CS-204: Design and Analysis of Algorithms

220001061, 220001046, 220001075

16 April , 2024

1 Linear Programming

Linear programming (LP) is a problem-solving technique in computer science and optimization that deals with the optimization (maximization or minimization) of a linear objective function subject to linear equality and inequality constraints.

1.1 General linear programs

In the general linear-programming problem, we wish to optimize a linear function subject to a set of linear inequalities. Given a set of real numbers a1,a2.....an and a set of variables x1,x2.....xn we define a linear function f on those variables by

$$f(x_1, x_2, \ldots, x_n) = a_1x_1 + a_2x_2 + \ldots + a_nx_n = \sum_{j=1}^{n} a_jx_j$$

If b is a real number and f is a linear function, then the equation

$$f(x_1, x_2, \ldots, x_n) = b$$

is a linear equality and the inequalities

$$f(x_1, x_2, \ldots, x_n) <= b \; and \; f(x_1, x_2, \ldots, x_n) >= b$$

In linear programming, a problem may be framed as either a minimization or maximization task. When the goal is to find the smallest possible value of a linear objective function, subject to linear constraints, it's referred to as a minimization linear program.

In order to keep costs as small as possible, you would like to minimize the amount spent on advertising. That is, you want to minimize the expression $x_1 + x_2$
Although negative advertising often occurs in political campaigns, there is no such thing as negative-cost advertising. Consequently, you require that

$$x_1 >= 0; \quad \text{and} \quad x_2 >= 0$$

We can format this problem tabularly as minimize $x_1 + x_2$ subject to

$$5x_1 - 2x_2 >= -2$$

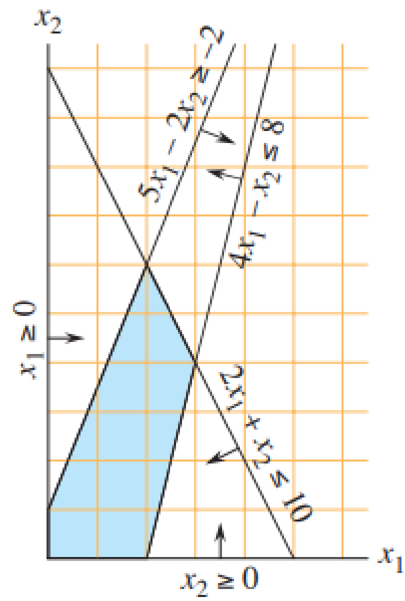

Figure 1: he highlighted area indicates the domain of variables x1 and x2

$$4x_1 - x_2 <= 8$$

$$2x_1 + x_2 <= 10$$

$$x_1, x_2 >= 0$$

The solution to this linear program yields your optimal strategy. The graph illustrates the domain within which we have the flexibility to select values for both x1 and x2. By analyzing the graph, we can strategically choose values for x1 and x2 to either minimize or maximize the sum of x1 and x2.

1.2 Integer Linear Programming

Integer Linear Programming (ILP) occurs when all variables in a linear programming problem are restricted to integer values. This limitation distinguishes it from standard linear programming, where variables can be any real number. ILP is crucial in scenarios where decisions must be made in discrete units or whole quantities.

2 Graph Coloring

Let $G(V, E)$ be a graph with vertex set V and edge set E. We define binary variables $X_{v,i}$ and C_i as follows:

$$X_{v,i} = \begin{cases} 1 & \text{if vertex } v \text{ is assigned color } i \\ 0 & \text{otherwise} \end{cases}$$

$$C_i = \begin{cases} 1 & \text{if color } i \text{ has been used} \\ 0 & \text{otherwise} \end{cases}$$

The objective function is to minimize the total number of colors used:

$$\text{minimize } \sum_{i=1}^{n} C_i$$

Subject to the following constraints:
The constraint that the summation of $X_{v,i}$ from $i = 1$ to n is equal to 1 for each v (n-constraints):

$$\sum_{i=1}^{n} X_{v,i} = 1 \quad \forall v \in V$$

The constraint that for all (u, v) belonging to E, they will not have the same color:

$$X_{u,i} + X_{v,i} \leq i \quad \forall (u, v) \in E, \forall i$$

For the decision variables $X_{v,i}$ and C_i belonging to $\{0, 1\}$, you can simply state that these variables are binary.

$$\sum_{i=1}^{n} C_i$$

minimize $\sum_{i=1}^{n} C_i$
The constraint that the summation of $x_{v,i}$ from $i = 1$ to n is equal to 1 for each v (n-constraints):

$$\sum_{i=1}^{n} x_{v,i} = 1 \quad \forall v \in V$$

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3 A problem to convert into ILP

Problem Description

You have four items (1, 2, 3, 4) that need to be assigned to four containers (1, 2, 3, 4). Each item can only be assigned to one container, and each container has a capacity limit. The objective is to minimize the total cost of assigning items to containers.

Pseudocode

Variables: X_{ij} (binary variable): Represents whether item i is assigned to container j.
Objective Function: Minimize: $C1 + C2 + C3 + C4$
Constraints: Each item must be assigned to exactly one container:

$$X_{11} + X_{12} + X_{13} + X_{14} = 1$$
$$X_{21} + X_{22} + X_{23} + X_{24} = 1$$
$$X_{31} + X_{32} + X_{33} + X_{34} = 1$$
$$X_{41} + X_{42} + X_{43} + X_{44} = 1$$

Each container has a capacity constraint:

$$X_{11} + X_{21} \leq C1$$
$$X_{12} + X_{22} \leq C2$$
$$X_{13} + X_{23} \leq C3$$
$$X_{14} + X_{24} \leq C4$$

Solution Steps:

1. Create a binary variable X_{ij} for each item i and container j.
2. Define the objective function to minimize $C1 + C2 + C3 + C4$.
3. Add constraints to ensure each item is assigned to exactly one container and that each container does not exceed its capacity.
4. Solve the ILP problem to find the optimal assignment of items to containers.