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1. Note
$$\mathbb{R}^* := \mathbb{R} \cup \{-\infty, \infty\}$$
is called the extended real number system.
2. $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ is called the extended complex number system.
3. There is only one infinity in the extended complex number system.
4. We define it as $|z| \to \infty$ or $z \to \infty$.
5. Infinite in complex plane means that we are travelling far away from the origin in any direction.

Unboundedness of a complex function means $|f(x)|$ is unbounded

Let $f: \mathbb{C} \to \mathbb{C}$ be a complex function (non-constant)
then if derivatives of all order exist for f, then f is unbounded.
(always)

If first order derivative exists for a complex function $f: \mathbb{C} \to \mathbb{C}$
then all higher order derivatives exist for the complex func^n

1. A real, non-constant, smooth function (derivatives of all order exist) on $\mathbb{R}$, then the range of f can leave infinitely many values from $\mathbb{R}$.
2. Example: Range of $f(x) = \exp(x)$ doe not contain the negative real axis including zero.
3. The range of any non-constant complex function (derivatives of all order exist) on $\mathbb{C}$, can leave **at most one value**. This is called Picard's Little Theorem (Emile Picard, at the age of 22, 1856-1941).
4. Examples: Polynomial
$P(\mathbb{C}) = \sin(\mathbb{C}) = \cos(\mathbb{C}) = \sinh(\mathbb{C}) = \cosh(\mathbb{C}) = \mathbb{C}$
5. But $e^{(\mathbb{C})} = \mathbb{C} - \{0\}$.

Infinity in complex numbers means infinite in any direc^n
$z \to \infty$ or $|z| \to \infty$ are same thing

$$\left.\begin{aligned} |z_1 + z_2| &\leq |z_1| + |z_2| \\ |z_1 - z_2| &\geq \big||z_1| - |z_2|\big| \end{aligned}\right\} \text{Triangle Inequalities}$$

Any complex number (except $z=0$) has a unique magnitude and direc^n

$z=0$ has $|z|=0$ but its direction is not unique.

ie $\arg(z)$ when $z=0$ is undefined.

$\mathrm{Arg}(z) \Rightarrow$ Principle argument $\in (-\pi, \pi]$

$\arg(z) \Rightarrow$ argument. Can take any value.

$\arg(z) = \mathrm{Arg}(z) \pm 2n\pi \qquad n = 0, \pm 1, \pm 2 \ldots$

$\arg(z) = \text{Arg}(z) \pm 2n\pi \qquad n = 0, \pm 1, \pm 2 \dots$

↳ countably infinite values.

De Moiver's Formula.

if $z = r(\cos\theta + i \sin\theta)$

then $z^n = r^n \left(\cos(n\theta) + i \sin(n\theta) \right)$ for $n \in I$

for $n \in R - I$ formule maynot be valid.

Straight line

$\bar{a} z + a \bar{z} + b = 0 \qquad a \in \mathbb{C} \qquad b \in \mathbb{R}$

Figures:

1) Annulus $\quad 1 < |z - z_0| < 2 \quad$ disc missing

2) Punctured disc $\quad 0 < |z - z_0| < 2 \quad$ one point missing

Neighbourhood

$N(z_0, \varepsilon) = \{ z : |z - z_0| < \varepsilon \}$ for any small ε

contains z_0. Used for continuity.

Deleted Neighbourhood.

$\hat{N}(z_0, \varepsilon) = \{ z : 0 < |z - z_0| < \varepsilon \}$ for any small ε

does not contain z_0. Used for limit.

Limit Point

z_0 is limit point of set S if $\quad \hat{N}(z_0, \varepsilon) \cap S \neq \emptyset$

Interior Point ⇁ z_0 if $N(z_0, \varepsilon) \subset S$

Boundary Point ⇁ z_0 if Some points of $N(z_0, \varepsilon) \subset S$

Open set: $S \in \mathbb{C}$ is called open set if all points of S are interior points of S

Closed Set: $S \in \mathbb{C}$ is called closed set if it contains all of its boundary points too.

