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Frobenius Method

$$x^2 y'' + x\,p(x)\,y' + Q\,y = 0 \quad \text{--- (1)}$$

$x = 0 \longrightarrow$ R.S.P , $P(x)$ & $Q(x)$ are analytic for $|x| < R$
$R > 0.$

Trial sol^n

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \quad \text{--- (1)}$$

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

$$y = a_0 + a_1 (x - x_0)^1 + a_2 (x - x_0)^2$$

$$y' = \sum_{n=1}^{\infty}$$

$$y'(x) = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Let

$$P(x) = c_0 + c_1 x + c_2 x^2 + \dots$$

$$Q(x) = d_0 + d_1 x + d_1 x^2 + \dots$$

Substituting the values of y, y', y'' in (1)

$$\text{Eq(1)} \Rightarrow \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2+2} + (c_0 + c_1 x + c_2 x^2 + \dots) \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1+1}$$

$$+ (d_0 + d_1 x + d_2 x^2 + \dots) \sum_{n=0}^{\infty} a_n x^{n+r} = 0 \quad \text{--- (3)}$$

Since eq (3) is an indentity, we can equate to zero the coefficient of various power of x

$\therefore$ The smallest of x is x^r and the correspondly equation is

$$\underline{[r(r-1) + c_0 r + d_0]}\, a_0 = 0$$

Since $a_0 \neq 0$ therefore $r(r-1) + c_0 r + d_0 = 0$

$$\Rightarrow \underbrace{r^2 + (c_0 - 1) r + d_0 = 0}$$

$\downarrow$

This is known as indicial equation of (1)

$\rightarrow$ Roots, we consider as r_1 & r_2

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→ Roots, we consider as r_1 & r_2

Then based on r_1 & r_2 there will be four different possibilities, which are discussed in following:

Case I   Roots (r_1 & r_2) of indicial equation not equal and difference is not an integer

r_1 & r_2 $r_1 - r_2 \neq$ integer

Then the computed solution is given by

$$y(x) = A\,[y(x)]_{r=r_1} + B\,[y(x)]_{r=r_2} \qquad 0<x<R$$

A, B are arbitrary constants.

Case II

r_1 & r_2 and $r_1 = r_2$

$$y(x) = A\,[y(x)]_{r=r_1} + B\left(\frac{\partial y(x)}{\partial r}\right)_{r=r_1} \qquad 0<x<R$$

Case III   If roots are unequal and differing by an integer and making a coefficient of y infinite.

r_1 & r_2 $r_1 - r_2 =$ integer,

if some of the coefficient of $y(x)$ becomes infinite when $r = r_1$.

we modify the form of $y(x)$ by replacing a_0 by $b_0(r-r_1)$.

Then we obtain two independent soln by putting $r=r_1$ in the modified form of $y(x)$ and $\frac{\partial y}{\partial x}$, $0<x<R$

→ The result of putting $r=r_2$ in $y(x)$ gives a numerical multiple of that obtained result by putting $r=r_1$ and hence, we reject the soln obtained by putting $r=r_2$ in $y(x)$

Ex $$2x\frac{d^2y}{dx^2} + (x+1)\frac{dy}{dx} + 3y = 0 \longrightarrow \text{F. M.}$$

$\Rightarrow \quad 2x \frac{d^2y}{dx^2} + (x+1)\frac{dy}{dx} \ldots \longrightarrow$ F. M.

Sol^n $\underline{x=0}$ is a regular point

$$P = \frac{x+1}{2x} \Rightarrow xP(x) = \frac{x+1}{2}$$

$$Q = \frac{3y}{2x} \Rightarrow x^2 Q(x) = \frac{3x}{2}$$

$$\left\{ \begin{array}{l} (x-x_0)P(x) \\ (x-x_0)^2 Q(x) \end{array} \right.$$

$$y = \sum_{n=0}^{\infty} C_n x^{\rho+n} \qquad \text{here } \underline{r=\rho} \quad \underline{0<x<\infty}$$

$$y' = \frac{dy}{dx} = \sum_{n=0}^{\infty} (\rho+n) C_n x^{\rho+n-1}$$

$$y'' = \sum_{n=0}^{\infty} (\rho+n)(\rho+n-1) C_n x^{\rho+n-2}$$

$$2xy'' + (x+1)y' + 3y = 0$$

$$\Rightarrow 2\sum_{n=0}^{\infty} C_n (\rho+n)(\rho+n-1) x^{n+\rho-1} + \sum_{n=0}^{\infty} C_n (\rho+n) x^{\rho+n} + \sum_{n=0}^{\infty} C_n (\rho+n) x^{\rho+n-1} + 3\sum_{n=0}^{\infty} C_n x^{\rho+n} = 0$$

Collecting the coefficient of $x^{\rho-1}$

$$2C_0\rho(\rho-1) + C_0\rho = 0$$

$$\Rightarrow C_0\{2\rho^2 - 2\rho + \rho\} = 0$$

$$\Rightarrow C_0\rho(2\rho-1) = 0 \Rightarrow \underline{C_0 \neq 0} \quad \rho = 0, \ \rho = \frac{1}{2}$$

$$r_1 = \rho = 0 \qquad r_2 = \rho = \frac{1}{2}$$

Coefficient of x^{ρ} and $x^{\rho+n}$

$x^{\rho+n}$

$$= \frac{C_{n+1}}{C_n} = -\frac{\rho+n+3}{(\rho+n+1)(2\rho+2n+1)}$$

x^{ρ}

$$\frac{C_1}{C_0} = \frac{-\rho+3}{(\rho+1)(2\rho+1)}$$

$$n = 1 \qquad C_2 = -\frac{(\rho+4)}{(\rho+2)(2\rho+3)} C_1$$

$$= \frac{(\rho+3)(\rho+4)}{(\rho+1)(\rho+2)(2\rho+3)(2\rho+1)} c_0$$

Now $y = c_0 x^\rho \left[1 + \frac{c_1}{c_0} x + \frac{c_2}{c_0} x^2 + \cdots\right]$

$$= c_0 x^\rho \left[1 - \frac{\rho+3}{(\rho+1)(2\rho+1)} x + \frac{(\rho+3)(\rho+4)}{(\rho+1)(\rho+2)(2\rho+1)(2\rho+3)} x^2 + \cdots\right] \quad (7)$$

$y = A y_1 + B y_2$ ✓

$= A [y(x)]_{r = r_1 = 0} + B [y]_{r = r_2 = \frac{1}{2}}$

Where

$[y]_{\rho=0} = y_1 = c_0 \left[1 - 3x + 2x^2 + \cdots\right]$

$[y]_{\rho=1/2} = y_2 = x^{1/2} c_0 \left[1 - \frac{7}{6} x + \frac{21}{40} x^2 + \cdots\right]$