Exact Differential eqⁿ:

$M\,dx + N\,dy = 0$ is called exact if there exist a funcⁿ $F(x,y)$ such that $F(x,y)$ is continuous in the domain and

$$\frac{\partial F(x,y)}{\partial x} = M \qquad \frac{\partial F(x,y)}{\partial y} = N$$

Condⁿ for exactness: $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ for $M\,dx + N\,dy = 0$

Solⁿ to exact D.E.

→ First check if its an exact D.E.

→ write $\dfrac{\partial F(x,y)}{\partial x} = M(x,y)$ $\quad \dfrac{\partial F(x,y)}{\partial y} = N(x,y)$

→ first integrate wrt x

$$\frac{\partial F(x,y)}{\partial x} = M(x,y) \rightarrow F(x,y) = \left(\int M(x,y)\,dx\right) + \phi(y)$$

→ Now diff wrt y and compare with $\dfrac{\partial F(x,y)}{\partial y}$ ie $N(x,y)$

→ you'll get an eqⁿ $\dfrac{\partial \phi(y)}{\partial y} = \dots$; solve that and obtain $\phi(y)$

→ At the end, we will get a solⁿ $F(x,y) = c$

→ Exact D.E. is in the form of $dF(x,y) = 0$ always

$$x\,dy + y\,dx = d\left(\frac{x^2+y^2}{2}\right) \qquad \text{short cut}$$

Reduction to Exact DE:

if a DE is not exact DE; then by multiplying something, we can make it exact.

$$M\,dx + N\,dy = 0$$

$$\mu(x,y)\,M\,dx + \mu(x,y)\,N\,dy = 0 \qquad \mu(x,y) = \text{integrating factor.}$$

if $\mu(x,y)$ = integrating factor

then $k\,\mu(x,y)$ = integrating factor too.

An exact D.E. will have one-parameter solⁿ only.

Whenever we multiply by IF

Imp → we may gain/loss solⁿs to the org. DE; Hence check solⁿs at the end too.

if $\dfrac{1}{N}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right) = f(x)$ → funcⁿ of x $\boxed{\text{ALONE}}$

then $\text{IF} = e^{\int f(x)\,dx}$

if $-\dfrac{1}{M}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right) = f(y)$ → funcⁿ of y $\boxed{\text{ALONE}}$

if $-\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(y)$ ALONE

then $IF = e^{\int f(y)\,dy}$

When a DE is given, first find whether it is exact. Then proceed accordingly.

$f(x)\,dx + g(y)\,dy = 0$ is always an exact DE

SEPARABLE EQUATIONS

if $\underbrace{F(x)\,G(y)\,dx}_{M} + \underbrace{f(x)\,g(y)\,dy}_{N} = 0$ then eq^n is called separable D.E.

$IF = \frac{1}{G(y)\,f(x)}$ Cond^n: $\boxed{G(y) \neq 0 \quad f(x) \neq 0}$

general sol^n: $\int \frac{F(x)}{f(x)}\,dx + \int \frac{g(y)}{G(y)}\,dy = C$

Singular sol^n may arise out of $G(y) = 0$.

if for some y_0's $G(y_0) = 0$

substitute y_0 into the org. eq^n & if it satisfies the D.E, then it is a particular sol^n.

Homogeneous Equations:

$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$ ← $M\,dx + N\,dy = 0$

test/cond^n: $F(tx, ty) = t^n F(x, y)$ Homogeneous Equation of degree $= n$

↳ for a func^n $F(x,y)$ to be Homogeneous

Imp if $M(x,y)$ & $N(x,y)$ are homogeneous func^{n}s with the same degree; then $M\,dx + N\,dy = 0$ is Homogeneous Eq^n.

Imp if $\frac{dy}{dx} = \frac{y \pm \sqrt{x^2+y^2}}{x}$

$= \frac{y}{x} \pm \frac{\sqrt{x^2+y^2}}{\sqrt{x^2}}$

$= \frac{y}{x} \pm \sqrt{1 + \left(\frac{y}{x}\right)^2}$

Theorem: if $M(x,y)\,dx + N(x,y)\,dy = 0$ is a Homogeneous Eq^n

ie $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$

then substituting $y = vx$ makes it into a separable DE.

final form: $\left[v - f(v)\right] dx + x\,dv = 0$

First Order Linear Differential Eq^n:

Standard form: $\frac{dy}{dx} + P(x)\,y = Q(x)$

IF: $e^{\int P(x)dx}$

$$y\,e^{\int P(x)dx} = \int Q(x)\,e^{\int P(x)dx}\,dx + C$$

constant to be included.

