(1)

ADI Method : (This part is taken from E. Kreyszig, 6th Edition)

A matrix is called tridiagonal matrix if it has all its nonzero entries on the main diagonal and on the two sloping parallels immediately above or below the diagonal. In this case Gauss elimination is particularly simple.

In the solution of Dirichlet problem for the Laplace or Poisson eq^n is it possible to obtain a system of equations whose coefficient matrix is tridiagonal.

Yes, in this direction a popular method known as 'ADI Method' (alternating direction implicit method) was developed by Paceman and Rachford.

Idea: The stencil in $\left\{\begin{matrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{matrix}\right\}$ shows ~~that~~ we could obtain a ~~tridiagonal~~ tridiagonal matrix if there were only three points in a row (or only three points in a column). This suggests ~~that~~ we write, the five point formulae,

$$u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{ij} = 0 \quad (1)$$

in the form,

$$u_{i-1,j} - 4u_{i,j} + u_{i+1,j} = -u_{i,j-1} - u_{i,j+1} \quad (2)$$

so that the left side belongs to y-Row j only and the right side to x-Column i.

② Indeed, we can also write ① in the form,

$$\cancel{u_{i-1,j} - 4u_{i,j}}$$

$$u_{i,j-1} - 4u_{i,j} + u_{i,j+1} = -u_{i-1,j} - u_{i+1,j} \quad \text{③}$$

so that the left side belongs to Column i and the right side to Row j.

In the ADI method we proceed by iteration. In one step we use iteration formulae resulting from ② and in the next we use the one resulting from ③ and so on in alternating order.

To illustrate more, suppose u_{ij}^{m} has already been computed. To obtain u_{ij}^{m+1} we solve,

$$u_{i-1,j}^{m+1} - 4u_{ij}^{m+1} + u_{i+1,j}^{m+1} = -u_{i,j-1}^{m} - u_{i,j+1}^{m} \quad \text{④}$$

— Note that we use ④ for a fixed j. i.e. row is fixed.

We solve ④ by Gauss-elimination.

Then we go to the next row, obtain another system of N equations and solve it by Gauss and so on, until all rows are done.

In the Next step we alternate direction, (ie) we compute the next approximation u_{ij}^{m+2} column by column from the u_{ij}^{m+1} and the given boundary values, using a formula obtained from ③:

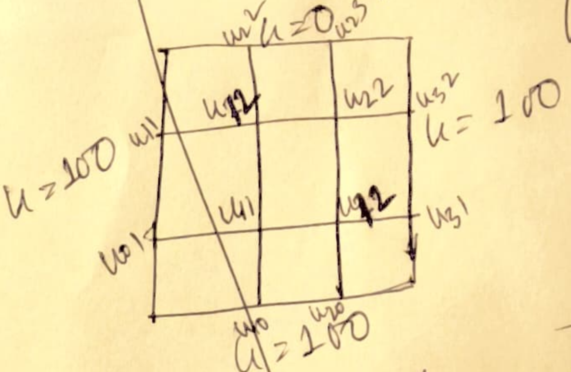
$$u_{i,j-1}^{m+2} - 4u_{ij}^{m+2} + u_{i,j+1}^{m+2} = -u_{i+1,j}^{m+1} - u_{i-1,j}^{m+1} \quad (5)$$

For each fixed i, that is, for each column, this is a system of M equations (M = number of internal mesh points per column) in M unknowns, which we solve by Gauss elimination.

Then we go to the next column and so on, until the columns are done.

Example (Dirichlet problem. ADI Method)

Recall Example 1 from notes of last week.



Take the same starting value

$$\boxed{u_{11}^0 = u_{21}^0 = u_{22}^0 = u_{21}^0 = 100}$$

From (4), for $m=0$, $j=1$ (first row) we have:

$$\boxed{\begin{array}{ll} \cancel{(i=1)} & \cancel{u_{01} - 4u_{11} + u_{21}} = -u_{21}^{(0)} - u_{10} \\ (i=2) & \cancel{u_{11} - 4u_{21} + u_{31} = -} \end{array}}$$

$$\begin{array}{ll} (i=1) & u_{01} - 4u_{11}^1 + u_{21}^1 = -u_{10} - u_{21}^0 \\ (i=2) & u_{11}^1 - 4u_{21}^1 + u_{31} = -u_{20} - u_{22}^0 \end{array}$$

