MA203: Module 3 - Partial Differential Equations

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These notes have largely been taken from the textbook *Advanced Engineering Mathematics* by Erwin Kreyszig (10th Edition, John Wiley & Sons, 2011).

1 Wave equation

1.0.1 d'Alembert's solution of the one-dimensional wave equation

Consider the one-dimensional wave equation

$$u_{tt} = c^2 u_{xx}. \tag{1}$$

Let us consider the change of variables $\xi = x - ct$ and $\eta = x + ct$. This implies that

$$\frac{\partial \xi}{\partial x} = 1, \qquad \frac{\partial \xi}{\partial t} = -c, \qquad \frac{\partial \eta}{\partial x} = 1, \qquad \frac{\partial \eta}{\partial t} = c.$$

We assume that all the partial derivatives of u involved are continuous, and apply the chain rule to obtain

$$\begin{aligned}
u_x &= u_\xi \frac{\partial \xi}{\partial x} + u_\eta \frac{\partial \eta}{\partial x} = u_\xi + u_\eta, \\
u_t &= u_\xi \frac{\partial \xi}{\partial t} + u_\eta \frac{\partial \eta}{\partial t} = u_\xi(-c) + u_\eta \times c = c(-u_\xi + u_\eta), \\
u_{xx} &= \frac{\partial (u_\xi + u_\eta)}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial (u_\xi + u_\eta)}{\partial \eta}\frac{\partial \eta}{\partial x} = u_{\xi\xi} + u_{\xi\eta} + u_{\eta\xi} + u_{\eta\eta} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}, \\
u_{tt} &= c\left[\frac{\partial (-u_\xi + u_\eta)}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial (-u_\xi + u_\eta)}{\partial \eta}\frac{\partial \eta}{\partial t}\right] = c\big[(-u_{\xi\xi} + u_{\xi\eta})(-c) + (-u_{\eta\xi} + u_{\eta\eta})c\big] \\
&= c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}).
\end{aligned}$$

Substituting these in the one-dimensional wave equation (1), we obtain

$$c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}) = c^2(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}) \qquad \Longrightarrow \qquad \boxed{u_{\xi\eta} = 0}$$

We shall now integrate both sides of the above equation with respect to ξ and η successively. On integrating with respect to ξ, we obtain

$$u_\eta = \bar{\psi}(\eta),$$

where $\bar{\psi}$ is an arbitrary function of η only. Next, integrating this equation with respect to η, we obtain

$$u(\xi, \eta) = \int \bar{\psi}(\eta)\,\mathrm{d}\eta + \phi(\xi),$$

where ϕ is an arbitrary function of ξ only. Writing $\displaystyle\int \bar{\psi}(\eta)\,\mathrm{d}\eta$ as $\psi(\eta)$, we obtain the solution of the wave equation as

$$u(\xi, \eta) = \phi(\xi) + \psi(\eta), \tag{2}$$

where ϕ and ψ are two arbitrary functions of ξ and η, respectively. Rewriting ξ and η in terms of x and t, we obtain from (2)

$$\boxed{u(x, t) = \phi(x - ct) + \psi(x + ct)} \tag{3}$$

This is referred to as the *d'Alembert's solution* of the one-dimensional wave equation (1).

Example (d'Alembert's solution satisfying the initial conditions)**:** Determine the solution of the wave equation

$$u_{tt} = c^2 u_{xx}, \qquad t > 0, \quad -\infty < x < \infty \tag{4}$$

in an infinite one-dimensional medium subject to the initial conditions

$$u(x,0) = f(x), \quad u_t(x,0) = g(x) \quad \text{for} \quad -\infty < x < \infty. \tag{5}$$

Solution: We know that the general solution of the wave equation (4) is given by the d'Alembert's solution

$$u(x,t) = \phi(x - ct) + \psi(x + ct). \tag{6}$$

Therefore,

$$u_t(x,t) = -c\,\phi'(x - ct) + c\,\psi'(x + ct), \tag{7}$$

where prime denotes the derivatives with respect to the entire arguments $x - ct$ and $x + ct$, respectively.

