Basics of General Theory of Relativity for Beginners

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Abstract

We present a basics of the Einstein General Theory of Relativity. In the first part of this review we derive relations of Riemann geometry which are used in the General Relativity. In the second part we discuss Einstein Equations and some of its consequences (The Schwarzschild solution, gravitational waves, Friedman Equations etc). In the Appendix we briefly discuss a history of the discovery of the Einstein Equations.

1   Introduction

General Theory of Relativity is a great theory, confirmed by all existing data (see, for example, "Experimental Tests of Gravitational Theory" by T. Damour in PDG [1]) It is based on the requirement of invariance under the general transformations of coordinates in a curved Riemann space. The General Theory of Relativity, apparently, support a suggestion[1] that in a correct theory the simplest possibilities are realized. Recent discovery of the predicted by GTO gravitational waves opened a new and very powerful way of the investigation of the Universe. In the book "Classical Theory of Fields" L.D. Landau and E.M. Lifshitz wrote : "The General Theory of Relativity which was created by Einstein (and finally formulated by him in 1916) is, apparently, the most beautiful of all existing physical theories. It is remarkable that it was built by Einstein in a purely deductive way and only later was confirmed by astronomical observations" In this review I tried to present the basics of the General Theory of Relativity and some of its consequences in such a way that all derivations can be easily followed by a reader.

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[1]"Simplicity is a guide to the theory choice" A. Einstein.

2   Curved Riemann space. Vectors, Tensors

The General Theory of Relativity (GTR) is based on the requirement of the invariance of basic equations under transformations of the coordinates in a general coordinate system[2] In order to formulate the equations of the General Theory of Relativity A. Einstein used mathematical methods of the non Euclidian geometry which we briefly consider now.

Let x^α ($\alpha = 0, 1, 2, 3$) be coordinates of a point in a general coordinate system. The square of the length between infinitesimally close points $x^\alpha + dx^\alpha$ and x^α (interval) has the form

$$ds^2 = g_{\alpha\beta}(x) \ dx^\alpha dx^\beta. \tag{1}$$

It is obvious that $g_{\alpha\beta}(x) = g_{\beta\alpha}(x)$. $g_{\alpha\beta}(x)$ is called the metric tensor.

Let x' and x be coordinates of a point in different reference systems. The coordinates x' are functions of x (and vice versa). Thus, we have

$$dx'^\alpha = \frac{\partial x'^\alpha}{\partial x^\rho} dx^\rho. \tag{2}$$

If we multiply (2) by $\frac{\partial x^\sigma}{\partial x'^\alpha}$ and sum over α we find the inverse transformation

$$dx^\sigma = \frac{\partial x^\sigma}{\partial x'^\alpha} dx'^\alpha. \tag{3}$$

Contravariant vector $A^\alpha(x)$ transforms as the differential of coordinates:

$$A'^\alpha(x') = \frac{\partial x'^\alpha}{\partial x^\rho} A^\rho(x). \tag{4}$$

Interval ds^2 is an invariant. From (1) and (2) we find

$$ds^2 = g_{\alpha\beta}(x) \ dx^\alpha dx^\beta = g_{\alpha\beta}(x) \ \frac{\partial x^\alpha}{\partial x'^\rho} \frac{\partial x^\beta}{\partial x'^\sigma} \ dx'^\rho dx'^\sigma = g'_{\rho\sigma}(x') \ dx'^\rho dx'^\sigma. \tag{5}$$

Thus, the metric tensor $g_{\alpha\beta}$ is transformed as follows

$$g'_{\rho\sigma}(x') = \frac{\partial x^\alpha}{\partial x'^\rho} \frac{\partial x^\beta}{\partial x'^\sigma} \ g_{\alpha\beta}(x). \tag{6}$$

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[2]For textbooks see, for example, [2, 3, 4].

From (6) we find the inverse transformation

$$g_{\alpha\beta}(x) = \frac{\partial x'^\rho}{\partial x^\alpha} \frac{\partial x'^\sigma}{\partial x^\alpha} \, g'_{\rho\sigma}(x') \tag{7}$$

The scalar product of two vectors A^α and B^α is determined as follows

$$A(x) \cdot B(x) = g_{\alpha\beta}(x) A^\alpha(x) B^\beta(x). \tag{8}$$

The scalar product is an invariant. In fact, taking into account (6) and (4), we have

$$A(x) \cdot B(x) = g_{\alpha\beta}(x) \frac{\partial x^\alpha}{\partial x'^\rho} \frac{\partial x^\beta}{\partial x'^\sigma} A'^\rho(x) B'^\sigma(x) = g'_{\rho\sigma}(x') A'^\rho(x') B'^\sigma(x') = A'(x') \cdot B'(x'). \tag{9}$$

Covariant vector A_α is determined by the relation

$$A_\alpha = g_{\alpha\beta} \, A^\beta. \tag{10}$$

Thus, the scalar product has the form

$$A \cdot B = A_\alpha B^\alpha. \tag{11}$$

Further, we have

$$A_\alpha(x) B^\alpha(x) = A_\alpha(x) \frac{\partial x^\alpha}{\partial x'^\rho} B'^\rho(x') = A'_\rho(x') B'^\rho(x'). \tag{12}$$

