Complex Analysis

Contents

1 The Complex Numbers

Definition 1.1 (Complex Number)**.** A *complex number* denoted by z is an ordered pair (x, y) with $x \in \mathbb{R}, y \in \mathbb{R}$. For $z = (x, y)$, x and y are respectively called the real and imaginary part of z. In symbol we write $\text{Re}z = x$, and $\text{Im}z = y$. The set of all complex numbers is denoted by $\mathbb{C}$. We write x for the complex number $(x, 0)$ and i for $(0, 1)$. In fact, the mapping $x \mapsto (x, 0)$ defines a *field isomorphism* of $\mathbb{R}$ onto a subset of $\mathbb{C}$, and hence we may consider $\mathbb{R}$ as a subset of $\mathbb{C}$.

Addition and multiplication of complex numbers are defined as follows:

$$\begin{aligned} (a, b) + (c, d) &:= (a + c, b + d) \\ (a, b)(c, d) &:= (ac - bd, bc + ad). \end{aligned}$$

The following are easy to check directly from definitions:

1. $z_1 + z_2 = z_2 + z_1$.
2. $z_1 z_2 = z_2 z_1$.
3. $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$.

Note that $z = (x, y)$ can be written as $z = x + iy$ by identifying $(x, 0)$ and $(y, 0)$ with x and y respectively. Moreover, $i^2 = -1$.

Example 1.1. Find a root of the equation $z^2 + 1 = 0$ in $\mathbb{C}$.

We can define division of complex numbers also. If $z \neq 0$, then we define

$$\frac{1}{z} = \frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2}.$$

From this we get

$$\frac{x_1 + iy_1}{x_2 + iy_2} = (x_1 + iy_1)\left(\frac{x_2 - iy_2}{x_2^2 + y_2^2}\right) = \frac{(x_1x_2 + y_1y_2) + i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}.$$

Equal Complex Numbers: Two complex numbers (x_1, y_1) and (x_2, y_2) are said to be *equal* if both their real parts and imaginary parts are equal, that is,

$$(x_1, y_1) = (x_2, y_2) \text{ if and only if } x_1 = x_2 \text{ and } y_1 = y_2.$$

Complex Plane: A complex number $z = x + iy$ is defined by the pair of real numbers x and y, so it is natural to assume a one-to-one correspondence between the complex number $z = x + iy$ and the point (x, y) in the xy-plane. We refer to that plane as the *complex plane* or the z-plane.

In the complex plane, any point that lies on the x-axis represents a real number. Therefore, the x-axis is termed the *real axis*. Similarly, any point on the y-axis represents an imaginary number, so the y-axis is called the *imaginary axis*.

Complex Conjugate: The complex conjugate $\overline{z}$ of a given complex number $z = x + iy$ is defined by

$$\overline{z} = x - iy.$$

Conjugation has the following properties which follows easily from the definition:

1. $\text{Re}z = \frac{1}{2}(z + \overline{z})$ and $\text{Im}z = \frac{1}{2i}(z - \overline{z})$.
2. $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$.
3. $\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$, and hence for a real number α, $\overline{\alpha z} = \alpha \overline{z}$.

Polar Form of Complex Numbers: Let $z = x + iy$ be a non-zero complex number. Then there exist unique $r \in (0, \infty)$, and $\theta \in (-\pi, \pi]$ such that $z = r(\cos\theta + i\sin\theta)$. r and θ are related to z by the relations

$$r = \sqrt{x^2 + y^2} \text{ and } \theta = \tan^{-1}\left(\frac{y}{x}\right).$$

Thus r is the distance of $z = (x, y)$ from the origin and θ is the angle between the positive x-axis and the line segment joining the origin and the point (x, y) (see Figure 1). r is called the *modulus* of z, and is denoted by $|z|$. θ is called the *principal argument* of z and is usually written as $\theta = Arg\ z$. Note that if $\phi = Arg\ z + 2k\pi$, k being an integer, then $z = |z|(\cos\phi + i\sin\phi)$. The representation $|z|(\cos\phi + i\sin\phi)$ of z is called the *polar representation* of z, and ϕ is called an *argument* of z. Note that if θ is an argument of z then so is $\theta + 2k\pi$, k being an integer, and $Arg\ z$ is an argument of z lying in the interval $(-\pi, \pi]$.

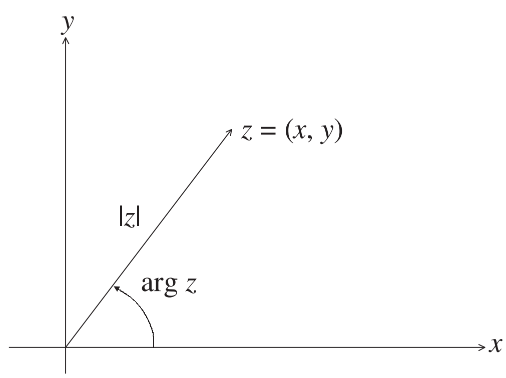

Figure 1:

Remark 1.1. As a note of caution, $\tan^{-1}\left(\frac{y}{x}\right)$ returns a value in the interval $(-\frac{\pi}{2}, \frac{\pi}{2}]$. Therefore, we adjust the value for Arg z by adding π to $\tan^{-1}\left(\frac{y}{x}\right)$ if (x, y) lies in the second quadrant or subtracting π from $\tan^{-1}\left(\frac{y}{x}\right)$ if (x, y) lies in the third quadrant.

Example 1.2. $Arg\ 1 = 0$, $Arg\ (-1) = \pi$, $Arg\ (-1 + i) = \frac{3\pi}{4}$, $Arg\ (1 - i) = \frac{-\pi}{4}$.

Problem 1.1. Let $z_k = r_k(\cos\theta_k + i\sin\theta_k)$, $k = 1, 2, \ldots, n$. Show that

1. $z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$.
2. $z_1 z_2 \cdots z_n = r_1 r_2 \cdots r_n[\cos(\theta_1 + \theta_2 + \cdots + \theta_n) + i\sin(\theta_1 + \theta_2 + \cdots + \theta_n)]$.
3. **De Moivers formula:**
$$z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta),$$
where n is an integer.

Remark 1.2. De Moivers formula fails when n is not an integer. For instance, consider $r = 1, \theta = 2\pi$ and $n = \frac{1}{2}$. Then De Moivre's formula gives $(1)^{\frac{1}{2}} = -1$, which is not true.

Problem 1.2. Use polar representation to explain why product of two negative real numbers is a positive real number.

Problem 1.3 (nth Root)**.** Given a nonzero complex number z_0 and a natural number $n \in \mathbb{N}$, find all distinct complex numbers w such that $z_0 = w^n$.

Solution. Let $z_0 = r(\cos\theta + i\sin\theta)$. Let $w = \rho(\cos\alpha + i\sin\alpha)$, and we determine ρ and α in terms of r and θ. By De Moivers formula, we have $w^n = \rho^n(\cos n\alpha + i\sin\alpha)$. Since $|z_0| = |w^n|$, we obtain $\rho = r^{\frac{1}{n}}$. Moreover, as $r(\cos\theta + i\sin\theta) = r(\cos n\alpha + i\sin n\alpha)$, we obtain

$$\begin{aligned} & n\alpha = \theta + 2k\pi \\ \Rightarrow\quad & \alpha = \frac{\theta + 2k\pi}{n}, \quad k = 0, \pm 1, \pm 2, \pm 3, \dots . \end{aligned}$$

Now we notice that the distinct values of w is given by

$$|z_0|^{\frac{1}{n}}\left(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right), \quad k = 0, 1, 2, \dots, n-1.$$

That is, other values of k will give complex numbers which is already obtained.
Note that the n many complex numbers we got lie on the circle of radius $|z_0|^{\frac{1}{n}}$ about the origin and constitute the vertices of a regular polygon of n sides.

Problem 1.4. Find the curve or region in the complex plane represented by each of the following equations or inequalities:

1. $\mathrm{Re}z = \mathrm{Im}z$
2. $|z| = 2$.
3. $|z - z_0| = 2$.
4. $|z - z_0| < 2$.
5. $|z - z_0| \leq 2$.
6. $1 < |z| < 3$.

Solution. (1) $\mathrm{Re}z = \mathrm{Im}z \Leftrightarrow x = y$, where $z = x + iy$.
The desired subset consists of the points of the straight line $y = x$ (Figure 2).
(2) $|z| = 2 \Leftrightarrow \sqrt{x^2 + y^2} = 2$.
The desired subset consists of the points on the circle with center at $(0, 0)$ and radius $r = 2$.

(3) Let $z_0 = (x_0, y_0)$.
$|z - z_0| = 2 \Leftrightarrow \sqrt{(x - x_0)^2 + (y - y_0)^2} = 4$.
The desired subset consists of the points on the circle with center at $z_0 = (x_0, y_0)$ and radius $r = 2$.

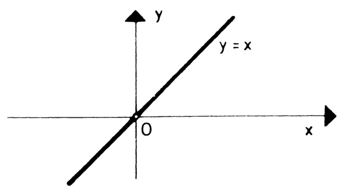

Figure 2:

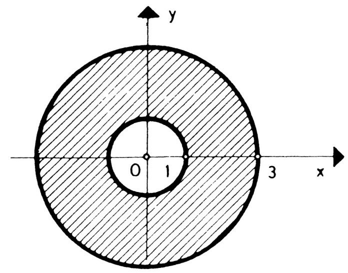

Figure 3:

(4) The desired subset consists of the points inside the circle with center at $z_0 = (x_0, y_0)$ and radius $r = 2$.

(5) The desired subset consists of the points inside and on the circle with center at $z_0 = (x_0, y_0)$ and radius $r = 2$.

(6) The desired subset is the annulus between the circles $|z| = 1$ and $|z| = 3$ without these circles (Figure 3).

1.1 Some Topological Definitions

Neighborhood: As we have seen above, the set of points z such that $|z - z_0| < \epsilon$, where $z_0 \in \mathbb{C}$, $\epsilon \in \mathbb{R}$ contains points that are inside the circle centered at z_0 and with radius ϵ. We call it a *neighborhood of* z_0 and denote it by $N(z_0; \epsilon)$. A deleted neighborhood of z_0 is the set $N(z_0; \epsilon) \setminus \{z_0\}$. We write it as $\widehat{N}(z_0; \epsilon)$.

Limit Point: Let S be a subset of $\mathbb{C}$. A point z_0 in $\mathbb{C}$ is called an *accumulation point* or a *limit point* of S if every neighborhood of z_0 contains a point of S other than z_0. That is, $\widehat{N}(z_0; \epsilon) \cap S \neq \emptyset$ for any $\epsilon > 0$.

Example 1.3. Consider the set $\{z : |z| < 2\}$. The limit points are points on and inside the circle $|z| = 2$. All the points of the set $S = \{x + iy : x = y\}$ are limit points of S (see Figure 2). Zero is the only limit point of the set $\{\frac{i}{n} : n \in \mathbb{Z} \setminus \{0\}\}$. Be aware that a limit point z_0 of S may or may not belong to the set S.

Problem 1.5. Let z_0 be a limit point of S. Then prove that every neighbourhood of z_0 contains infinitely many points of S.

Solution. Left as an exercise.

Interior Point: Let S be a subset of $\mathbb{C}$. A point z_0 in S is called an interior point of S if there exists a neighborhood of z_0, all points of which belong to S.

Example 1.4. The interior points of the set $\{z : |z| < 2\}$ and $\{z : |z| \leq 2\}$ are points inside the circle $|z| = 2$. The set $\{x + iy : x = y\}$ (cf. Figure 2) does not have any interior point.

Boundary Point and Boundary: Let S be a subset of $\mathbb{C}$. If every neighborhood of $z_0 \in \mathbb{C}$ contains points of S and also points not belonging to S, then z_0 is called a boundary point. The set of all boundary points of the set S is called the boundary of S.

Example 1.5. The boundary of the set $\{z : |z| < 2\}$ and $\{z : |z| \leq 2\}$ are given by the circle $|z| = 2$. The boundary of the set $S = \{x + iy : x = y\}$ is S itself.

Open Set: Let $S \subseteq \mathbb{C}$. S is said to be an open set if each point of S is an interior point of S.

Closed Set: A subset $S \subseteq \mathbb{C}$ is said to be closed if it contains all its boundary points.

Example 1.6. The sets $\{z : |z| < 2\}$ and $\{z : \text{Re}z > 0\}$ are open set while the sets $\{z : |z| \leq 2\}$ and $\{x + iy : x = y\}$ are closed sets. The set $\{z : |z| < 2\} \cup \{2\}$ is neither open nor closed. $\mathbb{C}$ is both open and closed.

Proposition 1.1. *A subset $S \subseteq \mathbb{C}$ is open if and only if its complement $S^c = \{z : z \notin S\}$ is closed.*

Closure: Let $S \subseteq \mathbb{C}$. The closure of S, denoted as $\overline{S}$, is the closed set that contains all points in S together with the whole boundary of S.

Example 1.7. The closure of a closed set S is S itself. The closed set $\{z : |z| \leq 2\}$ is the closure of the set $\{z : |z| < 2\}$.

Bounded and Unbounded Set: A bounded set is one that can be contained in a large enough circle centered at the origin. That is, there exists a sufficiently large real constant M such that $S \subseteq \{z : |z| < M\}$. An unbounded set is one that is not bounded.

Example 1.8. The set $\{\frac{1}{n} + i\frac{1}{n} : n \in \mathbb{Z} \setminus \{0\}\}$ is bounded as it is contained in the circle $|z| = \sqrt{2}$. The set $\{x + iy : x = y\}$ is not bounded.

Definition 1.2.

Connected Set: A set S is said to be connected if any two points of S can be joined by a continuous curve lying entirely inside S. For example, a neighborhood $N(z_0; \epsilon)$ is connected.

Domain: An open connected set is called an domain. For example, the set $\{z : \text{Re}z \geq 0\}$ is not a domain since it is not open. The set $\{z : 0 < \text{Re} < 1 \text{ or } 2 < \text{Re} < 3\}$ is also not a domain since it is open but not connected.

