x) + f(x_2)\ell_2(x) \\ &= f(0)\frac{(x-1)(x-2)}{2} + f(1)x(2-x) + f(2)\frac{x(x-1)}{2} \\ &= 3^0\frac{(x-1)(x-2)}{2} + 3x(2-x) + 3^2\frac{x(x-1)}{2} \\ &= \frac{(x-1)(x-2)}{2} + 3x(2-x) + 9\frac{x(x-1)}{2}. \end{aligned}$$

□

Solution (b). Because f is thrice differentiable over $\mathbb{R}$, the Lagrange Interpolation Remainder Theorem states that for every $x \in [0, 2]$ there exists some $z_x \in (0, 2)$

$$f(x) - p(x) = \frac{1}{3!}f'''(z_x)x(x-1)(x-2).$$

Remark. We can obtain a uniform error bound over $[0,2]$ by using calculus to find the maximum value of $|x(x-1)(x-2)|$ over $[0,2]$. This maximum value is found to be $\frac{2}{9}\sqrt{3}$, whereby

$$\begin{aligned}|f(x)-p(x)| &< \tfrac{3}{2}(\log(3))^3|x(x-1)(x-2)| \\ &\le \tfrac{1}{3}\sqrt{3}(\log(3))^3\,.\end{aligned}$$

We can obtain a much cruder uniform error bound without calculus by noting that $|x| \le 2$, $|x-1| \le 1$, and $|x-2| \le 2$ for every $x \in [0,2]$, whereby

$$|f(x)-p(x)| < \tfrac{3}{2}(\log(3))^3|x(x-1)(x-2)| \le 6(\log(3))^3\,.$$

This uniform bound is $6\sqrt{3}$ times larger than the previous one. This kind of crude uniform bound is presented in the book.

(2) Let $f(x) = 3^x$ for every $x \in \mathbb{R}$. Let $p(x)$ be the polynomial of degree at most two that agrees with this function at the points $x_0 = 0$, $x_1 = 1$, and $x_2 = 2$.
   (a) Use divided differences to construct $p(x)$.
   (b) Use the Neville algorithm to evaluate $p(\frac{1}{2})$.

Solution (a). The one-point divided differences are

$$f[x_0] = 3^0 = 1\,, \qquad f[x_1] = 3^1 = 3\,, \qquad f[x_2] = 3^2 = 9\,.$$

The two-point divided differences are

$$\begin{aligned}f[x_0,x_1] &= \frac{f[x_1]-f[x_0]}{x_1-x_0} = \frac{3-1}{1} = 2\,,\\ f[x_1,x_2] &= \frac{f[x_2]-f[x_1]}{x_2-x_1} = \frac{9-3}{1} = 6\,.\end{aligned}$$

The three-point divided difference is

$$f[x_0,x_1,x_2] = \frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0} = \frac{6-2}{2} = 2\,.$$

Therefore the divided-difference table is

x_i	$f[x_i]$	$f[x_i,x_{i+1}]$	$f[x_0,x_1,x_2]$
0	1		
		2	
1	3		2
		6	
2	9		

The coefficients for the Newton forward difference formula (as read off from the top entry of each column) are 1, 2, and 2, whereby

$$p(x) = 1 + 2x + 2x(x-1)\,.$$

Alternatively, the coefficients for the Newton backward difference formula (as read off from the bottom entry of each column) are 9, 6, and 2, whereby

$$p(x) = 9 + 6(x-2) + 2(x-2)(x-1)\,.$$

Either answer is correct. They are just different ways to write the same polynomial — namely, the polynomial $p(x) = 1 + 2x^2$. $\square$

It is easy to check that $LU = A$. □

(6) For a function f the Newton divided-difference table is

x_i	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_0, x_1, x_2]$
0	0		
		3	
1	?		3
		?	
2	?		

(a) Determine the missing entries in the table.
(b) Give the interpolating polynomial $p(x)$.

