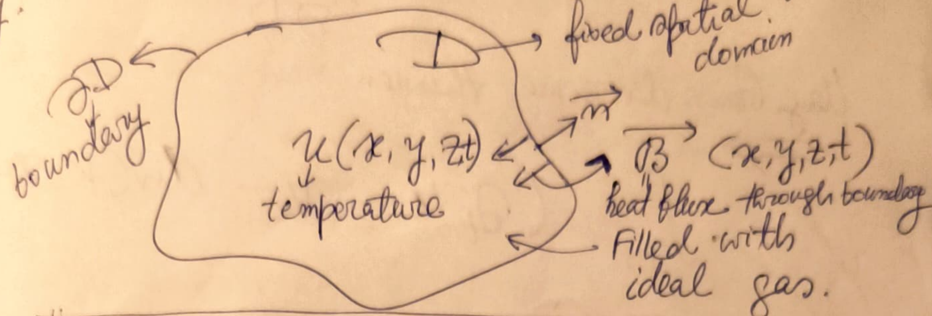
Derivation of Heat Equation:

The basic idea that guided Fourier to derive heat equation was conservation of Energy.



The internal energy of ideal gas:

$$U = c_V T,$$

where c_V is specific heat (at constant volume)

Let us normalize, $c_V = 1$

Change in the internal energy stored in D between time t and $t+\Delta t$

$$(*)\quad \begin{cases} = \displaystyle\int_D [u(x,y,z,t+\Delta t) - u(x,y,z,t)]\, dV \\[2ex] = \displaystyle\int_t^{t+\Delta t}\int_D q(x,y,z,t,u)\, dV\, dt \\[2ex] \quad - \displaystyle\int_t^{t+\Delta t}\int_{\partial D} \vec{B}(x,y,z,t)\cdot\vec{n}\, dS\, dt. \end{cases}$$

where, q is the rate of heat production in D.

Fourier's law of heat conduction:

$$\vec{B} = -K(x,y,z)\nabla u$$

$K \to$ heat conduction coefficient

Here dividing both sides of $(*)$ by Δt and passing $\Delta t \to 0$, we obtain

$$\int_D \partial_t u \, dV = \int_D q(x,y,z,t)\, dV + \int_{\partial D} K(x,y,z) \nabla u \cdot \vec{n}\, dS.$$

Using Gauss divergence theorem.

$$\Longrightarrow \int_D \left(\partial_t u - q - \operatorname{div}(K \nabla u)\right) dV = 0.$$

> **Lemma:** Let $h(x,y,z)$ be a continuous function satisfying $\int_\Omega h(x,y,z)\, dV = 0$ for every domain Ω. Then $h \equiv 0$
>
> **Proof:** Home-work

Since D is arbitrary, assuming enough regularity of u, we derive.

$$\partial_t u = q + \operatorname{div}(K \nabla u)$$

When, ~~$k=1$ and~~ $q=0$, and k a constant, we have classical heat eqn.

$$\boxed{\partial_t u = K \Delta u}$$

Classifications of PDEs and Canonical Forms:

2nd order PDEs:

General form of 2nd order PDE in two independent variables:

$$A\,\partial_x^2 u + B\,\partial_{xy}^2 u + C\,\partial_y^2 u + D\,\partial_x u + E\,\partial_y u + Fu = G. \quad \text{--- } (1)$$

where, A, B and C are functions of x and y and do not vanish simultaneously.

D, E and F are also functions of x and y.

- We shall assume that the ~~equations~~ function $u(x,y)$ and the coefficients are twice continuously differentiable in some domain Ω.

Classification:

The classification of 2nd order PDEs depend on the leading part of (1) consisting of the 2nd order terms.

So, for simplicity we combine the lower order terms of (1) in order to write the following:-

$$A(x,y)\,\partial_{xx}u + B(x,y)\,\partial_{xy}u + C(x,y)\,\partial_{yy}u = \Phi(x, y, u, \partial_x u, \partial_y u). \quad \text{--- } (2)$$

Fundamentally there are three types of PDEs:

- Hyperbolic: modelling the transport of physical quantity, such as, fluid density or wave propagation.

