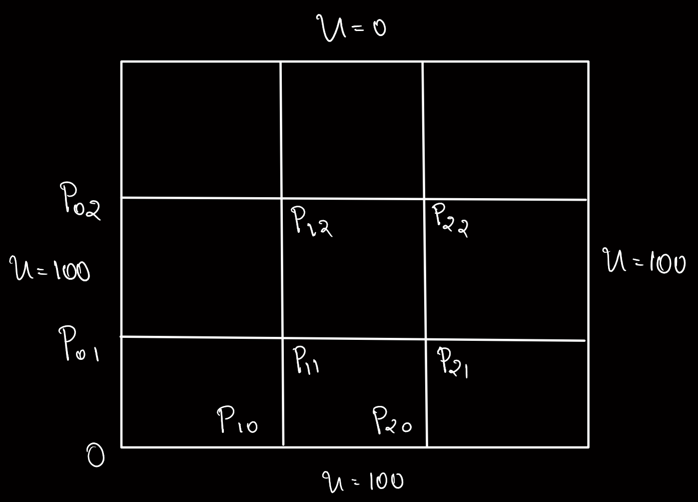
1) The four sides of a square plate of side 12 cm made of homogeneous material are kept at constant temperature 0° c and 100° c as shown in the following figure. Using a grid of 4 cm and applying ADI Scheme (Alternate direction implicit Scheme) find the steady-state temperature at the mesh points P_{11}, P_{21}, P_{12} and P_{22}. Calculate upto 6 iterates after considering the initial guess $u_{11}^{0} = u_{12}^{0} = u_{21}^{0} = u_{22}^{0} = 100$.



ans:-

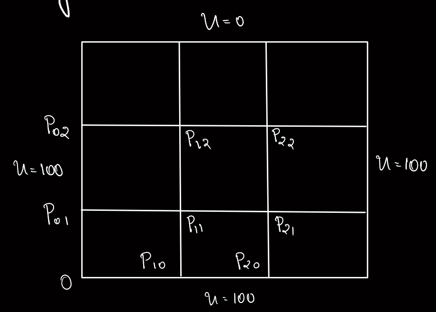
$$u(x-h,y) + u(x+h,y) + u(x,y-h) + u(x,y+h) - 4u(x,y) = 0.$$

$$u_{i-1,j}^{(m+1)} - 4u_{i,j}^{(m+1)} + u_{i+1,j}^{(m+1)} = -u_{i,j-1}^{(m)} - u_{i,j+1}^{(m)} \quad \text{(For a fixed row } j\text{)} \text{ — } ①$$

$$u_{i,j-1}^{(m+2)} - 4u_{i,j}^{(m+2)} + u_{i,j+1}^{(m+2)} = -u_{i-1,j}^{(m+1)} - u_{i+1,j}^{(m+1)} \quad \text{(For a fixed Column } i\text{)} \text{ — } ②$$

Finding the first approximations $u_{11}^{(1)}, u_{21}^{(1)}, u_{12}^{(1)}, u_{22}^{(1)}$ Using ①.

For the first row (ie, $u_{11}^{(1)}$ and $u_{21}^{(1)}$)



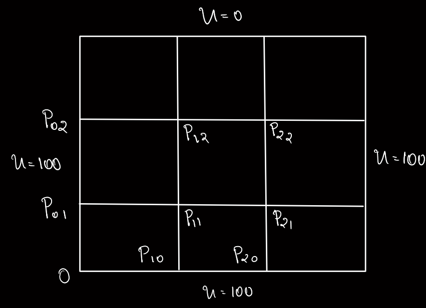
$$u_{01}^{(1)} - 4u_{11}^{(1)} + u_{21}^{(1)} = -u_{10}^{(0)} - u_{12}^{(0)}$$

$$u_{11}^{(1)} - 4u_{21}^{(1)} + u_{31}^{(1)} = -u_{20}^{(0)} - u_{22}^{(0)}$$

$$\Rightarrow -4u_{11}^{(1)} + u_{21}^{(1)} = -300 \qquad \Rightarrow u_{11}^{(1)} = 100 \text{ and } u_{21}^{(1)} = 100$$

$$u_{11}^{(1)} - 4u_{21}^{(1)} = -300$$

For the Second row ($j=2$)



