MANNA MA

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Approximating Polynomials. (Lagrange's Form)

$$P(x) = \sum_{i=1}^{k} L(x_i)\, y_i$$

$$\int_{x_0}^{x_k} P(x)\,dx = \int_{x_0}^{x_k} \sum_{i=1}^{k} L(x_i)\, y_i\, dx = \sum_{i=1}^{k} \int_{x_0}^{x_k} L(x_i)\, y_i\, dx$$

* Equally spaced abssicca's $h = x_i - x_{i-1}$ $i = 1, \dots n$

$$P(p) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2}\Delta y_0 \dots \frac{p(p-1)(p-2)\dots(p-k+1)}{k!}(\Delta y_0)^k + R_n(x)$$

$$P(x) = y_0 + (x-x_0)\Delta y_0 + \frac{(x-x_0)(x-x_1)}{2!\,h^2}(\Delta y_0)^2 + \dots \frac{(x-x_0)\dots(x-x_{k-1})}{k!\,h^k}\Delta y_0^k$$

$x = x_0 + ph$

$x_n = x_0 + nh$

$$\int_{x_0}^{x_k} P(x)\,dx = \int_{x_0}^{x_k} f(x)\,dx = h\int_{x_0}^{x_k} P(p)\,dp$$

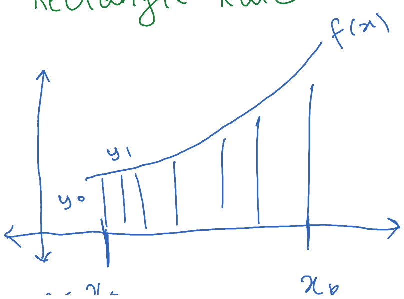
$$\text{Error } E = \int_{x_0}^{x_k} R(x)\,dx \qquad R(x) = \frac{p(p-1)\dots(p-k)}{(k+1)!} = \frac{(x-x_0)\dots(x-x_k)}{(k+1)!} f^{k+1}(\xi)$$

$a \le \xi \le b$

$$\therefore E = \int_{x_0}^{x_k} \frac{(x-x_0)\dots(x-x_k)}{(k+1)!} f^{k+1}(\xi)\,dx$$

$$= \frac{h^{k+2} f^{k+1}(\xi)}{(k+1)!} \int_0^k p(p-1)\dots(p-k)\,dp$$

Rectangle Rule



$$\int_{x_0}^{x_1} f(x)\,dx = \int_{x_0}^{x_1} y_0\,dx = y_0(x_1 - x_0) = h y_0$$

$$\therefore \underbrace{\int_{x_0}^{x_k} f(x)\,dx = h(y_0 + y_1 + \dots y_{k-1})}_{}$$

composite formula for n integral .. spaced

x_0

Composite formula for n integral

increasing $f(x)$ $\quad \int_a^b f(x)\,dx \geq h\,(y_0 + \cdots\, y_{n-1})$

decreasing $f(x)$ $\quad \int_a^b f(x)\,dx \leq h\,(y_0 + \cdots\, y_{n-1})$

* for equally spaced x_i's

$h = x_i - x_{i-1}$

Approximating Integral values

$$I(f) = \int_a^b f(x)\,dx = \int_a^b P(x)\,dx$$

$$I(f) \approx \int_a^b (\text{simpler func}^n)\,dx + R_n$$

Numerical integrator
Quadrature Rule.
} Process of approximating an integral

Construct polynomial f for n+1 nodes

$x_0 \quad x_1 \cdots\cdots x_n$

$$I(f) \approx I(P_n) = \int_a^b \sum f(x_i)\,L_i\,dx.$$

Newton's FD formula for equally spaced absssisca's

$$I(f) = \int_a^b f(x)\,dx = \int_{x_0}^{x_n} f(x)\,dx = h\int_0^n f(x_0 + ph)\,dp$$

$$= h\int_0^n E^p f(x_0)\,dp = h\int_0^n (1+\Delta)^p \underbrace{f(x_0)}_{y_0}\,dp$$

$$= h\int_0^n \left[1 + p\,\Delta y_0 + \frac{p(p-1)}{2!}\,\Delta y_0^2 + \cdots\right] dp$$

