INDIAN INSTITUTE OF TECHNOLOGY INDORE

MA 203: Complex Analysis and Differential Equations-II

Autumn Semester 2023

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Tutorial−1 (Differential Equations-II)

1. Test the convergence of the given series

   (a) $1+\frac{3}{2!}+\frac{5}{3!}+\frac{7}{4!}+\ldots$

   (b) $1+\frac{1}{1!}+\frac{2^2}{2!}+\frac{3^3}{3!}+\ldots$

   (c) $\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2}+\left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3}+\ldots$

   (d) $\frac{1}{2}+\frac{1}{3}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{2^3}+\frac{1}{3^3}+\ldots$

   (e) $1+\frac{1}{2^3}+\frac{1}{2^2}+\frac{1}{2^5}+\frac{1}{2^4}+\ldots$

   (f) $\sum\frac{x^n}{n},\quad x>0$

2. Find the radius of convergence of the given power series

   (a) $\frac{x}{2}+\frac{1\cdot 3}{2\cdot 5}x^2+\frac{1\cdot 3\cdot 5}{2\cdot 5\cdot 8}x^3+\ldots$

   (b) $\sum_{n=1}^{\infty}\left(2^n+3^n\right)x^n$

   (c) $\frac{1}{3}-x+\frac{x^2}{3^2}-x^3+\frac{x^4}{3^4}-x^5+\ldots$

   (d) $x+\frac{a\cdot b}{1\cdot c}x^2+\frac{a\cdot(a+1)\cdot b\cdot(b+1)}{1\cdot 2\cdot c\cdot(c+1)}x^3+\ldots$

   (e) $\sum_{n=0}^{\infty}a_nx^n$, where $a_n=\begin{cases}\frac{2^n}{n}, & n \text{ is odd}\\ 0, & n \text{ is even}\end{cases}$

3. Determine the convergence set for

$$\sum_{n=0}^{\infty}\frac{2^{-n}}{n+1}(x-1)^n$$

4. (a) Find the condition for convergence of the series,

$$\sum_{n=0}^{\infty}\left(\frac{az+b}{cz+d}\right)^n \quad \text{for } |a|=|c|>0, \quad z \text{ is complex number.}$$

   (b) If the power series $\sum_n a_nx^n$ has radius of convergence R, then find the radius of convergence of $\sum_n a_nx^{mn}$ for any positive integer m.

   (c) $\sum_{n=0}^{\infty}a_nx^n$ is a power series with a radius of convergence $R\,(>0)$, construct a power series $\sum_{n=0}^{\infty}b_nx^n$ other than $\sum x^n$ such that the radius of convergence of the power series $\sum_{n=0}^{\infty}a_nb_nx^n$ is also R.

(d) $\sum_{n=0}^{\infty} a_n x^n$ is a power series with the radius of convergence $R\,(>0)$, construct a power series $\sum_{n=0}^{\infty} b_n x^n$ other than $\sum \frac{x^n}{2^n}$ such that the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n b_n x^n$ is $2R$.

5. Find a power series expansion about $x = 0$ for a general solution to the given differential equation. The answer should include a general formula for the coefficients.

$$\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = 0.$$

6. Find the nature of the singular points of the following differential equations:

   (a) $x^4 y''(x) + x^3 y'(x) + (\sin x)\, y(x) = 0$

   (b) $y''(x) + \frac{x+a}{(x+p)^3} y(x) = 0, \quad a, p \in \mathbb{R}$

   (c) $\left(x^2 - 1\right)^2 y''(x) + (x+1)\, y'(x) - y(x) = 0$

   (d) $(x \sin x)\, y''(x) + 3y'(x) + x\, y(x) = 0$

   (e) $(\log x)^2\, y''(x) + (x-1)\, y'(x) + y(x) = 0$

   (f) $x^2 y''(x) + 4xy'(x) + 2y(x) = 0,$ at $x = \infty$

   (g) $(1 - x^2) y''(x) - 2xy'(x) + 6y(x) = 0,$ at $x = \infty$

7. Compute the power series for fg with $f = e^{-x}$ and $g = (1+x)^{-1}$

8. Find at least the first four nonzero terms in a power series expansion about $x = 0$ for the solution to the given initial value problem.

$$\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} - y = 0, \quad y(0) = 2, \quad \frac{dy(0)}{dx} = 0.$$

9. Find a power series solution of the differential equation

$$\frac{d^y}{dx^2} + \frac{3x}{x^2-1}\frac{dy}{dx} + \frac{x}{x^2-1}y = 0,$$

given that $y(0) = 4$ and $y'(0) = 6$.

10. Find the power series solution in powers of $(x-1)$ of the initial value problem

$$xy'' + y' + 2y = 0, \qquad y(1) = 1, \quad y'(1) = 2$$