$$u_{02}^{(1)} - 4u_{12}^{(1)} + u_{22}^{(1)} = -u_{11}^{(0)} - u_{13}^{(0)}$$

$$u_{12}^{(1)} - 4u_{22}^{(1)} + u_{32}^{(1)} = -u_{21}^{(0)} - u_{23}^{(0)}$$

$$\Rightarrow 100 - 4u_{12}^{(1)} + u_{22}^{(1)} = -100$$

$$u_{12}^{(1)} - 4u_{22}^{(1)} + 100 = -100$$

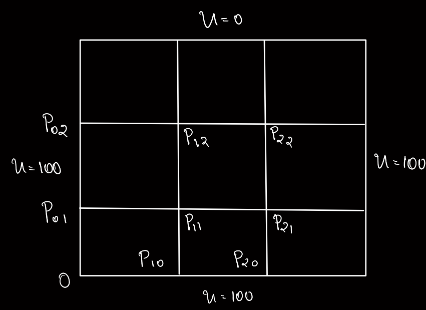
$$\Rightarrow -4u_{12}^{(1)} + u_{22}^{(1)} = -200$$

$$u_{12}^{(1)} - 4u_{22}^{(1)} = -200$$

$$\Rightarrow u_{12}^{(1)} = 66.667$$

$$u_{22}^{(1)} = 66.667$$

Finding Second approximations ($u_{11}^{(2)}$, $u_{21}^{(2)}$, $u_{12}^{(2)}$ and $u_{23}^{(2)}$ using ②)



For the first Column

$$u_{10}^{(2)} - 4u_{11}^{(2)} + u_{12}^{(2)} = -u_{01}^{(1)} - u_{21}^{(1)}$$

$$u_{11}^{(2)} - 4u_{12}^{(2)} + u_{13}^{(2)} = -u_{02}^{(1)} - u_{22}^{(1)}$$

$$\Rightarrow 100 - 4u_{11}^{(2)} + u_{12}^{(2)} = -100 - 100$$

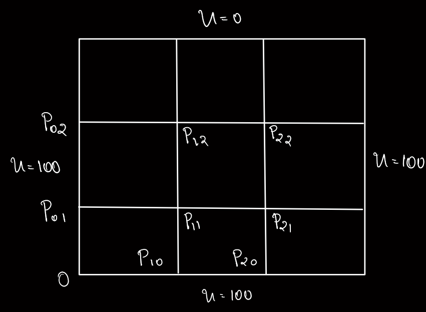
$$u_{11}^{(2)} - 4u_{12}^{(2)} + 0 = -100 - 66.667$$

$$\Rightarrow -4u_{1}^{(2)} + u_{12}^{(2)} = -300$$

$$u_{11}^{(2)} - 4u_{12}^{(2)} = -166.667 \Rightarrow$$

$$\Rightarrow u_{11}^{(2)} = 91.11, \quad u_{12}^{(2)} = 64.44$$

For the Second Column



$$u_{20}^{(2)} - 4u_{21}^{(2)} + u_{22}^{(2)} = -u_{11}^{(1)} - u_{31}^{(1)}$$

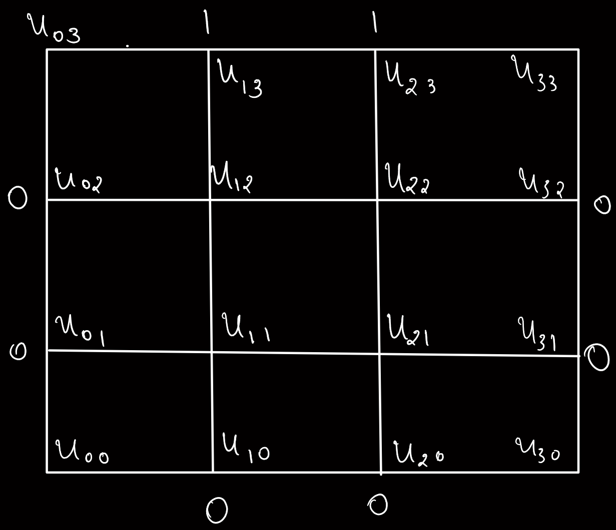
$$u_{21}^{(2)} - 4u_{22}^{(2)} + u_{23}^{(2)} = -u_{12}^{(1)} - u_{32}^{(1)}$$