Quadrature Formula

$$I = \int_a^b f\,dx \approx \sum_{k=0}^{n} \omega_k f(x_k) = \overbrace{\sum_{k=0}^{n} \omega_k f(x_k) + R_n(f)}^{\text{form } A}$$

x_k = abscisses

$f(x_k)$ = coordinates

ω_k = weights

$$R_n(f) = I - \sum_{k=0}^{n} \omega_k f(x_k)$$

Integration methods of form A is said to be of order p if $R_n = 0\ \forall n \leq p$

ie $f(x) = 1, x, x^2, \cdots x^p$

Error term for x^{p+1} is $E = \int_a^b \underbrace{c(x)}_{1}\, x^{p+1}\,dx - \sum_{k=0}^{n} \omega_k x_k^{p-1}$

weight factor $c(x)=1$

$$R_n(f) = \int_a^b f(x)\,dx - \sum w_k f(x_k)$$

$$= \frac{c}{(p+1)!} f^{p+1}(\xi) \qquad a \le \xi \le b$$

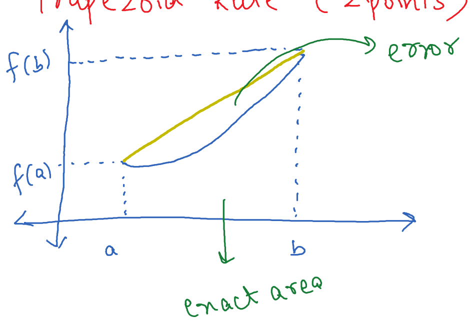
$$\therefore \quad |R_n(f)| \le \frac{|c|}{(p+1)!} \max_{a \le x \le b} |f^{p+1}(x)|$$

for uniform mesh grids. $a = x_0 \quad b = x_n \quad h = \frac{b-a}{n}$

$$I = \int_a^b f(x)\,dx = \sum w_k f(x_k)$$

$$= w_0 f(x_0) + w_1 f(x_1) \ldots \Big] \text{ Newton Cotes Quadrature formula.}$$

Trapezoid Rule (2 points)



$$f(x) = f(a) + \frac{x-a}{b-a}\left[f(b) - f(a)\right]$$

approx funcⁿ

$$I = \int_a^b f(x)\,dx = \frac{b-a}{2}\left[f(b) + f(a)\right]$$

This rule gives correct ans for polynomials with deg ≤ 1 $\quad f(x) = 1, x.$

ie $R(f, x) = 0$

$\therefore$ Order of trapezoid rule is one.

for $f(x) = x^2$

$$C = \int_a^b f(x) - \sum w_k f(x_k)$$

$$= \int_a^b x^2\,dx - \frac{b-a}{2}\left[b^2 + a^2\right]$$

$$= -\frac{1}{6}(b-a)^3$$

$$R_n(f, x) = \frac{c}{2!} f''(\xi) = -\frac{1}{12}(b-a)^3 f''(\xi) \quad a \le \xi \le b$$

$$\therefore \quad |R_n(f, x)| \le \frac{1}{12}|b-a|^3 \max_{a \le x \le b} |f''(x)|$$

$$I = \int_{x_0}^{x_n} y_n(x)\,dx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \dots\right]$$

$$y_n(x) = \underbrace{y_0}_{T_0} + \underbrace{p\Delta y_0}_{T_1} + \underbrace{\frac{p(p-1)}{2!}\Delta^2 y_0}_{T_2} + \dots$$

ignore terms after T_2

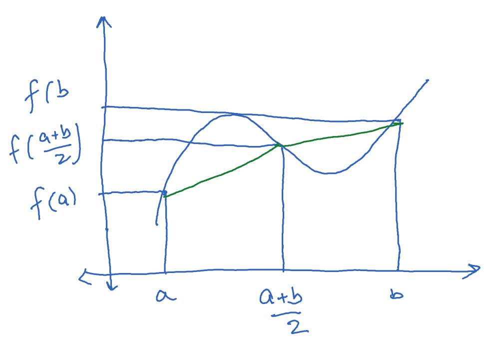
$$E_{Trap} = \int_a^b T_2\,dx = \frac{f''(p)}{2}\int_0^1 p(p)\,h\,dp = -f''(p)\frac{h}{12}$$

$$f''(p) = h^2 f''(x)$$

$$\therefore E_{trap} = \frac{-h^3}{12} f''(\xi) \qquad a \le \xi \le b$$

Simpson's 1/3 Rule

$$f(x) = f(x_0) + \frac{x-x_0}{h}\Delta f(x_0) + \frac{1}{2h^2}(x-x_0)(x-x_1)\Delta^2 f(x_0)$$



$$g(x) = f(x_0) + p\,\Delta f(x_0) + \frac{p(p-1)}{2!}\Delta^2 f(x_0)$$

$$x_0 = a \qquad x_1 = \frac{a+b}{2} \qquad x_2 = b$$

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_2} g(x)\,dx \qquad \text{Newton Cotes formula}$$

$$x_1 = x_0 + h$$
$$x_2 = x_0 + 2h$$

$$\Delta f(x_0) = f(x_1) - f(x_0)$$
$$\Delta^2 f(x_0) = f(x_0) - 2f(x_1) + f(x_2)$$

$$\therefore I = \int_a^b f(x)\,dx = \frac{b-a}{6}\left[f(a) + f(b) + 4f\left(\frac{a+b}{2}\right)\right]$$

