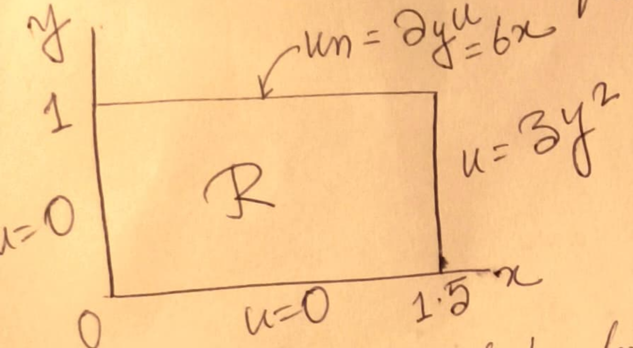
Neumann and Mixed Problems: Irregular Boundary

Example (Mixed Boundary Value Problem for a Poisson Eqⁿ).

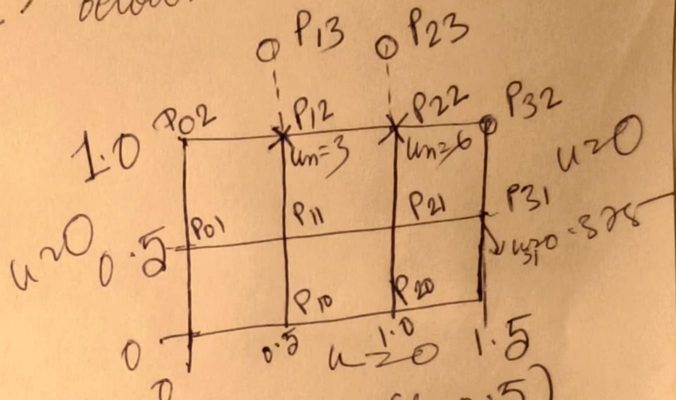
Solve the mixed boundary value problem for the Poisson eqⁿ.

$$\Delta u = \partial_{xx} u + \partial_{yy} u = f(x,y) = 12xy. \quad \text{—} (*)$$

shown in the figure below.



(a) Region R and boundary values



Grid ($h=0.5$)

<u>Solution.</u> Since, $h = 0.5$, we have,

the R.H.S of $(*)$ =

$$h^2 f(x,y) = 0.5^2 \times 12xy = 3xy.$$

Further the boundary data at grid points are computed as,

$$(b) \text{ — } \quad u_{31} = 0.375, \quad u_{32} = 3,$$

$$\partial_y u_{12} = 6 \times 0.5 = 3.$$

$$\partial_y u_{22} = 6 \times 1 = 6.$$

Now the five point formula applied to the mesh points P_{11} and P_{21}:

$$⑦ \left\{ \begin{array}{l} -4u_{11} + \underbrace{u_{21} + u_{12}}_{\text{unknown}} = 12 \cdot (0.5 \times 0.5) \cdot (0.5)^2 - 0 = 0.75 \\ \\ u_{11} - 4u_{21} + \underbrace{u_{22}}_{\text{unknown}} = 12 \cdot (1 \times 0.5) \cdot (0.5)^2 - 0.375 = 1.125 \end{array} \right.$$

<u>Idea</u>: We imagine the region R to be extended above to the first row of external mesh points (corresponding to $y = 1.5$), and we assume that the Poisson equation also holds in the extended region. Then we can write down ~~two~~ more equations at P_{12} and P_{22} using five-point formulae:

$$⑧ \left\{ u_{11} - 4u_{12} + u_{22} + u_{13} = 12 \cdot (0.5 \times 1) \frac{1}{4} - 0 = \right.$$

(8) $u_{41} - 4u_{42} + u_{22} + u_{43} = 12 \times (0.5) \times 1 \times \frac{1}{4} - 0$
$= 1.5$

$u_{21} + u_{42} - 4u_{22} + u_{23} = 12 \times 1 \times 1 \times \frac{1}{4} - 3$
$= 0$

Notice that we have introduced two new unknowns, u_{43} and u_{23}. To get rid of them we can use the following central difference formulae and the information about $\partial_y u$ at u_{42} and u_{22};