$$\Rightarrow 100 - 4u_{21}^{(2)} + u_{22}^{(2)} = -100 - 100$$

$$u_{21}^{(2)} - 4u_{22}^{(2)} + 0 = -66.667 - 100$$

$$\Rightarrow u_{21}^{(2)} = 91.11, \quad u_{22}^{(2)} = 64.44.$$

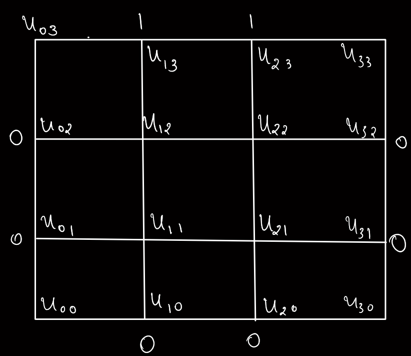
2) Solve the Laplace equation $\partial_{xx} + \partial_{yy} = 0$ at the mesh points of the domain using ADI Scheme. Find upto u_{ij}^{2} after considering the initial guess $u_{12}^{0} = u_{22}^{0} = 1$ and $u_{11}^{0} = u_{21}^{0} = 0$.



ans

Finding the first approximations $u_{11}^{(1)}, u_{21}^{(1)}, u_{12}^{(1)}$ and $u_{22}^{(1)}$.

Along the first row



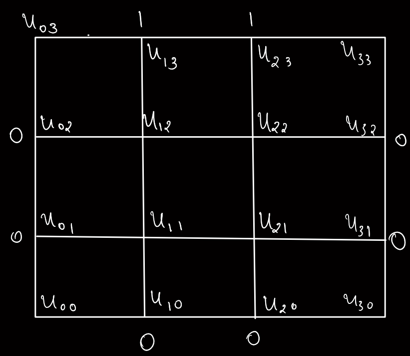
$$u_{01}^{(1)} - 4u_{11}^{(1)} + u_{21}^{(1)} = -u_{10}^{(0)} - u_{12}^{(0)}$$

$$u_{11}^{(1)} - 4u_{21}^{(1)} + u_{31}^{(1)} = -u_{20}^{(0)} - u_{22}^{(0)}$$

$$\Rightarrow -4u_{11}^{(1)} + u_{21}^{(1)} = -1 \qquad \Rightarrow u_{11}^{(1)} = u_{21}^{(1)} = \frac{1}{3} = 0.333.$$

$$u_{11}^{(1)} - 4u_{21}^{(1)} = -1$$

Along the Second row



$$u_{02}^{(1)} - 4u_{12}^{(1)} + u_{22}^{(1)} = -u_{11}^{(0)} - u_{12}^{(0)}$$

$$u_{12}^{(1)} - 4u_{22}^{(1)} + u_{32}^{(1)} = -u_{21}^{(0)} - u_{23}^{(0)}$$

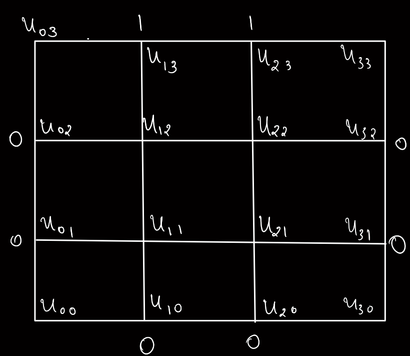
$$\Rightarrow \quad -4u_{12}^{(1)} + u_{22}^{(1)} = 0$$

$$u_{12}^{(1)} - 4u_{22}^{(1)} = 1$$

$$\Rightarrow u_{12}^{(1)} = u_{22}^{(1)} = \frac{1}{3} = 0.333$$

Finding the Second approximations $u_{11}^{(2)}, u_{12}^{(2)}, u_{21}^{(2)}, u_{22}^{(2)}$

Along the first Column



$$u_{10}^{(2)} - 4u_{11}^{(2)} + u_{12}^{(2)} = -u_{01}^{(1)} - u_{21}^{(1)}$$

$$u_{11}^{(2)} - 4u_{12}^{(2)} + u_{13}^{(2)} = -u_{02}^{(1)} - u_{22}^{(1)}$$

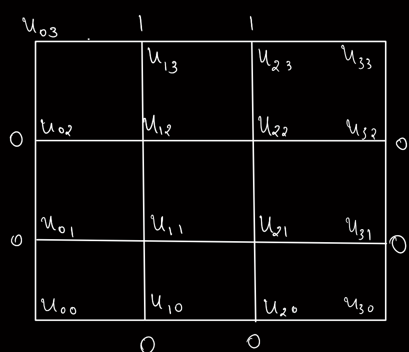
$$\Rightarrow \quad -4u_{11}^{(2)} + u_{12}^{(2)} = \frac{1}{3}$$