- Parabolic: Generally arise in the mathematical analysis of diffusion phenomena, as in the heating of a slab.

- Elliptic: in large time solution of Parabolic PDEs converge to a steady state / minimum energy state modelled by elliptic equations.

□ Mathematical classification
(This classification is even possible for quasi-linear 2nd order PDEs as well, i.e. PDEs which are linear in the 2nd derivatives only).

The classification depends on the sign of the discriminant,

$$\Delta(x_0,y_0) \equiv \begin{vmatrix} B & 2A \\ 2C & B \end{vmatrix} = B(x_0,y_0)^2 - 4A(x_0,y_0)C(x_0,y_0). \quad (3)$$

□ If, $\Delta(x_0,y_0) > 0$, the equation is hyperbolic,
□ If, $\Delta(x_0,y_0) = 0$, " " " parabolic
□ If, $\Delta(x_0,y_0) < 0$, " " " elliptic.

Note: A PDE can show different behaviour at different points.

Example

$$\partial_x^2 u + x\,\partial_y^2 u = 0$$

is hyperbolic when $x<0$,

parabolic when $x=0$,

and elliptic when $x>0$.

Definition: A PDE is hyperbolic/parabolic/elliptic in a region Ω if the PDE is hyperbolic/parabolic/elliptic at each point of Ω.

Change of co-ordinates and discriminant:

Let us define the new independent variables,

$$\xi = \xi(x,y), \quad \eta = \eta(x,y). \quad \text{(4)}$$

where, ξ and η are twice continuously differentiable and the Jacobian,

$$J = \frac{\partial(\xi,\eta)}{\partial(x,y)} = \begin{vmatrix} \partial_x \xi & \partial_y \xi \\ \partial_x \eta & \partial_y \eta \end{vmatrix} \neq 0 \quad \text{(5)}$$

in the region under consideration.

Note: The non-vanishing of the Jacobian of the transformation ensures that a one-to-one transformation exists between the new and old variables.

This simply means that the new independent variables can serve as new co-ordinate variables.

Now define,

$$w(\xi,\eta) = u(x(\xi,\eta), y(\xi,\eta)).$$

$$\Leftrightarrow u(x,y) = w(\xi(x,y), \eta(x,y)).$$

Implying:

$$(6) \text{ ——— } \left\{ \begin{aligned} \partial_x u &= \partial_\xi w \, \partial_x \xi + \partial_\eta w \, \partial_x \eta \\ \partial_y u &= \partial_\xi w \, \partial_y \xi + \partial_\eta w \, \partial_y \eta \\ \partial_{xx} u &= \partial_{\xi\xi} w (\partial_x \xi)^2 + 2 \partial_{\xi\eta} w \, \partial_x \xi \, \partial_x \eta + \partial_\eta w \, \partial_{xx}\eta \\ &\quad + \partial_{\eta\eta} w (\partial_x \eta)^2 + \partial_\xi w \, \partial_{xx}\xi \\ \partial_{yy} u &= \partial_{\xi\xi} w (\partial_y \xi)^2 + 2 \partial_{\xi\eta} w \, \partial_y \xi \, \partial_y \eta \\ &\quad + \partial_{\eta\eta} w (\partial_y \eta)^2 + \partial_\xi w \, \partial_{yy} \xi \\ &\quad + \partial_\eta w \, \partial_{yy} \eta. \\ \partial_{xy} u &= \partial_{\xi\xi} w \, \partial_x \xi \, \partial_y \xi + \partial_{\xi\eta} w (\partial_x \xi \, \partial_y \eta + \partial_y \xi \, \partial_x \eta) \\ &\quad + \partial_{\eta\eta} w \, \partial_x \eta \, \partial_y \eta + \partial_\xi w \, \partial_{xy} \xi \\ &\quad + \partial_\eta w \, \partial_{xy} \eta. \end{aligned} \right.$$

