# Tut-3.3

1.

i) Gauss-Jacobi

$$\mu = \max_{1\le i\le n}\left\{\alpha_i + \beta_i\right\} \quad \text{where } \alpha_i = \sum_{j=1}^{i-1}\frac{|a_{ij}|}{|a_{ii}|}, \quad \beta_i = \sum_{j=i+1}^{n}\frac{|a_{ij}|}{|a_{ii}|}$$

$$\Rightarrow A = \begin{bmatrix} 4 & 0 & 2 \\ 0 & 5 & 2 \\ 5 & 4 & 10 \end{bmatrix} \Rightarrow \begin{array}{l|l} \alpha_1 = 0 & \beta_1 = 0.5 \\ \alpha_2 = 0 & \beta_2 = 0.4 \\ \alpha_3 = 0.9 & \beta_3 = 0 \end{array}$$

$$\Rightarrow \underline{\mu = 0.9}$$

$$\|e^{(k)}\| = \|x - x^{(k)}\| \le \frac{\mu^k}{1-\mu}\|x^{(1)} - x^{(0)}\| < 10^{-5}$$

Let us take $x^{(0)} = 0$ (as not specified any value)

$$x^{1} = -D^{-1}(L+U)x^{(0)} + D^{-1}b \qquad D^{-1} = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{10} \end{bmatrix} = \begin{bmatrix} \frac{1}{a_{11}} & 0 & 0 \\ 0 & \frac{1}{a_{22}} & 0 \\ 0 & 0 & \frac{1}{a_{33}} \end{bmatrix}$$

$$= D^{-1}b$$

$$= \begin{bmatrix} 1 \\ -3/5 \\ 1/5 \end{bmatrix}$$

$$\rightarrow \frac{\mu^k}{1-\mu}\left[\left\|\begin{matrix} 1 \\ -3/5 \\ 1/5 \end{matrix}\right\|\right] < 10^{-5}$$

$$\frac{\mu^k}{1-\mu} < 10^{-5}$$

$$\mu^k < 10^{-6}$$

$$k \ge \frac{\log 10^{-6}}{\log 0.9} < 131.126$$

$$\rightarrow \underline{k \ge 132}$$

ii) Gauss-Seidel

$$\eta = \max_{1 \le i \le n} \left\{ \frac{\beta_i}{1-\alpha_i} \right\} \qquad \alpha_i = \sum_{j=1}^{n-1} \frac{|\alpha_{ij}|}{|\alpha_{ii}|}, \quad \beta_i = \sum_{j=i+1}^{n} \frac{|\alpha_{ij}|}{|\alpha_{ii}|}$$

$\cancel{\alpha_1 = 0}$
$\cancel{\alpha_2 =}$ $\Rightarrow \eta = 0.5$

$$\|e^{(k)}\| = \|x - x^{(k)}\| \le \frac{\eta^k}{1-\eta} \|x^{(1)} - x^{(0)}\|$$

Ly $x^{(0)} = 0$

$$(L+D)x^{(1)} = -Ux^{(0)} + b$$

$$\Rightarrow (L+D)x^{(1)} = b$$

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 5 & 4 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix}$$

$x_1 = 1,\ x_2 = -3/5$

$5 - \frac{12}{5} + 10x_3 = 2$

$x_3 = -3/50$

$$x^{(1)} = \begin{bmatrix} 1 \\ -3/5 \\ -3/50 \end{bmatrix}$$

$$\therefore \frac{0.5^k}{0.5} \le 10^{-5}$$

$$\left(\frac{1}{2}\right)^{k-1} \le 10^{-5} \Rightarrow 2^{k-1} \ge 10^5$$

$$k \ge 1 + \frac{5}{\log 2} \ge 17.6$$

$$\Rightarrow k \ge 18$$

Note:- $\mu = \max(\alpha_i + \beta_i)$, $\eta = \max\left(\frac{\beta_i}{1-\alpha_i}\right)$

$$\alpha_i = \sum_{j=1}^{i-1} \frac{|\alpha_{ij}|}{|\alpha_{ii}|}, \quad \beta_i = \sum_{j=i+1}^{n} \frac{|\alpha_{ij}|}{|\alpha_{ii}|}$$

$$\|e^{(k)}\| = \|x - x^{(k)}\| \le \frac{e^k}{1-e} \|x^{(1)} - x^{(0)}\|, \quad e = \mu \text{ (or) } \eta.$$