$$u_{11}^{(2)} - u_{12}^{(2)} + 1 = \frac{1}{3}$$

$$\Rightarrow \quad u_{11}^{(2)} = \frac{8}{45} = 0.1778$$

$$u_{12}^{(2)} = 17/45 = 0.778$$

Along the Second Column



$$u_{20}^{(2)} - 4u_{21}^{(2)} + u_{22}^{(2)} = -u_{11}^{(1)} - u_{31}^{(1)}$$

$$u_{21}^{(2)} - 4u_{22}^{(2)} + u_{23}^{(2)} = -u_{12}^{(1)} - u_{32}^{(1)}$$

$$\Rightarrow \quad -4u_{21}^{(2)} + u_{22}^{(2)} = \frac{-1}{3}$$

$$u_{21}^{(2)} - 4u_{22}^{(2)} + 1 = \frac{-1}{3}$$

$$\Rightarrow \quad u_{21}^{(2)} = 0.1778 \text{ and } u_{2,2}^{(2)} = 0.3778$$

3) Solve the mixed boundary value problem for the Poisson equation

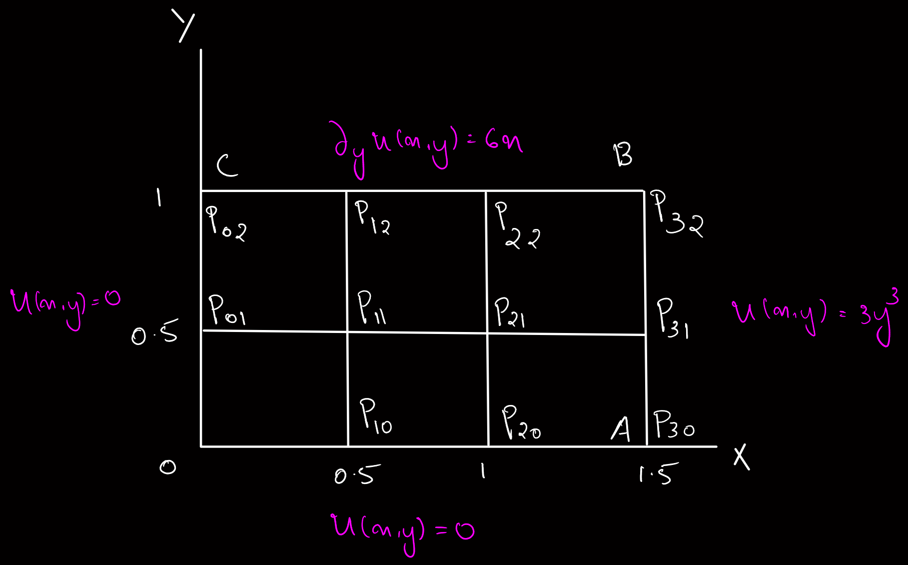
$$\Delta u = \partial_{xx} u + \partial_{yy} u = f(x,y) = 12xy$$

With boundary conditions

$$u(x,y) = 0 \text{ on } \overline{OA},\ \overline{OC},\quad u(x,y) = 3y^3 \text{ on } \overline{AB} \text{ and } \partial_y u(x,y) = 6 \text{ on } \overline{BC}$$

by using domain extension technique.

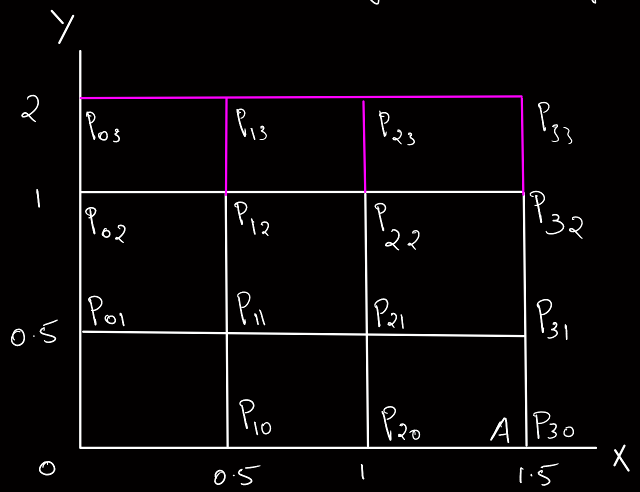
$$\boxed{u(x+h,y) + u(x,y+h) + u(x-h,y) + u(x,y-h) - 4u(x,y) = h^2 f(x,y).}$$



