Tutorial

① ~~claim: $J_n(x) = \frac{1}{2}\int_0^{\pi} \cos(n\theta - x\sin\theta)\,d\theta$ n being an integer~~

④ claim: a) $P_n(-1) = (-1)^n$

We know that the generating function formula:

$$(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x) \quad , |z|<1 \; , |x|\le 1$$

Part 1 Put $x=1$ then we get

$$(1-2z+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(1) \Rightarrow (1-z)^{2\times(-1/2)} = \sum_{n=0}^{\infty} z^n (P_n(1)) \Rightarrow (1-z)^{-1} = \sum_{n=0}^{\infty} z^n (P_n(1))$$

Note $(1-z)^{-1} = 1+z+z^2+\cdots$ valid when $|-z|<1$ or $|z|<1$

Since $|z|<1$, Using binomial theorem we expand $(1-z)^{-1}$

$$1+z+z^2+\cdots+z^n+\cdots = \sum_{n=0}^{\infty} z^n P_n(1)$$

Equating Coefficients of z^n from both side

$$P_n(1) = 1$$

Part 2 put $x=-1$ then we have

$$(1+2z+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P(-1) \quad \text{or} \quad (1+z)^{-1} = \sum_{n=0}^{\infty} z^n P_n(-1)$$

or $$1-z+z^2 - \cdots + (-1)^n z^n + \cdots = \sum_{n=0}^{\infty} z^n P_n(-1)$$

equating the coefficient of z^n from both sides

$$\boxed{P_n(-1) = (-1)^n}$$

ⓑ $P_n'(-1) = (-1)^{n-1} \times \frac{1}{2} n(n+1)$ $\qquad A P_n(x) + B Q_n(x)$

$P_n(x)$ satisfies Legendre's Equation.

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

we get $(1-x^2)P_n'' - 2xP_n' + n(n+1)P_n = 0$

put $x=1$ & use $P_n(1)=1$ we get

$$0 - 2P_n'(1) + n(n+1) = 0 \Rightarrow P_n'(1) = \tfrac{1}{2} n(n+1)$$

put $x=-1$ & use $P_n(-1) = (-1)^n$

$$0 + 2P_n'(-1) + n(n+1)(-1)^n = 0$$

or $P_n'(-1) = -(-1)^n \times \frac{1}{2} n(n+1)$ $\qquad [\because -(-1)^n = -(-1)^{n-1}(-1) = (-1)^{n-1}]$

$$\boxed{P_n'(-1) = (-1)^{n-1} \frac{1}{2} n(n+1)}$$

Proved

(c) Proof: $$P_{2m}(0) = (-1)^m \frac{(2m)!}{2^{2m}(m!)^2}$$

We have $$\sum_{n=0}^{\infty} z^n P_n(x) = (1-2xz+z^2)^{-1/2}, \quad |z|<1,\ |x|\le 1$$

Put $x=0$, then
$$\sum_{n=0}^{\infty} z^n P_n(0) = (1+z^2)^{-1/2}$$

Note
$$(1+z^2)^{-1/2} = \cdots$$
$$(1+x)^{-n} = 1 + (-n)x + \frac{(-n)(-n-1)}{2!}x^2 + \cdots \frac{(-n)(-n-1)\cdots(-n-r+1)}{r!}x^r$$

$$\therefore \sum_{n=0}^{\infty} z^n P_n(0) = 1 + \frac{(-\frac{1}{2})z^2}{1!} + \frac{(-\frac{1}{2})(-\frac{1}{2}-1)}{2!}(z^2)^2 + \cdots \frac{(-\frac{1}{2})\cdots(-\frac{1}{2}-r+1)}{r!}(z^2)^r$$

$$= \sum_{n=0}^{\infty} \frac{(-\frac{1}{2})(-\frac{1}{2}-1)\cdots(-\frac{1}{2}-n+1)}{n!}(z^2)^n$$

or $$\sum_{n=0}^{\infty} z^n P_n(0) = \sum_{n=0}^{\infty} (-1)^n \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n n!} z^{2n}$$ → This contains only every power of z alone. So equating coefficient z^{2m+1} from both sides we have $P_{2m+1} = 0$.