2. By Gershgorin's theorem:-

$$|\lambda - a_{ii}| \le \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ij}| \quad \text{atleast for some } 1 \le i \le n.$$

$i=1 \rightarrow |\lambda - 1| \le 3 \Rightarrow \lambda \in [-2, 4]$

$i=2 \rightarrow |\lambda - 2| \le 3 \Rightarrow \lambda \in [-1, 5]$ — ~~unio~~ union $\Rightarrow$ $\underline{\lambda \in [-2, 5]}$

$i=3 \rightarrow |\lambda - 3| \le 2 \Rightarrow \lambda \in [1, 5]$

3. Orthogonal vectors:- $u_i = a_i - \sum_{j=1}^{n-1} \langle a_i, e_j \rangle e_j$, where $e_j = \frac{u_j}{\|u_j\|_2}$

$u_1 = a_1 = [-3\ \ 6\ \ -6]^T$ $\qquad e_1 = [-\frac{1}{3}\ \ \frac{2}{3}\ \ -\frac{2}{3}]^T$

$u_2 = a_2 - \langle a_2, e_1 \rangle e_1$ $\qquad \underline{\langle a_1, e_1 \rangle = 9} = [1+4+4]$

$= [-5\ \ 4\ \ 2]^T - 3[-\frac{1}{3}\ \ \frac{2}{3}\ \ -\frac{2}{3}]^T$ $\qquad \underline{\langle a_2, e_1 \rangle = 3}$, $\underline{\langle a_3, e_1 \rangle = 0}$

$= [-5\ \ 4\ \ 2]^T - [-1\ \ 2\ \ -2]^T$ $\qquad e_2 = [-\frac{2}{3}, \frac{1}{3}, \frac{2}{3}]^T$

$= [-4\ \ 2\ \ 4]^T = [-4\ \ 2\ \ 4]^T$ $\qquad \underline{\langle a_2, e_2 \rangle = 6}$

$\underline{\langle a_3, e_2 \rangle = 9}$

$u_3 = a_3 - \langle a_3, e_2 \rangle e_2 - \langle a_3, e_1 \rangle e_1$ $\qquad e_3 = [-\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3}]^T$

$= [-8, 1, 5]^T - 9[-2/3, 1/3, 2/3]^T - 0$ $\qquad \underline{\langle a_3, e_3 \rangle = 3}$

$= [-2, -2, -1]^T$

$$Q_1 = \frac{1}{3}\begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}, \quad R_1 = \begin{bmatrix} 9 & 3 & 0 \\ 0 & 6 & 9 \\ 0 & 0 & 3 \end{bmatrix}$$

$$A_0 = \begin{bmatrix} -3 & -5 & -8 \\ 6 & 4 & 1 \\ -6 & 2 & 5 \end{bmatrix}$$

$$A_1 = \cancel{Q_2 R}\ R_0 Q_0 = \begin{bmatrix} 9 & 3 & 0 \\ 0 & 6 & 9 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1/3 & -2/3 & -2/3 \\ 2/3 & 1/3 & -2/3 \\ -2/3 & 2/3 & -1/3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -5 & -8 \\ -2 & 8 & -7 \\ -2 & 2 & -1 \end{bmatrix}$$

$\lambda_1' = -1$

$\lambda_2' = 8$

$\lambda_3' = -1$

diagonals

$a_1 = [-1\ -2\ -2]^T$

$a_2 = [-5\ 8\ 2]^T$

$a_3 = [-8\ -7\ -1]^T$

Now QR decomposition of $A_1 = Q_1 R_1$

$u_1 = [-1\ -2\ -2]^T$

$e_1 = \dfrac{u_1}{\|u_1\|_2} = \left[-\frac{1}{3}, -\frac{2}{3}, -\frac{2}{3}\right]^T$

$\langle a_1, e_1 \rangle = 3$

$\langle a_2, e_1 \rangle = \frac{1}{3}[5-16-4] = -5$

$\langle a_3, e_1 \rangle = \frac{1}{3}[8+14+2] = 8$

$u_2 = a_2 - \langle a_2, e_1 \rangle e_1$

$\cancel{= [-5, 8, 2]^T - 8[-1, -2, -2]^T = [-4, 10, 4]^T}$

$\langle a_2, e_2 \rangle = \frac{1}{\sqrt{33}}[10+40+4] = \frac{54}{\sqrt{33}}$

$e_2 = \dfrac{u_2}{\|u_2\|_2} = \frac{1}{\sqrt{33}}[-2, 5, 2]^T$

$= [-5, 8, 2]^T - \left[\frac{5}{3}, -\frac{16}{3}, \frac{-4}{3}\right]^T = \left[-\frac{20}{3}, \frac{40}{3}, \frac{10}{3}\right]^T$ ✗

