Trapezoidal Rule: $\int_a^b f(x)\,dx = \frac{h}{2}[f(x_0)+f(x_1)]$

(2 ordinates & exact for poly of deg one.)

Simpson's Rule: $\int_a^b f(x)\,dx = \frac{h}{3}[f(x_0)+4f(x_1)+f(x_2)]$

(3 ordinates & exact for poly of deg ≤ 2)

[equally spaced]

In Newton Methods

If you are given information at $f(x_i)$ points

then you are able to obtain an interpolate with polynomial by the method is exact for polynomial of deg $\le n$

Now. Can we derive an integration Method s.t. the method is exact for poly of deg. $> n$?

Gaussian Quadrature. → Nodes need be equally spaced.

$$\int_a^b f(x)\,dx = \sum_{k=1}^{n} c_k f(x_k) \quad (*)$$

$c_1, c_2, \dots, c_n$ — n ; $x_1, x_2, \dots, x_n$ — n unknown. $m=2n$

By the polynomial of degree $\le 2n-1$

for eg. Suppose $n=2$, exact for poly of deg $\le 2(2)-1$
$= 3$

How to work this method?

Note: Legendre polynomials

$(1-x^2)y'' - 2xy' + n(n+1)y = 0$
$n \in$ an integers

and we had derived Rodrigues formula

$$y(x) = P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}[(x^2-1)^n]$$

Observation by poly of degree all are odd poly
n n n even n even

Ex. $P_0(x) = 1$
$P_1(x) = x$
$P_2(x) = \frac{1}{2}(3x^2-1)$
$P_3(x) = \frac{1}{2}(5x^3-3x)$
$\vdots$

& $P_n(1) = 1$

Further Orthogonality property

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0 \quad m \ne n$$
$$= \frac{2}{2n+1} \quad m = n$$

$$\int_{-1}^{1} P_n(x)\,x^m\,dx = 0 \quad , m < n$$

roots of $P_n(x) = 0$ lies with in the interval $(-1,1)$

Now, Gaussian Quadrature formula for $n=2$

Let $n=2$ in $(*)$

$$\int_{-1}^{1} f(x)\,dx = \sum_{k=1}^{n} c_k f(x_k) = c_1 f(x_1) + c_2 f(x_2) \quad (**)$$

find c_1, c_2, x_1, x_2 such that $(**)$ is exact for polynomials of degree $\le 2(2)-1 = 3$

for $f(x) = 1, x, x^2, x^3$

$$\int_{-1}^{1} dx = x\Big|_{-1}^{1} = 2 = c_1 + c_2 \quad (1)$$

$$\int_{-1}^{1} x\,dx = \frac{x^2}{2}\Big|_{-1}^{1} = 0 = c_1 x_1 + c_2 x_2 \quad (2)$$

$$\int_{-1}^{1} x^2\,dx = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{2}{3} = c_1 x_1^2 + c_2 x_2^2 \quad (3)$$

$$\int_{-1}^{1} x^3\,dx = \frac{x^4}{4}\Big|_{-1}^{1} = 0 = c_1 x_1^3 + c_1 x_2^3 \quad (4)$$

[Multiply eq(3) $\times \frac{3}{2}$] − [eq(1) $\times \frac{1}{2}$] ⇒

$$\int_{-1}^{1} \frac{3}{2}x^2\,dx - \int_{-1}^{1} \frac{1}{2}\,dx = \frac{3}{2}c_1 x_1^2 + \frac{3}{2}c_2 x_2^2 - \frac{1}{2}c_1 - \frac{1}{2}c_2$$

$$\Rightarrow \int_{-1}^{1} \frac{1}{2}(3x^2-1)\,dx = \frac{c_1}{2}(3x_1^2-1) + \frac{c_2}{2}(3x_2^2-1)$$

$\int_{-1}^{1} f(x)\,dx$

$$\Rightarrow \int_{-1}^{1} P_2(x)\,dx = c_1 P_2(x_1) + c_2 P(x_2)$$

∴ We have.

Now $\int_{-1}^{1} P_1(x)P_0(x)\,dx = 0 = c_1 P_2(x_1) + c_2 P_2(x_2)$

$$\boxed{\therefore\ c_1 P_2(x_1) + c_2 P_2(x_2) = 0} \quad (5)$$

Again

[eq(1) $\times \frac{3}{2}$] + [eq(4) $\times \left(-\frac{1}{2}\right)$] ⇒

$$\int_{-1}^{1} \left(\frac{3}{2}x^2 - \frac{1}{2}\right)dx = c_1\frac{3}{2}x_1^3 - c_1\frac{1}{2}x_1 + c_2\frac{3}{2}x_2^3 - c_2\frac{1}{2}x_2$$
$$= c_1 x_1\left(\frac{3}{2}x_1^2 - \frac{1}{2}\right) + c_2 x_2\left(\frac{3}{2}x_2^2 - \frac{1}{2}\right)$$
$$= c_1 x_1 P_2(x_1) + c_2 x_2 P_2(x_2).$$