Let us now apply the given initial conditions. The condition $u(x,0) = f(x)$ implies

$$\phi(x) + \psi(x) = f(x) \tag{8}$$

and $u_t(x,0) = g(x)$ implies the condition

$$-c\,\phi'(x) + c\,\psi'(x) = g(x). \tag{9}$$

Dividing the above equation by $(-c)$ and integrating with respect to x, we obtain

$$\phi(x) - \psi(x) = -\frac{1}{c}\int_{x_0}^{x} g(s)\,\mathrm{d}s + k, \tag{10}$$

where $k = \phi(x_0) - \psi(x_0)$. Adding (8) and (10), and dividing the resulting equation by 2, we obtain

$$\phi(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_{x_0}^{x} g(s)\,\mathrm{d}s + \frac{k}{2}. \tag{11}$$

Substituting this in (8), we obtain

$$\psi(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_{x_0}^{x} g(s)\,\mathrm{d}s - \frac{k}{2}. \tag{12}$$

From (11) and (12),

$$\phi(x - ct) = \frac{1}{2}f(x - ct) - \frac{1}{2c}\int_{x_0}^{x-ct} g(s)\,\mathrm{d}s + \frac{k}{2}, \tag{13}$$

$$\psi(x + ct) = \frac{1}{2}f(x + ct) + \frac{1}{2c}\int_{x_0}^{x+ct} g(s)\,\mathrm{d}s - \frac{k}{2}. \tag{14}$$

From the above equations, the solution of the given problem is

$$\begin{aligned} u(x,t) &= \phi(x - ct) + \psi(x + ct) \\ &= \frac{1}{2}\big[f(x - ct) + f(x + ct)\big] + \frac{1}{2c}\left(\int_{x-ct}^{x_0} g(s)\,\mathrm{d}s + \int_{x_0}^{x+ct} g(s)\,\mathrm{d}s\right) \end{aligned}$$

or

$$\boxed{u(x,t) = \frac{1}{2}\big[f(x - ct) + f(x + ct)\big] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,\mathrm{d}s} \tag{15}$$

Remark: Notice that if the initial velocity is zero, solution (15) reduces to

$$u(x,t) = \frac{1}{2}\big[f(x-ct) + f(x+ct)\big].$$

Example (d'Alembert's solution satisfying the initial conditions)**:** Determine the solution of the wave equation

$$u_{tt} = c^2 u_{xx}, \qquad t > 0, \quad -\infty < x < \infty \tag{16}$$

in an infinite one-dimensional medium subject to the initial conditions

$$u(x,0) = 0, \quad u_t(x,0) = \sin x \quad \text{for} \quad -\infty < x < \infty. \tag{17}$$

Solution: From the previous example, the general solution of the wave equation (16) with the initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$ is given by the d'Alembert's solution

$$u(x,t) = \frac{1}{2}\big[f(x-ct) + f(x+ct)\big] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,\mathrm{d}s \tag{18}$$

Here, $f(x) = 0$ and $g(x) = \sin x$. Therefore, the solution for the given problem is

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \sin s\,\mathrm{d}s = \frac{1}{2c}(-\cos s)\Big|_{x-ct}^{x+ct} = \frac{1}{2c}\Big[\cos(x-ct) - \cos(x+ct)\Big].$$

Example: Let $y(x,t)$ represents transverse displacement in a long stretched string one end of which is attached to a ring (of negligible diameter and weight) that can slide along the y-axis. The other end is so far out on the positive x-axis that it may be considered to be infinitely far from the origin. The ring is initially at the origin and is then moved along the y-axis (see figure 1) so that $y = f(t)$ when $x = 0$ and $t \geq 0$, where f is a given continuous function with $f(0) = 0$. Assume that the string is initially at rest on the x-axis.