Bernoulli Differential Equation:

$$\frac{dy}{dx} + P(x)\,y = Q(x)\,y^{\alpha} \qquad \alpha \in \mathbb{R}$$

for $\alpha = 0,1$ Linear Eq^n

$$y^{-\alpha}\frac{dy}{dx} + P(x)\,y^{1-\alpha} = Q(x)$$

for $\alpha \neq 0,1$ not a linear Eq^n

Let $v = y^{1-\alpha}$

$\boxed{\text{Assuming } y^{\alpha} \neq 0}$

$$\frac{dv}{dx} = (1-\alpha)\,y^{-\alpha}\frac{dy}{dx}$$

$$\therefore \frac{1}{(1-\alpha)}\frac{dv}{dx} + P(x)\,v = Q(x)$$

transformed into a linear Equation.

Imp: At the end check if $y = 0$ is a sol^n

↳ Singular sol^n.

if $\frac{dy}{dx} = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$ POI $= (h, k)$

then $X = x - h$ $Y = y - k$

$$\frac{dY}{dX} = \frac{dy}{dx}$$

Substitute X & Y it will transform into a homogeneous Equation.

But if lines are //; then $v = ax+by$ substitution.

then eqⁿ converts to separable eqⁿ.

Q) If $\frac{dy}{dx} = F(ax+by+c)$ $b \neq 0$

Substituting $v = ax+by+c$.

Method to find differential equation of the family

$$f(x,y,c) = 0 \quad (9)$$

Step 1: Differentiate $f(x,y,c) = 0$ implicitly with respect to x to get a relation of the form

$$g\left(x, y, \frac{dy}{dx}, c\right) = 0. \quad (10)$$

Step 2: Eliminate the parameter c from (9) and (10) to obtain

$$F\left(x, y, \frac{dy}{dx}\right) = 0$$

as the desired differential equation.

ORTHOGONAL TRAJECTORY

$f(x,y,c) = 0$

diff wrt x $\quad g\left(x, y, \frac{dy}{dx}, c\right) = 0$

from $f(x,y,c) = 0$ find c and substitute in $g\left(x,y,\frac{dy}{dx},c\right) = 0$

to obtain $g\left(x, y, \frac{dy}{dx}\right) = 0 \longrightarrow$ DE of $f(x,y,c) = 0$

Now replace $\frac{dy}{dx} \longrightarrow \frac{1}{-\frac{dy}{dx}}$ & solve the DE again.

if we want trajectory inclined at α degree.

$$\tan\alpha = \frac{m_1 - m_2}{1 + m_1 m_2}$$

replace $\frac{dy}{dx} \rightarrow \dfrac{\frac{dy}{dx} \mp \tan\alpha}{1 \pm \frac{dy}{dx}\tan\alpha}$

$$\therefore \quad m_2 = \frac{m_1 \mp \tan\alpha}{1 \pm m_1 \tan\alpha}$$

Lipschitz Condition.

$f(x,y)$ defined in a domain D in xy plane

$$\left| f(x,y_1) - f(x,y_2) \right| \leq k\left|y_1 - y_2\right| \qquad k \geq 0 \quad (x,y_1)\ (x,y_2) \in D$$

ie for each fixed x_0; the funcⁿ $f(x_0, y)$ is continuous in y.

→ Any two points connected in D, the line must remain in D

→ $\frac{\partial f}{\partial y}$ must be bounded for all $(x, y) \in D$

Steps:

→ first prove any two points connected lie in the domain.

→ $\frac{\partial f}{\partial y}$ exist & it is bounded

→ $\left| \frac{\partial f(x_0, y_0)}{\partial y} \right|$ = Something ≤ bounded number for all $(x_0, y_0) \in D$

Imp If it doesn't have $\frac{\partial f}{\partial y}$; then go with the original eq^n

PICARD'S THEOREMS :

EXISTENCE OF A SOLN OF IVP at $y(x_0) = y_0$

Existence Theorem

Hypothesis:

1. $R := \{(x, y) : |x - x_0| \le a, |y - y_0| \le b\}, \ (a, b > 0).$
2. $f(x, y)$ is continuous in R.