Bounded Set: $S \in \mathbb{C}$ is bounded if $S \subseteq \{ z : |z| \leq M \}$ for some $M \in \mathbb{R}$

Unbounded: $S \in \mathbb{C}$ if $|z| \leq M$ for no $M \in \mathbb{R}$

Connected set: A set S be a subset of $\mathbb{C}$ is said to be connected if any two points in S can be joined by a polygonal line (pieces of lines joined end to end) or by any **continuous curve** which lies entirely on S.

Domain

i) Open connected set

ii) Two different separate open set are disconnected sets.

Region

Domain with some, all, none of its boundary points is called region.

doubts tut-2 Q 10, Q 9

FUNCTIONS:

$w \in \mathbb{C} \quad z \in \mathbb{C}$

We donate a complex func^n as $\quad w = f(z) \quad f : \mathbb{C} \to \mathbb{C}$

We can also represent this as $\quad w = f(z) = u + iv \quad u, v \in \mathbb{R}$

$\therefore \quad z = x + iy$

$w = f(x,y) = u(x,y) + i\, v(x,y) \qquad x, y, u, v \in \mathbb{R}$

We can also denote them in polar

$u = u(r,\theta) \qquad v = v(r,\theta) \qquad w = re^{i\theta} = f(z)$

1. Geometrically, elements of S can be represented in the complex plane $\mathbb{C}$.
2. The elements of $f(z)$ is presented in a different complex plane i.e., on the w-plane here.
3. The set S is called the domain of definition of the function f or simply domain of f.
4. The collection of all values of w is called the range of f.
5. Mathematically, range denoted by $R(f) = \{f(z) : z \in S\}$.
6. Domain of definition and domain (set) are different terminology. "Domain of definition" could be a region as well.
7. Often we say "domain of a function". Note that "domain of a function" is not necessarily a "domain". Avoid confusion.

LIMITS

Let D be a domain set $\subset \mathbb{C}$ and $z_0 \in \mathbb{C}$ be a limit point of D

Let $f : D \to \mathbb{C}$ be a func^n (z_0 may not belong to D)

The func^n is said to have a limit $l \in \mathbb{C}$ as z approaches z_0 if for EACH $\varepsilon > 0$ (sufficiently small) there exists a $\delta > 0$ such that

$$|f(z) - f(z_0)| < \varepsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta \quad z \in D$$

Imp: If a limit exist for a func^n, then it will have a fixed value irrespective of the direc^n in which it is approached

So if two directions of approach give different values, then the limit does not exist.

For calculating & proving the existence of a limit, first take a suitable direc^n like $y = x$ or $y = 0$ or $x = 0$ and find its value. Once the value is been found, use the $\varepsilon - \delta$ definition to prove its existence

For calculating limits, use $|z|^2 = z\bar{z}$ and $|a+b| \le |a| + |b|$

Like $\quad |u(x,y) + i\, v(x,y)| \le |u(x,y)| + |v(x,y)|$

LIMITS IN 2 VARIABLES

$f(x,y) = u(x,y) + i\,v(x,y) = f(z) \qquad z = x+iy \in \mathbb{C}$

Limit of $f(z)$ exists at $z_0 = x_0 + iy_0$ if limits of $u(x,y)$ and $v(x,y)$ exist at (x_0, y_0)

$$\lim_{z \to z_0} f(z) = \lim_{(x,y)\to(x_0,y_0)} f(x,y) = \lim_{(x,y)\to(x_0,y_0)} u(x,y) + i \lim_{(x,y)\to(x_0,y_0)} v(x,y)$$

CONTINUITY :

Let D be a domain $\subset \mathbb{C}$

and $z_0 \in D \subset \mathbb{C}$

$$|z - z_0| < \delta \qquad |f(z) - f(z_0)| < \varepsilon \qquad z \in D$$

then continuous.