Substituting (6) into (2) we obtain the transformed PDE as

$$(7) \text{ ——— } a\, \partial_{\xi\xi} w + b\, \partial_{\xi\eta} w + c\, \partial_{\eta\eta} w = \phi(\xi, \eta, w, \partial_\xi w, \partial_\eta w)$$

where Φ becomes ϕ and the new coefficients of the higher order terms a, b and c are expressed via the original coefficients and the change of variables formulas as follows:

$$(8)\;\begin{cases} a = A(\partial_x \xi)^2 + B\,\partial_x \xi\, \partial_y \xi + C(\partial_y \xi)^2 \\ b = 2A\,\partial_x \xi\, \partial_x \eta + B(\partial_x \xi\, \partial_y \eta + \partial_y \xi\, \partial_x \eta) \\ \qquad + 2C\,\partial_y \xi\, \partial_y \eta \\ c = A(\partial_x \eta)^2 + B\,\partial_x \eta\, \partial_y \eta + C\,\partial_y \eta^2 \end{cases}$$

(8) can be rewritten as,

$$\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} = \begin{pmatrix} \partial_x \xi & \partial_y \xi \\ \partial_x \eta & \partial_y \eta \end{pmatrix} \begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix} \begin{pmatrix} \partial_x \xi & \partial_x \eta \\ \partial_y \xi & \partial_y \eta \end{pmatrix}$$

$\Rightarrow$ Taking determinant;

$$\begin{vmatrix} a & b/2 \\ b/2 & c \end{vmatrix} = \begin{vmatrix} A & B/2 \\ B/2 & C \end{vmatrix} J^{-2}$$

where J is the Jacobian of the change of variables given by (5).

(ie)

$$(b^2 - 4ac) = J^{-2}(B^2 - 4AC) \Rightarrow \delta = J^2 \Delta \quad (9)$$

where, $\delta = (b^2 - 4ac)$ is the discriminant of (7).

Since, $J \neq 0$ (see (5)), it is clear ~~that~~ any real non-singular transformation does not change the type of PDE.

Canonical forms:

Let us construct transformations which will make one / possibly two of the coefficients of the leading second order terms of eqn (7) to vanish, thus reducing the equation to a simpler form, called 'canonical form'.

One has the following obvious choices of the new coefficients a, b and c which renders canonical forms:

- $a = c = 0$ corresponds to the first canonical form of hyperbolic PDE given by,

$$\partial_{\xi\eta} w = \psi(\xi, \eta, w, \partial_\xi w, \partial_\eta w).$$

(note, $\delta = b^2 - 4ac = b^2 = 1 > 0$)

- $b = 0,\ c = -a$ corresponds to the second canonical form of hyperbolic PDE given by,

$$\cancel{\partial_{\alpha\alpha} w - \partial_{\beta\beta} w = \psi(}$$

$$\partial_{\xi\xi} w - \partial_{\eta\eta} w = \psi(\xi, \eta, w, \partial_\xi w, \partial_\eta w).$$

(note, $\delta = b^2 - 4ac = 4c^2 > 0$)

- $a = b = 0$ corresponds to the canonical form of parabolic PDE given by,

$$\partial_{\eta\eta} w = \psi(\xi, \eta, w, \partial_\xi w, \partial_\eta w).$$

(note, $\delta = b^2 - 4ac = 0$)

- $b = 0,\ c = a$ corresponds to the canonical form of elliptic PDE given by,

$$\partial_{\xi\xi} w + \partial_{\eta\eta} w = \psi(\xi, \eta, w, \partial_\xi w, \partial_\eta w)$$

(note, $\delta = b^2 - 4ac = -4a^2 < 0$).

Hyperbolic equations:

Let us deduce the first canonical form of hyperbolic PDE:

$$\boxed{\partial_{\xi\eta} w = \Psi(\xi, \eta, w, \partial_\xi w, \partial_\eta w)}$$

(i) we need to choose new variables ξ and η such that $a = c = 0$ in (7).

