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## Sequence

mapping from $\mathbb{Z}^+$ to $\mathbb{C}$ $\{f(n)\}$

## Convergence

$\lim_{n\to\infty} z_n \to c$ $c \in \mathbb{C}$ and finite and unique

$|z_n - c| < \varepsilon$ $\forall\ n \geq N$ for every $\varepsilon > 0$

$z_n = x_n + i y_n$ Converges to $c = a + ib$ iff

$\{x_n\} \to a$ $\{y_n\} \to b$

## Series

$S = \sum_{n=1}^{\infty} z_n$ $S_n = \sum_{n=1}^{n} z_n$

if $\lim_{n\to\infty} S_n \to c$ (const) then series is convergent

ie $\{S_n\}$ sequence is convergent

if series is convergent then $\lim_{n\to\infty} z_n \to 0$

## Necessary Cond$^n$ (but not sufficient)

$\lim_{n\to\infty} z_n \to 0$ is necessary

$\sum_{1}^{\infty} \frac{1}{n}$ is divergent but $\lim_{n\to\infty} \frac{1}{n} \to 0$ is still true.

## Absolute Convergence

if $\sum_{n=1}^{\infty} |z_n|$ is convergent then $\sum_{n=1}^{\infty} z_n$ is called absolute convergent

* If series is absolutely convergent, then the series converges surely,

* if series converges but not absolutely, then it is called conditionally convergent. eg $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$

$z_n = x_n + i y_n$ $\sum z_n$ converges iff both $\sum x_n$ & $\sum y_n$ converges

# Series Test

1) Ratio test, Root test

$$\lim_{n \to \infty} \left| \frac{z_{n+1}}{z_n} \right| = L$$

$L > 1$ diverges
$L < 1$ converges
$L = 1$ cannot say

$$\lim_{n \to \infty} (|z_n|)^{\frac{1}{n}} = L$$

2) Comparison Test

$\sum b_n$ = converging series $b_n \geq 0$

then a series $\sum z_n$ with $|z_n| \leq b_n$ also converges

# Power Series

$$S = \sum_{n=1}^{\infty} a_n (z - z_0)^n \quad \text{around } z_0$$

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$$ for $|z| < 1$ converges absolutely

# Radius of Convergence.

$|z - z_0| < R$

$\sum_{n=1}^{\infty} \frac{z^n}{n}$ converges at $z = -1$ by Leibniz test but diverges elsewhere.

$$\sum_{n=1}^{\infty} (-1)^n a_n \text{ converges iff}$$

$|a_n|$ is decreasing

$\lim_{n \to \infty} |a_n| \to 0$

* Always check at boundary values ie $L = 1$

Differentiating or integrating a power series will give the same R.O.C = R

first check if series converges, then check for ROC

* $f(z) = \sum a_n (z - z_0)^n$ $\quad$ $g(z) = \sum b_n (z - z_0)^n$

ROC = $r_1$ $\quad$ ROC = $r_2$

$f(z) \pm g(z)$ may have ROC $r \geq \min\{r_1, r_2\}$

eg: $\sum (1+2^{-n}) z_n$ ROC $= 1$
$\sum -z_n^n$ ROC $= 1$ $\Big\} \to + \to \sum 2^{-n} z_n$ ROC $= 2$

$S(z) = f(z) g(z)$ absolutely converges for each $z$ belonging to the ROC of either series.

$$c_n = a_n b_0 + a_{n-1} b_1 \cdots\cdots a_0 b_n$$

## Taylor Series

A func$^n$ $f(z)$ analytic on $D = \{ z : |z - z_0| < R \}$ can be represented as power series

$$f(z) = \sum_{n=1}^{\infty} a_n (z - z_0)^n$$

$$a_n = \frac{f^n(z_0)}{n!} = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

C = positively oriented
closed simple contour
enclosing $z_0$
completly lying inside of D