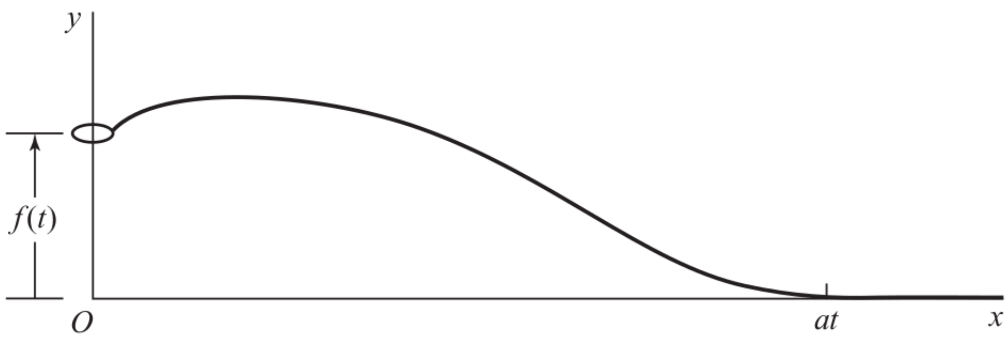

Figure 1: Schematic of the string in the given problem.

(a) Write down the boundary value problem describing the given problem.
Hint: Look at the figure carefully; the boundary value problem should include 'a' in it.

(b) The general solution of the partial differential equation identified in part (a) is given by $y(x,t) = \phi(x+at)+\psi(x-at)$, where ϕ and ψ are two arbitrary functions that need to be determined using the given conditions. Apply the *initial* condition(s) identified in part (a) to show that there is a constant c such that

$$\phi(x) = c \quad \text{and} \quad \psi(x) = -c \qquad (x \geq 0).$$

Then apply the *boundary* condition(s) identified in part (a) to show that

$$\psi(-x) = f\left(\frac{x}{a}\right) - c \qquad (x \geq 0),$$

where c is the same constant.

(c) With the aid of the results obtained in part (b), show that the solution of given problem is

$$y(x,t) = \begin{cases} 0 & x \geq at, \\ f\left(t - \dfrac{x}{a}\right) & x \leq at. \end{cases}$$

(d) What can you infer from this solution about the displacement in the string due to the movement of the ring?

Solution: (a) The boundary value problem, which describes the given problem is as follows.

$$y_{tt}(x,t) = a^2\, y_{xx}(x,t) \qquad (x > 0, \quad t > 0), \tag{19}$$

$$y(x,0) = 0 \quad \text{and} \quad y_t(x,0) = 0 \qquad (x \geq 0), \tag{20}$$

$$y(0,t) = f(t) \qquad (t \geq 0), \tag{21}$$

where $y(x,t)$ is the transverse displacement in the string and a is the wave speed in the string. The initial conditions are given by (20) and the boundary condition by (24).

(b) The general solution of (19) is given by (the d'Alembert's solution)

$$y(x,t) = \phi(x+at) + \psi(x-at). \tag{22}$$

This implies that

$$y_t(x,t) = a\,\phi'(x+at) - a\,\psi'(x-at) = a\big[\phi'(x+at) - \psi'(x-at)\big]. \tag{23}$$

Let us now apply the initial conditions (20), which hold for $x \geq 0$.

$$y(x,0) = 0 \quad \Longrightarrow \quad \phi(x) + \psi(x) = 0 \quad \Longrightarrow \quad \phi(x) = -\psi(x), \tag{24}$$

$$y_t(x,0) = 0 \quad \Longrightarrow \quad a\big[\phi'(x) - \psi'(x)\big] = 0 \quad \Longrightarrow \quad \phi'(x) = \psi'(x). \tag{25}$$

Differentiating (24) with respect to x and adding in (25), we obtain

$$\phi'(x) = 0 \quad \Longrightarrow \quad \phi(x) = c,$$

where c is a constant of integration. With this, eq. (24) implies $\psi(x) = -c$. Thus, applying after applying the initial conditions (20), we have

$$\boxed{\phi(x) = c} \quad \text{and} \quad \boxed{\psi(x) = -c} \qquad (x \geq 0). \tag{26}$$