Equating the coefficients of z^{2m} from both sides
$$P_{2m}(0) = (-1)^m \frac{1\cdot 3\cdot 5\cdots(2m-1)}{2^m m!}$$

$$= (-1)^m \frac{1\cdot 2\cdot 3\cdot 5\cdots(2m-1)\,2m}{2^m\cdot m!\ 2\cdot 4\cdot 6\cdots(2m)}$$

$$= (-1)^m \frac{(2m)!}{2^m m!\,(2\cdot 1)\cdot(2\cdot 2)(2\cdot 3)(2\cdot 4)\cdots(2\cdot m)}$$

$$= (-1)^m \frac{(2m)!}{2^m\cdot m!\ 2^m\cdot m!} = (-1)^m \frac{(2m)!}{2^{2m}(m!)^2}$$

Proved

$\cancel{f'(0) = 0 = f''(0) \cdots f^{n-1}(0)}$

(5) We know that $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n$

$\int uv\,dx = u\int v\,dx - \int u'(\int v\,dx)\,dx$

Now $\int_{-1}^{1} f(x) P_n(x)\,dx = \frac{1}{2^n n!} \int_{-1}^{1} \underbrace{\frac{d^n}{dx^n}(x^2-1)^n}_{f_1} \underbrace{f(x)}_{f_2}\,dx$

$= \frac{1}{2^n n!}\left[\left[\underbrace{\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n}_{\int f_1 dx} \underbrace{f(x)}_{f_2}\right]_{-1}^{1} - \int_{-1}^{1} \underbrace{\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n}_{\int f_1 dx} \underbrace{f'(x)}_{f_2'}\,dx\right]$

(On integration by parts)

Also, $\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n = \frac{d^{n-1}}{dx^{n-1}}(x-1)^n(x+1)^n$ {By Leibnitz theorem}

$= (x-1)^n (n-1)!(x+1) + {}^{n-1}C_1\, n(x-1)^{n-1}(n-2)!(x+1)^2 + \dots$

$+ (n-1)!(x-1)(x+1)^n$

[Since $(uv)_n = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \dots u v_n$

where $u_n = \frac{d^n u}{dx^n}$]

clearly $\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n$ will be zero at $x=1$ & $x=-1$

$\int_{-1}^{1} f(x) P_n(x)\,dx = -\int_{-1}^{1} \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n f'(x)\,dx$

$= (-1)^n \frac{1}{2^n n!} \int_{-1}^{1} (x^2-1)^n f^{(n)}(x)\,dx$

$= (-1)^n(-1)^n \frac{1}{2^n n!} \int_{-1}^{1} (1-x^2)^n f^{(n)}(x)\,dx$

$= \frac{1}{2^n n!} \int_{-1}^{1} (1-x^2)^n f^{(n)}(x)\,dx$ ##

(6) Proof! $\int_{-1}^{1} x P_n(x) P_{n-1}(x)\,dx = \frac{2n}{4n^2-1}$

from recurrence relation! $xP_n(x) = \frac{n+1}{2n+1} P_{n+1}(x) + \frac{n}{2n+1} P_{n-1}(x)$.

Multiplying both sides by $P_{n-1}(x)$ & then integrating w.r.t. x from -1 to 1.

we get

$\int_{-1}^{1} x P_n(x) P_{n-1}(x)\,dx = \frac{n+1}{2n+1}\int_{-1}^{1} P_{n+1}(x) P_{n-1}(x)\,dx + \frac{n}{2n+1}\int_{-1}^{1} [P_{n-1}(x)]^2\,dx$

But $\int_{-1}^{1} P_m(x) P_n(x)\,dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{2}{2n+1} & \text{if } m = n \end{cases}$

then we have

$\int_{-1}^{1} x P_n(x) P_{n-1}(x)\,dx = 0 + \frac{n}{2n+1} \times \frac{2}{2(n-1)+1} = \frac{2n}{(2n+1)(2n-1)}$ or $\frac{2n}{4n^2-1}$

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(7) $\int_{-1}^{1}(1-x^2)P_m'(x)P_n'(x)\,dx = \begin{cases} 0, & \text{when } m \neq n \\ \dfrac{2m(m+1)}{2m+1} & \text{when } m = n \end{cases}$