Simpson's 1/3 rule

Error: $R(f, x) = 0$ for $f = 1, x, x^2, x^3$

for $f(x) = x^4$ $\quad R(f,x) = \frac{c}{4!} f^4(\xi)$

$\int^b \dots 4] \quad (b-a)^5$

for $f(x) = x$ $\quad K(t,x) = \frac{c}{4!} f^4(\xi)$

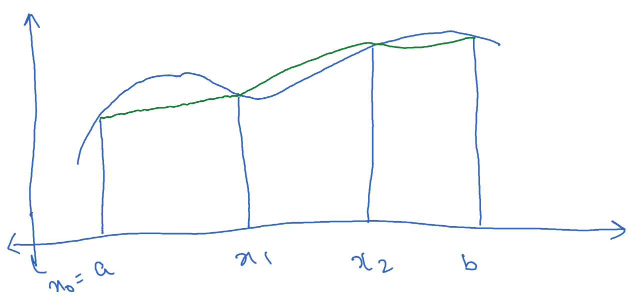
where $c = \int_a^b x^4 dx - \frac{b-a}{6}\left[a^4 + b^4 + 4\left(\frac{a+b}{2}\right)^4\right] = \frac{-(b-a)^5}{120}$

$\therefore R(f,x) = \frac{-(b-a)^5}{2880} f^4(\xi) = \frac{-h^5}{90} f^4(\xi) \qquad h = \frac{b-a}{2}$

a	b	n	Closed Newton-Cotes Formula	h	Truncation Error
x_0	x_1	1	$h \cdot \frac{f(x_0) + f(x_1)}{2}$	$\frac{(b-a)}{1}$	$-\frac{1}{12} h^3 f''(\xi)$
x_0	x_2	2	$\frac{1}{3} \cdot h \cdot [f(x_0) + 4f(x_1) + f(x_2)]$	$\frac{(b-a)}{2}$	$-\frac{1}{90} h^5 f^{(iv)}(\xi)$
x_0	x_3	3	$\frac{3}{8} \cdot h \cdot [f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)]$	$\frac{(b-a)}{3}$	$-\frac{3}{80} h^5 f^{(iv)}(\xi)$
x_0	x_4	4	$\frac{2}{45} \cdot h \cdot [7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)]$	$\frac{(b-a)}{4}$	$-\frac{8}{945} h^7 f^{(vi)}(\xi)$

$\frac{\Delta^n y_0}{h^n} = \frac{d^n y}{dx^n}$

Simpson's 3/8 Rule



$f(x) = f(x_0) + \frac{x - x_0}{h} \Delta f(x_0) + \frac{(x-x_0)(x-x_1)}{2h^2} \Delta^2 f(x_0) + \frac{(x-x_0)(x-x_1)(x-x_2)}{6h^2} \Delta^3 f(x_0)$

$I = \int_a^b f(x)dx = \int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right]$

$h = \frac{b-a}{3}$

$R(f,x) = \frac{-3h^5}{80} f^4(\xi)$

1. $\Delta^2 y_0 = \Delta(\Delta y_0) = \Delta(y_1 - y_0) = \Delta y_1 - \Delta y_0$
$= (y_2 - y_1) - (y_1 - y_0)$
$= y_2 - 2y_1 + y_0$

2. $\nabla^2 y_n = \nabla(\nabla y_n) = \nabla(y_n - y_{n-1})$
$= \nabla y_n - \nabla y_{n-1}$
$= (y_n - y_{n-1}) - (y_{n-1} - y_{n-2})$
$= y_n - 2y_{n-1} + y_{n-2}$

3. $E^2 y_0 = E(E y_0) = E y_1 = y_2$

4. $\delta^2 y_x = \delta\left[y(x+\frac{h}{2}) - y(x-\frac{h}{2})\right] = \delta y_{x+\frac{h}{2}} - \delta y_{x-\frac{h}{2}}$
$= (y_{x+h} - y_x) - (y_x - y_{x-h})$
$= y_{x+h} - 2y_x + y_{x-h}$

$\Delta y_0 = y_1 - y_0 \Rightarrow \Delta y_r = y_{r+1} - y_r$
$\nabla y_n = y_n - y_{n-1} \Rightarrow \nabla y_1 = y_1 - y_0$
$E y_0 = y_1 \quad E y_r = y_{r+1}$ etc
$\delta(f(x)) = f(x+\frac{h}{2}) - f(x-\frac{h}{2})$
$\delta = [E^{1/2} - E^{-1/2}]$
So, $\Delta f(x_r) = f(x_r + h) - f(x_r)$

