MA 204 Numerical Methods

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Lecture-7

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Contents

- Solution of a nonlinear equation, bisection and secant methods, Newton's method, rate of convergence.

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- Solution of a nonlinear equation, bisection and secant methods, Newton's method, rate of convergence.
- Interpolation by polynomials, divided differences, error of the interpolating polynomial, piecewise linear and cubic spline interpolation.

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- big error in certain intervals (esp. near the ends);
- no convergence result;
- heavy to compute for alrge n.

Suggestion: use piecewise polynomial interpolation.

Usage:

- visualization of discrete data
- graphic design

Requirement:

- interpolation
- certain degree of smoothness

Problem Setting

x	t_0	t_1	$\cdots t_n$
y	y_0	y_1	$\cdots y_n$

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Find a function $\mathcal{S}(x)$ which interpolates the point $(t_i, y_i)_{i=0}^n$. The set $t_0 < t_1 < \cdots < t_n$ are called knots. Note that they need to be ordered. $\mathcal{S}(x)$ consists of piecewise polynomial

$$\mathcal{S}(x) = \begin{cases} \mathcal{S}_0(x), & t_0 \leq x \leq t_1 \\ \mathcal{S}_1(x), & t_1 \leq x \leq t_2 \\ \vdots \\ \mathcal{S}_{n-1}(x), & t_{n-1} \leq x \leq t_n. \end{cases} \tag{1}$$

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$$\begin{aligned} \mathcal{S}_{i-1}(t_i) &= \mathcal{S}_i(t_i), \\ \mathcal{S}'_{i-1}(t_i) &= \mathcal{S}'_i(t_i), \\ &\vdots \\ \mathcal{S}^{(k-1)}_{i-1}(t_i) &= \mathcal{S}^{(k-1)}_i(t_i). \end{aligned}$$

Commonly used splines:

- $n = 1$: linear spline (simplest)
- $n = 2$: quadratic spline (less popular)
- $n = 3$: cubic spline (most used)

Problem 1. Determine whether this function is a first-degree spline function:

$$\mathcal{S}(x) = \begin{cases} x, & x \in [-1, 0], \\ 1 - x, & x \in (0, 1), \\ 2x - 2, & x \in [1, 2]. \end{cases} \tag{2}$$

Answer: Check all the properties of a linear spline.

- Linear polynomial for each piece: OK!

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- $\mathcal{S}(x)$ is continuous at inner knots:
  At $x = 0$, $\mathcal{S}(x)$ is discontinuous, because from the left we get 0 and from the right we get 1.
  Therefore, this is NOT a linear spline.

Problem 1. Determine whether the following function is a quadratic spline:

$$\mathcal{S}(x) = \begin{cases} x^2, & x \in [-10, 0], \\ -x^2, & x \in (0, 1), \\ 1 - 2x, & x \geq 1. \end{cases} \tag{3}$$

Answer: Let us label each piece:

$$Q_0(x) = x^2, \quad Q_1(x) = -x^2, \quad Q_2(x) = 1 - 2x.$$

We now check all the conditions, i.e., the continuity of Q and Q' at inner knots $0, 1$:

$$\begin{aligned} Q_0(0) = 0, &\quad Q_1(0) = 0, \quad \text{OK!} \\ Q_1(1) = -1, &\quad Q_2(1) = -1, \quad \text{OK!} \\ Q_0'(0) = 0, &\quad Q_1'(0) = 0, \quad \text{OK!} \\ Q_1'(1) = -2, &\quad Q_2'(1) = -2, \quad \text{OK!} \end{aligned}$$

It passes all the test, so it is a quadratic spline.

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Requirements:

$$\mathcal{S}_0(t_0) = y_0 \tag{4}$$

$$\mathcal{S}_{i-1}(t_i) = \mathcal{S}_i(t_i) = y_i, \quad i = 1, 2, \cdots, n-1 \tag{5}$$

$$\mathcal{S}_{n-1}(t_n) = y_n. \tag{6}$$

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Easy to find: write the equation for a line through two points: (t_i, y_i) and (t_{i+1}, y_{i+1})

$$\mathcal{S}_i(x) = y_i + \frac{y_{i+1} - y_i}{t_{i+1} - t_i}(x - t_i), \quad i = 0, 1, \cdots, n-1. \tag{7}$$

Accuracy Theorem for linear spline

- Assume $t_0 < t_1 < \cdots < t_n$, and let $h_i = t_{i+1} - t_i$, $h = \max_i h_i$.
- $f(x)$: given function, $\quad \mathcal{S}(x)$: a linear spline
- $\mathcal{S}(t_i) = f(t_i), \quad i = 0, 1, \cdots, n.$

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We have the following, for $x \in [t_0, t_n]$.

(a) If f'' exists and is continuous, then,

$$|f(x) - \mathcal{S}(x)| \le \max\left\{\frac{1}{8}h_i^2, \max_{t_i \le x \le t_{i+1}} |f''(x)|\right\} \le \frac{1}{8}h^2 \max_x |f''(x)|.$$

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(b) If f' exists and is continuous, then

$$|f(x) - \mathcal{S}(x)| \leq \max_i \left\{ \frac{1}{2} h, \max_{t_i \leq x \leq t_{i+1}} |f'(x)| \right\} \leq \frac{1}{2} h \max_x |f'(x)|.$$

To minimize the error, it is obvious that one should add more knots where the function has large first or second derivative.