(9)
$$3 = \partial_y u_{42} \approx \frac{u_{43} - u_{41}}{2h} = u_{43} - u_{41}$$
$$\Rightarrow u_{43} = u_{41} + 3$$

and
$$6 = \partial_y u_{22} \approx \frac{u_{23} - u_{21}}{2h} = u_{23} - u_{21}$$
$$\Rightarrow u_{23} = u_{21} + 6.$$

Substituting (9) into (8) one has:

$$(10)\quad \begin{cases} 2u_{41} - 4u_{42} + u_{22} = 1.5 - 3 = -1.5 \\ 2u_{21} + u_{42} - 4u_{22} = 3 - 3 - 6 = -6 \end{cases}$$

(7) and (10) together yields:

$$\begin{pmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 2 & 0 & -4 & 1 \\ 0 & 2 & 1 & -4 \end{pmatrix} \begin{pmatrix} u_{41} \\ u_{21} \\ u_{42} \\ u_{22} \end{pmatrix} = \begin{pmatrix} 0.75 \\ 1.125 \\ 1.5-3 \\ 0-6 \end{pmatrix} = \begin{pmatrix} 0.75 \\ 1.125 \\ -1.5 \\ -6 \end{pmatrix}.$$

(11)

The solution of (11) obtained by Gauss-elimination is as follows (Home-work):

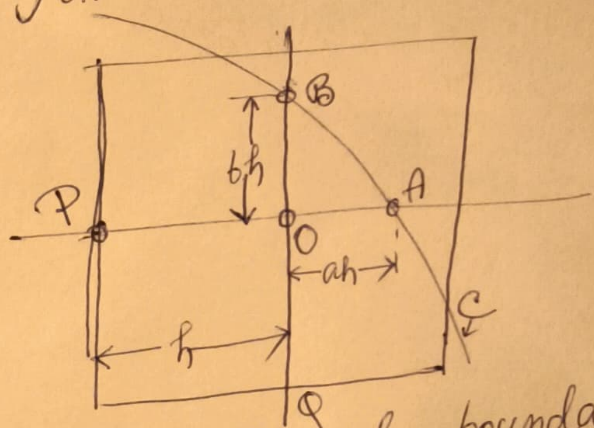
$$\begin{cases} u_{42} = 0.866, & u_{22} = 1.812 \\ u_{41} = 0.077, & u_{21} = 0.191 \end{cases}$$

Irregular Boundary:

So far we have obtained difference formulae for Laplace/Poisson equation applicable in ~~si~~ a simple Geometry (ie) square/rectangle.

What if the boundary C of R doesnot intersect the grid at points that are not mesh points.?

For instance lets consider:



$$\boxed{0 < a, b < 1}$$

Figure: Curved boundary C of a region R, a mesh point O near C, and neighbors A, B, P, Q.

Using Taylor's formulae:

(12)
(a) $u_A = u_0 + ah\,\partial_x u_0 + \frac{1}{2}(ah)^2\,\partial_{xx} u_0 + \cdots$

(b) $u_P = u_0 - h\,\partial_x u_0 + \frac{1}{2}h^2\,\partial_{xx} u_0 + \cdots$

Disregarding higher powers of h and adding (12a) and (12b)

$$\Rightarrow \quad u_A + a u_p \approx (1+a) u_O + \frac{1}{2} a(a+1) h^2 \partial_{xx} u_O$$

$$\Rightarrow \partial_{xx} u_O = \frac{\partial^2 u_O}{\partial x^2} \approx \frac{2}{h^2}\left[\frac{1}{a(1+a)} u_A + \frac{1}{1+a} u_p - \frac{1}{a} u_O\right] \quad (13)$$

Similarly, by considering the points O, B and Q:

$$\partial_{yy} u_O \approx \frac{2}{h^2}\left[\frac{1}{b(1+b)} u_B + \frac{1}{1+b} u_Q - \frac{1}{b} u_O\right]. \quad (14)$$

Adding (13) and (14):

$$\Delta u_O \approx \frac{2}{h^2}\left[\frac{u_A}{a(1+a)} + \frac{u_B}{b(1+b)} + \frac{u_p}{(1+a)} + \frac{u_Q}{(1+b)} - \frac{(a+b) u_O}{ab}\right].$$