Case 1 : If $m \neq n$

$$\int_{-1}^{1}\underbrace{\left[(1-x^2)P_m'(x)\right]}_{f}\underbrace{P_n'(x)}_{g}\,dx = \underbrace{\left[(1-x^2)P_m'\cdot P_n\right]_{-1}^{1}}_{=0} - \int_{-1}^{1}\left[(1-x^2)P_m''-2xP_m'\right]P_n\,dx \quad \text{--- (1)}$$

$\int fg\,dx = f\int g\,dx - \int\left[f'\int g\,dx\right]dx$

Since, P_m satisfies the Legendre eq^n

$(1-x^2)y''-2xy'+m(m+1)y=0$

hence, $(1-x^2)P_m''-2xP_m'+m(m+1)P_m=0 \Rightarrow (1-x^2)P_m''-2xP_m' = -m(m+1)P_m$ --- (2)

But $\int_{-1}^{1}P_mP_n\,dx = 0$ if $m \neq n$

Put eq^n (2) in (1) then we have

$$\int_{-1}^{1}(1-x^2)P_m'P_n'\,dx = m(m+1)\int_{-1}^{1}P_mP_n\,dx = 0$$

Case 2: If $m = n$ then

$$\int_{-1}^{1}(1-x^2)(P_m')^2\,dx = \int_{-1}^{1}\left[(1-x^2)P_m'\right]P_m'\,dx$$

$$= \left[(1-x^2)P_m'P_m\right]_{-1}^{1} - \int_{-1}^{1}\left[(1-x^2)P_m''-2xP_m'\right]P_m\,dx$$

$$= 0 + m(m+1)\int_{-1}^{1}P_m^2\,dx$$

$$= m(m+1)\cdot\frac{2}{2m+1}$$

$\left[\because \int_{-1}^{1}P_m^2\,dx = \dfrac{2}{2m+1} \text{ if } m=n\right.$

$$= \frac{2m(m+1)}{2m+1}$$

Hence, we have

$$\int_{-1}^{1}(1-x^2)P_m'P_n'\,dx = \frac{2m(m+1)}{2m+1}\delta_{mn}$$

Ans

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Claim: all roots of $P_n(x)$ are distinct.

If possible, let the roots of $P_n(x)$ be not all different. Then atleast two roots must be equal. Let α be repeated root then we know that

$$P_n(\alpha)=0 \ \& \ P_n'(\alpha)=0 \qquad \text{--- (1)}$$

Since $P_n(x)$ satisfies the Legendre's equation,

$$(1-x^2)P_n'' - 2xP_n' + n(n+1)P_n = 0$$

Differentiating r times and using Leibnitz theorem,

$$(1-x^2)\frac{d^{r+2}}{dx^{r+2}}P_n(x) + {}^rC_1(-2x)\times\frac{d^{r+1}}{dx^{r+1}}P_n(x) + {}^rC_2(-2)\frac{d^r}{dx^r}P_n(x)$$

$$-2\left[x\frac{d^{r+1}}{dx^{r+1}}P_n(x) + {}^rC_1\times 1\times\frac{d^r}{dx^r}P_n(x)\right] + n(n+1)\frac{d^r}{dx^r}P_n(x)=0$$

or $$(1-x^2)\frac{d^{r+2}}{dx^{r+2}}P_n(x) - 2x\left({}^rC_1+1\right)\frac{d^{r+1}}{dx^{r+1}}P_n(x) - \left\{2\times{}^rC_2 + 2\times{}^rC_1 - n(n+1)\right\}$$

$$\frac{d^r}{dx^r}P_n(x)=0 \qquad \text{--- (3)}$$

Putting $r=0$ and $x=\alpha$ in (3) and using (1), we get

$$(1-\alpha^2)P_n''(\alpha) - 0 - 0 = 0$$

or $$P_n''(\alpha)=0$$

Next putting $r=1$ & $x=\alpha$ in (3) & using (1) & (4), we get

$$(1-\alpha^2)P_n'''(\alpha) - 0 - 0 = 0 \quad \text{or} \quad P_n'''(\alpha)=0$$