Let us now apply the boundary condition (21), which holds for $t \geq 0$.

$$y(0,t) = f(t) \quad \implies \quad \phi(at) + \psi(-at) = f(t).$$

Let us apply the change of variable $at = x$ to obtain

$$\phi(x) + \psi(-x) = f\left(\frac{x}{a}\right) \qquad (x \geq 0).$$

After applying the initial conditions, we obtained $\phi(x) = c$. Therefore, the above equation yields

$$\boxed{\psi(-x) = f\left(\frac{x}{a}\right) - c} \qquad (x \geq 0). \tag{27}$$

(c) The results obtained in part (b) above are (26) and (27) and they hold for $x \geq 0$. Note from the general solution (22) that we need to determine the values of $\phi(x+at)$ and $\psi(x-at)$. From (26) and (27), it is clear that $\phi(x) = c$ for all $x \geq 0$. Therefore, $\phi(x+at) = c$ for all $x \geq 0$ because $x + at \geq x \geq 0$.

Now, we need to determine $\psi(x - at)$ from (26) and (27). From (26), we have

$$\psi(x-at) = -c \qquad (x - at \geq 0 \quad \text{or} \quad x \geq at) \tag{28}$$

and from (27), we have

$$\psi(x-at) = f\left(\frac{at-x}{a}\right) - c = f\left(t - \frac{x}{a}\right) - c \qquad (at - x \geq 0 \quad \text{or} \quad x \leq at). \tag{29}$$

Consequently, the general solution $y(x,t) = \phi(x+at) + \psi(x-at)$ of the above problem is

$$y(x,t) = c + (-c) = 0 \quad \text{for} \quad x \geq at$$

and

$$y(x,t) = c + \left[f\left(t - \frac{x}{a}\right) - c\right] = f\left(t - \frac{x}{a}\right) \quad \text{for} \quad x \leq at.$$

Combining the above two results, the solution of the given problem is

$$y(x,t) = \begin{cases} 0 & x \geq at, \\ f\left(t - \frac{x}{a}\right) & x \leq at. \end{cases}$$

(d) The solution of the given problems reveals that the part of the string to the right of the point $x = at$ on the x-axis is unaffected by the movement of the ring <u>prior to time t</u>, as also shown in figure 1.

[Lecture 10]

1.1 Vibration of a circular membrane

Circular membranes are encountered in many engineering applications, such as in drums, pumps, microphones, etc. Whenever a circular membrane is plane and its material is elastic, but offers no resistance to bending (e.g., not a metallic membrane!), its vibrations are governed by the two-dimensional wave equation. Since the membrane is circular, it is convenient to use the polar coordinates defined by $x = r\cos\theta$ and $y = r\sin\theta$.

The two-dimensional wave equation in the polar coordinates reads

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right). \tag{30}$$

We shall consider a membrane of radius R with fixed end (figure 2) and determine solutions $u(r,t)$ that are radially symmetric (i.e., those solutions which do not depend on θ). In this case, $u_{\theta\theta} = 0$ and the two-dimensional wave equation (30) reduces to

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right). \tag{31}$$

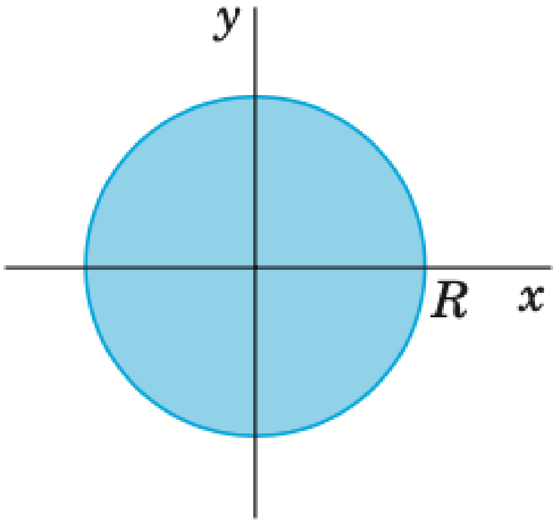

Figure 2: Circular membrane

Boundary Condition: Since the membrane is fixed along the boundary $r = R$, we have the boundary condition

