VINAY SIR

Heat Equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \qquad (u = \text{Temp})$$

$$\alpha^2 = \frac{k}{\rho s}$$

Method to solve :- separation of variable

$$u(x,t) = X(x)T(t)$$

Assumption that the temp is the product of function of position & time. Now this should satisfy the DE.

$$X(x)\frac{\partial T(t)}{\partial t} = \alpha^2 T(t) \frac{\partial^2 X(x)}{\partial x^2}$$

$$XT' = \alpha^2 T X''$$

$$\frac{T'}{T} = \alpha^2 \frac{X''}{X} \Rightarrow \underbrace{\frac{1}{\alpha^2}\frac{T'}{T}}_{\text{fn of time}} = \underbrace{\frac{X''}{X}}_{\text{fn. of position}}$$

These two are equal $\Rightarrow$ Both equal to same constant

Let $\frac{1}{\alpha^2}\frac{T'}{T} = \frac{X''}{X} = k$

Case I

For $k=0$

$T' = 0$ and $X'' = 0$

$T = C_1 \qquad X = C_2 X + C_3$

Thus $u(x,t) = C_1(C_2 X + C_3)$

$= A_1 X + B_1 \quad , \quad A_1 = C_1C_2$ & $B_1 = C_1C_3$

Case II

$k > 0$

$k = \lambda^2, \quad \lambda \neq 0$

$T' = \lambda^2\alpha^2 T$ and $X'' = \lambda^2 X$

$T = C_4 e^{\lambda^2\alpha^2 t} \qquad X = C_5 e^{\lambda x} + C_6 e^{-\lambda x}$

$u(x,t) = C_4 e^{\lambda^2\alpha^2 t}(C_5 e^{\lambda x} + C_6 e^{-\lambda x})$

$$= e^{\lambda^2\alpha^2 t}\left(A_2 e^{\lambda x} + B_2 e^{-\lambda x}\right)$$

<u>Case 3</u> $\therefore K < 0$

$$K = -\lambda^2, \quad \lambda \neq 0$$

$$T' = -\lambda^2\alpha^2 T \qquad X'' + \lambda^2 X = 0$$

$$\Rightarrow \quad u(x,t) = e^{-\lambda^2\alpha^2 t}\left(A_2\cos\lambda x + B_2\sin\lambda x\right)$$

Q $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2}$

$x = 0$ $\quad x = L$

$u(0,t) = 0 \quad u(L,t) = 0$

$$u(x,0) = f(x)$$

<u>Soln</u> $u(x,t) = X(x)\,T(t)$

$$\Rightarrow \frac{1}{k}\frac{T'}{T} = \frac{X''}{X} = \beta$$

I For $\beta = 0$

$$u(x,t) = A_1 x + B_1$$

$$u(0,t) \Rightarrow B_1 = 0$$

$$u(L,t) \Rightarrow A_1 L = 0 \Rightarrow A_1 = 0$$

$$\Rightarrow \boxed{u(x,t) = 0} \text{ Trivial solution}$$

Cannot satisfy the initial condn

II $\beta > 0 \Rightarrow \beta = \lambda^2, \ \lambda \neq 0$

$$\Rightarrow u(x,t) = e^{\lambda^2 k t}\left(A_2 e^{\lambda x} + B_2 e^{-\lambda x}\right)$$

Apply the boundary condn.

$$u(0,t) = e^{\lambda^2 k t}\left(A_2 + B_2\right) = 0$$

$$\Rightarrow \boxed{A_2 = -B_2}$$

$$u(L,t) = e^{\lambda^2 k t}\left(A_2 e^{\lambda L} + B_2 e^{-\lambda L}\right)$$

$$= A_2 e^{\lambda^2 k t}\left(e^{\lambda L} - e^{-\lambda L}\right) = 0$$

$$\Rightarrow A_2 = 0$$

$$\therefore B_2 = 0$$

$$\Rightarrow u(x,t) = 0 \Rightarrow \text{Trivial}$$

$\Rightarrow \beta > 0$ is also not possible

III $\beta < 0 \Rightarrow \beta = -\lambda^2$, $\lambda \neq 0$

$\rightarrow u(x,t) = e^{-\lambda^2 kt}(A_3 \cos \lambda x + B_3 \sin \lambda x)$

Apply Boundary condn

$u(0,t) = e^{\lambda^2 kt}(A_2 + B_2) = 0$

$\Rightarrow A_2 = -B_2$

$u(L,t) = e^{\lambda^2 kt}(A_2 e^{\lambda L} + B_2 e^{-\lambda L})$

$= A_2 e^{\lambda^2 kt}(e^{\lambda L} -$

$\Rightarrow u(x,t) = e^{-\lambda^2 kt}(A_3 \cos \lambda x + B_3 \sin \lambda x)$

$u(0,t) = e^{-\lambda^2 kt} A_3 = 0$

$\Rightarrow A_3 = 0$

$u(L,t) = 0$

$\Rightarrow e^{-\lambda^2 kt} B \sin \lambda L = 0$

B_3 could be zero but it will lead to trivial solution.

Thus lets consider

$\sin \lambda L = 0$

$\lambda L = n\pi$, $n = \pm 1, \pm 2$

$n \neq 0$ ($\because$ We ommitted $\lambda = 0$ case)

$\Rightarrow \lambda = \frac{n\pi}{L}$

Thus solution

$u(x,t) = B_3 \underbrace{e^{-\frac{n^2\pi^2}{L^2} kt} \sin\left(\frac{n\pi}{L} x\right)}_{\text{eigen function.}}$

Let $u_n(x,t) = e^{-\frac{n^2\pi^2}{L^2} kt} \sin\left(\frac{n\pi x}{L}\right)$

$n = 1, 2, 3$

Thus whole solution.

$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-\frac{n^2\pi^2}{L^2} kT} \sin\left(\frac{n\pi x}{L}\right)$

The -ve values of n are not required as they give same eigen fn. with a -ve sign which will be taken into account by new constant C_n

* Now for $t = 0$

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{l}\right).$$

This fourier sine series for $f(x)$ in $0 < x < l$

$$C_n = \frac{2}{L}\int_0^l f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

Thus complete solution can be shown as

$$u(x,t) = \sum_{n=1}^{\infty}\left(\frac{2}{L}\int_0^l f(x)\sin\left(\frac{n\pi x}{L}\right)dx\right)e^{-\frac{n^2\pi^2 kt}{L^2}}\sin\left(\frac{n\pi x}{L}\right)$$

Q $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2}$, $t > 0$, $0 < x < l$

$u(0,t) = T_1$ $\quad$ $u(L,t) = T_2$

$u(x,0) = f(x)$ $\quad$ $0 < x < l$

In the steady state, $(t \to \infty)$, $u(x,t) = V(x)$

Substituting this in the DE

we get $\quad V''(x) = 0 \Rightarrow V(x) = C_1 x + C_2$

$V(0) = T_1$

$V(L) = T_2$

$T_1 = C_2$

$T_2 = C_1 l + C_2 \Rightarrow C_1 = \dfrac{T_2 - T_1}{L}$

$$\boxed{V(x) = \frac{T_2 - T_1}{L}x + T_1}$$

steady state solution

Assumption

$$u(x,t) = V(x) + w(x,t)$$

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \Rightarrow \frac{\partial w}{\partial t} = k\left(V''(x) + \frac{\partial^2 w}{\partial x^2}\right)$$

$$\Rightarrow \frac{\partial w}{\partial t} = k\frac{\partial^2 w}{\partial x^2}$$

Now we need to find boundary condn. for w

$f(x) = u(x,0) = V(x) + w(x,0)$

$w(x,0) = f(x) - V(x)$

$u(0,t) = T_1 \Rightarrow u(0,t) = V(0) + w(0,t)$

$T_1 = T_1 + w(0,t) \Rightarrow w(0,t) = 0$

$u(L,t) = T_2 \Rightarrow u(L,t) = V(L) + w(L,t)$

$T_2 = T_2 + w(L,t) \Rightarrow w(L,t) = 0.$

Now we have boundary condns for

$$\frac{\partial w}{\partial t} = k\frac{\partial^2 w}{\partial x^2}$$

$$w(x,t) = \sum_{n=1}^{\infty}\left(\frac{2}{L}\int_0^L (f(x) - V(x)) \sin\frac{n\pi x}{L}\,dx\right) e^{-\frac{n^2\pi^2 kt}{L^2}} \sin\frac{n\pi x}{L}$$

$$u(x,t) = \frac{T_2 - T_1}{L}x + T_1 + \sum_{n=1}^{\infty}\left(\frac{2}{L}\int_0^L \left(f(x) - \frac{T_2 - T_1}{L}x - T_1\right)\sin\frac{n\pi x}{L}\,dx\right) e^{-\frac{n^2\pi^2 kT}{L^2}} \sin\frac{n\pi x}{L}$$

* $q = -X\frac{\partial T}{\partial X}$ (Heat flux).

Q $\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2}$ $\quad u_x(0,t) = 0 \quad u_x(L,t) = 0 \quad \forall\, t > 0$

$t > 0 \quad 0 < x < L \qquad u(x,0) = f(x) \quad \forall\, x \in (0,L)$

ends are insulated.

