Indian Institute of Technology Indore

MA-204 Numerical Methods

Assignment -2- System of Linear equations

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1. GAUSS JACOBI METHOD – (1,0,0) – INITIAL GUESS

1. Solve the following system of equations by Gauss-Jacobi method with initial approximation other than $(1, 1, 1)$:

$$\begin{aligned} 8x + 2y - 2z &= 8 \\ x - 8y + 3z &= -4 \\ 2x + y + 9z &= 12\,. \end{aligned}$$

Hint: use 4.29 or 4.27 to find first three iterations.

6. Solve Equations 8x+2y-2z=8,x-8y+3z=-4,2x+y+9z=12 using Gauss Jacobi method

Solution:
Total Equations are 3

$8x + 2y - 2z = 8$

$x - 8y + 3z = -4$

$2x + y + 9z = 12$

From the above equations

$$x_{k+1} = \frac{1}{8}\left(8 - 2y_k + 2z_k\right)$$

$$y_{k+1} = \frac{1}{-8}\left(-4 - x_k - 3z_k\right)$$

$$z_{k+1} = \frac{1}{9}\left(12 - 2x_k - y_k\right)$$

Initial gauss $(x, y, z) = (1, 0, 0)$

Solution steps are
1^{st} Approximation

$$x_1 = \frac{1}{8}[8 - 2(0) + 2(0)] = \frac{1}{8}[8] = 1$$

$$y_1 = \frac{1}{-8}[-4 - (1) - 3(0)] = \frac{1}{-8}[-5] = 0.625$$

$$z_1 = \frac{1}{9}[12 - 2(1) - (0)] = \frac{1}{9}[10] = 1.1111$$

2^{nd} Approximation

$$x_2 = \frac{1}{8}[8 - 2(0.625) + 2(1.1111)] = \frac{1}{8}[8.9722] = 1.1215$$

$$y_2 = \frac{1}{-8}[-4 - (1) - 3(1.1111)] = \frac{1}{-8}[-8.3333] = 1.0417$$

$$z_2 = \frac{1}{9}[12 - 2(1) - (0.625)] = \frac{1}{9}[9.375] = 1.0417$$

3^{rd} Approximation

$$x_3 = \frac{1}{8}[8 - 2(1.0417) + 2(1.0417)] = \frac{1}{8}[8] = 1$$

$$y_3 = \frac{1}{-8}[-4 - (1.1215) - 3(1.0417)] = \frac{1}{-8}[-8.2465] = 1.0308$$

$$z_3 = \frac{1}{9}[12 - 2(1.1215) - (1.0417)] = \frac{1}{9}[8.7153] = 0.9684$$

4^{th} Approximation

$$x_4 = \frac{1}{8}[8 - 2(1.0308) + 2(0.9684)] = \frac{1}{8}[7.8751] = 0.9844$$

$$y_4 = \frac{1}{-8}[-4 - (1) - 3(0.9684)] = \frac{1}{-8}[-7.9051] = 0.9881$$

$$z_4 = \frac{1}{9}[12 - 2(1) - (1.0308)] = \frac{1}{9}[8.9692] = 0.9966$$

5^{th} Approximation

$$x_5 = \frac{1}{8}[8 - 2(0.9881) + 2(0.9966)] = \frac{1}{8}[8.0169] = 1.0021$$

$$y_5 = \frac{1}{-8}[-4 - (0.9844) - 3(0.9966)] = \frac{1}{-8}[-7.9741] = 0.9968$$

$$z_5 = \frac{1}{9}[12 - 2(0.9844) - (0.9881)] = \frac{1}{9}[9.0431] = 1.0048$$

6^{th} Approximation

$$x_6 = \frac{1}{8}[8 - 2(0.9968) + 2(1.0048)] = \frac{1}{8}[8.016] = 1.002$$

$$y_6 = \frac{1}{-8}[-4 - (1.0021) - 3(1.0048)] = \frac{1}{-8}[-8.0165] = 1.0021$$