From this relation we find that a covariant vector is transformed in the following way

$$A'_\rho(x') = \frac{\partial x^\alpha}{\partial x'^\rho} A_\alpha(x). \tag{13}$$

From (13) we find the inverse transformation

$$A_\beta(x) = \frac{\partial x'^\rho}{\partial x^\beta} A'_\rho(x'). \tag{14}$$

Tensors are transformed as product of vectors. For example, the second rank tensors $T^{\rho\sigma}$ and T^ρ_σ are transformed , respectively, as product of two contravariant vectors and product of contravariant and covariant vectors:

$$T'^{\alpha\beta}(x') = \frac{\partial x'^\alpha}{\partial x^\rho} \frac{\partial x'^\beta}{\partial x^\sigma} T^{\rho\sigma}(x), \quad T'^\alpha_\beta(x') = \frac{\partial x'^\alpha}{\partial x^\rho} \frac{\partial x^\sigma}{\partial x'^\beta} T^\rho_\sigma(x). \tag{15}$$

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The tensor of the third rank $T_\nu^{\rho\sigma}$ is transformed as the product of two contravariant vectors and covariant vector:

$$T_\mu'^{\alpha\beta}(x') = \frac{\partial x'^\alpha}{\partial x^\rho} \frac{\partial x'^\beta}{\partial x^\sigma} \frac{\partial x^\nu}{\partial x'^\mu} T_\nu^{\rho\sigma}(x), \tag{16}$$

etc. From (6) it follows that $g_{\alpha\beta}$ is the tensor of the second rank.

Let us put in (16) $\beta = \mu$ and sum up over μ. We have

$$T_\mu'^{\alpha\mu}(x') = \frac{\partial x'^\alpha}{\partial x^\rho} \delta_\sigma^\nu T_\nu^{\rho\sigma}(x) = \frac{\partial x'^\alpha}{\partial x^\rho} T_\sigma^{\rho\sigma}(x). \tag{17}$$

Thus, $T_\sigma^{\rho\sigma}$ is a vector.

The metric tensor $g^{\rho\alpha} = g^{\alpha\rho}$ is determined as follows

$$g^{\rho\alpha} g_{\alpha\beta} = \delta_\beta^\rho. \tag{18}$$

From (10) and (18) we find

$$A^\rho = g^{\rho\alpha} A_\alpha. \tag{19}$$

Thus, with the help of the tensor $g_{\rho\alpha}$ ($g^{\rho\alpha}$) we can lower (raise) indexes. For the scalar product we have

$$A \cdot B = g_{\alpha\beta} g^{\alpha\rho} A_\rho g^{\beta\sigma} B_\sigma = g^{\rho\sigma} A_\rho B_\sigma. \tag{20}$$

In the flat Minkowski space the derivative of a vector is a tensor. Let us consider the derivative of a vector A^α. From (4) we find

$$\frac{\partial A'^\alpha(x')}{\partial x'^\beta} = \frac{\partial x'^\alpha}{\partial x^\rho} \frac{\partial x^\sigma}{\partial x'^\beta} \frac{\partial A^\rho(x)}{\partial x^\sigma} + \frac{\partial^2 x'^\alpha}{\partial x^\rho \partial x^\sigma} \frac{\partial x^\sigma}{\partial x'^\beta} A^\rho(x) \tag{21}$$

Thus, in a curved Riemann space the derivative of the vector is not a tensor. This is connected with the fact that at different points x and $x + dx$ vector A^α is transformed differently.

In the next section we will introduce a notion of *a covariant derivative.* The covariant derivative will be introduced in such a way that the derivative of a vector is a tensor, derivative of a tensor is a tensor of higher rank etc. Covariant derivatives allow to formulate invariant under general transformations of coordinates basic equations of the General Theory of Relativity.

3   Parallel Displacement of Vectors. Christoffel Symbols. Covariant Derivative

In order to introduce the covariant derivative we need first to define the operation of parallel displacement of a vector in a curved space from a point x to the point $x + dx$. For that let us imbed a four-dimensional curved space in a flat space of a dimension $N > 4$. Coordinates of a point in this space will be denoted z^n. The square of distance between points $z^n + dz^n$ and z^n is given by

$$ds^2 = h_{nm}dz^n dz^m, \tag{22}$$

where the metric tensor h_{nm} is a constant. We can write down ds^2 also in the form

$$ds^2 = dz^n dz_n, \tag{23}$$

where $dz_n = h_{nm}dz^m$.

