MA 203 Complex Analysis and Differential Equations-II

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Lecture-2

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Recap

- Periodic functions of period 2π
- Orthogonality of the trigonometric system
- The *Fourier series* representation of f
- A Fourier series for f does NOT always converge to f;
- Piecewise smooth and representation by a Fourier series
- An example

Let $f : Dom(f) \subset \mathbb{R} \to \mathbb{R}$ be a periodic function with period $2L$.

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$$g : Dom(g) \subset \mathbb{R} \to \mathbb{R} \quad \text{by} \quad g(y) = f\left(\frac{L}{\pi}y\right).$$

Now, g is a periodic function with period 2π. Hence, the Fourier series for the function $g(y) = f\left(\frac{L}{\pi}y\right)$ is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos ny + b_n \sin ny), \tag{1}$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(y)\,\cos ny\,\mathrm{d}y, \qquad n = 0, 1, 2, 3, \ldots \tag{2a}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(y)\,\sin ny\,\mathrm{d}y, \qquad n = 1, 2, 3, \ldots \tag{2b}$$

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Now, using the scale $y = \pi x/L$, we have $\mathrm{d}y = (\pi/L)\,\mathrm{d}x$, the Fourier series for the function $f(x) = g\left(\frac{\pi x}{L}\right)$ is given by [use the scaling in (1) and (2)]

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right] \tag{3}$$

with the Fourier coefficients given by

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(y)\cos ny\,\mathrm{d}y = \frac{1}{\pi}\int_{-L}^{L} g\left(\frac{\pi}{L}x\right)\cos\left(\frac{n\pi}{L}x\right)\times\frac{\pi}{L}\mathrm{d}x \\
&= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)\mathrm{d}x, \qquad n = 0, 1, 2, 3, \ldots
\end{aligned}
\tag{4a}
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Here, the fact $g(\pi x/L) = f(x)$ has been used.
This leads to the following definition.

Definition 1 (Fourier series of a function with period $p = 2L$)

Let $f(x)$ be a periodic function with period $2L$. The series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right) \right], \tag{5}$$

with the coefficients

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi}{L}x\right) \mathrm{d}x, \qquad n = 0, 1, 2, 3, \ldots \tag{6a}$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi}{L}x\right) \mathrm{d}x, \qquad n = 1, 2, 3, \ldots, \tag{6b}$$

is referred to as the *Fourier series* of $f(x)$. The coefficients $a_0, a_1, a_2, a_3, \ldots, b_1, b_2, b_3, \ldots$ in (5) are again referred to as the *Fourier coefficients* of $f(x)$.

Definition 2

A function $f : [a, b] \to \mathbb{R}$ is said to be piecewise smooth (or sectionally smooth) if this interval can be divided into a finite number of subintervals such that

1. f has a continuous derivative f' in the interior of each of these subintervals,
2. and both $f(x)$ and $f'(x)$ approach finite limits as x approaches either endpoint of each of these subintervals from its interior.

In other words, we may say that f is piecewise smooth on $[a, b]$ if both f and f' are piecewise continuous on $[a, b]$.

Theorem 3 (Representation by a Fourier series)

Let $f : [-L, L] \to \mathbb{R}$ be periodic with period $2L$ and be piecewise smooth in the interval $[-L, L]$. Then, the Fourier series of f, i.e., $\mathcal{S}_f(x)$ converges at every point x to the value

$$\frac{f(x+) + f(x-)}{2} \tag{7}$$

where $f(x+)$ is the right hand limit of f at x and $f(x-)$ is the left hand limit of f at x. In particular, if f is also continuous at x, the value (7) reduces to $f(x)$ and $\mathcal{S}_f(x) = f(x)$.

Fourier series of even and odd functions

Definition 4

A function $f : Dom(f) \subset \mathbb{R} \to \mathbb{R}$ is said to be an even function if $f(-x) = f(x)$ for all x and a function $g : Dom(g) \subset \mathbb{R} \to \mathbb{R}$ is said to be an odd function if $g(-x) = -g(x)$ for all x.

1. For an even function f and an odd function g, we have the following:

$$\int_{-L}^{L} f(x)\,\mathrm{d}x = 2\int_{0}^{L} f(x)\,\mathrm{d}x \qquad \text{and} \qquad \int_{-L}^{L} g(x)\,\mathrm{d}x = 0.$$

Therefore, it is not difficult to obtain the following theorem.