Thus from (8), we have.

$$a = A(\partial_x \xi)^2 + B \partial_x \xi \, \partial_y \xi + C(\partial_y \xi)^2 = 0 \quad \text{(13a)}$$

$$c = A(\partial_x \eta)^2 + B \partial_x \eta \, \partial_y \eta + C(\partial_y \eta)^2 = 0. \quad \text{(13b)}$$

$$\Rightarrow \begin{cases} A\left(\dfrac{\partial_x \xi}{\partial_y \xi}\right)^2 + B\left(\dfrac{\partial_x \xi}{\partial_y \xi}\right) + C = 0 & \text{(14a)} \\[2ex] A\left(\dfrac{\partial_x \eta}{\partial_y \eta}\right)^2 + B\left(\dfrac{\partial_x \eta}{\partial_y \eta}\right) + C = 0 & \text{(14b)} \end{cases}$$

Since the above system has atmost two distinct roots, we may without loss of generality assume that,

$$\begin{cases} \mu_1(x,y) = \dfrac{\partial_x \xi}{\partial_y \xi} = \dfrac{-B - \sqrt{B^2 - 4AC}}{2A}. & \text{(15a)} \\[2ex] \mu_2(x,y) = \dfrac{\partial_x \eta}{\partial_y \eta} = \dfrac{-B + \sqrt{B^2 - 4AC}}{2A}. & \text{(15b)} \end{cases}$$

Leading to the following first-order differential eqns:

$$\begin{cases} \partial_x \xi - \mu_1(x,y)\, \partial_y \xi = 0 & \text{(16a)} \\ \partial_x \eta - \mu_2(x,y)\, \partial_y \eta = 0 & \text{(16b)} \end{cases}$$

These are the equations that define the new coordinate variables ξ and η that are necessary to make $a = c = 0$ in (7).

Along the new coordinate line,

$$\xi(x,y) = \text{constant}$$

One has, the total differential $d\xi = 0$

(ie) $d\xi = \partial_x \xi \, dx + \partial_y \xi \, dy = 0$

$$\Rightarrow \frac{dy}{dx} = -\frac{\partial_x \xi}{\partial_y \xi}.$$

Similarly along the coordinate line, $\eta(x,y) = \text{const.}$

$$\frac{dy}{dx} = -\frac{\partial_x \eta}{\partial_y \eta}.$$

Hence, using (14),

$$A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C = 0 \quad \text{--- (17)}.$$

This is called the characteristic polynomial of PDE (2) whose roots are given by,

$$\text{(18a) ---} \quad \frac{dy}{dx} = \frac{B + \sqrt{B^2 - 4AC}}{2A} = \lambda_1(x,y)$$

$$\text{(18b) ---} \quad \frac{dy}{dx} = \frac{B - \sqrt{B^2 - 4AC}}{2A} = \lambda_2(x,y).$$

Note that (18) represent differential equations for families of curves in the xy-plane along which $\xi = \text{constant}$ and $\eta = \text{constant}$.

(18) $\Rightarrow$

$$y - \frac{B + \sqrt{B^2 - 4AC}}{2A}\, x = C_1 \quad \text{--- (19a)}$$

and

$$y - \frac{B - \sqrt{B^2 - 4AC}}{2A}\, x = C_2 \quad \text{--- (19b)}$$

for constants C_1 and C_2.

For constant A, B and C (19) represents a ~~pair of parallel straight lines~~ two distinct families of parallel straight lines.

Since, C_1 and C_2 are arbitrary constants and the characteristic curves in xy-plane are given by,

$\xi =$ constant and $\eta =$ constant, one has,

$$\text{(20)} \quad \xi = y - \frac{B + \sqrt{B^2 - 4AC}}{2A}\, x = y - \lambda_1 x$$

$$\text{(21)} \quad \eta = y - \frac{B - \sqrt{B^2 - 4AC}}{2A}\, x = y - \lambda_2 x$$