Putting $r=2,3,\dots,n-3,n-2$ in (3) then finally we arrive

$$P_n^{(n)}(\alpha)=0 \quad \text{i.e} \quad \left.\frac{d^n}{dx^n}P_n(x)\right|_{x=\alpha}=0$$

But $$P_n(x)=\frac{1\cdot3\cdot5\cdots(2n-1)}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \dots\right]$$

$$\frac{d^n}{dx^n}P_n(x) = \frac{1\cdot3\cdot5\cdots(2n-1)}{n!}\times n!$$

at $x=\alpha$

$$\frac{d^n}{dx^n}[P_n(x)] \neq 0$$

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(10) Let $P_n(x)$ be the Legendre Polynomial of degree n

We know that $(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x)$ — (1)

Replacing x by $-x$

$$(1+2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(-x) \quad — (2)$$

Next Replace z by $-z$ in (1)

$$(1+2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} (-z)^n P_n(x) \quad — (3)$$

from (2) & (3)

$$\sum_{n=0}^{\infty} z^n P_n(-x) = \sum_{n=0}^{\infty} (-1)^n z^n P_n(x)$$

Equating the coefficient of z^n from both sides we get

$$P_n(-x) = (-1)^n P_n(x)$$

Replacing x by 1 & noting that $P_n(1) = 1$ & $P_n(-1) = (-1)^n$.

$$P_n(-1) = (-1)^n P_n(1) = (-1)^n$$

~~if n is even~~

When n is odd, $(-1)^n = -1$ then

$P_n(-x) = -P_n(x)$ [$P_n(x)$ is a odd f^n when n is odd]

$|||^y$ $P_n(x)$ is even function when n is even.

(11) Using Rodrigue's formula:

$$\frac{d}{dx}\left\{(1-x^2)\frac{d}{dx}P_n(x)\right\} + n(n+1)P_n(x) = 0$$

Soln: Rodrigue's formula:

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n \quad — (1)$$

Let $y = (x^2-1)^n$ — (2)

Diff w.r.t. x

$y_1 = 2nx(x^2-1)^{n-1}$

or $(x^2-1)y_1 = 2nx(x^2-1)^n$ — (3)

Using (2)

$(x^2-1)y_1 = 2nxy$

~~Differen w.r.t. x~~

~~$(x^2-1)y_2 + 2x \cdot n \cdot 2xy_1$~~

Diff. w.r.t. x again

$(x^2-1)y_2 + 2xy_1 = 2ny + 2nxy_1$

$\Rightarrow (x^2-1)y_2 + 2(1-n)xy_1 - 2ny = 0$

After n times derivating w.r.t. x both sides

$D^n\{(x^2-1)y_2\} + 2(1-n)D^n xy_1 - 2nD^n(y) = 0$

where $D^n \equiv \frac{d^n}{dx^n}$

Using Leibnitz theorem:

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$y_{n+2}(x^2-1) + {}^nC_1\, y_{n+1}(2x) + {}^nC_2\, y_n \cdot 2 + 2(1-n)\left(y_{n+1}x + {}^nC_1\, y_n \cdot 1\right) - 2ny_n = 0$

or $(x^2-1)(y_{n+2}) + 2xy_{n+1} + \{n(n-1) + 2n(1-n) - 2n\} y_n = 0$

or $(1-x^2) y_{n+2} - 2xy_{n+1} + n(n+1) y_n = 0$

or $\frac{d}{dx}\left\{(1-x^2) y_{n+1}\right\} + n(n+1) y_n = 0$

or $\frac{d}{dx}\left\{(1-x^2) \times \frac{dy_n}{dx}\right\} + n(n+1) y_n = 0$

or $\frac{d}{dx}\left\{(1-x^2)\frac{d}{dx}\left(\underbrace{\frac{d^n}{dx^n}(x^2-1)^n}_{y_n}\right)\right\} + n(n+1)\frac{d^n}{dx^n}(x^2-1)^n = 0$

