INDIAN INSTITUTE OF TECHNOLOGY INDORE

MA 203: Complex Analysis and Differential Equations-II

Autumn Semester

Tutorial −4 (Complex Analysis)

1. Consider the function $f(x+iy) = y^3 - ix^3$. Find the subsets of $\mathbb{C}$, where the function f is
   (a) continuous;
   (b) differentiable;
   (c) analytic.

   Further determine $f'(z)$ on the set where f is differentiable. Justify your answers.

2. Consider the function $f(z) = |z|^2 = x^2+y^2, z = x+iy$. The function f can also be thought of as a function from $\mathbb{R}^2$ to $\mathbb{R}$ mapping (x, y) to x^2+y^2. Moreover, since the partial derivatives of f are continuous throughout $\mathbb{R}^2$, it follows that f is differentiable everywhere on $\mathbb{R}^2$. Show that $f(z)$ is not complex differentiable at any non-zero point z_0.

3. Suppose f is analytic in a domain D such that $|f|$ is constant in D. Then show that f is a constant in D.

4. Let $f = u + iv$ be a non-constant function such that $\bar{f} = u - iv$ be analytic in a domain D. Show that f cannot be analytic in D.

5. Find which of the following functions can be real or imaginary part of a complex function f which is differentiable in the region $|z| < 1$.
   (a) $x^2 - axy + y^2$ No
   (b) $e^x \cos y + xy$ Yes

6. If $f(z) = u + iv$ is an analytic function of $z = x + iy$, and $u - v = e^x(\cos y - \sin y)$, find $f(z)$ in terms of z.
   Ans: $f(z) = e^z + c$.

7. Consider the function $f = u + iv$ defined on $\mathbb{C}$, where $u(x, y) = x^2$, $v(x, y) = y^2$. Consider the set $D := \{x + iy \in \mathbb{C} : x = y\}$. Note that u, v satisfies the C-R equations in D, and u_x, u_y, v_x, v_y are also continuous in D. Prove that f is not analytic on D. Does this fact contradict the related Theorem? If not, explain why.

8. Suppose $f_1(z)$ is analytic at z_0, while $f_2(z)$ is non-analytic at z_0. Then show that $f_1(z)+f_2(z)$ is not analytic at z_0. Give an example to show that sum of two non-analytic function can be analytic.

9. Suppose f is analytic in a domain D. If $f'(z) = 0$ for all $z \in D$, then f is constant on D. Will the result hold if we take D to be any set instead of domain?

10. Is it possible to have an analytic function F in a domain D such that $F'(z) = |z|^2$ for all $z \in D$? Give reason for your answer.

11. If $f(z)$ is an entire function, then show that $e^{f(z)}$ also an entire function.

12. Given an analytic function
$$w = f(z) = u(x, y) + iv(x, y), z = x + iy,$$
the equations $u(x, y) = \alpha$ and $v(x, y) = \beta$, α and β are constants, define two families of curves in the complex plane. Show that the two families are mutually orthogonal to each other.

13. Let $f(z) = z^3$. For $z_1 = 1$ and $z_2 = i$, show that there do not exist any point c on the line $y = 1 - x$ joining z_1 and z_2 such that

$$\frac{f(z_1) - f(z_2)}{z_1 - z_2} = f'(c)$$

(Mean value theorem does not extend to complex derivatives).

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