Conclusion:

1. Existence: The IVP (8) has at least one solution $y = y(x)$.
2. Domain of Solution:
   - There exists a number K such that

$$|f(x, y)| \le K \quad \text{for all } (x, y) \in R. \quad \text{(why?)}$$

   - $\min\{a, \frac{b}{K}\} = \beta$ (say)
   - Domain: **At least** $[x_0 - \beta, x_0 + \beta] \subseteq [x_0 - a, x_0 + a].$

Imp

$\frac{dy}{dx} = f(x, y)$

Example

- Consider the initial-value problem
  $\frac{dy}{dx} = (x-1)^2 + (y-3)^2, \ y(1) = 3.$
- Here $f(x, y) = (x-1)^2 + (y-3)^2$, and $(x_0, y_0) = (1, 3)$.
- $R := \{(x, y) : |x - 1| \le 1, |y - 3| \le \frac{1}{2}\}.$
- f is continuous in R.
- The IVP has at least one solution $y = y(x)$.
- Domain of Solution:
  - $|f(x, y)| \le |x-1|^2 + |y-3|^2 \le 1 + \frac{1}{4} = \frac{5}{4}$ for all $(x, y) \in R$.
  - $\min\{a, \frac{b}{K}\} = \min\{1, \frac{2}{5}\} = \frac{2}{5}$
  - Domain: **At least** $[x_0 - \frac{2}{5}, x_0 + \frac{2}{5}] = [\frac{3}{5}, \frac{7}{5}].$

UNIQUENESS OF A SOLN IVP

Uniqueness Theorem

Hypothesis:

1. $R := \{(x, y) : |x - x_0| \le a, |y - y_0| \le b\}, \ (a, b > 0).$
2. $f(x, y)$ is continuous in R.
3. f satisfies a **Lipschitz condition** (with respect to y) in R.

Conclusion:

1. Then the initial value problem (8) has one and only one solution $y = y(x)$.

Corollary

Hypothesis:

1. $R := \{(x, y) : |x - x_0| \le a, |y - y_0| \le b\}, \ (a, b > 0).$
2. *$f(x, y)$ is continuous in R.*
3. *$\frac{\partial f}{\partial y}$ exists and is continuous in R.*
   *f satisfies a **Lipschitz condition** (with respect to y) in R.*

Conclusion:

1. *The IVP (8) has **one and only one solution** $y = y(x)$.*
2. *Domain: Proceed as in the Existence Theorem*

Once domain is decided & func^n is continuous in that domain; There exists a sol^n; Now proceed to identify uniqueness of the sol^n.

Theorem

Hypothesis:

1. $R := \{(x,y) : a \le x \le b, -\infty < y < \infty\}$, $(a > 0)$.
2. (x_0, y_0) is any point of the strip R.
3. $f(x,y)$ is continuous and satisfies a **Lipschitz condition** (with respect to y) in R.

Conclusion:

1. Then the initial value problem

$$y' = f(x,y) \quad y(x_0) = y_0$$

has one and only one solution $y = y(x)$ on the interval $a \le x \le b$.

$\min\left(a, \frac{b}{k}\right) = a$ always

So a solⁿ in strip $a \le x \le b$ is Confirmed

Basically we need a space around (x_0, y_0) where the funcⁿ is Continuous so that we can have a existence & later confirm the uniqueness of the solⁿ.

for $\mu(x,y)$ to be IF of $M\,dx + N\,dy = 0$

$$\frac{\partial}{\partial y}\left(\mu(x,y)\, M(x,y)\right) = \frac{\partial}{\partial x}\left(\mu(x,y)\, N(x,y)\right)$$

if form $\mu(x^2+y^2)$ is asked

substitute $$\frac{\partial}{\partial y}\left((x^2+y^2)^m M(x,y)\right) = \frac{\partial}{\partial x}\left((x^2+y^2)^m N(x,y)\right)$$

SECOND ORDER DE

$y'' + P(x)\, y' + Q(x)\, y = R(x)$ $\qquad R(x) = 0$ homogeneous eqⁿ

$R(x) \neq 0$ non homogeneous eqⁿ

to solve, first solve

$$y'' + P(x)y' + Q(x)y = 0$$

$f_1(x)$ & $f_2(x)$ ⇒ two funcⁿ which are L.I.

then $c_1 f_1(x) + c_2 f_2(x) = 0$ in $x \in [a,b]$ $\quad c_1 = c_2 = 0$

if $y_1(x)$ & $y_2(x)$ are two LI solⁿs; then $c_1 y_1 + c_2 y_2$ is a solⁿ too.

