Indian Institute of Technology Indore
MA-204 Numerical Methods
Assignment -1-System of Linear Equations

1. Solve the following systems of equations by converting the coefficient matrix to row reduced echelon form.

(a) $$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 3 \end{bmatrix}$$

(b) $$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 20 \\ 14 \end{bmatrix}$$

(c) $$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & -2 \\ 3 & 1 & 2 \\ 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 3 \\ 8 \\ 7 \end{bmatrix}$$

(d) $$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 14 \\ 20 \\ 14 \end{bmatrix}$$

2. Solve the following systems of equations by Gauss elimination method:

(a) x+2y+z=0
2x+2y+3z=3
x+3y= -2

(b) x+2y+3z=14
2x+3y+4z=20
3x+4y+z=14

(c) 2x+3y+z=9
x+2y+3z=6
3x+y+2z=8

3. Find the inverse of the following matrices by the Gauss-Jordan elimination method:

(a) $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{pmatrix}$

4. Solve the following linear systems by Gauss-Jordan elimination method, with **partial pivoting** if necessary (but without scaling):

(a) 4x+y+z=4
x+4y-2z=4
3x+2y-4z= 6

(b) x+y-z=2
2x+3y+5z=-3
3x+2y-3z=6

(c) 2x+3y+z=9
x+2y+3z=6
3x+y+2z=8

5. Solve the following linear systems by Gauss elimination method, with **total pivoting** if necessary (but without scaling):

(a) 3x+5y+2z=8
8y+2z=-7
6x+2y+8z=26

(b) 2x+y-z=0
x+y+z=9
2x+5y+7z=52

(c) x+y+z=2
2x+2y+3z=7
5x-y+13z=0

6. Solve the following systems of equations by Doolittle's and Crout's methods:

(a) 10x+y+z=12
x+10y+z=12
x+y+10z= 12

(b) x+y=0
y+z=1
x+z= 3

(c) x+y+z=2
2x+2y+3z=7
5x-y+13z=0

7. Verify whether the following matrices are positive definite:

(a) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$ (c) $\begin{pmatrix} 15 & 4 & 2 & 9 & 0 \\ 4 & 7 & 1 & 1 & 1 \\ -2 & 1 & 18 & 6 & 6 \\ 9 & 1 & 6 & 19 & 3 \\ 0 & 1 & 6 & 3 & 11 \end{pmatrix}$

8. Solve the following systems of equations by Cholesky's method, if the method is applicable. If it is not applicable, give the reason.

(a) $$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

(b) $$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 20 \\ 14 \end{bmatrix}$$

Solutions:

- [1a] $\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 3 \end{bmatrix}$, We start with writing augmented matrix $[A|b]$.

  $$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 9 \\ 1 & 2 & 3 & 6 \\ 1 & 1 & -2 & 3 \end{array}\right] \xrightarrow[R_{23(-1)}]{R_{13(-2)}} \left[\begin{array}{ccc|c} 0 & 1 & 5 & 3 \\ 0 & 1 & 5 & 3 \\ 1 & 1 & -2 & 3 \end{array}\right] \xrightarrow[R_{12(-1)}]{R_{32(-1)}} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & 5 & 3 \\ 1 & 0 & -7 & 0 \end{array}\right] \xrightarrow{R_{13}} \left[\begin{array}{ccc|c} 1 & 0 & -7 & 0 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

  Hence the solution is $x_2 = 3 - 5x_3$, and $x_1 = 7x_3$, for all $x_3 \in \mathbb{R}$.

- [1b] $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 14 \\ 2 & 3 & 4 & 20 \\ 1 & 3 & 5 & 14 \end{array}\right] \xrightarrow[R_{21(-2)}]{R_{31(-1)}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 14 \\ 0 & -1 & -2 & -8 \\ 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow{R_{32(1)}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 14 \\ 0 & -1 & -2 & -8 \\ 0 & 0 & 0 & 8 \end{array}\right]$

  Hence the system is not consistent, and there is no solution.

- [1d] $\left[\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 14 \\ 2 & 3 & 4 & 1 & 20 \\ 3 & 4 & 1 & 3 & 14 \end{array}\right] \xrightarrow[R_{21(-2)}]{R_{31(-3)}} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 14 \\ 0 & -1 & -2 & -1 & -8 \\ 0 & -2 & -8 & 0 & -28 \end{array}\right] \xrightarrow{R_{32(-2)}} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 14 \\ 0 & -1 & -2 & -1 & -8 \\ 0 & 0 & -4 & 2 & -12 \end{array}\right]$

- [2b] $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 14 \\ 2 & 3 & 4 & 20 \\ 1 & 3 & 1 & 14 \end{array}\right] \xrightarrow[R_{21(-2)}]{R_{31(-1)}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 14 \\ 0 & -1 & -2 & -8 \\ 0 & 1 & -2 & 0 \end{array}\right] \xrightarrow{R_{32(1)}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 14 \\ 0 & -1 & -2 & -8 \\ 0 & 0 & -4 & 8 \end{array}\right]$

