PROBLEM SET I FOR NUMERICAL PARTIAL DIFFERENTIAL EQUATIONS
COURSE TITLE: NUMERICAL METHODS,
COURSE CODE: MA 204
IIT INDORE

(1) Classify the following PDEs into hyperbolic, parabolic or elliptic.

$$
\begin{aligned}
&(a) \quad 2\,\partial_{xx}u + 4\,\partial_{xy}u + 3\,\partial_{yy}u = 0,\\
&(b) \quad \partial_{xx}u + 4\,\partial_{xy}u + 4\,\partial_{yy}u = 0,\\
&(c) \quad 2\,\partial_{xx}u + 6\,\partial_{xy}u + 3\,\partial_{yy}u = 0,\\
&(d) \quad t\,\partial_{tt}u + 2\,\partial_{xt}u + x\,\partial_{xx}u + \partial_{x}u = 0,\\
&(e) \quad \partial_{tt}u + t\,\partial_{xt}u + x\,\partial_{xx}u + 2\,\partial_{t}u + \partial_{x}u + 6u = 0.
\end{aligned}
$$

(2) (a) Choose appropriate characteristic variables to reduce a general second order parabolic PDE (as introduced in class) into canonical form.

(b) Further show that the one dimensional heat equation:

$$\partial_t u = \alpha\,\partial_{xx}u,\ \ \alpha > 0 \tag{1}$$

is parabolic and reduce (1) into canonical form using characteristic variables.

(3) (d'Alembert's solution satisfying the initial conditions): Recall the general solution formula of the wave equation obtained in class and next determine the solution of the following

$$\partial_{tt}u = c^2\partial_{xx}u,\ \ t > 0,\ \ -\infty < x < \infty \tag{2}$$

in an infinite one-dimensional medium subject to the initial conditions

$$u(x,0) = f(x),\ \ \partial_t u(x,0) = g(x) \text{ for } -\infty < x < \infty. \tag{3}$$

(4) Show that the Laplace's operator is invariant under rotation in two dimensions.

Hints: Perform the change of variables:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{4}$$

and next use chain rule of derivatives to rewrite the Laplace's operator in the newly defined variables.

(5) Solve Laplace equation (approximately) for the square region shown below (Figure 1) using Gauss-Seidel iteration scheme applied to the five point formula. Note that the boundary values of the unknown are as indicated in the figure.

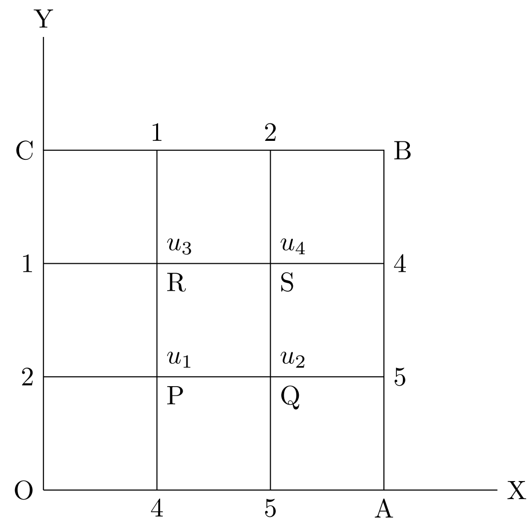

FIGURE 1. Domain Figure 1.

(6) Solve the Poisson equation (approximately) by using Gauss-Seidel iteration scheme applied to the five point formula

$$\partial_{xx}u + \partial_{yy}u = -10(x^2 + y^2 + 10) \tag{5}$$

in the following domain (Figure 2). Begin with the initial iterate $u_4^0 = u_2^0 = u_3^0 = 0$.

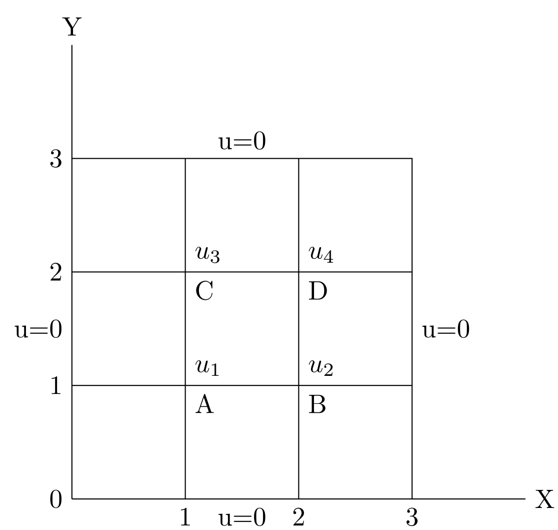

FIGURE 2. Domain Figure 2.

(7) Write a Matlab code to solve the Laplace equation (approximately) at the interior mesh points of the following figure by using Gauss-Seidel iteration scheme applied to the five point formula. (*Recall:* We already have solved for the first iterate during one of the lecture sessions)

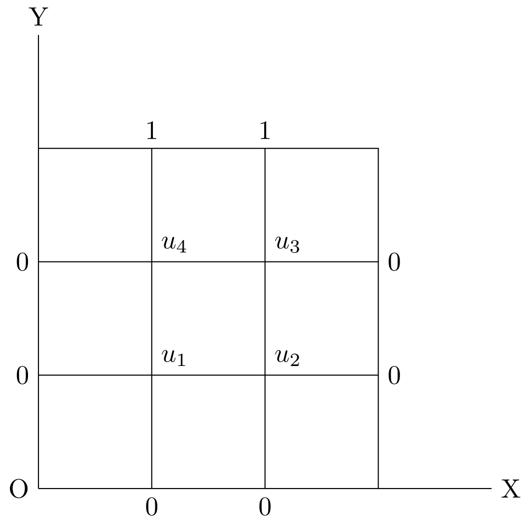

FIGURE 3. Domain Figure 3.