1)
a)

Sol. (i). We shall use the following results :

$$\left.\begin{aligned}\int_0^{\pi}\cos m\phi \cos n\phi \, d\phi = \int_0^{\pi}\sin m\phi \sin n\phi \, d\phi &= \pi/2 \text{ when } m = n \\ &= 0 \text{ when } m \neq n\end{aligned}\right\} \quad ...(1)$$

$$\cos (x \sin \phi) = J_0 + 2J_2 \cos 2\phi + 2J_4 \cos 4\phi + ... \quad ...(2)$$

and
$$\sin (x \sin \phi) = 2J_1 \sin \phi + 2J_3 \sin 3\phi + 2J_5 \sin 5\phi + ... \quad ...(3)$$

Multiplying both sides of (2) by $\cos n\phi$ and then integrating w.r.t. 'ϕ' between limits 0 to π and using (1), we have

$$\begin{aligned}\int_0^{\pi}\cos (x \sin \phi) \cos n\phi \, d\phi &= 0, \text{ if } n \text{ is odd} \quad ...(4) \\ &= \pi J_n, \text{ if } n \text{ is even.} \quad ...(5)\end{aligned}$$

Again, multiplying both sides of (3) by $\sin n\phi$ and then integrating w.r.t. 'ϕ' between limits 0 to π and using (1), we get

$$\begin{aligned}\int_0^{\pi}\sin (x \sin \phi) \sin n\phi \, d\phi &= \pi J_n, \text{ if } n \text{ is odd} \quad ...(6) \\ &= 0, \text{ if } n \text{ is even.} \quad ...(7)\end{aligned}$$

Let n be odd. Adding (4) and (6), we get

$$\int_0^{\pi}[\cos (x \sin \phi) \cos n\phi + \sin (x \sin \phi) \sin n\phi] \, d\phi = \pi J_n.$$

or $\displaystyle\int_0^{\pi}\cos (n\phi - x \sin \phi) \, d\phi = \pi J_n$ or $\displaystyle J_n(x) = \frac{1}{\pi}\int_0^{\pi}\cos (n\phi - x \sin \phi) \, d\phi$. ...(8)

Next, let n be even. Then adding (5) and (7) as before, we again get (8). Thus (8) holds for each positive integer (even as well as odd).

Part (ii). Let n be any integer. Then as in part (*i*), if n is positive integer, we have

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi}\cos (n\phi - x \sin \phi) \, d\phi \quad ...(9)$$

Next, let n be a negative integer so that $n = -m$, where m is a positive interger. To prove the required result for a negative integer, we prove that

$$J_{-m}(x) = \frac{1}{\pi}\int_0^{\pi}\cos (-m\phi - x \sin \phi) \, d\phi. \quad ...(10)$$

Let $\phi = \pi - \theta$ so that $d\phi = -d\theta$. Then, we have

R.H.S. of (10)

$$= \frac{1}{\pi}\int_{\pi}^{0} \cos \{-m(\pi - \theta) - x \sin (\pi - \theta)\} (-d\theta) = \frac{1}{\pi}\int_0^{\pi} \cos [(m\theta - x \sin \theta) - m\pi] \, d\theta$$

$$= \frac{1}{\pi}\int_0^{\pi} [\cos (m\theta - x \sin \theta) \cos m\pi + \sin(m\theta - x \sin \theta) \sin m\pi] \, d\theta$$

$$= \frac{1}{\pi}\int_0^{\pi} (-1)^m \cos(m\theta - x \sin \theta) \, d\theta \qquad [\because \sin m\pi = 0 \text{ and } \cos m\pi = (-1)^m]$$

$$= \frac{1}{\pi}(-1)^m\int_0^{\pi}\cos (m\phi - x \sin \phi) \, d\phi = (-1)^m J_m(x) \qquad [\text{Using (9) as } m \text{ is + ve integer}]$$

$$= J_{-m}(x) = \text{L.H.S. of (10)} \qquad [\because J_{-m}(x) = (-1)^m J_m(x)]$$

Thus, (10) is true. (9) and (10), show that the required result holds for each integer.

b) **Sol. (i)** From Ex. 9 above, $$\frac{d}{dx}(J^2_n + J^2_{n+1}) = 2\left(\frac{n}{x}J_n^2 - \frac{n+1}{x}J_{n+1}^2\right). \qquad \text{...(1)}$$

Replacing n by 0, 1, 2, 3 ... successively in (1), we get

$$\frac{d}{dx}(J_0^2 + J_1^2) = 2\left(0 - \frac{1}{x}J_1^2\right)$$
$$\frac{d}{dx}(J_1^2 + J_2^2) = 2\left(\frac{1}{x}J_1^2 - \frac{2}{x}J_2^2\right)$$
$$\frac{d}{dx}(J_2^2 + J_3^2) = 2\left(\frac{2}{x}J_2^2 - \frac{3}{x}J_3^2\right)$$
... ...

