MA 203 Tut-3

1) $u_t - \alpha^2 u_{xx} = 7e^{-2x}$, $t>0,\ 0<x<L$

BC $\rightarrow u(0,t)=0;\ u(L,t)=0$ for all $t \geq 0$

IC $\rightarrow u(x,0) = f(x)$ $0<x<L$

$\Rightarrow u(x,t) = v(x) + w(x,t)$

$w_t = (\alpha^2)(w_{xx})$

$w(0,t)=0,\ w(\cancel{L},t)=0,\ w(x,0)=f(x)$

$u_t = \alpha^2 u_{xx} + 7e^{-2x}$

$u_t = w_t,\quad u_{xx} = v_{xx} + w_{xx}$

$\Rightarrow w_t = \alpha^2(v_{xx} + w_{xx}) + 7e^{-2x}$

$\Rightarrow w_t - \alpha^2 w_{xx} = \alpha^2 \cancel{w}_{xx} + 7e^{-2x}$

$\Rightarrow \alpha^2 v_{xx} = -7e^{-2x}$

$\Rightarrow \int \alpha^2 v_x = \int \frac{7}{2} e^{-2x} + c_1$

$\Rightarrow \int v_x \Rightarrow v = \frac{-7}{4\alpha^2} e^{-2x} + c_1 x + c_2$

$v(0)=0,\ v(L)=0$

$\Rightarrow c_2 = \frac{7}{4\alpha^2},\quad c_1 = \frac{1}{L}\left[\frac{7}{4\alpha^2}(e^{-2L}-1)\right)$

2) $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ $t>0,\ 0<x<L$

$u_{tt} = c^2 u_{xx}$, $t>0,\ 0<x<L$

$u(0,t)=0,\ u(L,t)=0,\ t\geq 0$

$u(x,0) = f(x)$, $u_t(x,0) = g(x)$

$\Rightarrow$ Let $u(x,t) = X(x)\,T(t)$

$u_{tt} = X(x)\,T''(t)$

$u_{xx} = X''(x)\,T(t)$

$X(x)\,T''(t) = c^2 X''(x)\,T(t)$

$$\frac{T''(t)}{c^2\,T(t)} = \frac{X''(x)}{X(x)} = k$$

$X''(x) - kX(x) = 0$ & $T''(t) - kc^2 T(t) = 0$

Case I : when $k=0$

$X''(x) = 0 \Rightarrow X(x) = c_1 x + c_2$

$T''(t) = 0 \Rightarrow T(t) = c_3 t + c_4$

$u(x,t) = (c_1 x + c_2)(c_3 t + c_4)$

$u(0,t) = 0 \Rightarrow c_2 = 0$

$u(L,t) = 0 \Rightarrow c_1 L(c_3 t + c_4) = 0 \Rightarrow c_1 = 0$

$\Rightarrow u = 0$ $\hookrightarrow$ Trivial

Case II : When $k > 0$, let $k = \lambda^2$, $\lambda > 0$

$X''(x) - \lambda^2 X(x) = 0$ & $T''(t) - \lambda^2 c^2 T(t) = 0$

$X(x) = c_1 e^{\lambda x} + c_2 e^{-\lambda x}$, $T(t) = c_3 e^{\lambda c t} + c_4 e^{-\lambda c t}$

$u(x,t) = (c_1 e^{\lambda x} + c_2 e^{-\lambda x})(c_3 e^{\lambda c t} + c_4 e^{-\lambda c t})$

$u(0,t) = 0 \Rightarrow c_1 + c_2 = 0 \Rightarrow c_1 = -c_2$

$u(L,t) = c_1(e^{\lambda L} - e^{-\lambda L})(c_3 e^{-\lambda ct} + c_4 e^{\lambda ct}) = 0$

$\Rightarrow c_1 = 0 \Rightarrow u = 0 \rightarrow$ Trivial

Case III : when $k < 0 \Rightarrow k = -\lambda^2, \lambda > 0$

$X''(x) + \lambda^2 X(x) = 0$ & $T''(t) + c^2\lambda^2 T(t) = 0$

$X(x) = c_1 \cos \lambda x + c_2 \sin \lambda x$

$T(t) = c_3 \cos(\lambda ct) + c_4 \sin(\lambda ct)$

$u(x,t) = (c_1 \cos \lambda x + c_2 \sin \lambda x)(c_3 \cos(\lambda ct) + c_4 \sin(\lambda ct))$

