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A function $f : Dom(f) \subset \mathbb{R} \to \mathbb{R}$ is said to be a *periodic function* if there is some positive number p, such that for all $x \in Dom(f)$ we have

$$x + p \in Dom(f) \quad \text{and} \quad f(x+p) = f(x).$$

A Fourier series is an expansion of a periodic function into a sum of trigonometric functions.

Two distinct functions $f, g : [a, b] \to \mathbb{R}$ are said to be *orthogonal* on this interval if

$$\int_a^b f(x)\, g(x)\, \mathrm{d}x = 0.$$

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, \mathrm{d}x = 0 \qquad (m \neq n)$$

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, \mathrm{d}x = 0 \qquad (m \neq n)$$

$$\int_{-\pi}^{\pi} \sin mx \cos nx \, \mathrm{d}x = 0 \qquad (m \neq n \text{ or } m = n).$$

Fourier Series

Let $f : Dom(f) \subset \mathbb{R} \to \mathbb{R}$ be a periodic function with period 2π. The *Fourier series* representation of f is given by

$$\mathcal{S}_f(x) := \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \tag{2}$$

The coefficients $a_0, a_1, a_2, a_3, \ldots, b_1, b_2, b_3, \ldots$ are referred to as the *Fourier coefficients* of f and are given by

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, \mathrm{d}x, \qquad n = 0, 1, 2, 3, \ldots \tag{3a}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, \mathrm{d}x, \qquad n = 1, 2, 3, \ldots \tag{3b}$$

Since n is always an integer , we can use the integer property of trig functions

A Fourier series for $f(x)$ does NOT always converge to $f(x)$, the sum of the series at some specific point $x = x_0$ may differ from the value $f(x_0)$ of the function at $x = x_0$.

Peice wise Smooth function

A function $f : [a, b] \to \mathbb{R}$ is said to be piecewise smooth (or sectionally smooth) if this interval can be divided into a finite number of subintervals such that

1. f has a continuous derivative f' in the interior of each of these subintervals,
2. and both $f(x)$ and $f'(x)$ approach finite limits as x approaches either endpoint of each of these subintervals from its interior.

In other words, we may say that f is piecewise smooth on $[a, b]$ if both f and f' are piecewise continuous on $[a, b]$.

Convergence.

→ periodic func^n with $T = 2\pi$

→ piecewise smooth func^n in $[-\pi, \pi]$

→ $S_f(x) = \dfrac{f(x^+) + f(x^-)}{2} = f(x)$ $\quad S_f(x)$ converges to $f(x)$

Change of Scale

$f(x)$ = func^n with period $2L$

$g(y) = f\left(\frac{L}{\pi} x\right)$ $\quad g(y)$ = periodic func^n with period 2π

$$S_g(y) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos ny + b_n \sin ny$$

1. **Even function of period** 2π: If $f(x)$ is even and $L = \pi$, the Fourier series of $f(x)$ is the Fourier cosine series, given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \tag{12}$$

with coefficients

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, \mathrm{d}x, \qquad n = 0, 1, 2, 3, \ldots \tag{13}$$

$$S_g(y) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos ny + b_n \sin ny$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(y) \cos ny \, dy = \frac{1}{L}\int_{-L}^{L} f(x) \cos\left(\frac{n\pi}{L} x\right) dx$$ N = 0,1,2,3....

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(y) \sin ny \, dy = \frac{1}{L}\int_{-L}^{L} f(x) \sin\left(\frac{n\pi}{L} x\right) dx$$ N = 1,2,3....

$$S_f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right)$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx \, dx, \quad n = 0, 1, 2, 3, \ldots \tag{13}$$

1. **Odd function of period** 2π: If $f(x)$ is odd and $L = \pi$, the Fourier series of $f(x)$ is the Fourier sine series, given by

$$\sum_{n=1}^{\infty} b_n \sin nx \tag{14}$$

with coefficients

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx, \quad n = 1, 2, 3, \ldots \tag{15}$$

Convergence (general)

→ periodic funcⁿ $f(x)$ with $T = 2L$

→ piecewise smooth in $[-L, L]$ interval

→ $S_f(x) = \frac{f(x^+) + f(x^-)}{2} = f(x)$ Converges to $f(x)$

Partial Differential Equations.