Adding these columnwise and noting that $J_n \to 0$ as $n \to \infty$, we get

$$\frac{d}{dx}[J_0^2 + 2(J_1^2 + J_2^2 + \ldots)] = 0$$

Integrating, $$J_0^2(x) + 2[J_1^2(x) + J_2^2(x) + \ldots] = C. \qquad \text{...(2)}$$

Replacing x by 0 in (2) and noting that $J_0(0) = 1$ and $J_n(0) = 0$ for $n \geq 1$, we get

$1 + 2(0 + 0 + \ldots) = C$ or $C = 1$. Hence (2) becomes $J_0^2 + 2(J_1^2 + J_2^2 + \ldots) = 1$ ...(3)

2) **Part (ii).** From (3), $$J_0^2 = 1 - 2(J_1^2 + J_2^2 + \ldots + J^2_{n-1} + J^2_n + J^2_{n+1} + \ldots.) \qquad \text{...(4)}$$

Since $J_1^2, J_2^2, J_3^2 \ldots$ are all positive or zero, (4) gives $J_0^2 \leq 1$ so that $|J_0(x)| \leq 1$.

b) **Part (iii).** Solving (4) for J_n^2, we have

$$J_n^2 = (1/2) \times (1 - J_0^2) - (J_1^2 + J_2^2 + \ldots + J^2_{n-1} + J^2_{n+1} + \ldots). \qquad \text{...(5)}$$

Since $J_0^2, J_1^2, J_2^2 \ldots$ are all positive or zero, (5) gives $J^2_n \leq 1/2$ or $|J_n(x)| \leq 2^{-1/2}$, where $n \geq 1$.

3) **Example 17.4** Express $J_5(x)$ in terms of $J_0(x)$ and $J_1(x)$.

Solution: From the recurrence relation-III, we have

$$J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x)$$

Putting $n = 4, 3, 2, 1$ we get

$$J_5(x) = \frac{8}{x}J_4(x) - J_3(x), \qquad J_4(x) = \frac{6}{x}J_3(x) - J_2(x)$$
$$J_3(x) = \frac{4}{x}J_2(x) - J_1(x) \quad \text{and } J_2(x) = \frac{2}{x}J_1(x) - J_0(x).$$

$$\begin{aligned}
\text{Now } J_5(x) &= \frac{8}{x}J_4(x) - J_3(x) = \frac{8}{x}\left(\frac{6}{x}J_3(x) - J_2(x)\right) - J_3(x) = \frac{48}{x^2}J_3(x) - \frac{8}{x}J_2(x) - J_3(x) \\
&= \left(\frac{48}{x^2} - 1\right)J_3(x) - \frac{8}{x}J_2(x) = \left(\frac{48}{x^2} - 1\right)\left(\frac{4}{x}J_2(x) - J_1(x)\right) - \frac{8}{x}J_2(x) \\
&= \left(\frac{192}{x^3} - \frac{4}{x}\right)J_2(x) - \left(\frac{48}{x^2} - 1\right)J_1(x) - \frac{8}{x}J_2(x) = \left(\frac{192}{x^3} - \frac{12}{x}\right)J_2(x) - \left(\frac{48}{x^2} - 1\right)J_1(x) \\
&= \left(\frac{192}{x^3} - \frac{12}{x}\right)\left(\frac{2}{x}J_1(x) - J_0(x)\right) - \left(\frac{48}{x^2} - 1\right)J_1(x) = \left(\frac{192}{x^3} - \frac{12}{x}\right)\frac{2}{x}J_1(x) - \left(\frac{192}{x^3} - \frac{12}{x}\right)J_0(x) \\
&- \left(\frac{48}{x^2} - 1\right)J_1(x) = \left(\frac{384}{x^4} - \frac{72}{x^2} + 1\right)J_1(x) - \left(\frac{192}{x^3} - \frac{12}{x}\right)J_0(x)
\end{aligned}$$

4) a)

Sol. The generating function formula is $(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x), \ |z|<1, |x| \le 1.$ ...(1)

Part (i). Putting $x = 1$ in (1), we have

$$(1-2z+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(1) \qquad \text{or} \qquad (1-z)^{-1} = \sum_{n=0}^{\infty} z^n P_n(1).$$

Since $|z| < 1$, the binomial theorem can be used for expansion of $(1-z)^{-1}$.

$$\therefore \qquad 1 + z + z^2 + ... + z^n + ... = \sum_{n=0}^{\infty} z^n P_n(1). \qquad ...(2)$$

Equating the coefficient of z^n from both sides, (2) gives $\qquad P_n(1) = 1$

Part (ii). Putting $x = -1$ in (1), we have as before

$$(1+2z+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(-1) \qquad \text{or} \qquad (1+z)^{-1} = \sum_{n=0}^{\infty} z^n P_n(-1)$$

or
$$1 - z + z^2 ... + (-1)^n z^n + ... = \sum_{n=0}^{\infty} z^n P_n(-1). \qquad ...(3)$$

Equation the coefficients of z^n from both sides, (3) gives $\qquad P_n(-1) = (-1)^n.$

b)

Part (iii). Since $P_n(x)$ satisfies Legendre's equation $\qquad (1-x^2)y'' - 2xy' + n(n+1)y = 0,$ we get

$$(1-x^2)\, P''_n(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0. \qquad ...(4)$$

Putting $x = 1$ in (4) and using $P_n(1) = 1$, we get

$$0 - 2P'_n(1) + n(n+1) = 0 \qquad \text{or} \qquad P'_n(1) = \tfrac{1}{2} n(n+1).$$