Physical space is a "surface" in the N-dimensional space. To each point x of the surface corresponds a point $y^n(x)$ in N-dimensional flat space. We have

$$\delta y^n = y^n_{,\alpha}\, \delta x^\alpha, \tag{24}$$

where the following notation

$$\frac{\partial y^n}{\partial x^\alpha} = y^n_{,\alpha} \tag{25}$$

is used. The square of the distance between points $x + dx$ and x on the surface is given by

$$\delta s^2 = h_{nm}\delta y^n \delta y^m = h_{nm}y^n_{,\alpha}y^m_{,\beta}\delta x^\alpha \delta x^\beta = y^n_{,\alpha}y_{n,\beta}\delta x^\alpha \delta x^\beta. \tag{26}$$

Notice, that we take into account in (26) that h_{nm} is a constant. The equation (26) has the standard form

$$\delta s^2 = g_{\alpha\beta}\delta x^\alpha \delta x^\beta, \tag{27}$$

where the metric tensor $g_{\alpha\beta}$ is given by

$$g_{\alpha\beta} = y^n_{,\alpha}y_{n,\beta}. \tag{28}$$

Let us consider a vector $A^\alpha(x)$ in the four-dimensional space. From (24) follows that the corresponding vector in the N-dimensional space is given by the relation

$$A^n(x) = y^n_{,\alpha}(x)A^\alpha(x). \tag{29}$$

Let us perform the parallel displacement of the vector A^n from the point $y^n(x)$ to the point $y^n(x + dx)$. The transferred vector in general will not lie down on the surface. We can present it in the form

$$A^n = A^n_{\tan} + A^n_{\text{nor}}, \tag{30}$$

where $A^n_{\tan}$ is the projection of A^n on the surface and A^n_{nor} is orthogonal to $A^n_{\tan}$ vector:

$$A^n_{\tan}(A_n)_{\text{nor}} = 0. \tag{31}$$

If a vector $K^\alpha(x + dx)$ on the surface corresponds to the vector $A^n_{\tan}$ we have

$$A^n_{\tan} = y^n_{,\alpha}(x + dx)K^\alpha(x + dx). \tag{32}$$

Let us now multiply (32) by $y_{n,\beta}(x + dx)$. Using (28), we find

$$g_{\alpha\beta}(x + dx)K^\alpha(x + dx) = K_\beta(x + dx) = A^n_{\tan}y_{n,\beta}(x + dx). \tag{33}$$

From (31) and (32) we find that

$$A^n_{\text{nor}}y_{n,\beta}(x + dx) = 0. \tag{34}$$

Taking into account this relation, from (33) we have[3]

$$
\begin{aligned}
K_\beta(x + dx) &= A^n y_{n,\beta}(x + dx) = A^\alpha(x)y^n_{,\alpha}(x)(y_{n,\beta}(x) + y_{n,\beta\sigma}(x)dx^\sigma) \\
&= A_\beta(x) + \delta A_\beta(x)
\end{aligned} \tag{35}
$$

Here

$$\delta A_\beta = A^\alpha(x)y^n_{,\alpha}(x)y_{n,\beta\sigma}(x)dx^\sigma. \tag{36}$$

Thus, δA_β is a change of the covariant vector A_β under the parallel displacement.

The term $y^n_\alpha y_{n,\beta\sigma}$ can be expressed through derivatives of the metric tensor $g_{\alpha\beta}$. In fact, from (28) we have

$$g_{\alpha\beta,\sigma} = y^n_{,\alpha\sigma}y_{n,\beta} + y^n_{,\alpha}y_{n,\beta\sigma}. \tag{37}$$

Let us change $\beta \leftrightarrows \sigma$ in (37). We find the relation

$$g_{\alpha\sigma,\beta} = y^n_{,\alpha\beta}y_{n,\sigma} + y^n_{,\alpha}y_{n,\sigma\beta}. \tag{38}$$

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[3]Notice that we took into account that components of a vector in the flat N-dimensional space are not changed under the parallel displacement.

It is obvious that $y_{n,\beta\sigma} = y_{n,\sigma\beta}$. Thus, the last two terms in (37) and (38) are equal. Let us now change in (37) $\alpha \leftrightarrows \sigma$. We find the relation

$$g_{\sigma\beta,\alpha} = y^n_{,\sigma\alpha} y_{n,\beta} + y^n_{,\sigma} y_{n,\beta\alpha} \tag{39}$$

It is obvious that this term is equal to the sum of first terms of (37) and (38). Thus, we obtain the following relation

$$g_{\alpha\beta,\sigma} + g_{\alpha\sigma,\beta} = g_{\sigma\beta,\alpha} + 2y^n_{,\alpha} y_{n,\beta\sigma}. \tag{40}$$

Finally we have

$$y^n_\alpha y_{n,\beta\sigma} = \frac{1}{2}(g_{\alpha\beta,\sigma} + g_{\alpha\sigma,\beta} - g_{\sigma\beta,\alpha}). \tag{41}$$

The right-hand part of this equation is called the Christoffel symbol:

$$\Gamma_{\alpha\beta\sigma} = \frac{1}{2}(g_{\alpha\beta,\sigma} + g_{\alpha\sigma,\beta} - g_{\sigma\beta,\alpha}) \tag{42}$$

From (36) and (41) we have

$$\delta A_\beta = A^\alpha \Gamma_{\alpha\beta\sigma} dx^\sigma. \tag{43}$$

It is obvious that δA_β can be also presented in the form

$$\delta A_\beta = \Gamma^\alpha_{\beta\sigma} A_\alpha dx^\sigma, \tag{44}$$

where

$$\Gamma^\alpha_{\beta\sigma} = g^{\alpha\rho} \Gamma_{\rho\beta\sigma}. \tag{45}$$

We conclude from (42) and (43) that the parallel displacement of a vector in four-dimensional non Euclidian space is determined by derivatives of the metric tensor $g_{\alpha\beta}$ (there are no more references to the N-dimensional flat space).

