Indian Institute of Technology Indore
MA203 Complex Analysis and Differential Equations-II
(Autumn Semester 2023)
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Tutorial Sheet 3

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1. [Nonhomogeneous heat equation] Can you think of an idea—based on what we have learned in the class—to solve a nonhomogeneous heat equation

$$u_t - \alpha^2 u_{xx} = 7\mathrm{e}^{-2x}, \qquad t > 0, \quad 0 < x < L \tag{$*$}$$

with the boundary conditions

$$u(0,t) = 0, \qquad u(L,t) = 0, \quad \text{for all } t \geq 0$$

and the initial condition

$$u(x,0) = f(x), \qquad 0 < x < L,$$

where $f(x)$ is a given function? [The term on the right-hand side of the nonhomogeneous heat equation ($*$) may represent heat loss in the bar.]

2. Find the solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad t > 0, \quad 0 < x < L,$$

satisfying the boundary conditions

$$u(0,t) = 0, \quad u(L,t) = 0 \quad \text{for} \quad t \geq 0$$

and the initial conditions

$$u(x,0) = f(x), \quad u_t(x,0) = g(x) \quad \text{for} \quad 0 \leq x \leq L,$$

by directly using the method of separation of variables.

3. Determine the solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad t > 0, \quad 0 < x < 1,$$

satisfying the boundary conditions

$$u(0,t) = 1, \quad u(1,t) = 0 \quad \text{for} \quad t \geq 0$$

and the initial conditions

$$u(x,0) = 1 - x, \quad u_t(x,0) = 0 \quad \text{for} \quad 0 \leq x \leq 1.$$

4. (a) Show that the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

can be reduced to the form $u_{\xi\eta} = 0$ by the change of variables $\xi = x - ct$ and $\eta = x + ct$.

(b) Show that $u(x,t)$ can be written as

$$u(x,t) = \phi(x - ct) + \psi(x + ct),$$

where ϕ and ψ are arbitrary functions.

5. Consider the wave equation

$$u_{tt} = c^2 u_{xx}$$

in an infinite one-dimensional medium subject to the initial conditions

$$u(x,0) = f(x), \quad u_t(x,0) = 0 \quad \text{for} \quad -\infty < x < \infty.$$

Using the form of the solution obtained in problem 5, show that the solution of the given problem is

$$u(x,t) = \frac{1}{2}\big[f(x - ct) + f(x + ct)\big].$$

6. Show that the solution of problem 2 with $g(x) = 0$ obtained with the method of separation of variables can be written in the form

$$u(x,t) = \frac{1}{2}\big[h(x - ct) + h(x + ct)\big].$$