## EXAMPLES

1) $\dfrac{1}{1-z} = \sum z^n \quad |z| < 1$ diff wrt $z$

$$\frac{1}{(1-z)^2} = \sum n z^{n-1} \quad \forall \quad |z| < 1$$

2) $\dfrac{1}{1+z} = \dfrac{1}{1-(-z)} = \displaystyle\sum_{n=1}^{\infty} (-z)^n \quad |-z| < 1$

3) $\dfrac{1}{z} = \dfrac{1}{1-(1-z)} = \sum (1-z)^n \quad |1-z| < 1$

IMPORTANT

if $f(z)$ is not analytic at $z = z_0$ in $D = \{z : |z - z_0| < R\}$

then series representation of $f(z) = \sum a_n (z-z_0)^n$ is not

valid as $\sum a_n (z-z_0)^n$ is analytic at $z = 0$

then we write $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$

## Laurett Series

$f(z)$ analytic on $D = \{ z : r_1 < |z - z_0| < r_2 \}$ then

$$f(z) = \sum_{0}^{\infty} a_n (z-z_0)^n + \underbrace{\sum_{1}^{\infty} b_n (z-z_0)^{-n}}_{\text{principal part}}$$

or $f(z) = \sum_{-\infty}^{\infty} a_n (z-z_0)^n \quad n = 0, \pm 1, \pm 2 \ldots$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz \qquad b_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{-n+1}} dz$$

$C$ = positively oriented (anticlockwise)
simple closed contour
enclosed in $D$

## Uniqueness

Uniqueness:- The Laurent series of a given analytic function $f(z)$ in its annulus of convergence is unique.
However, $f(z)$ may have different Laurent series in two annuli with the same center.

But it is hard computing through integral formula. Hence we compute it using other predefined series.

## Cauchy Gorsat

$$\oint_C f(z)\,dz = 0$$

$f(z)$ is analytic on and inside $C$

## Methods.

1) Maclaurin Series.

$$\sin z = \sum^{\infty} (-1)^n \frac{z^{2n+1}}{\ldots} \qquad \forall\, z \in \mathbb{C}$$

$$\sin z = \sum_{0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \quad \forall z \in \mathbb{C}$$

$$\therefore z^{-5} \sin z = \sum_{0}^{\infty} \frac{(-1)^n z^{2n+1-5}}{(2n+1)!} \quad \forall |z| > 0$$

## 2) Substitution

$$e^z = \sum \frac{z^n}{n!} \quad \forall z \in \mathbb{C}$$

$$z^2 e^{\frac{1}{z}} = \sum z^2 \times \frac{1}{n!} \left(\frac{1}{z}\right)^n \quad \forall |z| > 0$$

## 3) Manipulation

$$f(z) = \frac{1}{1-z}$$

$$\frac{1}{1-z} = \sum_{0}^{\infty} z^n \quad |z| < 1$$

$$\frac{1}{1-z} = \frac{-1}{z\left(1 - \frac{1}{z}\right)} = \frac{-1}{z} \frac{1}{\left(1 - \frac{1}{z}\right)} = \frac{-1}{z} \sum_{0}^{\infty} \left(\frac{1}{z}\right)^n \quad \forall \left|\frac{1}{z}\right| < 1$$

## 4) Partial Fractions

$$f(z) = \frac{-2z+3}{z^2 - 3z + 2} = -\left(\frac{1}{z-1} + \frac{1}{z-2}\right)$$

$$\frac{-1}{1-z} = \begin{cases} \sum z^n & |z| < 1 \\ \sum \frac{1}{z^{n+1}} & |z| > 1 \end{cases}$$

$$\frac{-1}{2-z} = \begin{cases} \frac{1}{2} \sum \left(\frac{z}{2}\right)^n & |z| < 2 \\ \frac{-1}{z} \sum \left(\frac{2}{z}\right)^n & |z| > 2 \end{cases}$$

$$\frac{-1}{2-z} = \begin{cases} \frac{1}{2} \sum \left(\frac{z}{2}\right)^n & |z| < 2 \\ -\frac{1}{z} \sum \left(\frac{2}{z}\right)^n & |z| > 2 \end{cases}$$

Now make cases: $|z| < 1$

$1 < |z| < 2$

$|z| > 2$

# Zeros And Singularities

Regular / ordinary pt $\Rightarrow$ $z = z_0$ $f(z_0)$ exists and $f(z)$ is analytic at $z = z_0$

Zero pt $\Rightarrow$ $z = z_0$ $f(z_0) = 0$ Taylor Series

singular pt $\Rightarrow$ $z = z_0$ $f(z_0)$ ceases to exist Laurett series

ie $f(z)$ not analytic or not defined at $z = z_0$

isolated singularity $\Rightarrow$ $z_0$ is a singular pt, but in its neighbourhood there is no other singuler point.