It is easy to see that under the change $\alpha \leftrightarrows \beta$ the first term in the right-hand side of the Christoffel symbol (42) is symmetric and the last two terms are antisymmetric. Thus, we have the relation

$$\Gamma_{\alpha\beta\sigma} + \Gamma_{\beta\alpha\sigma} = g_{\alpha\beta,\sigma}. \tag{46}$$

It is also obvious that

$$\Gamma_{\alpha\beta\sigma} = \Gamma_{\alpha\sigma\beta}. \tag{47}$$

Under the parallel displacement the length of a vector is not changed. In fact, we have

$$
\begin{aligned}
\delta(g^{\alpha\beta}A_\alpha A_\beta) &= g^{\alpha\beta}\delta A_\alpha A_\beta + g^{\alpha\beta}A_\alpha \delta A_\beta + \delta g^{\alpha\beta}A_\alpha A_\beta \\
&= [A^\rho A^\alpha(\Gamma_{\rho\alpha\sigma} + \Gamma_{\alpha\rho\sigma}) + g^{\alpha\beta}_{,\sigma}A_\alpha A_\beta]dx^\sigma
\end{aligned}
\tag{48}
$$

Further, using relation (46) we find

$$
\delta(g^{\alpha\beta}A_\alpha A_\beta) = [g_{\alpha\rho,\sigma}A^\alpha A^\rho + g^{\alpha\beta}_{,\sigma}A_\alpha A_\beta]dx^\sigma.
\tag{49}
$$

We have

$$
g^{\alpha\beta}g_{\alpha\rho} = \delta^\beta_\rho.
\tag{50}
$$

From this relation we find

$$
g^{\alpha\beta}_{,\sigma}g_{\alpha\rho} + g^{\alpha\beta}g_{\alpha\rho,\sigma} = 0.
\tag{51}
$$

If we multiply this relation by $g_{\beta\tau}$, we obtain the following relation

$$
g_{\tau\rho,\sigma} = -g^{\alpha\beta}_{,\sigma}g_{\alpha\rho}g_{\beta\tau}
\tag{52}
$$

From (52) we have

$$
g_{\tau\rho,\sigma}A^\tau A^\rho = -g^{\alpha\beta}_{,\sigma}g_{\alpha\rho}g_{\beta\tau}A^\rho A^\sigma = -g^{\alpha\beta}_{,\sigma}A_\alpha A_\beta.
\tag{53}
$$

Thus, finally, we find

$$
\delta(g^{\alpha\beta}A_\alpha A_\beta) = 0.
\tag{54}
$$

Let us consider the vector $A + \lambda B$, where λ is an arbitrary constant. From the relation

$$
\delta(A_\alpha + \lambda B_\alpha)(A^\alpha + \lambda B^\alpha) = 0
\tag{55}
$$

we have

$$
\delta(A_\alpha B^\alpha) = 0.
\tag{56}
$$

Thus, the scalar product $(A_\alpha B^\alpha)$ is not changed under the parallel displacement of vectors A and B.

From (56) we find

$$
\delta A_\alpha B^\alpha + A_\alpha \delta B^\alpha = 0
\tag{57}
$$

Further, using (43) from (57) we obtain the following relation

$$
A_\rho \delta B^\rho = -B^\alpha \Gamma_{\beta\alpha\sigma}dx^\sigma A^\beta = -B^\alpha \Gamma^\rho_{\alpha\sigma}dx^\sigma A_\rho
\tag{58}
$$

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Thus, for the contravariant vector we have

$$\delta B^\rho = -\Gamma^\rho_{\alpha\sigma} B^\alpha dx^\sigma. \tag{59}$$

Let us consider now the difference of the vector $A_\alpha(x + dx)$ and the vector which is obtained by the parallel displacement of the vector $A_\alpha(x)$ from the point x to the point $x + dx$. Taking into account (58) we have

$$A_\alpha(x+dx) - [A_\alpha(x) + \Gamma^\beta_{\alpha\sigma}(x)A_\beta(x)dx^\sigma] = [A_{\alpha,\sigma}(x) - \Gamma^\beta_{\alpha\sigma}(x)A_\beta(x)]\, dx^\sigma. \tag{60}$$

The covariant derivative of the vector A_α is denoted by $A_{\alpha:\sigma}$. From (60) follows that it is given by the relation

$$A_{\alpha:\sigma} = A_{\alpha,\sigma} - \Gamma^\beta_{\alpha\sigma} A_\beta \tag{61}$$

The covariant derivative $A_{\alpha:\sigma}$ is a tensor. This follows from the fact that the left-hand side of (60) is a vector.