A func^n is said to be continuous in a region if the func^n is continuous at all points in that region.

$\omega = f(z) = u(x,y) + i\,v(x,y)$

A func^n is said to be continuous at $z_0 = x_0 + iy_0$ if $u(x,y)$ & $v(x,y)$ are continuous at (x_0, y_0)

DIFFERENTIABILITY

Let $D \subset \mathbb{C}$ be a domain and z_0 be limit point of D.

Let $f : D \to \mathbb{C}$ be a func^n

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \qquad h \in \mathbb{C}$$

Cauchy Rieman Equations

$f(z) = u(x_0, y_0) + i\,v(x_0, y_0) \qquad z_0 = x_0 + iy_0$

If $f(z)$ is diff at z_0

then partial derivatives of u & v exist at (x_0, y_0)

$u_x(x_0, y_0) = v_y(x_0, y_0)$

$u_y(x_0, y_0) = -\,v_x(x_0, y_0)$

$f'(z_0) = u_x(x_0, y_0) + i\,v_x(x_0, y_0)$

Imp If C-R eq^n is not satisfied, then $f(z)$ is not diff at $z = z_0$

$f'(z_0) = u_x(x_0, y_0) + i\, v_x(x_0, y_0)$

Imp If C-R eq^n is not satisfied, then $f(z)$ is not diff at $z = z_0$
But if C-R eq^n is satisfied at $z = z_0$, it doesn't confirm that $f(z)$ is diff at $z = z_0$

Sufficient Conditions:

$f(z) = u(x_0, y_0) + i\, v(x_0, y_0) \qquad z = x_0 + i y_0$

i) $f(z)$ is defined in $N(z_0, \varepsilon)$

ii) $u_x \; u_y \; v_x \; v_y$ are defined & continuous in $N(z_0, \varepsilon)$

iii) if u & v satisfy C-R eq^{n}s at (x_0, y_0)

If all above cond^{n}s are satisfied, then $f(z)$ is diff at $z = z_0$

$f'(z_0) = u_x(x_0, y_0) + i\, v_x(x_0, y_0)$

Analytic Functions

1) At a point: A func^n f is said to be analytic at a point z_0 if there exists a $N(z_0, \varepsilon)$ $\varepsilon > 0$ such that f is diff at every $z \in N(z_0, \varepsilon)$

2) On a set: A func^n f is said to be analytic on a set D if it is diff at every point of some open set containing D.

i) If f is diff at all points on a set D, then f may not be analytic on D.
ii) if f is diff at all points on a open set D, then f is analytic on set D.
iii) If f is analytic at all points of a set D, then it is analytic on set D.

Imp iii) If f is diff on all points on a set D, then it does not mean that f will be analytic on D. ($f(x,y) = x^2 + i y^2$ is diff only on $y = x$)

Necessary And Sufficient Condns

$f(z) = f(x+iy) = u(x,y) + i\, v(x,y)$ is analytic on set D if $u(x,y)$ & $v(x,y)$ satisfy C-R eq^{n}s on D and $u_x \; u_y \; v_x \; v_y$ are continuous on D.

Entire Function

A func^n analytic on entire complex plane is called Entire func^n.

Props:

i) f is analytic on D such that $|f| =$ const. Then f is constant in D.

ii) if $\bar{f}$ is analytic on D, then f is not analytic on D. (provided f is non-const func^n)

iii) if $f'(z) = 0$ everywhere in a domain D; $f(z) =$ const. in that D.

Problem: Suppose f is analytic in a domain D. If any of $\operatorname{Re} f$, $\operatorname{Im} f$ is constant in D, then f is constant in D.

Imp When $f(z)$ is defined only using z (ie no $x+iy$ def)

then we $f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$ definition and substitute $z \to (z+h)$

Orthogonal

$f(x+iy)$ = analytic curve

$f(x+iy) = u(x,y) + i\,v(x,y)$

$u(x,y) = \alpha$ curve 1

$v(x,y) = \beta$ curve 2

curve 1 & curve 2 are orthogonal

α, β = constants.