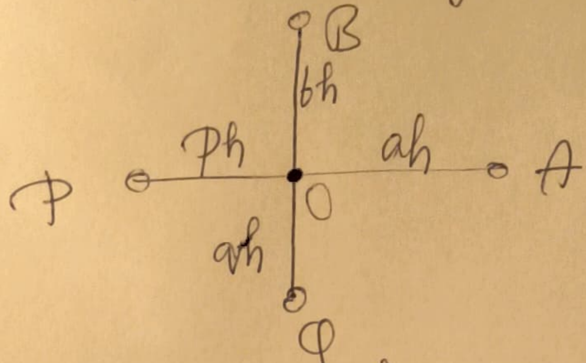
For example, if $a = 1/2$, $b = 1/2$, instead of the usual stencil

$$\begin{Bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{Bmatrix}$$

we now have,

$$\begin{Bmatrix} & 4/3 & \\ 2/3 & -4 & 4/3 \\ & 2/3 & \end{Bmatrix}.$$

Hence, for a more general setup-



On has the following approximation :

$$\Delta u_O \approx \frac{2}{h^2}\left[\frac{u_A}{a(a+p)} + \frac{u_B}{b(b+q)} + \frac{u_P}{p(p+a)} + \frac{u_Q}{q(q+b)} - \frac{ap+bq}{abpq}\, u_O\right].$$

Dirichlet Problem for the Laplace Equation.
Curved Boundary:

Find the potential in the following region with the boundary values given.

Here the curved portion of the boundary is an arc of the circle of radius 10 about (0,0). Use the grid in the figure.

Solution:

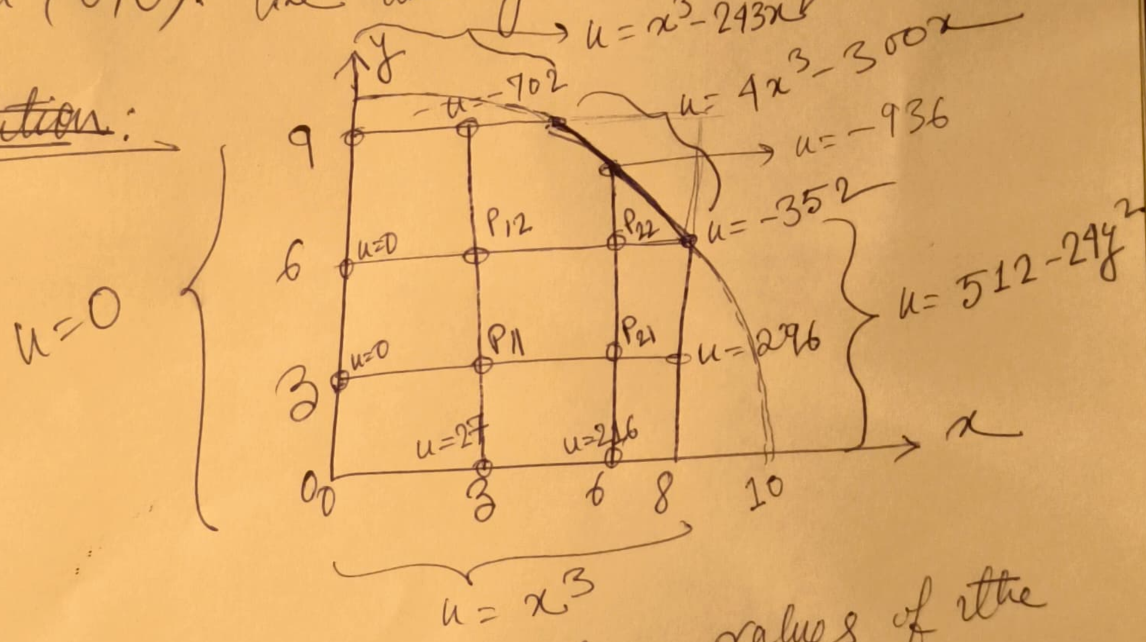


Figure: Region, boundary values of the potential and grid points.

