MA 203 Complex Analysis and Differential Equations-II

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Lecture-3

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Contents

- Classification of linear second order PDE's in two variables.
- Laplace equation using separation of variables.

Why PDE?

- Partial differential equations are vital in mathematically oriented scientific fields, such as physics and engineering. For instance, they are foundational in the modern scientific understanding of sound, heat, diffusion, fluid dynamics, elasticity, general relativity, and quantum mechanics (Schrödinger equation etc.).

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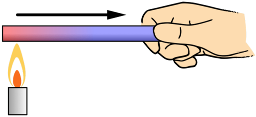

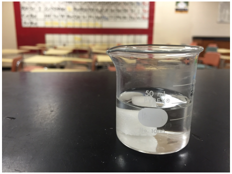

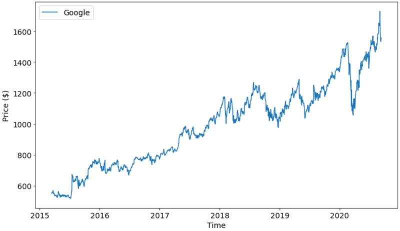

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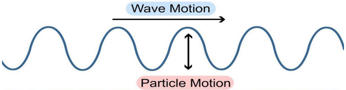

Definition 1

A *partial differential equation* (PDE) is an equation involving one or more partial derivatives of an (unknown) function, let us say u, that depends on two or more variables, often time t and one or several variables in space.

The independent variables will be denoted by x and y, while the dependent variable by u, i.e., by $u = u(x, y)$.

Example 2

$$\frac{\partial u}{\partial t} = u\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^3 u}{\partial z^3}, \quad u_{xx} + u\, u_y + u_{yz} = x^2 + y^2 + u$$

Definition 3

The **order** of a partial differential equation is defined as the order of the highest partial derivative occuring in the partial differential equation.

Definition 4

A PDE is said to be **linear** if the dependent variable and its partial derivatives occur only in the first degree and are not multiplied. A partial differential equation which is not linear is called a **non-linear**.

Definition 5

A PDE is said to be **semilinear** if the highest order terms are linear and the coefficients of the highest order derivatives are functions of independent variables only.

Definition 6

A PDE is said to be **quasi-linear** if the highest derivative power is linear but coefficients of highest order derivatives involve the dependent variable u or its lower order derivative.

Example 7

1. Linear PDE: $a(x,y)u_x + b(x,y)u_y + c(x,y)u = f(x,y)$
2. Semi-linear PDE: $a(x,y)u_x + b(x,y)u_y = f(x,y,u)$
3. Quasi-linear PDE: $a(x,y,u)u_x + b(x,y,u)u_y = f(x,y,u)$

Definition 8

A **linear** PDE is said to be *homogeneous* if each of its terms contains either the unknown function u or one of its partial derivatives. Otherwise, the PDE is called *nonhomogeneous* or *inhomogeneous*.

Example 9

(i)	$\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$	**One-dimensional wave equation**
(ii)	$\dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}$	**One-dimensional heat equation**
(iii)	$\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$	**Two-dimensional Laplace equation**
(iv)	$\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = f(x, y)$	**Two-dimensional Poisson equation**
(v)	$\dfrac{\partial^2 u}{\partial t^2} = \left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right)$	**Two-dimensional wave equation**
(vi)	$\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$	**Three-dimensional Laplace equation**

PDEs (i)–(iii), (v) and (vi) are homogeneous while (iv) is nonhomogeneous for $f(x, y) \neq 0$.

Remark 1

Second-order PDEs are the most important ones in applications. Our syllabus contains only linear second-order homogeneous PDEs in two variables. These are one-dimensional wave equation, one-dimensional heat equation and two-dimensional Laplace equation.

A PDE is formed by two methods.

- **Method 1: By eliminating arbitrary constants:** **Find the PDE of all sphere whose centre lie on z-axis and given by equations** $x^2 + y^2 + (z - a)^2 = b^2$**; a,b being constants.**

- **Method 2: By eliminating arbitrary functions:**

Proof.