Harmonic Functions:

Let $\phi(x,y)$ be a real func^n in xy plane.

$\phi(x,y)$ is harmonic in domain $D \in \mathbb{R}^2$ if

i) Partial derivatives upto 2-orders are continuous

ii) $\phi_{xx}(x,y) + \phi_{yy}(x,y) = 0 \quad \forall \ (x,y) \in D$

Harmonic Conjugate

$\phi(x,y)$ $\psi(x,y)$ are harmonic func^n

$\psi(x,y)$ is called the HC of $\phi(x,y)$ if:

i) $\phi_x = \psi_y \qquad \phi_y = -\psi_x$ in D

Condition Of Analyticity

$f(x+iy) = u(x,y) + i\,v(x,y)$

$f(x+iy)$ is analytic if $v(x,y)$ is HC of $u(x,y)$

Property

w, v are HC of u

then $w - v = K \quad K \in \mathbb{R}$ const.

Exponential Function Props:

$z = x+iy \qquad e^{iz} = \cos(z) + i\sin(z)$

i) $|e^z| = e^x$

ii) e^z is an entire func^n $\frac{d}{dz}(e^z) = e^z$

iii) $e^z \neq 0$ for $\forall\, z \in \mathbb{C}$

dz

iii) $e^z \neq 0$ for $\forall z \in \mathbb{C}$

iv) e^z is periodic with FUNDAMENTAL period $2\pi i$

v) e^z is not injective (not one to one)

vi) $H = \{ z : x \in \mathbb{R},\ y \in (-\pi, \pi] \}$ $\qquad \mathbb{C}^* = \mathbb{C} - \{0\}$

e^z is bijective from $H \to \mathbb{C}^*$

injective (one-one) + surjective (onto)

TRIGNOMETRIC FUNCTION

1) $\sin(y) = \dfrac{e^{iy} - e^{-iy}}{2i} \qquad \cos(y) = \dfrac{e^{iy} + e^{-iy}}{2}$

2) $\sin(z)$ $\cos(z)$ are entire func^{n}s

3) $\dfrac{d}{dz} \sin z = \cos z \qquad \dfrac{d}{dz} \cos z = -\sin z$

4) $\sin z = \sin(x)\cosh(y) + i \cos(x) \sinh(y)$

$\cos z = \cos(x)\cosh(y) - i \sin(x)\sinh(y)$

$\sinh(x) = \dfrac{e^x - e^{-x}}{2} \qquad \cosh(x) = \dfrac{e^x + e^{-x}}{2}$

5) $|\sin z| = \sqrt{\sin^2 x + \sinh^2 y}$

$|\cos z| = \sqrt{\cos^2 x + \sinh^2 y}$

6) $\sin(z)$ $\cos(z)$ are unbounded in z

7) $\left.\begin{array}{l} \sin(z) = 0 \text{ then } z = k\pi \text{ only.} \\ \cos(z) = 0 \text{ then } z = k\pi + \frac{\pi}{2} \end{array}\right\} k \in \mathbb{Z}$

1. $\sin(-z) = -\sin z$ and $\cos(-z) = \cos z$.
2. $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$.
3. $\sin 2z = 2 \sin z \cos z$.
4. $\sin(z + \pi) = -\sin z$, $\sin(z + 2\pi) = \sin z$
5. $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$.

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5. $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$.
6. $\cos 2z = \cos^2 z - \sin^2 z$.

COMPLEX FUNC :

1) complex sinh(z) cosh(z) are :

$\sinh(z) = \dfrac{e^z - e^{-z}}{2}$ $\cosh(z) = \dfrac{e^z + e^{-z}}{2}$

2) $-i \sinh(iz) = \sin(z)$ $\sinh(-z) = -\sinh(z)$

$\cosh(iz) = \cos(z)$ $\cosh(-z) = \cosh(z)$

COMPLEX LOGARITHM .