Part (iv). Putting $x = -1$ in (4) and using $P_n(-1) = (-1)^n$, we get

$$0 + 2P'_n(-1) + n(n+1)(-1)^n = 0 \qquad \text{or} \qquad P'_n(-1) = -(-1)^n \times \tfrac{1}{2} n(n+1).$$

or
$$P_n'(-1) = (-1)^{n-1} \times \tfrac{1}{2} n(n+1) \qquad [\because -(-1)^n = -(-1)^{n-1}(-1) = (-1)^{n-1}]$$

c)

Sol. We have
$$\sum_{n=0}^{\infty} z^n P_n(x) = (1-2xz+z^2)^{-1/2}, \ |z|<1, \ |x| \le 1. \qquad ...(1)$$

Putting $x = 0$ in (1),
$$\sum_{n=0}^{\infty} z^n P_n(0) = (1+z^2)^{-1/2}, \qquad i.e.,$$

$$\sum_{n=0}^{\infty} z^n P_n(0) = \sum_{n=0}^{\infty} \frac{(-\frac{1}{2})(-\frac{1}{2}-1)....(-\frac{1}{2}-n+1)}{n!}(z^2)^n \quad \text{or} \quad \sum_{n=0}^{\infty} z^n P_n(0) = \sum_{n=0}^{\infty} (-1)^n \frac{1 \cdot 3 \cdot 5 ...(2n-1)}{2^n \, n!} z^{2n}.$$
...(2)

Part (i). Note that the R.H.S. of (2) consists of even powers of z alone. So equating the coefficients of z^{2m+1} from both sides of (2), we have $\qquad P_{2m+1}(0) = 0.$ ...(3)

Part (ii). Equating the coefficients of z^{2m} from both sides of (2), we get

$$P_{2m}(0) = (-1)^m \frac{1 \cdot 3 \cdot 5 ...(2m-1)}{2^m \, m!} = (-1)^m \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 ...(2m-1)\,(2m)}{2^m \, m! \, 2 \cdot 4 \cdot 6 ...(2m)}$$

$$= (-1)^m \frac{(2m)!}{2^m \, m! \, (2 \cdot 1)\,(2 \cdot 2)\,(2 \cdot 3) ...(2 \cdot m)} = (-1)^m \frac{(2m)!}{2^m \, m!} \cdot \frac{1}{2^m \, m!} = (-1)^m \frac{(2m)!}{2^{2m} \, (m!)^2} \qquad ...(4)$$

5) **Example 15.5** Show that for any function $f(x)$, for which the $n-th$ derivative is continuous

$$\int_{-1}^{1} f(x)P_n(x)dx = \frac{1}{2^n n!}\int_{-1}^{1}(1-x^2)^n f^{(n)}(x)dx$$

Solution: We know that $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$. Now

$$\int_{-1}^{1} f(x)P_n(x)dx = \frac{1}{2^n n!}\int_{-1}^{1}\frac{d^n}{dx^n}(x^2-1)^n f(x)dx$$

$$= \frac{1}{2^n n!}\left[\left[\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n f(x)\right]_{-1}^{1} - \int_{-1}^{1}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n f'(x)dx\right] \qquad (15.48)$$

[On Intigration by parts]

Also $$\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n = \frac{d^{n-1}}{dx^{n-1}}(x-1)^n(x+1)^n$$

$$= (x-1)^n(n-1)!(x+1) + {}^{(n-1)}C_1 n(x-1)^{n-1}(n-2)!(x+1)^2 + \cdots + (n-1)!(x-1)(x+1)^n$$

[Since $(uv)_n = u_n v + {}^n C_1 u_{n-1}v_1 + {}^n C_2 u_{n-2}v_2 + \cdots + uv_n$, where $u_n = \frac{d^n u}{dx^n}$].

Now we can easily seen that $\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n$ will be zero at $x=1$ and $x=-1$. So first part of (15.48) must be zero, so from (15.48) reduces to

$$\int_{-1}^{1} f(x)P_n(x)dx = -\int_{-1}^{1}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n f'(x)dx\Big] = (-1)^n\frac{1}{2^n n!}\int_{-1}^{1}(x^2-1)^n f^{(n)}(x)dx$$

$$= (-1)^n(-1)^n\frac{1}{2^n n!}\int_{-1}^{1}(1-x^2)^n f^{(n)}(x)dx = \frac{1}{2^n n!}\int_{-1}^{1}(1-x^2)^n f^{(n)}(x)dx.$$

6) *Prove that* $\int_{-1}^{1} x\,P_n(x)\,P_{n-1}(x)\,dx = \frac{2n}{4n^2-1}$.

