Lecture-3

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Existence Theorem for Analytic Coefficients

Let x_0 be any real number and suppose that the coefficients $p_0(x), p_1(x), \dots, p_{n-1}(x)$ in

$$f(D)y = y^n(x) + p_{n-1}(x)y^{n-1}(x) + p_{n-2}(x)y^{n-2}(x) + \cdots + p_0(x)y(x)$$

have convergent power series expansions in powers of $x - x_0$ on an interval

$$|x - x_0| < r \quad , \quad r > 0$$

If $\alpha_1, \alpha_2, \dots \alpha_n$ are any n constants, then $\exists$ a solution ϕ of the problem

$$f(D)y = 0, \quad y(x_0) = \alpha_1,$$
$$y'(x_0) = \alpha_2$$
$$\vdots$$
$$y^{n-1}(x_0) = \alpha_n$$

With a power series expansion

$$\boxed{\phi(x) = \sum_{k=0}^{\infty} c_k (x - x_0)^k}$$

Convergent for $|x - x_0| < R$ where the radius of convergence $R \geq r$

Theorem:

Suppose x_0 is an ordinary point of the n^{th} order linear ODE

$$y^n(x) + p_{n-1}(x)y^{n-1}(x) + \cdots + p_0(x)y(x) = f(x)$$

where the coefficients $p_0(x), p_1(x), \dots, p_{n-1}(x)$ and $f(x)$ are analytic at $x = x_0$ and each can be expressed as a power series about x_0 convergent for $|x - x_0| < r$, $r > 0$.

Then every solution of this diff. equ. can be expressed in one and only one way as a power series in $(x - x_0)$

$$\boxed{y_i(x) = \sum_{n=0}^{\infty} a_{i,n}(x - x_0)^n}, \quad |x - x_0| < R$$

where the radius of convergence $R \geq r$

In particular, the series solution about the ordinary point $x = x_0$ of 2nd order ODE.

Series Solutions of ODE's

Theorem:

The power series representation $\boxed{y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n}$ about an ordinary point $x = x_0$ of the differential equation $\boxed{\frac{d^2y}{dx^2} + p_1(x)\frac{dy}{dx} + p_0(x)y = 0}$ always converges.

The maximum possible radius of convergence R is the distance from x_0 to the nearest singular point of the differential equation and the interval of convergence is $(x_0 - R, x_0 + R)$.

Ex ① Solve in series the equation

$$(x^2+1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$$

Sol^n The given D.E. can be written as

$$\frac{d^2y}{dx^2} + \frac{x}{x^2+1}\frac{dy}{dx} - \frac{y}{x^2+1} = 0 \quad \text{—①}$$

Comparing the above equation with $y'' + P_1(x)y' + P_0 y = 0$

we have

$$p_1(x) = \frac{x}{x^2+1}$$
$$P_0(x) = -\frac{1}{x^2+1}$$

Since, all the coefficients $P_0(x)$ and $p_1(x)$ are analytic at $x = 0$, i.e., $p_i(x)$ for $i = 0, 1$ can be expressed as power series about $x = 0$ that are convergent for $-1 < x < 1$ i.e., for $i = 0, 1$.

$$p_i(x) = (-1)^{i+1} x^i (1 + x^2)^{-1}$$
$$= (-1)^{i+1} x^i [1 - x^2 + x^4 - x^6 + \cdots], \quad -1 < x < 1$$

So, $x = 0$ is the ordinary point of the D.E and let

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} a_n x^n, \quad -1 < x < 1 \quad \text{—②}$$

be the series solution of ①.

Differentiating twice in a succession ② gives

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad -1 < x < 1$$

$$\& \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}, \quad -1 < x < 1$$

Substituting these values y, y', y'' in the given equation, we get

$$(x^2+1)\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n$$
$$- \sum_{n=0}^{\infty} a_n x^n = 0$$

We shift the index of summation in the 2nd series by 2, replacing n with $n+2$ and using the initial value $n = 0$, and we shift the index of summation in the third series by 1, replacing n with $n+1$ and using the initial value $n = 0$.

$$2a_2 - a_0 + 6a_3 x + \sum_{n=2}^{\infty} \{n(n-1)a_n + (n+1)(n+2)a_{n+2}$$
$$+ n a_n - a_n\} x^n = 0$$

Equating the coefficient of various power of x to zero, we get

$$2a_2 - a_0 = 0 \Rightarrow a_2 = \frac{a_0}{2}$$
$$6a_3 = 0 \Rightarrow a_3 = 0$$
and $$n(n-1)a_n + (n+1)(n+2)a_{n+2} + n a_n - a_n = 0$$
$$\Rightarrow a_{n+2} = \frac{-(n-1)}{(n+2)} a_n \quad \forall \; n \geq 2$$

Now putting $n = 2, 3, 4, \dots$ successively in the above recurrence relation we get

$$a_4 = -\frac{1}{4}a_2 = -\frac{1}{8}a_0$$
$$a_5 = -\frac{2}{3}a_3 = 0$$
$$a_6 = -\frac{1}{2}a_4 = \frac{1}{16}a_0$$
$$a_7 = -\frac{4}{7}a_5 = 0$$
$$a_8 = -\frac{5}{8}a_6 = -\frac{5}{128}a_0 \quad \text{and so on}$$

Substituting the values of $a_0, a_1, a_2, \dots$ in ② we get the required solution as

$$y(x) = a_0\left\{1 + \frac{x^2}{2} - \frac{x^4}{8} + \frac{x^6}{16} - \frac{5x^8}{128} + \cdots\right\}$$
$$+ a_1 x, \quad -1 < x < 1$$

where a_0 & a_1 are arbitrary constants.