Lecture-2

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Power Series :

is an infinite Series of the form

$$\sum_{n=0}^{\infty} C_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + \quad \text{—(1)}$$

Convergence of a Power Series :

$\lim_{n\to\infty} S_n$ = finite & unique $\longrightarrow$ Convergent

$\lim_{n\to\infty} S_n = +\infty$ or $-\infty$ $\longrightarrow$ Divergent

where $S_n = \sum_{n=0}^{\infty} C_n = C_0 + C_1 + C_2 + \cdots$

$S_0 = a_0$
$S_1 = a_0 + a_1$
$S_2 = a_1 + a_2 + a_2 + \cdots$

Radius of Convergences :

$R = \frac{1}{L}$ where $\lim_{n\to\infty} \left| \frac{C_{n+1}}{C_n} \right| = L$

OR

$\lim_{n\to\infty} |C_n|^{1/n} = L$

There are three very important power Series

$\rightarrow \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n , \quad -1 < x < 1$

$\rightarrow e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} , \quad -\infty < x < \infty$

$\rightarrow \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} , \quad -\infty < x < \infty$

(1) $\sum_{n=0}^{\infty} \frac{(x-6)^n}{n^n}$ $\quad L=0$, $R=\infty$, Int of Convergence $-\infty < x < \infty$

(2) $\sum \frac{n!\,(5x+3)^n}{(n+1)^2+4n}$ $\quad L=\infty$, $R=0$, Interval of Convergence $x=-\frac{3}{5}$ and no where else

(3) $\sum_{n=0}^{\infty} \frac{(3x+4)^n}{(n^3+2)3^n}$ $\quad$ Here $R=1$, $\left(-\frac{7}{3}, -\frac{1}{3}\right)$

Now for any x in the interval of Convergence, the Sum of the Series defines a function

$$f(x) = \sum_{n=0}^{\infty} C_n (x-a)^n$$

then f is Continuous, differentiable ?

is the Series differentiable or integrated term by term etc. ?

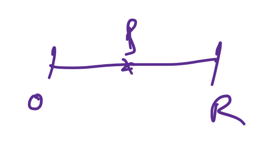
$\Rightarrow$ Answer of these questions depend on whether the Series Converges uniformly ?

Theorem: A power Series Converges uniformly to its limit in the interval of Convergence.

> This means that if $f(x)$ is the some function of the Series and radius of Convergence of the P.S. is R and the center of the series is a then the series always Converges uniformly in the open interval
> $$a-R < x < a+R$$

Proof $\sum_{n=0}^{\infty} C_n (x-a)^n$ & let R be its +ve radius of Convergence
[if $R=0$ then series Converges only at the center, so uniform Converge does not work]

Let us chose a number ρ s.t. $0 < \rho < R$



For any x in the interval $|x-a| < \rho$ then

$$|C_n|\,|x-a|^n < |C_n|\rho^n = M_n$$

then the Series $\sum_{n=0}^{\infty} M_n$ is a Convergent Series

by Root test

$$\lim_{n\to\infty} M_n^{1/n} = \lim_{n\to\infty} |C_n|^{1/n} (\rho^n)^{1/n} = \lim_{n\to\infty} |C_n|^{1/n} \rho = L\rho = \frac{\rho}{R}$$

Since $R > 0$, $\frac{\rho}{R}$ is finite quantity

$\therefore \sum_{n=0}^{\infty} M_n$ Converges by root test

So, by Weierstrass M test, the series $\sum_{n=0}^{\infty} C_n (x-a)^n$ Converges uniformly in the interval

$$|x-a| < \rho \quad \text{or} \quad a-\rho < x < a+\rho$$

Since, ρ is arbitrary therefore the given series Converges uniformly in the interval $(a-R, a+R)$

Hence, we can conclude that the limit function $f(x)$ is Continuous in $(a-R, a+R)$

(*) Result

Suppose $f(x) = \sum_{n=0}^{\infty} C_n (x-a)^n$ Converges uniformly for $|x-a| < R$, then if $-R < \alpha < \beta < R$ we have

$$\int_{\alpha}^{\beta} f(x)\,dx = \sum_{n=0}^{\infty} C_n \int_{\alpha}^{\beta} (x-a)^n\,dx.$$

α, β is a close interval, lying in $|x-a| < R$

Result 2 (term by term diff of a power Series)

$f(x) = \sum_{n=0}^{\infty} C_n (x-a)^n$ Converges for $|x-a| < R$, then the Series

$\sum_{n=1}^{\infty} n C_n (x-a)^{n-1}$ has precisely the same radius of Convergence.

So, if the given series has radius of Convergence R, then the differentiated Series also has the radius of Convergence R.

Operations of Power Series

Suppose functions $f(x)$ & $g(x)$ Can be expanded into power Series as

$$f(x) = \sum_{n=0}^{\infty} \alpha_n (x-a)^n \quad \text{for } |x-a| < R_1$$

$$g(x) = \sum_{n=0}^{\infty} \beta_n (x-a)^n \quad \text{for } |x-a| < R_2$$

then for $|x-a| < R$, $R = \min(R_1, R_2)$

$$f(x) \pm g(x) = \sum_{n=0}^{\infty} (\alpha_n \pm \beta_n)(x-a)^n$$

i.e., the power series of the sum or difference of the functions can be obtained by termwise addition and Substraction.

$\rightarrow$ Multiplication

$$f(x)\,g(x) = \left[\sum_{n=0}^{\infty} \alpha_n (x-a)^n\right]\left[\sum_{m=0}^{\infty} \beta_m (x-a)^n\right]$$

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{\infty} \alpha_n \beta_{n-m}\right)(x-a)^n$$