Relation
$E = 1 + \Delta$
$\Delta = E - 1$
$E = (1-\nabla)^{-1}$
$\delta = E^{1/2} - E^{-1/2}$
$\Delta\nabla = \Delta - \nabla$
& $\Delta\nabla = \Delta - \nabla = \delta^2$

if we reach x from x_0 through p steps then we must reach y from y_0 through p steps

$$y = E^p y_0 = (1+\Delta)^p y_0 = \left[1 + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \frac{p(p-1)(p-2)}{3!}\Delta^3 \dots\right] y_0$$

Gauss Forward Formula

$$y = y_0 + G_1 \Delta y_0 + G_2 \Delta^2 y_{-1} + G_3 \Delta^3 y_{-1} + \dots$$

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_1 + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-1}$$

$$\Delta^2 y_{-1} = \Delta^2 E^{-1} y_0 = \Delta^2 (1+\Delta)^{-1} y_0$$
$$= \Delta^2 (1 - \Delta + \Delta^2 - \Delta^3 + \Delta^4 \dots) y_0$$
$$= \Delta^2 y_0 - \Delta^3 y_0 - \Delta^5 y_0 \dots$$

For N.F :
$y_p = y_0 + {}^pC_1 \Delta y_0 + {}^pC_2 \Delta^2 y_0 + {}^pC_3 \Delta^3 y_0 + \dots$

N.B. : $y_p = y_0 + {}^pC_1 \Delta y_{-1} + {}^{(p+1)}C_2 \Delta^2 y_{-2} + \dots$

G.F. $y_p = y_0 + {}^pC_1 \Delta y_0 + {}^pC_2 \Delta^2 y_{-1} + {}^{(p+1)}C_3 \Delta^3 y_{-1} + \dots$

G.B. $y_p = y_0 + {}^pC_1 \Delta y_{-1} + {}^{(p+1)}C_2 \Delta^2 y_{-1} + {}^{(p+1)}C_3 \Delta^3 y_{-2} + \dots$

Stirling: $y_p = y_0\left(\frac{1+1}{2}\right) + \frac{\Delta y_1 + \Delta y_0}{2}\, {}^pC_1 + \Delta^2 y_{-1}\left({}^{p+1}C_3 + {}^pC_2\right)$
$+ {}^{p+1}C_3\left(\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2}\right) + \dots$

Bessel $y_p = \frac{y_0 + y_1}{2} + \Delta y_0\left(\frac{{}^pC_1 + {}^{p+1}C_1}{2}\right) + {}^pC_2\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right)$
$+ \Delta^3 y_{-1}\left({}^{p+1}C_3 + {}^pC_3\right) + \dots$

Gaussian Quadrature.

$$\int_a^b f(x)\,dx = \sum_{k=1}^{n} c_k f(x_k)$$ exact for polynomial with degree $\leq 2n-1$

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \longrightarrow \text{sol}^n \text{ is } P_n(x)$$

$$y(x) = P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left((x^2-1)^n\right)$$

$$\int_{-1}^{1} P_m(x)\,P_n(x)\,dx = 0 \qquad n \neq m$$

$$= \frac{2}{2n+1} \qquad n = m$$

$$\int_{-1}^{1} P_n(x)\,x^m\,dx = 0 \qquad m < n$$

$$\int_{-1}^{1} f(x)\,dx = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$ open type formula

exact for poly with deg $\leq 2(2)-1$
$= 3$

$$\int_a^b f(x)\,dx \longrightarrow \int_{-1}^{1} f(x)\,dx \qquad X = \frac{b-a}{2}x + \frac{b+a}{2}$$

Gauss Legendre Formulas

1 point: $$\int_{-1}^{1} f(x)\,dx = 2f(0)$$

2 point $$\int_{-1}^{1} f(x)\,dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

3 point $$\int_{-1}^{1} f(x)\,dx = \frac{5}{9}f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9}f(0) + \frac{5}{9}f\left(\sqrt{\frac{3}{5}}\right)$$

2 point: $$\int_0^{\infty} e^{-x} f(x)\,dx = \frac{2+\sqrt{2}}{4}f(2-\sqrt{2}) + \frac{2-\sqrt{2}}{4}f(2+\sqrt{2})$$

Picards Method of Successive Approximation.

$$\frac{dy}{dx} = f(x, y) \qquad y(x_0) = y_0$$

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\, dx$$

$|y_{k+1}(x) - y_k(x)| \leq \varepsilon$ then we conclude that it converged.