Solution (a). Because

$$3 = f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{f[x_1, x_2] - 3}{2 - 0} = \tfrac{1}{2} f[x_1, x_2] - \tfrac{3}{2},$$

we see that $f[x_1, x_2] = 9$. Because

$$3 = f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = \frac{f[x_1] - 0}{1 - 0} = f[x_1],$$

we see that $f[x_1] = 3$. Because

$$9 = f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1} = \frac{f[x_2] - 3}{2 - 1} = f[x_2] - 3,$$

we see that $f[x_2] = 12$. Therefore the table is

x_i	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_0, x_1, x_2]$
0	0		
		3	
1	3		3
		9	
2	12		

□

Solution (b). The interpolating polynomial $p(x)$ is

$$p(x) = 0 + 3 \cdot x + 3 \cdot x(x - 1) = 3x^2 .$$

□

Remark. It is easier to do part (b) first. Then we can read off that $f[x_1] = p(1) = 3$, $f[x_2] = p(2) = 12$, and

$$f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1} = \frac{12 - 3}{2 - 1} = 9 .$$

9. For the function $f(x) = \cos x$, let $x_0 = 0$,
$x_1 = 0.6$, $x_2 = 0.9$.

The basic Lagrange polynomials are:

$$l_0(x) = \frac{(x-0.6)(x-0.9)}{(0-0.6)(0-0.9)}$$

$$l_1(x) = \frac{x(x-0.9)}{0.6(0.6-0.9)}$$

$$l_2(x) = \frac{x(x-0.6)}{0.9(0.9-0.6)}$$

$$\therefore\ p(x) = l_0(x) + \cos(0.6)\, l_1(x) + \cos(0.9)\, l_2(x)$$

$$\therefore\ p(0.45) = 0.898100747$$

$$\cos(0.45) = 0.9004471024$$

Actual error at $x = 0.45$:

$$\cos(0.45) - p(0.45) \approx 0.0023$$

$\boxed{P-1}$

Error bound:

$$E_2(x,f) = \frac{f'''(\xi(x))}{3!}\, x(x-0.6)(x-0.9)$$

We want to compute

$$\max_x |E_2(x,f)|$$

Step I: Find $\max_{a \le x \le b} |f'''(x)|$

$\therefore f(x) = \cos x$; $f'(x) = -\sin x$

$f''(x) = -\cos x$

Now, $|\sin x|$ is increasing over $[0, 0.9]$.

Hence: $\max_{a \le x \le b} |\sin(x)| \le |\sin(0.9)|$.

Step II: Find $\max_{a \le x \le b} |x(x-0.6)(x-0.9)|$

Let $g(x) = x(x-0.6)(x-0.9)$

~~Let~~ $\therefore g'(x) = 3(x^2 - x + 0.18) = 0$

$\Rightarrow p_1 = 0.2354248689$

and $p_2 = 0.7645751311$

$\therefore |g(x)| \le |g(p_1)| = 0.05704$

and $|g(p_2)| = 0.01704 05784$.

$$\therefore |E_2(x, f)| = \frac{|\sin(\xi x)|}{6} |x(x-0.6)(x-0.9)|$$

$$\le \frac{|\sin 0.9|}{6} 0.05704$$

$$= 0.0074468.$$

10.

10. $f(x) = \sin(\ln x)$

Soln: We construct the basic Lagrange polynomials:

$$l_0(x) = \frac{(x-2.4)(x-2.6)}{(2-2.4)(2-2.6)}$$

$$l_1(x) = \frac{(x-2)(x-2.6)}{(2.4-2)(2.4-2.6)}$$

$$l_2(x) = \frac{(x-2)(x-2.4)}{(2.6-2)(2.6-2.4)}$$

$$\therefore\ p(x) = \sin(\ln 2)\, l_0(x) + \sin(\ln 2.4)\, l_1(x) + \sin(\ln 2.6)\, l_2(x).$$

The error is given by:

$$E_2(x, f) = \frac{f'''(\xi(x))}{3!}(x-2)(x-2.4)(x-2.6)$$

Now, $f'''(x) = \dfrac{3\sin(\ln x) + \cos(\ln x)}{x^3}$.

$$\max_{2 \le x \le 2.6} |f'''(x)| \le |f'''(2)|$$

$$= \frac{3\sin(\ln 2) + \cos(\ln 2)}{8}$$

$$= 0.335765.$$

Next we bound $|g(x)|$.

$$g(x) = (x-2)(x-2.4)(x-2.6).$$

To find the maximum of $|g(x)|$, we need to take the derivative:

$$g'(x) = 3x^2 - 14x + 16.24 = 0$$

$$\Rightarrow p_1 = 2.157 \quad \text{and} \quad p_2 = 2.5$$

$$|g(x)| \le |g(2.157)| = 0.0169$$

$$|g(2.5)| = 0.005$$

$\boxed{P-5}$

$$|E_2(x;f)| = \frac{|f'''(\xi_x)|}{3!}\,|(x-2)(x-2.4)(x-2.6)|$$

$$\leq \frac{0.335765}{6} \times 0.0169$$

$$= 9.457 \times 10^{-4}$$