Region: The set obtained from a domain by joining some, none, or all of its boundary points, is called a region.

Definition 1.3.

Curve: A *curve* C in the complex plane $\mathbb{C}$ is given by a function

$$\gamma : [a, b] \to \mathbb{C}, \ \gamma(t) = x(t) + iy(t)$$

with $x, y : [a, b] \to \mathbb{R}$ being continuous. The curve C is then the set

$$C = \{\gamma(t) : t \in [a, b]\}.$$

The function $\gamma : [0, 2\pi] \to \mathbb{C}$ given by $\gamma(t) = z_0 + r(\cos t + i \sin t)$, where z_0 is a fixed complex number, gives the circle with center at z_0 and radius r. The function $\gamma : [0, 1] \to \mathbb{C}$ given by $\gamma(t) = tz_1 + (1 - t)z_0$ gives the line segment joining z_0 and z_1.

Smooth Curve: A curve C is called a *smooth curve* if $\gamma'(t) = x'(t) + iy'(t)$ is continuous and nonzero for all t. Geometrically this means that C has everywhere a continuously turning tangent.

Closed Curve: A curve C is called *closed* if its terminal point coincides with its initial point, that is $\gamma(a) = \gamma(b)$.

Contour: A *contour* is a curve that is obtained by joining finitely many smooth curves end to end.

Simple Contour: A contour is called *simple* if it does not cross itself (if initial point and the final point are same they are not considered as non simple). For instance, a circle is simple, but the curve of the shape '8' is not simple (cf. Figure 4).

Curve with Reverse Orientation: Let $C : \gamma : [a, b] \to \mathbb{C}$ be a curve. Then the curve with the reverse orientation, denoted as $-C : \gamma_- : [a, b] \to \mathbb{C}$ and is defined as

$$\gamma_-(t) = \gamma(b + a - t).$$

Thus for a contour C, the contour with the negative orientation $-C$ make sense.

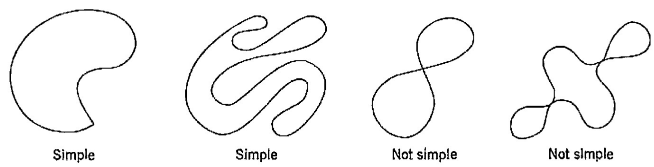

Figure 4:

Notation 1.1. Let $C_1, C_2, \ldots, C_n$ are contours such that the terminal point of C_k coincides with the initial point of C_{k+1}. Then $C = C_1 + C_2 + \cdots + C_n$ will be used to denote the contour obtained by joining the contours $C_1, C_2, \ldots, C_n$ end to end.

Definition 1.4.

Simply connected domain: A domain D is called *simply connected* if every simple closed contour within it encloses points of D only. A domain D is called multiply connected if it is not simply connected.

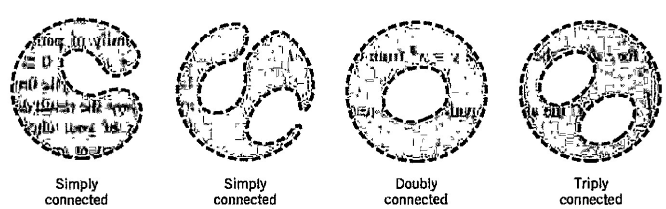

Figure 5:

According to the definition, it is then always possible to construct some simple closed contour inside a multiply connected domain in such a manner that one or more points inside the contour do not belong to the domain. Intuitively, there are holes contained inside some simple closed contour lying completely in the domain. We have one hole in a *doubly connected domain* and two holes in a *triply connected domain* (see Figure 5). For example, the domain $\{z : 1 < |z| < 2\}$ is doubly connected.

2 Complex Functions

Let S be a set of complex numbers in the complex plane. For every point $z = x + iy \in S$, we specify the rule for assigning a corresponding complex number $w = u+iv$. This defines a function of the complex variable z, and the function is denoted by

$$w = f(z).$$

The set S is called the domain of definition of the function f and the collection of all values of w is called the range of f.

A complex function of the complex variable z may be visualized as a pair of real functions of the two real variables x and y, where $z = x + iy$. Let $u(x, y)$ and $v(x, y)$ be the real and imaginary parts of $f(z)$, respectively. We may write

$$f(z) = u(x, y) + iv(x, y), \quad z = x + iy.$$

For example, consider the function

$$f(z) = z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy;$$

its real and imaginary parts are the real functions

$$u(x, y) = x^2 - y^2 \text{ and } v(x, y) = 2xy,$$

respectively.

Definition 2.1 (Limit)**.** Let $z_0 \in \mathbb{C}$ be a limit point of D. A function $f : D \to \mathbb{C}$ is said to have the limit l as z approaches z_0, written $\lim_{z\to z_0} f(z) = l$, if for a given $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(z) - l| < \epsilon \text{ whenever } z \in D \text{ and } 0 < |z - z_0| < \delta.$$

The limit l, if exists, must be unique. The value of l is independent of the direction along which $z \to z_0$. For example, consider the limit $\lim_{z\to 0} \frac{z}{\overline{z}}$.
Along the positive direction of x-axis, we obtain $\lim_{z\to 0} \frac{z}{\overline{z}} = 1$.
Along the positive direction of y-axis, we obtain $\lim_{z\to 0} \frac{z}{\overline{z}} = -1$.
Thus, it follows that $\lim_{z\to 0} \frac{z}{\overline{z}}$ does not exist.

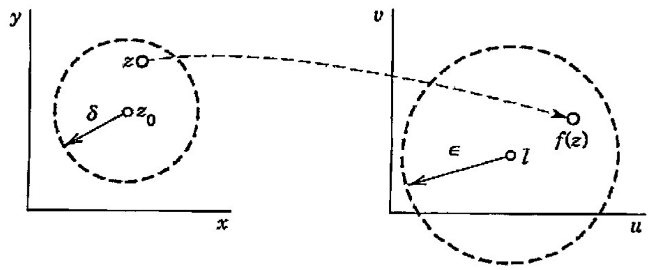

Figure 6: Limit

Definition 2.2 (Continuity)**.** A function $f : D \to \mathbb{C}$ is said to be continuous at $z_0 \in D$ if

$$\lim_{z \to z_0} f(z) = f(z_0).$$

A complex function is said to be continuous in a region R if it is continuous at every point in R.

It is not difficult to prove the following.

Proposition 2.1. *Let $f = u + iv$, that is $u(x, y)$ and $v(x, y)$ be the real and imaginary parts of the function $f(z)$. Then f is continuous at $z_0 = x_0 + iy_0$ if and only if $u(x, y)$ and $v(x, y)$ are both continuous at (x_0, y_0).*

Example 2.1. Consider the function $f(x+iy) = \mathrm{e}^x \cos y + i\mathrm{e}^x \sin y$. Then $f = u+iv$, where $u(x, y) = \mathrm{e}^x \cos y$, and $v(x, y) = \mathrm{e}^x \sin y$. Since both $u(x, y)$ and $v(x, y)$ are continuous at any point (x_0, y_0) in the xy-plane, we conclude that $f(z)$ is continuous at any point $z_0 = x_0 + iy_0$ in $\mathbb{C}$.
Consider the function $f(z) = |z|^2$. Then $f = u+iv$, where $u(x, y) = x^2+y^2$ and $v(x, y) = 0$. Here u and v are continuous at every point in the xy-plane, and hence $f(z)$ is continuous everywhere on $\mathbb{C}$.

Theorems on real continuous functions can be extended to complex continuous functions. For instance, if two complex functions are continuous at a point, then their sum, difference and product are also continuous at that point; and their quotient is continuous at any point where the denominator is non-zero.

3 Analytic Functions

3.1 Differentiability

Definition 3.1. Let $f : D \to \mathbb{C}$ be a function, and $z_0 \in D$ be a limit point of D. Suppose that the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

or equivalently

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h},$$

exists. Then we say that f is differentiable at z_0 and the limit denoted by $f'(z_0)$ is called the derivative of f at z_0.

The following results about derivatives follow exactly as in the case of reals.

Proposition 3.1.

1. *Derivative of a constant function is zero and* $\dfrac{\mathrm{d}}{\mathrm{d}z}z^n = nz^{n-1}$, $n \in \mathbb{Z}$.
2. *If* $\alpha, \beta \in \mathbb{C}$, *then* $(\alpha f + \beta g)' = \alpha f' + \beta g'$.
3. **(Chain Rule)** $\dfrac{\mathrm{d}}{\mathrm{d}z} f(g(z)) = f'(g(z))g'(z)$ *whenever all the terms make sense.*

Like real calculus, we have the following.

Proposition 3.2. *If f is differentiable at z_0 then f is continuous at z_0.*

Proof. Since $f(z)$ is differentiable at z_0, the limit $\lim\limits_{z\to z_0} \dfrac{f(z)-f(z_0)}{z-z_0}$ exists. Therefore,

$$\lim_{z\to z_0} f(z) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}(z-z_0) + f(z_0) = f(z_0).$$

$\square$

Remark 3.1. Continuity of $f(z)$ at a point z_0 may not imply the differentiability of $f(z)$ at z_0. For example, consider the function $f(z) = |z|$.

f(z) is continuous at z = 0: Note that $f = u + iv$, where $u = \sqrt{x^2+y^2}$, and $v(x,y) = 0$. As $u(x,y)$ and $v(x,y)$ are continuous at $(0,0)$, it follows that $f(z)$ is continuous at $z = 0$.

f(z) is NOT differentiable at z = 0: We will show that the limit $\lim\limits_{h\to 0} \dfrac{f(0+h)-f(0)}{h}$ does not exist.

Suppose h approaches 0 along the straight line $y = mx$, $x > 0$, then in this case

$$\begin{aligned}
\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} &= \lim_{h\to 0} \frac{|h|}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{h_1^2+h_2^2}}{h_1+ih_2}, \quad h = h_1 + ih_2 \\
&= \lim_{h_1\to 0} \frac{\sqrt{h_1^2+m^2h_1^2}}{h_1+imh_1} \\
&\quad \text{(putting } h_2 = mh_1 \text{ as } h \text{ approaches 0 along } y = mx) \\
&= \frac{\sqrt{1+m^2}}{1+im}.
\end{aligned}$$

Since the limit depends on m, $\lim\limits_{h\to 0} \dfrac{f(0+h)-f(0)}{h}$ does not exist, and hence $f(z)$ is not differentiable at $z = 0$.

Problem 3.1. Consider the function $f(z) = |z|^2 = x^2+y^2$, $z = x+iy$. The function f can also be thought of as a function from $\mathbb{R}^2$ to $\mathbb{R}$ mapping (x, y) to $x^2 + y^2$. Moreover, since the partial derivatives of f are continuous throughout $\mathbb{R}^2$, it follows that f is differentiable everywhere on $\mathbb{R}^2$. Show that $f(z)$ is not complex differentiable at any non-zero point z_0.

Solution. Left as an exercise.

Theorem 3.3 (C-R Equations)**.** *Let $f(z) = u(x, y)+iv(x, y)$, $z = x+iy$, be differentiable at $z_0 = x_0 + iy_0$. Then the partial derivatives of u and v exist at the point (x_0, y_0), and*

$$u_x(x_0, y_0) = v_y(x_0, y_0) \quad \textit{and} \quad v_x(x_0, y_0) = -u_y(x_0, y_0). \tag{3.1}$$

Further,

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) = v_y(x_0, y_0) - iu_y(x_0, y_0). \tag{3.2}$$

The Equations in (3.2) are called *Cauchy-Riemann equations* (C-R equations).

Example 3.1. Consider the function $f(z) = z^2+i$. Let $f = u+iv$. Then $u(x, y) = x^2-y^2$, $v(x, y) = 1 + 2xy$. f is differentiable at every $z \in \mathbb{C}$. Moreover, for $z_0 = x_0 + iy_0$,

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) = 2x_0 + i2y_0 = 2z_0.$$

Problem 3.2. Use C-R equations to prove that $f(z) = |z|^2$ is not differentiable at any non-zero point z_0.

Solution. Here $f = u + iv$, where $u(x, y) = x^2 + y^2$, and $v(x, y) = 0$. Then

$$\begin{array}{ll} u_x(x_0, y_0) = 2x_0 & v_x(x_0, y_0) = 0 \\ u_y(x_0, y_0) = 2y_0 & v_y(x_0, y_0). \end{array}$$

Therefore, it follows that the C-R equations

$$u_x(x_0, y_0) = v_y(x_0, y_0) \quad \text{and} \quad v_x(x_0, y_0) = -u_y(x_0, y_0)$$

are satisfied ONLY at the origin $(0, 0)$, and therefore by Theorem 3.3, $f(z) = |z|^2$ is not differentiable at any non-zero point z_0.

Example 3.2 (Example of a function which satisfies C-R equations at a point, but is not differentiable at that point)**.** Consider the function $f(z) = f(x + iy) = \sqrt{|xy|}$. Then $f = u + iv$, where $u(x, y) = \sqrt{|xy|}$, and $v(x, y) = 0$ for all $(x, y) \in \mathbb{R}^2$. We note the following fact:

u and v satisfies C-R equations at the origin: Since $v(x, y) = 0$ for all $(x, y) \in \mathbb{R}^2$, we have

$$v_x(0, 0) = v_y(0, 0) = 0.$$

Also,

$$\begin{aligned} u_x(0, 0) &= \lim_{h\to 0} \frac{u(h, 0) - u(0, 0)}{h} = 0 \\ u_y(0, 0) &= \lim_{h\to 0} \frac{u(0, h) - u(0, 0)}{h} = 0. \end{aligned}$$

Thus u and v satisfies the Cauchy-Riemann equations.
From this fact, can we conclude that $f(z)$ is differentiable at the origin? Answer is no. In fact, $f(z)$ is not differentiable at the origin as shown below.

f(z) is not differentiable at the origin: We will show that the limit $\lim_{z\to 0}\dfrac{f(z)-f(0)}{z-0}$ does not exist.
Suppose z approaches 0 along the straight line $y = mx$, $x > 0$, $m \neq 0$. Then in this case

$$\begin{aligned}
\lim_{z\to 0}\frac{f(z)-f(0)}{z-0} &= \lim_{z\to 0}\frac{\sqrt{|xy|}}{x+iy}, \quad z = x+iy \\
&= \lim_{z\to 0}\frac{\sqrt{|mx^2|}}{x+imx} \\
&\quad \text{(putting } y = mx \text{ as } z \text{ approaches 0 along } y = mx) \\
&= \frac{\sqrt{|m|}}{1+im}.
\end{aligned}$$

Since the limit depends on m, $\lim_{z\to 0}\dfrac{f(z)-f(0)}{z-0}$ does not exist, and hence $f(z)$ is not differentiable at $z = 0$.