(ie) from last step

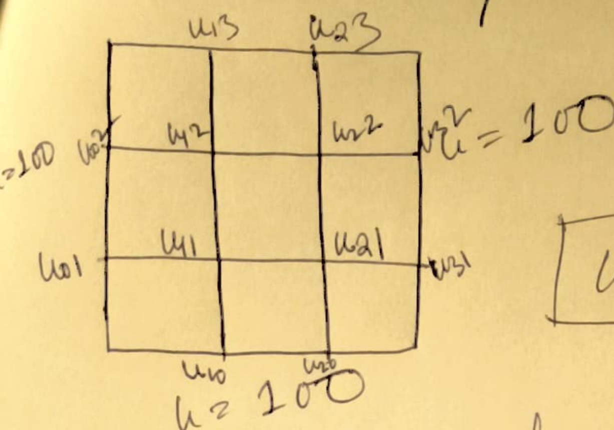
$$-4u_{11}^1 + u_{21}^1 = -100 - 100 - 100$$

$$u_{11}^1 - 4u_{21}^1 = -100 - 100 - 100$$

$$\Rightarrow u_{11}^1 = u_{21}^1 = 100$$

Example (Dirichlet Problem, ADI Method)

Recall Example 1 from notes of last week



Take the same initial values.

$$\boxed{u_{11}^0 = u_{21}^0 = u_{22}^0 = u_{21}^0 = 100}$$

From ④, for $m=0$, $j=1$ (first row) we have:

$(i=1)$ $\quad u_{01} - 4u_{11}^1 + u_{21}^1 = -u_{10} - u_{12}^0$

$(i=2)$ $\quad u_{11}^1 - 4u_{21}^1 + u_{31} = -u_{20} - u_{22}^0$

$$\Rightarrow \quad -4u_{11}^1 + u_{21}^1 = -100-100-100=-300$$

$$u_{11}^1 - 4u_{21}^1 = -100-100-100=-300$$

$$\Rightarrow \quad u_{11}^1 = u_{21}^1 = 100.$$

From ④, for $m=0$, $j=2$ (second row) we obtain:

$(i=1)$ $\quad u_{02} - 4u_{12}^1 + u_{22}^1 = -u_{13} - u_{11}^0$

$(i=2)$ $\quad u_{12}^1 - 4u_{22}^1 + u_{32} = -u_{23} - u_{21}^0$

$$\Rightarrow \quad -4u_{12}^1 + u_{22}^1 = -0-100-100=-200$$

$$u_{12}^1 - 4u_{22}^1 = -0-100-100=-200$$

$$\Rightarrow \quad u_{12}^1 = u_{22}^1 = 66.667.$$

Second approximations:

$u_{11}^2, u_{21}^2, u_{12}^2, u_{22}^2$ will now be obtained from (5) with $m=1$ by using the first approximations just computed and the boundary values.

For $i=1$ (first column) we have:

$(j=1)$ $\quad u_{10} - 4u_{11}^2 + u_{12}^2 = -u_{01} - u_{21}^1$

$(j=2)$ $\quad u_{11}^2 - 4u_{12}^2 + u_{13} = -u_{02} - u_{22}^1$

(ie) $\quad -4u_{11}^2 + u_{12}^2 = -100 - 100 - 100 = -300$

$\quad u_{11}^2 - 4u_{12}^2 = -100 - 66.667 - 0 = -166667$

$\Rightarrow$ $u_{11}^2 = 91.11, \quad u_{12}^2 = 64.44$

Now, applying (5) with $m=1$ and $i=2$ (2nd column) we have,

$(j=1)$ $\quad u_{20} - 4u_{21}^2 + u_{22}^2 = -u_{11}^1 - u_{31}$.

$(j=2)$ $\quad u_{21}^2 - 4u_{22}^2 + u_{23} = -u_{12}^1 - u_{32}$.

$\Rightarrow$ $u_{21}^2 = 91.11, \quad u_{22}^2 = 64.44$

Hence first approximation:

Method	u_{11}	u_{21}	u_{12}	u_{22}
ADI	91.11	91.11	64.44	64.44

$\rightarrow$ First approximation