MA 204 Numerical Methods

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Lecture-2

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Contents

- Solution of a nonlinear equation, bisection and secant methods, Newton's method, rate of convergence, solution of a system of nonlinear equations.

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- Interpolation by polynomials, divided differences, error of the interpolating polynomial, piecewise linear and cubic spline interpolation.

Motivation

Nonlinearity

One of the most frequent problem in engineering and science is to find the root(s) of a non-linear equation

$$f(x) = 0. \tag{1}$$

Here,

- $f : [a, b] \to \mathbb{R}$ is a nonlinear function in x;
- $f \in C^1[a, b]$;
- Roots are **isolated**.

Root of the equation

Definition 1

Given a nonlinear function $f : [a, b] \to \mathbb{R}$, find a value of r for which $f(r) = 0$. Such a solution value for r is called a **root** of the equation

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Approximation to a root

A point $x^\star \in \mathbb{R}$ such that

- $|r - x^\star|$ is **very small**, and
- $f(x^\star)$ is **very close** to 0.

Types of Iterative Methods

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   Advantages: Always converges

   Disadvantages: Locating the root initially
2. Closed Domain Methods (Non- Bracketing Methods)

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    - Secant method
    - Fixed point theorem
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2. Closed Domain Methods (Non- Bracketing Methods)
   - Secant method
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   Advantages: No need to locate the root initially

   Disadvantages: May not converge

For each of the method, we study the following:

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- Computational Scheme
- Application and example

Secant Method: Basic idea

This method is based on Mean Value Theorem.

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- Equation of the secant line

$$y - f(x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_1) \tag{2}$$

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$$\Longrightarrow \boxed{x_2 = x_1 - \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x_2 - x_1)}. \tag{3}$$

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- Having found x_2, we can drop x_0 and use x_1, x_2 as a new set of approximate value for α. This leads to an improved value x_3; and this process can be continued indefinitely.

Secant Method: General Iteration

- The general iteration formula for the secant method is

$$x_{n+1} = x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \quad n \geq 1. \tag{4}$$

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- The sequence of iterates does not need to converge to root of the function. In fact it might also diverge.Then, **why does one use secant method instead of bisection method, which gives the security of convergence?**
- It is called a two-point method, since two approximation values are needed to obtain an improved value.
- The Bisection method is also a two-point method, but the Secant method will almost always converge faster than bisection.

- Note that only one function evaluation is needed per step for the Secant method after x_2 has been determined. In contrast, each step of Newton's method requires an evaluation of both the function and its derivative.

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- Newton's method or the Secant method is often used to refine an answer obtained by another technique, such as the bisection method, since these methods requires good first approximation but generally give rapid convergence.

How fast is the convergence?

Definition 2

Let $\{x_n\}_{n\geq 1}$ be a sequence that converges to α. If positive constants λ and p exist with

$$\lim_{n\to\infty} \frac{|x_{n+1}-\alpha|}{|x_n-\alpha|^p} = \lambda,$$

then $\{x_n\}_{n\geq 1}$ is said to **converge to α of order p, with assymptotic error constant** λ. If $p=1$, the method is called **linear**. If $p=2$, the method is called **quadratic**.

How fast is the convergence?

- Can we find the exponent p such that

$$|x_{n+1} - \alpha| \approx |x_n - \alpha|^p? \tag{5}$$

- **Answer:** $p = \frac{1}{2}(1 + \sqrt{5}) \approx 1.62$. This is called **super linear convergence** $(1 < p < 2)$.

Error analysis of the iteration and convergence

- The general iteration formula for the secant method is

$$x_{n+1} = x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \quad n \geq 1. \tag{6}$$

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- Let $\varepsilon_n = x_n - \alpha$ and $\varepsilon_{n+1} = x_{n+1} - \alpha$. So, ε_n and ε_{n+1} denote the errors in the root at the nth and (n+1)th iterations.

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- (6) $\implies$

$$\varepsilon_{n+1} = \varepsilon_n - \frac{f(\alpha + \varepsilon_n)(\varepsilon_{n+1} - \varepsilon_n)}{f(\alpha + \varepsilon_n) - f(\alpha + \varepsilon_{n-1})} \tag{7}$$

Error analysis of the iteration and convergence

- Assume that f is twice differentiable and $f'(\alpha), f''(\alpha) \neq 0$. By Taylor's formula (with very small ε)

$$f(\alpha+\varepsilon) = f(\alpha) + \varepsilon f'(\alpha) + \frac{\varepsilon^2}{2} f''(\alpha) + R_2(\varepsilon).$$

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  Now, $f(\alpha) = 0$, and ε is very small, and $R_2(\varepsilon)$ is the remainder term. $R_2(\varepsilon)$ vanishes as $\varepsilon \to 0$ at a faster rate than ε^2. Therefore,

$$f(\alpha+\varepsilon) \approx \varepsilon f'(\alpha) + \frac{\varepsilon^2}{2} f''(\alpha).$$

Error analysis of the iteration and convergence

- Let $M = \dfrac{f''(\alpha)}{2f'(\alpha)}$.
- $$f(\alpha + \varepsilon_n) \approx \varepsilon_n f'(\alpha)\Big(1 + \varepsilon_n M\Big).$$

$$\varepsilon_{n+1} \approx \frac{f''(\alpha)}{2f'(\alpha)} \varepsilon_{n-1}\varepsilon_n \tag{8}$$

- (8) tells us that, as $n \to \infty$, the error tends to zero faster than a linear function but not quadratically!

Error Estimate

By Mean Value Theorem,

$$f(\alpha) - f(x_n) = f'(c_n)(\alpha - x_n) \tag{9}$$

where c_n lies between x_n and α. So, if $x_n \to \alpha$, then $c_n \approx x_n$ for large n, and we have

$$\begin{aligned}
\alpha - x_n &\approx -\frac{f(x_n)}{f'(c_n)} \\
&\approx -\frac{f(x_n)}{f'(x_n)} \\
&\approx -f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \\
&\approx x_{n+1} - x_n.
\end{aligned}$$

Thus, $\boxed{\alpha - x_n \approx x_{n+1} - x_n}$.