CS-204: Design and Analysis of Algorithms

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1 Some common NP-complete problems

1. SAT 2. Vertex Cover 3. Independent Set 4. Clique 5. Graph Colouring 6. Subset Sum 7. Travelling Salesman 8. Hamiltonian Cycle 9. 0-1 knapsack

2 Independent set

For an undirected graph G(V,E), V' $\subseteq$ V is said to be an independent set if $\forall$ u,v $\in$ V', (u,v) $\notin$ E.

2.1 Optimisation problem

Find out the maximal independent set V_{max} such that $\forall$ V' $\in$ V, $|V_{max}| \geq |V'|$ where V' and V_{max} are independent sets

Now we have to prove that the given optimisation problem is NP-complete. A decision problem L is in NP-complete if:
1. The decision problem is in NP
2. Every problem in NP is reducible to L in polynomial time

Now to prove that a decison problem is in NP is simple, we just need to show that a certificate of the problem is verifiable in polynomial time.
To prove the second point that every problem in NP is reducible to L is practically impossible.
But if we are able to reduce a known NP-complete problem to the given decision problem then we can give the argument that any problem in NP is reducible to the given decision problem because:
(Any problem in NP) $\rightarrow$(known NP-complete problem)$\rightarrow$(decision problem)

We can reduce an problem in NP to known NP-complete problem and then we can reduce the known NP-complete problem to given decision problem. By following these steps, we can reduce any problem in NP to our decision problem.

2.2 Given optimisation problem is in NP

If the problem is verifiable in polynomial time for every certificate, then the given problem is in NP.
certificate :- $V' \subseteq V$, $|V'| \geq k$
$|V'| = O(n)$
Verification :-
for $u \in V'$
    if$(u,v) \in E$
        if $(v) \in V'$
            not an independent set
Thus, the problem is verifiable in polynomial time, thus the problem is in NP.

2.3 Given optimisation problem is in NP-complete

We have proved that the given problem is in NP. Now in order to show that the problem is in NP-complete we need to be able to find a decision problem corresponding to this optimisation problem such that decision problem can be solved using the given optimisation problem.

Decision Problem :- $\{\langle G,K \rangle \mid$ G has an independent set of size at least k$\}$
It should be noted that the following decision problem can be solved by solving the optimisation problem.
Say that we are able to find a solution to the optimisation problem. Let S be the set of maximal independent set. if $|S|$ is less than k (in decision problem) then the answer to the decision problem is no, and otherwise, yes.

Now consider the vertex cover decision problem discussed earlier (if a graph has a vertex cover of at most k vertices)
We know that the vertex cover decision problem is NP-complete If we are able to reduce the vertex cover decision problem to the independent set decision problem then we will be able to prove that the independent set decision problem is also NP-complete
(i/p vertex cover = $\{\langle$ G,n-k $\rangle\}$) $\rightarrow$ (i/p independent set = $\{\langle$ G,k $\rangle\}$)
where $n = |V|$
Note: reduction function of vertex cover problem to independent set problem is O(1)(which is also polynomial) as we are not making any change in graph or anywhere.
Argument : If the independent set decision problem is solved then vertex cover problem is also solved
If there exists an independent set of size atleast k, let the set be S. There is no edge (u,v) for $u \in S$ and $v \in S$. Thus, for each $u \in S$ and $(u,v) \in E$, $v \in V \setminus S$. Or we can say that all edges in E has atleast one end point in $V \setminus S$. Thus, $V \setminus S$ is a vertex cover of size $n-|S|$. Also, $n-|S| \leq n-k$ as $|S| \geq k$.
Thus, since vertex cover decision problem can be reduced to independent set

decision problem and vertex cover problem is a NP-complete problem, thus independent set decision problem is also NP-complete problem.

It can be noted that since independent set decision problem is in NP, the independent set decision problem can also be reduced to vertex cover decision problem.
(i/p vertex cover = $\{\langle$ G,k $\rangle\}$) $\leftarrow$ (i/p independent set = $\{\langle$ G,n-k $\rangle\}$).
Let set of vertices of vertex cover = S with |S|$\leq$k. Consider set of V\S. If there is edge (u,v) in E where u,v $\in$ V\S, then u,v $\notin$ V. It means the edge (u,v) not covered by vertex cover, thus V is not vertex cover. Contradiction. Thus, there is no edge (u,v) where u,v $\in$ V\S. thus V\S is an independent set of size |V|-|S| $\geq$ k.