Remark 3.2. Example 3.2 shows that the satisfaction of the C-R equations by the real and imaginary parts of a function f at a point z_0 does not give the differentiability of the function f at z_0. But, in addition of satisfying the C-R equations, if partial derivatives of the real and imaginary parts of f are also continuous at z_0, then we will obtain the differentiability of f at z_0. More precisely, we have the following result.

Theorem 3.4 (Converse of C-R Equations)**.** *Suppose $f = u + iv$ is defined on some neighborhood $N(z_0;\epsilon)$ of $z_0 = x_0 + iy_0$ such that u_x, u_y, v_x, v_y exist on $N(z_0;\epsilon)$ and are continuous at (x_0, y_0). If u, v satisfies the C-R equations at (x_0, y_0), then f' exist at z_0 and $f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$.*

Problem 3.3. Using Theorem 3.4 show that the following functions are differentiable everywhere in the complex plane:

1. $f(x+iy) = x^3 - 3xy^2 + i(3x^2y - y^3)$
2. $f(x+iy) = \mathrm{e}^{-y}\cos x + i\mathrm{e}^{-y}\sin x$.

Solution. (1) Here $f = u + iv$ where $u(x,y) = x^3 - 3xy^2$, and $v(x,y) = 3x^2y - y^3$.
Moreover,

$$\begin{aligned}
u_x(x,y) &= 3x^2 - 3y^2 \qquad & u_y(x,y) &= -6xy \\
v_y(x,y) &= 3x^2 - 3y^2 \qquad & v_x(x,y) &= 6xy.
\end{aligned}$$

Therefore, u_x, u_y, v_x, v_y are all continuous on $\mathbb{R}^2$, and u, v satisfies the C-R equations $u_x(x,y) = v_y(x,y)$, $u_y(x,y) = -v_x(x,y)$ at each point $(x,y) \in \mathbb{R}^2$. Therefore, by Theorem 3.4, f is differentiable everywhere in the complex plane. Moreover,

$$f'(z) = u_x(x,y) + iv_x(x,y) = 3(x^2 - y^2) + i6xy, \ z = x + iy.$$

(2) Left as an exercise.

Problem 3.4. Find all the points where the function $f(x+iy) = x^2+iy^2$ is differentiable.

Solution. Here $f = u + iv$, where $u(x, y) = x^2$, and $v(x, y) = y^2$. Therefore,

$$\begin{aligned} u_x(x,y) &= 2x & u_y(x,y) &= 0 \\ v_y(x,y) &= 2y & v_x(x,y) &= 0. \end{aligned}$$

Then $u_x = v_y$ and $u_y = -v_x$ gives $x = y$. Thus, C-R equations are satisfied only when $x = y$. Therefore, by Theorem 3.3, f is not differentiable at a point which does not lie on the straight line $x = y$. Moreover, as u_x, u_y, v_x, v_y are all continuous at each point on the line $x = y$, it follows from Theorem 3.4 that f is differentiable at each point on the line $x = y$. Thus f is differentiable only at the points on the line $x = y$.

Problem 3.5. Using $x = r\cos\theta$, $y = r\sin\theta$ and the chain rule, prove that the C-R equation is equivalent to

$$u_r = \frac{1}{r}v_\theta,\ v_r = -\frac{1}{r}u_\theta. \tag{3.3}$$

Equations in (3.3) are the C-R equations in polar form.

Solution. Left as an exercise.

3.2 Analyticity

Definition 3.2. A function f is said to be analytic at the point z_0 if there exists a neighborhood $N(z_0, \epsilon)$ of z_0, $\epsilon > 0$ such that f is differentiable at every point $z \in N(z_0, \epsilon)$. Similarly, f is said to be analytic (or, regular, or holomorphic) on a set D if it is differentiable at every point of some open set containing D.

We note the following obvious facts:

1. If f is differentiable at all points of an **open set** D, then f is analytic on D.
2. If f is **differentiable** at all points of a set D, then it **does not** mean that f will be analytic on D (see Example 3.3).
3. If f is **analytic** at all points of a set D, then f is analytic on D.

Compare the Item 2 with the Items 1 and 3 and note the differences.

Example 3.3. Consider the function $f(x + iy) = x^2 + iy^2$, and the set $D := \{x + iy \in \mathbb{C} : x = y\}$. As shown in Problem 3.4, f is differentiable at all points of D, but f is not differentiable at any point lying outside D. Therefore, f is not analytic on D as any open set containing D will contain a point lying outside D, and hence contain a point where f is not be differentiable.

Suppose two complex functions are analytic in some domain D. Then their sum, difference and product are all analytic in D. Also, their quotient is analytic in D given that the denominator function is non-zero at all points in D. The composition of two analytic functions is also analytic.

Problem 3.6. Show that polynomials are analytic on the set $\mathbb{C}$ of all complex numbers.

Solution. Left as an exercise.

Theorem 3.5 (Necessary and Sufficient Condition for Analyticity)**.** *A function $f = u + iv$ is analytic in a **domain**[1] D if and only if u, v satisfies C-R equations in D, and u_x, u_y, v_x, v_y are continuous in D.*

Example 3.4. Consider the function $f = u + iv$ defined on $\mathbb{C}$, where $u(x, y) = x^2$, $v(x, y) = y^2$. Consider the set $D := \{x + iy \in \mathbb{C} : x = y\}$. Note that u, v satisfies the C-R equations in D, and u_x, u_y, v_x, v_y are also continuous in D. But, as shown in Example 3.3, f is not analytic on D. Does this fact contradict Theorem 3.5? If not, explain why.

Definition 3.3 (Entire Function)**.** A function analytic on the entire complex plane is called an *entire function.*

Problem 3.7. Suppose $f_1(z)$ is analytic at z_0, while $f_2(z)$ is non-analytic at z_0. Then show that $f_1(z) + f_2(z)$ is not analytic at z_0. Give an example to show that sum of two non-analytic function can be analytic.

Solution. Left as an exercise.

3.3 Applications of C-R Equations

Proposition 3.6. *Suppose f is analytic in a domain D. If $f'(z) = 0$ for all $z \in D$, then f is constant on D.*

Solution. Let $f = u + iv$. Since f is analytic in D, we have

$$f'(x + iy) = u_x(x, y) + iv_x(x, y) = v_y(x, y) - iu_y(x, y), \; x + iy \in D.$$

Therefore, $f'(z) = 0$ for all $z \in D$ gives

$$\begin{aligned} &u_x(x, y) = 0 \text{ and } u_y(x, y) = 0 \text{ for all } (x, y) \in D \;\Rightarrow u \text{ is a constant function in } D \\ &v_x(x, y) = 0 \text{ and } v_y(x, y) = 0 \text{ for all } (x, y) \in D \;\Rightarrow v \text{ is a constant function in } D. \end{aligned}$$

This implies that $f = u + iv$ is a constant function in D.

Proposition 3.7. *Suppose f is analytic in a domain D. If any of* $\mathrm{Re}f$, $\mathrm{Im}f$ *is constant in D, then f is constant in D.*

Solution. Let $f = u + iv$. First we assume that $\mathrm{Re}f$ is constant on D, that is $u(x, y)$ is a constant function in D. Thus, $u_x(x, y) = u_y(x, y) = 0$ for all $(x, y) \in D$. Now, since f is analytic in D, we must have for all $(x, y) \in D$,

$$v_y(x, y) = u_x(x, y) = 0 \text{ and } v_x(x, y) = -u_y(x, y) = 0.$$

This implies that $v(x, y)$ is a constant function on D.
Therefore, f is a constant function in D as both u and v is obtained as constant functions in D.

[1]Recall that domain is an open connected set

Problem 3.8. Suppose f is analytic in a domain D such that $|f|$ is constant in D. Then show that f is a constant in D.

Solution. Left as an exercise.

Example 3.5. The functions $|z|$, $|z|^2$, Re z, Im z are non-constant real functions (i.e. imaginary part of f is zero) defined on $\mathbb{C}$. Hence, by Proposition 3.7 they are nowhere analytic.

Problem 3.9. Let $f = u + iv$ be a non-constant function such that $\overline{f} = u - iv$ be analytic in a domain D. Show that f cannot be analytic in D.

Solution. Left as an exercise.

Problem 3.10. Given an analytic function

$$w = f(z) = u(x, y) + iv(x, y), z = x + iy,$$

the equations $u(x, y) = \alpha$ and $v(x, y) = \beta$, α and β are constants, define two families of curves in the complex plane. Show that the two families are mutually orthogonal to each other.

Solution. Left as an exercise.

3.4 Harmonic Functions

Definition 3.4 (Harmonic Function)**.** A real-valued function $\phi(x, y)$ of two real variables x and y is said to be harmonic in a given domain D in the xy-plane if ϕ has continuous partial derivatives up to the second order in D and satisfies the Laplace equation

$$\phi_{xx}(x, y) + \phi_{yy}(x, y) = 0.$$

Definition 3.5 (Conjugate Harmonic Function)**.** If two harmonic functions $\phi(x, y), \psi(x, y)$ satisfy C-R equations, namely

$$\phi_x = \psi_y \text{ and } \phi_y = -\psi_x$$

in a domain D, then ψ is called conjugate harmonic function of ϕ.

Remark 3.3. Note that harmonic conjugacy is not a symmetric relation, that is, if ψ is conjugate harmonic function of ϕ, then it does not mean that ϕ will be conjugate harmonic function of ψ. This is due to the minus sign in the second CauchyRiemann relation.

The following theorem gives the close link between analyticity and harmonic conjugacy.

Theorem 3.8. *A complex function $f(z) = u(x, y) + iv(x, y)$, $z = x + iy$, is analytic in a domain D if and only if v is a harmonic conjugate of u in D.*

Proof. We need one result to prove the above claim. It will be shown in Section 5 that if a complex function is analytic at a point, then its real and imaginary parts have continuous partial derivatives of all orders at that point (cf. Remark 5.2).

($\Rightarrow$) We assume that $f = u + iv$ is analytic in D, and we prove that v is a harmonic conjugate of u in D.
Since f is analytic in D, by the above mentioned fact, u and v have continuous partial derivatives of all orders in D, and

$$u_x(x, y) = v_y(x, y), \quad u_y(x, y) = -v_x(x, y) \text{ for all } (x, y) \in D.$$

Therefore, for all $(x, y) \in D$

$$\begin{aligned}
u_{xx}(x, y) &= v_{yx}(x, y) = v_{xy}(x, y) \quad \text{(since the partial derivatives are all continuous)} \\
u_{yy}(x, y) &= -v_{xy}(x, y) \\
v_{xx}(x, y) &= u_{yx}(x, y) = -u_{xy}(x, y) \quad \text{(since the partial derivatives are all continuous)} \\
v_{yy}(x, y) &= u_{xy}(x, y).
\end{aligned}$$

This gives $u_x x(x, y) + u_y(x, y) = 0$ and $v_x x(x, y) + v_y(x, y) = 0$ for all $(x, y) \in D$. Thus, we have shown that v is a harmonic conjugate of u in D.

($\Leftarrow$) We now assume that v is a harmonic conjugate of u in D, and we prove that $f = u+iv$ is analytic in D. In fact, this follows directly from Theorem 3.5. □

Proposition 3.9. *Any two harmonic conjugates v, w of u in a domain D differ by a constant, that is, $v(x, y) - w(x, y) = K$ for all $(x, y) \in D$, where K is a real constant.*

Proof. By the given conditions, the functions $f_1 = u + iv$, and $f_2 = u + iw$ are both analytic in D, and hence $f_1 - f_2 = i(v - w)$ is analytic in D. Note that $\mathrm{Re}(f_1 - f_2) = 0$, and hence by Proposition 3.7, $f_1 - f_2$ is constant in D. This gives $v - w$ as a real constant in D. □

Problem 3.11. Let v be harmonic conjugate of u in a domain D. Then show that $v + K$ is also a harmonic conjugate of u in D, where K is a real constant.

Solution. Left as an exercise.

Given harmonic function $u(x, y)$ in a *simply connected domain*[2] D, it is always possible to obtain its harmonic conjugate $v(x, y)$ of u in D as illustrated by the following problem.

Problem 3.12. Find a conjugate harmonic function of $u(x, y) = x^2 - y^2 - y$ in $\mathbb{C}$.

Solution. Let v be harmonic conjugate of u in $\mathbb{C}$. Then for all $(x, y) \in \mathbb{C}$,

$$v_y(x, y) = u_x(x, y) = 2x \text{ and} \tag{3.4}$$
$$v_x(x, y) = -u_y(x, y) = 2y + 1. \tag{3.5}$$

From (3.4), we obtain

$$v(x, y) = 2xy + \phi(x) \tag{3.6}$$

[2]Defined in Section 5

Differentiating (3.6) w.r.t x, we obtain

$$v_x(x,y) = 2y + \phi'(x) \tag{3.7}$$

From (3.5) and (3.7), we obtain

$$\begin{aligned} & 2y + \phi'(x) = 2y + 1 \\ \Rightarrow\ & \phi'(x) = 1 \\ \Rightarrow\ & \phi(x) = x + K \end{aligned}$$

Thus for each value of the real constant K (for instance $K = 2$), we obtain a conjugate harmonic function of u as $v(x,y) = 2xy + K$.