From (59) follows that the covariant derivative of the contravariant vector is given by

$$A^\alpha_{:\sigma} = A^\alpha_{,\sigma} + \Gamma^\alpha_{\beta\sigma} A^\beta. \tag{62}$$

For the covariant derivative of a product of two vectors we obviously find

$$
\begin{aligned}
(A_\alpha B_\beta)_{:\sigma} &= A_{\alpha:\sigma} B_\beta + A_\alpha B_{\beta:\sigma} = (A_{\alpha,\sigma} - \Gamma^\rho_{\alpha\sigma} A_\rho)B_\beta + A_\alpha(B_{\beta,\sigma} - \Gamma^\rho_{\beta\sigma} B_\rho) \\
&= (A_\alpha B_\beta)_{,\sigma} - \Gamma^\rho_{\alpha\sigma} A_\rho B_\beta - \Gamma^\rho_{\beta\sigma} A_\alpha B_\rho.
\end{aligned} \tag{63}
$$

Taking into account that a product of two vectors is transformed as a tensor of the second rank, for the covariant derivative of a second rank tensor $T_{\alpha\beta}$ we have

$$T_{\alpha\beta:\sigma} = T_{\alpha\beta,\sigma} - \Gamma^\rho_{\alpha\sigma} T_{\rho\beta} - \Gamma^\rho_{\beta\sigma} T_{\alpha\rho}. \tag{64}$$

It is obvious that the covariant derivative of a scalar $Y(x)$ is equal to the normal derivative:

$$Y_{:\sigma} = Y_{,\sigma}. \tag{65}$$

Finally, let us calculate the covariant derivative of the metric tensor. Using (64), we have

$$g_{\alpha\beta:\sigma} = g_{\alpha\beta,\sigma} - \Gamma_{\beta\alpha\sigma} - \Gamma_{\alpha\beta\sigma}. \tag{66}$$

Taking into account (46), from this relation we find

$$g_{\alpha\beta:\sigma} = 0. \tag{67}$$

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4 The Riemann Curvature Tensor

Let us consider a second covariant derivative of a vector $A_{\alpha:\beta:\rho}$. In a flat space the order of differentiation is not important. This is not the case in a curved Riemann space. In fact, we have

$$
\begin{aligned}
A_{\alpha:\beta:\rho} &= (A_{\alpha:\beta})_{:\rho} = A_{\alpha,\beta\rho} - \Gamma^{\sigma}_{\alpha\beta,\rho}A_{\sigma} - \Gamma^{\sigma}_{\beta\rho}A_{\sigma,\rho} - \Gamma^{\sigma}_{\alpha\rho}A_{\sigma,\beta} \\
&+ \Gamma^{\tau}_{\alpha\rho}\Gamma^{\sigma}_{\tau\beta}A_{\sigma} - \Gamma^{\sigma}_{\beta\rho}A_{\alpha,\sigma} + \Gamma^{\sigma}_{\beta\rho}\Gamma^{\tau}_{\alpha\sigma}A_{\tau}.
\end{aligned}
\tag{68}
$$

From this relation we find

$$
A_{\alpha:\beta:\rho} - A_{\alpha:\rho:\beta} = (\Gamma^{\sigma}_{\alpha\rho,\beta} - \Gamma^{\sigma}_{\alpha\beta,\rho} + \Gamma^{\tau}_{\alpha\rho}\Gamma^{\sigma}_{\tau\beta} - \Gamma^{\tau}_{\alpha\beta}\Gamma^{\sigma}_{\tau\rho})A_{\sigma}.
\tag{69}
$$

The difference $A_{\alpha:\beta:\rho} - A_{\alpha:\rho:\beta}$ characterizes the curvature of a Riemann space. The tensor of the fourth rank

$$
R^{\sigma}_{\alpha\beta\rho} = \Gamma^{\sigma}_{\alpha\rho,\beta} - \Gamma^{\sigma}_{\alpha\beta,\rho} + \Gamma^{\tau}_{\alpha\rho}\Gamma^{\sigma}_{\tau\beta} - \Gamma^{\tau}_{\alpha\beta}\Gamma^{\sigma}_{\tau\rho}
\tag{70}
$$

is called *the Riemann curvature tensor*.[4]

From (69) It is obvious from (69) and (69) that

$$
R^{\sigma}_{\alpha\beta\rho} = -R^{\sigma}_{\alpha\rho\beta}.
\tag{71}
$$

Further, taking into account the relation $\Gamma^{\sigma}_{\alpha\rho} = \Gamma^{\sigma}_{\rho\alpha}$ (see (47)) it is easy to check that the Riemann curvature tensor satisfy the following cyclic relation

$$
R^{\sigma}_{\alpha\beta\rho} + R^{\sigma}_{\beta\rho\alpha} + R^{\sigma}_{\rho\alpha\beta} = 0.
\tag{72}
$$

We have

$$
R_{\sigma\alpha\beta\rho} = g_{\sigma\tau}R^{\tau}_{\alpha\beta\rho}.
\tag{73}
$$

From (70) and (73) we find

$$
R_{\sigma\alpha\beta\rho} = g_{\sigma\tau}\Gamma^{\tau}_{\alpha\rho,\beta} + \Gamma^{\mu}_{\alpha\rho}\Gamma_{\sigma\mu\beta} - (\rho \leftrightarrows \beta).
\tag{74}
$$

Using (46), for the first term in the right-hand side of (74) we obviously have

$$
g_{\sigma\tau}\Gamma^{\tau}_{\alpha\rho,\beta} = \Gamma_{\sigma\alpha\rho,\beta} - g_{\sigma\tau,\beta}\Gamma^{\tau}_{\alpha\rho} = \Gamma_{\sigma\alpha\rho,\beta} - (\Gamma_{\sigma\tau\beta} + \Gamma_{\tau\sigma\beta})\Gamma^{\tau}_{\alpha\rho}.
\tag{75}
$$

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[4]Notice that it is possible to show that if the curvature tensor is equal to zero the space is flat. Thus, if the curvature tensor is different from zero, a four-dimensional space is curved.