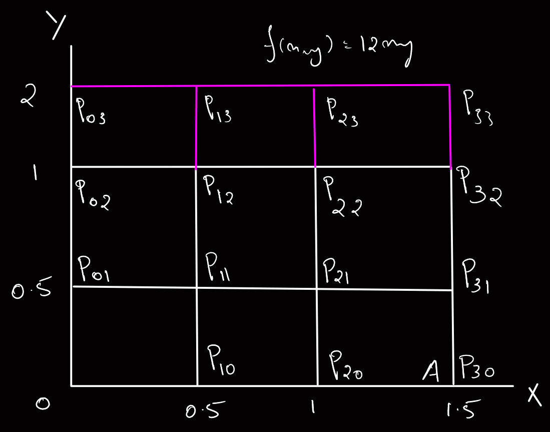
Using the boundary Conditions we've

$$u_{10} = u_{20} = u_{30} = 0 \qquad u_{31} = 3 \qquad u_{32} = 24$$

$$\frac{\partial u_{02}}{\partial y} = 0 \qquad \frac{\partial u_{12}}{\partial y} = 3 \qquad \frac{\partial u_{22}}{\partial y} = 6 \qquad \frac{\partial u_{32}}{\partial y} = 9$$



Assume that the Poisson equation also holds in the extended region. Then we've



$$u_{02} + u_{22} + u_{11} + u_{13} - 4u_{12} = h^2 f(0.5, 1)$$

$$\Rightarrow 0 + u_{22} + u_{11} + u_{13} - 4u_{12} = 0.25 \times 12 \times 0.5 = 1.5 \quad \text{---} \ (1)$$

and

$$u_{12} + u_{32} + u_{21} + u_{23} - 4u_{22} = h^2 f(1,1)$$

$$\Rightarrow u_{12} + 3 + u_{21} + u_{23} - 4u_{22} = 3$$

$$\Rightarrow u_{12} + u_{21} + u_{23} - 4u_{22} = 0 \quad \text{---} \ (2)$$

$3y^2$

Also we've $$3 = \frac{\partial u_{12}}{\partial u_y} \approx \frac{u_{13} - u_{11}}{2h} = u_{13} - u_{11}$$

$$\Rightarrow u_{13} \approx u_{11} + 3$$

and

$$6 = \frac{\partial u_{22}}{\partial u_y} \approx \frac{u_{23} - u_{21}}{2h} = u_{23} - u_{21}$$

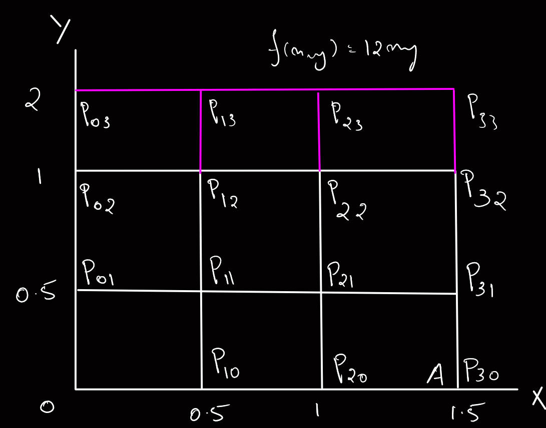
$$\Rightarrow u_{23} \approx u_{21} + 6$$

$$\therefore (1) \Rightarrow u_{11} - 4u_{12} + u_{22} + u_{11} + 3 = 1.5 \Rightarrow 2u_{11} - 4u_{12} + u_{22} = -1.5$$

$$(2) \Rightarrow u_{12} - 4u_{22} + u_{21} + u_{21} + 6 = 0 \Rightarrow u_{12} - 4u_{22} + 2u_{21} = -6$$