Hence with respect to the new coordinates ξ and η given by (20) and (21), we can write ~~(2) in~~ an hyperbolic equation (i.e. $B^2 - 4AC > 0$ in (2)) ~~the of~~ in the form,

$$\text{(22)} \quad \partial_{\xi\eta} w = \psi(\xi, \eta, w, \partial_\xi w, \partial_\eta w)$$

where, $\psi = \frac{\phi}{b}$ and b is calculated from (8) as:

$$b = 2A\, \partial_x \xi\, \partial_x \eta + B(\partial_x \xi\, \partial_y \eta + \partial_y \xi\, \partial_x \eta) + 2C\, \partial_y \xi\, \partial_y \eta$$

$$= 2A\left(\frac{B^2 - (B^2 - 4AC)}{4A^2}\right) + B\left(-\frac{B}{2A} - \frac{B}{2A}\right) + 2C$$

$$\Rightarrow \boxed{b = 4C - \frac{B^2}{A} = -\Delta/A} \quad \text{--- (23)}$$

Example :

Show that the one-dimensional wave eqn,

$$\partial_t^2 u - c^2 \partial_x^2 u = 0$$

is hyperbolic, and find an equivalent canonical form ~~and then~~ obtain the general solution.

Solution :

Here,

$$A = 1, \quad B = 0, \quad C = -c^2$$

$\Rightarrow \Delta = 4c^2 > 0$ and hence the PDE is hyperbolic.

The roots of the characteristic polynomial are given by :

$$\lambda_1 = \frac{B + \sqrt{\Delta}}{2A} = c \quad \text{and} \quad \lambda_2 = \frac{B - \sqrt{\Delta}}{2A} = -c$$

Hence, from the characteristic eqns (18a) and (18b), we have,

$$\frac{dx}{dt} = c, \quad \frac{dx}{dt} = -c.$$

Integrate the above ODEs to obtain the characteristics of ~~the~~ wave equation,

$$x = ct + K_1, \quad x = -ct + K_2$$

where, K_1 and K_2 are the constants of integration.

$\Rightarrow$ The following transformation,

$$\xi = x - ct, \quad \eta = x + ct$$

reduces the wave eqn to the canonical form.

We, further have, with respect to the new coordinates ξ and η,

$$a = 0, \quad c = 0, \quad b = \frac{-\Delta}{A} = -4c^2.$$

$\therefore$ In terms of the characteristic variables, the wave eq^n. reduces to the following canonical form,

$$\boxed{\partial_{\xi\eta} w = 0} \quad \text{--- (24)}$$

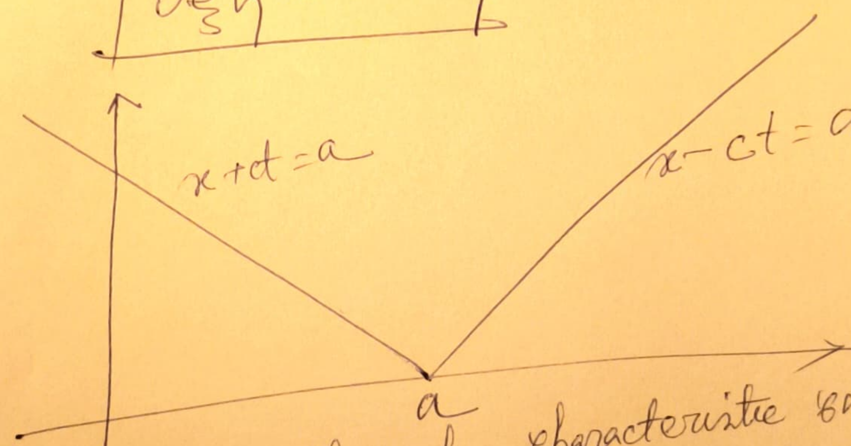


Figure: The pair of characteristic curves for wave equation.

From (24):

$$\partial_\eta w = h(\eta) \quad \text{--- (25)}$$

where $h(\eta)$ is an arbitrary function of η.

(25) $\Rightarrow$

$$w(\xi,\eta) = \int h(\eta)\,d\eta + f(\xi)$$
$$= f(\xi) + g(\eta)$$

where f and g are arbitrary twice differentiable functions and g is just the integral of the arbitrary function h.

Hence the general solutions of the wave equation in terms of its original variable x and t are then given by:

$$\boxed{u(x,t) = f(x-ct) + g(x+ct)}$$

$\Downarrow$

Interpretation: