Hence the general solutions of the wave equation in terms of its original variable x and t are then given by:

$$\boxed{u(x,t) = f(x-ct) + g(x+ct)}$$

$\Downarrow$

Interpretation: A general solution of wave equation can be expressed as the sum of two waves, one travelling to the right with constant velocity c and the other travelling to the left with the same velocity c.

· Difference Equations for the Laplace and Poisson Equations

Let us consider the Laplace equation

$$\Delta u = \partial_{xx} u + \partial_{yy} u = 0 \qquad (1)$$

and the Poisson equation

$$\Delta u = \partial_{xx} u + \partial_{yy} u = f(x,y) \qquad (2)$$

To obtain methods of numeric solution, we replace the partial derivatives by corresponding difference quotients, as follows:

~~By f~~

Recall Taylor's Formula: (T.M. Apostol, Mathematical Analysis, P. 113)

Let f be a function having finite nth derivative f^{n} everywhere in an open interval (a,b) and assume that $f^{(n-1)}$ is continuous on the closed interval $[a,b]$. Assume ~~that~~ $c \in [a,b]$. Then, for every $x \in [a,b]$, $x \neq c$, $\exists$ a point x_1 interior to the interval joining x and c such that

$$f(x) = f(c) + \sum_{k=1}^{n-1} \frac{f^{k}(c)}{k!}(x-c)^{k} + \frac{f^{(n)}(x_1)}{n!}(x-c)^{n}$$

By the Taylor formula,

$$(3a)\quad u(x+h,y) = u(x,y) + h\,\partial_x u(x,y) + \frac{1}{2}h^2 \partial_{xx} u(x,y) + \frac{1}{6}h^3 \partial_{xxx}(x,y) + \cdots$$

$$(3b)\quad u(x-h,y) = u(x,y) - h\,\partial_x u(x,y) + \frac{1}{2}h^2 \partial_{xx} u(x,y) - \frac{1}{6}h^3 \partial_{xxx} u(x,y) + \cdots$$

We subtract (3b) from (3a), neglect terms in $h^3, h^4, \ldots$ and solve for $\partial_x u$. Then,

$$\partial_x u(x,y) \approx \frac{1}{2h}\left[u(x+h,y) - u(x-h,y)\right]. \quad \text{(4a)}$$

Similarly,

$$u(x,y+k) = u(x,y) + k\,\partial_y u(x,y) + \frac{1}{2}k^2 \partial_{yy} u(x,y) + \cdots$$

and

$$u(x,y-k) = u(x,y) - k\,\partial_y u(x,y) + \frac{1}{2}k^2 \partial_{yy} u(x,y) + \cdots$$

Similar to (4a) we obtain,

$$\partial_y u(x,y) \approx \frac{1}{2k}\left[u(x,y+k) - u(x,y-k)\right].$$

Now, let us turn to second derivatives.

Adding (3a) and (3b) and neglecting terms in $h^4, h^5, \ldots$, we obtain,

$$u(x+h,y) + u(x-h,y) \approx 2u(x,y) + h^2 \partial_{xx} u(x,y).$$

$$\Rightarrow \partial_{xx} u(x,y) \approx \frac{1}{h^2}\left[u(x+h,y) - 2u(x,y) + u(x-h,y)\right]. \quad \text{(5a)}$$

Similarly,

$$\partial_{yy} u(x,y) \approx \frac{1}{k^2}\left[u(x,y+k) - 2u(x,y) + u(x,y-k)\right]. \quad \text{(5b)}$$

Next choosing $h = k$ and substituting (5a) and (5b) into the Poisson equation (2):

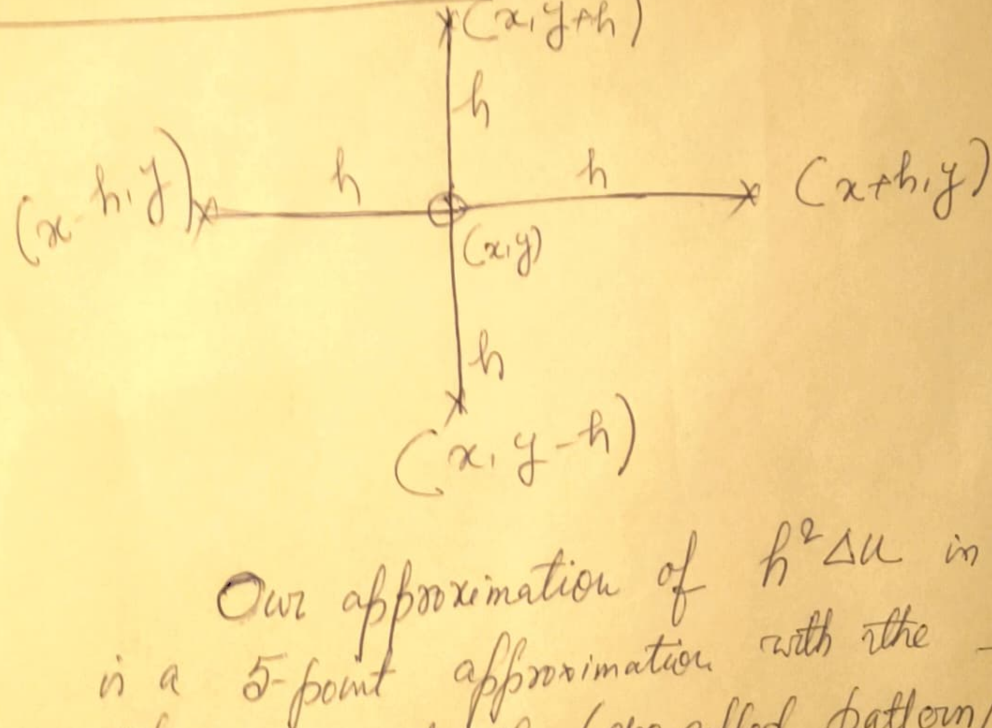
$$u(x+h,y) + u(x,y+h) + u(x-h,y) + u(x,y-h) - 4u(x,y) = h^2 f(x,y). \quad \text{(6)}$$

This is a difference equation corresponding to (2). Hence for the Laplace eq^n (1), the corresponding difference equation is,

$$u(x+h,y)+u(x,y+h)+u(x-h,y)+u(x,y-h)-4u(x,y)=0. \quad \text{—— (7)}$$

h — is called the <u>mesh size</u>.

Interpretation: u at (x,y) equals the mean of the values of u at the four neighboring points.



Our approximation of $h^2\Delta u$ in (6) and (7) is a 5-point approximation with the <u>coefficient scheme</u> or <u>stencil</u> (also called <u>pattern/molecule/star</u>)

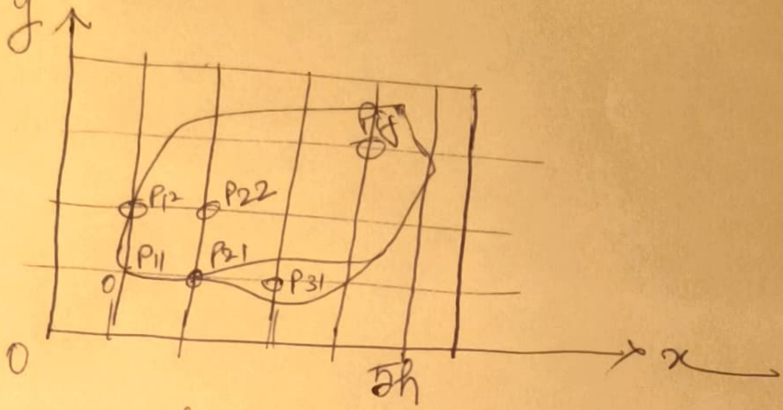
$$(8) \text{ —— } \left\{\begin{matrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{matrix}\right\}.$$

We may now write (6) as,

$$\left\{\begin{matrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{matrix}\right\} u = h^2 f(x,y)$$

Dirichlet Problem:

In numerics for the Dirichlet problem in a region R we choose an h and introduce a square grid of horizontal and vertical straight lines of distance h. This interactions are called mesh points (or lattice points / nodes).