Definition 8

The **order** of a partial differential equation is defined as the order of the highest partial derivative occuring in the partial differential equation.

Definition 9

A PDE is said to be **linear** if the dependent variable and its partial derivatives occur only in the first degree and are not multiplied. A partial differential equation which is not linear is called a **non-linear**.

Definition 10

A PDE is said to be **semilinear** if the highest order terms are linear and the coefficients of the highest order derivatives are functions of independent variables only.

Definition 11

A PDE is said to be **quasi-linear** if the highest derivative power is linear but coefficients of highest order derivatives involve the dependent variable u or its lower order derivative.

Example 12

1. Linear PDE: $a(x, y)u_x + b(x, y)u_y + c(x, y)u = f(x, y)$
2. Semi-linear PDE: $a(x, y)u_x + b(x, y)u_y = f(x, y, u)$
3. Quasi-linear PDE: $a(x, y, u)u_x + b(x, y, u)u_y = f(x, y, u)$

Definition 13

A **linear** PDE is said to be *homogeneous* if each of its terms contains either the unknown function u or one of its partial derivatives. Otherwise, the PDE is called *nonhomogeneous* or *inhomogeneous*.

Special Equations

(i)	$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$	**One-dimensional wave equation**	Hyperbollic
(ii)	$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$	**One-dimensional heat equation**	Parabollic
(iii)	$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$	**Two-dimensional Laplace equation**	(potential equation) Elliptic
(iv)	$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$	**Two-dimensional Poisson equation**	
(v)	$\frac{\partial^2 u}{\partial t^2} = \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$	**Two-dimensional wave equation**	
(vi)	$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$	**Three-dimensional Laplace equation**	

Methods to convert to PDE

1) eliminate arbitary constants in equation.
2) eliminate arbitary func's like $f(x^2-y^2)$ etc.

General 2nd order PDE

$$A u_{xx} + B u_{xy} + C u_{yy} + \underbrace{D u_x + E u_y + F u = G}$$

$A, B, C, D, E, F, G = f(x,y)$ only

$$A u_{xx} + B u_{xy} + C u_{yy} + I(x, y, u_x, u_y, u) = 0$$

$$\Delta(x,y) = B(x,y)^2 - 4A(x,y)\, C(x,y)$$

$\downarrow$

Discriminant

At a point (x_0, y_0), the second order PDE is called

- → Elliptic $\Delta(x_0, y_0) < 0$
- → Parabollic $\Delta(x_0, y_0) = 0$
- → Hyperbolic $\Delta(x_0, y_0) > 0$

Lagrange's Equation.

→ quasi - linear PDE

→ $Pp + Qq = R$ $\quad P, Q, R = f(x,y,z)$

Steps:

→ convert to Lagrange form $Pp + Qq = R$

→ Lagrange Auxillary Eq^n $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$

→ form two func^{n}s $u(x,y,z) = c_1$
$v(x,y,z) = c_2$

→ ans: $\phi(u,v) = 0$ $\quad \phi \Rightarrow$ arbitary func^n.

Laplace Equation (Method of separation of variables)

$$\Delta^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \Delta^2 u = 0$$

We assume $u(x,y) = X(x)\,Y(y)$ and substitute

$$\underbrace{\frac{X''}{X}}_{\text{func}^n \text{ of } x} = -\underbrace{\frac{Y''}{Y}}_{\text{func}^n \text{ of } y} = k \rightarrow \text{const.}$$

Case 1 : $k=0$

$$\frac{X''}{X} = 0 \qquad X = Ax+B$$

$$\frac{Y''}{Y} = 0 \qquad Y = Cy+D$$

$$u(x,y) = (Ax+B)(Cy+D)$$

Case 2 : $k>0$, $k=\lambda^2$

$$X'' - \lambda^2 X = 0 \qquad X = Ae^{\lambda x} + Be^{-\lambda x}$$

$$Y'' + \lambda^2 Y = 0 \qquad Y = C\cos\lambda y + D\sin\lambda y$$

$$u(x,y) = \left[Ae^{\lambda x} + Be^{-\lambda x}\right]\left[C\cos\lambda y + D\sin\lambda y\right]$$