$0.25 \times 12 \times 0.25$

Also we've



$$u_{01} + u_{21} + u_{10} + u_{12} - 4u_{11} = h^2 f(0.5, 0.5)$$

$$\Rightarrow u_{21} + u_{12} - 4u_{11} = 3/4$$

and

$$u_{11} + u_{31} + u_{20} + u_{22} - 4u_{21} = h^2 f(1, 0.5)$$

$$\Rightarrow u_{11} + \frac{3}{8} + 0 + u_{22} - 4u_{21} = 0.25 \times 6$$

$$\Rightarrow u_{11} + u_{22} - 4u_{21} = \frac{9}{8}$$

.75
4 | 30
28
20
16
40

12×0.5

$3y^3 = 3 \times 0.5^3$

3×0.25

Hence we've a System of equations

$$2u_{11} - 4u_{12} + u_{22} = -1.5$$

$$u_{12} - 4u_{22} + 2u_{21} = -6$$

$$u_{21} + u_{12} - 4u_{11} = 3/4$$

$$u_{11} + u_{22} - 4u_{21} = \frac{9}{8}$$

$$\begin{array}{cccc} u_{11} & u_{21} & u_{12} & u_{22} \\ \end{array}$$

$$\begin{bmatrix} 2 & 0 & -4 & 1 \\ 0 & 2 & 1 & -4 \\ -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \\ u_{12} \\ u_{22} \end{bmatrix} = \begin{bmatrix} -1.5 \\ -6 \\ 3/4 \\ 9/8 \end{bmatrix}$$

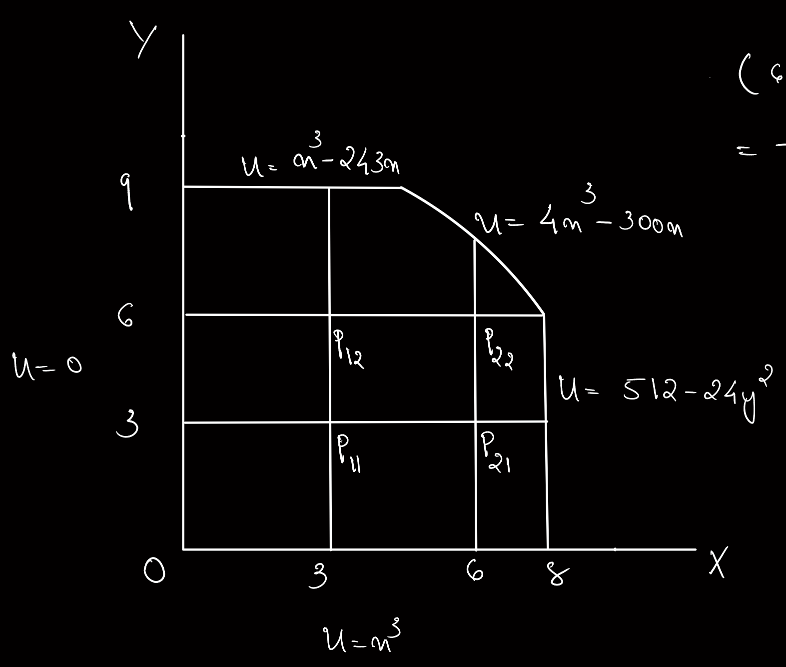
$\Rightarrow u_{11} = 235/1288$

$u_{21} = 331/644$

$u_{12} = 311/322$

$u_{22} = 1287/644$.

4) Find the potential u (ie, u Solves Laplace equation) in the region that has the boundary value given in that figure, here the curved portion of the boundary is an arc of the circle of radius 10 about $(0,0)$. After achieving a set of linear equations, use Gauss elimination to Solve the Same.



$(6,8)$

$= -936$

$(3,9)$

$3^3 - 243 \times 3$

$(8,3)$

$(8,3)$

$512 - 24 \times 3^2$

$= 512 - 24 \times 9$

$(3,0)$

Solution

u_{11}

$$u_{01} + u_{21} + u_{10} + u_{12} - 4u_{11} = 0$$

$$\Rightarrow \quad u_{21} + 27 + u_{12} - 4u_{11} = 0$$

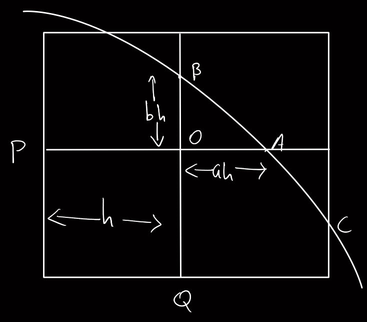
$$\Rightarrow \quad -4u_{11} + u_{12} + u_{21} = -27$$