[Purvanchal 2007; Gulbarga 2005; Sagar 2004; Kanpur 2007]

Sol. From recurrence relation *I*, $\quad xP_n(x) = \frac{n+1}{2n+1}P_{n+1}(x) + \frac{n}{2n+1}P_{n-1}(x)$. ...(1)

Multiplying both sides of (1) by $P_{n-1}(x)$ and then integrating w.r.t. x from -1 to 1, we get

$$\int_{-1}^{1} x\,P_n(x)\,P_{n-1}(x)\,dx = \frac{n+1}{2n+1}\int_{-1}^{1}P_{n+1}(x)\,P_{n-1}(x)\,dx + \frac{n}{2n+1}\int_{-1}^{1}[P_{n-1}(x)]^2 dx. \qquad ...(2)$$

But, $$\int_{-1}^{1} P_m(x)\,P_n(x)\,dx = \begin{cases}0, & if\ m \neq n\\ 2/(2n+1), & if\ m = n\end{cases} \qquad ...(3)$$

Making use of (3), (2) becomes

$$\int_{-1}^{1} xP_n(x)\,P_{n-1}(x)\,dx = 0 + \frac{n}{2n+1}\times\frac{2}{2(n-1)+1} = \frac{2n}{(2n+1)(2n-1)} \quad \text{or} \quad \frac{2n}{4n^2-1}.$$

7)

Sol. Case I. Let $l \neq m$. Then integrating by parts, we have

$$\int_{-1}^{1} [(1-x^2)\,P_l']\,P_m'\,dx = \Big[(1-x^2)P_l'.P_m\Big]_{-1}^{1} - \int_{-1}^{1} [(1-x^2)\,P_l'' - 2x\,P_l']\,P_m\,dx = -\int_{-1}^{1} [(1-x^2)\,P_l'' - 2x\,P_l']\,P_m\,dx. \quad ...(1)$$

Since P_l satisfies Legendre's equation $\quad (1-x^2)y'' - 2xy' + l(l+1)y = 0$, hence

$$(1-x^2)\,P_l'' - 2xP_l' + l(l+1)P_l = 0 \qquad \text{or} \qquad (1-x^2)P_l'' - 2xP_l' = -l(l+1)P_l. \quad ...(2)$$

But
$$\int_{-1}^{1} P_l\,P_m\,dx = 0, \text{ if } l \neq m \quad ...(3)$$

Using (2), (1) reduces to

$$\int_{-1}^{1} (1-x^2)\,P_l'\,P_m'\,dx = l(l+1)\int_{-1}^{1} P_l\,P_m\,dx = 0, \text{ using (3).} \quad ...(4)$$

Case II. Let $l = m$. Then the required result takes the form

$$\int_{-1}^{1} (1-x^2)\,(P_l')^2\,dx = \frac{2l(l+1)}{2l+1}. \qquad \textbf{[Agra 2010]} \quad ...(5)$$

We have, by using integration by parts,

$$\int_{-1}^{1} (1-x^2)\,(P_l')^2\,dx = \int_{-1}^{1} [(1-x^2)\,P_l']\,.\,P_l'\,dx$$

$$= \Big[(1-x^2)\,P_l' P_l\Big]_{-1}^{1} - \int_{-1}^{1} \Big[(1-x^2)P_l'' - 2xP_l'\Big]\,P_l\,dx = 0 + l(l+1)\int_{-1}^{1} (P_l)^2\,dx, \text{ using (2)}$$

$$= l(l+1)\,.\,\frac{2}{2l+1} = \frac{2l(l+1)}{2l+1}.$$

Combining (4) and (5) and using symbol δ_{lm}, we get $\quad \displaystyle\int_{-1}^{1} (1-x^2)\,P_l' P_m'\,dx = \frac{2l(l+1)}{2l+1}\,\delta_{lm}.$

Ex. 10. *Prove that all the roots of $P_n(x)$ are distinct.*

Sol. If possible, let the roots of $P_n(x) = 0$ be not all different. Then at least two roots must be equal. Let α be the repeated root, then from the theory of equations, we have

$$P_n(\alpha) = 0 \qquad \text{and} \qquad P'_n(\alpha) = 0. \quad ...(1)$$

Since $P_n(x)$ satisfies Legendre's equation, $\quad (1-x^2)P''_n - 2xP'_n + n(n+1)P_n = 0. \quad ...(2)$

Differentiating r times and using Leibnitz theorem, (2) gives

$$(1-x^2)\,\frac{d^{r+2}}{dx^{r+2}}P_n(x) + {}^rC_1 \times (-2x) \times \frac{d^{r+1}}{dx^{r+1}}P_n(x) + {}^rC_2 \times (-2) \times \frac{d^r}{dx^r}P_n(x)$$

$$-2\left[x\frac{d^{r+1}}{dx^{r+1}}P_n(x) + {}^rC_1 \times 1 \times \frac{d^r}{dx^r}P_n(x)\right] + n(n+1)\frac{d^r}{dx^r}P_n(x) = 0$$

or $\quad \displaystyle (1-x^2)\,\frac{d^{r+2}}{dx^{r+2}}P_n(x) - 2x({}^rC_1 + 1)\,\frac{d^{r+1}}{dx^{r+1}}P_n(x) - \{2 \times {}^rC_2 + 2 \times {}^rC_1 - n(n+1)\}\,\frac{d^r}{dx^r}P_n(x) = 0$

....(3)

Putting $r = 0$ and $x = \alpha$ in (3) and using (1), we get

$$(1-\alpha^2)P''_n(\alpha) - 0 - 0 = 0 \qquad \text{or} \qquad P''_n(\alpha) = 0. \quad ...(4)$$

Next, putting $r = 1$ and $x = \alpha$ in (3) and using (1) and (4), we get

$$(1-\alpha^2)P_r'''(\alpha) - 0 - 0 = 0 \qquad \text{or} \qquad P'''_n(\alpha) = 0. \quad ...(5)$$