$e_2 =$

$u_2 = a_2 - \langle a_2, e_1 \rangle e_1 = [-5, 8, 2]^T + 5\left[-\frac{1}{3}, -\frac{2}{3}, -\frac{2}{3}\right]^T$

$= \left[-\frac{20}{3}, \frac{14}{3}, \frac{-4}{3}\right]^T =$

$\langle a_2, e_2 \rangle = \frac{1}{3\sqrt{17}}[50+56-4]$

$= 2\sqrt{17}$

$e_2 = \frac{1}{3\sqrt{17}}[-10, 7, -2]^T$

$\langle a_3, e_2 \rangle = \frac{1}{3\sqrt{17}}[80-49+2] = \frac{11}{\sqrt{17}}$

$$u_3 = a_3 - \langle a_3, e_2\rangle e_2 - \langle a_3, e_1\rangle e_1$$

$$= [-8,-7,-1]^T - \frac{11}{3\sqrt{17}}[-10,7,-2]^T - \frac{8}{3}[-1,-2,-2]$$

$$= \left[-\frac{54}{17}, -\frac{54}{17}, \frac{81}{17}\right]$$

$$\langle a_3, e_3\rangle = \frac{16}{\sqrt{17}} + \frac{14}{\sqrt{17}} - \frac{3}{\sqrt{17}} = \frac{27}{\sqrt{17}}$$

$$e_3 = \frac{u_3}{27}\sqrt{17}$$

$$= \frac{[-54,-54,81]^T}{27\sqrt{17}} = \frac{1}{\sqrt{17}}[-2,-2,3]$$

$$Q_1 = \begin{bmatrix} -\frac{1}{3} & \frac{-10}{3\sqrt{17}} & -\frac{2}{\sqrt{17}} \\ -\frac{2}{3} & \frac{7}{3\sqrt{17}} & -\frac{2}{\sqrt{17}} \\ -\frac{2}{3} & \frac{-2}{3\sqrt{17}} & \frac{3}{\sqrt{17}} \end{bmatrix} \quad R_1 = \begin{bmatrix} 3 & -5 & 8 \\ 0 & 2\sqrt{17} & \frac{11}{\sqrt{17}} \\ 0 & 0 & \frac{27}{\sqrt{17}} \end{bmatrix}$$

$$A_2 = R_1 Q_1 = \begin{bmatrix} 3 & -5 & 8 \\ 0 & 2\sqrt{17} & \frac{11}{\sqrt{17}} \\ 0 & 0 & \frac{27}{\sqrt{17}} \end{bmatrix} \begin{bmatrix} -\frac{1}{3} & \frac{-10}{3\sqrt{17}} & -\frac{2}{\sqrt{17}} \\ -\frac{2}{3} & \frac{7}{3\sqrt{17}} & -\frac{2}{\sqrt{17}} \\ -\frac{2}{3} & -\frac{2}{3\sqrt{17}} & \frac{3}{\sqrt{17}} \end{bmatrix}$$

$$= \begin{bmatrix} -3 & -\frac{27}{\sqrt{17}} & \frac{28}{\sqrt{17}} \\ -\frac{30}{\sqrt{17}} & \frac{72}{17} & -\frac{35}{17} \\ -\frac{18}{\sqrt{17}} & -\frac{18}{17} & \frac{81}{17} \end{bmatrix} = \begin{bmatrix} -3 & -6.5484 & 6.79099 \\ -7.276 & 4.235294 & -2.05882 \\ -4.36564 & -1.05882 & 4.764705 \end{bmatrix}$$

$$\lambda_1^{(2)} = -3$$

$$\lambda_2^{(2)} = 4.235294$$

$$\lambda_3^{(2)} = 4.764705$$

4. Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ are eigen values & $V_1, V_2, V_3, \ldots, V_m$ are eigen vectors

Then any vector $z$ can be written as

$$z = \alpha_1 V_1 + \alpha_2 V_2 + \cdots + \alpha_m V_m$$ where $\alpha_1, \alpha_2, \ldots, \alpha_m$ ~~are constants~~ $\in R$.