u_{12}

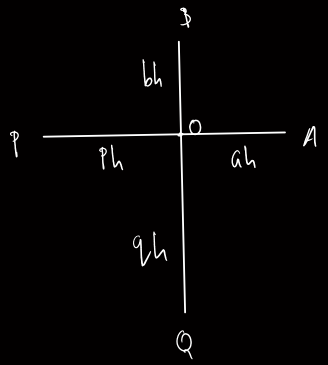
$$u_{02} + u_{22} + u_{11} + u_{13} - 4u_{12} = 0$$

$$\Rightarrow \quad u_{22} + u_{11} - 702 - 4u_{12} = 0$$

$$\Rightarrow \quad u_{11} - 4u_{12} + u_{22} = 702$$



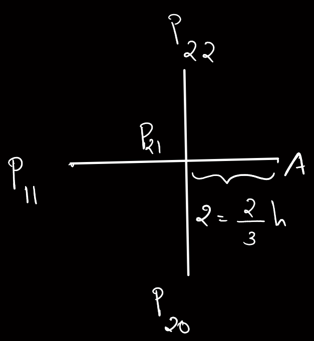
$$\nabla^2 u_0 \approx \frac{2}{h^2}\left[\frac{u_A}{a(1+a)} + \frac{u_B}{b(1+b)} + \frac{u_P}{1+a} + \frac{u_Q}{1+b} - \frac{(a+b)u_0}{ab}\right]$$



$$\nabla^2 u_0 \approx \frac{2}{h^2}\left[\frac{u_A}{a(a+p)} + \frac{u_B}{b(b+q)} + \frac{u_P}{p(p+a)} + \frac{u_Q}{q(q+b)} - \frac{ap+bq}{abpq}u_0\right]$$

u_{21}

$h = 3$



$$\frac{2}{3} \times 3$$

$$\frac{\frac{2}{3} + 1}{\frac{2}{3}} = \frac{5}{3} \times \frac{3}{2}$$

$$a = \frac{2}{3} \quad , \; b = p = q = 1$$

$$\left(\frac{u_A}{\frac{2}{3}\left(\frac{2}{3}+1\right)} + \frac{u_{22}}{2} + \frac{u_{11}}{1+\frac{2}{3}} + \frac{u_{20}}{2} - \frac{5}{2}u_{21}\right)\frac{2}{3^2} = 0$$

$$\frac{2}{3} \times \frac{5}{3} = \frac{10}{9}$$

$$\Rightarrow \quad \frac{9}{10}u_A + \frac{u_{22}}{2} + \frac{3}{5}u_{11} + \frac{u_{20}}{2} - \frac{5}{2}u_{21} = 0$$