Single Step Method

$$y' = \frac{dy}{dx} = f(x, y) \qquad y(x_0) = y_0$$

$y(x_{n+1}) = F(x_n, y_n, y_n', h)$ process depending only on one past value

Taylor Series Method

$$\frac{dy}{dx} = f(x, y) \quad \text{— (a)} \qquad y(x_0) = y_0 \qquad x \in [x_0, b]$$

assumptions (1) Eq^n (a) has unique sol^n on $[x_0, b]$

(2) $y(x)$ has continuous partial derivatives of order $p+1$ on $[x_0, b]$ $p \geq 1$

$$y(x) = y(x - x_0 + x_0)$$

$$= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2!} y''(x_0) \ldots$$

$$+ \underbrace{\frac{(x - x_0)^{p+1}}{(p+1)!} f^{p+1}(\xi_n)}_{R_n} \qquad x_0 < \xi < x$$

$$y(x_{n+1}) = y(x_n) + h y'(x_n) + \frac{h^2}{2!} y''(x_n) + \frac{h^3}{3!} y'''(x_n) + \ldots \frac{h^p}{p!} y^p(x_n) + R_n$$

$$x_{n+1} = h + x_n$$

$$y''(x_n) = \frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x} = \left[\frac{\partial f}{\partial x} + f \frac{\partial y}{\partial x}\right]$$

$$y''(x_n) = \frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x} = \left[\frac{\partial f}{\partial x} + f\frac{\partial y}{\partial x}\right]_{x_n, y_n}$$

$$R_n = \frac{h^{p+1}}{(p+1)!} y^{p+1}(\xi_n) = \frac{h^{p+1}}{(p+1)!} f^{p}(\xi_n, y(\xi_n)) = \varepsilon$$

R_n → Residual error

ε → Taylor Remainder

Euler Method

→ Taylor method too complicated, therefore this is used. Used for small h

$$y_{n+1} = y_n + h f(x_n, y_n) \qquad \text{Error} = \frac{h^2}{2!} f(x_0, y_0) \qquad O(h^2)$$

Modified Euler Method

→ average slope

$$y_{n+1} = y_n + h\left[\frac{f(x_n, y_n) + f(x_n + h, y_n + h f(x_n, y_n))}{2}\right]$$

↑ Predictor - Corrector Method

When h not given, $h = 0.1$

Runge-Kutta Method (RK Method)

1^{st} order RK

$$y_{n+1} = y_n + \underbrace{h f(x_n, y_n)}_{K_1} \qquad x_{n+1} = x_n + h$$

Error order $O(h^2)$ $\qquad y(x_{n+1}) = y_{n+1}$

2^{nd} order RK

$$y(x_{n+1}) = y(x_n) + h\left[\frac{f(x_n, y_n) + \overbrace{f(x_n + h, y_0 + \underbrace{h f(x_n, y_n)}_{K_1})}^{K_2/h}}{2}\right]$$

$$\therefore y(x_{n+1}) = y(x_n) + h\left[\frac{\frac{K_1}{h} + \frac{K_2}{h}}{2}\right]$$

$$K_2 = h f(x_n + h, y_0 + K_1)$$

$$y(x_{n+1}) = y(x_n) + \frac{1}{2}[K_1 + K_2]$$

order of Error $O(h^3)$

3^{rd} order RK Method

$$k_1 = h f(x_n, y_n)$$

$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = h f(x_n + h, y_0 - k_1 + 2k_2)$$

$$y(x_{n+1}) = y_n + \frac{1}{6}(k_1 + 4k_2 + k_3)$$

Error order: $O(h^4)$

4^{th} order RK Method

$$k_1 = h f(x_0, y_0)$$

$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = h f(x_0 + h, y_0 + k_3)$$

$$y(x_{n+1}) = y(x_n) + \frac{1}{6}\left[k_1 + 2(k_2 + k_3) + k_4\right]$$

order of error $= O(h^5)$

Multi Step Methods (Predictor-Corrector formulas)

$$\frac{dy}{dx} = f(x, y) \qquad y(x_0) = y_0$$

first we predict y_1 from a predictor formula $y_1^{(P)}$

then we correct the prediction with a corrector formula $y_1^{(C)}$

we can use corrector formula multiple times.

Euler Predictor Corrector Method

$$y_1^{(P)} = y_0 + h f(x_0, y_0)$$

$$y_1^{(C)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_0 + h, y_1^{(P)})\right]$$

$$y_1 = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_0+h, y_1) \right]$$

$$y_1^{(n+1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(n)}) \right]$$

Error $= O(h^3)$ Error $= -\frac{h^3}{12} f''(\xi)$

Milne's Predictor- Corrector Method

$\frac{dy}{dx} = f(x,y)$ $y(x_0) = y_0$

We need 4 pair values (x_i, y_i) $i = 0$ to 3 to calc y_4

predictor formula:

$$\int_{y_0}^{y_n} dy = \int_{x_0}^{x_n} f(x,y)\,dx \qquad f(x,y) = f_0 + n\Delta f_0 + \frac{n(n-1)}{2!}\Delta^2 f_0 \ldots$$