Laurett Series: (for isolated pt) $z_0$ = isolated pt

then for $0 < |z - z_0| < r$

$$f(z) = \sum_0^{\infty} a_n (z - z_0)^n + \sum_1^{\infty} b_n (z - z_0)^{-n}$$

Case 1 : principal part has infinite terms

$z_0$ is called essential isolated singularity of $f(z)$

Case 2 : principal part does not exist ie $b_i = 0$

$z_0$ is called removable singularity of $f(z)$

$z_0$ = non isolated singularity

case 3: principal part has finite terms

$$\sum' b_n (z-z_0)^{-n} = \frac{b_1}{z-z_0} + \cdots\cdots \frac{b_m}{(z-z_0)^m}$$

$z = z_0$ called pole with order $m$

pole = isolated singularity

## Theorem (Pole)

$f(z)$ is analytic and has a pole at $z = z_0$ then

$$\lim_{z \to z_0} |f(z)| \to \infty$$

## Picard's Theorem (isolated essential singularity)

$f(z)$ is analytic $z_0$ = isolated essential singularity

$\lim_{z \to z_0} |f(z)|$ takes every value in $\mathbb{C}$ with at most one exceptional value.

# Zeros

$f(z_0) = 0$ and $z_0 \in D$

$z_0$ = zero of order $n$ if $f^n(z_0) = 0 \quad \forall n = 0, 1, \cdots n$

if only $f(z_0) = 0$ then simple zero

## Taylor Series (At $z = z_0$ zero)

if $z_0$ is n-th order zero, then the coeff of taylor series of $f(z)$ around $z_0$ will have coeff = 0 for

$n = 0, 1, \ldots n-1$ ie

$$f(z) = 0 (z-z_0)^0 + 0 (z-z_0)^1 + \cdots + a_n (z-z_0)^n + \cdots$$

ie $f(z) = (z-z_0)^n \left[ a_n + a_{n+1} (z-z_0) + \cdots \right]$

$$f(z) = \varphi(z) \quad (z-z_0)^n \qquad |z-z_0| < r$$

$$\underbrace{f(z)}_{\text{func}^n \text{ with } n\text{-th order zero at } z_0} = \underbrace{\varphi(z)}_{\text{analytic at } z_0 \text{ and } \varphi(z_0) \neq 0} (z - z_0)^n \qquad |z - z_0| < r$$

## Theorem:

Each zero of an analytic func$^n$ $(f(z_0) = 0)$ is an isolated point ie in some neighbourhood of $z_0$, there is no such $z$ such that $f(z) = 0$ $\forall$ $|z - z_0| < \varepsilon$ for some $\varepsilon \in \mathbb{R}$

## Poles and Zeros theorem.

$f(z)$ has a zero of $n$-th order at $z = z_0$

$\frac{1}{f(z)}$ has a pole of $n$-th order at $z = z_0$

## METHOD

Want to investigate $f(z)$ at $|z| \to \infty$

take $w = \frac{1}{z}$ $\quad f(z) = f\left(\frac{1}{w}\right) = g(w) \quad w \to 0$ as $z \to \infty$

we define $g(0) = \lim_{w \to 0} g(w)$ if it exists.