$$z_6 = \frac{1}{9}[12 - 2(1.0021) - (0.9968)] = \frac{1}{9}[8.999] = 0.9999$$

7^{th} Approximation

$$x_7 = \frac{1}{8}[8 - 2(1.0021) + 2(0.9999)] = \frac{1}{8}[7.9957] = 0.9995$$

$$y_7 = \frac{1}{-8}[-4 - (1.002) - 3(0.9999)] = \frac{1}{-8}[-8.0017] = 1.0002$$

$$z_7 = \frac{1}{9}[12 - 2(1.002) - (1.0021)] = \frac{1}{9}[8.9939] = 0.9993$$

8^{th} Approximation

$$x_8 = \frac{1}{8}[8 - 2(1.0002) + 2(0.9993)] = \frac{1}{8}[7.9982] = 0.9998$$

$$y_8 = \frac{1}{-8}[-4 - (0.9995) - 3(0.9993)] = \frac{1}{-8}[-7.9974] = 0.9997$$

$$z_8 = \frac{1}{9}[12 - 2(0.9995) - (1.0002)] = \frac{1}{9}[9.0009] = 1.0001$$

Solution By Gauss Jacobi Method.
$x = 0.9998 \cong 1$

$y = 0.9997 \cong 1$

$z = 1.0001 \cong 1$

Iterations are tabulated as below

Iteration	x	y	z
1	1	0.625	1.1111
2	1.1215	1.0417	1.0417
3	1	1.0308	0.9684
4	0.9844	0.9881	0.9966
5	1.0021	0.9968	1.0048
6	1.002	1.0021	0.9999
7	0.9995	1.0002	0.9993
8	0.9998	0.9997	1.0001

2. GAUSS JACOBI METHOD – (0,0,1) – INITIAL GUESS

2. Solve the following linear system by Jacobi method with $X^{(0)} = (0,0,1)^T$. Find out three iterations:

$$10x_1 + 3x_2 + x_3 = 14$$
$$2x_1 - 10x_2 + 3x_3 = -5$$
$$x_1 + 3x_2 + 10x_3 = 14\,.$$

Hint: use 4.29 or 4.27 to find iterations.

Solution:

Total Equations are 3

$10x + 3y + z = 14$

$2x - 10y + 3z = -5$

$x + 3y + 10z = 14$

From the above equations

$$x_{k+1} = \frac{1}{10}\left(14 - 3y_k - z_k\right)$$

$$y_{k+1} = \frac{1}{-10}\left(-5 - 2x_k - 3z_k\right)$$

$$z_{k+1} = \frac{1}{10}\left(14 - x_k - 3y_k\right)$$

Initial gauss $(x, y, z) = (0, 0, 1)$

Solution steps are
1^{st} Approximation

$$x_1 = \frac{1}{10}[14 - 3(0) - (1)] = \frac{1}{10}[13] = 1.3$$

$$y_1 = \frac{1}{-10}[-5 - 2(0) - 3(1)] = \frac{1}{-10}[-8] = 0.8$$

$$z_1 = \frac{1}{10}[14 - (0) - 3(0)] = \frac{1}{10}[14] = 1.4$$

2^{nd} Approximation

$$x_2 = \frac{1}{10}[14 - 3(0.8) - (1.4)] = \frac{1}{10}[10.2] = 1.02$$

$$y_2 = \frac{1}{-10}[-5 - 2(1.3) - 3(1.4)] = \frac{1}{-10}[-11.8] = 1.18$$

$$z_2 = \frac{1}{10}[14 - (1.3) - 3(0.8)] = \frac{1}{10}[10.3] = 1.03$$

3^{rd} Approximation

$$x_3 = \frac{1}{10}[14 - 3(1.18) - (1.03)] = \frac{1}{10}[9.43] = 0.943$$

$$y_3 = \frac{1}{-10}[-5 - 2(1.02) - 3(1.03)] = \frac{1}{-10}[-10.13] = 1.013$$

$$z_3 = \frac{1}{10}[14 - (1.02) - 3(1.18)] = \frac{1}{10}[9.44] = 0.944$$

4^{th} Approximation

$$x_4 = \frac{1}{10}[14 - 3(1.013) - (0.944)] = \frac{1}{10}[10.017] = 1.0017$$

$$y_4 = \frac{1}{-10}[-5 - 2(0.943) - 3(0.944)] = \frac{1}{-10}[-9.718] = 0.9718$$

$$z_4 = \frac{1}{10}[14 - (0.943) - 3(1.013)] = \frac{1}{10}[10.018] = 1.0018$$

5^{th} Approximation

$$x_5 = \frac{1}{10}[14 - 3(0.9718) - (1.0018)] = \frac{1}{10}[10.0828] = 1.0083$$

$$y_5 = \frac{1}{-10}[-5 - 2(1.0017) - 3(1.0018)] = \frac{1}{-10}[-10.0088] = 1.0009$$

$$z_5 = \frac{1}{10}[14 - (1.0017) - 3(0.9718)] = \frac{1}{10}[10.0829] = 1.0083$$