$$y_n - y_0 = \int_{x_0}^{x_n} \left(f_0 + n\Delta f_0 + \frac{n(n-1)}{2!}\Delta^2 f_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 f_0 + \ldots \right) dx$$

x_n $n = 4$ $x_4 = x_0 + 4h$ $dx = h\,dn$

$$y_4 - y_0 = h\int_0^4 (f_0 + \cdots)\,dn$$

predictor formula

$$\boxed{y_4 = y_0 + \frac{4h}{3}\left[2f_1 - f_2 + 2f_3 \right]}$$

$$\text{Error} = O(h^5) = \frac{14}{45} h^5 f^4(\xi)$$

corrector formula:

$$\int_{y_0}^{y_2} dy = \int_{x_0}^{x_2} f(x,y)\,dx = h\int_0^2 \left(f_0 + n\Delta f_0 + \frac{n(n-1)}{2!}\Delta^2 f_0 \ldots \right) dn$$

$$y_2 = y_0 + \frac{h}{3}\left[f_0 + 4f_1 + f_2 \right]$$

corrector

$$y_2 = y_0 + \frac{h}{3}\left[f_0 + 4f_1 + f_2\right]$$

$$\therefore \boxed{y_4 = y_2 + \frac{h}{3}\left[f_2 + 4f_3 + f_4\right]}$$

corrector formula

f_4 → from predictor formula

Error $= O(h^5)$ $\quad -\frac{h^5}{90} y_0^{(4)}(\xi)$

* Note: If $(x_0, y_0) \to (x_3, y_3)$ are not give, calc using Euler, modified Euler, RK method etc.

Simultaneous Diff Equation.

$$\frac{dy}{dx} = f_1(x, y, z) \qquad \frac{dz}{dx} = f_2(x, y, z)$$

$$y(x_0) = y_0 \qquad z(x_0) = z_0$$

$$y(x_0 + h) = ? \qquad z(x_0 + h) = ?$$

We want $y(x_0+h)$ and $z(x_0+h)$

$k_1 =$, k_2, k_3, k_4

$$\begin{cases} k_1 = hf_1(x_0, y_0, z_0) \\ m_1 = hf_2(x_0, y_0, z_0) \end{cases}$$

$$\begin{cases} k_2 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{m_1}{2}\right) \\ m_2 = hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{m_1}{2}\right) \end{cases}$$

$$k_3 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{m_2}{2}\right)$$

$$m_3 = hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{m_2}{2}\right)$$

$$k_4 = hf_1(x_0 + h, y_0 + k_3, z_0 + m_3)$$

$$m_4 = hf_2(x_0 + h, y_0 + k_3, z_0 + m_3)$$

$$y(x_0 + h) = y_0 + \frac{1}{6}\left[k_1 + 2(k_2 + k_3) + k_4\right]$$

$$z(x_0 + h) = z_0 + \frac{1}{6}\left[m_1 + 2(m_2 + m_3) + m_4\right]$$

Accuracy of the method $O(h^5)$

LINEAR BOUNDARY PROBLEMS (Forward Diff method)

Numerical Differentiation.

$$f'(x) = D^{+}f(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \qquad y_i'(x_i) = \frac{y_i(x_i+h) - y_i(x_i)}{h} + O(h)$$

↳ Error

$$f'(x) = D^{-}f(x) = \lim_{h\to 0} \frac{f(x) - f(x-h)}{h} \qquad y_i'(x_i) = \frac{y_i(x_i) - y_i(x_i-h)}{h} + O(h)$$

$$f'(x) = D^{0}f(x) = \lim_{h\to 0} \frac{f(x+h) - f(x-h)}{2h} \qquad y_i'(x_i) = \frac{y_i(x_i+h) - y_i(x_i-h)}{2h} + O(h)$$

Since error is of order $O(h)$, we say FDM is of order 1

Dirichlet Problem.

$$f_1(x)\, y'' + f_2(x)\, y' + f_3(x)\, y = \gamma(x)$$

$$y(x_0) = y_0 \qquad y(x_n) = y_n.$$

Divide $x_0 \to x_n$ in n subinterval

$$y(x_i+h) = y(x_i) + h\, y'(x_i) + \frac{h^2}{2} y''(\xi) \quad \text{— } (1)$$

$$y(x_i-h) = y(x_i) - h\, y'(x_i) + \frac{h^2}{2} y''(\xi') \quad \text{— } (2)$$

Sub (1) & (2) $\quad y'(x_i) = \dfrac{y_{i+1} - y_{i-1}}{2h} = \dfrac{y(x_i+h) - y(x_i-h)}{2h}$

Add (1) & (2) $\quad y''(x_i) = \dfrac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$

substitute in equation & find out values

Type 2 equation.