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

if $y_2 = k y_1$ ie Linearly dependent

if y_1 & y_2 ⇒ solⁿ $W(y_1, y_2) = 0$ on whole $[a,b]$

$\neq 0$ on whole $[a,b]$

proof : ① Assume $y_2 = c y_1$ & solve $\qquad W(y_1, y_2) = 0$

② since $W(y_1, y_2) = 0$ $\quad \left(\frac{y_2}{y_1}\right)' = \frac{y_1 y_2' - y_1' y_2}{y_1^2} = 0$

Hence $y_2 = k y_1$

If a Solⁿ is Known

if $y_1(x)$ is a solⁿ for $y'' + y' P(x) + y\, Q(x) = R(x)$

then $\quad y_2(x) = v\, y_1(x)$

Substitute everything & find $v(x)$

$$v(x) = \int \frac{1}{y_1^2} e^{-\int P dx} dx$$

Therefore $\quad y_2(x) = v\, y_1(x)$

HOMOGENEOUS ODE WITH CONST COEFF

$y'' + p y' + q y = 0 \qquad y = e^{mx}$ solⁿ.

$\hookrightarrow \left(m^2 + pm + q\right) e^{mx} = 0$

① $m =$ distinct roots m_1 & m_2

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$ solⁿ

② $m =$ complex roots $= a \pm ib$

$y = e^{ax}\left(c_1 \cos bx + c_2 \sin bx\right)$

③ equal roots m use $y_2 = v y_1$ for another.

$y = c_1 e^{mx} + c_2 x e^{mx}$

NON HOMOGENEOUS

$y'' + y' P(x) + y\, Q(x) = R(x)$

i) find homogeneous solⁿ y_g → from previous $y = e^{mx}$ method.

ii) find particular solⁿ y_p → from other methods

iii) general solⁿ $= y_p + y_g$

UNDETERMINED COEFFICIENTS

depending on $R(x)$, choose a solⁿ for y_p ; substitute & compare

if the solⁿ comes similar to y_g ; solⁿ $= x\, y_p$

if the solⁿ comes as repeated root of y_g; solⁿ $= x^2 y_p$

$R(x) = e^{ax} \quad$ then $\quad y_p = A e^{ax}$

substitute ; $A = \dfrac{1}{a^2 + pa + q} \rightarrow \neq 0$

… $A e^{ax}$

$A = \frac{1}{a^2+pa+q} \neq 0$

if $a \neq$ root of auxillary eq^n $\quad y_p = Ae^{ax}$

if $a =$ root of auxillary eq^n $\quad y_p = A x e^{ax}$

if $a =$ repeated root of auxillary eq^n $\quad y_p = A x^2 e^{ax}$

If $R(x) = \sin bx \,/\, \cos bx$

$y_p = A \sin bx + B \cos x$ substitute

if y_p is also the sol^n of homogeneous eq^n

$y_p = x(A \sin bx + B \cos bx)$

If $R(x) = a_0 + a_1 x + \dots a_n x^n$

$y_p = A_0 + A_1 x + \dots A_n x^n$ substitute & compare coeff.