Dividing by $2^n n!$

$\frac{d}{dx}\left\{(1-x^2)\frac{d}{dx} P_n(x)\right\} + n(n+1) P_n(x) = 0$

Q1 a) $J_n(x) = \frac{1}{2}\int_0^{\pi} \cos(n\theta - x \sin n\theta)\, d\theta$; $n \in \mathbb{Z}$

$\int_0^{\pi} \cos m\theta \cos n\theta \, d\theta = \int_0^{\pi} \sin m\theta \sin n\theta \, d\theta = \begin{cases} \pi/2 & m = n \\ 0 & m \neq n \end{cases}$ — (*)

Trigonometric Bessel identities

$\cos(x \sin\theta) = J_0 + 2J_2 \cos 2\theta + 2J_4 \cos 4\theta + \dots$

$\sin(x \sin\theta) = 2J_1 \sin\theta + 2J_3 \sin 3\theta + 2J_5 \sin 5\theta + \dots$

(using gen. func

$e^{\frac{x}{2}(z - \frac{1}{z})} = J_0 + (z - z^{-1})J_1 + (z^2 + z^{-2})J_2 + (z^3 - z^{-3})J_3 + \dots$

put $z = e^{i\theta}$

$e^{\frac{x}{2}(e^{i\theta} - e^{-i\theta})} = e^{xi\sin\theta}$

$= \cos(x\sin\theta) + i \sin(x\sin\theta)$)

first take n +ve int

Multiply both sides by $\cos n\theta$ & integ θ $0 \to \pi$

$\int_0^{\pi} \cos(x\sin\theta) \cos n\theta \, d\theta = \begin{cases} 0 & ; n \text{ odd} \\ \pi J_n & ; n \text{ even} \end{cases}$ (using (*))

Illy multiply second one by $\sin n\theta$

$\int_0^{\pi} \sin(x\sin\theta) \sin n\theta \, d\theta = \begin{cases} \pi J_n & ; n \text{ odd} \\ 0 & ; n \text{ even} \end{cases}$

Let n odd then $\int_0^{\pi} [\underbrace{\cos(x\sin\theta)}_{B}\underbrace{\cos n\theta}_{A} + \underbrace{\sin(x\sin\theta)}_{B}\underbrace{\sin n\theta}_{A}]\, d\theta = \pi J_n$

$\Rightarrow J_n = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta)\, d\theta$ — (1)

Let n negative int $n = -m$, $m > 0$

we need to prove $J_m(x) = \frac{1}{\pi}\int_0^{\pi} \cos(-m\theta - x\sin\theta)\, d\theta$

Let $\theta = \pi - \phi \qquad d\theta = -d\phi$

then solving RHS

$$= -\frac{1}{\pi}\int_{\pi}^{0} \cos[-m(\pi-\phi) - x\sin(\pi-\phi)]\,d\phi = \frac{1}{\pi}\int_0^{\pi} \cos[(m\phi - x\sin\phi) - m\pi]\,d\phi$$

$$= \frac{1}{\pi}\int_0^{\pi} [\cos(m\phi - x\sin\phi)\cos m\pi + \sin(m\phi - x\sin\phi)\underbrace{\sin m\pi}_{0}]\,d\phi$$

$$= \frac{1}{\pi}\int_0^{\pi} (-1)^m \cos(m\phi - x\sin\phi)\,d\phi$$

$$= \frac{(-1)^m}{\pi}\int_0^{\pi} \cos(m\phi - x\sin\phi)\,d\phi = (-1)^m J_m$$

$$= J_{-m} \qquad [\because J_{-m}(x) = (-1)^m J_m]$$

Q1 b) $J_0^2 + 2(J_1^2 + J_2^2 + \dots) = 1$

Let us first prove $\frac{d}{dx}(J_n^2 + J_{n+1}^2) = 2\left(\frac{n}{x}J_n^2 - \frac{n+1}{x}J_{n+1}^2\right)$ — (*)

for this use reccurence relⁿ

$$J_n'(x) = -\frac{n}{x}J_n + J_{n-1} \quad \text{—} ①$$

$$J_n' = \frac{n}{x}J_n - J_{n+1} \quad \text{—} ②$$

$n \to n+1$ in ① $\Rightarrow J_{n+1}' = -\frac{n+1}{x}J_{n+1} + J_n$

$$\frac{d}{dx}(J_n^2 + J_{n+1}^2) = 2J_nJ_n' + 2J_{n+1}J_{n+1}'$$

$$= 2J_n\left(\frac{n}{x}J_n - J_{n+1}\right) + 2J_{n+1}\left(-\frac{n+1}{x}J_{n+1} + J_n\right)$$

$$= 2\left(\frac{n}{x}J_n^2 - \frac{n+1}{x}J_{n+1}^2\right)$$

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put $n=0$ $\quad \frac{d}{dx}(J_0^2+J_1^2) = 2\left(0 - \frac{1}{x}J_1^2\right)$