  The solution is $x_3 = -2, x_2 = 8 - 2x_3 = 12, x_1 = 14 - 2x_2 - 3x_3 = 14 - 24 + 6 = -4$

- [3c] $\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 3 & 4 & 0 & 1 & 0 \\ 1 & 4 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_{21(-1)}]{R_{31(-1)}} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 2 & 0 & -1 & 0 & 1 \end{array}\right] \xrightarrow[R_{23(-\frac{1}{2})}]{R_{13(-1)}} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 2 & 0 & 1 \\ 0 & 0 & 1 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 2 & 0 & -1 & 0 & 1 \end{array}\right]$

  $$\xrightarrow[R_{12(-3)}]{R_{3(\frac{1}{2})}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{2} & -3 & \frac{3}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right] \xrightarrow[R_{12}]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{2} & -3 & \frac{3}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & 1 & -\frac{1}{2} \end{array}\right]$$

- [4b] $\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 2 & 3 & 5 & -3 \\ 3 & 2 & -3 & 6 \end{array}\right] \xrightarrow{R_{31}} \left[\begin{array}{ccc|c} 3 & 2 & -3 & 6 \\ 2 & 3 & 5 & -3 \\ 1 & 1 & -1 & 2 \end{array}\right] \xrightarrow[R_{21(-\frac{2}{3})}]{R_{31(-\frac{1}{3})}} \left[\begin{array}{ccc|c} 3 & 2 & -3 & 6 \\ 0 & \frac{5}{3} & 7 & -7 \\ 0 & \frac{1}{3} & 0 & 0 \end{array}\right] \xrightarrow{R_{32(-\frac{1}{5})}}$

  $$\left[\begin{array}{ccc|c} 3 & 2 & -3 & 6 \\ 0 & \frac{5}{3} & 7 & -7 \\ 0 & 0 & -\frac{7}{5} & \frac{7}{5} \end{array}\right] \xrightarrow[R_{13(-\frac{15}{7})}]{R_{23(5)}} \left[\begin{array}{ccc|c} 3 & 2 & 0 & 3 \\ 0 & \frac{5}{3} & 0 & 0 \\ 0 & 0 & \frac{7}{5} & -\frac{7}{5} \end{array}\right] \xrightarrow[R_{12(-\frac{6}{5})}]{R_{3(\frac{5}{7})}} \left[\begin{array}{ccc|c} 3 & 0 & 0 & 3 \\ 0 & \frac{5}{3} & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow[R_{1(\frac{1}{3})}]{R_{2(\frac{3}{5})}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

  Note that for partial pivoting we first search for the pivot element in the first column with largest magnitude, next we consider the possible pivot entries in the second column and again consider the largest in magnitude.

- [5] Suppose if we want to solve [4b] by Gauss elimination with total pivoting, we first need to search for the first pivot entry as of the largest magnitude in the matrix, which 5, placed in the third column and the second row, since we can do only row transformations we should first interchange the second and third row to get pivot entry at a_{33}.

  $$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 2 & 3 & 5 & -3 \\ 3 & 2 & -3 & 6 \end{array}\right] \xrightarrow{R_{32}} \left[\begin{array}{ccc|c} 3 & 2 & -3 & 6 \\ 1 & 1 & -1 & 2 \\ 2 & 3 & 5 & -3 \end{array}\right] \xrightarrow[R_{13(\frac{3}{5})}]{R_{23(\frac{1}{5})}} \left[\begin{array}{ccc|c} \frac{21}{5} & \frac{19}{5} & 0 & \frac{21}{5} \\ \frac{7}{5} & \frac{8}{5} & 0 & \frac{7}{5} \\ 2 & 3 & 5 & -3 \end{array}\right] \xrightarrow{R_{21}} \left[\begin{array}{ccc|c} \frac{7}{5} & \frac{8}{5} & 0 & \frac{7}{5} \\ \frac{21}{5} & \frac{19}{5} & 0 & \frac{21}{5} \\ 2 & 3 & 5 & -3 \end{array}\right]$$

  $\xrightarrow{R_{12(-\frac{8}{19})}} \left[\begin{array}{ccc|c} \frac{7}{5} - \frac{21\times 8}{5\times 19} & 0 & 0 & \frac{7}{5} - \frac{21\times 8}{5\times 19} \\ \frac{21}{5} & \frac{19}{5} & 0 & \frac{21}{5} \\ 2 & 3 & 5 & -3 \end{array}\right]$ Now we can find the solution by forward substitution method.

- $[7c]$ and $[8a]$ are not symmetric, hence not positive definite.