$u(x,t) = X(x)T(t)$

$XT' = \alpha^2 TX'' \Rightarrow \frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T} = k.$

Case I $\quad k = 0$

$u(x,t) = A_1 x + B_1$

$u_x(x,t) = A_1 \Rightarrow A_1 = 0$ {boundary condn}

$u(x,t) = B_1$

'1' is the eigen function in this case.

Case II $\quad k > 0, \quad k = \lambda^2, \quad \lambda \neq 0$

$u(x,t) = e^{-\alpha^2\lambda^2 t}(A_2 e^{\lambda x} + B_2 e^{-\lambda x})$

$u_x(x,t) = \lambda e^{-\alpha^2\lambda^2 t}(A_2 e^{\lambda x} - B_2 e^{-\lambda x})$

$u_x(0,t) = 0 \Rightarrow \lambda e^{-\alpha^2\lambda^2 t}(A_2 - B_2) = 0$

$\Rightarrow A_2 - B_2$

$$u_x(l,t) \Rightarrow A_2 \lambda e^{-\alpha^2\lambda^2 t}(e^{\lambda l} - e^{-\lambda l}) = 0$$
$$\Rightarrow A_2 = 0$$
$$\therefore B_2 = 0$$

$$u(x,t) = 0$$
(Discard)

<u>Case III</u> $R < 0, \; R = -\lambda^2, \; \lambda \neq 0$

$$u(x,t) = e^{-\alpha^2\lambda^2 t}(A_3 \cos \lambda x + B_3 \sin \lambda x)$$
$$u_x(x,t) = \lambda e^{-\alpha^2\lambda^2 t}(-A_3 \sin \lambda x + B_3 \cos \lambda x)$$

$u_x(0,t) = 0 \Rightarrow \lambda e^{-\alpha^2\lambda^2 t}(B_3) = 0 \Rightarrow B_3 = 0$

$u_x(l,t) = 0 \Rightarrow \lambda e^{-\alpha^2\lambda^2 t}(-A_3 \sin \lambda l) = 0$

[$A_3 = 0$ leads to trivial soln]

$$\sin \lambda L = 0$$
$$\lambda = \frac{n\pi}{L} \qquad n = \pm 1, \pm 2, \pm 3$$

$$\underbrace{u(x,t) = A_3 e^{-\frac{\alpha^2 n^2 \pi^2}{L^2} t} \cos \frac{n\pi x}{L}}_{\text{eigensolution}} = u_n(x,t)$$

Final soln
$$u(x,t) = C_0 + \sum_{n=1}^{\infty} C_n e^{-\frac{\alpha^2 n^2 \pi^2}{L^2} t} \cos \frac{n\pi x}{L}$$

{-ve n gives same soln}

$\Downarrow$ from Case I

{multiplying eigen fn with some constant then taking linear combination}

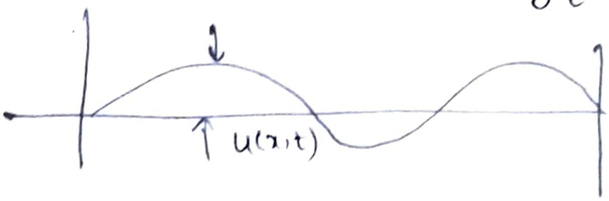
Initial condn

$$u(x,0) = f(x)$$
$$f(x) = C_0 + \sum_{n=1}^{\infty} C_n \cos \frac{n\pi x}{L}$$
$$C_0 = \frac{1}{L} \int_0^L f(x)\,dx$$
$$C_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\,dx$$

Wave Function

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad t>0,\ 0<x<l$$

(Hyperbolic)



- $u(x,t)$ is vertical displacement in the string
- $c^2 = T/\rho$ — $T \rightarrow$ tension in string; $\rho \rightarrow$ mass per unit length of string; $c \rightarrow$ Velocity

- Method of separation variables

$$u(x,t) = X(x)\,T(t)$$

$$\frac{X''(x)}{X(x)} = \frac{1}{c^2}\frac{T''(t)}{T(t)} = k.$$

$$\frac{d^2X}{dx^2} - kX = 0 \quad \text{and} \quad \frac{d^2T}{dt^2} - kc^2T = 0$$

Case I $\boxed{k=0}$ $\quad u(x,t) = (c_1x + c_2)(c_3t + c_4)$

Case II $\quad k>0,\ k=\beta^2,\ \beta \neq 0.$

$$u(x,t) = \left(c_5e^{\beta x} + c_6e^{-\beta x}\right)\left(c_7e^{\beta ct} + c_8e^{-\beta ct}\right)$$

Case III $\quad k<0,\ k=-\beta^2,\ \beta \neq 0$

$$u(x,t) = \left(c_9\cos\beta x + c_{10}\sin\beta x\right)\left(c_{11}\cos\beta ct + c_{12}\sin\beta ct\right)$$

→ we need two initial condn. & 2 boundary condn

Q $\quad \dfrac{\partial^2 u}{\partial t^2} = c^2\dfrac{\partial^2 u}{\partial x^2} \qquad t>0,\ 0<x<l$

$u(0,t) = 0 \qquad u(l,t) = 0 \quad$ for $t>0$

$u(x,0) = f(x) \qquad \underbrace{u_t(x,0) = 0}_{\text{initial velocity is zero}} \quad$ for $0 \le x \le l$

- If given $0 \le x \le L$, then consider $f(0) = 0$, $f(L) = 0$

$$u(x,t) = X(x)\,T(t)$$

I $\;k = 0$

$$u(x,t) = (c_1 x + c_2)(c_3 t + c_4)$$

$u(0,t) = 0 \Rightarrow c_2(c_3 t + c_4) = 0 \Rightarrow c_2 = 0$

$u(L,t) = 0 \Rightarrow c_1 L(c_3 t + c_4) = 0 \Rightarrow c_1 = 0$

$u(x,t) = 0$

$k = 0$ is not possible.

II $\;k > 0$, $k = \beta^2$, $\beta \neq 0$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$XT'' = c^2 X'' T \Rightarrow \frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = k$$

$X'' - \beta^2 X = 0$ $\qquad$ $T'' - \beta^2 c^2 T = 0$

$T'' - \lambda^2 T = 0 \qquad (\lambda = \beta c)$

$X(0)T(t) = 0$

$X(0) = 0$

$$u(x,t) = (c_1 e^{\beta x} + c_2 e^{-\beta x})(c_3 e^{\lambda t} + c_4 e^{-\lambda t})$$

$u(0,t) = 0 \Rightarrow c_2 = -c_1$

$u(L,t) = 0 \Rightarrow c_1(e^{\beta L} - e^{-\beta L})(c_3 e^{\lambda t} + c_4 e^{-\lambda t}) = 0$

$c_1 = 0$

$\therefore c_2 = 0$

$X(L) = 0$

$\boxed{u(x,t) = 0}$

discard

III $\;k < 0$, $k = -\beta^2$, $\beta \neq 0$

$$u(x,t) = (c_1 \cos \beta x + c_2 \sin \beta x)(c_3 \cos \lambda t + c_4 \sin \lambda t)$$

$u(0,t) \Rightarrow c_1 = 0$

$$u(x,t) = \sin \beta x\,(A \cos \lambda t + B \sin \lambda t)$$

$u(L,t) = 0 \Rightarrow \sin \beta L = 0 \Rightarrow \beta = \frac{n\pi}{L}$

$n = \pm 1, \pm 2, \pm 3 \ldots$

$$\lambda = \frac{n\pi c}{L}$$

↑ eigen values

$$u(x,t) = \sin \beta x \,(A\cos \lambda t + B \sin \lambda t)$$

$u_t(x,0)$ $\qquad u_t(x,t) = \lambda \sin\beta x\,(-A\sin\lambda t + B\cos\lambda t)$

$\Rightarrow B = 0$

$$\boxed{u(x,t) = A \sin\beta x \cos\lambda t}$$

frequency (w) of simple harmonic motion

$$\underbrace{u(x,t) = A \sin\frac{n\pi x}{L} \cos\frac{n\pi ct}{L}}_{u_n(x,t)} \qquad n = 1, 2, \ldots$$

$u_n(x,t)$ eigen functions

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}$$

$u(x,0) = f(x)$

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}$$

$V(x,t)$ ↑

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.$$

$$u(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx\right)\sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}$$

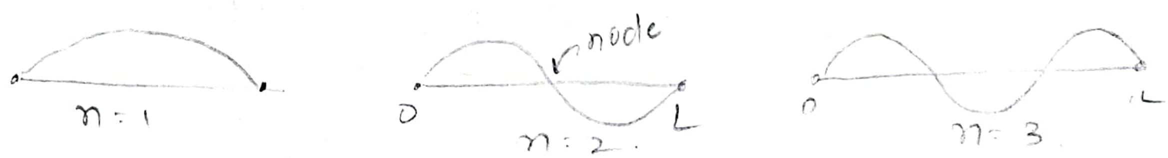
* each of u_n is representing a simple harmonic motion with frequency $\frac{\lambda_n}{2\pi} = \frac{nc}{2L}$

(fundamental modes & overtones)

For what values of x u_n will be zero?