$n=1$ $\quad \frac{d}{dx}(J_1^2+J_2^2) = 2\left(\frac{1}{x}J_1^2 - \frac{2}{x}J_2^2\right)$

$n=2$ $\quad \frac{d}{dx}(J_2^2+J_3^2) = 2\left(\frac{2}{x}J_2^2 - \frac{3}{x}J_3^2\right)$

— — — —

Add columnwise $\quad (J_n \to 0$ as $n \to \infty)$

$$\frac{d}{dx}\left[J_0^2 + 2(J_1^2 + \dots)\right] = 0$$

Integrate $\quad J_{0(x)}^2 + 2\left[J_{1(x)}^2 + J_{2(x)}^2 + \dots\right] = C$

replace $x=0$, $\quad J_0(0)=1$, $J_n(0)=0 \ \forall n \geq 1$

$$\Rightarrow \quad J_0^2 + 2(J_1^2 + J_2^2 \dots) = 1$$

Q2 a) $\boxed{|J_0(x)| \leq 1}$

from above, $\quad J_0^2 = 1 - 2\underbrace{(J_1^2 + J_2^2 + \dots)}_{\text{all } \geq 0}$

$$\Rightarrow \quad J_0^2 \leq 1 \quad \Rightarrow \quad |J_0(x)| \leq 1$$

also $J_0^2 \geq 0$

b) $\boxed{|J_n(x)| \leq 2^{-1/2};\ n \geq 1}$

$$J_n^2 = \frac{1}{2} \times \underbrace{(1 - J_0^2)}_{\geq 0} - \underbrace{(J_1^2 + J_2^2 + \dots J_{n-1}^2 + J_{n+1}^2 + \dots)}_{\text{each } \geq 0}$$

$$J_n^2 \leq 1/2$$

$$|J_n(x)| \leq \frac{1}{\sqrt{2}} \qquad n \geq 1$$

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Q3 Express J_5 in terms of $J_0(x)$ & $J_1(x)$

Using $J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)$

put $n = 4, 3, 2, 1$

$J_5(x) = \frac{8}{x} J_4 - J_3$; $J_4 = \frac{6}{x} J_3(x) - J_2(x)$

$J_3(x) = \frac{4}{x} J_2 - J_1$; $J_2 = \frac{2}{x} J_1 - J_0$

Back Substitution

$$J_5 = \left(\frac{384}{x^4} - \frac{72}{x^2} + 1\right) J_1(x) - \left(\frac{192}{x^3} - \frac{12}{x}\right) J_0(x)$$

Q12 Laguange diff eq^n

$$xy'' + (1-x)y' + ny = 0 \quad (1), \quad x \neq 0$$

in form of Sturm-Liouville eq^n

Sol: General form SLP

$$\frac{d}{dx}\left[p(x) \cdot \frac{dy}{dx}\right] + [q(x) + \lambda \underset{\text{weight func.}}{r(x)}]y = 0$$

(1) $\Rightarrow$ $y'' + \left(\frac{1-x}{x}\right) y' + \frac{n}{x} y = 0$

Integrating factor $e^{\int\left(\frac{1}{x}-1\right)dx} = x \cdot e^{-x}$

multiply both sides by If

$$xe^{-x} y'' + e^{-x}(1-x)y' + ne^{-x} y = 0$$

$$\frac{d}{dx}[xe^{-x} y''] + ne^{-x} y = 0$$

$\frac{d}{dx} + py = Q$

$IF = \int p\,dx$

$$\frac{d}{dx}\left[x e^{-x}\frac{dy}{dx}\right] + \left[0 + n e^{-x}\right] y = 0$$

