$$I(f) = \int_a^b f(x)\,dx \approx \int f(x)\,dx$$

The process of approximating the value of $I(f)$ is referred to as numerical integration or quadrature rule.

i.e. $\int_a^b f(x)\,dx \approx \int$ (simple function) $dx \quad + R_n$

For this we construct polynomial of $n+1$ nodes $x_0, x_1, x_2, \ldots, x_n$

then $I(f) \approx I(L)$

And then $I(f) \approx I(L) = \sum_{i=0}^{n} f(x_i) L_i(x)$

$\Rightarrow \int_a^b I(f)\,dx = \sum_{i=0}^{n} f(x_i) \int_a^b L_i(x)\,dx$

On the other hand we apply Newton's F.D. formula for equally spaced abscissa and then

By $x = x_0 + th$, t is variable, depends on x

when $x = x_0 \Rightarrow t = 0$

when $x = x_n = x_0 + nh \Rightarrow x_n = x_0 + n\cdot h$

$\Rightarrow t = n$

$$I = \int_{x_0}^{x_n} f(x)\,dx$$

Change the variable

$$I = h\int_0^n f(x_0 + th)\,dt$$

$$= h\int_0^n P_n(t)\,dt \quad \text{(ability formula)}$$

$$= h\int_0^n \left(1 + t\Delta + \frac{t(t-1)}{2!}\Delta^2 + \cdots\right)dt \quad [\because P_n(t) = 1]$$

$$= h\int_0^n \left(1 + t\Delta + \frac{t(t-1)}{2!}\Delta^2 + \frac{t(t-1)(t-2)}{3!}\Delta^3 + \cdots\right)dt$$

$$= h\int_0^n \left[1 + t\Delta + \frac{t^2}{2}\Delta^2 + \frac{t^3}{6} - \frac{t^2}{2}\Delta^3 + \cdots\right]dt$$

Integrating and including the limits

$$I = h\left[n + \frac{n^2}{2}\Delta + \frac{n^3 - n^2}{12}\Delta^2 + \frac{n^4 - 2n^3}{24}\Delta^3 + \cdots\right]$$

$$= \frac{n^2 - n^3 + n^4}{24}\Delta^3 + \cdots\Big]$$

Note: Rectangular Rule, $n=1$

$$\int_{x_0}^{x_1} f(x)\,dx = y_0(x_1 - x_0) = hy_0$$

$$\text{Error} = \int_0^{n+1} h\,P_n(t)\,dt$$

$$= h\int_0^1 [1 + t\Delta]\,dt$$

$$= \frac{h}{2}f(t) \Big|_0^1 = \frac{h}{2}[f(1) - f(0)]$$

For Lagrange's formula: $f(t) = y_0$ means $\frac{x - x_1}{x_0 - x_1} y_0 + \frac{x - x_0}{x_1 - x_0} y_1 = y_0$

$$P(x) = \frac{x - x_1}{x_0 - x_1} y_0 + \frac{x - x_0}{x_1 - x_0} y_1 = y_0 \checkmark$$

Error $R(x) = (x - x_0) f'(\xi)$ $\quad x_0 \le \xi \le x_1$

$$\int_{x_0}^{x_1} f(x) = y_0(x_1 - x_0) = hy_0$$

$$\int_{x_0}^{x_1} R(x)\,dx = \int_{x_0}^{x_1} (x - x_0) f'(\xi)\,dx = f'(\xi)\cdot\frac{h^2}{2}, \text{ as } f'(\xi) = \frac{1}{h^2} f'(\xi)$$

$$\int_{x_0}^{x_1} (x - x_0)\,dx = \frac{(x - x_0)^2}{2}\Big|_{x_0}^{x_1} = \frac{h^2}{2}$$

Quadrature Formula

The integral is approximated by a linear combination of the values of $f(x)$ at tabular point at,

$$I = \int_a^b f(x)\,dx \simeq \sum_{k=0}^{n} w_k f(x_k) = \sum_{k=0}^{n} w_k f(x_k) + R_n(f)$$

$$= w_0 f(x_0) + w_1 f(x_1) + w_2 f(x_2) + \cdots + w_n f(x_n) \quad \text{(A)}$$

$(x_0, f(x_0))$, $(x_1, f(x_1))$, $(x_n, f(x_n))$

The points x_k's are called abscissas

$f(x_k)$'s are called coordinates

& w_k's are called weights.

Quadrature formula

Error of quadrature formula

$$R_n(f) = \int_a^b f(x)\,dx - \sum_{k=0}^{n} w_k f(x_k)$$

Order of a method.

An integration method of the form (A) is said to be order p, if $R_n = 0$ $\forall$ polynomial of degree $\le p$

i.e. it produces exact result for $f(x) = 1, x, x^2, x^3, \ldots, x^p$

$$\Rightarrow R_n(x^m) = \int_a^b x^m\,dx - \sum_{k=0}^{n} w_k x_k^m = 0$$

for $m = 0, 1, 2, \ldots, p$

The error term for x^{p+1} is defined as

$$E = \int_a^b (x)\,x^{p+1}\,dx - \sum_{k=0}^{n} w_k x_k^{p+1}$$

error constant $\quad C(x) = 1$ when w_k weight factor

After solving error term the value of w_k & x_k is known

rather x^{p+1} have change?

Such C kya hai?