$\sin\frac{n\pi x}{L} = 0 \qquad 0 < x < L$

$x = \frac{L}{n}, \frac{2L}{n}, \ldots, \frac{(n-1)L}{n}.$



Q $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ $t>0$ $0<x<L$

$u(0,t) = u(L,t) = 0$
$u(x,0) = 0$ $\Bigg\}$ $0<x<L$
$u_t(x,0) = g(x)$

$u = X(x)\,T(t)$

$\dfrac{X''}{X} = \dfrac{1}{c^2}\dfrac{T''}{T} = k.$

III $k<0$, $k=-\beta^2$, $\beta \neq 0$.

$u(x,t) = (c_1 \cos\beta x + c_2 \sin\beta x)(c_3 \cos\lambda t + c_4 \sin\lambda t)$

$\lambda = \beta c$

$u(0,t) \Rightarrow c_1 = 0$

$u(x,0) \Rightarrow c_3 = 0$

$\therefore u(x,t) = A \sin\beta x \sin\lambda t$

$u(L,t) = A \sin\beta L \sin\lambda t = 0.$

$\therefore \sin\beta L = 0$

$\beta L = n\pi$

$\beta = \dfrac{n\pi}{L}$, $n = \pm 1, \pm 2$

$u(x,t) = \underbrace{A \sin\dfrac{n\pi x}{L} \sin\dfrac{n\pi c t}{L}}_{u_n(x,t)}$

Actual soln. is a linear combination

$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\dfrac{n\pi x}{L} \sin\dfrac{n\pi c t}{L}$

$u_t(x,0) = g(x)$ $\quad u_t(x,t) = \dfrac{\pi c}{L} \sum_{n=1}^{\infty} n A_n \sin\dfrac{n\pi x}{L} \sin\dfrac{n\pi c t}{L}$

$\rightarrow$ $g(x) = \sum_{n=1}^{\infty} \underbrace{\dfrac{n\pi c}{L} A_n}_{\text{fourier coefficient}} \sin\dfrac{n\pi x}{L}$ $\quad 0<x<L$

$$A_n \frac{n\pi c}{L} = \frac{2}{L}\int_0^L g(x) \sin \frac{n\pi x}{L}\, dx$$

$$A_n = \frac{2}{n\pi c}\int_0^L g(x) \sin \frac{n\pi x}{L}\, dx$$

$w(x,t)$

$$u(x,t) = \frac{2}{\pi c}\sum_{n=1}^{\infty}\left(\frac{1}{n}\int_0^L g(x) \sin\frac{n\pi x}{L}\,dx\right)\sin\frac{n\pi x}{L}\,\sin\frac{n\pi ct}{L}$$

Q $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ $t>0$ $0<x<L$.

$u(0,t) = u(L,t) = 0$

$\left.\begin{array}{l} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{array}\right\} 0<x<L.$

$\frac{\partial^2 v}{\partial t^2} = c^2 \frac{\partial v^2}{\partial x^2}$	$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial w^2}{\partial x^2}$
$v(0,t) = v(L,L) = 0$	$w(0,t) = w(L,t) = 0$
$v(x,0) = f(x)$	$w(x,0) = 0$
$v_t(x,0) = 0$	$w_t(x,0) = g(x)$

$u(x,t) = v(x,t) + w(x+t)$

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 (v+w)}{\partial t^2} - c^2 \frac{\partial^2 (v+w)}{\partial x^2}$$

$$= \left(\frac{\partial^2 v}{\partial t^2} - c^2 \frac{\partial^2 v}{\partial x^2}\right) + \left(\frac{\partial^2 w}{\partial t^2} - c^2\frac{\partial w^2}{\partial x^2}\right)$$

$= 0$

$u(0,t) = v(0,t) + w(0,t)$

$= 0 + 0 = 0$

$u_t(x,0) = v_t(x,0) + w_t(x,0) = g(x) + 0 = g(x)$

$u(x,0) = v(x,0) + w(x,0) = 0 + f(x) = f(x)$

#

d'Alembert's solution of the wave Equation

$$u_{tt} = c^2 u_{xx}$$

$$\eta = x + ct$$

$$\xi = x - ct$$

$$\frac{\partial \xi}{\partial x} = 1, \quad \frac{\partial \xi}{\partial t} = -c, \quad \frac{\partial \eta}{\partial x} = 1, \quad \frac{\partial \eta}{\partial t} = c$$

$$u_t = \frac{\partial u}{\partial t} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t} = c(-u_\xi + u_\eta)$$

$$u_{tt} = c\left[(-u_\xi + u_\eta)_\xi \frac{\partial \xi}{\partial t} + (-u_\xi + u_\eta)_\eta \frac{\partial \eta}{\partial t}\right]$$

$$= c^2\left[-(-u_{\xi\xi} + u_{\eta\xi}) + (u_{\xi\eta} + u_{\eta\eta})\right]$$

$$= c^2\left[u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}\right]$$

$$u_x = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{d\eta}{\partial x} = u_\xi + u_\eta$$

$$u_{xx} = (u_\xi + u_\eta)_\xi \frac{d\xi}{dx} + (u_\xi + u_\eta)_\eta \frac{d\eta}{dx}$$

$$= u_{\xi\xi} + u_{\eta\xi} + u_{\xi\eta} + u_{\eta\eta} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$

$$u_{tt} = c^2 u_{xx}$$

$$c^2\left[u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}\right] = c^2\left[u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}\right]$$

$$\Rightarrow u_{\xi\eta} = 0$$

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0 \Rightarrow \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \eta}\right) = 0$$

integrate w.r.t ξ

$$\frac{\partial u}{\partial \eta} = \psi(\eta)$$

$$u = \int \psi(\eta)\, d\eta + \phi(\xi)$$

* $u = \psi(\eta) + \phi(\xi)$

$u(x,t) = \psi(x+ct) + \phi(x-ct)$

Debopriya

$u_{tt} = c\, u_{xx}^2$

$t = 0$ Cauchy condition.

Change of variable

$\xi(x+ct)$ $\qquad \eta(x-ct)$

* Method of characteristic

$u_{tt} = c^2 u_{xx}$

$\xi(x+ct)$, $\eta = x-ct$

$\tilde{u}(\xi,\eta) = u(x,t)$

$\tilde{u}_{\xi\eta} = \dots$

$u(x,t) = \phi(x-ct) + \psi(x+ct)$ $\qquad \phi, \psi$ are arbitrary functions.

Remark :- Solution is a superposition of two functions (waves) moving in the opposite direction at speed "c".

Q $u_{tt} = c^2 u_{xx}$ $\qquad t>0, \quad -\infty < x < \infty$

$u(x,0) = f(x), \quad u_t(x,0) = g(x)$ $\qquad -\infty < x < \infty$

$u(x,t) = \phi(x-ct) + \psi(x+ct)$

$u_t(x,t) = -c\phi'(x-ct) + c\psi(x+ct)$

$u(x,0) = f(x) \Rightarrow \quad \phi(x) + \psi(x) = f(x)$

$u_t(x,0) = g(x) \Rightarrow \quad -c\phi'(x) + c\psi'(x) = g(x)$

Dividing above eqn. by $(-c)$ & integrating ~~with x~~ wrt x

$$\phi(x) - \psi(x) = -\frac{1}{c}\int_x^{x_0} g(s)\,ds + K$$

$\hookrightarrow \phi(x_0) - \psi(x_0)$

$$\phi(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_{x_0}^{x} g(s)\,ds + \frac{K}{2}$$

$$\psi(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_{x_0}^{x} g(s)\,ds - \frac{K}{2}$$

From above equation

$$u(x,t) = \phi(x-ct) + \psi(x+ct)$$

$$= \frac{1}{2}\left(f(x-ct) + f(x+ct)\right) + \frac{1}{2c}\left(\int_{x-ct}^{x} g(s)\,ds + \int_{x_0}^{x+ct} g(s)\,ds\right)$$

$$u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds.$$

Q(a) Think of a string of ∞ length. One side is attached to a ~~ring~~ ring $\rightarrow$ can move vertically along ~~g~~ y-axis

$y(x,t) =$ displacement at position ~~of~~ x & time t.

At $t=0$, $x=0$

$y(x,0)=0$ $\quad\quad y_t(x,0)=0$.

~~$y(0,0)=0$~~ $\quad y(0,t) = f(t) \quad (t \geq 0)$.

$$y(x,t) = \phi(x+at) + \psi(x-at)$$

(b) $\quad y(x,t) = \phi(x-at) + \psi(x+at)$

$y(x,0) = 0 = \phi(x) + \psi(x) \quad$ —(i).