Putting $r = 2, 3, ..., n-3, n-2$ in (3) and doing as above stepwise, we finally arrive at

$$P_n^{(n)}(\alpha) = 0 \qquad i.e. \qquad \left[\frac{d^n}{dx^n}P_n(x)\right]_{x=\alpha} = 0. \quad ...(6)$$

But
$$P_n(x) = \frac{1\cdot 3\cdot 5\ldots(2n-1)}{n\,!}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \ldots\right]$$

$$\therefore \qquad \frac{d^n}{dx^n}P_n(x) = \frac{1\cdot 3\cdot 5\ldots(2n-1)}{n\,!} \times n\,! \quad \Rightarrow \quad \left[\frac{d^n}{dx^n}P_n(x)\right]_{x=\alpha} \neq 0. \quad ...(7)$$

***Rolle's theorem:** If $f(x)$ vanishes for $x = a$ and $x = b$, then $f'(x)$ vanishes at least once for some value of x between a and b.

8)

Prove that all the roots of $P_n(x)$ are distinct.

Sol. If possible, let the roots of $P_n(x) = 0$ be not all different. Then at least two roots must be equal. Let α be the repeated root, then from the theory of equations, we have

$$P_n(\alpha) = 0 \quad \text{and} \quad P'_n(\alpha) = 0. \quad ...(1)$$

Since $P_n(x)$ satisfies Legendre's equation, $(1 - x^2)P''_n - 2xP'_n + n(n + 1)P_n = 0.$ ...(2)

Differentiating r times and using Leibnitz theorem, (2) gives

$$(1-x^2)\frac{d^{r+2}}{dx^{r+2}}P_n(x) + {}^rC_1 \times (-2x) \times \frac{d^{r+1}}{dx^{r+1}}P_n(x) + {}^rC_2 \times (-2) \times \frac{d^r}{dx^r}P_n(x)$$

$$-2\left[x\frac{d^{r+1}}{dx^{r+1}}P_n(x) + {}^rC_1 \times 1 \times \frac{d^r}{dx^r}P_n(x)\right] + n(n+1)\frac{d^r}{dx^r}P_n(x) = 0$$

or $$(1-x^2)\frac{d^{r+2}}{dx^{r+2}}P_n(x) - 2x({}^rC_1+1)\frac{d^{r+1}}{dx^{r+1}}P_n(x) - \{2\times{}^rC_2 + 2\times{}^rC_1 - n(n+1)\}\frac{d^r}{dx^r}P_n(x) = 0$$

....(3)

Putting $r = 0$ and $x = \alpha$ in (3) and using (1), we get

$$(1 - \alpha^2)P''_n(\alpha) - 0 - 0 = 0 \quad \text{or} \quad P''_n(\alpha) = 0. \quad ...(4)$$

Next, putting $r = 1$ and $x = \alpha$ in (3) and using (1) and (4), we get

$$(1 - \alpha^2)P_r'''(\alpha) - 0 - 0 = 0 \quad \text{or} \quad P'''_n(\alpha) = 0. \quad ...(5)$$

Putting $r = 2, 3, ..., n - 3, n - 2$ in (3) and doing as above stepwise, we finally arrive at

$$P_n^{(n)}(\alpha) = 0 \quad \textit{i.e.} \quad \left[\frac{d^n}{dx^n}P_n(x)\right]_{x=\alpha} = 0. \quad ...(6)$$

But $$P_n(x) = \frac{1\cdot 3\cdot 5 ...(2n-1)}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + ...\right]$$

$$\therefore \quad \frac{d^n}{dx^n}P_n(x) = \frac{1\cdot 3\cdot 5 ...(2n-1)}{n!} \times n! \quad \Rightarrow \quad \left[\frac{d^n}{dx^n}P_n(x)\right]_{x=\alpha} \neq 0. \quad ...(7)$$

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*__Rolle's theorem:__ If $f(x)$ vanishes for $x = a$ and $x = b$, then $f'(x)$ vanishes at least once for some value of x between a and b.

Since (6) and (7) are contradictory results, it follows that our assumption about not distinct roots of $P_n(x)$ is absurd. Hence all the roots of $P_n(x) = 0$ must be distinct.

10)

Part (v). Replacing x by $-x$ in (1), $$(1 + 2xz + z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(-x). \quad ...(5)$$

Next, replacing z by $-z$ in (1), $$(1 + 2xz + z)^{-1/2} = \sum_{n=0}^{\infty} (-z)^n P_n(x). \quad ...(6)$$

From (5) and (6), $$\sum_{n=0}^{\infty} z^n P_n(-x) = \sum_{n=0}^{\infty} (-1)^n z^n P_n(x). \quad ...(7)$$

Equating the coefficients of z^n from both sides of (8), we get

$$P_n(-x) = (-1)^n P_n(x). \quad ...(8)$$

Deduction. Replacing x by 1 and noting that $P_n(1) = 1$, (8) gives $P_n(-1) = (-1)^n$.

Note. When n is odd, $(-1)^n = -1$ and so (8) becomes $P_n(-x) = -P_n(x)$. Thus, $P_n(x)$ is an odd function of x when n is odd. Similarly, $P_n(x)$ is an even function of x when n is even.