Theorem 5

(i) *The Fourier series of an even function f of period $2L$ is a* **Fourier cosine series**

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right), \tag{8}$$

with the coefficients

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)\mathrm{d}x, \qquad n = 0, 1, 2, 3, \ldots \tag{9a}$$

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(ii) *The Fourier series of an odd function f of period $2L$ is a* **Fourier sine series**

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right), \tag{10}$$

with the coefficients

$$b_n = \frac{2}{L}\int_0^L f(x)\,\sin\left(\frac{n\pi}{L}x\right)\mathrm{d}x, \qquad n = 1, 2, 3, \ldots \tag{11}$$

Even function of period 2π

1. **Even function of period** 2π: If $f(x)$ is even and $L = \pi$, the Fourier series of $f(x)$ is the Fourier cosine series, given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \tag{12}$$

with coefficients

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, \mathrm{d}x, \quad n = 0, 1, 2, 3, \ldots \tag{13}$$

Odd function of period 2π

1. **Odd function of period** 2π: If $f(x)$ is odd and $L = \pi$, the Fourier series of $f(x)$ is the Fourier sine series, given by

$$\sum_{n=1}^{\infty} b_n \sin nx \tag{14}$$

with coefficients

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, \mathrm{d}x, \qquad n = 1, 2, 3, \ldots \tag{15}$$

Contents

- Classification of linear second order PDE's in two variables.
- Laplace, wave, and heat equations using separation of variables.
- D' Alembert solution to the wave equations.
- Vibration of a circular membrane.
- Heat equation in the half space.

Why PDE?

- Partial differential equations are vital in mathematically oriented scientific fields, such as physics and engineering. For instance, they are foundational in the modern scientific understanding of sound, heat, diffusion, fluid dynamics, elasticity, general relativity, and quantum mechanics (Schrödinger equation etc.).

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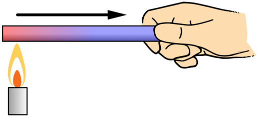

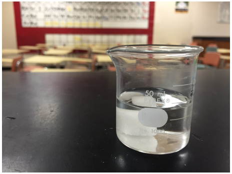

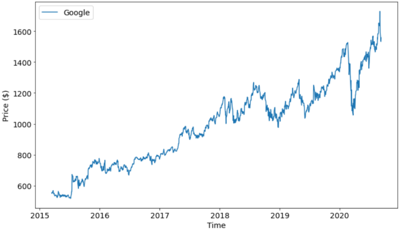

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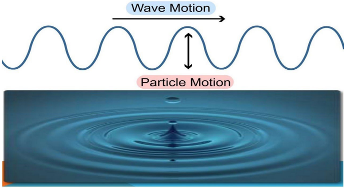
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PDE

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Example 7

$$\frac{\partial u}{\partial t} = u\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^3 u}{\partial z^3}, \quad u_{xx} + u\, u_y + u_{yz} = x^2 + y^2 + u$$

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Definition 9

A PDE is said to be **linear** if the dependent variable and its partial derivatives occur only in the first degree and are not multiplied. A partial differential equation which is not linear is called a **non-linear**.

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Example 12

1. Linear PDE: $a(x,y)u_x + b(x,y)u_y + c(x,y)u = f(x,y)$
2. Semi-linear PDE: $a(x,y)u_x + b(x,y)u_y = f(x,y,u)$
3. Quasi-linear PDE: $a(x,y,u)u_x + b(x,y,u)u_y = f(x,y,u)$

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Definition 13

A **linear** PDE is said to be *homogeneous* if each of its terms contains either the unknown function u or one of its partial derivatives. Otherwise, the PDE is called *nonhomogeneous* or *inhomogeneous*.

Example 14

(i) $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$ **One-dimensional wave equation**

(ii) $$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$ **One-dimensional heat equation**

(iii) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$ **Two-dimensional Laplace equation**

(iv) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$$ **Two-dimensional Poisson equation**

(v) $$\frac{\partial^2 u}{\partial t^2} = \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$ **Two-dimensional wave equation**

(vi) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$ **Three-dimensional Laplace equation**

PDEs (i)–(iii), (v) and (vi) are homogeneous while (iv) is nonhomogeneous for $f(x, y) \neq 0$.

Method of forming PDEs

Remark 1

Second-order PDEs are the most important ones in applications. Our syllabus contains only linear second-order homogeneous PDEs in two variables. These are one-dimensional wave equation, one-dimensional heat equation and two-dimensional Laplace equation.

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Proof.

