# MA 204 Numerical Methods

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Lecture-1

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## Contents

- Solution of a nonlinear equation, bisection and secant methods, Newton's method, rate of convergence, solution of a system of nonlinear equations.

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- Solution of a nonlinear equation, bisection and secant methods, Newton's method, rate of convergence, solution of a system of nonlinear equations.
- Interpolation by polynomials, divided differences, error of the interpolating polynomial, piecewise linear and cubic spline interpolation.

# Motivation

## Nonlinearity

One of the most frequent problem in engineering and science is to find the root(s) of a non-linear equation

$$f(x) = 0. \tag{1}$$

Here,

- $f : [a, b] \to \mathbb{R}$ is a nonlinear function in $x$;
- $f \in C^1[a, b]$;
- Roots are **isolated**.

# Root of the equation

**Definition 1**

Given a nonlinear function $f : [a, b] \to \mathbb{R}$, find a value of $r$ for which $f(r) = 0$. Such a solution value for $r$ is called a **root** of the equation

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### Approximation to a root

A point $x^\star \in \mathbb{R}$ such that

- $|r - x^\star|$ is **very small**, and
- $f(x^\star)$ is **very close** to 0.

# Examples

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## Example 3

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$$f(x) = 0 \implies x^2 + 4x + 10 = 0 \implies (x+2)^2 + 6 = 0.$$

As $(x+2)^2 + 6 \geq 6 \ \forall \ x \in \mathbb{R}$. $\boxed{\text{So, } f(x) = 0 \text{ has no real roots.}}$

**Example 4**

$f(x) = x^2 + \cos(x) + e^x + \sqrt{x+1}$

The equation $f(x) = 0$ might have real root/roots but the point is that it is very difficult to find the analytic expression of $x$.

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   **Advantages:** No need to locate the root initially

   **Disadvantages:** May not converge

For each of the method, we study the following:

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- Geometry of the method (iteration) process
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- Application and example

# Bisection Method: Basic idea

This method is based on Intermediate Value Theorem.

**Theorem**

If $f \in C[a, b]$ and $K$ is any number between $f(a)$ and $f(b)$, then there exists $c \in (a, b)$ such that $f(c) = K$.

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**Bisection Method**

- Suppose that $f(x)$ is continuous on given interval $[a, b]$.
- The function $f$ satisfies the property $f(a)f(b) < 0$ with $f(a) \neq 0$ and $f(b) \neq 0$.
- By Intermediate Value Theorem, there exists a number $c$ such that $f(c) = 0$.

# Bisection Method: Description of the method

The Bisection method consists of the following steps:

**Step 1:** Given an initial interval $[a_0, b_0]$, set $n = 0$.

**Step 2:** Define $c_n = \dfrac{(a_n + b_n)}{2}$, the mid-point of interval $[a_n, b_n]$.

**Step 3:**
- If $f(c_n) = 0$, then $x^\star = c_n$ is the root.
- If $f(c_n) \neq 0$, then either

$$f(a_n)f(c_n) < 0 \quad \text{or} \quad f(a_n)f(c_n) > 0.$$

- If $f(a_n)f(c_n) < 0$, then $a_{n+1} = a_n$, $b_{n+1} = c_n$ and the root $x^\star \in [a_{n+1}, b_{n+1}]$.
- If $f(a_n)f(c_n) > 0$, then $f(b_n)f(c_n) < 0$, this implies $a_{n+1} = c_n$, $b_{n+1} = b_n$ and the root $x^\star \in [a_{n+1}, b_{n+1}]$.

**Step 4:** Repeat

**Step 5:** If the root is not achieved in **Step 3**, then, find the length of new reduced interval $[a_{n+1}, b_{n+1}]$. If the length of the interval $b_{n+1} - a_{n+1}$ is less than a recommended positive number $\varepsilon$, then take the mid-point of this interval $(x^{\star} = (b_{n+1} - a_{n+1})/2)$ as the approximate root of the equation $f(x) = 0$, otherwise go to Step 2.

# Convergence and error in Bisection Method

Let $[a_0, b_0] = [a, b]$ be the initial interval with $f(a)f(b) < 0$. Define the approximate root as $c_n = (a_n + b_n)/2$. Then, there exists a root $x^\star \in [a, b]$ such that

$$|c_n - x^\star| \leq (\frac{1}{2})^n (b - a). \tag{2}$$

Moreover, to achieve the accuracy of $|c_n - x^\star| \leq \varepsilon$, it is sufficient to take

$$\frac{|b-a|}{2^n} \leq \varepsilon \quad \text{i.e.} \quad n \geq \frac{log(|b-a|) - log(\varepsilon)}{log 2}. \tag{3}$$

# Error analysis in Bisection Method

$$b_{n+1} - a_{n+1} = \frac{1}{2}(b_n - a_n), \quad n \geq 1. \tag{4}$$

$$\begin{aligned} b_n - a_n &= \frac{1}{2}(b_{n-1} - a_{n-1}) \\ &= \frac{1}{2^2}(b_{n-2} - a_{n-2}) = \frac{1}{2^{n-1}}(b_1 - a_1). \end{aligned} \tag{5}$$

$$\begin{aligned} |c_n - x^\star| \leq c_n - a_n = b_n - c_n &= \frac{1}{2}(b_n - a_n) \\ &= (\frac{1}{2})^n(b_1 - a_1) = (\frac{1}{2})^n(b - a). \end{aligned} \tag{6}$$

Therefore, $|c_n - x^\star| \leq (\frac{1}{2})^n(b - a)$. This implies that the iterates $c_n$ converge to $x^\star$ as $n \to \infty$.

# How many iterations?

Our goal is to have $|c_n - x^\star| \leq \varepsilon$. This will be satisfied if

$$\begin{aligned}
(\frac{1}{2})^n(b-a) \leq \varepsilon &\implies 2^n \geq \frac{b-a}{\varepsilon} \\
&\implies nlog_{10}2 \geq log_{10}(\frac{b-a}{\varepsilon}) \\
&\implies n \geq \frac{log(|b-a|) - log(\varepsilon)}{log2}.
\end{aligned} \tag{7}$$

## Stop criteria

First select some tolerance $\varepsilon > 0$.

1. Small enough interval i.e., $b_n - a_n \leq \varepsilon$;
2. Small enough difference of consecutive approximations i.e.,

$$|c_{n+1} - c_n| \leq \varepsilon \quad \text{or} \quad \frac{|c_{n+1} - c_n|}{|c_n|} \leq \varepsilon;$$

3. Small enough functional value $|f(c_n)| \leq \varepsilon$;
4. Maximum number of iterations;
5. Any combination of the above.

# Pros and Cons

## Pros

1. This method is very easy to understand.
2. The sequence forms a cauchy sequence, and always converge to a solution.
3. It is often used as a starter for the more efficient methods.

## Cons

1. This method is relatively slow to converge.
2. Choosing a guess close to the root may result in requiring many iterations to converge.