if $g(w)$ analytic at $w = 0$ then $f(z)$ analytic at $|z| \to \infty$

if $g(w)$ singular at $w = 0$ then $f(z)$ singular at $|z| \to \infty$

## Pole Shortcut:

if $f(z) = \dfrac{\varphi(z)}{(z - z_0)^n}$ $\quad \varphi(z_0) \neq 0$ & analytic in $0 < |z - z_0| < r$

then $z = z_0$ is a pole of order $n$

## Zero Shortcut

$f(z) = \varphi(z)(z - z_0)^n$ $\quad \varphi(z)$ analytic, $\varphi(z_0) \neq 0$

then $z = z_0$ zero of order $n$

<u>Theorem</u> (Finding the order of a pole): Let $z_0$ be a pole of order $m$. Then for all positive integers $k$, we have

$$\lim_{z \to z_0} (z - z_0)^k f(z) = \begin{cases} \ell, & k = m \\ 0, & k > m \\ \infty, & k < m \end{cases}$$

then $z = z_0$ zero of order $n$ $\quad \varphi(z_0) \neq 0$, $\; |z - z_0| < r$

Theorem ... for all positive integers $k$, we have order $m$. Then

$$\lim_{z \to z_0} (z - z_0)^k f(z) = \begin{cases} \ell, & k = m \\ 0, & k > m \\ \infty, & k < m \end{cases}$$

Leading order ⑦

# Residue.

$$\oint_C f(z)\,dz = 0 \quad \text{if } f(z) \text{ analytic inside } C$$

But if $z_0 \in C$ is a singular point for $f(z)$ then we apply Residue theorem

$$f(z) = \sum a_n (z - z_0)^n + \sum \frac{b_n}{(z - z_0)^n} \quad \forall \; 0 < |z - z_0| < r$$

Coeff of $\frac{1}{z - z_0}$ ie $b_1$ is called residue of isolated singularity $z_0$

$$b_1 = \text{Res}(f : z_0) = \underset{z = z_0}{\text{Res}} = \frac{1}{2\pi i} \oint f(z)\,dz$$

## Residue of simple Pole

$f(z)$ analytic in $0 < |z - z_0| < r$ and $z = z_0$ is simple pole

$$\text{Res}(f : z_0) = \lim_{z \to z_0} (z - z_0) f(z) = b_1$$

Let $f(z) = \dfrac{g(z)}{z - z_0}$ $\quad g(z_0) \neq 0$, $g(z)$ analytic $0 < |z - z_0| < r$

$$\text{Res}(f : z_0) = \lim_{z \to z_0} g(z) = g(z_0)$$

* $f(z) = \dfrac{p(z)}{q(z)}$ $\quad p, q$ analytic at $z_0$ but $p(z_0) \neq 0$ $\quad q(z_0) = 0$ and $q'(z_0) \neq 0$

$$\text{Res}(f : z_0) = \frac{p(z_0)}{q'(z_0)}$$

# Residue of a pole of order n

$$\text{Res}(f : z_0) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left[ (z - z_0)^m f(z) \right]$$

# Cauchy Residue Theorem

$\oint_C f(z)\,dz$ $\quad$ $f(z)$ has isolated singularities at $z_1\ z_2 \cdots z_n$ inside $C$

$$\oint_C f(z)\,dz = 2\pi i \left[ \sum_{i=1}^{n} \text{Res}(f : z_i) \right]$$

# Evaluation of Real Integrals

$$I = \int_0^{2\pi} f(\cos\theta, \sin\theta)\,d\theta$$

$f(x,y)$ defined inside $|z| < 1$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

use $z = e^{i\theta}$

$dz = i e^{i\theta} d\theta$

$$I = \oint_C \frac{1}{iz} f\left( \frac{z + \frac{1}{z}}{2}, \frac{z - \frac{1}{z}}{2i} \right) dz$$

$$= 2\pi i \left[ \text{sum of residues of } \frac{1}{iz} f\left( \frac{z + \frac{1}{z}}{2}, \frac{z - \frac{1}{z}}{2i} \right) \right]$$

inside $|z| < 1$

# Improper Integrals

$$I = \int_{-\infty}^{\infty} f(x)\,dx$$

$$I = \lim_{R \to \infty} \int_{-R}^{R} f(z)\,dz = \underbrace{\int_{-R}^{0} f(z)\,dz + \int_{0}^{R} f(z)\,dz}_{\text{if these limits exists.}}$$