Now $R_n(f) = \int_a^b f(x)\,dx - \sum_{k=0}^{n} w_k x_k^{p+1}$

$$= \frac{c}{(p+1)!} f^{(p+1)}(\xi) \quad a \le \xi \le b$$

The bound for the error term is

$$\boxed{|R_n(f)| \le \frac{|c|}{(p+1)!} \max_{a \le \xi \le b} |f^{(p+1)}(x)|}$$

Now For uniform mesh grids

$x_0 = a, \quad x_n = b, \quad h = \frac{b-a}{n}$

$$I = \int_a^b f(x)\,dx = \sum_{k=0}^{n} w_k f(x_k)$$

$$I = w_0 f(x_0) + w_1 f(x_0 + h) + w_2 f(x_0 + 2h) + \cdots + w_n f(x_0 + nh)$$

$\Downarrow$

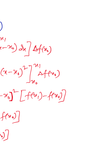
Newton-Cotes quadrature formula

Trapezoidal Rule $\Rightarrow$ for 2 points formula

Simpson's $1/3$ rule $\Rightarrow$ for 3 points formula

Simpson's $3/8$ rule $\Rightarrow$ for 4 points formula

For Trapezoidal Rule



Let the curve $y = f(x)$, $a \le x \le b$ be approximated by a line joining the points P, Q.

Using the Newton's forward difference formula, the linear interpolating polynomial passes through $P(a, f(a))$ & $Q(b, f(b))$ is given by

$$f(x) = f(x_0) + \frac{1}{h}(x - x_0)\Delta f(x_0)$$

$$I = \int_a^b f(x)\,dx = f(x_0)\int_{a=x_0}^{b=x_1} dx + \frac{1}{h}\int_{x_0}^{x_1} (x - x_0)\,dx\,\Delta f(x_0)$$

$$= (x_1 - x_0) f(x_0) + \frac{1}{h}\left[\frac{1}{2}(x - x_0)^2\right]_{x_0}^{x_1} \Delta f(x_0)$$

$$= (x_1 - x_0) f(x_0) + \frac{1}{2h}(x_1 - x_0)^2 [f(x_1) - f(x_0)]$$

$$= h f(x_0) + \frac{h^2}{2h}[f(x_1) - f(x_0)]$$

$$= h f(x_0) + \frac{h}{2}[f(x_1) - f(x_0)]$$

$$= \frac{h}{2}[f(x_0) + f(x_1)]$$

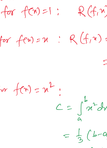
$$\boxed{I = \frac{b-a}{2}[f(b) + f(a)]}$$

Form (A) $\quad n = 1, \quad x_0 = a, \quad x_1 = a + h = b$

$$I = \int_a^b f(x)\,dx = \int_{x_0}^{x_1} w_0 f(x)\,dx = h\left[\frac{1}{2} + \frac{1}{2}\Delta x_1\right]$$

$$= \frac{h}{2}[f(x_0) + f(x_1)]$$

$$= \frac{b-a}{2}[f(a) + f(b)]$$



$$= (b - a) f(x_0) + \frac{1}{2}(b - a)[f(x_1) - f(x_0)]$$

$$= \frac{b-a}{2}\left[2f(x_0) + f(x_1) - f(x_0)\right]$$

$$= \frac{b-a}{2}[f(x_0) + f(x_1)]$$

For error estimation

One can prove Trapezoidal rule gives the exact value of the integration for poly of degree ≤ 1

i.e. $R(f, x) = 0$ for $f(x) = 1, x$.

$f(x) = 1$, $f'(x) = 0$, $f''(x) = 0$

i.e.

for $f(x) = 1$: $\quad R(f, x) = \int_a^b 1\,dx - \frac{(b-a)}{2}[f(x_0) + f(x_1)] + R_n(f)$

$$= \int_a^b 1\,dx - \frac{(b-a)}{2}[1 + 1] + R_n(f) \quad \text{(b-a)} - \text{(b-a)} = 0$$

for $f(x) = x$: $\quad R(f, x) = \int_a^b x\,dx - \frac{b-a}{2}\cdot(a + b)$

$$= \frac{1}{2}(b^2 - a^2) - \frac{1}{2}(b^2 - a^2) = 0 = R_n(f)$$

For $f(x) = x^2$:

$$c = \int_a^b x^2\,dx - \frac{b-a}{2}[a^2 + b^2]$$

$$= \frac{1}{3}(b - a)^3 - \frac{1}{2}(b^2 + a^2 + 2ab - a^2 - b^2)$$

$$= \frac{1}{6}(a^3 - 3a^2 b + 3ab^2 - b^3)$$

$$= -\frac{1}{6}(b - a)^3$$

$$\boxed{\text{The order of trapezoidal rule is one.}}$$

Using the definition of error the error is given by

$$R(f, x) = \frac{c}{p!} f^{(p)}(\xi) = -\frac{(b-a)^3}{12} f''(\xi); \quad a \le \xi \le b$$

The bound of the error is given by

$$|R(f, x)| \le \frac{(b-a)^3}{12} \max_{a \le \xi \le b} |f''(\xi)|$$

If the length of the interval $[a, b]$ is large, then $(b - a)$ is also large and so the error term in expression $(1, 10)$ will be large.

Thus the method becomes meaningless.

Q. $\int_0^1 \frac{x}{1+x}\,dx$, $\quad n = 1$

$f(x) = \frac{x}{1+x}$

$$= \frac{b-a}{2}\left[f(a) + f(b)\right]$$

$$= \frac{1}{2}\left[0 + \frac{1}{2}\right] = \frac{1}{4} = \frac{1}{2}$$

$$= 0.25$$

$$\int_0^1 \frac{x}{1+x}\,dx = \int_0^1 \left(1 - \frac{1}{1+x}\right)dx$$

$$= \left[x - \log(1+x)\right]_0^1 = 1 - \log 2 = 1 - 0.3010$$

$$= 0.6990$$