Case 3 : $k<0$, $k=-\lambda^2$

$$X'' + \lambda X = 0 \qquad X = A\sin\lambda y + B\cos\lambda y$$

$$Y'' - \lambda Y = 0 \qquad Y = Ce^{+\lambda y} + De^{-\lambda y}$$

$$u(x,y) = (A\sin\lambda y + B\cos\lambda y)(Ce^{\lambda y} + De^{-\lambda y})$$

* Now from the boundary cond^{n}s, we can check which of the cases give non trivial sol^{n}s.

for eg: $\sin\lambda b = 0 \qquad \lambda b = n\pi \qquad b = \frac{n\pi}{\lambda}$

$\therefore\ u_n(x,y) = f(n)$

$\therefore\ u(x,y) = \sum_{n=1}^{\infty} f(n)$

apply fourier series with boundary cond^n for coeffs.

Heat Solutions

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \qquad \begin{array}{ll} 0<x<L & \text{distance of rod} \\ t>0 & \text{time} \end{array}$$

$u(x,t) = X(x)\,T(t)$ method of separation

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T} = k \qquad \begin{array}{l} X'' - kX = 0 \\ T' - \alpha^2 k T = 0 \end{array}$$

Case 1 $k=0$ $\quad u(x,t) = Ax+B$

Example 3 (Bar with insulated ends)

Find the solution of the heat equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \qquad t>0, \quad 0<x<L$$

satisfying the following boundary conditions and initial condition

$$u_x(0,t) = 0, \quad u_x(L,t) = 0, \quad \text{for} \quad t>0$$

$$u(x,0) = f(x) \quad \text{for} \quad 0 \le x \le L.$$

Case 2 $K > 0$, $K = \lambda^2$ $\quad u(x,t) = \left(Ae^{\lambda x} + Be^{-\lambda x}\right) e^{\lambda^2 \alpha^2 t}$

Case 3 $K < 0$, $K = -\lambda^2$ $\quad u(x,t) = \left(A\cos\lambda x + B\sin\lambda x\right) e^{-\lambda^2\alpha^2 t}$

Wave Equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad t > 0 \qquad 0 < x < L \qquad c^2 = \frac{T}{\rho}$$

$$u_{tt} = c^2 u_{xx} \qquad u(x,t) = X(x)\,T(t)$$

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = K \qquad \begin{array}{l} X'' - KX = 0 \\ T'' - c^2 K T = 0 \end{array}$$

Case 1 $K = 0$ $\quad u(x,t) = (Ax + B)(Ct + D)$

Case 2 $K > 0$, $K = \lambda^2$ $\quad u(x,t) = \left(Ae^{\lambda x} + Be^{-\lambda x}\right)\left(Ce^{\lambda c t} + De^{-\lambda c t}\right)$

Case 3 $K < 0$, $K = -\lambda^2$ $\quad u(x,t) = \left(A\cos\lambda x + B\sin\lambda x\right)\left(C\cos\lambda ct + D\sin\lambda ct\right)$

D'Alembert's Solution of Wave Equation.

$$u_{tt} = c^2 u_{xx} \qquad \eta = x + ct \qquad \xi = x - ct$$

$$\frac{\partial \xi}{\partial x} = 1 \qquad \frac{\partial \eta}{\partial x} = 1 \qquad \frac{\partial \xi}{\partial t} = -c \qquad \frac{\partial \eta}{\partial t} = c$$

$$u_{tt} = \left[u_{\xi\xi} - 2u_{\eta\xi} + u_{\eta\eta}\right]$$

$$u_{xx} = \left[u_{\xi\xi} + 2u_{\eta\xi} + u_{\eta\eta}\right]$$

$$\therefore u_{tt} = c^2 u_{xx} \longrightarrow u_{\eta\xi} = 0$$

$$\frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \eta}\right) = 0 \longrightarrow \frac{\partial u}{\partial \eta} = \bar{\psi}(\eta)$$

integrating wrt η

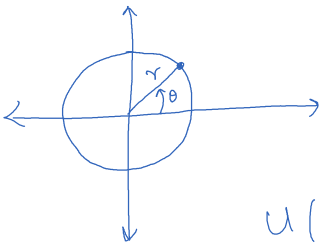
$$u(\xi, \eta) = \underbrace{\int \bar{\psi}(\eta)\, d\eta}_{\psi(\eta)} + \phi(\xi)$$

$$\therefore\ u(\xi,\eta) = \psi(\eta) + \phi(\xi)$$

$$\therefore\ u(x,t) = \psi(x+ct) + \phi(x-ct)$$

Circular Membrane.