$\rightarrow$ ~~$B^K z = B - \lambda I$~~ $(B-\lambda I)z = 0 \Rightarrow Bz = \lambda z$

$$B^K z = \alpha_1 \lambda_1^K V_1 + \alpha_2 \lambda_2^K V_2 + \cdots + \alpha_m \lambda_m^K V_m$$

$\therefore$ $\boxed{\lambda_1 = \lim_{K\to\infty} \frac{\langle B^{K+1} z, u\rangle}{\langle B^K z, u\rangle}}$ & $\lim_{K\to\infty} \lambda_1^{-K} B^K z = \alpha_1 V_1$

$$\boxed{\Rightarrow V_1 \propto \lim_{K\to\infty} B^K z}$$

We can take any $u$ but $\begin{bmatrix}1\\0\end{bmatrix}$ to be simple. ($u$ is not $\perp$ to $V_1, z$).

$$B = \begin{pmatrix}3 & 4\\1 & 2\end{pmatrix}, \quad z = \begin{pmatrix}1\\2\end{pmatrix}, \quad u = \begin{pmatrix}1\\0\end{pmatrix}$$

5
2
$\lambda_1 + \lambda_2 = 5$
$\lambda_1\lambda_2 = 2$
$\lambda_2 = \frac{2}{\lambda_1}$
$\lambda_1 + \frac{2}{\lambda_1} = 5$
$\lambda_1^2 - 5\lambda_1 + 2 = 0$

$Bz = \begin{pmatrix}3 & 4\\1 & 2\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}11\\5\end{pmatrix} \longrightarrow \lambda_1^{(1)} = \xrightarrow{53/11} 4.8181$ ~~2.2~~, $v_1^{(1)} = \begin{pmatrix}2.2\\1\end{pmatrix}$ (11/5)

$B^2 z = \begin{pmatrix}3 & 4\\1 & 2\end{pmatrix}\begin{pmatrix}11\\5\end{pmatrix} = \begin{pmatrix}53\\21\end{pmatrix} \longrightarrow \lambda_1^{(2)} = \xrightarrow{243/53} 4.5849$, $v_1^{(2)} = \begin{pmatrix}2.5238\\1\end{pmatrix} = \begin{pmatrix}53/21\\1\end{pmatrix}$

$B^3 z = \begin{pmatrix}3 & 4\\1 & 2\end{pmatrix}\begin{pmatrix}53\\21\end{pmatrix} = \begin{pmatrix}243\\95\end{pmatrix} \longrightarrow \lambda^{(3)} = 4.5637$, $v^{(3)} = \begin{pmatrix}2.55789\\1\end{pmatrix} \to \frac{243}{95}$

$B^4 z = \begin{pmatrix}3 & 4\\1 & 2\end{pmatrix}\begin{pmatrix}243\\95\end{pmatrix} = \begin{pmatrix}1109\\433\end{pmatrix} \longrightarrow \lambda^{(4)} = 4.5617$, $v^{(4)} = \begin{pmatrix}2.5612\\1\end{pmatrix}$

$B^5 z = \begin{pmatrix}3 & 4\\1 & 2\end{pmatrix}\begin{pmatrix}1109\\433\end{pmatrix} = \begin{pmatrix}5059\\1975\end{pmatrix} \longrightarrow \lambda^{(5)} = 4.5615$, $v^{(5)} = \begin{pmatrix}2.5615\\1\end{pmatrix}$

$B^6 z = \begin{pmatrix}3 & 4\\1 & 2\end{pmatrix}\begin{pmatrix}5059\\1975\end{pmatrix} = \begin{pmatrix}23077\\9009\end{pmatrix} \longrightarrow \lambda^{(6)} = 4.56155$, $v^{(6)} = \begin{pmatrix}2.5615\\1\end{pmatrix}$

$\lambda_1 = 4.56155$, $\lambda_2 = Tr(B) - \lambda_1 = 0.43844$

$z = \alpha_1 V_1 + \alpha_2 V_2$ $\quad V_1 = \begin{pmatrix}2.5615\\1\end{pmatrix}$, $V_2 = \begin{pmatrix}\cdot & \cdot & \cdot\end{pmatrix}$

5.