~~$y_t(x,0) = \phi$~~

diff. $\quad y_t(x,t) = a\left[\phi'(x+at) - \psi(x-at)\right]$

$y_t(x,0) = a\left[\phi'(x) + \psi(x)\right] = 0 \quad$ —(ii)

$\phi'(x) = \psi'(x)$ $\quad\quad \phi(x) = -\psi(x). = C$

$\therefore \phi(x) = c$

$\therefore \psi(x) = -c.$

(c) $y(0,t) = f(t)$

$\phi(at) + \psi(at) = f(t)$

$\phi(t) + \psi(-t) = f(t/a).$

~~$\phi(t)$~~ $\psi(-t) = f(t/a) - c \quad$ fo

$\ast \quad y(x,t) = 0 \quad x \geq at \quad \left[\begin{array}{l} x+at \geq 0, \; x \geq 0 \\ \phi(x+at) = c \end{array}\right]$

Tutorial 3

① Let $u = v(x) + w(x,t)$

$w_t = \alpha^2 w_{xx}$

$w(0,t) = 0 = w(L,t)$

$w(x,0) = f(x) - v(x)$

$u(0,t) = 0$

$u(L,t) = 0$

$u(x,0) = f(x)$

$u_t - \alpha^2 u_{xx} = 7e^{-2x}$

$u_t = w_t$

$u_{xx} = v_{xx} + w_{xx}$

$w_t - \alpha^2 (v_{xx} + w_{xx}) = 7e^{-2x}$

$\underbrace{(w_t - \alpha^2 w_{xx})}_{w_t \;=\; 0} - \alpha^2 v_{xx} = 7e^{-2x}$

$v_x = -\dfrac{7e^{-2x}}{\alpha^2}$

$v(x) = -\dfrac{7e^{-2x}}{4\alpha^2} + c_1 x + c_2$

$v(0) = 0$

$v(L) = 0$

$c_2 = 7/4\alpha^2$

$c_1 = \dfrac{-7}{4\alpha^2 L}(e^{-2L} - 1)$??

② $u_{tt} = c^2 u_{xx}$, $t > 0$, $0 < x < L$

$u(0,t) = 0$, $u(L,t) = 0$

$u(x,0) = f(x)$, $u_t(x,0) = g(x)$

$\Rightarrow$ $u(x,t) = X(x).T(t)$

$u_{xx} = X''(x).T(t)$

$u_{tt} = X(x).T''(t)$

$X(x)T''(t) = c^2 X''(x).T(t)$

$\dfrac{X''(x)}{X(x)} = \dfrac{1}{c^2}\dfrac{T''(t)}{T(t)} = k$

Case I $k = 0$

$X''(x) = 0$ & $T''(t) = 0$

$X(x) = c_1 x + c_2$

$T(t) = c_3 t + c_4$.

$u = (c_1 x + c_2)(c_3 t + c_4)$

$u(0,t) = c_2 (c_3 t + c_4) = 0$

$c_2 = 0$

$u(L,t) = C_1 L (C_3 t + C_4) = 0$

$C_1 = 0$

case $K > 0$ $\quad K = \lambda^2 , \lambda > 0$

$X''(x) - \lambda^2 X(x) = 0 \quad \& \quad T''(t) - \lambda^2 c^2 T(t) = 0$

$X(x) = C_1 e^{-\lambda x} + C_2 e^{\lambda x} \qquad T(t) = C_3 e^{-\lambda ct} + C_4 e^{\lambda ct}$

$$u = (C_1 e^{-\lambda x} + C_2 e^{\lambda x})\,(C_3 e^{-\lambda ct} + C_4 e^{\lambda ct})$$

$$u(0,t) = (C_1 + C_2)\,(C_3 e^{-\lambda ct} + C_4 e^{\lambda ct}) = 0$$

$C_2 = -C_1$

$$u(L,t) = (C_1 e^{-\lambda L} - C_1 e^{\lambda L})(C_3 e^{-\lambda ct} + C_4 e^{\lambda ct}) = 0$$

$$C_1 (e^{-\lambda L} - e^{\lambda L})\,(C_3 e^{-\lambda ct} + C_4 e^{\lambda ct}) = 0$$

$C_1 = 0$

Case $K < 0$ $\quad K = -\lambda^2 , \lambda > 0$

$X''(x) + \lambda^2 X(x) = 0 \quad \& \quad T''(t) + \lambda^2 c^2 T(t) = 0$

$X(x) = C_1 \sin \lambda x + C_2 \cos \lambda x \qquad T(t) = C_3 \sin \lambda ct + C_4 \cos \lambda tc$

$u(0,t) \Rightarrow C_2 = 0$

$u(L,t) \Rightarrow C_1 \neq 0$

$\sin \lambda L = 0$

$\lambda L = n\pi \qquad n = \pm 1, \ldots$

$L = \frac{n\pi}{L} \qquad n = \pm 1, \pm 2 \ldots$

$$X_n(t) = \sin\left(\frac{n\pi x}{L}\right)$$

$$T_n(t) = A_n \sin\left(\frac{n\pi c}{L} t\right) + B_n \cos\left(\frac{n\pi c}{L} t\right)$$

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[A_n \sin \frac{n\pi c t}{L} + B_n \cos \frac{n\pi c t}{L}\right]$$

$u(x,0) = f(x)$

$$f(x) = \sum_{n=1} B_n \sin\left(\frac{n\pi x}{L}\right)$$

Multiplying both sides by $\sin\left(\frac{m\pi x}{L}\right)$

$$f(x)\sin\frac{m\pi x}{L} = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)$$

$$\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx = \sum_{n=1}^{\infty} B_n \int_0 \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)$$

$$= \sum_{n=1}^{\infty} B_n x \begin{cases} L/2 & m=n \\ 0 & m \neq n \end{cases} = B_n \frac{L}{2}$$

$$B_m = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)$$

$$A_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)$$

(3.) $u(x,t) = V(x) + w(x,t)$

$u(x,0) = 1-x$

$w_{tt} = c^2 w_{xx}$

$u_t(x,0) = 0$

$w(0,t) = 0, \quad w(L,t) = 0$

$w(x,0) = \cancel{1}\, x$

$u_{tt} = c^2 u_{xx}$

$u(0,t) = V(0,t) + w(0,t)$

$w_{tt} = c^2(V_{xx} + u_{xx})$

$0 = c^2 V_{xx}$

$V_{xx} = 0$

$\Rightarrow V = Ax + B.$ $\quad B = 1$

$A = -1.$

$= 1-x.$

(6.) $$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\cdot\cos\left(\frac{n\pi ct}{L}\right)$$

$$= \frac{1}{2}\left(h(x+ct) + h(x-ct)\right)$$

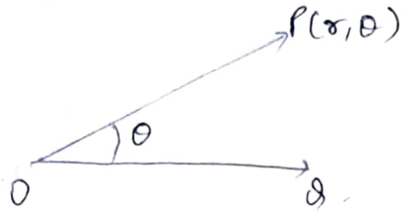
$$u(x,t) = \sum_{n=1}^{\infty} B_n \left[\sin\left(\frac{n\pi}{l}(x+ct)\right) + \sin\left(\frac{n\pi}{l}(x-ct)\right) \right]$$

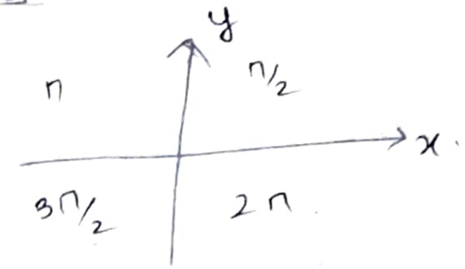
$$h(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right)$$

$$u = \frac{1}{2}\left[h(x+ct) + h(x-ct) \right]$$

Vibration of Circular Membranes

→ Polar coordinate





$x = r\cos\theta \qquad y = r\sin\theta \qquad \theta \in [0, 2\pi]$

* 2D wave eqn

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right)$$

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right)$$

$u_{\theta\theta} = 0$ (symmetrical)

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right)$$

At boundary $u(R,t) = 0$

Initial condition

$u(r,0) = f(r)$ and $u_t(r,0) = g(r) \qquad 0 \le r \le R$

$x = r\cos\theta \qquad y = r\sin\theta$

$u(r,t) = T(t)\,R(r)$

$$W(r)\ddot{T}(t) = c^2\left[W''(r)T(t) + \frac{1}{r}W'(r)T(t)\right]$$

$$\Rightarrow \frac{1}{c^2}\frac{\ddot{T}(t)}{T(t)} = \frac{W''(r)}{W(r)} + \frac{1}{r}\frac{W'(r)}{W(r)} = K.$$

2 ODES

$$\frac{d^2T}{dt^2} - Kc^2T = 0$$

$$W'' + \frac{1}{r}W' - KW = 0.$$

$K > 0, \quad K < 0, \quad K = 0$

$\boxed{k=0}$ $\quad T(t) = c_1 t + c_2$ (linear)

$\boxed{k>0}$ $\quad T(t) = c_1 e^{\sqrt{ck}\,t} + c_2 e^{-\sqrt{ck}\,t}$ (exponential)

$\boxed{k<0}$ $\quad$ as $t \to \infty$ $\quad \to \infty$ (grows) $\quad \to 0$ (decay)

$\boxed{k<0}$ $\quad T(t) = c_1 \sin\sqrt{ck}\,t + c_2 \cos\sqrt{ck}\,t$ (periodic)