From (74) and (75) we find

$$R_{\sigma\alpha\beta\rho} = \Gamma_{\sigma\alpha\rho,\beta} - \Gamma^{\mu}_{\alpha\rho}\Gamma_{\mu\sigma\beta} - (\rho \leftrightarrows \beta). \qquad (76)$$

Further, using (42) and (76) we find the following expression for the Riemann curvature tensor

$$R_{\sigma\alpha\beta\rho} = \frac{1}{2}(g_{\sigma\rho,\alpha\beta} - g_{\rho\alpha,\sigma\beta} - g_{\beta\sigma,\alpha\rho} + g_{\beta\alpha,\sigma\rho}) - \Gamma^{\mu}_{\alpha\rho}\Gamma_{\mu\sigma\beta} + \Gamma^{\mu}_{\alpha\beta}\Gamma_{\mu\sigma\rho}. \qquad (77)$$

From (77) follows that the tensor $R_{\sigma\alpha\beta\rho}$ is antisymmetric under the exchange of indexes $\sigma \leftrightarrows \alpha$ or $\beta \leftrightarrows \rho$ and is symmetric under the exchange of pairs of indexes $(\sigma\alpha) \leftrightarrows (\beta\rho)$:

$$R_{\sigma\alpha\beta\rho} = -R_{\alpha\sigma\beta\rho}, \quad R_{\sigma\alpha\beta\rho} = -R_{\sigma\alpha\rho\beta}, \quad R_{\sigma\alpha\beta\rho} = R_{\beta\rho\sigma\alpha}. \qquad (78)$$

5   Bianci identity

The Bianci identity play an important role in the General Relativity. In this section we will derive this identity.

Let us consider the second covariant derivative of product of two vectors. We have

$$(A_\alpha B_\beta)_{:\rho:\sigma} = A_{\alpha:\rho:\sigma}B_\beta + A_{\alpha:\rho}B_{\beta:\sigma} + A_{\alpha:\sigma}B_{\beta:\rho} + A_\alpha B_{\beta:\rho:\sigma}. \qquad (79)$$

From this relation, taking into account (69) (70), we find

$$(A_\alpha B_\beta)_{:\rho:\sigma} - (A_\alpha B_\beta)_{:\sigma:\rho} = R^{\tau}_{\alpha\rho\sigma}A_\tau B_\beta + R^{\tau}_{\beta\rho\sigma}A_\alpha B_\tau \qquad (80)$$

The product $A_\alpha B_\beta$ is transferred as a tensor of a second rank. For any second rank tensor we have the following relation

$$(T_{\alpha\beta})_{:\rho:\sigma} - (T_{\alpha\beta})_{:\sigma:\rho} = R^{\tau}_{\alpha\rho\sigma}T_{\tau\beta} + R^{\tau}_{\beta\rho\sigma}T_{\alpha\tau}. \qquad (81)$$

In particular, for the tensor $A_{\alpha:\beta}$ we find

$$A_{\alpha:\beta:\rho:\sigma} - A_{\alpha:\beta:\sigma:\rho} = R^{\tau}_{\alpha\rho\sigma}A_{\tau:\beta} + R^{\tau}_{\beta\rho\sigma}A_{\alpha:\tau}. \qquad (82)$$

Let us perform in (82) the cyclic permutation of the indexes $\beta\rho\sigma$. We obtain two additional relations. The sum of all three relations is given by

$$(A_{\alpha:\beta:\rho:\sigma} - A_{\alpha:\beta:\sigma:\rho}) + (A_{\alpha:\sigma:\beta:\rho} - A_{\alpha:\rho:\beta:\sigma}) + (A_{\alpha:\rho:\sigma:\beta} - A_{\alpha:\sigma:\rho:\beta}) =$$
$$(A_{\alpha:\rho:\sigma} - A_{\alpha:\sigma:\rho})_{:\beta} + (\text{cyc. perm.}) = (R^{\tau}_{\alpha\rho\sigma}A_\tau)_{:\beta} + (\text{cyc. perm.}) =$$
$$R^{\tau}_{\alpha\rho\sigma}A_{\tau:\beta} + R^{\tau}_{\beta\rho\sigma}A_{\alpha:\tau} + (\text{cyc. perm.}) \qquad (83)$$

From this relation we find

$$R^\tau_{\alpha\rho\sigma:\beta} A_\tau + (\text{cyc. perm.}) = R^\tau_{\beta\rho\sigma} A_{\alpha:\tau} + (\text{cyc. perm.}). \tag{84}$$

Finally, from (84), taking into account the cyclic relation (72), we obtain the famous Bianci identity

$$R^\tau_{\alpha\rho\sigma:\beta} + R^\tau_{\alpha\sigma\beta:\rho} + R^\tau_{\alpha\beta\rho:\sigma} = 0. \tag{85}$$

6   The Ricci curvature tensor

If we contract a pair of indexes of the Riemann curvature tensor we will obtain the tensor of the second rank. Let us determine the rank two Ricci curvature tensor as follows

$$g^{\rho\sigma} R_{\sigma\alpha\beta\rho} = R^\rho_{\alpha\beta\rho} = R_{\alpha\beta}. \tag{86}$$