6^{th} Approximation

$$x_6 = \frac{1}{10}[14 - 3(1.0009) - (1.0083)] = \frac{1}{10}[9.9891] = 0.9989$$

$$y_6 = \frac{1}{-10}[-5 - 2(1.0083) - 3(1.0083)] = \frac{1}{-10}[-10.0414] = 1.0041$$

$$z_6 = \frac{1}{10}[14 - (1.0083) - 3(1.0009)] = \frac{1}{10}[9.9891] = 0.9989$$

7^{th} Approximation

$$x_7 = \frac{1}{10}[14 - 3(1.0041) - (0.9989)] = \frac{1}{10}[9.9887] = 0.9989$$

$$y_7 = \frac{1}{-10}[-5 - 2(0.9989) - 3(0.9989)] = \frac{1}{-10}[-9.9945] = 0.9995$$

$$z_7 = \frac{1}{10}[14 - (0.9989) - 3(1.0041)] = \frac{1}{10}[9.9887] = 0.9989$$

8^{th} Approximation

$$x_8 = \frac{1}{10}[14 - 3(0.9995) - (0.9989)] = \frac{1}{10}[10.0028] = 1.0003$$

$$y_8 = \frac{1}{-10}[-5 - 2(0.9989) - 3(0.9989)] = \frac{1}{-10}[-9.9943] = 0.9994$$

$$z_8 = \frac{1}{10}[14 - (0.9989) - 3(0.9995)] = \frac{1}{10}[10.0028] = 1.0003$$

9^{th} Approximation

$$x_9 = \frac{1}{10}[14 - 3(0.9994) - (1.0003)] = \frac{1}{10}[10.0014] = 1.0001$$

$$y_9 = \frac{1}{-10}[-5 - 2(1.0003) - 3(1.0003)] = \frac{1}{-10}[-10.0014] = 1.0001$$

$$z_9 = \frac{1}{10}[14 - (1.0003) - 3(0.9994)] = \frac{1}{10}[10.0014] = 1.0001$$

Solution By Gauss Jacobi Method.

$x = 1.0001 \cong 1$

$y = 1.0001 \cong 1$

$z = 1.0001 \cong 1$

Iterations are tabulated as below

Iteration	x	y	z
1	1.3	0.8	1.4
2	1.02	1.18	1.03
3	0.943	1.013	0.944
4	1.0017	0.9718	1.0018
5	1.0083	1.0009	1.0083
6	0.9989	1.0041	0.9989
7	0.9989	0.9995	0.9989
8	1.0003	0.9994	1.0003
9	1.0001	1.0001	1.0001

3. GAUSS-SEIDEL METHOD – (0,0,0) – INITIAL GUESS

3. Solve the system of equations

$$4x + y + 2z = 4$$
$$3x + 5y + z = 7$$
$$x + y + 3z = 3$$

by the Jacobi and Gauss-Seidel methods. In each case continue the iteration up to three steps starting with initial approximation $x = 0, y = 0, z = 0$. Hint: use 4.40 or 4.41 to find three iterations.