→ since oscillatory nature of drum thus periodic

$\boxed{k<0}$ $\quad w'' + \frac{1}{r} w' - k w = 0$

The equation for W can be reduced to Bessel function

$$[x^2 y'' + x y' + (x^2 - \nu^2) y = 0]$$

$s = \beta r \qquad w(r) = w(\frac{s}{\beta})$

$w' = \beta \frac{dw}{ds} \qquad s = \beta r \quad r \mapsto s \qquad = w \circ g(s)$

$\qquad\qquad\qquad\qquad w \mapsto \tilde{w} \qquad = \tilde{w}(s)$

$w'' = \beta^2 \frac{dw}{ds}$

$$s^2 \frac{d^2 w}{ds^2} + s \frac{dw}{ds} + s^2 w = 0 \quad \text{---} (1)$$

(1) is the Bessel equation with $\nu^2 = 0$, $w(r)$, $s = \beta r$

$\tilde{W}(s) = J_0(s)$ or $Y_0(s)$ (or linear combination of them)

$Y_0(s) = \infty$ at $s = 0$. Thus cannot be a part of solution

$w(r) = w(s/\beta) = \tilde{w}(s) = J_0(s) = J_0(\beta r)$

$W(r) = J_0(\beta r)$

Boundary condn $\quad J_0(\beta r) = 0$

↳ ∞ many +ve solutions

zeros $\alpha_1, \alpha_2, \alpha_3$

$J_0(\beta R) = 0 = J_0(\alpha_n)$; $n = 1, 2, \ldots$

$\alpha_n = \beta R \Rightarrow \beta_n = \frac{\alpha_n}{R}$; $n = 1, 2$

~~$W(r)$~~

$W_n(r) = J_0(\beta_n r) \qquad n = 1, 2, \ldots$

$= J_0\left(\frac{\alpha_n r}{R}\right)$

are the soln of eqn of w for each n

eigen fn. & Eigen value

$\lambda_n = \beta_n r$

$u(r, t) = T(t)\, W(r)$

$u_n(r, t) = T(t)\, W_n(r)$

$= \cancel{J_0(\beta} \; (c_n \cos \lambda_n t + d_n \sin \lambda_n t)\; J_0(\beta_n r)$

VINAY SIR

Sequence :- $f: I^+ \to \mathbb{C}$

$\{f(n)\}$ or $\{z_n\}$ $n \in I^+$

Ex $\{i^n\}$, $\{2^n\}$, $\left\{\frac{i^n}{n}\right\}$

Convergent Sequence: $\{z_n\}$ is convergent if $\lim_{n\to\infty} z_n = c$ (c is unique)

For a given $\epsilon > 0$ there $\exists$ N such that $|z_n - c| < \epsilon$ for $n > N$.

$\{z_n\}$, where $z_n = x_n + iy_n$

$\downarrow$

$c = a+ib$ iff $\begin{cases} \{x_n\} \to a \\ \{y_n\} \to b \end{cases}$

Eg. $\left\{\frac{i^n}{n}\right\} = \left\{i, \frac{-1}{2}, \frac{-i}{3}, \frac{1}{4}, \dots\right\}$ converges to zero

$\lim_{n\to\infty} \frac{i^n}{n} = 0$.

Eg. $\{i^n\}$ diverges

Eg $\left\{1 - \frac{1}{n^2} + i\left(2 + \frac{4}{n}\right)\right\}$ $z_n = \underbrace{1 - \frac{1}{n^2}}_{x_n} + i\underbrace{\left(2+\frac{4}{n}\right)}_{y_n}$

converges to $1 + 2i = c$.

$\quad\quad\quad\quad\ \ \downarrow\ \ \downarrow$

$\quad\quad\quad\quad\ \ a\ \ \ b$

$\{z_n\} \to A$ $\{w_n\} \to B$

$\lim_{n\to\infty} (z_n \pm w_n) = A \pm B$

$\lim_{n\to\infty} (z_n w_n) = AB$

$\lim_{n\to\infty} \frac{z_n}{w_n} = \frac{A}{B}$

Series - $\{z_n\}$

The expression of the form $z_1 + z_2 + z_3 + \dots$ $\sum_{n=1}^{\infty} z_n$ is called a series

- Partial Sums

$$S_n = z_1 + z_2 + \dots + z_n \quad (n^{th} \text{ partial sum})$$

- Convergence : $\sum_{n=1}^{\infty} z_n$.

If $\lim_{n\to\infty} S_n = S$ for some S, then $\sum_{n=1}^{\infty} z_n$ converges and S is called the sum of the series

$$\sum_{n=1}^{\infty} z_n = S$$

Thm (Necessary condition for convergence)

If $\sum_{n=1}^{\infty} z_n$ converges, then $\lim_{n\to\infty} z_n = 0$

* The condition is not sufficient

Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$ diverge.

Absolute convergence : A series $\sum_{n=1}^{\infty} z_n$ is absolutely convergent if $\sum_{n=1}^{\infty} |z_n|$ is convergent.

Absolute convergence $\Rightarrow$ Convergence.

* If $\sum_{n=1}^{\infty} z_n$ converges but $\sum_{n=1}^{\infty} |z_n|$ is divergent then $\sum_{n=1}^{\infty} z_n$ is said to be conditionally convergent.

E.g. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ conditionally convergent

$\sum_{n=1}^{\infty} z_n$ where $z_n = x_n + iy_n$.

Eg $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

converges

Eg $\sum_{n=1}^{\infty} \left[1 + \frac{i(-1)^n}{n}\right]$ diverges

1. Comparison Test :-

Given $\sum_{n=1}^{\infty} z_n$. Suppose we can find a converging series $\sum_{n=1}^{\infty} b_n$ with non negative real terms s.t.

$$|z_n| \leq b_n \quad \text{for } n = 1, 2, 3.$$

then the given series $\sum_{n=1}^{\infty} z_n$ converges, even absolutely.

2. Ratio Test

$\sum_{n=1}^{\infty} z_n$ with $z_n \neq 0$ (for $n = 1, 2, 3$)

such that $\lim_{n \to \infty} \left| \frac{z_{n+1}}{z_n} \right| = L$ then

(a) the series is convergent if $L < 1$

(b) " " " divergent if $L > 1$

(c) fails if $L = 1$

3. Root Test

$\sum_{n=1}^{\infty} z_n$ such that $\lim_{n \to \infty} \sqrt[n]{|z_n|} = L$ then

Cauchy product $\sum_{n=0}^{\infty} c_n (z-z_0)^n$ where

$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \dots + a_n b_0$

within the circle of convergence of both the series

$S(z) = f(z) \cdot g(z)$

Taylor Series

$\sum_{n=0}^{\infty} a_n (z-z_0)^n$ with r.o.c. R represents a fn. analytic in the domain $|z-z_0|<R$

Taylor's Thm. Let f be analytic on $D = \{z : |z-z_0| < R\}$

Then $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ for $z \in D$

where for $n = 0, 1, 2, 3$

$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz$$

simple closed contour

any +vely oriented contour enclosing z_0 and lying completely in D.

* Taylor series is unique

If $z_0 = 0$; Maclaurine series

E.g. around $z=0$

$f(z) = e^z$

(find domain where it is analytic)

$\sum_{n=0}^{\infty} \frac{f^n(0)}{n!} z^n$

$f^n(z) = e^z$

$f^n(0) = 1$

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n, \quad z \in \mathbb{C}$$

② $f(z) = 1/1-z$

$f^n(z) = \frac{n!}{(1-z)^{n+1}}$

$f^n(0) = \cancel{n+} n!$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad |z|<1$$

E.g. ③ $1/1-z^2 = \sum_{n=1}^{\infty} n z^{n-1}, \quad |z|<1$

similarly $\frac{1}{1+z}$ $\quad \frac{1}{1-(-z)}$

$$R = \frac{1}{\mu} = 1$$

$$\lim_{n\to\infty} \left| \frac{z^{n+1}/(n+1)}{z^n/n} \right| = \lim_{n\to\infty} \left| \frac{z\, n}{n+1} \right|$$

$$= |z| \lim_{n\to\infty} \frac{n}{n+1}$$

converges for $|z| < 1$

E.g ① $\sum_{n=1}^{\infty} z^n$ $\quad R=1$

② $\sum_{n=1}^{\infty} n!\, z^n$ $\quad R=0$

③ $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2} z^n$ $\quad R=\frac{1}{e}$

* Termwise differentiation
* Termwise integration

convergent in the same radius

$$\sum_{n=0}^{\infty} a_n z^n \longrightarrow R = r_1$$

$$\sum_{n=0}^{\infty} b_n z^n \longrightarrow R = r_2$$

$$\sum_{n=0}^{\infty} (a_n + b_n) z^n \longrightarrow R \geq (\min\{r_1, r_2\})$$

Eg. $\sum_{n=0}^{\infty} (1+2^{-n}) z^n$ $\quad R = 1 = r_1$

$\sum_{n=0}^{\infty} (-z^n)$ $\quad R = 1 = r_2$

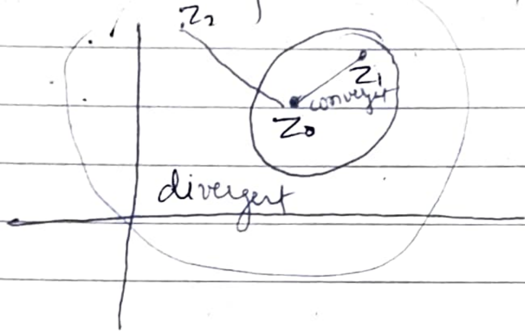
$\sum_{n=0}^{\infty} 2^{-n} z^n$ $\quad R = 2$

Multiplication $\quad \underbrace{\sum_{n=0}^{\infty} a_n z^n}_{r_1} \quad \underbrace{\sum_{n=0}^{\infty} b_n z^n}_{r_2}$

Convergence $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ — (*)

Suppose for $z=z_1$, (*) is convergent

for $z=z_2$, (*) is divergent



circle around z_0

If R is radius of convergence then circle, $|z-z_0|=R$ is called the circle of convergence

→ The series converges for $|z-z_0|<R$

diverges for $|z-z_0|>R$

① $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ converges at $z=1$

② $\sum_{n=1}^{\infty} \frac{z^n}{n}$ diverges at at $z=1$. but converges at $z=-1$

③ $\sum_{n=1}^{\infty} z^n$ diverges everywhere.