All other contractions of two indexes of the Riemann curvature tensor give zero or (up to a sign) the Ricci tensor. In fact, taking into account relations (78), we have[5]

$$g^{\sigma\alpha} R_{\sigma\alpha\beta\rho} = 0, \quad g^{\sigma\beta} R_{\sigma\alpha\beta\rho} = -R_{\alpha\rho}, \quad g^{\alpha\beta} R_{\sigma\alpha\beta\rho} = R_{\sigma\rho},$$
$$g^{\alpha\rho} R_{\sigma\alpha\beta\rho} = -R_{\sigma\beta}, \quad g^{\beta\rho} R_{\sigma\alpha\beta\rho} = 0. \tag{87}$$

Further, from (78) we find

$$g^{\rho\sigma} R_{\sigma\alpha\beta\rho} = g^{\sigma\rho} R_{\rho\beta\alpha\sigma}. \tag{88}$$

Thus, the tensor Ricci is a symmetric tensor

$$R_{\alpha\beta} = R_{\beta\alpha}. \tag{89}$$

Contracting two indexes of the Ricci tensor, we obtain the scalar curvature

$$g^{\alpha\beta} R_{\beta\alpha} = R^\alpha_\alpha = R. \tag{90}$$

From (70) we find that the Ricci curvature tensor is given be the relation

$$R_{\alpha\beta} = \Gamma^\sigma_{\alpha\sigma,\beta} - \Gamma^\sigma_{\alpha\beta,\sigma} + \Gamma^\tau_{\alpha\sigma}\Gamma^\sigma_{\tau\beta} - \Gamma^\tau_{\alpha\beta}\Gamma^\sigma_{\tau\sigma}. \tag{91}$$

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[5]Notice that in literature exist different definitions of the Ricci tensor. For example, in [3] the Ricci tensor is determined by the contraction $g^{\sigma\beta} R_{\sigma\alpha\beta\rho}$. It is equal to $-R_{\alpha\rho}$.

The Bianci identity for the Ricci curvature tensor plays a fundamental role in the General Theory of Relativity. In order to obtain this identity let us contract two pairs of indexes in the Bianci identity for the Riemann tensor (84). Taking into account that the covariant derivative of the metric tensor $g_{\alpha\beta}$ is equal to zero, from (84) we obtain the following relation

$$(g^{\alpha\beta} R^{\tau}_{\alpha\beta\rho})_{:\tau} + (g^{\alpha\beta} R^{\tau}_{\alpha\rho\tau})_{:\beta} + (g^{\alpha\beta} R^{\tau}_{\alpha\tau\beta})_{:\rho} = 0. \tag{92}$$

For the different terms in this relation we obviously have

$$g^{\alpha\beta} R^{\tau}_{\alpha\rho\tau} = g^{\alpha\beta} R_{\alpha\rho} = R^{\beta}_{\rho}, \quad g^{\alpha\beta} R^{\tau}_{\alpha\tau\beta} = g^{\alpha\beta} g^{\tau\sigma} R_{\sigma\alpha\tau\beta} = -g^{\tau\sigma} R_{\sigma\tau} = -R,$$
$$g^{\alpha\beta} R^{\tau}_{\alpha\beta\rho} = g^{\alpha\beta} g^{\tau\sigma} R_{\sigma\alpha\beta\rho} = g^{\tau\sigma} R_{\sigma\rho} = R^{\tau}_{\rho}. \tag{93}$$

From (92) and (93) we find

$$2 R^{\tau}_{\rho:\tau} - R_{:\rho} = 0. \tag{94}$$

Finally, if we multiply (94) by $g^{\rho\mu}$, the Bianci identity takes the form

$$(R^{\mu\tau} - \frac{1}{2} g^{\mu\tau} R)_{:\tau} = 0. \tag{95}$$

7 The Equation of Motion of a Particle in a Riemann Space (Geodesics)

We will derived here the equation of motion of a particle in a curved Riemann space. Let $x^{\alpha}(s)$ be a coordinate of a particle on a trajectory. We assume that the velocity of a particle u^{α} is a timelike vector and choose a proper time s as a parameter. We have $u^{\alpha} = \frac{dx^{\alpha}}{ds}$. Taking into account that $ds^2 = dx^{\alpha} dx_{\alpha}$ we have

$$u^{\alpha} u_{\alpha} = 1. \tag{96}$$

Let us assume that along the trajectory a velocity is changing by the parallel displacement. We have in this case

$$u^{\alpha}(s + ds) = u^{\alpha}(s) + \delta u^{\alpha}(s), \tag{97}$$

where

$$\delta u^{\alpha} = -\Gamma^{\alpha}_{\beta\rho} u^{\beta} dx^{\rho} = -\Gamma^{\alpha}_{\beta\rho} u^{\beta} u^{\rho} ds. \tag{98}$$

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From (97) and (98) we find the following equation

$$\frac{du^\alpha}{ds} + \Gamma^\alpha_{\beta\rho} u^\beta u^\rho = 0. \tag{99}$$

We will show now that (99) is the equation of motion of a particle in a curved Riemann space.