3 Clique

A clique in an undirected graph G=(V,E) is a subset $V' \subseteq V$ of vertices, each pair of which is connected by an edge in E. In other words, a clique is a complete subgraph of G. The size of a clique is the number of vertices it contains. The clique problem is the optimization problem of finding a clique of maximum size in a graph. The corresponding decision problem asks simply whether a clique of a given size k exists in the graph. The formal definition is

$\{\langle G,k\rangle : G\ is\ a\ graph\ containing\ a\ clique\ of\ size\ k\}$

A naive algorithm for determining whether a graph G=(V,E) with $|V|$ vertices contains a clique of size k lists all k-subsets of V and checks each one to see whether it forms a clique. The running time of this algorithm is $\omega(k^2nk)$, which is polynomial if k is a constant. In general, however, k could be near $|V|/2$, in which case the algorithm runs in superpolynomial time. Indeed, an effecient algorithm for the clique problem is unlikely to exist.

The clique problem is NP-complete.

3.1 Optimization Problem

Find out maximum clique G'=(V',E') s.t.V'⊆ V ,E'⊆ E and ∀ (u,v)∈ V'×V', (u,v)∈E', i.e. G' is an induced complete subgraph of G.

3.2 Given optimization problem in NP

If the problem is verifiable in polynomial time for every certificate, then the given problem is in NP.
Certificate:- V' ⊆ V,$|V'|$≥K
$|V'|$=o(n)
Verification:-
for v ∈ V'
    if v ∈ V'
        checkif (u,v) ∈ E

Thus, the problem is verifiable in polynomial time, thus the problem is in NP.

3.3 Given optimisation problem is in NP-complete

Proof:- We have shown that $clique \in NP$ by showing that a certificate can be verified in polynomial time.
Since, We already know that Vertex Cover is a known NP-complete problem we will reduce this vertex cover decision problem to our clique decision problem.

Clique Decision Problem :- $\{\langle G, k\rangle : G\ has\ a\ clique\ of\ size\ at\ least\ k\}$

It should be noted that the following decision problem can be solved by solving the optimisation problem. Say that we are able to find a solution to the optimisation problem. Let S be the set of maximal clique set. if S is less than k (in decision problem) then the answer to the decision problem is no, and otherwise, yes.

Now consider the vertex cover decision problem discussed earlier (if a graph has a vertex cover of at most k vertices) We know that the vertex cover decision problem is NP-complete If we are able to reduce the vertex cover decision problem to the clique decision prob lem then we will be able to prove that the clique decision problem is also NP-complete

(i/p vertex cover = $\{\langle G', n-k\rangle\}$) $\rightarrow$ (i/p Clique = $\{\langle G, k\rangle\}$)

we will reduce vertex cover decision problem to clique decision problem via independent set decision problem. First we will reduce vertex cover to independent set problem then we will reduce independent set to clique decision problem (vertex cover = $\{\langle G', n-k\rangle\}$) $\rightarrow$ (independent set = $\{\langle G', k\rangle\}$) $\rightarrow$ (i/p Clique = $\{\langle G, k\rangle\}$)
We have already shown that we can reduce vertex cover decision problem to independent set decision problem now we can show that we can reduce independent set $\{\langle G', k\rangle\}$ to clique $\{\langle G, k\rangle\}$ as below.

Let V' be the independent set of G' = (V,E)
$\Rightarrow$ $\forall$ (u,v) $\in$ V'xV' does not form an edge in G' (by definition)
$\Rightarrow$ $\forall$ (u,v) $\in$ V'xV' forms an edge in G[as per definition of G]
$\Rightarrow$ V' forms a complete induced Sub-graph in G

Now, if size of independent set |V'| $\geq$ k then size of clique is also $\geq$ k. Thus the independent set decision problem is reducible to clique decision problem
We can see that reduction function of independent set decision problem to clique is polynomial as we are just considering complement of graph G. We can covert G to G' in polynomial time.

Thus, vertex cover decision problem is reucible to clique decision problem
Thus, Clique Decision Problem is NP-complete and thus, Clique optimisation problem is NP-complete.
Hence Proved.

4 Graph Colouring

Given an undirected graph G(V,E). G is called K-colorable if there exists C=(C1,C2,......Ck) and f:V$\rightarrow$C such that f(v)$\neq$f(u) if (u,v) $\in$ E.
i.e. we conclude that two adjacent vertices will not have the same color.

(Note:The chromatic number of a graph is the minimum number of colors needed to color the vertices of a graph so that no two adjacent vertices have the same color.)

4.1 Optimization Problem

To determine minimum number of colors needed to color the graph such that no two adjacent vertices have same color.

certificate: { (v1,c1),(v2,c2),......(vn,cn) where c1,c2,.....cn $\in$ C } To verify that the certificate is k-colorable just do simple bfs or dfs and check if any adjacent edge has same color . This verification can be done in polynomial time. Thus, the optimisation problem is in NP.

To show that the problem is in NP-hard, show that SAT is reducible to k-coloring problem (SAT $\leq$ k-coloring)