11) *Using Rodrigue's formula, show that $P_n(x)$ satisfies*

$$\frac{d}{dx}\left\{(1-x^2)\frac{d}{dx}P_n(x)\right\}+n(n+1)P_n(x)=0$$ **[CDLU 2004]**

Sol. Rodrigue's formula is $$P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n \quad \text{... (1)}$$

Let $$y=(x^2-1)^n \quad \text{... (2)}$$

Differentiating (2) w.r.t. 'x', $y_1=2nx(x^2-1)^{n-1}$ so that $(x^2-1)y_1=2nx\,(x^2-1)^n$

or $$(x^2-1)\,y_1=2nxy, \text{ using (2)} \quad \text{... (3)}$$

Differentiating (3) w.r.t. 'x', $(x^2-1)y_2+2xy_1=2n\,(xy_1+y)$

or $$(x^2-1)\,y_2+2(1-n)xy_1-2ny=0 \quad \text{... (4)}$$

Differentiating both sides of (4) w.r.t. 'x' n times, we have

$$D^n\{(x^2-1)y_2\}+2(1-n)\,D^n(xy_1)-2nD^n(y)=0, \text{ where } D^n\equiv d^n/dx^n \quad \text{... (5)}$$

Using Leibnitz' theorem, (5) yields

$$y_{n+2}\,(x^2-1)+{}^nC_1\,y_{n+1}(2x)+{}^nC_2\,y_n\cdot 2+2(1-n)\,(y_{n+1}x+{}^nC_1\;y_n\cdot 1)-2ny_n=0$$

or $$(x^2-1)\,y_{n+2}+2x\,y_{n+1}+\{n(n-1)+2n(1-n)-2n\}\,y_n=0$$

or $$(1-x^2)\,y_{n+2}-2x\,y_{n+1}+n(n+1)\,y_n=0$$

or $$\frac{d}{dx}\{(1-x^2)y_{n+1}\}+n(n+1)y_n=0 \qquad \text{or} \qquad \frac{d}{dx}\left\{(1-x^2)\times\left(\frac{dy_n}{dx}\right)\right\}+n(n+1)y_n=0$$

or $$\frac{d}{dx}\left\{(1-x^2)\frac{d}{dx}\left(\frac{d^n}{dx^n}(x^2-1)^n\right)\right\}+n(n+1)\frac{d^n}{dx^n}(x^2-1)^n=0, \text{ using (2)}$$

Dividing by $2^n n!$ and using (1), we get $$\frac{d}{dx}\{(1-x^2)\frac{d}{dx}P_n(x)\}+n(n+1)P_n(x)=0$$

12) $$xy''+(1-x)y'+ny=0 \qquad x\neq 0 \quad \text{— (1)}$$

General form of sturm - Liouville equation

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+\left[q(x)+\lambda r(x)\right]y=0$$

$r(x)$ ↳ Weight function

$$y''+\frac{(1-x)y'}{x}+\frac{n}{x}y=0 \quad \leftarrow \text{ Second order ODE} \quad \text{— (2)}$$

Integrating factor

$$I.F. = e^{\int(1/x-1)dx} = e^{\ln x - x} = x\cdot e^{-x}$$

Multiply by Integrating factor in eq^n (2)

$$xe^{-x}y''+e^{-x}x(1-x)y'+ne^{-x}y=0$$

$$xe^{-x}y''+e^{-x}(1-x)y'+ne^{-x}y=0$$

$$\frac{d}{dx}\left[xe^{-x}\frac{dy}{dx}\right]+ne^{-x}y=0$$

$$\boxed{\frac{d}{dx}\left[xe^{-x}\frac{dy}{dx}\right]+\left[0+ne^{-x}\right]y=0}$$

$$p(x)=xe^{-x}, \quad q(x)=0, \quad r(x)=e^{-x}, \quad \lambda=n$$

13 a) $(1-x^2)y'' - xy' + n^2 y = 0$

$\rightarrow$ strum Liouville

$$\frac{d}{dx}\left[r(x)\frac{dy}{dx}\right] + \left[q(x) + \lambda \underbrace{P(x)}_{\text{weight function}}\right] y = 0$$

$$y'' - \frac{x}{(1-x^2)} y' + \frac{n^2}{(1-x^2)} y = 0 \quad \text{— (1)}$$

Integrating factor

$$I.F. = \int \frac{-x}{1-x^2}\,dx = \sqrt{1-x^2}$$

Multiplying by $\sqrt{1-x^2}$ in equation (1)

$$\sqrt{1-x^2}\, y'' - \frac{x}{\sqrt{1-x^2}} y' + \frac{n^2 \cancel{(1-x^2)}}{\cancel{(1-x^2)\sqrt{1-x^2}}\ \sqrt{1-x^2}} \cdot y = 0$$

$$\Rightarrow \boxed{\frac{d}{dx}\left[\sqrt{1-x^2}\, y'\right] + \frac{n^2}{\sqrt{1-x^2}} y = 0}$$

$$P(x) = \frac{1}{\sqrt{1-x^2}}$$

(b) $y'' - 2xy' + 2ny = 0$ — (1)

