# Natural Cubic Splines: Derivation of Algorithm

Define $z_i = S''(t_i)$, $i = 1, 2, \ldots, n-1$, $z_0 = z_n = 0$.

<u>Note</u>: These $z_i$'s are our unknowns.

Let $h_i = t_{i+1} - t_i$.

Lagrange form for $S_i''$:

$$S_i''(x) = \frac{z_{i+1}}{h_i}(x - t_i) - \frac{z_i}{h_i}(x - t_{i+1}).$$

Then,

$$S_i'(x) = \frac{z_{i+1}}{2h_i}(x - t_i)^2 - \frac{z_i}{2h_i}(x - t_{i+1})^2 + C_i - D_i$$

$$S_i(x) = \frac{z_{i+1}}{6h_i}(x - t_i)^3 - \frac{z_i}{6h_i}(x - t_{i+1})^3 + C_i(x - t_i) - D_i(x - t_{i+1}).$$

Interpolating properties:

(1) $s_i(t_i) = y_i$ gives

$$y_i = -\frac{z_i}{6h_i}(-h_i)^3 - D_i(-h_i)$$

$$= \frac{1}{6} z_i h_i^2 + D_i h_i$$

$$\Rightarrow D_i = \frac{y_i}{h_i} - \frac{h_i}{6} z_i$$

(2). $s_i(t_{i+1}) = y_{i+1}$ gives

$$y_{i+1} = \frac{z_{i+1}}{6h_i} h_i^3 + C_i h_i$$

$$\Rightarrow C_i = \frac{y_{i+1}}{h_i} - \frac{h_i}{6} z_{i+1}.$$

We see that, once $z_i$'s are known, then $(C_i, D_i)$'s are known, and so $S_i, S_i'$ are known.

$$S_i(x) = \frac{z_{i+1}}{6h_i}(x-t_i)^3 - \frac{z_i}{6h_i}(x-t_{i+1})^3$$

$$+\left(\frac{y_{i+1}}{h_i} - \frac{h_i}{6} z_{i+1}\right)(x-t_i)$$

$$-\left(\frac{y_i}{h_i} - \frac{h_i z_i}{6}\right)(x-t_{i+1}).$$

$$S_i'(x) = \frac{z_{i+1}}{2h_i}(x-t_i)^2 - \frac{z_i}{2h_i}(x-t_{i+1})^2$$

$$+ \frac{y_{i+1}-y_i}{h_i} - \frac{z_{i+1}-z_i}{6} h_i.$$

Continuity of $S'(x)$ requires:

$S'_{i-1}(t_i) = S_i'(t_i)$, $i = 1, 2, \ldots, n-1$.

$$S_i'(t_i) = -\frac{z_i}{2h_i}(-h_i)^2 + \underbrace{\frac{y_{i+1}-y_i}{h_i}}_{b_i} - \frac{z_{i+1}-z_i}{6} h_i$$

$$= -\frac{1}{6} h_i z_{i+1} - \frac{1}{3} h_i z_i$$

$$+ b_i$$

$$S'_{i-1}(t_i) = \frac{1}{6} z_{i-1} h_{i-1} + \frac{1}{3} z_i h_{i-1} + b_{i-1}$$

Set them equal to each other, we get

$$\left.\begin{aligned} &\text{(i)}\ \ h_{i-1} z_{i-1} + 2(h_{i-1}+h_i) z_i + h_i z_{i+1} \\ &\qquad = 6(b_i - b_{i-1}) \quad , i = 1, 2, \dots, n-1 \\ &\text{(ii)}\ \ z_0 = z_n = 0 . \end{aligned}\right\}$$

In matrix-vector form: $H\vec{z} = \vec{b}$

where

$$H = \begin{bmatrix} 2(h_0+h_1) & h_1 & & O \\ h_1 & 2(h_1+h_2) & h_2 & \\ & h_2 & 2(h_2+h_3) & \cdots \\ O & \ddots & \ddots & \\ & h_{n-3} & 2(h_{n-3}+h_{n-2}) & h_{n-2} \\ & & h_{n-2} & 2(h_{n-2}+h_{n-1}) \end{bmatrix}$$

$$= \begin{bmatrix} 2(h_0+h_1) & h_1 & & & \\ h_1 & 2(h_1+h_2) & h_2 & & \\ & h_2 & 2(h_2+h_3) & h_3 & \\ & & \ddots & \ddots & \\ & & h_{n-3} & 2(h_{n-3}+h_{n-2}) & h_{n-2} \\ & & & h_{n-2} & 2(h_{n-2}+h_{n-1}) \end{bmatrix}$$

H: tridiagonal symmetric and diagonal dominant

$$2|h_{i-1} + h_i| > |h_i| + |h_{i-1}|.$$

which implies unique solution.

$$\vec{z} = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_{n-2} \\ z_{n-1} \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 6(b_1 - b_0) \\ 6(b_2 - b_1) \\ 6(b_3 - b_2) \\ \vdots \\ 6(b_{n-2} - b_{n-3}) \\ 6(b_{n-1} - b_{n-2}) \end{pmatrix}$$

Summarizing the algorithm:

(1) Set up the matrix-vector equation and solve for $z_i$.

(2) Compute $S_i(x)$ using these $z_i$'s.