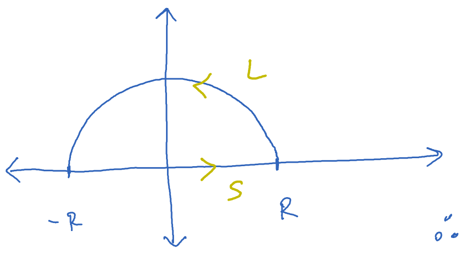
Cauchy principal value

$\int^{\infty} x\,dx = 0$ but $\int^{\infty}$ ...

principal value

$\int_{-\infty}^{\infty} x\,dx = 0$ but $\int_{0}^{\infty} x\,dx = \infty$ if these limits exists.

Assumption: $f(z)$ is a real function and denominator of $f(z)$ is $\neq 0$ $\forall\, x \in \mathbb{R}$ and denominator is atleast 2 degrees higher than degree of numerator.



$C = L + S$

$$\oint_C f(z)\,dz = \int_L f(z)\,dz + \int_S f(z)\,dz$$

$$= 2\pi i \sum \text{Res}(f : z_i)$$

$$\therefore \int_{-R}^{R} f(z)\,dz = 2\pi i \sum \text{Res}(f : z_i) - \int_L f(z)\,dz$$

assume $z = Re^{i\theta}$ $\quad \theta : 0 \to \pi \quad R \to \infty$

$|f(z)| \leq \dfrac{K}{|z|^2}$ for some $K > 0$, $|z| = R > R_0$

$$\therefore \int_S f(z)\,dz < \frac{K}{R^2}\pi R \quad \text{ie} < \frac{K\pi}{R}$$

for $R \to \infty$ $\quad \int_L f(z)\,dz \to 0$

Hence $\displaystyle\int_{-\infty}^{\infty} f(z)\,dz = 2\pi i \left(\text{Res}(f : z_i)\right)$ $\quad \text{Re}(z_i) \geq 0$

Conditions to check:

→ Denominator of $f(z)$ has a degree 2 more than num$^r$

→ Denominator of $f(z) \neq 0$ for all real $z$

Residue to Calculate:

→ Not all residue need to be calculated.

Only Residues from I and II quad are to be calculated ie $\text{Re}(z_i) \geq 0$

# Integrals with Trig Func$^n$s

$$I = \int_{-\infty}^{\infty} f(z)\cos(sz)\,dz \quad \text{or} \quad \int_{-\infty}^{\infty} f(x)\sin(sx)\,dx$$

Let $z = \alpha_1, \alpha_2 \ldots \alpha_n$ be the poles in the upper part of the graph of $f(z)$

$$\int_{-\infty}^{\infty} f(z)\, e^{isz}\,dz = 2\pi i\left[\sum \operatorname{Res}(f : z_i)\right]$$

$$\int_{-\infty}^{\infty} f(z)\cos(sz)\,dz + i\int_{-\infty}^{\infty} f(z)\sin(sz)\,dz$$

$$= 2\pi i\left(\sum \operatorname{Res}(f : z_i)\right)$$

$$\therefore \quad \int_{-\infty}^{\infty} f(z)\cos(sz)\,dz = \operatorname{Re}\left[2\pi i \sum \operatorname{Res}(f : z_i)\right]$$

$$i\int_{-\infty}^{\infty} f(z)\sin(sz)\,dz = \operatorname{Im}\left[2\pi i \sum \operatorname{Res}(f : z_i)\right]$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

LAURETT SERIES ONLY EXISTS FOR ISOLATED SINGULARITY POINTS NOT FOR NON-ISLOATED SINGULARITITES

8) $f(z) = z^3 - 3z^2 + 3z - 1 + \frac{1}{z-2}$ around $z = 2$

$$= (z-1)^3 + \frac{1}{z-2}$$

$$= (z-2+1)^3 + \frac{1}{z-2}$$

$$= (z-2)^3 + 3(z-2)^2 + 3(z-2) + 1 + \frac{1}{z-2}$$