<u>To Prove:-</u> $\|A\|_{\infty} = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|$

<u>Proof:-</u>

$$\|Ax\|_{\infty} = \max_{1\le i\le n} |(Ax)_i| = \max_{1\le i\le n} \left|\sum_{j=1}^{n} a_{ij}x_j\right| \le \max_{1\le i\le n}\left[\sum_{j=1}^{n} |a_{ij}| \left(\max_{1\le j\le n} |x_j|\right)\right]$$

But $\max_{1\le j\le n} |x_j| = \|x\|_{\infty} = 1$ $[\text{Let } \|x\|_{\infty} = 1]$,

$$\Rightarrow \|Ax\|_{\infty} \le \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|$$

and consequenty,

$$\|A\| = \max_{\|x\|=1} \|Ax\|$$

<u>Sub Proof:-</u> Let $\|x\|=1$ & $x = z/\|z\|$

$$\max_{\|x\|=1} \|Ax\| = \max_{z\ne 0} \left\|A\left(\frac{z}{\|z\|}\right)\right\| = \max_{z\ne 0} \frac{\|Az\|}{\|z\|}$$

& $\|Az\| \le \|A\| \cdot \|z\|$

$$\rightarrow \cancel{\max\|A} \ \|A\| \ge \max_{z\ne 0} \frac{\|Az\|}{\|z\|} \Rightarrow \|A\| = \max_{\|x\|=1} \|Ax\|$$

$$\Rightarrow \|A\|_{\infty} = \max_{\|x\|=1} \|Ax\|_{\infty}$$

$$\Rightarrow \|A\|_{\infty} = \max_{\|x\|_{\infty}=1} \|Ax\|_{\infty} \le \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}| \quad \text{——} (1)$$

Now we will show opposite ineq.

$$\sum_{j=1}^{n} |a_{pj}| = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|$$

and $x$ be the vector with components

$$x_j = \begin{cases} 1, & a_{pj} \geq 0 \\ -1, & a_{pj} < 0 \end{cases}$$

Then $\|x\|_\infty = 1$, $a_{pj} x_j = |a_{pj}| \ \forall j \in [1,n]$.

So $$\|Ax\|_\infty = \max_{1 \leq i \leq n} \left| \sum_{j=1}^{n} a_{ij} x_j \right| \geq \left| \sum_{j=1}^{n} a_{pj} x_j \right| = \left| \sum_{j=1}^{n} |a_{pj}| \right|$$
$$= \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|$$

$$\Rightarrow \|A\|_\infty = \max_{\|x\|_\infty = 1} \|Ax\|_\infty \geq \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}| \quad \text{— } (2)$$

from (1) & (2)

$$\|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|$$

6. $$A = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \qquad \eta = \max\left\{ \frac{\beta_i}{1-\alpha_i} \right\}$$

$$\alpha_1 = 0 \qquad \beta_1 = \frac{a_{21}}{a_{11}}$$
$$\alpha_2 = \frac{a_{12}}{a_{22}} \qquad \beta_2 = 0$$

$$\eta = \max\left[ \frac{a_{21}}{a_{11}}, 0 \right]$$

$$\eta = \frac{a_{21}}{a_{11}} > 1$$

$$\rightarrow a_{21} \geq a_{11}$$

$\therefore$ ~~$A = ($~~ for Positive definite $a_{21} = a_{12}$

$$a_{11} > 0$$

$$\Rightarrow A = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} \longleftarrow a_{11} a_{22} > a_{21}^2$$

~~$\Rightarrow a_{22} > a_{21}$~~

b) $A = \begin{pmatrix} a_{11} & a_{21} \\ a_{21} & a_{22} \end{pmatrix}$ $\mu = \max\{\alpha_i + \beta_i\}$

$\alpha_1 = 0$ $\beta_1 = \frac{a_{21}}{a_{11}}$

$\alpha_2 = \frac{a_{21}}{a_{22}}$ $\beta_2 = 0$

$$\mu = \max\left[\frac{a_{21}}{a_{11}}, \frac{a_{21}}{a_{22}}\right] \geq 1$$

$$\Rightarrow a_{21} \geq a_{11} \text{ (or) } a_{21} \geq a_{22}$$

$A = \begin{pmatrix} 1 & 3 \\ 3 & 5 \end{pmatrix}$