Problem 3.13. Given an analytic function

$$w = f(z) = u(x,y) + iv(x,y), z = x + iy,$$

the equations $u(x,y) = \alpha$ and $v(x,y) = \beta$, α and β are constants, define two families of curves in the complex plane. Show that the two families are mutually orthogonal to each other.

Solution. Left as an exercise.

Problem 3.14. Given the harmonic function

$$u(x,y) = e^x \cos y + xy,$$

find the family of curves that is orthogonal to the family

$$u(x,y) = \alpha, \quad \alpha \text{ is constant.}$$

Solution. Left as an exercise.

4 Elementary Functions

4.1 Exponential Function

Definition 4.1. Complex exponential function e^z is defined by

$$\mathrm{e}^z = \mathrm{e}^x(\cos y + i \sin y), \quad z = x + iy.$$

Therefore, for $z = iy$, $y \in \mathbb{R}$, we obtain

$$\mathrm{e}^{iy} = \cos y + i \sin y. \tag{4.1}$$

Equation (4.1) is known as *Euler formula.* Using this formula, the polar form of any complex number $z = r(\cos\theta + i\sin\theta)$ can be written as $z = r\mathrm{e}^{i\theta}$.

Proposition 4.1 (Properties)**.**

1. $|\mathrm{e}^z| = \mathrm{e}^x$, *and* $\arg \mathrm{e}^z = y \pm 2n\pi$, $(n = 0, 1, 2, \ldots)$, *where* $z = x + iy$.

2. *e^z is an entire function, and* $\dfrac{d}{dz}e^z = e^z$.
3. $e^{z_1+z_2} = e^{z_1}e^{z_2}$.
4. $e^z \neq 0$ *for all* z.
5. *e^z is periodic with the fundamental period $2\pi i$, that is*

$$e^{z+2k\pi i} = e^z,$$

   for any z and integer k.

   [Thus the complex exponential function is periodic while its real counterpart is not]

6. *e^z is NOT injective.*
7. *If $H = \{x + iy : -\pi < y \leq \pi\}$. Then e^z is bijective from H to $\mathbb{C} \setminus \{0\}$.*

Proof. **(1):** Obvious.

(2): Let u and v be the real and imaginary parts of the function e^z. Then

$$u(x,y) = e^x \cos y,\ \ v(x,y) = e^x \sin y.$$

Note that u and v satisfies the C-R equation $u_x = v_y,\ u_y = -v_x$ for all $(x,y) \in \mathbb{R}^2$. Moreover, u_x, u_y, v_x, v_y are all continuous on $\mathbb{R}^2$. Therefore, by Theorem 3.5, it follows that e^z is an entire function, and

$$\begin{aligned}\frac{d}{dz}e^z &= u_x(x,y) + iv_x(x,y) \quad \text{(By Theorem 3.3)}\\ &= e^x \cos y + ie^x e^x \sin y\\ &= e^z.\end{aligned}$$

(3): Left as an exercise.

(4): Note that $|e^z| = e^x \neq 0$ for all z. Therefore, $e^z \neq 0$ for all z.

(5): Left as an exercise.

(6): As e^z is periodic, it follows that e^z is not injective.

(7): Left as an exercise. □

Problem 4.1. Let $f(z)$ be a complex function which satisfies the following:

1. $f(z)$ is an entire function,
2. $f'(z) = f(z)$ for all z, and
3. $f(x) = e^x$ for real x.

Then show that $f(z) = e^z$.

Solution. Left as an exercise.

Problem 4.2. Find all the roots of the equation

$$e^z = i.$$

Solution. Left as an exercise.

4.2 Logarithmic Function

We use $\mathbb{C}^*$ and H to denote the set $\mathbb{C} \setminus \{0\}$, and $\{x + iy : -\pi < y \leq \pi\}$ respectively.

Definition 4.2. For $z \in \mathbb{C}^*$, we define

$$\log z = \ln |z| + i\, arg\ z.$$

Here $\ln |z|$ stands for the real logarithm of $|z|$. Since $arg\ z = Arg\ z + 2k\pi$, $k \in \mathbb{Z}$, it follows that $\log z$ is not well defined as a function. In fact, $\log z$ is an instance of multi-valued function. Therefore, we have the following definition.

Definition 4.3. For $z \in \mathbb{C}^*$ the principal value of the logarithm is defined as

$$Log\ z = \ln |z| + i\, Arg\ z.$$

Note that $Log\ \cancel{z} : \mathbb{C}^* \to H$ is well defined (now it is single valued). The connection between $\log z$ and $Log\ z$ is

$$Log\ z + i2k\pi = \log z \quad \text{for some } k \in \mathbb{Z}.$$

Theorem 4.2 (Properties)**.**

1. *For* $z \neq 0$, $\mathrm{e}^{\mathrm{Log}\ z} = z$.
2. *For* $z \in H$, $\mathrm{Log}\ \mathrm{e}^z = z$.
3. *For* $z \notin H$, $\mathrm{Log}\ \mathrm{e}^z \neq z$.
4. *For real* $x > 0$, $\mathrm{Log}\ x = \ln x$.
5. $\mathrm{Log}\ z$ *is not continuous on the negative real axis* $\mathbb{R}^- = \{z = x + iy : x < 0, y = 0\}$.

   [Unlike real logarithm, it is defined there, but useless]
6. $\mathrm{Log}\ z$ *is analytic on the set* $\mathbb{C}^* \setminus \mathbb{R}^-$.
7. $\dfrac{\mathrm{d}}{\mathrm{d}z}\mathrm{Log}\ z = \dfrac{1}{z}$.
8. *The identity* $\mathrm{Log}\ (z_1 z_2) = \mathrm{Log}\ z_1 + \mathrm{Log}\ z_2$ *is true iff* $\mathrm{Arg}\ z_1 + \mathrm{Arg}\ z_2 \in (-\pi, \pi]$.

Proof. **(1):** $\mathrm{e}^{Log\ z} = \mathrm{e}^{\ln |z| + i Arg\ z} = |z|(\cos Arg\ z + i \sin Arg\ z) = z$.

(2): Let $z = x + iy$, $-\pi < y \leq \pi$. Then

$$Log\ \mathrm{e}^z = Log\ (\mathrm{e}^x \cos y + i\mathrm{e}^x \sin y) = \ln \mathrm{e}^x + iy = x + iy.$$

(3): Let $z = x + iy \notin H$, and hence $y \notin (-\pi, \pi]$. Note that $Arg\ (\mathrm{e}^x \cos y + i\mathrm{e}^x \sin y) \neq y$ as $y \notin (-\pi, \pi]$. Therefore

$$Log\ \mathrm{e}^z = Log\ (\mathrm{e}^x \cos y + i\mathrm{e}^x \sin y) = \ln \mathrm{e}^x + i\, Arg\ (\mathrm{e}^x \cos y + i\mathrm{e}^x \sin y) \neq x + iy = z.$$

(4): Left as an exercise.

(5): Consider an arbitrary point $z = -\alpha \in \mathbb{R}^-$, $\alpha > 0$, and we show that $Log\ z$ is not continuous at α. Consider the sequences

$$\{a_n = \alpha e^{i(\pi - \frac{1}{n})}\} \text{ and } \{b_n = \alpha e^{i(-\pi + \frac{1}{n})}\}.$$

Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = z,$$

but

$$\begin{aligned} \lim_{n\to\infty} Log\ a_n &= \lim_{n\to\infty} \ln\alpha + i(\pi - \frac{1}{n}) = \ln\alpha + i\pi, \quad \text{and} \\ \lim_{n\to\infty} Log\ b_n &= \lim_{n\to\infty} \ln\alpha + i(-\pi + \frac{1}{n}) = \ln\alpha - i\pi. \end{aligned}$$

Since $\lim_{n\to\infty} Log\ a_n \neq \lim_{n\to\infty} Log\ b_n$, it follows that $Log\ z$ is not continuous at α.

(6): Consider the domain $D = \mathbb{C}^* \setminus \mathbb{R}^- = \{z : z \neq 0, Arg\ z \in (-\pi, \pi)\}$, and let $Log\ z = \ln r + i\theta = u(r,\theta) + iv(r,\theta)$, where $z = r(\cos\theta + i\sin\theta) \in D$. Therefor $u(r,\theta) = \ln r$, $v(r,\theta) = \theta$.
Then

$$u_r = \frac{1}{r}v_\theta = \frac{1}{r} \text{ and } v_r = -\frac{1}{r}u_\theta = 0.$$

Thus the C-R equations are satisfied. Since $u_r, u_\theta, v_r, v_\theta$ are continuous in D, it follows that $Log\ z$ is analytic in D.
Note: Note that the function $f(z) = Arg\ z$ is discontinuous at each point α on the negative x-axis. In fact, if z approaches α through the upper half plane, then $\lim_{z\to\alpha} f(z) = \pi$, and if z approaches α through the lower half plane, then $\lim_{z\to\alpha} f(z) = -\pi$. Therefore, above argument will not give analyticity of $Log\ z$ in $\mathbb{C}^*$.

(7): Left as an exercise.

(8): Left as an exercise. □

4.3 Trigonometric and Hyperbolic Functions

Using the Euler formula

$$e^{iy} = \cos y + i\sin y$$

the real sine and cosine functions can be expressed in terms of e^{iy} and e^{-iy} as follows:

$$\sin y = \frac{e^{iy} - e^{-iy}}{2i} \quad \text{and} \quad \cos y = \frac{e^{iy} + e^{-iy}}{2}.$$

It is natural to define the complex sine and cosine functions in terms of the complex exponential functions e^{iz} and e^{-iz} in the same manner as for the real functions. Thus, we have the following definition.

Definition 4.4. The functions $\sin z$ and $\cos z$ are defined as follows:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

Theorem 4.3 (Properties)**.**

1. *$\sin z$ and $\cos z$ are entire functions.*
2. $\dfrac{\mathrm{d}}{\mathrm{d}z}\sin z = \cos z$, *and* $\dfrac{\mathrm{d}}{\mathrm{d}z}\cos z = -\sin z$.
3. *Real and imaginary parts of $\sin z$ and $\cos z$ are given by*

$$\begin{aligned}\sin z &= sinx \cosh y + i\cos x \sinh y\\ \cos z &= \cos x\cosh y - i\sin x\sinh y,\end{aligned}$$

   where $\sinh x = \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2}$ *and* $\cosh x = \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2}$.

4. *Modulii of $\sin z$ and $\cos z$ are given by*

$$\begin{aligned}|\sin z| &= \sqrt{\sin^2 x + \sinh^2 y}\\ |\cos z| &= \sqrt{\cos^2 x + \sinh^2 y}.\end{aligned}$$

5. *$\sin z$ and $\cos z$ are unbounded.*

   [Recall that real sine and cosine functions are bounded. This is an important difference between real and complex sine and cosine functions]

6. *$\sin z = 0$ iff $z = k\pi$, and $\cos z = 0$ iff $z = k\pi + \frac{\pi}{2}$, where $k \in \mathbb{Z}$.*

Proof. **(1):** The complex sine and cosine functions are entire since they are formed by the linear combination of the entire functions e^{iz} and e^{-iz}.

(2): Left as an exercise.

(3): Note that for $z = x + iy$,

$$\begin{aligned}\mathrm{e}^{iz} &= \mathrm{e}^{-y+ix} = \mathrm{e}^{-y}(\cos x + i\sin x)\\ \mathrm{e}^{-iz} &= \mathrm{e}^{y-ix} = \mathrm{e}^{y}(\cos x - i\sin x).\end{aligned}$$

Therefore,

$$\begin{aligned}\sin z &= \frac{\mathrm{e}^{iz} - \mathrm{e}^{-iz}}{2i}\\ &= \frac{i\sin x(\mathrm{e}^{y} + \mathrm{e}^{-y})}{2i} + \frac{\cos x(\mathrm{e}^{-y} - \mathrm{e}^{y})}{2i}\\ &= \sin x\cosh y + \cos x\sinh y.\end{aligned}$$

Similarly, one can prove $\cos z = \cos x\cosh y - i\sin x\sinh y$.

(4): Left as an exercise.

(5): Since $\sinh y$ is unbounded at large values of y, from (4) it follows that $\sin z$ and $\cos z$ are unbounded.

(6): Let $z = x + iy$.

$$\begin{aligned} & \sin z = 0 \\ \Rightarrow \quad & \frac{e^{iz} - e^{-iz}}{2i} = 0 \\ \Rightarrow \quad & e^{2iz} = 1 \\ \Rightarrow \quad & e^{-2y} \cos 2x = 1 \text{ and } e^{-2y} \sin 2x = 0. \end{aligned}$$

Since $e^{-2y} \neq 0$, we obtain $\sin 2x = 0$, and hence $x = \frac{1}{2} n\pi$, $n \in \mathbb{Z}$.
Note that for $x = \frac{1}{2} n\pi$, $n \in \mathbb{Z}$, we obtain $\cos 2x \in \{-1, 1\}$, and hence $e^{-2y} \cos 2x = 1$ gives $e^{-2y} = 1$, and $\cos 2x = 1$. This gives $y = 0$, and $x = n\pi$, $k \in \mathbb{Z}$. Therefore $z = k\pi$. Similarly one can prove that $\cos z = 0$ iff $z = k\pi + \frac{\pi}{2}$. $\square$

The following theorem follows just by applying the definitions.

Theorem 4.4.

1. $\sin(-z) = -\sin z$ *and* $\cos(-z) = \cos z$.
2. $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$.
3. $\sin 2z = 2 \sin z \cos z$.
4. $\sin(z + \pi) = \sin z$, $\sin(z + 2\pi) = \sin z$.
5. $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$.
6. $\cos 2z = \cos^2 z - \sin^2 z$.

Now using sine and cosine we can define $\tan z, \sec z, \operatorname{cosec}\ z$ as in the real case. We can also define complex analogue of the hyperbolic functions $\sinh z = \frac{e^z - e^{-z}}{2}$ and $\cosh z = \frac{e^z + e^{-z}}{2}$. The functions $\tan z$ and $\sec z$ are analytic in any domain that does not include points where $\cos z = 0$. Similarly, the functions $\cot z$ and $\operatorname{cosec}\ z$ are analytic in any domain excluding those points z such that $\sin z = 0$. Derivative formulas for the complex trigonometric and hyperbolic functions are exactly the same as those for their real counterparts, and can be deduced easily using the derivative of e^z.