$$u(R,t) = 0, \qquad \text{for all} \quad t \geq 0. \tag{32}$$

Initial Conditions: We can obtain radially symmetric solutions only if the initial conditions do not depend on θ. Let us assume that the initial deflection in the membrane is $f(r)$ and the initial velocity of the membrane is $g(r)$. Therefore, the initial conditions are

$$u(r,0) = f(r) \quad \text{and} \quad u_t(r,0) = g(r), \qquad 0 \leq r \leq R. \tag{33}$$

We would like to solve the reduced wave equation (31) along with the boundary condition (32) and the initial conditions (33) using the method of separation of variables.

Solution: First Step: To find the ordinary differential equations:
Let the solution of this problem be $u(r,t) = W(r)\,T(t)$. Therefore, it satisfies (31). Substituting this ansatz in (31), we obtain

$$W\,\ddot{T} = c^2\left[W''\,T + \frac{1}{r}W'\,T\right], \tag{34}$$

where the time derivative has been denoted with dots while the spacial derivative has been denoted by primes. The above equation can be written as

$$\frac{1}{c^2}\frac{\ddot{T}}{T} = \frac{1}{W}\left(W'' + \frac{1}{r}W'\right). \tag{35}$$

Again, the left-hand side of the above equation is a function of t alone while the right-hand side is a function of r alone. Therefore, both of them must be equal to a constant, let us say, it is k. This gives two ordinary differential equations:

$$\frac{\mathrm{d}^2T}{\mathrm{d}t^2} - kc^2T = 0 \qquad \text{and} \qquad W'' + \frac{1}{r}W' - kW = 0. \tag{36}$$

The equation for $T(t)$ has solutions which grow or decay exponentially for $k > 0$, are linear or constant for $k = 0$, and are periodic for $k < 0$. Physically, it is expected that a solution to the problem of a vibrating membrane will be oscillatory in time, and this leaves only the third case $k < 0$; let $k = -\beta^2$, $\beta \neq 0$. With this, the above ordinary differential equations become

$$\frac{\mathrm{d}^2T}{\mathrm{d}t^2} + \lambda^2T = 0 \qquad \text{and} \qquad rW'' + W' + \beta^2rW = 0, \tag{37}$$

where $\lambda = \beta c$. The equation for W can be reduced to the Bessel equation, which is $x^2y'' + xy' + (x^2 - \nu^2)y = 0$ by a scaling $s = \beta r$. With this scaling,

$$\begin{aligned}
W' &= \frac{\mathrm{d}W}{\mathrm{d}r} = \frac{\mathrm{d}W}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}r} = \beta\frac{\mathrm{d}W}{\mathrm{d}s},\\
W'' &= \frac{\mathrm{d}^2W}{\mathrm{d}r^2} = \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}W}{\mathrm{d}r}\right)\frac{\mathrm{d}s}{\mathrm{d}r} = \frac{\mathrm{d}}{\mathrm{d}s}\left(\beta\frac{\mathrm{d}W}{\mathrm{d}s}\right)\beta = \beta^2\frac{\mathrm{d}^2W}{\mathrm{d}s^2},
\end{aligned}$$

and the equation for W becomes

$$\begin{aligned}
\beta^2r\frac{\mathrm{d}^2W}{\mathrm{d}s^2} + \beta\frac{\mathrm{d}W}{\mathrm{d}s} + \beta^2rW = 0 \quad &\Longrightarrow \quad \beta^2r^2\frac{\mathrm{d}^2W}{\mathrm{d}s^2} + \beta r\frac{\mathrm{d}W}{\mathrm{d}s} + \beta^2r^2W = 0\\
&\Longrightarrow \quad s^2\frac{\mathrm{d}^2W}{\mathrm{d}s^2} + s\frac{\mathrm{d}W}{\mathrm{d}s} + s^2W = 0,
\end{aligned} \tag{38}$$

which is the Bessel equation with $\nu = 0$.