Solution:

Total Equations are 3

$4x + y + 2z = 4$

$3x + 5y + z = 7$

$x + y + 3z = 3$

From the above equations

$$x_{k+1} = \frac{1}{4}\left(4 - y_k - 2z_k\right)$$

$$y_{k+1} = \frac{1}{5}\left(7 - 3x_{k+1} - z_k\right)$$

$$z_{k+1} = \frac{1}{3}\left(3 - x_{k+1} - y_{k+1}\right)$$

Initial gauss $(x, y, z) = (0, 0, 0)$

Solution steps are
1^{st} Approximation

$$x_1 = \frac{1}{4}[4 - (0) - 2(0)] = \frac{1}{4}[4] = 1$$

$$y_1 = \frac{1}{5}[7 - 3(1) - (0)] = \frac{1}{5}[4] = 0.8$$

$$z_1 = \frac{1}{3}[3 - (1) - (0.8)] = \frac{1}{3}[1.2] = 0.4$$

2^{nd} Approximation

$$x_2 = \frac{1}{4}[4 - (0.8) - 2(0.4)] = \frac{1}{4}[2.4] = 0.6$$

$$y_2 = \frac{1}{5}[7 - 3(0.6) - (0.4)] = \frac{1}{5}[4.8] = 0.96$$

$$z_2 = \frac{1}{3}[3 - (0.6) - (0.96)] = \frac{1}{3}[1.44] = 0.48$$

3^{rd} Approximation

$$x_3 = \frac{1}{4}[4 - (0.96) - 2(0.48)] = \frac{1}{4}[2.08] = 0.52$$

$$y_3 = \frac{1}{5}[7 - 3(0.52) - (0.48)] = \frac{1}{5}[4.96] = 0.992$$

$$z_3 = \frac{1}{3}[3 - (0.52) - (0.992)] = \frac{1}{3}[1.488] = 0.496$$

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4. Strictly row diagonally dominant

4. Give an example of a matrix

   (a) which is Strictly-Row Diagonally Dominate but not Positive Definite.
   Hint: A strictly row diagonally dominant matrix satisfies $|a_{ii}| > \sum_{j\neq i, j=1,2,\ldots,n} |a_{ij}|, \forall i$
   $$\begin{pmatrix} -9 & 1 \\ 2 & 7 \end{pmatrix}$$

   (b) which is Symmetric Positive Definite matrix but not Strictly-Row Diagonally Dominate.
   $$\begin{pmatrix} 5 & 6 \\ 6 & 9 \end{pmatrix}$$

Recap

Tests For Positive Definite matrices

- All diagonal elements must be positive.
- All the leading principal determinants must be positive.

Tests For Positive Semi-definite Matrices

- All diagonal elements are non-negative.
- All the principal determinants are non-negative.

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Two by two symmetric matrices

Example

Let $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ *be a symmetric* 2×2 *matrix. Then the leading principal minors are* $D_1 = a$ *and* $D_2 = ac - b^2$*. If we want to find all the principal minors, these are given by* $\Delta_1 = a$ *and* $\Delta_1 = c$ *(of order one) and* $\Delta_2 = ac - b^2$ *(of order two).*

Let us compute what it means that the leading principal minors are positive for 2×2 matrices:

Example

Let $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ *be a symmetric* 2×2 *matrix. Show that if* $D_1 = a > 0$ *and* $D_2 = ac - b^2 > 0$*, then* A *is positive definite.*

Definiteness and principal minors

Theorem

Let A *be a symmetric* $n \times n$ *matrix. Then we have:*

- A *is positive definite* $\Leftrightarrow D_k > 0$ *for all leading principal minors*
- A *is negative definite* $\Leftrightarrow (-1)^k D_k > 0$ *for all leading principal minors*
- A *is positive semidefinite* $\Leftrightarrow \Delta_k \geq 0$ *for all principal minors*
- A *is negative semidefinite* $\Leftrightarrow (-1)^k \Delta_k \geq 0$ *for all principal minors*

5. CONVERGENCE OF JACOBI ITERATION

5. Rewrite the following system of equations in two different ways so that the Jacobi iteration scheme converges:

$$\begin{aligned}
3x_1 - 5x_2 + 47x_3 + 20x_4 &= 18 \\
12x_1 + 16x_2 + 17x_3 + 50x_4 &= 25 \\
17x_1 + 65x_2 - 13x_3 + 7x_4 &= 84 \\
56x_1 + 23x_2 + 11x_3 - 19x_4 &= 36\,.
\end{aligned}$$

Hint: If the coefficient matrix A is Strictly-Row Diagonally Dominant, Jacobi iteration method converges.