Integrating factor $(I.F.) = \int e^{-2x\,dx} = e^{-x^2}$

Multiply by e^{-x^2} in equation (1)

$$e^{-x^2} y'' - 2x e^{-x^2} y' + 2e^{-x^2} n y = 0$$

$$\boxed{\frac{d}{dx}\left[e^{-x^2}\frac{dy}{dx}\right] + 2n e^{-x^2} y = 0}$$

$$P(x) = e^{-x^2}$$

(c) $xy'' + 2y' + (x+\lambda)y = 0$ $\qquad I.F. = e^{\int 2/x\, dx} = x^2$

$x^2 y'' + 2xy' + x(x+\lambda)y = 0$

$$\frac{d}{dx}\left[x^2 y'\right] + \left[x^2 + \lambda x\right] y = 0 \qquad \boxed{P(x) = x^2}$$

X (12) a) $(1-x^2)y'' - xy' + n^2y = 0$

$\rightarrow$ strum Liouville

$$\frac{d}{dx}\left[r(x)\frac{dy}{dx}\right] + \left[q(x) + \lambda p(x)\right]y = 0$$

$\hookrightarrow$ weight function

$$y'' - \frac{x}{(1-x^2)}y' + \frac{n^2}{(1-x^2)}y = 0 \quad \text{— (1)}$$

Integrating factor

$$\text{I.F.} = \int \frac{-x}{1-x^2}\,dx = \sqrt{1-x^2}$$

Multiplying by $\sqrt{1-x^2}$ in equation (1)

$$\sqrt{1-x^2}\,y'' - \frac{x}{\sqrt{1-x^2}}y' + \frac{n^2 \cancel{(1-x^2)}}{\cancel{(1-x^2)\sqrt{1-x^2}}\ \sqrt{1-x^2}}\,y = 0$$

$$\Rightarrow \boxed{\frac{d}{dx}\left[\sqrt{1-x^2}\,y'\right] + \frac{n^2}{\sqrt{1-x^2}}y = 0}$$

$$p(x) = \frac{1}{\sqrt{1-x^2}}$$

(b) $y'' - 2xy' + 2ny = 0$ — (1)

$$\text{Integrating factor (I.F.)} = \int e^{-2x\,dx} = e^{-x^2}$$

Multiply by e^{-x^2} in equation (1)

$$e^{-x^2}y'' - 2xe^{-x^2}y' + 2e^{-x^2}ny = 0$$

$$\boxed{\frac{d}{dx}\left[e^{-x^2}\frac{dy}{dx}\right] + 2ne^{-x^2}y = 0}$$

$$p(x) = \cancel{2}\,e^{-x^2}$$

(c) $xy'' + 2y' + (x+\lambda)y = 0$ $\quad$ $\text{I.F.} = e^{\int 2/x\,dx} = x^2$

$$x^2y'' + 2xy' + x(x+\lambda)y = 0$$

$$\frac{d}{dx}[x^2y'] + [x^2 + \lambda x]y = 0 \qquad \boxed{p(x) = x^2}$$

Q14 a) $\frac{d}{dx}[x^2y'] + \lambda y = 0$ $\quad y(1)=0 \quad y(b)=0$

$x^2y'' + 2xy' + \lambda y = 0$ — ①

Cauchy Euler eq^n

$x = e^t \qquad t = \ln x \qquad D = \frac{d}{dt} \quad D' = \frac{d}{dx}$

① becomes $\left(D(D-1) + 2D + \lambda\right) y = 0$

$(D^2 + D + \lambda) y = 0$

roots of auxiliary eq^n

$$m = \frac{-1 \pm \sqrt{1-4\lambda}}{2}$$

make cases for i) $1-4\lambda = 0$
ii) $1-4\lambda > 0$
iii) $1-4\lambda < 0$

In first 2 cases i) & ii), sol^n will be trivial (Try & show!)

In case iii)

$$m = \frac{-1 \pm i\sqrt{4\lambda-1}}{2}$$

$$y(t) = e^{-\frac{1}{2}t}\left(A\cos\left(\frac{\sqrt{4\lambda-1}}{2}t\right) + B\sin\left(\frac{\sqrt{4\lambda-1}}{2}t\right)\right)$$

Converting first boundary condition in t, using $t = \ln x$

$y(0) = 0 \Rightarrow A = 0$

$y(\ln b) = 0 \Rightarrow e^{-\frac{1}{2}\ln b}\left[B\sin\left(\frac{\sqrt{4\lambda-1}}{2}\ln b\right)\right] = 0$

for non trivial sol^n, $\sin\left(\frac{\sqrt{4\lambda-1}}{2}\ln b\right) = 0$

$\Rightarrow \frac{\sqrt{4\lambda-1}}{2}\ln b = n\pi \Rightarrow \boxed{\lambda_n = \frac{1}{4} + \left(\frac{n\pi}{\ln b}\right)^2}$