Problem 4.3. Show that $\overline{\cos z} = \cos \overline{z}$.

Solution. Left as an exercise.

5 Complex Integration

5.1 Integral of a Complex Valued Function of a Real Variable

Let $f : [a, b] \to \mathbb{C}$ be a piecewise continuous function. Then $f(t) = u(t) + iv(t)$ where $u, v : [a, b] \to \mathbb{R}$. We then define

$$\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt.$$

Theorem 5.1 (Properties). *Let $f : [a, b] \to \mathbb{C}$ be a piecewise continuous function.*

1. *If $F'(t) = f(t)$ for all $t \in [a, b]$, then*
$$\int_a^b f(t)\,dt = F(b) - F(a).$$
2. Re $\int_a^b f(t)\,dt = \int_a^b \text{Re } f(t)\,dt = \int_a^b u(t)\,dt.$
3. Im $\int_a^b f(t)\,dt = \int_a^b \text{Im } f(t)\,dt = \int_a^b v(t)\,dt.$
4. $\int_a^b [f(t) + g(t)]\,dt = \int_a^b f(t)\,dt + \int_a^b g(t)\,dt.$
5. $\int_a^b \alpha f(t)\,dt = \alpha \int_a^b f(t)\,dt$*, where α is any complex constant.*
6. *If $a < c < b$, then $\int_a^b f(t)\,dt = \int_a^c f(t)\,dt + \int_c^b f(t)\,dt$.*
7. $\left|\int_a^b f(t)\,dt\right| \le \int_a^b |f(t)|\,dt.$

Proof. Item (1) follows from the fundamental theorem of calculus. Proofs of Items (2)-(6) are obvious. We provide the proof of Item (7). Let $\phi = Arg\left(\int_a^b f(t)\,dt\right)$

$$\begin{aligned}
\left|\int_a^b f(t)\,dt\right| &= \mathrm{e}^{-i\phi}\int_a^b f(t)\,dt \qquad (\because |z| = \mathrm{e}^{-iArg\ z}z)\\
&= \int_a^b \mathrm{e}^{-i\phi} f(t)\,dt\\
&= Re\int_a^b \mathrm{e}^{-i\phi} f(t)\,dt \qquad \left(\because \int_a^b \mathrm{e}^{-i\phi} f(t)\,dt = \left|\int_a^b f(t)\,dt\right| \text{ is real }\right)\\
&= \int_a^b Re\left(\mathrm{e}^{-i\phi} f(t)\right)\,dt\\
&\quad \left[\text{Note that this is Riemann integral of the real function } Re\ \left(\mathrm{e}^{-i\phi} f(t)\right)\right]\\
&\le \int_a^b \left|\mathrm{e}^{-i\phi} f(t)\right|\,dt\\
&\quad \left[\text{ Using the property of Riemann integral with the fact that } Re\ \left(\mathrm{e}^{-i\phi} f(t)\right) \le \left|\mathrm{e}^{-i\phi} f(t)\right|\right]\\
&= \int_a^b |f(t)|\,dt.
\end{aligned}$$

$\square$

Problem 5.1. For $\alpha \in \mathbb{R}$, show that

$$\int_a^b \mathrm{e}^{i\alpha t}\,dt = \frac{\mathrm{e}^{i\alpha b} - \mathrm{e}^{i\alpha a}}{i\alpha}.$$

Solution. Left as an exercise.

5.2 Contour Integral

Definition 5.1 (Contour Integral)**.** Let $C = \gamma(t)$, $t \in [a,b]$ be a contour contained in a domain D, and $f : D \to \mathbb{C}$ be a piecewise continuous function defined on C. Then the contour integral of $f(z)$ along the contour C, denoted as $\int_C f(z)\,dz$ is defined as

$$\int_C f(z)\,dz = \int_a^b f\big(\gamma(t)\big)\gamma'(t)\,dt. \tag{5.1}$$

Proposition 5.2. *The contour integral $\int_C f(z)\,dz$ is independent of the parametrization of the curve C.*

Proof. Let $\gamma : [a,b] \to \mathbb{C}$ and $\mu : [c,d] \to \mathbb{C}$ be two parametrizations of the curve C. Then there exists a bijective and increasing function $g : [c,d] \to [a,b]$ such that $\mu = \gamma \circ g$. Now,

$$\begin{aligned}
\int_\gamma f(z)\,dz &= \int_a^b f(\gamma(u))\gamma'(u)\,du \\
&= \int_a^b f(\gamma(g(t)))\frac{d\gamma(g(t))}{du}g'(t)\,dt \quad (\text{Taking } u = g(t)) \\
&= \int_a^b f(\gamma(g(t)))\frac{d}{dt}(\gamma(g(t)))\,dt \;(\because \frac{d}{dt}(\gamma(g(t))) = \gamma'(g(t))g'(t)) \\
&= \int_a^b f(\mu(t))\mu'(t)\,dt
\end{aligned}$$

□

Problem 5.2. Find $\int_C \frac{1}{z-z_0}\,dz$, where C is a circle centered at z_0 and of radius r. The path is traced out once in the anticlockwise direction.

Solution. Here $f(z) = \frac{1}{z-z_0}$. A parametrization of C is given by $\gamma(\theta) = z_0 + r\mathrm{e}^{i\theta}$, $0 \le \theta \le 2\pi$, where r is the radius of the circle C. Therefore,

$$\begin{aligned}
\int_C \frac{1}{z-z_0}\,dz &= \int_C f(\gamma(\theta))\gamma'(\theta)\,d\theta \\
&= \int_0^{2\pi} f(z_0 + r\mathrm{e}^{i\theta})ir\mathrm{e}^{i\theta}\,d\theta \\
&= \int_0^{2\pi} \frac{1}{r\mathrm{e}^{i\theta}}ir\mathrm{e}^{i\theta}\,d\theta \\
&= \int_0^{2\pi} i\,d\theta \\
&= 2\pi i.
\end{aligned}$$

Theorem 5.3 (Properties)**.**

1. *$\int_C f(z)\,dz = -\int_{-C} f(z)\,dz$.*
2. *Let $C = C_1 + C_2 + \cdots + C_n$, where the terminal point of C_k coincides with the initial point of C_{k+1}, then*

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \cdots + \int_{C_n} f(z)\,dz.$$

3. *$\int_C K f(z)\,dz = K\int_C f(z)\,dz$, K is a complex constant.*
4. *$\int_C (f(z) + g(z))\,dz = \int_C f(z)\,dz + \int_C g(z)\,dz$.*

Proof. We provide the proof of Item 1 only. Rest of the proofs follow straightway from the definitions involved and left as exercise.
Let $C : \gamma(t),\ a \le t \le b$ be a parametrization of C. Then a parametrization of $-C$ is given by $-C : \gamma_-(t) = \gamma(b + a - t),\ a \le t \le b$. Therefore,

$$\begin{aligned}
\int_{-C} f(z)\,dz &= \int_a^b f(\gamma_-(t))\frac{\mathrm{d}}{\mathrm{d}t}(\gamma_-(t))\,dt \\
&= \int_a^b f(\gamma(b+a-t))\frac{\mathrm{d}}{\mathrm{d}t}(\gamma(b+a-t))\,dt \\
&= \int_a^b f(\gamma(s))\frac{\mathrm{d}}{\mathrm{d}t}(\gamma(s))\,dt, \quad s = b + a - t \\
&= -\int_b^a f(\gamma(s))\frac{\mathrm{d}}{\mathrm{d}s}(\gamma(s))\frac{\mathrm{d}s}{\mathrm{d}t}\,ds \\
&= \int_b^a f(\gamma(s))\frac{\mathrm{d}}{\mathrm{d}s}(\gamma(s))\,ds \\
&= -\int_a^b f(\gamma(s))\frac{\mathrm{d}}{\mathrm{d}s}(\gamma(s))\,ds \\
&= -\int_C f(z)\,dz.
\end{aligned}$$

□

Problem 5.3. Evaluate $I = \int_C z^2\,dz$, where

1. C is along x-axis from 0 to 1 and then along the line parallel to y-axis from 1 to $1 + 2i$.
2. C is the line segment from 0 to $1 + 2i$.
3. C is an arc of unit circle $|z| = 1$ traversed in the anticlockwise direction with initial point -1 and final point i.
4. C is an arc of unit circle $|z| = 1$ traversed in the clockwise direction with initial point -1 and final point i.

Solution. **(1):** Let C_1 be x-axis from 0 to 1 and C_2 be the line parallel to y-axis from 1 to $1+2i$. Then $C = C_1 + C_2$ (cf. Figure 7). Parametrizations of C_1 and C_2 are given by

$$\begin{aligned} C_1: &\quad \gamma_1(t) = t,\ 0 \le t \le 1 \\ C_2: &\quad \gamma_2(t) = (1-t) + t(1+2i) \\ &\qquad\quad = 1 + 2it,\ 0 \le t \le 1 \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{C_1} z^2\, dz &= \int_0^1 t^2\, dt = \frac{1}{3} \quad \text{(from (5.1))} \\ \int_{C_2} z^2\, dz &= \int_0^1 (1+2it)^2 2i\, dt = -4 - \frac{2}{3} i. \end{aligned}$$

Hence,

$$\int_C z^2\, dz = \int_{C_1} z^2\, dz + \int_{C_2} z^2\, dz = -\frac{11+2i}{3}.$$

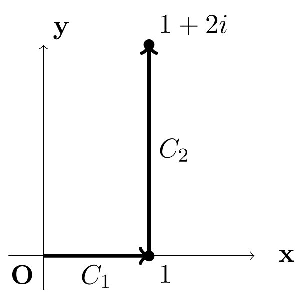

Figure 7:

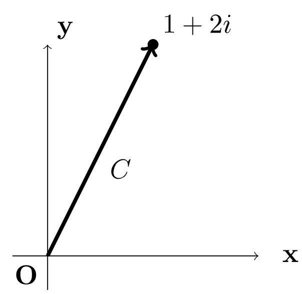

Figure 8:

(2): Left as an exercise. C is given by Figure 8.

(3): Let C_1, C_2, C_3 and C_4 are the arc of the unit circle $S : |z| = 1$ from 1 to i, i to -1, -1 to $-i$ and $-i$ to 1 respectively (cf. Figure 9). Then $C = C_3 + C_4 + C_1$. Therefore

$$\begin{aligned} \int_S z^2\, dz &= \int_{C_1} z^2\, dz + \int_{C_2} z^2\, dz + \int_{C_3} z^2\, dz + \int_{C_4} z^2\, dz \\ &= \int_C z^2\, dz + \int_{C_2} z^2\, dz \end{aligned}$$

Parametrizations of S and C_2 are given by

$$\begin{aligned} S: &\quad \gamma_1(t) = \mathrm{e}^{i\theta},\ 0 \le \theta \le 2\pi \\ C_2: &\quad \gamma_2(t) = \mathrm{e}^{i\theta},\ \frac{\pi}{2} \le \theta \le \pi. \end{aligned}$$

Therefore,

$$
\begin{aligned}
\therefore \int_C z^2\,dz &= -\int_{C_2} z^2\,dz + \int_S z^2\,dz \\
&= -\int_{\frac{\pi}{2}}^{\pi} \left(e^{i\theta}\right)^2 ie^{i\theta}\,d\theta + \int_0^{2\pi} \left(e^{i\theta}\right)^2 ie^{i\theta}\,d\theta \\
&= -\frac{1}{3}\left[e^{3i\theta}\right]_{\frac{\pi}{2}}^{\pi} + \frac{1}{3}\left[e^{3i\theta}\right]_0^{2\pi} \\
&= \frac{1}{3}(1-i)
\end{aligned}
$$

(4): C is shown in Figure 10. Let $D : \gamma(t) = e^{i\theta}$, $\frac{\pi}{2} \le \theta \le \pi$, be the arc of the unit circle

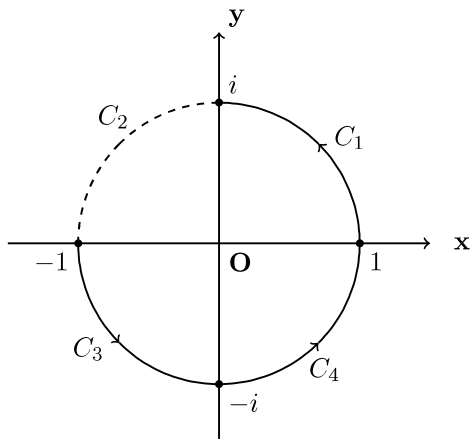

Figure 9:

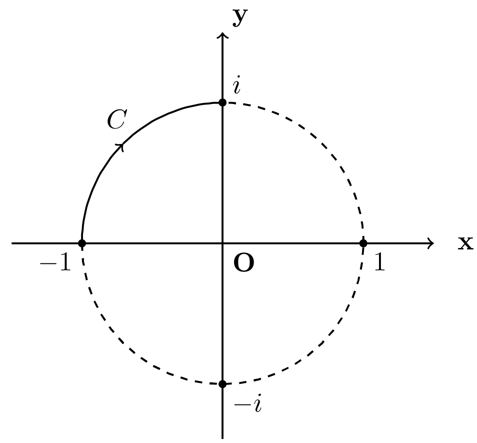

Figure 10:

$|z| = 1$ from i to -1. Then $-C = D$. Therefore

$$
\begin{aligned}
\int_C z^2\,dz &= -\int_{-C} z^2\,dz \quad \text{(by Item 1 of Theorem 5.3)} \\
&= -\int_{\frac{\pi}{2}}^{\pi} \left(e^{i\theta}\right)^2 ie^{i\theta}\,d\theta \\
&= -\frac{1}{3}\left[e^{3i\theta}\right]_{\frac{\pi}{2}}^{\pi} \\
&= -\frac{1}{3}(i-1) \\
&= \frac{1}{3}(1-i).
\end{aligned}
$$

The reason for obtaining same answer for the problems in Items 3 and 4 of Problem 5.3 is due to the following result.