$$\begin{aligned}
56x_1 + 23x_2 + 11x_3 - 19x_4 &= 36 \\
17x_1 + 65x_2 - 13x_3 + 7x_4 &= 84 \\
3x_1 - 5x_2 + 47x_3 + 20x_4 &= 18 \\
12x_1 + 16x_2 + 17x_3 + 50x_4 &= 25\,.
\end{aligned}$$

$$\begin{aligned}
56x_1 + 23x_2 - 19x_4 + 11x_3 &= 36 \\
17x_1 + 65x_2 + 7x_4 - 13x_3 &= 84 \\
12x_1 + 16x_2 + 50x_4 + 17x_3 &= 25 \\
3x_1 - 5x_2 + 20x_4 + 47x_3 &= 18
\end{aligned}$$

$AX = b,\ X = [x_1, x_2, x_4, x_3]^t$

6. GAUSS-SEIDEL METHOD – (0,0,1) – INITIAL GUESS

6. Solve the following system of equations by Gauss-Seidel iterative method.

$$\begin{aligned}
2x - y &= 7 \\
-x + 2y - z &= 1 \\
-y + 2z &= 1
\end{aligned}$$

Hint: use 4.40 or 4.41 to find three iterations.

7. Minimum Condition Number

7. If $A = \begin{pmatrix} \alpha/10 & \alpha/10 \\ 1 & 3/2 \end{pmatrix}$, then show that cond (A) is minimum if $\alpha = 12.5$ and the minimum is 11.

Hint: Note that for $\alpha = 0$, A is not invertible, hence no question of condition number. Assume $\alpha \neq 0$, $A^{-1} = \begin{pmatrix} 30/\alpha & -2 \\ 20/\alpha & 2 \end{pmatrix}$, note that for the case $|\alpha| \geq 12.5$: $\|A\|_\infty = |\alpha|/5$, and the condition number $=\|A\|_\infty \|A^{-1}\|_\infty = |\alpha|/2.5 + 6$, which is minimum when $|\alpha| = 12.5$ and the minimum value of condition number is 11. Note that $\|A^{-1}\|_\infty = 2 + (30/|\alpha|)$
further for the case $|\alpha| \leq 12.5$: $\|A\|_\infty = 2.5$, and the condition number $=\|A\|_\infty \|A^{-1}\|_\infty = 5 + (75/|\alpha|)$, which is minimum when $|\alpha| = 12.5$ and the minimum value of condition number is 11.

Recap

$\|A\|_\infty$ = Maximum row sum (Absolute)

$$= \max_{1 \le i \le n} \sum_{j=1}^{n} |A_{ij}|$$

$\|A\|_1$ = Max column sum (Absolute)

$$= \max_{1 \le j \le n} \sum_{i=1}^{n} |A_{ij}|$$

7 $\quad A = \begin{pmatrix} \alpha/10 & \alpha/10 \\ 1 & 3/2 \end{pmatrix}$

Solution: cond(A) is not defined if $\alpha = 0$

Now, $\alpha \neq 0$

$$\|A\|_\infty = \max\left\{ \left|\frac{\alpha}{10}\right| + \left|\frac{\alpha}{10}\right|,\ 1 + \frac{3}{2} \right\} = \max\left\{ \frac{|\alpha|}{5}, \frac{5}{2} \right\}$$

Case-1

$\|A\|_\infty = \frac{|\alpha|}{5}$ (i.e. $\frac{|\alpha|}{5} \geq \frac{5}{2}$) $\Rightarrow |\alpha| \geq 12.5$

$$A^{-1} = \begin{pmatrix} 30/\alpha & -2 \\ 20/\alpha & 2 \end{pmatrix}$$

$$\|A^{-1}\|_\infty = \max\left\{ \frac{30}{|\alpha|} + 2,\ \frac{20}{|\alpha|} + 2 \right\}$$

$$= \cancel{\max\left\{ \frac{30 + 2|\alpha|}{|\alpha|}, \frac{20 + 2|\alpha|}{|\alpha|} \right\}}$$

$$= 2 + \frac{30}{|\alpha|}$$

$$\text{Cond}(A) = \|A\|_\infty \|A^{-1}\|_\infty = \frac{|\alpha|}{5} \times \left(2 + \frac{30}{|\alpha|}\right)$$