* Formula for computing R

- If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \mu$ then $R=\frac{1}{\mu}$.
- If $\lim_{n\to\infty} |a_n|^{1/n} = \mu$ then $R=\frac{1}{\mu}$

E.g. $\sum_{n=1}^{\infty} \frac{z^n}{n}$ Find R

$a_n = \frac{1}{n}$ $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{n}{n+1}\right| = \lim_{n\to\infty} \frac{1}{1+1/n} =1$

- $\sum_{n=0}^{\infty} a_n (z - z_0)^n$
  $\uparrow$ variable

E.g. ① $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ $\quad \lim_{n\to\infty} \left| \frac{z^{n+1}/(n+1)!}{z^n/n!} \right| = \lim_{n\to\infty} \left| \frac{z}{n+1} \right|$

$= \lim_{n\to\infty} \frac{|z|}{n+1} = 0 < 1$

Convergent for all z. $\quad \forall z \in \mathbb{C}$

② $\sum_{n=0}^{\infty} n!\, z^n$ $\quad \lim_{n\to\infty} |(n+1)z|$

convergent only for $z=0$ (useless)

③ $\sum_{n=0}^{\infty} z^n$

$\lim_{n\to\infty} |z| = |z| < 1$ (converges by ratio test)

$|z| > 1$ (diverges)

$|z| = 1$ (fails)

$S_n = 1 + z + \dots + z^n$ — ①

$z S_n = \cancel{z}\, z + \dots + z^n + z^{n+1}$ — ②

① − ②

$\cancel{(1-z)} S_n = \frac{1 - z^{n+1}}{1 - z}$

$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{(1 - z^{n+1})}{(1 - z)}$

(for $|z|<1$)

$= \frac{1}{1-z}$.

$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \dots \quad |z| < 1$

Power series

$$\sum_{n=1}^{\infty} a_n (z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$$

$a_n \in \mathbb{C}$ coefficients of the series.

$z_0 \in \mathbb{C}$ centre of the series

If $z_0 = 0$ $\sum_{n=0}^{\infty} a_n z^n$.

every power series is convergent around its centre

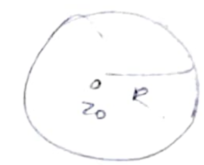
$z = z_1 \in \mathbb{C}$ $\sum_{n=0}^{\infty} a_n (z_1 - z_0)^n$ convergent

$z = z_2 \in \mathbb{C}$ $\sum_{n=0}^{\infty} a_n (z_2 - z_0)^n$ may be divergent

Given $f(z)$ is analytic in $D = \{z : |z - z_0| < R\}$

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad \text{Taylor series}$$

$$a_n = \frac{f^{(n)}(z_0)}{n!}$$



* when $f(z)$ not analytic at $z = 0$.

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \qquad \underline{\text{laurent series}}$$

Laurent's Theorem

Let f be analytic in an annulus $D = \{z : r_1 < |z - z_0| < r_2\}$. Then at each point $z \in D$, f can be represented as

$$> f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}}_{\text{laurent series of } f \text{ around } z_0}$$

Principal part of f

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} \, dz \quad \text{and}$$

$$b_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{-n+1}} \, dz$$

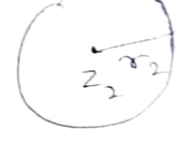
with C being any +vely oriented simple closed contour enclosing z_0 and lying completely in D.

$$* \quad f(z) = \sum_{-\infty}^{\infty} a_n (z - z_0)^n, \quad a_n = \frac{1}{2\pi i} \oint \frac{f(z)}{(z - z_0)^{n+1}} \, dz$$

** laurent series for a fn $f(z)$ is unique in given D

Remark: If $f(z)$ is analytic in the domain $D = \{z : |z - z_0| < r_2\}$

Then $$b_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{-n+1}} \, dz = 0$$



Laurent series will reduce to the Taylor series (By Cauchy - Goursat theorem)

$$g(z) = \frac{f(z)}{(z-z_0)^{-n+1}}$$

g is analytic c

$\oint_c g(z)\,dz = 0$

Q Laurent series for $z^{-5}\sin z$ with center 0:

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}, \quad z \in \mathbb{C}$$

$$z^{-5}\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n-4} \qquad |z|>0.$$

$$= \underbrace{\frac{1}{z^4} - \frac{1}{3!z^2}}_{\text{principal part}} + \frac{1}{5!} - \frac{z^2}{7!} + \frac{z^4}{9!} - \cdots \qquad |z|>$$

$\boxed{0<|z|<\infty}$ annulus.

Q $z^2 e^{1/z}$

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad z \in \mathbb{C}$$

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!\,z^n} \qquad |z|>0$$

$$z^2 e^{1/z} = \sum_{n=0}^{\infty} \frac{z^2}{n!\,z^n}$$

$$= z^2 + \frac{z}{1!} + \frac{1}{2!} + \frac{1}{3!z} + \frac{1}{4!\,z^2} + \cdots$$

$0<|z|<\infty$

Q Develop $1/(1-z)$ in

(a) non negative powers of z.

(b) negative powers of z.

(a) $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1+z+z^2+\dots$, $|z|<1$

(b) $|\frac{1}{z}|<1$ $\quad 1<|z|<\infty$

$|\frac{1}{z}|<1$

$$\frac{1}{1-z} = \frac{1}{z}\cdot\frac{1}{\frac{1}{z}-1} = -\frac{1}{z}\cdot\frac{1}{1-\frac{1}{z}}$$

$$= -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n \qquad \left|\frac{1}{z}\right|<1 \text{ or } |z|>1$$

$$= -\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}$$

$$= -\frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \dots \quad , |z|>1$$

Q find all taylor and laurent series of $f(z) = \frac{-2z+3}{z^2-3z+2}$ with centre 0.

$(z-2)(z-1)$

$$f(z) = \frac{-1}{(z-2)} - \frac{1}{(z-1)}.$$

laurent series 1 & 2

Taylor series

laurent series b/w 2 & ∞

I $|z|<1$

$$f(z) = \frac{1}{1-z} + \frac{1}{2}\cdot\frac{1}{1-\frac{z}{2}} \qquad \left|\frac{z}{2}\right|<1 \Rightarrow |z|<2$$

$\Downarrow$

$\frac{1}{z-2}$ is analytic

$$= \sum_{n=0}^{\infty} z^n + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n , \ |z|<1$$

$$= \frac{3}{2} + \frac{5}{4}z + \frac{9}{8}z^2 \dots \qquad |z|<1$$

II $1<|z|<2$ This is not analytic after $|z|=1$

$$f(z) = \frac{-1}{z-1} + \frac{1}{z-2}$$

$|z|>1$

$\left|\frac{1}{z}\right|<1$

$$= \frac{-1}{z}\left(\frac{1}{1-\frac{1}{z}}\right) - \frac{1}{2}\frac{1}{1-\frac{z}{2}}$$

$$= -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$

III $|z|>2$

$$f(z) = -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n - \sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}}$$

Tutorial 4

1. (a) $z_n = \dfrac{(100+75i)^n}{n!}$

Ratio test $\lim_{n\to\infty} \left|\dfrac{z_{n+1}}{z_n}\right| = \lim_{n\to\infty} \left(\dfrac{125}{n+1}\right) = 0 < 1$

conv

(b) $\Sigma\, a_n + b_n$

$a_n = \dfrac{i}{2^{3n}}$ $\qquad b_n = \dfrac{1}{2^{3n+1}}$

(i) $\left|\dfrac{1/2^{3n+1}}{i/2^{3n}}\right|$ & $\left|\dfrac{i/2^{3n}}{1/2^{3n-2}}\right|$

$(|b_n/a_n|)$ $\qquad |a_n/b_{n-1}|$

$\longrightarrow \frac{1}{2}$ $\qquad \longrightarrow \frac{1}{4}$.

(c) $\sum_{n=2}^{\infty} \dfrac{(-i)^n}{\ln n}$.

$\ln n < n$.

$\dfrac{1}{\ln n} > \dfrac{1}{n}$.