$u(0,t) = 0 \Rightarrow c_1 = 0$

$u(L,t) = 0 \Rightarrow c_2 \sin \lambda L = 0$

$c_2 \neq 0, \sin \lambda L = 0 \Rightarrow \lambda L = n\pi, \quad n \in \mathbb{Z} - \{0\}$

$\Rightarrow \lambda = \frac{n\pi}{L}, \quad n = \pm 1, \pm 2 \ldots$

$u_n(x,t) = X_n(x) \cdot T_n(t), \quad n = \pm 1, \pm 2 \ldots$

$X_n(x) = \sin\left(\frac{n\pi}{L} x\right)$

$T_n(t) = A_n \cos\left(\frac{n\pi c}{L} t\right) + B_n \sin\left(\frac{n\pi c}{L} t\right), \quad \forall n = \pm 1, \pm 2 \ldots$

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[A_n \cos\left(\frac{n\pi c}{L} t\right) + B_n \sin\left(\frac{n\pi c}{L} t\right)\right]$$

$u(x,0) = f(x), \quad u_t(x,0) = g(x)$

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$$

Multiply bs by $\sin\left(\frac{m\pi x}{L}\right)$

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$$f(x) \cdot \sin\left(\frac{m\pi x}{L}\right) = \sum_{A=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right)$$

Integrate bs 0 to L wrt x

$$\int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx = \sum_{n=1}^{\infty} A_n \int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx$$

$$= \sum_{n=1}^{\infty} A_n \begin{cases} \frac{L}{2} & n = m \\ 0 & n \neq m \end{cases}$$

$$A_n / A_m = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx$$

$$B_n = \frac{2}{n\pi c} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

3) $u = v + w$

$w(0,t) = 0, \quad w(L,t) = 0$

$v(0) = 1, \quad v(L) = 0$

$w(x,0) = f(x), \quad w_t(x,0) = g(x)$

$c^2 v_{xx} = 0$

$v = c_1 x + c_2$

$$B_n = \frac{2}{n\pi c} \int_0^L g(x) \cdot \sin\left(\frac{n\pi x}{L}\right) dx$$

4) In class

5) $u_{tt} = c^2 u_{xx}$, $u(x,0) = f(x)$

$u_t(x,0) = 0$

from Q.NO-4

$u(x,t) = \phi(x-ct) + \psi(x+ct)$

$t=0 \Rightarrow \quad f(x) = \phi(x) + \psi(x)$

$u_t(x,t) = -c\,\phi'(x-ct) + c\,\psi'(x+ct)$

$0 = -c\,\phi'(x) + c\,\psi'(x)$

$\Rightarrow \psi'(x) = \phi'(x) \Rightarrow \psi(x) = \phi(x) + \alpha$

$\Rightarrow 2\phi(x) + \alpha = f(x) \Rightarrow \phi(x) = \frac{1}{2}[f(x) - \alpha]$

$\Rightarrow \psi(x) = \frac{1}{2}[f(x) + \alpha]$

$\phi(x-ct) = \frac{1}{2}[f(x-ct) - \alpha]$

$\psi(x+ct) = \frac{1}{2}[f(x+ct) + \alpha]$

Adding, $u(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)]$

6) $g(x) = 0 \Rightarrow B_n = 0$

$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi c t}{L}\right)$

$2\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right) = \sin\left(\frac{n\pi}{L}(x+ct)\right) + \sin\left(\frac{n\pi}{L}(x-ct\right.$

$u(x,t) = \sum_{n=1}^{\infty} \frac{A_n}{2}\left[\sin\left(\frac{n\pi}{L}(x+ct)\right) + \sin\left(\frac{n\pi}{L}(x-ct)\right)\right.$

Let $h(x) = \sum_{n=1}^{\infty} \frac{A_n}{2}\sin\left(\frac{n\pi x}{L}\right)$