Second Step: Satisfying the boundary condition:
The boundary condition $u(R,t) = 0$ leads to $W(R)\,T(t) = 0$ and, hence, to

$$W(R) = 0 \tag{39}$$

because $T(t) = 0$ will result into the zero solution which is meaningless.

Solution of the Bessel equation (38) are the Bessel functions $J_0(s)$ and $Y_0(s)$ of the first and second kind, respectively. It turns out that $Y_0(s)$ becomes infinite at $s = 0$; therefore $Y_0(s)$ cannot be a part of the solution because the deflection of the membrane must always be finite. This leaves us with the solution $W(s) = J_0(s)$ or, in other words,

$$W(r) = J_0(\beta r). \tag{40}$$

The boundary condition (39) implies that

$$J_0(\beta R) = 0 \tag{41}$$

We can satisfy this condition because $J_0(s)$ has infinitely many positive zeros, $s = \alpha_1, \alpha_2, \alpha_3, \dots$ (see figure 3), with numerical values

$$\alpha_1 = 2.4048, \quad \alpha_2 = 5.5201, \quad \alpha_3 = 8.6537, \quad \alpha_4 = 11.7915, \quad \alpha_5 = 14.9309$$

and so on. These zeros are slightly irregularly spaced, as we can see in the figure.

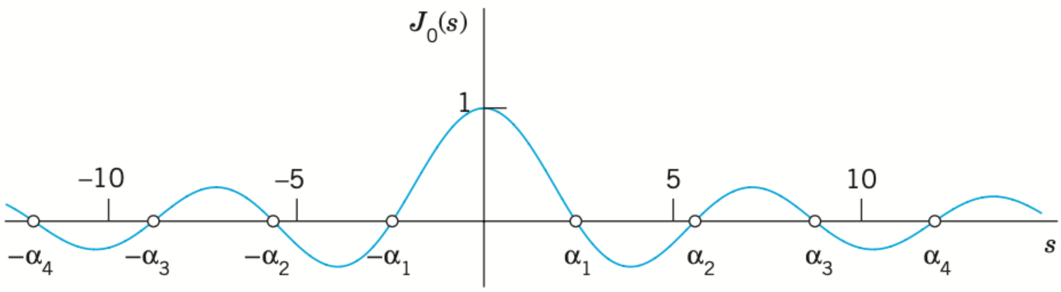

Figure 3: Bessel function $J_0(s)$

Equation (41) now implies that

$$\beta R = \alpha_n \quad \Longrightarrow \quad \beta = \beta_n = \frac{\alpha_n}{R}, \qquad n = 1, 2, 3, \dots \tag{42}$$

Hence, the functions

$$W_n(r) = J_0(\beta_n r) = J_0\left(\frac{\alpha_n}{R} r\right), \qquad n = 1, 2, 3, \dots \tag{43}$$

are solutions of (37) that vanish at $r = R$.

Eigenfunctions and eigenvalues: For W_n in (43), a corresponding general solution of $(37)_1$ with $\lambda = \lambda_n = \beta_n c = \alpha_n c/R$ is

$$T_n(t) = c_n \cos \lambda_n t + d_n \sin \lambda_n t. \tag{44}$$

Hence the functions

$$u_n(x,t) = W_n(r)\, T_n(t) = \Big(c_n \cos \lambda_n t + d_n \sin \lambda_n t\Big)\, J_0(\beta_n r) \tag{45}$$

with $n = 1, 2, 3, \dots$ are solutions of the wave equation (31) satisfying the boundary condition (32). These are the *eigenfunctions* of our problem. The corresponding *eigenvalues* are λ_n.