1) complex log $\log(z) = \ln|z| + i \arg(z)$ $z \in ¢^*$

↓

multivalued func^n

2) $\text{Log}(z) = \ln|z| + i \,\text{Arg}(z)$ $\text{Log} : ¢^* \to H$

↓

single valued .

1. For $z \neq 0$, $e^{\text{Log}\, z} = z$.
2. For $z \in H$, $\text{Log}\, e^z = z$.
3. For $z \notin H$, $\text{Log}\, e^z \neq z$.
4. For real $x > 0$, $\text{Log}\, x = \ln x$.
5. $\text{Log}\, z$ is not continuous on the negative real axis $\mathbb{R}^- = \{z = x + iy : x < 0, y = 0\}$.

Hence arg (z) is not continuous .

$z = x + i\varepsilon$ $\varepsilon \to 0$

$\varepsilon \to 0^+$ $\arg(z) = \pi - \delta$ $\delta \to 0$

$\varepsilon \to 0^-$ $\arg(z) = -\pi + \delta$ $\delta \to 0$

1. $\text{Log}\, z$ is analytic on the set $\mathbb{C}^* \setminus \mathbb{R}^-$.
2. $\dfrac{d}{dz} \text{Log}\, z = \dfrac{1}{z}$.
3. The identity $\text{Log}\,(z_1 z_2) = \text{Log}\, z_1 + \text{Log}\, z_2$ is true iff $\text{Arg}\, z_1 + \text{Arg}\, z_2 \in (-\pi, \pi]$.

CURVES

$\gamma : [a, b] \to ¢$ (denotion)

$\gamma(t) = x(t) + i\, y(t)$ $x, y \in \mathbb{R}$ and continuous .

Smooth curve :

i) $\gamma'(t) = x'(t) + i\, y'(t)$ exists & cont. & bounded

ii) $\gamma'(t) \neq 0$

Closed curve :

if $\gamma(a) = \gamma(b)$

Contour Curve:

comb^n of smooth curves.

Simple contour curve doesn't cross itself.

$\gamma : [a,b] \to \mathbb{C} \qquad \gamma(t_1) \neq \gamma(t_2) \quad \forall \; t_1, t_2 \in (a,b)$

Reverse Orientation

$C : \gamma : [a,b] \to \mathbb{C}$ → original curve

$\sim C : \quad \gamma_{\sim}(t) = \gamma(a+b-t)$ → negation curve.

Domain :

1. A domain D is called simply connected if every simple closed contour within it encloses points of D only.
2. A simply connected domain does not contain any hole.
3. A domain D is called multiply connected if it is not simply connected.
4. A multiply connected domain contains at least one hole.
5. A doubly connected domain contain EXACTLY one hole.
6. A triply connected domain contain EXACTLY two holes.

Complex valued func^n of a real variable.

1. Understand the meaning of **"complex valued function of a real variable"**.
2. Let $f : [a,b] \to \mathbb{C}$ be a piecewise continuous function. Then $f(t) = u(t) + iv(t)$ where $u, v : [a,b] \to \mathbb{R}$. We then define
$$\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt.$$
3. The above expression is called integral of a complex valued function of a real variable.

prop

$$\left| \int_a^b f(t)\,dt \right| \leq \int_a^b |f(t)|\,dt$$

$f : [a,b] \to \mathbb{C} \qquad c \in (a,b)$

$\int_a^b f(t)\,dt = f(c)(b-a)$

not necessary

$(\because |z| = e^{-i\text{Arg } z} z)$

Hence every complex func^n cannot be integrated directly. $f(x) = e^{ix}$

$\int f(x)\,dx = \int \cos x\,dx + i\int \sin x\,dx$

Complex Integrals

contour integral $\int_C f(z)\,dz \qquad C =$ contour curve

$z \in \mathbb{C}$

Riemann Sum $\qquad S(n) = \sum_{n=1}^{n} f(c_k)\,\Delta z_k$

$\Delta z_k = z_{k+1} - z_k$

$c_k \in (z_k, z_{k+1})$

$z_k =$ partitions of the contour curve.

$$\int_C f(z)\,dz = \lim_{n\to\infty} S(n) \quad \text{or} \quad \lim_{|\Delta z_k| \to 0} S(n)$$

Methods to solve.