$\sum_{n=2}^{\infty} \left|\dfrac{(-i)^n}{\ln n}\right| > \left|\dfrac{1}{n}\right| > \sum_{n=2} \dfrac{1}{n}$.

$\therefore$ Divergent

(d) $\sum_{n=0}^{\infty} \dfrac{(n+ni)^{2n+1}}{(2n+1)!}$

Ratio test $\lim_{n\to\infty} \left|\dfrac{z_{n+1}}{z_n}\right| = \lim_{n\to\infty} \left|\dfrac{n+ni}{2n+1}\right|$

$\longrightarrow 0$

convergent

Q2 (a) $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n!} \left(z-\frac{\pi}{2}\right)^{2n}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\left|\frac{(-1)^{n+1}}{(2n+2)!}\left(z-\frac{\pi}{2}\right)^{2n+2}\right|}{\left|\frac{(-1)^n}{(2n)!}\left(z-\frac{\pi}{2}\right)^{2n}\right|}$$

$$\lim_{n\to\infty}\left|\frac{\left(z-\frac{\pi}{2}\right)^2}{(2n+2)(2n+1)}\right| < 1$$

$$= \left(z-\frac{\pi}{2}\right)^2 < \lim_{n\to\infty}(2n+2)(2n+1)$$

$$= \underbrace{\left(z-\frac{\pi}{2}\right)^2}_{\text{centre}} < \underset{R}{\underset{\Downarrow}{\infty}}$$

(b) $\sum_{n=0}^{\infty}\left[1+(-1)^n+\frac{1}{2^n}\right]z^n.$

Ratio test fails

Root test (Take supremum)

<u>odd</u> $\lim_{n\to\infty}\left|\frac{1}{2^n}\right|^{1/n} = \frac{1}{2}$

<u>even</u> $\lim_{n\to\infty}\left|2+\frac{1}{2^n}\right|^{1/n} = 1$

$R = 1.$

when $\sum_{n=0}^{\infty} \overbrace{a_n z^{n}}^{b_n}$

then $R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$

when power not n.

$\lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| < 1$

(c) $\sum_{n=0}^{\infty} \frac{n(n-1)}{3^n}(z-i)^{2n}$

$$\lim_{n\to\infty}\left|\frac{(n+1)\cancel{(n)}(z-i)^2}{3\cancel{(n)}(n-1)}\right| < 1$$

$$\left|(z-i)^2\right| < \lim_{n\to\infty}\left|\frac{(1-1/n)}{(1+1/n)}\right|^3$$

$$|(z-i)^2| < 3$$

$$|z-i| < \sqrt{3}$$

$\sum a_n z^n$

coefficient can be anything

The whole should be less than 1 for convergence

(d) $R = 1$

Q3 (a) $\sin z$

$f(z) = \sin z$ $\qquad f'(0) = 1$

$f'(z) = \cos z$ $\qquad f''(0) = 0$

$f''(z) = -\sin z$ $\qquad f'''(0) = -1$

$f'''(z) = -\cos z$

$$f^n(0) = \begin{cases} 0 & n = 2k \\ (-1)^k & n = 2k+1 \end{cases}$$

Taylor series

$$f(z) = f(0) + \frac{f'(0)}{1!}(z-0) + \frac{f''(0)}{2!}(z-0)^2 + \dots$$

$$+ \frac{f^n(0)}{n!}(z-0)^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

$$= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots$$

(b) $1 - \frac{z^2}{2!} + \frac{z^4}{4!}$

(c) $\frac{1}{1+z^2} = \frac{1}{1-(-z^2)}$ (G.P.).

$$= \sum_{n=0}^{\infty} (-z^2)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n z^{2n}.$$

$$= 1 - z^2 + z^4 - \dots \quad , |z| < 1$$

(d) $f(z) = \tan^{-1}(z)$

$$f'(z) = \frac{1}{1+z^2}$$

$$= \sum_{n=0}^{\infty} (-1)^n z^{2n} , \quad |z|<1$$

~~$f''(z) =$~~

$$\int f'(z)\, dz = \int \sum_{n=0}^{\infty} (-1)^n z^{2n}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{2n+1} , \quad |z|<1$$

Also find the radius of convergence for each place.

4. $\frac{1}{(c-z)} = \frac{1}{c-z+z_0-z_0} = \frac{1}{(c-z_0)-(z-z_0)}$

$$= \frac{1}{(c-z_0)\left[1 - \frac{z-z_0}{c-z_0}\right]}$$

$$= \frac{1}{c-z_0}\left[\frac{1}{1-\left(\frac{z-z_0}{c-z_0}\right)}\right] \quad = \cancel{\frac{1}{c-z_0}}$$

$$= \frac{1}{c - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{c - z_0}\right)^n, \quad \left|\frac{z - z_0}{c - z_0}\right| < 1$$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \; (GP)$$

$$= \frac{1}{c - z_0} \left[1 + \frac{z - z_0}{c - z_0} + \left(\frac{z - z_0}{c - z_0}\right)^2 + \cdots \right]$$

Q5 (a) $\dfrac{2z^2 + 9z + 5}{z^3 + z^2 - 8z - 12}$

$$= \left[\frac{1}{(z+2)^2} + \frac{2}{z-3}\right]$$

$$= \frac{1}{(z-1+3)^2} + \frac{2}{(z-1-2)}.$$

$$\frac{1}{(3 + (z-1))^2}$$

$$= \frac{1}{9\left(1 + \left(\frac{z-1}{3}\right)\right)^2}$$

$$= \frac{1}{9}\left(1 + \left(\frac{z-1}{3}\right)\right)^{-2}$$

$$= \frac{1}{9} \sum_{n=0}^{\infty} \left({}^{-2}C_n\right) \left(\frac{z-1}{3}\right)^n$$

$\left|\frac{z-1}{3}\right| < 1.$ $\quad = \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)(z-1)^n}{3^{n+2}}$

$$\frac{2}{z-1-2} = \frac{2}{-2 + (z-1)}$$

$$= \frac{-2}{2\left(1 - \left(\frac{z-1}{2}\right)\right)}$$

$$= \frac{1}{1 - \left(\frac{z-1}{2}\right)}$$

$$= -1 \sum_{n=0}^{\infty} \left(\frac{z-1}{2}\right)^n,$$

$$\left|\frac{z-1}{2}\right| < 1$$

$$(1+x)^{-2}$$

$$= \sum_{n=0}^{\infty} \left({}^{-2}C_n\right) x^n$$

$$= 1 - 2x \ldots$$

$$\therefore \frac{1}{(1+x)^m} = (1+x)^{-m} = \sum_{n=0}^{\infty} \left({}^{-m}C_n\right) x^n$$

$$= 1 + (-m)x + \frac{(-m)(-m-1)}{2!}x^2 + \frac{(-m)(-m-1)(-m-2)}{3!}x^3$$

$${}^{n}C_2 = \frac{n(n-1)}{2!}$$

$${}^{-n}C_2 = \frac{n(-n+1)}{2!}$$

$|z-1|<2$, $|z-1|<3$

$\Rightarrow |z-1|<2$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(z-\frac{\pi}{2}\right)^{2n}$, $R=\infty$

(c) $2\left(z-\frac{i}{2}\right) + \frac{2^3}{3!}\left(z-\frac{i}{2}\right)^3 + \frac{2^5}{5!}\left(z-\frac{i}{2}\right)^5 \cdots$

$R = \infty$.

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (n+1)}{2^{n+2}\, i^{n+2}} (z-i)^n$.

1 Nov. 2023 Happy Birthday to me

Regular point :- If a fn. f is analytic at $z = z_0$ then z_0 is called an ordinary point or a regular point.

Singular point :- A function $f(z)$ is said to be SINGULAR or said to have a singularity at a point $z = z_0$ if $f(z)$ fails to be analytic at $z = z_0$ (perhaps not even defined)

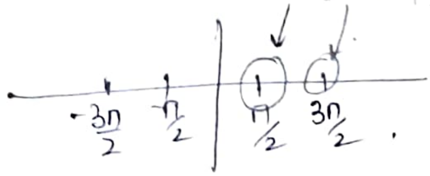
Ex ① $f(z) = \frac{1}{z}$ is singular at $z = 0$.

② $f(z) = |z|^2$ " " $z = 0$.

- Isolated singularity: A point $z = z_0$ is called an isolated singularity of $f(z)$ if $z = z_0$ has a nbd without further singularities.

E.g. ① $\tan z$, $\pm\frac{\pi}{2}$, $\pm\frac{3\pi}{2}$, $\pm\frac{5\pi}{2}$ ...

isolated singularity



② $f(z) = \tan\left(\frac{1}{z}\right)$ $0, \pm\frac{2}{\pi}, \pm\frac{2}{3\pi}, \pm\frac{2}{5\pi}$.

Non isolated singularity

* Let $z = z_0$ be an isolated singularity. Then $\exists$ a $r > 0$ such that f is analytic in the annulus $0 < |z - z_0| < r$. By Laurent's thm

Case I: The principle part is having infinitely many terms. In such a case $z = z_0$ is called ESSENTIAL SINGULARITY.