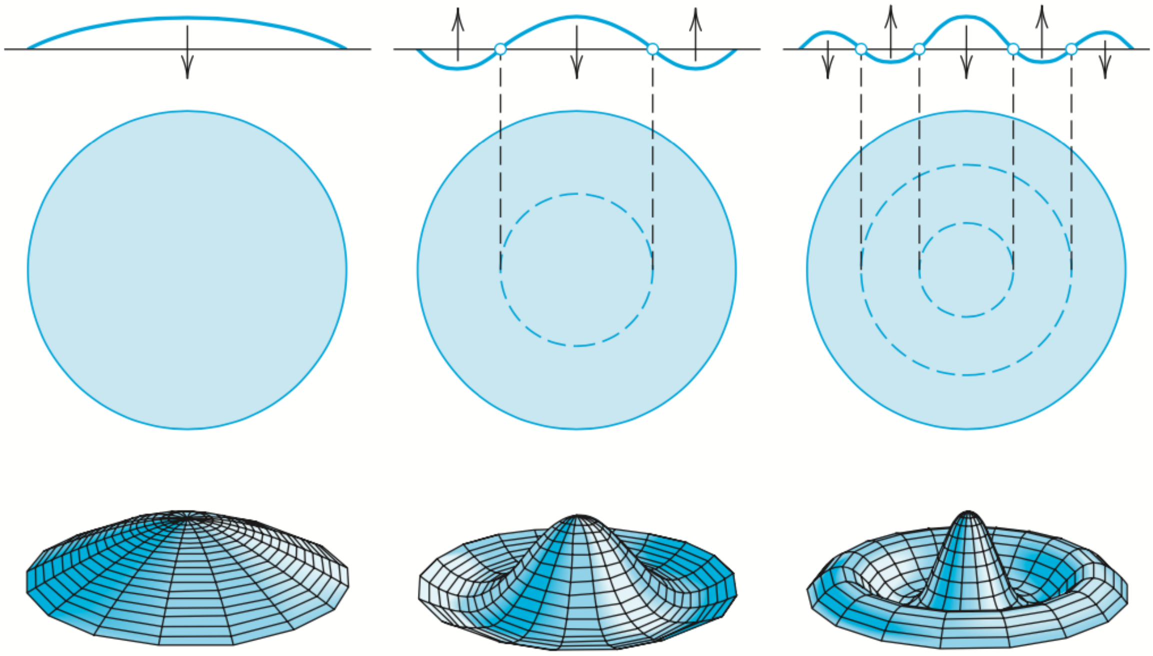

Figure 4: Normal modes of the circular membrane in the case of vibrations independent of the angle θ

Since the zeros of the Bessel function J_0 are not regularly spaced on the axis, the sound of a drum is entirely different from that of a violin. The forms of the normal modes can be easily obtained from figure 3 and are shown in figure 4.

Third Step: Solution of the entire problem: From (45), the most general solution of the wave equation (31) that satisfies the given boundary condition (32) is

$$u(r,t) = \sum_{n=1}^{\infty} u_n(r,t) = \sum_{n=1}^{\infty} \Big(c_n \cos \lambda_n t + d_n \sin \lambda_n t\Big)\, J_0\left(\frac{\alpha_n}{R} r\right). \tag{46}$$

Let us now apply the initial conditions (33). The initial condition $u(r,0) = f(r)$ gives

$$f(r) = \sum_{n=1}^{\infty} c_n\, J_0\left(\frac{\alpha_n}{R} r\right). \tag{47}$$

To obtain the coefficients c_n, we shall use the orthogonality of the Bessel functions, which is given by

$$\int_0^1 x\, J_p(ax)\, J_p(bx)\, \mathrm{d}x = \begin{cases} 0 & \text{if} \quad a \neq b \\ \dfrac{1}{2} J_{p+1}^2(a) & \text{if} \quad a = b, \end{cases} \tag{48}$$

where p is a non-negative integer, and a and b are the zeros of $J_p(x)$.