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial^2 u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right)$$



considering symmetric condⁿs will make

$$u_{\theta\theta} = 0$$

$$\therefore\ u_{tt} = c^2\left[u_{rr} + \frac{1}{r}u_r\right]$$

$$u(r,t) = W(r)\ T(t)$$

Boundary condⁿs maybe: $u(R,t) = 0 \qquad u(r,0) = f(r)$

$$u_t(r,0) = g(r)$$

$$\frac{1}{c^2}\frac{T''}{T} = \frac{1}{W}\left[W'' + \frac{1}{r}W'\right] = K$$

$$T'' - Kc^2T = 0 \qquad W'' + \frac{1}{r}W' - KW = 0$$

Case 1 $K = 0$
Case 2 $K = \beta^2$ } will give solⁿs not physically possible

Case 3 $K = -\beta^2$ $\qquad T'' + \lambda^2 T = 0 \qquad \lambda = \beta c$

$$rW'' + W' + \beta^2 r W = 0$$

Let $s = \beta r$ convert to Bezzels

$$W' = \frac{dW}{dr} = \frac{dW}{ds}\frac{ds}{dr} = \beta\frac{dW}{ds}$$

$$W'' = \frac{d^2W}{dr^2} = \frac{d}{ds}\left(\frac{dW}{dr}\right)\frac{ds}{dr} = \frac{d}{ds}\left(\beta\frac{dW}{ds}\right)\beta = \beta^2\frac{d^2W}{ds^2}$$

$$s^2\frac{d^2W}{ds^2} + s\frac{dW}{ds} + s^2W = 0$$

$$x^2y'' + xy + (x^2 - \nu^2)y = 0 \quad \text{Bezzel's} \quad \nu = 0$$

$$u(R,t) = 0 = W(R)\ T(t) \qquad \therefore\ W(R) = 0$$

$W(R)=0$ gives $J_0(s)$ and $Y_0(s)$ as sol^{n}s

But $Y_0(s)\to\infty$ as $s\to 0$ hence rejected

$\therefore\quad W(r) = J_0(\beta r)\qquad$ ie $\qquad W(r)=J_0(s)$

$\therefore\quad W(R) = J_0(\beta R) = 0\qquad\qquad J_0(\beta R)=0$

Let $J_0(s)=0$ has roots $\quad s=\alpha_1,\alpha_2,\alpha_3 \,---$

$\therefore\quad J_0(\beta R)=0\qquad \beta R = \alpha_1,\alpha_2,\alpha_3\,\cdots$

$\beta_n = \dfrac{\alpha_n}{R}\qquad n=1,2,\cdots$

$\therefore\quad W_n(r) = J_0(\beta_n r) = J_0\left(\dfrac{\alpha_n}{R} r\right)$

$\lambda = \beta c$ for $T''+\lambda^2 T = 0$

$\therefore\ \lambda_n = \beta_n c = \dfrac{\alpha_n}{R} c$

$T_n(t) = A\cos\lambda_n t + B\sin\lambda_n t$

$$\therefore\ \underbrace{u_n(r,t)}_{\text{eigen func}^n\text{s}} = \left(A\cos\lambda_n t + B\sin\lambda_n t\right) J_0(\beta_n r)$$

$$\therefore\ u(r,t) = \sum_{n=1}^{\infty} u_n(r,t) = \sum_{n=1}^{\infty}\left(A\cos\lambda_n t + B\sin\lambda_n t\right) J_0(\beta_n r)$$

Boundary cond^n 1 : $\quad u(r,0) = f(r)$

$$f(r) = \sum_{n=1}^{\infty} C_n J_0\left(\frac{\alpha_n}{R} r\right)$$

$$\int_0^1 x\, J_p(nx)\, J_p(mx)\, dx = \begin{cases} 0 \ ; & m\neq n \\ \frac{1}{2} J_{p+1}^2(a)\ ; & a=b \end{cases}$$

n, m = zero's of bezzels equation.