$p(x) = xe^{-x}$ $\quad q(x) = 0$ $\quad r(x) = e^{-x}$ $\quad \lambda = n$

Q13

$$\frac{d}{dx}\left[r(x)\frac{dy}{dx}\right] + \left[q(x) + \lambda P(x)\right] y = 0$$

a) $(1-x^2)y'' - xy' + n^2 y = 0$

$$\text{If} = e^{\int \frac{-x}{1-x^2}dx} = \sqrt{1-x^2}$$

$$\underbrace{\sqrt{1-x^2}\, y'' - \frac{x}{\sqrt{1-x^2}} y'}_{\frac{d}{dx}\left[\sqrt{1-x^2}\, y'\right]} + \frac{n^2}{\sqrt{1-x^2}} y = 0$$

$P(x) = \frac{1}{\sqrt{1-x^2}}$ $\quad \lambda = n^2$

b) $y'' - 2xy' + 2ny = 0$

$$\text{If} = e^{\int -2x\,dx} = e^{-x^2}$$

$$e^{-x^2}y'' - 2xe^{-x^2}y' + 2e^{-x^2}ny$$

$$\frac{d}{dx}\left[e^{-x^2}\frac{dy}{dx}\right] + 2n e^{-x^2} y = 0$$

$\lambda = 2n$ $\quad P(x) = e^{-x^2}$

c) $xy'' + 2y' + (x+\lambda)y = 0$

$IF = e^{\int 2/x \, dx} = x^2$

$x^2y'' + 2xy' + x(x+\lambda)y = 0$

$\frac{d}{dx}[x^2y'] + [x^2 + \lambda x]y = 0$

$P(x) = x$

Q14

a) $\frac{d}{dx}[x^2y'] + \lambda y = 0$ $\quad y(1) = 0$, $y(b) = 0$ $\quad 1 < x < b$

$x^2y'' + 2xy' + \lambda y = 0$ (Cauchy Euler eqn)

change of variables $x = e^t$ or $t = \ln x$

$y(x)$ & $x(t)$

$D = \frac{d}{dt} \quad D' = \frac{d}{dx}$

$y = \cancel{\frac{d}{dx}\left(\frac{dy}{dx}\cdot\frac{dx}{dt}\right)}$

$= \frac{d}{dx}\left(\frac{dy}{dt}\right)$

$\frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt}$ $\quad x$

$x = e^t$

$\frac{dx}{dt} = e^t$

$(D(D-1) + 2D + \lambda)y = 0$

$(D^2 + D + \lambda)y = 0$

$m = \frac{-1 \pm \sqrt{1-4\lambda}}{2}$

Cases i) $1 - 4\lambda = 0$

$m = -\frac{1}{2}, -\frac{1}{2}$

$y(t) = (A + Bt)e^{-1/2 t}$

$y(0) = 0 \Rightarrow A = 0$

$y(\ln b) = 0 \Rightarrow [B \ln b]e^{-1/2 \ln b} = 0 \Rightarrow B = 0$

$y = 0$ trivial soln

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Case ii) $1-4\lambda > 0$

$$m_1 = \frac{-1+\sqrt{1-4\lambda}}{2}, \quad m_2 = \frac{-1-\sqrt{1-4\lambda}}{2}$$

$$y(t) = A e^{m_1 t} + B e^{m_2 t}$$

$y(0) = 0$ & $y(\ln b) = 0 \Rightarrow A = B = 0$ (trivial)

Case iii) $1-4\lambda < 0$

$$m = \frac{-1 \pm i\sqrt{4\lambda-1}}{2}$$

$$y(t) = e^{-\frac{1}{2}t}\left(A\cos\left(\frac{\sqrt{4\lambda-1}}{2}t\right) + B\sin\left(\frac{\sqrt{4\lambda-1}}{2}t\right)\right)$$

