Problem: Find the root of

$$f(x) \equiv x^6 - x - 1 = 0 \quad \text{in } [1, 2]$$

accurate to within $\epsilon = 0.001$

Solution: $a = 1$, $b = 2$; $f(a) = f(1) = -1$

$f(b) = f(2) = 61$. $\therefore f(a)f(b) < 0$.

Therefore, by IVT, the continuous function f has a root in $[1, 2]$.

n	a_n	b_n	c_n	$f(c_n)$
1	1	2	1.5	8.8906
2	1	1.5	1.25	1.5647
3	1	1.25	1.125	-0.0977
4	1.125	1.25	1.1875	0.6167
5	1.125	1.1875	1.15625	0.2333
6	1.125	1.15625	1.4063	0.0616
7	1.125	1.14063	1.13281	-0.0196
8	1.13281	1.14063	1.13672	0.0206
9	1.13281	1.13672	1.13477	0.0004
10	1.3281	1.13477	1.13379	-0.0096

2. Find the root of the equation $x^2 - x - 3 = 0$ using Bisection method correct upto 3 decimal places.

3. $f(x) = x^3 + 4x^2 - 10$ has a root in $[1, 2]$. Using the Bisection method, find an approximation to the root that is accur[ate] to atleast within 10^{-4}.

$$n \geq \frac{\log_{10}\left(\frac{2-1}{10^{-4}}\right)}{\log_{10} 2} = \frac{4}{\log_{10} 2} \simeq 13.2877$$

We need to perform 14 iterations.

Problem (1): Find the root of $f(x) = x^6 - x - 1 = 0$ in $[1, 2]$. (Secant method)

n	x_n	$f(x_n)$
0	2	61
1	1	-1
2	1.016129032	-0.9153677138
3	1.190577769	0.6574656967
4	1.117655831	-0.1684911678
5	1.132531555	-0.02243728619
6	1.134816808	0.0009535640641
7	1.134723646	$-5.066165712 \times 10^{-6}$
8	1.134724138	$-1.134763172 \times 10^{-9}$
9	1.134724138	$1.110223025 \times 10^{-15}$

(2) Find an approximate solution to the non-linear equation

$$\sin x + x^2 - 1 = 0$$

using secant method.

<u>solution</u>: Note that the true value is

$r \approx 0.636733$.

Take $x_0 = 0$, $x_1 = 1$. Then the iterations from the secant method are given by:

n	x_n	ϵ
2	0.543044	0.093689
3	0.626623	0.010110
4	0.637072	0.000339
5	0.636732	0.000001

Newton Method

(1)

Problem: Using Newton's method, solve $f(x) \equiv x^6 - x - 1 = 0$ to find out the solution lying in $[1, 2]$.

Solution: By Newton's method,

$$x_{n+1} = x_n - \frac{x_n^6 - x_n - 1}{6x_n^5 - 1}, \quad n = 0, 1, 2, \ldots$$

Let $x_0 = 1.5$.

n	x_n	$f(x_n)$	$x_n - x_{n-1}$
0	1.5	8.890625	
1	1.300490884	2.537264143	−0.1995091164
2	1.181480416	0.5384585843	−0.1190104672
3	1.1394559	0.04923525701	−0.04202482613
4	1.134777625	0.0005503238584	−0.0004677965038
5	1.134724145	$7.113585232 \times 10^{-8}$	−0.00005347992089
6	1.134724138	$1.110223025 \times 10^{-15}$	$-6.914698369 \times 10^{-9}$
7	1.134724138	$1.110223025 \times 10^{-15}$	0

Thus, the root is $x_7 = 1.34724138$, which is accurate to 10 significant digits.

For accuracy of $\epsilon = 10^{-10}$, the bisection method would have taken 34 iterations.

(2) Find an approximate solution to the nonlinear equation:

$\sin x + x^2 - 1 = 0$ using Newton's Raphson's method.

Solution: Let $x_0 = 1$. Then the iterations from the Newton Raphson method gives:

n	x_n	Error
1	0.668752	0.032019
2	0.637068	0.000335
3	0.636733	0.000001

Recall that the exact solution is 0.636733.

Secant method: $x_3 \simeq 0.626623$,

Error $\simeq 0.010110$