Theorem 5.4 (Integration using Indefinite Integral)**.** *Let f be a continuous function defined on an open set D and there exist a function F defined on D such that $F' = f$. Let $z_1, z_2 \in D$. Then for any contour C lying in D starting from z_1, and ending at z_2,*

$$\int_C f(z)\,dz = F(z_2) - F(z_1).$$

Proof. Suppose that that C is given by a map $\gamma : [a, b] \to \mathbb{C}$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}F(\gamma(t)) = F'(\gamma(t))\gamma'(t).$$

Hence

$$\begin{aligned}
\int_C f(z)\,dz &= \int_a^b f(\gamma(t))\gamma'(t)\,dt \\
&= \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}F(\gamma(t))\,dt \\
&= \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}G(t)\,dt, \text{ where } G(t) = F(\gamma(t)) \\
&= G(b) - G(a) \quad \text{(by Item (1) of Theorem 5.1)} \\
&= F(\gamma(b)) - F(\gamma(a)) \\
&= F(z_2) - F(z_1).
\end{aligned}$$

□

Remark 5.1. In order to use Theorem 5.4 to evaluate $\int_C f(z)\,dz$, we need to find a simply connected domain D such that

1. f is continuous on D,
2. $F' = f$ on D, for some F,
3. C lies in D.

Compare Theorem 5.4 with the fundamental theorem of calculus.

Problem 5.4. Evaluate the following integrals.

1. $\int_{C_1} z^2\,dz$ and $\int_{C_2} z^2\,dz$ where C_1 and C_2 are arc of unit circle $|z| = 1$ with initial point -1 and final point i traversed in the clockwise and anticlockwise directions respectively.
2. $\int_C z^2\,dz$, where C is any contour starting from z_1 and ending at z_2.
3. $\int_C \cos z\,dz$, where C is any contour starting from $-i\pi$ and ending at $i\pi$.
4. $\int_C \frac{1}{z}\,dz$, where C is the straight line from $-i$ to 1 and then from 1 to i.

Solution. **(1):** Note that $\dfrac{\mathrm{d}}{\mathrm{d}z}\left(\dfrac{z^3}{3}\right) = z^2$ for all $z \in \mathbb{C}$. Therefore, we take $\mathbb{C}$ to be the simply connected domain D and then by Theorem 5.4, we obtain

$$\int_{C_1} z^2\,dz = \int_{C_2} z^2\,dz = \left[\frac{z^3}{3}\right]_{-1}^{i} = \frac{i^3 - (-1)^3}{3} = \frac{1-i}{3}.$$

This problem illustrates why we obtain same answer for the problems in Items 3 and 4 of Problem 5.3.

(2): Again using Theorem 5.4, we obtain

$$\int_C z^2\,dz = \left[\frac{z^3}{3}\right]_{z_1}^{z_2} = \frac{z_2^3 - z_1^3}{3}.$$

(3): Since $\dfrac{\mathrm{d}}{\mathrm{d}z}\sin z = \cos z$ for all $z \in \mathbb{C}$, and hence by Theorem 5.4, we obtain

$$\int_C \cos z\,dz = \Big[\sin z\Big]_{-i\pi}^{i\pi} = \sin(i\pi) - \sin(-i\pi) = 2\sin(i\pi).$$

(4): Let us take the domain D to be the set $\mathbb{C}\setminus\{x+iy : y=0, x \le 0\}$. Since $\dfrac{\mathrm{d}}{\mathrm{d}z}Log\, z = \dfrac{1}{z}$ for all $z \in \mathbb{D}$, and C lies in $\mathbb{D}$, we obtain from Theorem 5.4

$$\int_C \frac{1}{z}\,dz = \Big[Log\, z\Big]_{-i}^{i} = Log\,(i) - Log\,(-i) = i\frac{\pi}{2} - i\frac{-\pi}{2} = i\pi.$$

Problem 5.5. Explain why the following integrals cannot be evaluated using Theorem 5.4:

1. $\int_C \frac{1}{z}\,dz$, where C is the straight line from i to $-i$.
2. $\int_C |z|^2\,dz$, where C is an arc of unit circle $|z| = 1$ traversed in the clockwise direction with initial point -1 and final point i.

Problem 5.6. Evaluate $\int_C |z|^2\,dz$, where C is an arc of unit circle $|z| = 1$ traversed in the clockwise direction with initial point -1 and final point i.

Solution. Left as an exercise.

Definition 5.2 (Arc Length of a Contour)**.** The arc length of a contour $C : \gamma(t) = x(t) + iy(t)$, $a \le t \le b$ is given by

$$\begin{aligned} L(\gamma) &= \int_a^b |\gamma'(t)|\,dt \\ &= \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\,dt \end{aligned}$$

Example 5.1. Consider the circle with center at z_0 and radius r given by $\gamma(t) = z_0 + re^{it}$, $0 \le t \le 2\pi$. Then

$$L(\gamma) = \int_0^{2\pi} |ire^{it}|\,dt = \Big[rt\Big]_{t=0}^{2\pi} = 2\pi r.$$

Theorem 5.5 (L-M Formula)**.** *If L is the arc length of a contour $C : \gamma(t)$, $a \le t \le b$ and M is a positive number such that $|f(z)| \le M$ for all $z \in C$, then*

$$\left|\int_C f(z)\,dz\right| \le ML.$$

Proof.

$$
\begin{aligned}
\left|\int_C f(z)\,dz\right| &= \left|\int_a^b f(\gamma(t))\gamma'(t)\,dt\right| \\
&\leq \int_a^b |f(\gamma(t))| \left|\gamma'(t)\right| \,dt \\
&\leq \int_a^b M \left|\gamma'(t)\right| \,dt \\
&= M\int_a^b \left|\gamma'(t)\right| \,dt \\
&= ML
\end{aligned}
$$

□

5.3 Cauchy Integral Theorem

Theorem 5.6 (Cauchy Integral Theorem)**.** *Let $f(z)$ be a function such that*

1. *$f(z)$ is analytic* on and inside *a simple closed contour C, and*
2. *$f'(z)$ be continuous on and inside C.*

Then

$$\int_C f(z)\,dz = 0. \tag{5.2}$$

Proof. Let $f = u + iv$, and $C : \gamma(t) = x(t) + iy(t)$, $a \le t \le b$. Then

$$\begin{aligned}
\int_C f(z)\,dz &= \int_a^b f(\gamma(t))\gamma'(t)\,dt \\
&= \int_a^b \left[u\big(x(t),y(t)\big) + iv\big(x(t),y(t)\big)\right]\left[x'(t) + iy'(t)\right]\,dt \\
&= \int_a^b \Big[u\big(x(t),y(t)\big)x'(t) - v\big(x(t),y(t)\big)y'(t)\Big]\,dt + \\
&\qquad\qquad i\int_a^b \left(v\big(x(t),y(t)\big)x'(t) + u\big(x(t),y(t)\big)y'(t)\right)dt \\
&= \int_a^b \Big(u\big(x(t),y(t)\big), -v\big(x(t),y(t)\big)\Big)\cdot\Big(x'(t),y'(t)\Big)\,dt + \\
&\qquad\qquad i\int_a^b \Big(v\big(x(t),y(t)\big), u\big(x(t),y(t)\big)\Big)\cdot\Big(x'(t),y'(t)\Big)\,dt \\
&\quad \Big[\text{`}\cdot\text{' denotes the dot product, and the integrals here are the line integrals}\Big] \\
&= \int_C u\,dx - v\,dy + \int_C v\,dx + u\,dy \\
&= \iint_D \left(-v_x - u_y\right)\,dx\,dy + i\iint_D \left(u_x - v_y\right)\,dx\,dy, \\
&\quad \text{where } D \text{ is the closed region enclosed by } C \\
&\quad \Big[\text{Using Green's Theorem: } \iint_D \left(N_x - M_y\right)\,dx\,dy = \int_C M\,dx + N\,dy\Big] \\
&= 0 \quad \text{(Using C-R equations).}
\end{aligned}$$

□

In 1903, Goursat was able to obtain the same result as in eq. (5.2) without assuming the continuity of $f'(z)$. This stronger version is called the *Cauchy-Goursat Theorem* or *Goursat Theorem*, and is stated as follows.

Theorem 5.7 (Cauchy-Goursat Theorem)**.** *Let $f(z)$ be a function such that $f(z)$ is analytic* on and inside *a simple closed contour C. Then*

$$\int_C f(z)\,dz = 0.$$

The Cauchy-Goursat theorem can be stated in the following alternative form:

Theorem 5.8 (Alternative Form of Cauchy-Goursat Theorem). *Let $f(z)$ be a function analytic throughout a simply connected domain D and C be a simple closed contour lying completely inside D (cf. Figure 11). Then*

$$\int_C f(z)\,dz = 0.$$

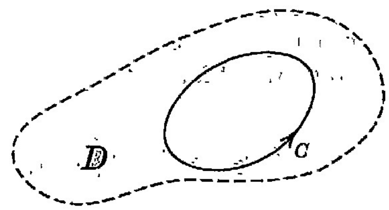

Figure 11:

Problem 5.7. Prove that the statements of the Cauchy-Goursat theorem given in Theorems 5.7 and 5.8 are equivalent.

Problem 5.8. Evaluate the following integrals where C denotes the circle of unit radius with center at zero.

1. $\int_C \sin z$.
2. $\int_C \mathrm{e}^{z^n}$.
3. $\int_C \frac{\mathrm{e}^{iz^2}}{z^2+4}\,dz$.

Solution. **(1)-(2):** Since $f(z) = \sin z$ and $g(z) = \mathrm{e}^{z^n}$ are both analytic on and inside C, it follows from Cauchy-Goursat Theorem 5.7 that

$$\int_C \sin z = 0 \text{ and } \int_C \mathrm{e}^{z^n} = 0.$$

(3): The function $f(z) = \frac{\mathrm{e}^{iz^2}}{z^2+4}$ is analytic on and inside C. Therefore, from Cauchy-GoursatTheorem 5.7 we have

$$\int_C \frac{\mathrm{e}^{iz^2}}{z^2+4}\,dz = 0.$$

Note that the integrand $\frac{\mathrm{e}^{iz^2}}{z^2+4}$ is not analytic at $z = \pm 2$ but that does not bother us as these points are outside C.

Problem 5.9. Evaluate $\int_C \operatorname{cosec}^2 z$, where C denotes the circle of unit radius with center at zero. Explain why Cauchy-Goursat's Theorem cannot be applied here to evaluate the integral.

Solution. It is obvious that $\operatorname{cosec}^2 z$ is not analytic at $z = k\pi, k \in \mathbb{Z}$. Let us take the domain $D^* = \{z : 0 < |z| < \pi\}$. Note that $\operatorname{cosec}^2 z$ is continuous in D^*, $\dfrac{\mathrm{d}}{\mathrm{d}z}(-\cot z) = \operatorname{cosec}^2 z$ for all $z \in D^*$, and C lies in D^*. Therefore, we obtain from Theorem 5.4 $\int_C \operatorname{cosec}^2 z = 0$ as initial and final points of C are same.
Goursat's Theorem cannot be applied to evaluate $\int_C \operatorname{cosec}^2 z$ as the function $\operatorname{cosec}^2 z$ is not analytic inside C (in fact, $\operatorname{cosec}^2 z$ is not analytic at origin).

Theorem 5.9 (Consequences of Cauchy-Goursat's Theorem)**.**

1. ***(Independence of Path):*** *Let D be a simply connected domain and f be an analytic function defined in D. Then the integral of $f(z)$ is independent of path in D. That is, if z_1, z_2 are two points in D and C_1 and C_2 be two contours inside D joining z_1 and z_2, then we have*

$$\int_{C_1} f(z)\, dz = \int_{C_2} f(z)\, dz.$$

2. ***(Existence of Antiderivative):*** *If f is an analytic function on a simply connected domain D then there exists a function F which is analytic on D such that $F' = f$.*

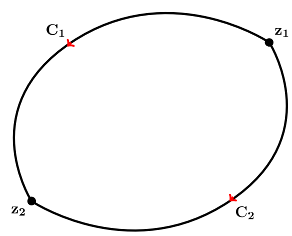

Figure 12:

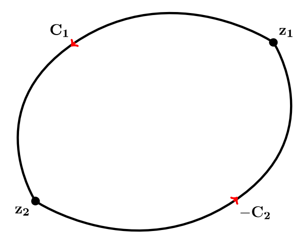

Figure 13:

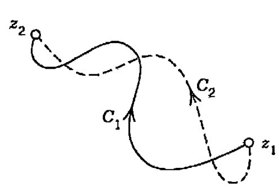

Figure 14:

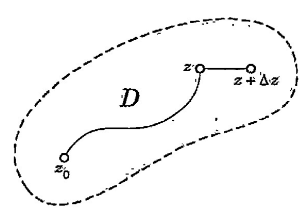

Figure 15:

Proof. **(1):** For paths C_1 and C_2 that have only the endpoints in common (cf. Figure 12), we consider closed curve $C := C_1 + (-C_2)$ obtain by joining C_1 and $-C_2$ as shown in

Figure 13. Then by Cauchy-Goursat's Theorem, we have

$$\begin{aligned}
&\int_C f(z)\,dz = 0\\
\Rightarrow\ &\int_{C_1} dz + \int_{-C_2} dz = 0\\
\Rightarrow\ &\int_{C_1} dz - \int_{C_2} dz = 0\\
\Rightarrow\ &\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.
\end{aligned}$$

For paths that have finitely many further common points, apply the above argument to each "loop" (portions of C_1 and C_2 between consecutive common points; four loops in Figure 14. For paths with infinitely many common points we would need additional argumentation not to be presented here.