$$\int_0^R r\, J_0\left(\frac{\alpha_m}{R} r\right) J_0\left(\frac{\alpha_n}{R} r\right) dr = \begin{cases} 0\ ; & \alpha_m\neq\alpha_n \\ \frac{1}{2} R^2 J_1^2(\alpha_m)\ ; & \alpha_m=\alpha_n \end{cases}$$

$$f(r) = \sum_{n=1}^{\infty} C_n J_0\left(\frac{\alpha_n}{R} r\right)$$

multiply by $r J_0\left(\frac{\alpha_n}{R} r\right)$ and integrate.

$$C_m \frac{1}{2} R^2 J_1\left(\frac{\alpha_n}{R} r\right) = \int_0^R r\, J_0\left(\frac{\alpha_n}{R} r\right) f(r)\, dr$$

$$C_n = \frac{2}{R^2 J_1(\alpha_m \ldots)} \int^R r\, J_0\left(\frac{\alpha_n}{R} r\right) f(r)\, dr$$

$$C_n = \frac{2}{R^2 J_1\left(\frac{\alpha_m}{R} r\right)} \int_0 r J_0\left(\frac{\alpha_n}{R} r\right) f(r)\, dr$$

Boundary Condition 2 : $u_t(r, 0) = g(r)$

$$u_t = \sum_{n=1}^{\infty} \lambda_n \left(- C_n \sin \lambda_n t + d_n \cos \lambda_n t\right) J_0\left(\frac{\alpha_m}{R} r\right)$$

$$g(r) = \sum_{n=1}^{\infty} \lambda_n d_n \cos \lambda_n t \; J_0\left(\frac{\alpha_m}{R} r\right)$$

$$d_n = \frac{2}{\alpha_m c R J_1^2(\alpha_m)} \int_0^R r\, g(r)\, J_0\left(\frac{\alpha_m}{R} r\right) dr$$

3) Maclaurian series.

c) $f(z) = \frac{1}{1+z^2} = \frac{1}{1-(-z^2)} = \sum_{n=0}^{\infty} (-z^2)^n \qquad |z|<1$

$= \sum_{n=0}^{\infty} (-1)^n z^{2n}$

d) $f(z) = \arctan z$

$f'(z) = \frac{1}{1+z^2} = \sum (-1)^n z^{2n}$

$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{2n+1} \qquad |z|<1$

Note : $\frac{1}{(1+z)^m} = \sum_{n=0}^{\infty} \binom{-m}{n} z^n$

$-{}^{m}C_n = 1 - mz + \frac{m(m+1)}{2!} z^2 - \frac{m(m+1)(m+2)}{3!} z^3 + \dots$

Conformal Mapping

→ preserves angles

$w = f(z)$ in domain D

if $f(z)$ is analytic in D, the it preserves the angles in D except at critical points ie $f'(z) = 0$

$z = z(t) \quad t \in [a, b] \quad$ curve C

$w = f(z) = f(z(t)) \qquad f =$ analytic on $\mathbb{C}$

then Γ is image of C under transformation $w = f(z(t))$

suppose $f(z)$ passes through a point z_0

$\therefore \; z_0 = z(t_0)$

$w(t_0) = f(z(t_0))$

$\therefore$

$w'(t_0) = f'(z(t_0))\, z'(t_0)$

taking arg on both sides

$\arg w'(z_0) = \underbrace{\arg f'(z(t_0))} + \underbrace{\arg(z'(t_0))}$

taking arg

$$\underbrace{\arg w'(z_0)}_{\phi_0} = \underbrace{\arg f'(z(t_0))}_{\psi_0} + \underbrace{\arg(z'(t_0))}_{\theta_0}$$

angle of rotation.

Let two curves C_1 C_2 pass through z_0

$$C_1 \to \theta_1 \quad C_2 \to \theta_2$$

Let ϕ_1 ϕ_2 be angle of inclination of Γ_1 Γ_2 ie images of C_1 C_2

$\therefore \quad \phi_1 = \psi_0 + \theta_1 \qquad \phi_2 = \psi_0 + \theta_2$

$\phi_1 - \theta_1 = \phi_2 - \theta_2$ Angles preserved.

Hence $f(z)$ conformal at z_0