$y(0) = 0 \Rightarrow A = 0$

$y(\ln b) = 0 \Rightarrow e^{-\frac{1}{2}\ln b}\left[B\sin\left(\frac{\sqrt{4\lambda-1}\ln b}{2}\right)\right] = 0$

for nontrivial soln $\sin\left(\frac{\sqrt{4\lambda-1}}{2}\ln b\right) = 0$

$$\frac{\sqrt{4\lambda-1}}{2}\ln b = n\pi$$

$$\lambda_n = \frac{1}{4} + \left(\frac{n\pi}{\ln b}\right)^2$$

corresp eigenfunc $\sin\left(\frac{n\pi}{\ln b}\ln x\right)$

14b) $\frac{d}{dx}[xy'] + \frac{\lambda}{x}y = 0 \qquad y(1) = 0 \quad y(e^{\pi}) = 0$

$x^2y'' + xy' + \lambda y = 0 \qquad x = e^t$

$D(D-1)y + Dy + \lambda y = 0$

$(D^2+\lambda)y = 0$

$(D^2+\lambda)y = 0$

$\cancel{D = \pm\sqrt{\lambda}}$

$\lambda = 0$, $\lambda \not< 0$ trivial case

$\underline{\lambda < 0}$

$y(t) = c_1 \cos\sqrt{\lambda}\, t + c_2 \sin\sqrt{\lambda}\, t$

$x = 1 \Rightarrow t = 0$ $\quad y(0) = 0 \quad c_1 = 0$

$x = e^{\pi} \Rightarrow t = \pi$ $\quad y(\pi) = 0 \quad \cdot\; c_2 \sin\sqrt{\lambda}\,\pi = 0$

$\sqrt{\lambda}\,\pi = n\pi$

$\sqrt{\lambda} = n \Rightarrow \lambda = n^2$ (eigenvalues)

$y_n(t) = c_2 \sin nt$

$y_n(x) = c_2 \sin(n \ln x)$

$c_2 = 1$ for conv~~inience~~

Q15 $\quad y'' + \lambda y = 0 \qquad y(0) = 0 \quad y(\pi) + y'(\pi) = 0$

$\lambda > 0$

$y = c_1 \cos\sqrt{\lambda}\, x + c_2 \sin\sqrt{\lambda}\, x$

$y(0) = 0 \quad \Rightarrow c_1 = 0$

$y' = \cancel{c_1} \cdot c_2 \sqrt{\lambda} \cos\sqrt{\lambda}\, x.$

$y(\pi) + y'(\pi) = c_2 [\sin\sqrt{\lambda}\,\pi + \sqrt{\lambda} \cos\sqrt{\lambda}\,\pi] = 0$

$\sin\sqrt{\lambda}\,\pi + \sqrt{\lambda}\cos\sqrt{\lambda}\,\pi = 0$

$\sqrt{\lambda} = -\tan\sqrt{\lambda}\,\pi$

eigenvalues satisfy this eqn

& eigenfn $\quad \sin(x\sqrt{\lambda_n}) = .\sin(-\tan\sqrt{\lambda}.\pi)x$

Q 16 $\quad 4(e^{-x}y')' + (1+\lambda)e^{-x}y = 0 \qquad y(0)=0,\ y(1)=0$

$$4[e^{-x}y'' + e^{-x}y'] + (1-\lambda)e^{-x}y = 0$$

$$4y'' - 4y' + (1+\lambda)y = 0$$

$$4m^2 - 4m + (1+\lambda) = 0$$

$$m = \frac{4 \pm \sqrt{16-16(1+\lambda)}}{8} = \frac{1 \pm \sqrt{\lambda}\, i}{2}$$

$$y = e^{x/2}\left[A\cos\left(\frac{\sqrt{\lambda}}{2}x\right) + B\sin\left(\frac{\sqrt{\lambda}}{2}x\right)\right]$$

$y(0)=0$

$\Rightarrow A = 0 \qquad \Rightarrow y = e^{x/2} B \sin\left(\frac{\sqrt{\lambda}}{2}x\right)$

$y(1) = 0 \qquad e^{1/2} B \sin\left[\frac{\sqrt{\lambda}}{2}\right] = 0$

$$\frac{\sqrt{\lambda}}{2} = n\pi \qquad \lambda = 4n^2\pi^2$$

eigenfn. $\qquad y_n = A_n\, e^{x/2} \sin(n\pi x)$