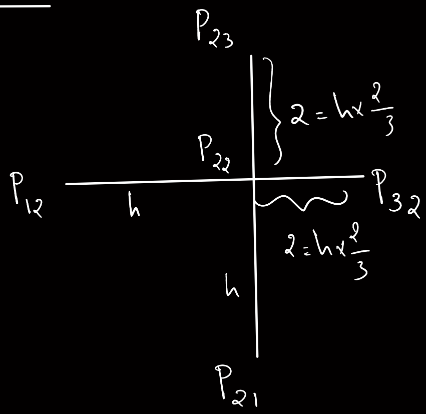
$$\frac{5}{3}$$

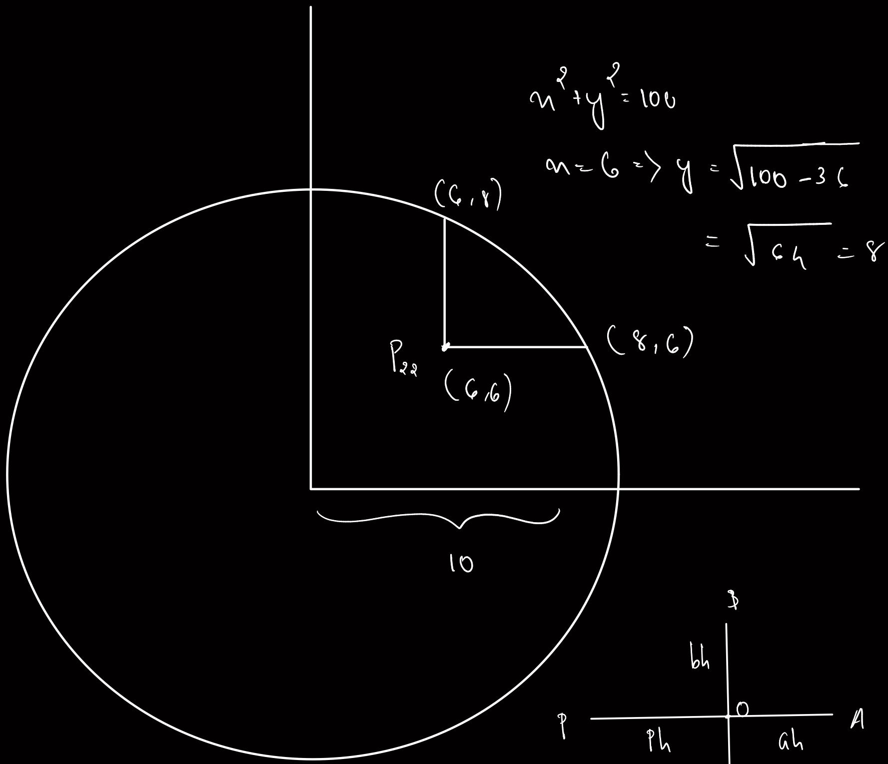
$$\Rightarrow \quad \frac{9}{10} \times 296 + \frac{u_{22}}{2} + \frac{3}{5}u_{11} + 4 - \frac{5}{2}u_{21} = 0$$

$$\Rightarrow \quad \frac{u_{22}}{2} + \frac{3}{5}u_{11} - \frac{5}{2}u_{21} = \frac{-1352}{5}$$

u_{22}

$3 \times \frac{2}{3}$





$x^2 + y^2 = 100$

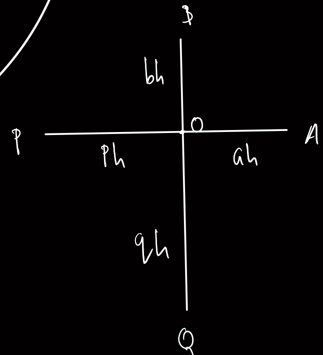
$x = 6 \Rightarrow y = \sqrt{100 - 36}$

$= \sqrt{64} = 8$

$a = \frac{2}{3} \quad b = \frac{2}{3} \quad p = 1 \quad q = 1$

$$\frac{u_{32}}{\frac{2}{3}\left(\frac{2}{3}+1\right)} + \frac{u_{23}}{\frac{2}{3}\left(\frac{2}{3}+1\right)} + \frac{u_{12}}{1+\frac{2}{3}}$$

$$+ \frac{u_{21}}{1+\frac{2}{3}} - \frac{2/3 + 2/3}{4/9} u_{22} = 0.$$



$$\Rightarrow -352 \times \frac{9}{10} - 936 \times \frac{9}{10} + u_{12} \times \frac{3}{5} + u_{21} \times \frac{3}{5}$$

$$- 3u_{22} = 0$$

$$\nabla^2 u_0 \approx \frac{2}{h^2}\left[\frac{u_A}{a(a+p)} + \frac{u_B}{b(b+q)} + \frac{u_P}{p(p+a)} + \frac{u_Q}{q(q+b)} - \frac{ap+bq}{abpq} u_0\right]$$

$$\Rightarrow \frac{3}{5}u_{12} + \frac{3}{5}u_{21} - 3u_{22} = \frac{5796}{5}$$

Hence we have

$$-4u_{11} + u_{12} + u_{21} = -27$$

$$u_{11} - 4u_{12} + u_{22} = 702$$

$$\frac{u_{22}}{2} + \frac{3}{5}u_{11} - \frac{5}{2}u_{21} = \frac{-1352}{5}$$

$$\frac{3}{5}u_{12} + \frac{3}{5}u_{21} - 3u_{22} = \frac{5796}{5}$$

$$\begin{array}{cccc} u_{11} & u_{21} & u_{12} & u_{22} \end{array}$$

$$\begin{bmatrix} -4 & 1 & 1 & 0 \\ 1 & 0 & -4 & 1 \\ \frac{3}{5} & \frac{-5}{2} & 0 & \frac{1}{2} \\ 0 & \frac{3}{5} & \frac{3}{5} & -3 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \\ u_{12} \\ u_{22} \end{bmatrix} = \begin{bmatrix} -27 \\ 702 \\ \frac{-1352}{5} \\ \frac{5796}{5} \end{bmatrix}$$