Convert the integral into complex valued func^n of real variable.

Convert the contour curve C into $\gamma(t) = x(t) + i\,y(t) \quad t \in [a,b]$

$$\int_C f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$

→ complex valued func^n with real variable

assume $f(z) = u + iv \qquad dz = dx + i\,dy$

$$\int_C f(z)\,dz = \int_C (u+iv)(dx + i\,dy)$$

→ cartesian form

$$= \int_a^b \phi(t) + i \int_a^b \psi(t)\,dt$$

Props:

1) $$\int_C f(z)\,dz = -\int_{-C} f(z)\,dz$$

2) $$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz \ldots \qquad C = C_1 + C_2 \ldots$$

Indefinite Integrals

Theorem: Let f be a continuous function defined on a domain D and there exists a function F defined on D such that $F' = f$. Let $z_1, z_2 \in D$. Then for any contour C lying in D starting from z_1, and ending at z_2,

$$\int_C f(z)\,dz = F(z_2) - F(z_1).$$

requirements:

1) $f(z)$ cont in D
2) $F(z)$ defined and cont in D such that $F' = f$
3) Curve C lies in Domain D.

Length of Curve

$$L(\gamma) = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

$\gamma(t) = x(t) + i\,y(t)$

$t \in [a,b]$

$$= \int_a^b |\gamma'(t)|\,dt$$

ML prop:

$f(z) \Rightarrow$ cont func^n

$L =$ arc length of $\gamma(t)$ $\quad t \in [a,b]$

$|f(z)| \leq M \quad \forall z \in$ arc $\quad M \in \mathbb{R}^+$

then $\left| \int_C f(z)\, dz \right| \leq ML$

Cauchy's Integral Theorem:

Requirements:

i) Closed contour C

ii) $f(z)$ is analytic on and inside C

iii) $f'(z)$ is cont on and inside C

then $$\int_C f(z)\, dz = 0$$

GREEN'S THEOREM.

$$\int_C (M\,dx + N\,dy) = \iint_R (N_x - M_y)\, dx\, dy$$

line C taken anti-clockwise.

Application of Green's theorem.

$$\int_C f(z)\, dz = \int_C (u+iv)(dx + i\,dy)$$

$$= \int_C (u\,dx - v\,dy) + i \int_C (v\,dx + u\,dy)$$

$$= \iint_R (-v_x - u_x)\, dx\, dy + \iint_R (-u_x + v_y)\, dx\, dy$$

$f(z)$ is analytic on C ; CR eq^{n}s $\quad u_x = v_y$

$u_y = -v_x$

$$\therefore \int_C f(z)\, dz = 0$$

Cauchy - Gorsat Theorem.

Req :-

1) $f(z)$ is analytic on and inside $C \to$ closed contour

then $\int_C f(z)\,dz = 0$

Let $f(z)$ be a function analytic throughout a simply connected domain D and C be a simple closed contour lying completely inside D. Then

$$\int_C f(z)\,dz = 0.$$

Imp

$\int_C \operatorname{cosec}^2 z\,dz \qquad C : |z| = 1$

Let domain D : $\{ 0 < |z| \leq \pi \}$

$f(z) = \operatorname{cosec}^2(z)$

$f(0)$ is Not def.