(2): Fix a point $z_0 \in D$, and define

$$F(z) = \int_{z_0}^{z} f(w)\,dw,$$

where the integral is considered as a contour integral over any curve lying in D and joining z with z_0. By the first part of this theorem, the integral does not depend on the curve we choose and hence the function F is well defined. We will show that $F' = f$.
We keep z fixed. Then we choose $z + \Delta z$ in D so that the whole line segment with endpoints z and $z + \Delta$ is in D (Figure 15). This can be done because D is a domain, hence it contains a neighborhood of z. Then

$$\begin{aligned}
F(z+\Delta z) - F(z) &= \int_{z_0}^{z+\Delta z} f(w)\,dw - \int_{z_0}^{z} f(w)\,dw\\
&= \int_{z}^{z+\Delta z} f(w)\,dw,
\end{aligned}$$

where the curve joining z and $z + \Delta$ can be considered as a straight line. Since z is fixed, we have $\int_z^{z+\Delta z} f(z)\,dw = \Delta z$, and hence we obtain

$$\left|\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z)\right| = \frac{1}{\Delta z}\left|\int_z^{z+\Delta z} \big[f(w) - f(z)\big]\,dw\right|. \tag{5.3}$$

Since f is continuous at z, given $\epsilon > 0$, there exist a $\delta > 0$ such that $|f(w) - f(z)| < \epsilon$ if $|w - z| < \delta$. Thus for $|\Delta z| < \delta$, we obtain

$$|f(w) - f(z)| < \epsilon,$$

for all w lying on the line segment with endpoints z and $z + \Delta z$. Therefore, by L-M formula (cf. Theorem 5.5), we obtain

$$\left|\int_z^{z+\Delta z} \big[f(w) - f(z)\big]\,dw\right| \le \epsilon \Delta z.$$

This together with eq. (5.3) gives for $|\Delta z| < \delta$,

$$\left|\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z)\right| \leq \epsilon.$$

This implies

$$F'(z) = \lim_{\Delta z \to 0} \frac{F(z+\Delta z) - F(z)}{\Delta z} = f(z).$$

$\square$

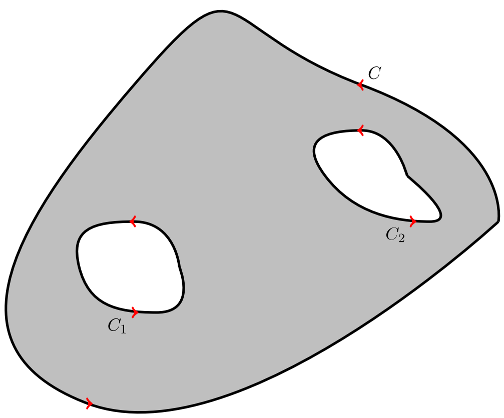

Figure 16:

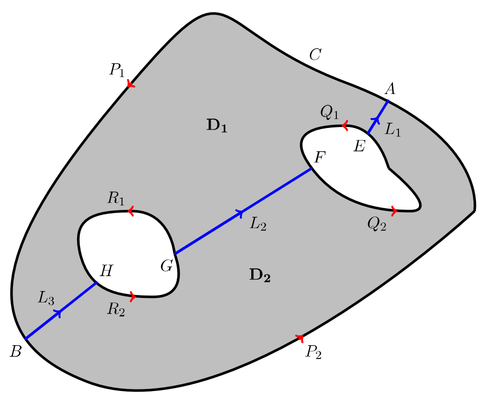

Figure 17:

Theorem 5.10 (Cauchy-Goursat's Theorem for Multiply Connected Domain)**.** *Let C be a positively oriented simple closed contour and C_k, $k = 1, 2, \ldots n$ denote a finite number of positively oriented simple closed contours all lying wholly within C, but each C_k lies in the exterior of every other whose interior have no points in common (cf. Figure 16 for $n = 2$). If a function f is analytic throughout the closed region D consisting of all points within and on C except for the points interior to each C_k (gray region of Figure 16 for $n = 2$). Then*

$$\int_C f(z)\, dz = \int_{C_1} f(z)\, dz + \int_{C_2} f(z)\, dz + \cdots + \int_{C_n} f(z)\, dz.$$

Proof. We provide the proof for the case when $n = 2$.
Let R_1 and R_2 be the portions of C_1 from G to H and H to G respectively as shown in Figure 17. Similarly, Q_1 and Q_2 are the portions of C_2 from E to F and F to E and P_1 and P_2 are the portions of C from A to B and B to A.
By three cuts L_1, L_2, L_3, we divide the domain D into two simply connected domains D_1 and D_2 (cf. Figure 17). Domain D_1 and D_2 are enclosed by the simple closed contours T_1 and T_2 respectively, where

$$\begin{aligned} T_1 &= P_1 + L_3 + (-R_1) + L_2 + (-Q_1) + L_1 \\ T_2 &= P_2 + (-L_1) + (-Q_2) + (-L_2) + (-R_2) + (-L_3). \end{aligned}$$

From Cauchy-Goursat Theorem 5.7, we obtain

$$\begin{aligned} \int_{T_1} f(z)\,dz &= 0 \\ \int_{T_2} f(z)\,dz &= 0 \end{aligned}$$

and hence

$$\begin{aligned} 0 &= \int_{T_1} f(z)\,dz + \int_{T_2} f(z)\,dz \\ &= \Big[\int_{P_1} f(z)\,dz + \int_{L_3} f(z)\,dz + \int_{-R_1} f(z)\,dz + \int_{L_2} f(z)\,dz + \int_{-Q_1} f(z)\,dz \\ &\quad + \int_{L_1} f(z)\,dz\Big] + \Big[\int_{P_2} f(z)\,dz + \int_{-L_1} f(z)\,dz + \int_{-Q_2} f(z)\,dz + \int_{-L_2} f(z)\,dz \\ &\quad + \int_{-R_2} f(z)\,dz + \int_{-L_3} f(z)\,dz\Big] \\ &= \Big[\int_{P_1} f(z)\,dz + \int_{L_3} f(z)\,dz - \int_{R_1} f(z)\,dz + \int_{L_2} f(z)\,dz - \int_{Q_1} f(z)\,dz \\ &\quad + \int_{L_1} f(z)\,dz\Big] + \Big[\int_{P_2} f(z)\,dz - \int_{L_1} f(z)\,dz - \int_{Q_2} f(z)\,dz - \int_{L_2} f(z)\,dz \\ &\quad - \int_{R_2} f(z)\,dz - \int_{L_3} f(z)\,dz\Big] \\ &= \int_{P_1} f(z)\,dz + \int_{P_2} f(z)\,dz - \int_{R_1} f(z)\,dz - \int_{R_2} f(z)\,dz - \int_{Q_1} f(z)\,dz - \int_{Q_2} f(z)\,dz \\ &= \int_{C} f(z)\,dz - \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz \end{aligned}$$

This gives

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz.$$

For domains of higher connectivity the idea remains the same. In fact, for $n = k$, we need $k+1$ cuts. □

Corollary 5.11. *Let C_1 and C_2 be two simple closed contours with* same orientation *such that C_2 completely lies inside C_1. Let f be a function analytic in the annulus region between C_1 and C_2 including the points on the curves. Then*

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$$

5.4 Cauchy's Integral Formula

Theorem 5.12 (Cauchy's Integral Formula)**.** *Let the function $f(z)$ be analytic on and inside a positively oriented simple closed contour C and z_0 be any point inside C. Then*

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz. \tag{5.4}$$

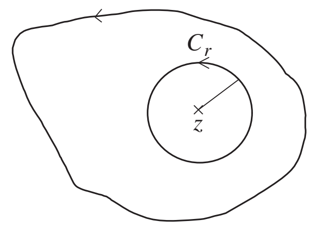

Figure 18:

Proof. We draw a circle C_r , with radius r, centered at the point z_0, small enough to be completely inside C (see Figure 18). Since $\frac{f(z)}{z-z_0}$ is analytic in the region lying between C_r and C, by Theorem 5.10, we have

$$\begin{aligned}\frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz &= \frac{1}{2\pi i}\int_{C_r} \frac{f(z)}{z-z_0}\,dz \\ &= \frac{1}{2\pi i}\int_{C_r} \frac{f(z)-f(z_0)}{z-z_0}\,dz + \frac{f(z_0)}{2\pi i}\int_{C_r} \frac{1}{z-z_0}\,dz \\ &= \frac{1}{2\pi i}\int_{C_r} \frac{f(z)-f(z_0)}{z-z_0}\,dz + f(z_0) \quad \text{(by Problem 5.2)}\end{aligned}$$

Therefore, to complete the proof, it suffices to show that

$$\frac{1}{2\pi i}\int_{C_r} \frac{f(z)-f(z_0)}{z-z_0}\,dz = 0.$$

Since f is continuous at z_0, for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(z) - f(z_0)| < \epsilon \quad \text{whenever} \quad |z - z_0| < \delta.$$

Now, suppose we choose $r < \delta$ and thus we ensure that C_r lies completely inside the contour C. Let $g(z) = \frac{f(z)-f(z_0)}{z-z_0}$. Note that

$$|g(z)| = \left|\frac{f(z)-f(z_0)}{z-z_0}\right| = \left|\frac{f(z)-f(z_0)}{r}\right| < \frac{\epsilon}{r} \quad \text{for all} \quad z \in C_r : |z - z_0| = r.$$

Thus, by LM-formula (Theorem 5.5), we have

$$\left|\int_{C_r} \frac{f(z)-f(z_0)}{z-z_0}\,dz\right| = \left|\int_{C_r} g(z)\,dz\right| \le \frac{\epsilon}{r} \times 2\pi r = 2\pi\epsilon. \tag{5.5}$$

Therefore,

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_{C_r} \frac{f(z)-f(z_0)}{z-z_0}\,dz\right| &= \frac{1}{2\pi}\left|\int_{C_r} \frac{f(z)-f(z_0)}{z-z_0}\,dz\right| \\ &\le \epsilon \quad \text{(by eq. (5.5))}\end{aligned}$$

Since the above inequality holds for every ϵ, it follows that

$$\begin{aligned}&\left|\frac{1}{2\pi i}\int_{C_r} \frac{f(z)-f(z_0)}{z-z_0}\,dz\right| = 0 \\ \Rightarrow\ &\frac{1}{2\pi i}\int_{C_r} \frac{f(z)-f(z_0)}{z-z_0}\,dz = 0. \qquad \square\end{aligned}$$

The Cauchy integral formula is a remarkable result. The value of $f(z)$ at any point inside the closed contour C is determined by the values of the function along the bounding contour C.

Problem 5.10. Evaluate the following integrals.

1. $\int_C \frac{e^z}{z}\,dz$, where C is the circle $|z-4|=5$ traversed in the anticlockwise direction.
2. $\int_C \frac{z^2}{z^2+1}\,dz$, where C is the circle $|z-i|=1$ traversed in the clockwise direction.
3. $\int_C \frac{e^z}{z(z-1)}\,dz$, where C is the circle $|z|=2$ traversed in the anticlockwise direction.

Solution. **(1):** Here $f(z)=e^z$ is analytic within and on C, and 0 lies inside C. Therefore, by Theorem 5.12, we obtain

$$\int_C \frac{e^z}{z}\,dz = 2\pi i \times f(0) = 2\pi i.$$

(2): Let C_1 be the circle $|z-i|=1$ traversed in the anticlockwise direction. Here $f(z)=\frac{z^2}{(z+i)}$ is analytic within and on D, and i lies inside C. Therefore, by Theorem 5.12, we obtain

$$\int_{C_1} \frac{z^2}{z^2+1}\,dz = 2\pi i \times f(i) = 2\pi i \times \frac{-1}{2i} = -\pi.$$

Therefore,

$$\int_C \frac{z^2}{z^2+1}\,dz = -\int_{C_1} \frac{z^2}{z^2+1}\,dz = \pi.$$

(3): Note that we cannot use directly Theorem 5.12 to evaluate this integral as $z=0$ and $z=1$ both lies inside the circle C. We can proceed in two ways:

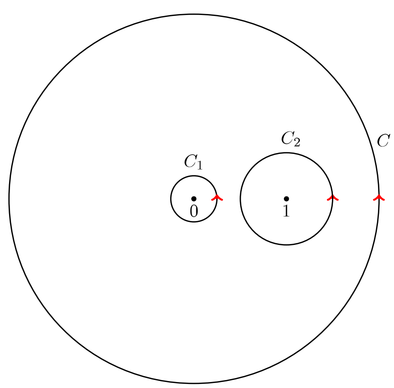

Figure 19:

Method 1: Let $C_1 : |z| = \frac{1}{4}$, and $C_2 : |z-1| = \frac{1}{2}$ (cf. Figure 19). Then by Theorem 5.10, we obtain

$$\begin{aligned}\int_C \frac{e^z}{z(z-1)}\,dz &= \int_{C_1} \frac{e^z}{z(z-1)}\,dz + \int_{C_2} \frac{e^z}{z(z-1)}\,dz \\ &= 2\pi i \times (-1) + 2\pi i \times e\end{aligned}$$

Method 2:

$$\begin{aligned}
\int_C \frac{e^z}{z(z-1)}\,dz &= \int_C e^z\left[\frac{1}{z-1}-\frac{1}{z}\right]dz \\
&= \int_C \frac{e^z}{z-1}\,dz - \int_C \frac{e^z}{z}\,dz \\
&= 2\pi i \times e^1 - 2\pi i \times e^0 \\
&= 2\pi i(e-1).
\end{aligned}$$

Theorem 5.13 (Derivative of Contour Integrals)**.** *If f is analytic on a simply connected domain D then f has derivatives of all orders in D (which are then analytic in D) and for any $z_0 \in D$, one has*

$$f^n(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz, \tag{5.6}$$

where C is a simple closed contour (oriented counterclockwise) around z_0 in D.