To use the orthogonality of the Bessel functions for the problem under consideration, let us replace p with 0, x with r/R, a with α_m and b with α_n. With this, the above orthogonality relation changes to

$$\int_0^R r\, J_0\left(\frac{\alpha_m}{R} r\right)\, J_0\left(\frac{\alpha_n}{R} r\right)\, \mathrm{d}r = \begin{cases} 0 & \text{if} \quad m \neq n \\ \dfrac{1}{2} R^2 J_1^2(\alpha_m) & \text{if} \quad m = n. \end{cases} \tag{49}$$

Now, to find the coefficients c_n in (47), we shall use the orthogonality relation (49). For that, let us multiply both sides of (47) with $rJ_0\left(\frac{\alpha_m}{R}r\right)$ for some fixed m $(m = 1, 2, 3, \dots)$ and integrate both sides of the resulting equation with respect to r in $(0, R)$. Using the orthogonality relation (49), the only term that will be nonzero in the right-hand side will be for $n = m$ and we would have

$$\int_0^R r\, f(r)\, J_0\left(\frac{\alpha_m}{R}r\right)\, \mathrm{d}r = \frac{1}{2}R^2 J_1^2(\alpha_m)c_m, \qquad m = 1, 2, 3, \dots \tag{50}$$

or

$$\boxed{c_m = \frac{2}{R^2 J_1^2(\alpha_m)} \int_0^R r\, f(r)\, J_0\left(\frac{\alpha_m}{R}r\right)\, \mathrm{d}r} \qquad m = 1, 2, 3, \dots \tag{51}$$

Let us now apply the remaining initial condition $u_t(r, 0) = g(r)$. For that we first differentiate solution (46) partially with respect to t to obtain

$$u_t(r, t) = \sum_{n=1}^{\infty} \lambda_n \Big(-c_n \sin \lambda_n t + d_n \cos \lambda_n t \Big)\, J_0\left(\frac{\alpha_n}{R}r\right). \tag{52}$$

The initial condition $u_t(r, 0) = g(r)$ gives

$$g(r) = \sum_{n=1}^{\infty} \lambda_n d_n\, J_0\left(\frac{\alpha_n}{R}r\right). \tag{53}$$

To obtain the coefficients d_n, let us multiply both sides of (53) with $rJ_0\left(\frac{\alpha_m}{R}r\right)$ for some fixed m $(m = 1, 2, 3, \dots)$ and integrate both sides of the resulting equation with respect to r in $(0, R)$. Using the orthogonality relation (49), the only term that will be nonzero in the right-hand side will be for $n = m$ and we would have

$$\int_0^R r\, g(r)\, J_0\left(\frac{\alpha_m}{R}r\right)\, \mathrm{d}r = \frac{1}{2}R^2 J_1^2(\alpha_m)\lambda_m d_m, \qquad m = 1, 2, 3, \dots \tag{54}$$

or

$$\boxed{d_m = \frac{2}{\alpha_m c R J_1^2(\alpha_m)} \int_0^R r\, g(r)\, J_0\left(\frac{\alpha_m}{R}r\right)\, \mathrm{d}r} \qquad m = 1, 2, 3, \dots, \tag{55}$$

where the relation $\lambda_m = \alpha_m c/R$ has been used. Therefore, the deflection in a (radially symmetric) vibrating membrane fixed at the boundary and satisfying the initial conditions (33) is given by

$$\boxed{u(r, t) = \sum_{n=1}^{\infty} \left[c_n \cos\left(\frac{\alpha_n c}{R}t\right) + d_n \sin\left(\frac{\alpha_n c}{R}t\right)\right]\, J_0\left(\frac{\alpha_n}{R}r\right)} \tag{56}$$

where

$$\boxed{c_n = \frac{2}{R^2 J_1^2(\alpha_n)} \int_0^R r\, f(r)\, J_0\left(\frac{\alpha_n}{R}r\right)\, \mathrm{d}r} \qquad n = 1, 2, 3, \dots \tag{57}$$

and

$$\boxed{d_n = \frac{2}{\alpha_n c R J_1^2(\alpha_n)} \int_0^R r\, g(r)\, J_0\left(\frac{\alpha_n}{R}r\right)\, \mathrm{d}r} \qquad n = 1, 2, 3, \dots \tag{58}$$