i) if $q = 0$

$y_p = x(A_0 + A_1 x + \dots A_n x^n)$

EULER CAUCHY EQUATION

form: $x^2 y'' + a x y' + by = 0$

substitute $y = x^m$

will get: $x^m (m^2 + (a-1)m + b) = 0$

$(m^2 + (a-1)m + b) = 0$

1. Distinct roots m_1, m_2

$y = C_1 x^{m_1} + C_2 x^{m_2}$

2. Double root $m = \frac{1-a}{2}$

$y_1 = x^m \qquad y_2 = v y_1 \qquad y_2 = y_1 \log x$

$\therefore y = (c_1 + c_2 \log x) x^m$

3. Complex root $m = a \pm ib$

$y = x^a (c_1 \cos(b \log x) + c_2 \sin(b \log x))$

OPERATOR METHOD

$Dy = \frac{dy}{\,}$ $\qquad D^2 y = d^2 y$

OPERATOR METHOD

$$Dy = \frac{dy}{dx} \qquad D^2 y = \frac{d^2 y}{dx^2}$$

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots\cdots = f(x)$$

$$\underbrace{(D^n + a_1 D^{n-1} + a_2 D^{n-2} \cdots\cdots)}_{P(D)} y = f(x)$$

$$y = \frac{f(x)}{P(D)}$$

$$y = \frac{1}{D} f(x) \Rightarrow y = \int f(x)\,dx$$

$$y = \frac{1}{D^2} f(x) \Rightarrow y = \iint f(x)\,dx$$

$$y = \frac{1}{D-r} f(x) \Rightarrow y = e^{rx} \int e^{-rx} f(x)\,dx$$

① if $f(x) = e^{kx} g(x)$

$$y = \frac{1}{P(D)} f(x) = \frac{1}{P(D)} e^{kx} g(x) = e^{kx} \frac{1}{P(D+k)} g(x)$$

② When $f(x) =$ polynomial convert to power series.

$$\frac{1}{P(D)} f(x) = (D^n + a_1 D^{n-1} \cdots\cdots) f(x)$$

③ When $P(D) = (D-r_1)(D-r_2)\cdots\cdots$ successive integration.

$$\frac{1}{P(D)} f(x) = \frac{1}{(D-r_1)} \cdot \frac{1}{(D-r_2)} \cdots\cdots \frac{1}{(D-r_n)} f(x)$$

Partial fracⁿ

or $$\frac{1}{P(D)} f(x) = \left(\frac{A_1}{(D-r_1)} + \frac{A_2}{(D-r_2)} + \cdots \right) f(x)$$

④ $$\frac{1}{1-r} = 1 + r + r^2 + r^3 \cdots\cdots$$ for $\frac{1}{1-D^n}$ form

$$\frac{1}{1+r} = 1 - r + r^2 - r^3 \cdots\cdots$$

METHOD OF VARIATION OF PARAMETERS

for $P(t)\,y'' + Q(t)\,y' + R(t)\,y = G(t)$

homogeneous solⁿ for $P(t)\,y'' + Q(t)\,y' + R(t)\,y = 0$

homogeneous sol^n for $P(t)\, y'' + Q(t)\, y' + R(t)\, y = 0$
are y_1 & y_2.

$y_p = u_1 y_1 + u_2 y_2$ $\quad$ u_1 & $u_2 \Rightarrow$ $func^n$ of t

$$u_1 = -\int \frac{y_2\, G(t)}{W(y_1, y_2)}\, dt \qquad u_2 = \int \frac{y_1\, G(t)}{W(y_1, y_2)}\, dt$$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

Assume $u_1' y_1 + u_2' y_2 = 0$
and $P(t) = 1$ and y_1 & y_2 are sol^n to homogeneous eq^n.

$y_p = u_1 y_1 + u_2 y_2$

$y_{general} = y_p + y_c$ $\qquad$ $y_c = c_1 y_1 + c_2 y_2$

METHOD FOR THIRD ORDER EQUATIONS

$$y''' + a(x)\, y'' + b(x)\, y' + c(x)\, y = r(x)$$

$y_1\ y_2\ y_3 \Rightarrow$ $sol^n s$ to homogeneous eq^n

$y_p = u_1 y_1 + u_2 y_2 + u_3 y_3$

$u_1' y_1 + u_2' y_2 + u_3' y_3 = 0$ $\quad$ $\boxed{\text{Assumption}}$.
$u_1' y_1' + u_2' y_2' + u_3' y_3' = 0$

$$W(y_1, y_2, y_3) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} \qquad W_2(y_1, y_2, y_3) = \begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & r(x) & y_3'' \end{vmatrix}$$

$$W_1(y_1, y_2, y_3) = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ r(x) & y_2'' & y_3'' \end{vmatrix}$$

$$u_1 = \int \frac{W_1}{W}\, dx \qquad u_2 = \int \frac{W_2}{W}\, dx \qquad u_3 = \int \frac{W_3}{W}\, dx$$

LAPLACE TRANSFORMATIONS

$$\mathcal{L}(f) = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$ Laplace transformation.

$$\mathcal{L}^{-1}\big(F(s)\big) = f(t)$$ inverse

$\mathcal{L}(1) = \frac{1}{s}$ $\mathcal{L}(e^{at}) = \frac{1}{s-a}$ if $s-a>0$

$\mathcal{L}\left(a f(g) + b g(t)\right) = a\mathcal{L}(f(t)) + b\mathcal{L}(g(t))$

$\mathcal{L}(\cos at) = \frac{s}{s^2+a^2}$ $\mathcal{L}(\sin at) = \frac{a}{s^2+a^2}$

$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$ $\mathcal{L}(t^a) = \frac{\Gamma(a+1)}{s^{a+1}}$ Gamma function $\Gamma(a+1) = \int_0^\infty e^{-x} x^a \, dx$.

$\mathcal{L}(e^{at} f(t)) = F(s-a)$ first shifting func^n

$\mathcal{L}(f^n) = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) \dots - f^{n-1}(0)$

$\mathcal{L}\left(\int_0^t f(v)\,dv\right) = \frac{F(s)}{s}$ $\mathcal{L}^{-1}\left(\frac{F(s)}{s}\right) = \int_0^t f(t)\,dt$

* to calc $\mathcal{L}(f)$ for a given complicated func^n
try seeing if f' or f'' are simple to evaluate.
using the help of $\mathcal{L}(f')$ & $\mathcal{L}(f'')$; calc $\mathcal{L}(f)$

CONDITION FOR LAPLACE'S

$f(t)$ doesn't grow too fast ie $|f(t)| \le M e^{kt}$
$\downarrow$ growth restriction.

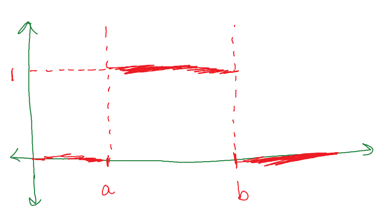
$f(t)$ is piecewise continuous.
$\downarrow$
in each interval continuous
limits finite at interval end

UNIT STEP FUNCTION

$$u(t-a) = \begin{cases} 1 & t>a \\ 0 & t \le a \end{cases}$$

$\mathcal{L}(u(t-a)) = \frac{e^{-as}}{s}$ $s>0$

$\mathcal{L}(f(t-a)\,u(t-a)) = e^{-as} F(s)$



$V_0 = V_0\left(u(t-a) - u(t-b)\right)$

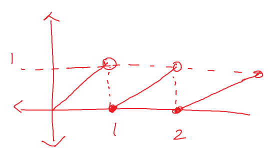
$$\mathcal{L}(t^n f) = (-1)^n F^n(s)$$

Periodic Function

$f(x+T) = f(x)$ $\quad$ T = period

$$F(s) = \frac{\int_0^T e^{-st} f(t)\,dt}{1 - e^{-sT}}$$

Sawtooth func^n $\quad f(t) = \begin{cases} 1; & 0 \le t < 1 \\ f(t-1); & t \ge 1 \end{cases}$



Theorem

$f(x) \Rightarrow$ continuous func^n of exponential order

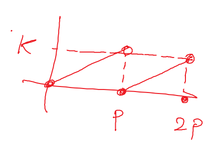
$\mathcal{L}(f) = F(s)$ $\quad\quad F(s) \to 0$ as $s \to \infty$

as $|f(x)| \le M e^{kt}$ for $\forall\, t \ge 0$

Any func^n without this behaviour cannot be the Laplace transform^n of a function.

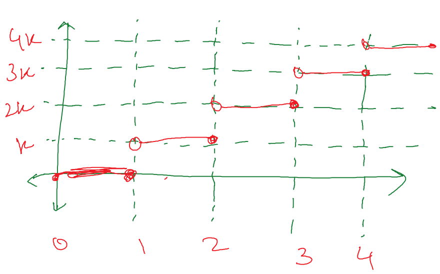
Formulas

Sawtooth func^n $\quad f(t) = \begin{cases} \frac{kt}{p}; & 0 \le t < 1 \\ f(t-1); & t \ge 1 \end{cases}$



ie $f(t+np) = f(t)$ and $f(t)_{max} = k$

$$\mathcal{L}(f) = \frac{k}{ps^2} - \frac{k e^{-sp}}{s(1-e^{-sp})}$$



$f(t) = \frac{kt}{p} - g(t) \longrightarrow$ sawtooth func^n.

$\mathcal{L}(f) = \mathcal{L}\left(\frac{kt}{p} - g(t)\right)$

$$= \frac{k}{p} \mathcal{L}(t) - \mathcal{L}(g(t))$$

$$= \frac{k\, e^{-sp}}{s(1-e^{-sp})}$$

CONVOLUTION THEOREM

$\mathcal{L}(f) = F(s)$ $\qquad$ $\mathcal{L}(g) = G(s)$

$h(t) = (f*g)t$ $\qquad$ $H(s) = \mathcal{L}(h) = \mathcal{L}(f*g)$

$$\mathcal{L}(f*g) = \int_0^\infty e^{-st} (f*g)t \, dt$$

$$\mathcal{L}(f*g) = \int_0^\infty e^{-st} \left(\int_0^t f(\tau) g(t-\tau) \, d\tau \right) dt$$

$$\mathcal{L}(f*g) = F(s)\, G(s) = H(s)$$

$$h(t) = (f*g)t = \int_0^t f(\tau)\, g(t-\tau)\, d\tau = \mathcal{L}^{-1}(F(s)G(s))$$

DIRAC DELTA FUNCTION

$$f_k(t-a) = \frac{1}{k} \qquad a \le t \le a+k$$

$$= 0 \; ; \quad t \notin [a, a+k]$$

$$\delta(t-a) = \lim_{k \to 0} f_k(t-a) \qquad \text{unit impulse function.}$$

$$\int_0^\infty \delta(t-a)\, dt = 1 \qquad \delta(t-a) = \begin{cases} \infty \; ; & t = a \\ 0 \; ; & t \ne a \end{cases}$$

$$\int_0^\infty g(t)\, \delta(t-a)\, dt = g(a) \qquad \text{Shifting property of } \delta$$

$$f_k(t-a) = \frac{1}{k} \left[u(t-a) - u(t-(a+k)) \right]$$

$$\mathcal{L}(f_k(t-a)) = \frac{e^{-as}(1-e^{-ks})}{ks}$$

when $k \to 0$ $\boxed{\mathcal{L}(\delta(t-a)) = e^{-as}}$

while taking $\mathcal{L}^{-1}$ and coming at a sol^n

check that $F(s) \to 0$ as $s \to \infty$ to eliminate 'c'

Steps:

$$\mathcal{L}^{-1}(F(s)\,G(s))$$

$$= \mathcal{L}^{-1}(F(s))\,\mathcal{L}^{-1}(G(s))$$

$$= (f*g)\,t$$

$$= \int_0^t f(\tau)\,g(t-\tau)\,d\tau$$

MULTIPLE FUNCTIONS

$$f*(g*h) = (f*g)*h$$

NON HOMOGENEOUS

$$y'' + ay' + by = r$$

$$\mathcal{L}(y'' + ay' + by) = \mathcal{L}(r)$$

$$Y(s) = \frac{s+a}{s^2+as+b}\,y(0) + \frac{y'(0)}{s^2+as+b} + \frac{R(s)}{s^2+as+b}$$

$$Q(s) = \frac{1}{s^2+as+b}$$ transfer func^n

if $y(0) = 0$ & $y'(0) = 0$

$$Y(s) = Q(s)\,R(s)$$

transfer^n $\leftarrow Q(s) = \frac{Y(s)}{}$

transfer func^n $\leftarrow Q(s) = \dfrac{Y(s)}{R(s)}$

$q(t) = L^{-1}(Q(s)) \rightarrow$ impulse func^n.

While solving for y_p; if the roots of auxillary equation do not match EXACTLY with the chosen y_p; do not multiply with the x or x^2.

$y'' + P_1 y' + Qy = 0 \qquad y_1\ y_2 \Rightarrow$ sol^n

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

$$W'(y_1, y_2) = y_1 y_2'' - y_2 y_1''$$

$$\left.\begin{aligned} &(y_2'' + P y_2' + Q y_2 = 0)\, y_1 \\ -&(y_1'' + P y_1' + Q y_1 = 0)\, y_2 \end{aligned}\right\} \Rightarrow (y_1 y_2'' - y_2 y_1'') + P(y_1 y_2' - y_2 y_1') = 0$$

$$\frac{dW}{dx} + PW = 0$$

$$\therefore W = c\, e^{-\int P dx}$$

Hence ① $W = 0$ if $c = 0$ OR ② never $= 0$ if $c \neq 0$.

$$\left.\begin{aligned} \cos x &= Re(e^{ix}) \\ \sin x &= Im(e^{ix}) \end{aligned}\right\} \text{ helpful during operator method.}$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$e^{ix} = \cos x + i \sin x$$

$L(e^{ix}) = L(\cos x + i \sin x)$ helpful method.

INDUCTION METHOD

i) first find for $x=0$ trully ie without use of formula.

ii) Confirm $x=0$ values satisfies the formula.

iii) We know $f(x)$; find $f(x+1)$ in terms of $f(x)$ TRULY

iv) Now substitute $f(x)$ value.

v) Compare this result with the formula.

PIECE WISE FUNCTIONS

whenever piecewise functions are given; use unit step func^n to convert all the partitions into one single formula.

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1}$$

STAIRCASE FUNCTION

you can represent staircase func^n as:

$$f(t) = u(t-1) + u(t-2) + u(t-3) \text{ - - - - - } u(t-\infty)$$

$$f(t) = \sum_{n=1}^{\infty} u(t-n)$$

$$L(f(t)) = L\left(\sum_{n=1}^{\infty} u(t-n)\right) = \frac{1}{s} \cdot \frac{1}{(e^{s}-1)}$$

USAGE OF FORMULA $L(t^n f) = (-1)^n F^n(s)$

When the given eq^n is too complicated to calc.; try using differentiation for $L^{-1}(\ )$

$$F(s) = \ln\left(\frac{s-a}{s-b}\right) = \ln(s-a) - \ln(s-b)$$

$$F'(s) = \frac{1}{s-a} - \frac{1}{s-b}$$ a known form

$$L\left(t\,f(t)\right) = (-1)\;F'(s)$$

$$\therefore t\,f(t) = L^{-1}\left(\frac{1}{s-b} - \frac{1}{s-a}\right)$$

$$t\;f(t) = e^{-bt} - e^{-at}$$

$$f(t) = \frac{e^{-bt}}{t} - \frac{e^{-at}}{t}$$

$L^{-1}(f(t-3))$ in equations involving other $L^{-1}(g(t))$ is not possible.

we $u = t-h$

substitute $t = u+h$ in equations

$\therefore y''(t)$ becomes $y''(t+h)$

now define $x(t) = y(t+h)$

$\therefore x''(t) = y''(t+h)$

solve the equation & find $x(t)$; then back substitute these equations.

Convolution theorem.

$$L\left(\int_0^t (g+f)\tau * y(\tau-t)\,d\tau\right) = \left(F(s) + G(s)\right)Y(s)$$

$L(g) = G$ $\quad L(f) = F$

NOTATION FOR CONVOLUTION

$$\mathcal{L}^{-1}\left(F(s)\, G(s) \right) = \mathcal{L}^{-1}(F(s)) * \mathcal{L}^{-1}(G(s))$$

$$= (f * g)\, t$$

$$\mathcal{L}^{-1}\left(F(s)\, G(s) \right) = \int_0^t f(v)\, g(t-v)\, dv$$

UNIQUENESS AND EXISTENCE

① if $\frac{\partial f}{\partial y}$ is bounded in D ; then LC satisfied

↳ for unbounded domain

② if $\frac{\partial f}{\partial y}$ is continuous in D ; then LC satisfied

↳ for bounded domain

$|x - x_0| \leq a \qquad |y - y_0| \leq b$

$K = f(x, y)_{max}$.

$$\beta = \min\left(a, \frac{b}{K} \right)$$

$|x - x_0| \leq \beta$ confirmed.

for undetermined coefficients.

$R(x)$ $\quad$ y_p

e^{kx} $\quad$ $A e^{kx}$

x^n $\quad$ $A_0 + A_1 x + A_2 x^2 + \cdots A_n x^n$

$\left.\begin{array}{l}\cos kx \\ \sin kx\end{array}\right\}$ $M \cos kx + N \sin kx$

in case of multiplication/ addition of func^n; do the same with y_p.

$$\mathcal{L}\left(\cosh(at)\right) = \frac{s}{s^2 - a^2} \qquad \mathcal{L}\left(\sinh(at)\right) = \frac{a}{s^2 - a^2}$$