$$f_1(x)\, y'' + f_2(x)\, y' + f_3(x)\, y = \gamma(x) \qquad y(x_0) = y_0$$

$$y'(x_0) = y'_0$$

Let $z = y'(x)$

$$\therefore \quad f_1(x)\, z' + f_2(x)\, z + f_3(x)\, y = \gamma(x)$$

$$z' = \frac{\gamma(x) - f_2(x)\, z - f_3(x)\, y}{f_1(x)} \quad \text{— } 1$$

$$y' = z \quad \text{— } 2$$

Let
$$\left.\begin{aligned} z' &= f(x, y, z) \\ y' &= g(x, y, z) \end{aligned}\right\}$$
Simultaneous diff eq^n using RK method

$$y_{n+1} = \frac{1}{6}\left[k_1 + 2(k_2 + k_3) + k_4\right] + y_n$$

$$\left.\begin{aligned} k_1 &= h\, f(x, y, z) \qquad m_1 = h\, g(x, y, z) \\ k_2 &= f\left(x + \frac{h}{2},\ y + \frac{k_1}{2},\ z + \frac{m_1}{2}\right) \end{aligned}\right\} \text{RK method}$$

Tan-1(x) expansion

$$\left.\begin{aligned} y_1^{(n+1)} &= y_0 + \frac{h}{2}\left(f(x_0, y_0) + f(x_1, y_1^{(n)})\right) \\ y_1^{(0)} &= y_0 + h \cdot f(x_0, y_0) . \\ y_2^{(n+1)} &= y_1 + \frac{h}{2}\left(f(x_1, y_1) + f(x_2, y_2^{(n)})\right) \\ y_2^{(0)} &= y_1 + h\, f(x_1, y_1) \\ &\vdots \end{aligned}\right\} \text{Iterati proces}$$

Modified euler iteratice corrector formula

Milne Predictor corrector method

$$y_4^{P} = y_0 + \frac{4h}{3}\left[2 f(x_1, y_1) - f(x_2, y_2) + 2 f(x_3, y_3)\right] \quad \text{Error: } \frac{14}{45} h^5 y^{iv}(\xi)$$

$$y_4^{C} = y_2 + \frac{h}{3}\left[f(x_2, y_2) + 4 f(x_3, y_3) + f(x_4, y_4^{P})\right] \quad \text{Error: } \frac{-h^5}{90} y_0^{(4)}(\xi)$$

(15) . Adams Moulton:

$$y_4^{P} = y_3 + \frac{h}{24}\left[55 f(x_3, y_3) - 59 f(x_2, y_2) + 37 f(x_1, y_1) - 9 f(x_0, y_0)\right]$$

$$y_4^{C} = y_3 + \frac{h}{24}\left[9 f(x_4, y_4) + 19 f(x_3, y_3) - 5 f(x_2, y_2) + f(x_1, y_1)\right]$$

$y(0.\cdot)$ ... $\quad$ $z(0.\cdot) = z(0.3)$

11) $k_1 = hf(x_n + \alpha h, y_n + \beta k_1)$

$= h\left[f(x_n, y_n) + \alpha h \frac{\partial f}{\partial x} + \beta k_1 \frac{\partial f}{\partial y} + \cdots \right]$

$= hf_n + \alpha h^2 \frac{\partial f}{\partial x} + h\beta \frac{\partial f}{\partial y} k_1 + \cdots$

$\therefore k_1 = hf_n + \alpha h^2 \frac{\partial f}{\partial x} + h\beta \frac{\partial f}{\partial y}\left(hf_n + \alpha^2 h^2 \frac{\partial f}{\partial x} + h\beta \frac{\partial f}{\partial y} k_1 + \cdots \right) + \cdots$

$\therefore y_{n+1} = y_n + w_1 \left\{ hf_n + \alpha h^2 \frac{\partial f}{\partial x} + h^2 \beta f_n \frac{\partial f}{\partial y} + O(h^3) \right\}$

Again, $y(x_{n+1}) = y(x_n) + hf(x_n, y_n) + \frac{h^2}{2}\left(\frac{\partial f}{\partial x} + f \frac{\partial f}{\partial y} \right) + \cdots$ (Taylor series)

$\Rightarrow w_1 = 1, \quad \alpha w_1 = 1/2, \quad \beta w_1 = 1/2 \quad \Rightarrow \alpha = \beta = 1/2$

$\therefore y_{n+1} = y_n + hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right), \quad k_1 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$

9) $y' = f(x, y) = \lambda y, \quad k_1 = hf(x_n, y_n) = h\lambda y_n$

$k_2 = hf(x_n + \alpha h, y_n + \beta k_1) = h\lambda(y_n + \beta k_1) = h\lambda(y_n + \beta \cdot \lambda h y_n)$