Corresp eigen fn is $\boxed{\sin\left(\frac{n\pi}{\ln b}\cdot \ln x\right)}$

Q14 b)

$$\frac{d}{dx}[xy'] + \frac{\lambda}{x} y = 0 \qquad y(1) = 0, \quad y(e^{\pi}) = 0$$

$$x^2y'' + xy' + \lambda y = 0$$

Convert into DE by substituting $x = e^t$

$$D' = \frac{d}{dx} \quad D = \frac{d}{dt}$$

Euler-cauchy eqn

$$D(D-1)y + Dy + \lambda y = 0$$

$$D^2 y + \lambda y = 0$$

$$y(t) = c_1 \cos\sqrt{\lambda}t + c_2 \sin\sqrt{\lambda}\, t$$

$$y(0) = 0 \Rightarrow c_1 = 0 \qquad (x=1 \Rightarrow t=0)$$

$$y(\pi) = 0 \Rightarrow c_2 \sin\sqrt{\lambda}\pi = 0 \qquad \boxed{x = e^{\pi} \Rightarrow t = \pi}$$

$$c_2 \neq 0 \qquad \sqrt{\lambda}\pi = n\pi$$

$$\sqrt{\lambda} = n$$

Eigenvalues $\quad \lambda = n^2$

$$y_n(t) = c_2 \sin nt$$

$$y_n(x) = c_2 \sin(n \ln x) \qquad c_2 = 1 \text{ for conv.}$$

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Q15 $y'' + \lambda y = 0$; $y(0) = 0$, $y(\pi) + y'(\pi) = 0$

Case → $\lambda > 0$

$$y = c_1 \cos\sqrt{\lambda}\, x + c_2 \sin\sqrt{\lambda}\, x$$

$$y(0) = 0 \Rightarrow c_1 = 0$$

$$y' = -c_1 \sqrt{\lambda} \sin\sqrt{\lambda} x + c_2 \sqrt{\lambda} \cos\sqrt{\lambda}\, x$$

$$y(\pi) + y'(\pi) = 0 \Rightarrow c_2\left[\sin\sqrt{\lambda}\pi + \sqrt{\lambda}\cos\sqrt{\lambda}\pi\right] = 0$$

$$\sin\sqrt{\lambda}\pi + \sqrt{\lambda}\cos\sqrt{\lambda}\pi = 0$$

$$\Rightarrow \sqrt{\lambda} = -\tan\sqrt{\lambda}\pi$$

So eigenvalues satisfy this

$$y = c_2 \sin(-\tan\sqrt{\lambda}\pi)x$$

(Show other 2 cases as done in class)

15 $4(e^{-x}y')' + (1+\lambda)e^{-x}y = 0$ $\quad y(0)=0,\ y(1)=0$

$\Rightarrow 4[e^{-x}y'' - e^{-x}y'] + (1+\lambda)e^{-x}y = 0$

$\Rightarrow 4y'' - 4y' + (1+\lambda)y = 0$

auxiliary equation

$$4m^2 - 4m + (1+\lambda) = 0$$

$$\Rightarrow m = \frac{4 \pm \sqrt{16 - 16(1+\lambda)}}{8}$$

$$= \frac{4 \pm \sqrt{-16\lambda}}{8} = \frac{1 \pm \sqrt{\lambda}\, i}{2}$$

Solution is

$$y = e^{x/2}\left[A\cos\left(\frac{\sqrt{\lambda}\,x}{2}\right) + B\sin\left(\frac{\sqrt{\lambda}\,x}{2}\right)\right]$$

$y(0) = 0$

$\Rightarrow A = 0$

then $y = e^{x/2} \cdot B \cdot \sin\left[\frac{\sqrt{\lambda}\,x}{2}\right]$

$y(1) = e^{1/2} \cdot B \cdot \sin\left[\frac{\sqrt{\lambda}}{2}\right] = 0$

for non trivial sol^n

$\Rightarrow \sin\left[\frac{\sqrt{\lambda}}{2}\right] = 0$

$\Rightarrow \frac{\sqrt{\lambda}}{2} = n\pi$

$\Rightarrow \boxed{\lambda = 4n^2\pi^2}$ $\quad n = 1, 2, 3 \ldots \rightarrow$ eigen values

& eigenfunction are $\boxed{y_n = A_n e^{x/2}\sin(n\pi x)}$ $\quad n = 1, 2, 3 \ldots$

Linear operator $L(u) = p_0 u'' + p_1 u' + p_2 u = 0$

(exact if $p_0'' - p_1' + p_2 = 0$)

Adjoint of L. $M[v] = [p_0(x)v]'' - [p_1(x)v]' + p_2(x)v = 0$

$$p_0 v'' + (2p_0' - p_1)v' + (p_0'' - p_1' + p_2)v = 0$$

Lagrange identity $vL[u] - uM[v] = \frac{d}{dx}[p_0(u'v - v'u) - (p_0' - p_1)uv]$

Necc & Suff condition for self adjoint:

$$2p_0' - p_1 = p_1 \quad \text{or} \quad p_0' = p_1$$

2^{nd} order LDE is self adj iff it has form

$$\frac{d}{dx}\left[p_0(x)\frac{du}{dx}\right] + p_2(x)u = 0$$

Q Show that operator $Lu = -(pu')_x + qu$; $u(0) = u(1) = 0$ is self adjoint.

Way 1 use necc & suff condition

$$-pu'' - p'u' + qu = 0$$

$$\underset{p_0'}{(-p)'} = \underset{p_1}{-p'}$$

$\therefore$ self adj.

Way 2 $\int (Lu)v = \int u L(v)$ (another defn of self adj.)

$$\int_0^1 [(-pu_x)_x + qu]v \, dx = \int_0^1 [(-pv_x)_x + qv]u \, dx$$

prove it.