Region in the xy-plane covered by a grid of mesh h, also showing mesh points

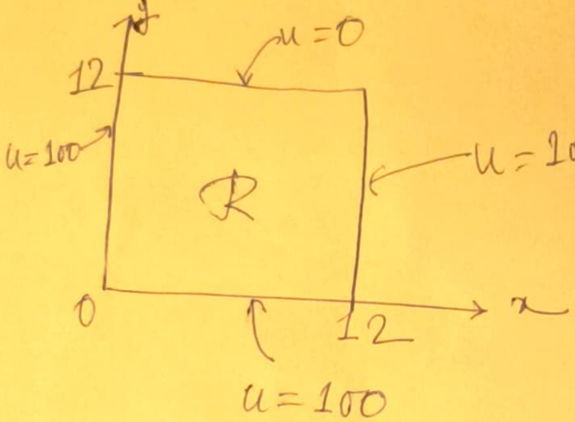
$$P_{11} = (h, h), \ldots, P_{ij} = (ih, jh), \ldots$$

With the notation ⑦ for any mesh point P_{ij} we can write,

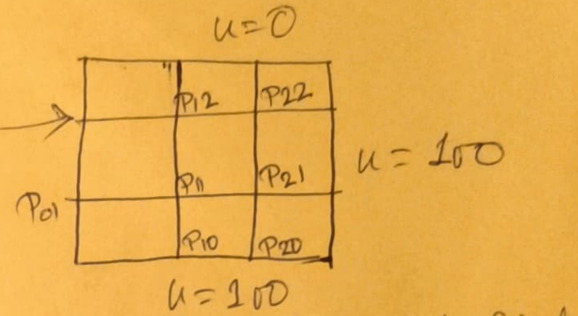
$$⑨ \longrightarrow \quad u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1} - 4u_{ij} = 0$$

Example 1.

The four sides of a square plate of side 12cm made of homogeneous material are kept at constant temperature 0°C and 100°C as in the figure below.



(a) Given problem



(b) Grid and mesh points.

Using a grid of mesh of 4cm and applying Gauss - Seidel iteration, find the (steady - state) temperature at the mesh points.

Solution: Recall, that the steady-state temperature distribution solves

$$\partial_{xx}u + \partial_{yy}u = 0$$

Hence applying (7) at all the four mesh points:

$$(10)\quad \left\{\begin{aligned} -4u_{11} + u_{21} + u_{12} \phantom{+u_{22}} &= -200 \\ u_{11} - 4u_{21} \phantom{+u_{12}} + u_{22} &= -200 \\ u_{11} \phantom{+u_{21}} - 4u_{12} + u_{22} &= -100 \\ u_{21} + u_{12} - 4u_{22} &= -100. \end{aligned}\right.$$

Assignment: Solve (10) by Gauss elimination to obtain, $u_{11} = u_{21} = 87.5$, $u_{12} = u_{22} = 62.5$.

Solving (10) by Gauss-Siedel:

Let us write (10) in the form:

$$(11)\quad \begin{cases} u_{11} = 0.25\,u_{21} + 0.25\,u_{12} + 50 \\ u_{21} = 0.25\,u_{11} + 0.25\,u_{22} + 50 \\ u_{12} = 0.25\,u_{11} + 0.25\,u_{22} + 25 \\ u_{22} = 0.25\,u_{21} + 0.25\,u_{12} + 25. \end{cases}$$

To solve (11) one implements the following itiration process:

$$(12b)\quad \begin{cases} u_{11}^{(n+1)} = 0.25\,u_{21}^{(n)} + 0.25\,u_{12}^{(n)} + 50 \\ u_{21}^{(n+1)} = 0.25\,u_{11}^{(n+1)} + 0.25\,u_{22}^{(n)} + 50 \\ u_{12}^{(n+1)} = 0.25\,u_{11}^{(n+1)} + 0.25\,u_{22}^{(n)} + 25 \\ u_{22}^{(n+1)} = 0.25\,u_{21}^{(n+1)} + 0.25\,u_{12}^{(n+1)} + 25 \end{cases}$$

and for the beginning itirate one assumes, (ie for $n=0$)

$$(12a)\quad \left\{ u_{11}^{(0)} = u_{12}^{(0)} = u_{21}^{(0)} = u_{22}^{(0)} = 100. \right.$$