We have,

$$x^2 + y^2 + (z-a)^2 = b^2. \tag{1}$$

(1) **contains two arbitrary constants a and b. Differentiating** (1) **partially with respect to x, we get**

$$2x + 2(z-a)\frac{\partial z}{\partial x} = 0 \implies \boxed{x + (z-a)p = 0} \quad \text{where} \quad p = \frac{\partial z}{\partial x}. \tag{2}$$

Again differentiating (1) **partially with respect to y, we get**

$$2y + 2(z-a)\frac{\partial z}{\partial y} = 0 \implies \boxed{y + (z-a)q = 0} \quad \text{where} \quad q = \frac{\partial z}{\partial y}. \tag{3}$$

(2) $\times\, q -$ (3) $\times\, p$, **we get**

$$xp - yq = 0 \implies \boxed{x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = 0}. \tag{4}$$

This represents PDE of all spheres whose centre lie on z-axis. □

Method 2: By eliminating arbitrary functions: Form the PDE from $z = f(x^2 - y^2)$.

Proof.

Differentiating the above equation partially with respect to x and y, we get

$$p = \frac{\partial z}{\partial x} = f'(x^2 - y^2)2x \tag{5}$$

$$q = \frac{\partial z}{\partial y} = f'(x^2 - y^2)(-2y). \tag{6}$$

Dividing (5) **by** (6) **we get**

$$\frac{p}{q} = -\frac{x}{y} \implies y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 0. \tag{7}$$

□

Definition 10

A *quasi-linear PDE* of order one, which is of the form $Pp + Qq = R$, where P, Q and R are functions of x, y and z. Such a PDE is known as Lagrange equation. For example, $xyp + yzq = zx$ is a Lagrange equation.

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- **Step I: Put the given linear PDE of the first order in the standard form:**

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- **Step IV: The general solution (or integral) of** (8) **is then written in one of the following forms:**

$$\phi(u, v) = 0, \quad u = \phi(v) \quad \textbf{or} \quad v = \phi(u), \quad \phi \text{ being an arbitrary function.}$$

Solve the PDE: yzp-xzq=xy.

- **Step I: By comparing this with Lagrange's equation ($Pp + Qq = R$), we get $P = yz, Q = -xz, R = xy$.**
- **Step II: The auxiliary equations are**

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \implies \frac{dx}{yz} = \frac{dy}{-xz} = \frac{dz}{xy}.$$

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- **Step III: Now taking first and second component** $\frac{dx}{yz} = \frac{dy}{-xz} \implies xdx + ydy = 0.$
  Integrating on both sides, $\boxed{x^2 + y^2 = a.}$ **Let the first solution be** $u(x, y, z) = a$. **Therefore,** $x^2 + y^2 = u$.

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  Integrating on both sides, $\boxed{x^2 + y^2 = a.}$ **Let the first solution be** $u(x, y, z) = a$. **Therefore,** $x^2 + y^2 = u$. **Now taking the second and third component** $\frac{dy}{-xz} = \frac{dz}{xy} \implies ydy + zdz = 0$. **Integrating both sides,** $\boxed{y^2 + z^2 = b}$. **Let the second solution be** $v(x, y, z) = b$. **Then,** $y^2 + z^2 = v$. **The general solution** $f(x^2 + y^2, y^2 + z^2) = 0$, **f being an arbitrary function.**

Classification of linear second-order PDEs in two variables

- **The general second-order linear PDE has the following form:**

$$A\, u_{xx} + B\, u_{xy} + C\, u_{yy} + D\, u_x + E\, u_y + F\, u = G, \tag{10}$$

where the coefficients A, B, C, D, F and the free term G are in general functions of the independent variables x and y, but do not depend on the unknown function u.

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- **The classification of second-order equations depends on the form of the leading part of the equations consisting of the second-order terms. So, for simplicity of notation, we combine the lower-order terms and rewrite the above equation in the following form**

$$A\,u_{xx} + B\,u_{xy} + C\,u_{yy} + I(x, y, u, u_x, u_y) = 0. \tag{11}$$

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- **The type of the above equation depends on the sign of the quantity**

$$\Delta(x, y) = B^2(x, y) - 4A(x, y)\,C(x, y), \tag{12}$$

which is called the *discriminant* **for** (11)**.**

The classification of second-order linear PDEs is given by the following.

Definition 11

At the point (x_0, y_0), the second-order linear PDE (11) is called

(i) *elliptic*, if $\Delta(x_0, y_0) < 0$

(ii) *parabolic*, if $\Delta(x_0, y_0) = 0$

(iii) *hyperbolic*, if $\Delta(x_0, y_0) > 0$

Remark 2

1. For each of these categories, equation (11) and its solutions have distinct features.
2. In general, a second order equation may be of one type at a specific point, and of another type at some other point.
3. The terminology is motivated from the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

which—for A, B, C, D, E, F being constants—represents a conic section in the xy-plane and the different types of conic sections arising are determined by $B^2 - 4AC$.