We have,

$$x^2 + y^2 + (z-a)^2 = b^2. \tag{16}$$

(16) contains two arbitrary constants a and b. Differentiating (16) partially with respect to x, we get

$$2x + 2(z-a)\frac{\partial z}{\partial x} = 0 \implies \boxed{x + (z-a)p = 0} \quad \text{where} \quad p = \frac{\partial z}{\partial x}. \tag{17}$$

Again differentiating (16) partially with respect to y, we get

$$2y + 2(z-a)\frac{\partial z}{\partial y} = 0 \implies \boxed{y + (z-a)q = 0} \quad \text{where} \quad q = \frac{\partial z}{\partial y}. \tag{18}$$

(17) $\times\, q -$ (18) $\times\, p$, we get

$$xp - yq = 0 \implies \boxed{x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = 0}. \tag{19}$$

This represents PDE of all spheres whose centre lie on z-axis. □

Method 2: By eliminating arbitrary functions: Form the PDE from $z = f(x^2 - y^2)$.

Proof.

Differentiating the above equation partially with respect to x and y, we get

$$p = \frac{\partial z}{\partial x} = f'(x^2 - y^2)2x \tag{20}$$

$$q = \frac{\partial z}{\partial y} = f'(x^2 - y^2)(-2y). \tag{21}$$

Dividing (20) **by** (21) **we get**

$$\frac{p}{q} = -\frac{x}{y} \implies y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 0. \tag{22}$$

□

Classification of linear second-order PDEs in two variables

The general second-order linear PDE has the following form:

$$A\,u_{xx} + B\,u_{xy} + C\,u_{yy} + D\,u_x + E\,u_y + F\,u = G, \tag{23}$$

where the coefficients A, B, C, D, F **and the free term** G **are in general functions of the independent variables** x **and** y**, but do not depend on the unknown function** u**. The classification of second-order equations depends on the form of the leading part of the equations consisting of the second-order terms. So, for simplicity of notation, we combine the lower-order terms and rewrite the above equation in the following form**

$$A\,u_{xx} + B\,u_{xy} + C\,u_{yy} + I(x, y, u, u_x, u_y) = 0. \tag{24}$$

The type of the above equation depends on the sign of the quantity

$$\Delta(x, y) = B^2(x, y) - 4A(x, y)\,C(x, y), \tag{25}$$

which is called the *discriminant* **for** (24)**. The classification of second-order linear PDEs is given by the following.**

Definition 15

At the point (x_0, y_0), the second-order linear PDE (24) is called

(i) *elliptic*, if $\Delta(x_0, y_0) < 0$

(ii) *parabolic*, if $\Delta(x_0, y_0) = 0$

(iii) *hyperbolic*, if $\Delta(x_0, y_0) > 0$

Remark 2

1. For each of these categories, equation (24) and its solutions have distinct features.
2. In general, a second order equation may be of one type at a specific point, and of another type at some other point.
3. The terminology is motivated from the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

which—for A, B, C, D, E, F being constants—represents a conic section in the xy-plane and the different types of conic sections arising are determined by $B^2 - 4AC$.

Remark 3

1. The canonical examples of the elliptic, parabolic and hyperbolic PDEs are the two-dimensional Laplace equation, one-dimensional heat equation and one-dimensional wave equations, i.e.,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \textbf{(Laplace equation)}$$

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \textbf{(Heat equation)}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \textbf{(Wave equation)}$$

Let u denotes the dependent variable in a boundary value problem (BVP).

Definition 16 (Dirichlet condition)

A condition that prescribes the values of u itself along a portion of the boundary is known as a *Dirichlet condition*.

Definition 17 (Neumann condition)

A condition that prescribes the values of the normal derivatives $\partial u/\partial\hat{n}$ on a portion of the boundary is known as a *Neumann condition*. Here, $\hat{n}$ denotes the unit outward normal to the boundary.

Definition 18 (Robin condition)

A condition that prescribes the values of $hu + \partial u/\partial\hat{n}$ at boundary points is known as a *Robin condition*. Here, h is either a constant or a function of the independent variables.

Definition 19 (Cauchy condition)

If a PDE in u is of second order with respect to one of the independent variables t (time) and if the values of both u and u_t are prescribed at $t = 0$, the boundary condition is known as a *Cauchy-type* condition with respect to t.

Definition 20 (Two-dimensional Laplace equation)

The two-dimensional Laplace equation is given by

$$\nabla^2 u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{26}$$

Note: the Laplace equation is also referred to as the *potential equation*.