Proof. Assuming that z_0 and z_0+h both lie inside C, we consider the expression

$$\begin{aligned}
&\frac{f(z_0+h)-f(z_0)}{h} - \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}\,dz \\
=\ & \frac{1}{h}\left\{\frac{1}{2\pi i}\int_C \frac{f(z)}{z-(z_0+h)}\,dz - \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz - h\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}\,dz\right\} \\
&\text{(by Theorem 5.12)} \\
=\ & \frac{1}{h2\pi i}\left\{\int_C \left[\frac{f(z)}{z-(z_0+h)} - \frac{f(z)}{z-z_0} - h\frac{f(z)}{(z-z_0)^2}\right]dz\right\} \\
=\ & \frac{1}{h2\pi i}\left\{\int_C h^2\frac{f(z)}{(z-z_0-h)(z-z_0)^2}\,dz\right\} \\
=\ & \frac{h}{2\pi i}\left\{\int_C \frac{f(z)}{(z-z_0-h)(z-z_0)^2}\,dz\right\}
\end{aligned}$$

Thus, we have

$$\frac{f(z_0+h)-f(z_0)}{h} - \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}\,dz = \frac{h}{2\pi i}\left\{\int_C \frac{f(z)}{(z-z_0-h)(z-z_0)^2}\,dz\right\}. \tag{5.7}$$

To prove the theorem, it now suffices to show that

$$\lim_{h\to 0}\frac{h}{2\pi i}\left\{\int_C \frac{f(z)}{(z-z_0-h)(z-z_0)^2}\,dz\right\} = 0.$$

To estimate the value of the last integral, we draw the circle $C_{2d} : |z-z_0| = 2d$ that lies completely inside the domain bounded by C and choose h such that $0 < |h| < d$. Every point z on the curve C is then outside the circle C_{2d} so that

$$\left.\begin{aligned} |z-z_0| > d \quad &\text{and} \\ |z-z_0-h| > d \quad &\text{for all } z \in C. \end{aligned}\right\} \tag{5.8}$$

Note that, as f is continuous it is bounded on C. Therefore, there exists an $M > 0$ such that

$$|f(z)| < M \quad \text{for all} \quad z \in C. \tag{5.9}$$

Let $g(z) = \frac{f(z)}{(z-z_0-h)(z-z_0)^2}$. Then from equations (5.8) and (5.9), we obtain

$$|g(z)| = \left|\frac{f(z)}{(z-z_0-h)(z-z_0)^2}\right| \le \frac{M}{d^3} \text{ for all } \ z \in C.$$

Thus, by LM-formula (Theorem 5.5), we have

$$\left|\int_C \frac{f(z)}{(z-z_0-h)(z-z_0)^2}\, dz\right| \le \frac{M}{d^3} L,$$

where L is the total arc length of C. Therefore,

$$\lim_{h\to 0}\left|\frac{h}{2\pi i}\int_C \frac{f(z)}{(z-z_0-h)(z-z_0)^2}\, dz\right| \le \lim_{h\to 0}\frac{|h|}{2\pi}\frac{M}{d^3}L = 0.$$

Hence, from eq. (5.7), we obtain

$$f'(z_0) = \lim_{h\to 0}\frac{f(z_0+h) - f(z_0)}{h} = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}\, dz.$$

By induction, the generalized Cauchy integral formula can be established as follows:

$$f^n(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz.$$

□

Theorem 5.13 can be equivalently formulated as follows:

Theorem 5.14 (Derivative of Contour Integrals)**.** *Let the function $f(z)$ be analytic on and inside a positively oriented simple closed contour C and z_0 be any point inside C. Then*

$$f^n(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz.$$

As a direct consequence of Theorem 5.13, we get the following important result which is very different from real analysis:

Theorem 5.15.

1. *If a function $f(z)$ is analytic at a point, then its derivatives of all orders are also analytic at the same point.*

2. *If a function $f(z)$ is analytic in a domain D, then its derivatives of all orders are also analytic in D.*

Remark 5.2. Suppose we express an analytic function inside a domain D as $f(z) = u(x,y) + iv(x,y), z = x + iy$. Since the derivatives of f of all orders are analytic functions, it then follows that the partial derivatives of $u(x,y)$ and $v(x,y)$ of all orders exist and are continuous. This result is consistent with the earlier claim on continuity of higher-order derivatives when we discuss the theory of harmonic functions in Section 3.4: the mere assumption of analyticity of f at a point would guarantee the continuity of all second-order derivatives of the real part and imaginary part of f.

Remark 5.3. For memorizing (5.6), it is useful to observe that these formulas are obtained formally by differentiating the Cauchy formula (5.4) under the integral sign with respect to z_0.

Theorem 5.16 (Cauchy's Integral Formula for Multiply Connected Domain)**.** *Let C be a positively oriented simple closed contour and C_k, $k = 1, 2, \dots n$ denote a finite number of positively oriented simple closed contours all lying wholly within C, but each C_k lies in the exterior of every other whose interior have no points in common (cf. Figure 16 for $n = 2$). Let the function f be analytic throughout the closed region D consisting of all points within and on C except for the points interior to each C_k (gray region of Figure 16 for $n = 2$). Then for z_0 belonging to interior of D,*

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}\,dz - \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{z - z_0}\,dz - \frac{1}{2\pi i}\int_{C_2} \frac{f(z)}{z - z_0}\,dz - \cdots - \frac{1}{2\pi i}\int_{C_n} \frac{f(z)}{z - z_0}\,dz.$$

Problem 5.11. Evaluate the following integrals.

1. $\int_C e^z z^{-3}\,dz$, where C is the circle $|z| = 1$ traversed in the anticlockwise direction.
2. $\int_C \frac{1}{(z-4)(z+1)^4}\,dz$, where C is the circle $|z-1| = \frac{5}{2}$ traversed in the clockwise direction.

Solution. **(1):** Here $f(z) = e^z$ is analytic within and on C, and $z = 0$ lies inside C. Therefore, by Theorem 5.14, we obtain

$$\int_C e^z z^{-3}\,dz = \frac{2\pi i}{2!} \times f''(0) = \pi i.$$

(2): Let C_1 be the circle $|z - 1| = \frac{5}{2}$ traversed in the anticlockwise direction. Here $f(z) = \frac{1}{(z-4)}$ is analytic within and on C_1, and $z = -1$ lies inside C_1. Therefore, by Theorem 5.14, we obtain

$$\int_{C_1} \frac{1}{(z-4)(z+1)^4}\,dz = \frac{2\pi i}{3!} \times f^{(3)}(-1) = \frac{\pi i}{3} \times \frac{-6}{5^4}.$$

Therefore,

$$\int_C \frac{1}{(z-4)(z+1)^4}\,dz = -\int_{C_1} \frac{1}{(z-4)(z+1)^4}\,dz = \frac{\pi i}{3} \times \frac{6}{5^4}.$$

Methods of Evaluation of Integrals:

To evaluate $\int_C f(z)\,dz$, where C is a positively oriented simple closed contour, we can proceed as follows:

1. If $f(z)$ is analytic within and on C, then $\int_C f(z)\,dz = 0$.

2. If $f(z) = \frac{g(z)}{(z-z_0)}$, where $g(z)$ is analytic within and on C, and z_0 lies inside C, then $\int_C f(z)\,dz = 2\pi i \times f(z_0)$.

3. If $f(z) = \frac{g(z)}{(z-z_0)^{n+1}}$, $n = 1, 2, 3, \dots$, where $g(z)$ is analytic within and on C, and z_0 lies inside C, then $\int_C f(z)\,dz = \frac{2\pi i}{n!} \times f^{(n)}(z_0)$.

4. Sometime we need to use partial fraction or Cauchy-Goursat's theorem for multiply connected domain (Theorem 5.10) before using results under Items 2 and 3 (see Problem 3).

5. If f is continuous in a domain D containing C, and there exists a function F such that $F'(z) = f(z)$ for all $z \in D$, then

$$\int_C f(z)\,dz = F(z_2) - F(z_1),$$

where z_1 and z_2 are respectively starting and ending points of C. (See Problem 5.4 for illustration).

6. Use the parametrization of the curve and apply the definition as illustrated in Problems 5.2 and 5.3.

5.5 Converse of Cauchy-Goursat's Theorem: Morera's Theorem

Suppose $f(z)$ is a function with domain D such that

$$\int_C f(z)\,dz = 0$$

for every closed contour C lying inside D. Can we conclude from this that f is analytic in D? Answer is NO as shown by the following example.

Example 5.2. Consider the domain $D := \{z : |z| < 1\}$, and $f(z) = \begin{cases} z & \text{if } z \in D \setminus \{\frac{1}{2}\} \\ 1 & \text{if } z = \frac{1}{2} \end{cases}$.
One can show that

$$\int_C f(z)\,dz = 0.$$

But, note that $f(z)$ is not analytic in D.

Example 5.2 shows that converse of Cauchy-Goursat's Theorem (Theorem 5.8) does not hold. But we have the following **partial** converse to Cauchy's theorem.

Theorem 5.17 (Morers's Theorem)**.** *Suppose $f(z)$ is continuous inside a simply connected domain D and*

$$\int_C f(z)\,dz = 0,$$

for any simple closed contour C lying inside D. Then $f(z)$ is analytic throughout D.

Proof. Fix a point $z_0 \in D$, and define

$$F(z) = \int_{z_0}^{z} f(w)\, dw,$$

where the integral is considered as a contour integral over any curve lying in D and joining z with z_0.

Claim 1: We claim that the above integral does not depend on the curve we choose and hence the function F is well defined.

Proof of claim: Proceeding as in the proof of Item 1 of Theorem 5.9, we obtain the result.

Claim 2: $F'(z) = f(z)$ for all $z \in D$.

Proof of claim: Proceeding as in the proof of Item 2 of Theorem 5.9, and using continuity of $f(z)$, we obtain the result.

From Claim 2, it follows that $F(z)$ is analytic in D, and hence by Theorem 5.15 it follows that $F'(z)$, that is, $f(z)$ is analytic in D. □

5.6 Consequences of the Cauchy Integral Formula

Theorem 5.18 (Cauchy Estimates). *Let $f(z)$ be analytic on and inside the circle $C : |z - z_0| = r$. If $|f(z)| \leq M$ for all $z \in C$, then for all $n \geq 0$,*

$$|f^{(n)}(z_0)| \leq \frac{n!M}{r^n}.$$

Proof. Note that

$$\left| \frac{f(z)}{(z - z_0)^{n+1}} \right| = \left| \frac{f(z)}{r^{n+1}} \right| \leq \frac{M}{r^{n+1}} \quad \text{for all } z \in C.$$

Therefore, by LM-formula (Theorem 5.5), we have

$$\left| \int_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz \right| \leq \frac{M}{r^{n+1}} \times 2\pi r = \frac{2\pi M}{r^n}. \tag{5.10}$$

Moreover, from Theorem 5.14, we obtain

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz.$$

$$\begin{aligned}
\Rightarrow \left| f^{(n)}(z_0) \right| &= \left| \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz \right| \\
&\leq \frac{n!}{2\pi} \frac{2\pi M}{r^n} \quad \text{(using eq. (5.10))} \\
&= \frac{n!M}{r^n}.
\end{aligned}$$

□

As a consequence of the above theorem we get the following miraculous result.

Theorem 5.19 (Liouville's Theorem)**.** *If f is analytic and bounded on the whole $\mathbb{C}$, then f is a constant function.*

Proof. To prove this, it is enough to show that f' is the zero function. Let z_0 be an arbitrary point in $\mathbb{C}$.

Claim: For any $\epsilon > 0$, $|f'(z_0)| < \epsilon$.

Proof of Claim: Since f is bounded on $\mathbb{C}$, there exists a M such that

$$|f(z)| \leq M \quad \text{for all } z \in \mathbb{C}. \tag{5.11}$$

Choose an $r > 0$ such that $\frac{M}{r} < \epsilon$. Then applying Theorem 5.18 on the circle $C : |z - z_0| = r$, we obtain

$$|f'(z_0)| \leq \frac{M}{r} < \epsilon.$$

This proves the claim.
Since $|f'(z_0)| < \epsilon$ for all $\epsilon > 0$, however small, it follows that $|f'(z_0)| = 0$, and hence $f'(z_0) = 0$.
Since z_0 is an arbitrary point in $\mathbb{C}$, it follows that $f'(z) = 0$ for all $z \in \mathbb{C}$. □

Problem 5.12. Prove that $\sin z$ and $\cos z$ are not bounded in $\mathbb{C}$.

Solution. Since $\sin z$ and $\cos z$ are both analytic and non-constant function on $\mathbb{C}$, it follows from Liouville Theorem that $\sin z$ and $\cos z$ are not bounded in $\mathbb{C}$.

Theorem 5.20 (Fundamental Theorem of Algebra)**.** *Every polynomial $P(z)$ of degree $n \geq 1$ has a root in $\mathbb{C}$.*

Proof. If possible, let $P(z)$ does not have any root in $\mathbb{C}$. Then the function $f(z) = \frac{1}{P(z)}$ is analytic in $\mathbb{C}$. Moreover, since $|P(z)| \to \infty$ as $z \to \infty$, we obtain $|f(z)| \to 0$ as $z \to \infty$. This implies that $f(z)$ is a bounded function in $\mathbb{C}$. Therefore, from Liouville's Theorem, we obtain $f(z)$ and hence $P(z)$ as a constant function. This is a contradiction. Hence $P(z)$ must have a root in $\mathbb{C}$. □

6 Sequence and Series

6.1 Sequence

Definition 6.1 (Sequence)**.** A mapping f from the set of positive integers to $\mathbb{C}$ is called a sequence in $\mathbb{C}$, and is denoted by $\{f(n)\}$. $\{\frac{i^n}{n}\}$, $\{i^n\}$, $\{2^n\}$ are examples of a few sequences.

In the rest of this section, by a sequence, we will mean a sequence in $\mathbb{C}$.

Definition 6.2 (Limit of a Sequence)**.** A complex number L is said to be the limit of a sequence $\{z_n\}$, denoted as $\lim\limits_{n\to\infty} z_n = L$, if for a given $\epsilon > 0$, there exists a positive integer K such that

$$|z_n - L| < \epsilon \text{ for all } n > k.$$

Proposition 6.1 (Uniqueness of Limit)**.** *If $\lim\limits_{n\to\infty} z_n = L$ and $\lim\limits_{n\to\infty} z_n = M$, then $L = M$.*