LHS: $\int_{-1}^{1} x[\frac{1}{2}(3x^2-1)]\,dx$

$$= \int_{-1}^{1} P_1(x)P_2(x)\,dx = 0$$

$$\boxed{\therefore\ c_1 x_1 P_2(x_1) + c_2 x_2 P_2(x_2) = 0} \quad (6)$$

∴ For any arbitrary choice of c_1 & c_2 the equations (5) & (6) will be identically satisfied if x_1 & x_2 are chosen as zeros of P_2, namely the roots of P_2.

$\frac{1}{2}(3x^2-1) = 0$ or $3x^2-1=0$
or $x^2 = \frac{1}{3}$
$x = \pm\frac{1}{\sqrt{3}}$

$\Rightarrow x = \pm\frac{1}{\sqrt{3}}$

$x_1 = \frac{1}{\sqrt{3}}$, $x_2 = -\frac{1}{\sqrt{3}}$

Now $c_1 + c_2 = 2$ (7)
$c_1 x_1 + c_2 x_2 = 0$ (8)
$c_1\left(\frac{1}{\sqrt{3}}\right) + c_2\left(\frac{1}{\sqrt{3}}\right) = 0$

$\boxed{c_1 = c_2}$

$2c_1 = 2$
$\Rightarrow c_1 = 1 = c_2$

$$\boxed{\therefore \int_{-1}^{1} f(x)\,dx = c_1 f(x_1) + c_2 f(x_2) = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)}$$

This method is exact for poly of deg ≤ 3
$2n-1$

→ Open-type formula

Suppose $\int_a^b f(x)\,dx$ transform $\int_{-1}^{1} f(x)\,dx$

Transformation $x = \frac{b-a}{2}x + \frac{b+a}{2}$ ⇒ $dx = \frac{b-a}{2}dx$

$x = a \Rightarrow a = \frac{b-a}{2}x + \frac{b+a}{2}$

$a - \left(\frac{b+a}{2}\right) = \frac{b-a}{2}x$

$-\left(\frac{b-a}{2}\right) = \frac{b-a}{2}x \Rightarrow x = -1$

If $x = b$, $b = \frac{b-a}{2}x + \frac{b+a}{2}$

$b - \left(\frac{b+a}{2}\right) = \left(\frac{b-a}{2}\right)x$

$\Rightarrow \frac{b-a}{2} = \frac{b-a}{2}x$

$\Rightarrow x = 1$

$$\therefore \int_a^b f(x)\,dx = \frac{b-a}{2}\int_{-1}^{1} f\left(\frac{b-a}{2}x + \frac{b+a}{2}\right)dx$$

$$= \frac{b-a}{2}\left[f\left(\frac{b-a}{2}\left(-\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}\right) + f\left(\frac{b-a}{2}\left(\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}\right)\right]$$

eg. $\int_{\frac{\pi}{2}}^{3\pi} \frac{\cos 2x}{1+\sin x}\,dx$ $\quad x = \frac{1}{2}x + \frac{3\pi}{2}$, $dx = \frac{1}{2}dx$

$a = \frac{\pi}{2}$, $b = 3\pi$

$$\int_{\frac{\pi}{2}}^{3\pi} \frac{\cos 2x}{1+\sin x}\,dx = \frac{1}{2}\int_{-1}^{1} \frac{\cos 2\left(\frac{x}{2}+\frac{\pi}{2}\right)}{1+\sin\left(\frac{x}{2}+\frac{\pi}{2}\right)}\,dx$$

$$= \frac{1}{2}\left[\frac{\cos\left(-\frac{1}{\sqrt{3}}+\frac{\pi}{2}\right)}{1+\sin\left(-\frac{1}{\sqrt{3}}+\frac{\pi}{2}\right)} + \frac{\cos\left(\frac{1}{\sqrt{3}}+\frac{\pi}{2}\right)}{1+\sin\left(\frac{1}{\sqrt{3}}+\frac{\pi}{2}\right)}\right]$$

$$= \frac{1}{2}\left[0.565583 - 0.158546872\right]$$

$$= 0.203508 42$$

Q3. $y(0) = 1$

Picard's Method of Successive approximation

$\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$ $\quad (x_0, y_0)$

$dy = f(x,y)\,dx$

$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x,y)\,dx$

$\Rightarrow y - y_0 = \int_{x_0}^{x} f(x,y)\,dx$

$\Rightarrow y = y_0 + \int_{x_0}^{x} f(x,y)\,dx$

$y = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx$

$y(0) = y_0$
$y(0) = 0$

$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$

$\vdots$

$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\,dx$