1. $f(z)$ is continuous on the modified domain.
2. The contour C lies entirely inside D.
3. Also $F'(z) = f(z)$ on D, where $F(z) = -\cot z$.
4. Since C is a closed contour with starting and end points z_1 and z_2. Then $z_1 = z_2$.
5. Hence $\int_C \operatorname{cosec}^2 z = F(z_2) - F(z_1) = 0$.

ugly curve $\Rightarrow$ parameter form not defined. Random.

if C is a ugly curve, but the func^n $f(z)$ is analytic on C and inside the domain enclosed by C then $\int_C f(z)\,dz = 0$

Imp

Theorem

anticlockwise

Let C_1 and C_2 be two simple closed **positively** oriented contours such that C_2 lies interior to C_1. If $f(z)$ is analytic in a domain D that contains both the contours and the region between them, then

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

1. Both the contours have positive orientation.
2. It is not necessary that the function needs to be analytic inside the domain enclosed by C_2.
3. In practical situations (examples), $f(z)$ is NOT analytic inside the domain enclosed by C_2.

Let C be a positively oriented simple closed contour and C_k, $k = 1, 2, \ldots n$ denote a finite number of positively oriented simple closed contours all lying wholly within C, but each C_k lies in the exterior of every other whose interior have no points in common. If a function f is analytic throughout the closed region D consisting of all points within and on C except for the points interior to each C_k. Then

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \cdots + \int_{C_n} f(z)\,dz.$$

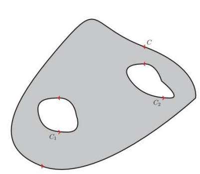

Figure: Think about the proof with for doubly connected domain

Cauchy's Integral Formula.

$f(z)$ is analytic on & inside a positively oriented curve C and Let z_0 be a point inside C

$$\int_C \frac{f(z)}{(z-z_0)^n}\,dz = \frac{2\pi i}{(n-1)!} f^{n-1}(z_0)$$

* C should be compulsorily positively oriented around z_0

Converse of Cauchy's Integral.

Problem statement: Suppose $f(z)$ is a function with domain D such that
$$\int_C f(z)\,dz = 0$$
for every closed contour C lying inside D. Can we conclude from this that f is analytic in D?

1. Answer is NO.
2. Consider the domain $D := \{z : |z| < 1\}$
3. $f(z) = \begin{cases} z & \text{if } z \in D \setminus \{\frac{1}{2}\} \\ 1 & \text{if } z = \frac{1}{2} \end{cases}$.
4. For every closed contour, one can see:
$$\int_C f(z)\,dz = 0.$$
5. But, note that $f(z)$ is NOT analytic in D.

But if $f(z)$ is continuous, then we can say that converse will hold. It is called Morera's Theorem.

Morera's Theorem.

Statement: Suppose $f(z)$ is continuous inside a simply connected domain D and
$$\int_C f(z)\,dz = 0,$$
for any simple closed contour C lying inside D. Then $f(z)$ is analytic throughout D.

1. We say that Morera's theorem is partially converse of Cauchy-Goursat's theorem.
2. Why partially?
3. Because: Instead of the function $f(z)$, we have imposed continuity property on $f(z)$ on the domain.

Cauchy's Estimates

$C: |z - z_0| = r$

1) $f(z)$ is analytic on and inside C
2) $|f(z)| \leq M \quad \forall z \in C$

then $\quad |f^n(z_0)| \leq \dfrac{n!\, M}{r^n}$

Liouville's Theorem:

1) $f(z)$ bounded
2) $f(z)$ is an entire func^n

then $f(z)$ is a constant func^n

Fundamental's of Algebra

$|P(z)| \to \infty$ as $|z| \to \infty$ for every polynomial $P(z)$ with degree $n \geq 1$

Theorem: every polynomial with $n \geq 1$ as a root in $\mathbb{C}$

Collorary: Every polynomial of degree $n \geq 1$ has exactly n roots (not necessarily distinct) in $\mathbb{C}$.

Tut Notes:

i) $f(z) = |z|^2$ $\quad f(x+iy) = x^2+y^2$

$f(x,y) = x^2+y^2$ is diff as its partial derivatives are cont

But $f(x+iy) = x^2+y^2$ is not complex differentiable

Hence there is a difference b/w real diff & complex diff.