Assignment: Write a Matlab code to solve the iterative equations (12a) - (12b) and to conclude ~~that~~,

$$u_{11} = u_{21} = 87.5$$

and $u_{12} = u_{22} = 62.5$.

Different iteration techniques ~~based on~~ for Laplace equation based on 'standard-five-point formulae':

Jacobi's Method:

Let $u_{i,j}^{(n)}$ denotes the nth ~~decimal~~ iterative value of $u_{i,j}$. An iterative procedure to solve

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{ij} = 0 \quad (*)$$

is

$$u_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i-1,j}^{(n)} + u_{i+1,j}^{(n)} + u_{i,j-1}^{(n)} + u_{i,j+1}^{(n)}\right]$$

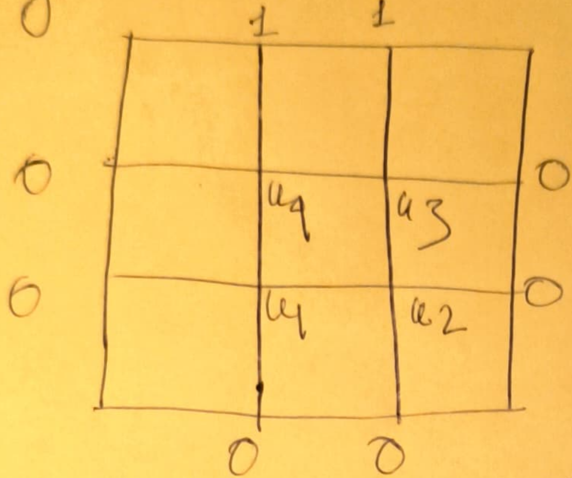
Gauss - Seidel Method:

This method uses the latest iterative values available and scans the mesh points systematically from left to right along successive rows. The iterative formulae is:

$$u_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i-1,j}^{(n+1)} + u_{i+1,j}^{(n)} + u_{i,j-1}^{(n+1)} + u_{i,j+1}^{(n)}\right].$$

Example:

Solve the eq^n $\partial_{xx}u + \partial_{yy}u = 0$ in the following domain



by ① Gauss–Seidel's method and ② ~~Gauss~~ Jacobi's iteration.

① Gauss-Seidel method:

$$u_1^{n+1} = \frac{1}{4}\left[0 + 0 + u_2^n + u_4^n\right]$$

Initial guess, $u_2^0 = 0$, $u_4^0 = 1$

$$\Rightarrow u_1^1 = 0.25$$

$$u_2^{n+1} = \frac{1}{4}\left[u_3^n + u_1^{n+1} + 0 + 0\right]$$

Initial guess, $u_3^0 = 1$

$$\Rightarrow u_2^1 = \frac{1}{4}\left[1 + 0.25\right] = 0.3125$$

$$u_4^{n+1} = \frac{1}{4}\left[1 + 0 + u_3^n + u_1^{n+1}\right]$$

Since, initial guess, $u_3^0 = 1$

$$u_4^1 = \frac{1}{4}\left[1 + 1 + 0.25\right] = 0.5625$$

$$u_3^{n+1} = \frac{1}{4}\left[u_4^{n+1} + u_2^{n+1} + 0 + 1\right]$$

$$\Rightarrow u_3^1 = \frac{1}{4}\left[0.5625 + 0.3125 + 1\right] = 0.46875$$

Exercise: ~~Continue using~~ $(u_1^1, u_2^1, u_3^1, u_4^1)$ ~~and~~ Write a Matlab Code to calculate upto 5 iterates of Gauss-Seidel starting from $(u_1^1, u_2^1, u_3^1, u_4^1)$.