$$= \frac{|\alpha|}{2.5} + 6 = 11 \text{ (min if } |\alpha| = 12.5)$$

Case-2 $\quad \|A\|_\infty = \frac{5}{2}$

$\Rightarrow |\alpha| \leq 12.5$

$$\text{Cond}(A) = \frac{5}{2} \times \left(2 + \frac{30}{|\alpha|}\right)$$

$$= 5 + \frac{75}{|\alpha|}$$

$$= 5 + \frac{75}{12.5} \quad \left(\text{min if } |\alpha| = 12.5\right)$$

$$= 5 + 6 = 11$$

8. Condition Number

8. If $A = \begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix}$, $\alpha \neq 1$, then find cond (A).

Hint: Consider $|\alpha| \neq 1$, $A^{-1} = \frac{1}{1-\alpha^2}\begin{pmatrix} 1 & -\alpha \\ -\alpha & 1 \end{pmatrix}$, $\|A^{-1}\|_\infty = \frac{1+|\alpha|}{|1-\alpha^2|}$, hence condition no$=\frac{(1+|\alpha|)^2}{|1-\alpha^2|} = \frac{(1+|\alpha|)}{|(1-|\alpha|)|}$

(8) $A = \begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix}$ $\alpha \neq 1$

Solution: if $|\alpha| = 1$ then Cond(A) is NOT defines

Take $|\alpha| \neq 1$

$$A^{-1} = \begin{pmatrix} \frac{1}{1-\alpha^2} & \frac{-\alpha}{1-\alpha^2} \\ \frac{-\alpha}{1-\alpha^2} & \frac{1}{1-\alpha^2} \end{pmatrix}$$

$$\|A\|_\infty = 1+|\alpha| \quad \& \quad \|A^{-1}\| = \frac{1}{|1-\alpha^2|} + \frac{|\alpha|}{|1-\alpha^2|}$$

$$= \frac{1+|\alpha|}{|1-\alpha^2|} =$$

$$\text{cond}(A) = (1+|\alpha|)\left(\frac{1+|\alpha|}{(1-\alpha^2)}\right)$$

$$= \frac{(1+|\alpha|)^2}{|1-|\alpha|^2|} = \frac{(1+|\alpha|)^{\cancel{2}}}{|(1-|\alpha|)(\cancel{1+|\alpha|})|}$$

$$= \frac{1+|\alpha|}{|(1-|\alpha|)|}$$

9. ILL-conditioned system

Recap:

4.18. Ill-conditioned matrix

We first solve the following system of two linear equations in two unknowns.

$$\begin{aligned} x_1 + 3x_2 &= 19 \\ 2.5x_1 + 7.857x_2 &= 47.499 \end{aligned}$$

The exact solution to this system is $x_1 = -3$ and $x_2 = 7$. But if we round off 47.499 by 47.500, then the solution changes drastically to $x_1 = 19$ and $x_2 = 0$. Or if we round off 7.857 by 7.86, the solution changes to $x_1 = 98.17$ and $x_2 = -26.39$.

We observe that a small change in the coefficient matrix A or the constant vector b leads to a large change in the solution vector. Such system is called ill-conditioned, otherwise the system is called well-conditioned.

Question

9. Verify that the system of equations:

$$\begin{aligned} 400x_1 - 201x_2 = 200 \\ -800x_1 + 401x_2 = -200 \end{aligned}$$

is an ill-conditioned system.

Hint: A system is ill conditioned if condition number is very big compared to 1, Show that condition number in this case is big.

$$A = \begin{bmatrix} 400 & -201 \\ -800 & 401 \end{bmatrix}$$

Solution:

$$|A| = \begin{vmatrix} 400 & -201 \\ -800 & 401 \end{vmatrix}$$

$$= 400 \times 401 - (-201) \times (-800)$$

$$= 160400 - 160800$$

$$= -400$$

Now, $A^{-1} = \frac{1}{|A|} \times Adj(A)$

$$= \frac{1}{-400} \times \begin{bmatrix} 401 & 201 \\ 800 & 400 \end{bmatrix}$$

$$= \begin{bmatrix} -1.0025 & -0.5025 \\ -2 & -1 \end{bmatrix}$$

Cond(A)=1201.3=3603

Hence, Ill-conditioned system.