CURVES

normal vector $\vec{n}$ to a curve $u(x,y) = c$ at (x_0, y_0) is

$$\vec{n} = \vec{\nabla} u(x_0,y_0) = \left.\frac{\partial u}{\partial x}\right|_{(x_0,y_0)} + i \left.\frac{\partial u}{\partial y}\right|_{(x_0,y_0)}$$

Remark 1.2. De Moivers formula fails when n is not an integer. For instance, consider $r = 1, \theta = 2\pi$ and $n = \frac{1}{2}$. Then De Moivre's formula gives $(1)^{\frac{1}{2}} = -1$, which is not true.

Definition 3.2. A function f is said to be analytic at the point z_0 if there exists a neighborhood $N(z_0, \epsilon)$ of z_0, $\epsilon > 0$ such that f is differentiable at every point $z \in N(z_0, \epsilon)$. Similarly, f is said to be analytic (or, regular, or holomorphic) on a set D if it is differentiable at every point of some open set containing D.

We note the following obvious facts:

1. If f is differentiable at all points of an **open set** D, then f is analytic on D.
2. If f is **differentiable** at all points of a set D, then it **does not** mean that f will be analytic on D (see Example 3.3).
3. If f is **analytic** at all points of a set D, then f is analytic on D.

Compare the Item 2 with the Items 1 and 3 and note the differences.

Example 3.3. Consider the function $f(x+iy) = x^2 + iy^2$, and the set $D := \{x + iy \in \mathbb{C} : x = y\}$. As shown in Problem 3.4, f is differentiable at all points of D, but f is not differentiable at any point lying outside D. Therefore, f is not analytic on D as any open set containing D will contain a point lying outside D, and hence contain a point where f is not be differentiable.

to prove $\sin(z) = 0$ $\quad z = k\pi$

$$\frac{e^{iz} - e^{-iz}}{2i} = 0$$

$$e^{2iz} = 1 \qquad e^{i(2z)} = e^{i(2x+2iy)} = e^{-2y+2ix}$$

$$e^{-2y}\left(\cos(2x) + i\sin(2x)\right) = 1$$

Imp points :

1) $\gamma'(t) \neq 0$ for $t \in (a,b)$ in a smooth curve

2) if $f(z), g(z)$ = analytic then $f(g(z))$ = analytic

We have $u_{xx} + u_{yy} = 0$

Let 'f' be any function of u

$$[f(u)]_x = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x}$$

$$[f(u)]_{xx} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x}\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial u}\right) \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial u} \cdot \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial u}\right) \cdot \left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial x}\right) + \frac{\partial f}{\partial u} \cdot \frac{\partial^2 u}{\partial x^2}$$

$$= \frac{\partial^2 f}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial f}{\partial u} \cdot \frac{\partial^2 u}{\partial x^2}$$

$$= f_{uu}(u_x)^2 + f_u \cdot u_{xx}$$

IIIly $[f(u)]_{yy} = f_{uu}(u_y)^2 + f_u u_{yy}$

$$\therefore [f(u)]_{xx} + [f(u)]_{yy} = f_{uu}(u_x)^2 + f_u(u_{xx}) + f_{uu}(u_y)^2 + f_u \cdot u_{yy}$$

$$= f_{uu}[(u_x)^2 + (u_y)^2] + f_u[\underbrace{u_{xx} + u_{yy}}_{=0}]$$

$$= f_{uu}[u_x^2 + u_y^2] \;//$$

$= 0 \Rightarrow f_{uu} = 0$ ($\because u_x = u_y = 0 \Rightarrow u$ is const/.]

$$\Rightarrow \frac{\partial^2 f}{\partial u^2} = 0 \Rightarrow \frac{\partial f}{\partial u} = A \Rightarrow \boxed{f = uA + B}$$

Polar

1) $$\frac{du}{dx} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}$$

similarly for v_x u_y v_y