Solution For, P_{11} and P_{12} we have the usual regular stencil

$$P_{11}, P_{12}: \left\{ \begin{matrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{matrix} \right\}$$

For, P_{21} and P_{22} we use (12) ~~to obtain the stencils~~,

Particularly, for, P_{21} : $P = 1$, $a = \frac{2}{3}$, $q = 1$,
~~$b = \frac{2}{3}$~~ $\cdot$ $b = 1$

~~Δu_{21}~~ Δu_{21}

$$\cancel{\boxed{\approx \frac{2}{9}\left[\frac{\overbrace{296}^{u_A}}{\frac{2}{3}\left(\frac{2}{3}+1\right)} + \frac{u_{22}}{\frac{2}{3}\left(\frac{2}{3}+1\right)} + \frac{u_{11}}{1(1+2/3)} + \frac{\overbrace{216}^{u_B}}{1(1+2/3)} - \frac{\frac{2}{3}\cdot 1 + \cancel{\frac{2}{3}}\cdot 1}{\frac{2}{3}\cdot\cancel{\frac{2}{3}}\cdot 1\cdot 1} u_{21}\right]}}$$

$$\Delta u_{21} \approx \frac{2}{9}\left[\frac{\overbrace{296}^{u_A}}{\frac{2}{3}\left(\frac{2}{3}+1\right)} + \frac{u_{22}}{1(1+1)} + \frac{u_{11}}{1(1+2/3)} + \frac{\overbrace{216}^{u_B}}{1(1+1)} - \frac{\frac{2}{3}\cdot 1 + 1\cdot 1}{\frac{2}{3}\cdot 1\cdot 1\cdot 1} u_{21}\right]$$

(ii) at P_{21} the stencil is:

$$P_{21} : \left\{\begin{array}{ccc} & 0.5 & \\ 0.6 & -2.5 & 0.9 \\ & 0.5 & \end{array}\right\}$$

A, P_{22} : $p = 1,\ b = 2/3,\ a = 2/3,\ q = 1.$

Hence applying (12)

$$\Delta u_{22} \approx \frac{2}{9}\left[\frac{\overbrace{-352}^{u_A}}{2/3(2/3+1)} + \frac{\overbrace{-936}^{u_B}}{2/3(2/3+1)} + \frac{u_{42}}{1(2/3+1)} + \frac{u_{21}}{1\cdot(2/3+1)} \cancel{-\ 2/3 \cdot 2/3 +} - \frac{2/3 \cdot 1 + 2/3 \cdot 1}{2/3 \cdot 2/3 \cdot 1.1}\, u_{22}\right]$$

(ii) at P_{22} the stencil is,

$$P_{22} : \left\{\begin{matrix} & 0.9 & \\ 0.6 & -3 & 0.9 \\ & 0.6 & \end{matrix}\right\}$$

Hence at the mesh points P_{11}, P_{21}, P_{12} and P_{22} we have the system of equations:

$$-4u_{11} + u_{21} + u_{12} = 0 - 27 = -27$$

$$0.6u_{11} - 2.5u_{21} + 0.5u_{22} = -0.9\cdot 296 - 0.5\cdot 216 = -374.4$$

$$u_{11} - 4u_{12} + u_{22} = 702 + 0 = 702$$

$$0.6u_{21} + 0.6u_{12} - 3u_{22} = 0.9\times 352 + 0.9\times 936 = 1159.2$$

$\Downarrow$

$$\begin{pmatrix} -4 & 1 & 1 & 0 \\ 0.6 & -2.5 & 0 & 0.5 \\ 1 & 0 & -4 & 1 \\ 0 & 0.6 & 0.6 & -3 \end{pmatrix} \begin{pmatrix} u_{11} \\ u_{21} \\ u_{12} \\ u_{22} \end{pmatrix} = \begin{pmatrix} -27 \\ -374.4 \\ 702 \\ 1159.2 \end{pmatrix}.$$

Exercise; Gauss-Elimination yields,

$$u_{11} \approx -55.6, \quad u_{21} \approx 49.2$$

$$u_{12} \approx -298.5, \quad u_{22} \approx -436.3.$$