Analytic Function

A function $f(x)$ defined in an interval I containing the point a i.e. $(x=a)$ is said to be analytic at $x=a$ if it can be expressed as power series of $(x-a)$ with Radius of Convergence $R>0$

i.e. $$f(x) = \sum_{n=0}^{\infty} \alpha_n (x-a)^n$$

OR

$f(x)$ is analytic at $x=a$ if it's Taylor's series $\sum_{m=0}^{\infty} f^m(a) \frac{(x-a)^m}{m!}$ exists and converges to $f(x)$.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots = 1 + (x-0) + \frac{(x-0)^2}{2!} + \frac{(x-0)^3}{3!} + \dots$$

Sin x
Cos x

ex $f(x) = x+3$ is it analytic at $x=0$

$f(0) = 3$

$f(x) = \frac{1}{x+2}$ is it analytic at $x=-2$?

$f(-2) = \frac{1}{0} = \infty$

So, $\frac{f(x)}{g(x)}$ is analytic if $g(x) \neq 0$

ex $f(x) = \frac{x}{x^2-5x+6} = \frac{x}{(x-2)(x-3)}$

Ordinary Points

Consider the n^{th} order linear differential equation

$$y^n(x) + P_{n-1}(x) y^{n-1}(x) + P_{n-2}(x) y^{n-2}(x) + \dots + P_0(x) y(x) = f(x) \quad \text{--- (1)}$$

$$\boxed{y^n(x) = \frac{d^n y}{dx^n}}$$

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Def A point $x = x_0$ is called an ordinary point of the given differential equation (1) if each of the coefficients $P_0(x), P_1(x), P_2(x), \ldots, P_{n-1}(x)$ & $f(x)$ are analytic at $x = x_0$ i.e.

$P_i(x)$, $i = 0, 1, 2, \ldots n-1$, and $f(x)$ can be expressed as power series about $x = x_0$ that converges for $|x - x_0| < R$, $R > 0$

Singular Points

If $P_{n-1}, P_{n-2}, \ldots$ gives infinity at $x = x_0$ then $x = x_0$ is not an ordinary point.

Def A point $x = x_0$ is said to be singular point of (1) if it is not an ordinary point. i.e., not all of the coefficients $P_0, P_1, \ldots P_{n-1}(x)$ are analytic at $x = x_0$

Singular Point

- Regular Singular Point: A point $x = x_0$ is called R.S.P. if both $(x-x_0)P_1(x)$ and $(x-x_0)^2 P_2(x)$ possess derivative of all order in nbd of $x = x_0$
- Irregular Singular point: If $(x-x_0)P_1(x)$ & $(x-x_0)^2 P_2(x)$ gives infinity value at $x = x_0$ then x_0 is Ir.S.P.

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0$$

$$\Rightarrow \frac{d^2y}{dx^2} + \frac{a_1(x)}{a_0(x)}\frac{dy}{dx} + \frac{a_2(x)}{a_0(x)}y = 0$$

$$\Rightarrow \boxed{\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0}$$

where $P_1(x) = \dfrac{a_1(x)}{a_0(x)}$

$P_2(x) = \dfrac{a_2(x)}{a_0(x)}$

ex $(x^2-1)^2 y''(x) + (x+1)y'(x) - y(x) = 0$

$$y'' + P_1(x)y' + P_2(x)y = 0$$

Sol^n $P_1(x) = \dfrac{x+1}{(x^2-1)^2}$ $\quad P_2(x) = \dfrac{-1}{(x^2-1)^2}$

$\therefore$ $x = \pm 1$ are singular points of the given equation.

For $x = 1$

$$\lim_{x\to 1} (x-1) P_1(x) = \lim_{x\to 1} \frac{(x-1)(x+1)}{(x+1)^2 (x-1)^2}$$

$$= \text{does not exist}$$

$$\lim_{x\to 1} (x-1)^2 P_2(x) = \lim_{x\to 1} -(x-1)^2 \frac{1}{(x^2-1)^2}$$

$$= -\frac{1}{4}$$

$\therefore$ $x = 1$ is Ir. S. P.

Vanishing of all coefficients

If a power series has a (+)ve radius of convergence and a sum that is identically zero throughout it's interval of convergence then each coefficient of the series must be zero.

$$f(x) = \sum_{n=0}^{\infty} c_n (x-x_0)^n \quad , \quad |x-x_0| < R$$

Now it is given $f(x) = 0$ $\forall x$ satisfy $|x-x_0| < R$

$$c_n = \frac{f^n(x_0)}{n!} \quad \forall \; n = 0, 1, 2, \ldots$$

Since $f(x) = 0$ $\forall x$ we have $f^n(x_0) = 0$ $\forall$ $n = 0, 1, 2, 3, \ldots$

$\therefore$ $c_n = 0$, $\forall$ $n = 0, 1, 2, 3$

Q.

$$\frac{d^2 x(t)}{dt^2} + x(t) = 0 \quad \text{—— (1)}$$

Sol.

$$x(t) = \sum_{n=0}^{\infty} c_n t^n \qquad |t| < R$$

$$x'(t) = \sum_{n=1}^{\infty} n c_n t^{n-1}$$

$$x''(t) = \sum_{n=2}^{\infty} n(n-1)\, c_n t^{n-2}$$

$$\sum_{n=2}^{\infty} n(n-1)\, c_n t^{n-2} + \sum_{n=0}^{\infty} c_n t^n = 0$$

Let us replace $n = j+2$

$$\sum_{j=0}^{\infty} (j+1)(j+2)\, C_{j+2}\, t^{j} + \sum_{n=0}^{\infty} c_n t^n = 0$$

$j = n$

$$\sum_{n=0}^{\infty} \left\{ (n+1)(n+2)\, C_{n+2} + C_n \right\} t^n = 0 \qquad \forall\ t \text{ satisfy } |t| < R$$

$$(n+1)(n+2)\, C_{n+2} + C_n = 0$$

$$\Rightarrow \quad C_{n+2} = -\frac{c_n}{(n+1)(n+2)}$$