Case II: If all coefficients $b_1, b_2, b_3, \ldots$ of the principal part are zero, then $z = z_0$ is called REMOVABLE SINGULARITY.

Case III: The principle part is containing only finitely many terms i.e. NOT ALL $b_1, b_2, \ldots$ are zero.

Principle part will be of the form

$$\frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_m}{(z-z_0)^m} \qquad b_m \neq 0$$

In this case, the singularity $z = z_0$ of $f(z)$ is called a pole of order m.

Q $$f(z) = \frac{1}{z(z-2)^5} + \frac{3}{(z-2)^2}$$

for $z = 2$

$$\frac{1}{z} = \frac{1}{2 + (z-2)} = \frac{1}{2\left(1 + \frac{z-2}{2}\right)}$$

$$= \frac{1}{2}\left[1 - \frac{z-2}{2} + \left(\frac{z-2}{2}\right)^2 - \left(\frac{z-2}{2}\right)^3 \cdots\right]$$

$$\left|\frac{z-2}{2}\right| < 1$$

$$0 < |z-2| < 2$$

$$f(z) = \frac{1}{2}\left[\frac{1}{(z-2)^5} - \frac{1}{2(z-2)^4} + \frac{1}{4(z-2)^3} \cdots\right] + \frac{3}{(z-2)^2}$$

E.g. $f(z) = e^{1/z}$ has an essential singularity at $z=0$.

E.g. $f(z) = \frac{\sin z}{z}$ (Not defined at $z=0$ and find a laurent series around $z=0$)

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}, \quad z \in \mathbb{C}$$

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n+1)!}, \quad |z| > 0.$$

$z=0$, for $f(z) = \frac{\sin z}{z}$ is a removable singularity

$$f(z) = \begin{cases} \frac{\sin z}{z}, & |z| > 0 \\ 1 & z = 0. \end{cases}$$

E.g. $f(z) = \frac{1}{z^2}$ has a pole of order 2 at $z=0$

$$\lim_{z \to z_0} |f(z)| \to \infty$$

$\boxed{\text{Thm}}$ If $f(z)$ is analytic and has a pole at $z = z_0$ then $|f(z)| \to \infty$ as $z \to z_0$ in any manner

Ex. $f(z) = e^{1/z}$

Picard's Theorem :— If $f(z)$ is analytic and has an essential singularity at z_0. it takes on every value with at most one exceptional value, in an arbitrarily small nbd of z_0

Zeroes of an Analytic function

A zero of an analytic function $f(z)$ in a domain D is a point $z = z_0$ in D such that $f(z_0) = 0$.

E.g. $f(z) = 1 + z^2$ has zeroes at $z = \pm i$

A zero has an order n if not only f but also $f', f'', \dots, f^{(n-1)}$ are all zero at $z = z_0$ but $f^{(n)}z_0 \neq 0$

zeroes. $z = \pm 1, \pm i$

E.g. $f(z) = (1 - z^4)^2$

$f'(z) = 2(-4z^3)(1 - z^4) = -8z^3(1 - z^4)$

$f''(z) = -24z^2(1 - z^4) + 32z^6$] ⟶ No equal to zero at $\pm 1, \pm i$

So all zeroes are of order 2

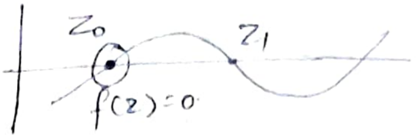
E.g. $f(z) = e^z$ (no zero).

E.g. $f(z) = \sin z$. $z = 0, \pm\pi, \pm 2\pi$.

$f(z) = 1 - \cos z$ $z = 0, \pm 2\pi, \pm 4\pi$

(second order zeroes)

Theorem : Zeros of an analytic function are isolated.



Q How zeroes of an analytic fn is realated to Taylor series?

$f(z)$ is analytic in D

$$\Rightarrow f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n, \quad z_0 \in D$$

$$a_n = \frac{f^{(n)}(z_0)}{n!}$$

If z_0 is a zero of f of order m.

$$\Rightarrow f^n(z_0) = 0 \text{ for } n = 0, 1, 2, \ldots, n-1$$

$$f^m(z_0) \neq 0$$

$$f(z) = \sum_{n=m}^{\infty} a_n (z-z_0)^n = a_m (z-z_0)^m + a_{m+1}(z-z_0)^{m+1} + \ldots$$

$$= (z-z_0)^m \left[a_m + a_{m+1}(z-z_0) + \ldots\right]$$

$$f(z) = (z-z_0)^m \psi(z)$$

$\psi(z)$ is analytic in D and $\psi(z_0) \neq 0$.

$\boxed{\text{Theorem}}$ A point $z_0 \in \mathbb{C}$ is a zero of order of m of an analytic fn f iff

$$f(z) = (z-z_0)^m \phi(z) \qquad |z-z_0| < r$$

for som $r > 0$, where ψ is analytic at z_0 and $\psi(z_0) \neq 0$!

Thm. : A point z_0 is a pole of f order m iff f can be expressed in the form

$$f(z) = \frac{\psi(z)}{(z-z_0)^m} \qquad 0<|z-z_0|<\gamma$$

for some $\gamma > 0$ where ψ is analytic at z_0 and $\psi(z_0) \neq 0$.

Thm : Let z_0 be an isolated singularity of $f(z)$ and let z_0 be a pole of order m of $f(z)$ iff $\frac{1}{f(z)}$ is analytic at z_0 and has a zero order m.

Proof : z_0 is a pole of f of order m

$\iff f(z) = \frac{\psi(z)}{(z-z_0)^m}$, ψ is analytic at z_0 and $\psi(z_0) \neq 0$.

$\iff \frac{1}{f(z)} = (z-z_0)^m \frac{1}{\psi(z)}$, $1/\psi(z)$ is " " z_0.

$\iff$ z_0 is a zero of order m for $1/f(z)$

Remark :- The same holds for $\frac{h(z)}{f(z)}$ provided $h(z)$ is analytic at z_0 and $h(z_0) \neq 0$.

Thm :- Let z_0 be a pole of order m. Then for all +ve integer k, we have

$$\lim_{z \to z_0} (z-z_0)^k f(z) = \begin{cases} l & k = m \quad \text{for some } l \neq 0 \\ 0 & k > m \\ \infty & k < m \end{cases}$$

E.g. $f(z) = \frac{1}{z(z-2)^5} + \frac{3}{(z-2)^2}$

$\lim_{z \to 2} (z-2)^5 \left(\frac{1}{z(z-2)^5} + \frac{3}{(z-2)^2} \right)$

$\lim_{z \to 0} (z-2)^5 \left(\frac{1}{z(z-2)^5} + \frac{3}{(z-2)^2} \right)$

Remark: To investigate $f(z)$ for large $|z|$

$z = \frac{1}{w}$ $\quad f(z) = f\left(\frac{1}{w}\right) = g(w)$

we define $f(z)$ to be analytic or singular at infinity if $g(w)$ is analytic or singular

$g(0) = \lim_{w \to 0} g(w)$ if the limit exists

* $f(z)$ has a nth order zero at singularity ∞ if $f\left(\frac{1}{w}\right) = g(w)$ has nth order zero at $w = 0$

similarly for poles & essential singularities.

E.g. $f(z) = \frac{1}{z^2}$ $\quad z \to \frac{1}{w}$

$g(w) = f\left(\frac{1}{w}\right) = w^2$.

zeros $(\pm\infty)$ is a second order.

E.g. $f(z) = z^3$

- singular ; $g(w) = \frac{1}{w^3}$.
- Pole of order 3 at $z = \infty$.

E.g. $f(z) = e^z$

- essential singularity at $z = \infty$.
- ∞ly many terms in principle part

Residue

Let $z = z_0$ be an isolated singularity of $f(z)$. There exists deleted neighbourhood $0 < |z - z_0| < r$ for some $r > 0$ s.t. f is analytic everywhere in this deleted nbd and hence.

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

$$0 < |z - z_0| < r$$

The coefficient of $\frac{1}{z - z_0}$, i.e., b_1 is called the residue of f at the isolated singularity $z = z_0$. $Res(f : z_0)$ or $\underset{z = z_0}{Res} f(z)$

$$Res(f; z_0) = b_1 = \frac{1}{2\pi i} \oint_C f(z)\, dz$$

+vely oriented simple closed contour.

Q $\oint_C z^{-4} \sin z\, dz$

+vely oriented $|z| = 1$

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \qquad z \in \mathbb{C}$$

$$\frac{\sin z}{z^4} = \frac{1}{z^4} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \qquad |z| > 0$$

$$= \frac{1}{z^4} \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} \dots \right)$$

$$b_1 = Res(f; 0) = -\frac{1}{6} = \frac{1}{2\pi i} \oint_C f(z) dz$$

$$\oint_C \frac{\sin z}{z^4}\, dz = \frac{-\pi i}{3}$$

Ex (1) $\text{Res}\left(\frac{1}{(z-z_0)^k}; z_0\right) = \begin{cases} 1 & k=1 \\ 0 & k \neq 1 \end{cases}$

(2) $\text{Res}\left(e^{1/z}; 0\right) = 1$

(3) $\text{Res}\left(\frac{1}{(z-1)(z-2)}; 1\right)$