$= h\lambda y_n(1 + \beta\lambda h)$

$\therefore y_{n+1} = y_n + \left(1 - \frac{1}{2\alpha}\right)\lambda h y_n + \frac{1}{2\beta}\lambda h(1 + \beta\lambda h) y_n$

$= \left[1 + \lambda h\left(1 - \frac{1}{2\alpha}\right) + \frac{\lambda h}{2\beta}(1 + \beta\lambda h) \right] y_n$

$= \left[1 + \lambda h\left(1 - \frac{1}{2\beta}\right) + \frac{\lambda h}{2\beta}(1 + \lambda h\beta) \right] y_n \quad$ [Choosing $\alpha = \beta$]

$= \left(1 + \lambda h + \frac{\lambda^2 h^2}{2}\right) y_n$, the sequence will converge

$\therefore E(\lambda h) = 1 + \lambda h + \frac{\lambda^2 h^2}{2}$, for stability $\left| 1 + \lambda h + \frac{\lambda^2 h^2}{2} \right| \leq 1$

$\Rightarrow \lambda h \in (-2, 0)$

if $h = \frac{1}{4}, \lambda = 3$, the method is not stable

$h = \frac{1}{2}, \lambda = -2$, the method is stable.

Best accuracy rk method 4th order

Q.12

Solve $y' = -2xy^2$, $y(0) = 1$ with $h = 0.3$ using 2nd order implicit Runge-Kutta method.

$y_{n+1} = y_n + k_1$

$k_1 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$

$f(x, y) = -2xy^2, \quad x_0 = 0, \quad y_0 = 1$

$k_1 = h\left[-2\left(x_n + \frac{h}{2}\right)\left(y_n + \frac{k_1}{2}\right)^2 \right]$

$= -h(2x_n + h)\left(y_n + \frac{k_1}{2}\right)^2$ which is an implicit equation for k_1 and one may use any iterative method.

define $F(k_1) = k_1 + h(2x_n + h)\left(y_n + \frac{k_1}{2}\right)^2$

$= k_1 + 0.3(2x_n + 0.3)\left(y_n + \frac{k_1}{2}\right)^2$

let us propose to use Newton-Raphson method

$$k_1^{(l+1)} = k_1^{(l)} - \frac{F(k_1^{(l)})}{F'(k_1^{(l)})}, \quad l = 0, 1, 2, \ldots$$

define $F(k_1) = k_1 + h(2x_n + h)\left(y_n + \frac{k_1}{2}\right)^2$

$= k_1 + 0.3(2x_n + 0.3)\left(y_n + \frac{k_1}{2}\right)^2$

let us propose to use Newton-Raphson method

$$\boxed{k_1^{(l+1)} = k_1^{(l)} - \frac{F(k_1^{(l)})}{F'(k_1^{(l)})}, \quad l = 0, 1, 2 \dots}$$

assume $k_1^{(0)} = h f(x_0, y_0) = -h 2 x_0^2 y_0^2$

$= -2(0.3)(0) = 0$

$$F(k_1) = k_1 + 0.3(2x_n + 0.3)\left(y_n + \frac{k_1}{2}\right)^2$$

$$F'(k_1) = 1 + 0.3(2x_n + 0.3)\, 2\left(y_n + \frac{k_1}{2}\right)\frac{1}{2}$$

$$= 1 + 0.3(2x_n + 0.3)\left(y_n + \frac{k_1}{2}\right)$$

$$F'(k_1^{(0)}) = 1 + 0.3(2(0) + 0.3)(1 + 0) = 1.09$$

$$F(k_1^{(0)}) = 0 + 0.3(2(0) + 0.3)(1 + 0)^2 = 0.09$$

$F(k_1^{(0)}) = 0.09$; $F'(k_1^{(0)}) = 1.09$

$$\therefore k_1^{(1)} = k_1^{(0)} - \frac{F(k_1^{(0)})}{F'(k_1^{(0)})} = 0 - \frac{0.09}{1.09}.$$

$$= -0.0825688$$

$F(k_1^{(1)}) = 0.00015151$

$F'(k_1^{(1)}) = 1.08628$ $\Rightarrow k_1^{(2)} = -0.0826994$

one proceeds until $|k_1^{(l+1)} - k_1^{(l)}| < \epsilon$ (preassigned)

$y(0.3) \approx y_1 = 1 + (-0.08269) = \cancel{0.8}\ 0.9173006.$

Eulers method: $y_{n+1} = y_n + h f(x_n, y_n)$, $\quad y' = f(x, y)$

Total solution error: $e_{n+1} = e_n\left[1 + h f_y(x_n, \xi_n)\right] + L_{n+1}$, $\quad e_0 = 0$

$$L_{n+1} = -\frac{h^2}{2} y''(\tau_n)$$