The equation of motion of a particle follows from the variational principle

$$\delta \int_P^Q ds = 0, \tag{100}$$

where P and Q are fixed points on the trajectory. From

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta \tag{101}$$

we find

$$\delta ds^2 = 2 ds \delta ds = \delta g_{\alpha\beta} dx^\alpha dx^\beta + 2 g_{\alpha\beta} dx^\alpha \delta dx^\beta. \tag{102}$$

Further, using $\delta dx^\alpha = d\delta x^\alpha$, we have

$$\delta \int_P^Q ds = \int_P^Q [\frac{1}{2} g_{\alpha\beta,\rho} u^\alpha u^\beta \delta x^\rho - \frac{d}{ds}(g_{\alpha\rho} u^\alpha)] \delta x^\rho ds + \int_P^Q d(g_{\alpha\rho} u^\alpha \delta x^\beta) \tag{103}$$

Taking into account that $\delta x^\rho(P) = \delta x^\rho(Q) = 0$, from (100) and (103) we obtain the equation

$$\delta \int_P^Q ds = \int_P^Q [\frac{1}{2} g_{\alpha\beta,\rho} u^\alpha u^\beta - g_{\alpha\rho,\beta} u^\alpha u^\beta - g_{\alpha\rho} \frac{du^\alpha}{ds}] \delta x^\rho ds$$
$$= -\int_P^Q [g_{\alpha\rho} \frac{du^\alpha}{ds} + \Gamma_{\rho\alpha\beta} u^\alpha u^\beta] \delta x^\rho ds = 0, \tag{104}$$

where

$$\Gamma_{\rho\alpha\beta} = \frac{1}{2}(g_{\rho\alpha,\beta} + g_{\rho\beta,\alpha} - g_{\alpha\beta,\rho}) \tag{105}$$

is the Christoffel symbol.

Thus, from variational principle we find the equation

$$g_{\alpha\rho} \frac{du^\alpha}{ds} + \Gamma_{\rho\alpha\beta} u^\alpha u^\beta = 0. \tag{106}$$

If we multiply this equation by $g^{\sigma\rho}$ we obviously come to the equation (99).

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The equation (99) can be written in the form

$$\frac{d^2 x^\alpha}{ds^2} = -\Gamma^\alpha_{\beta\rho} \frac{dx^\beta}{ds} \frac{dx^\rho}{ds}. \tag{107}$$

Hence, the motion of the particle in curved Riemann space is determined by the Christoffel symbols, $-m\Gamma^\alpha_{\beta\rho} u^\beta u^\rho$ (m is a mass) is a force and components of the metric tensor play a role of potential.

Summarizing, between two fixed points in a curved Riemann space a particle is moving along the shortest trajectory. Such track of a particle is called geodesic. As we will see later, curvature of a space is determined by the gravitational field. Thus, in a gravitational field a particle is moving along the geodesic which is determined by the equation (99).

8   On the Equivalence Principle

Since 1907 for Einstein the *Equivalence Principle* was a guide to the Theory of General Relativity. According to the Equivalence Principle in any infinitesimal region of a four-dimensional space, in which the gravitational force can be considered as a constant, exist such a coordinate system in which the gravitational force does not affect movement of a body and all other physical processes. This principle is based on the equality of inertial and gravitational masses which ensure that in the gravitational field all bodies have the same acceleration.

In such a system and in all other systems, which can be obtained from it by the Lorenz transformation, all physical laws have the same form. The square of an invariant distance between points $\xi^\mu + d\xi^\mu$ and ξ^μ in cartesian coordinates is given by

$$ds^2 = \eta_{\mu\nu} d\xi^\mu d\xi^\nu, \tag{108}$$

where

$$\eta_{00} = 1, \quad \eta_{ii} = -1, \quad \eta_{\mu\nu} = 0 \quad (\mu \neq \nu). \tag{109}$$

Let us consider *any other coordinate system.* Coordinates of the same point in such a system x^α are functions of ξ^β. For invariant ds^2 we find

$$ds^2 = \eta_{\mu\nu} \frac{\partial \xi^\mu}{\partial x^\alpha} \frac{\partial \xi^\nu}{\partial x^\beta} dx^\alpha dx^\beta = g_{\alpha\beta} dx^\alpha dx^\beta. \tag{110}$$

Here

$$g_{\alpha\beta} = \eta_{\mu\nu} \frac{\partial \xi^\mu}{\partial x^\alpha} \frac{\partial \xi^\nu}{\partial x^\beta} \tag{111}$$

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is the metric tensor.

The equation of the free motion of a particle in the initial inertial Galileo system (no gravitational force) is given by

$$\frac{d^2\xi^\mu}{ds^2} = 0. \tag{112}$$

We will shaw now that in any other system Eq.(112) become the equation for geodesic (99), considered in the previous section. From (112) we have

$$\frac{d^2\xi^\mu}{ds^2} = \frac{d}{ds}\left(\frac{\partial\xi^\mu}{\partial x^\rho}\frac{dx^\rho}{ds}\right) = \frac{\partial^2\xi^\mu}{\partial x^\rho\partial x^\beta}\frac{dx^\rho}{ds}\frac{dx^\beta}{ds} + \frac{\partial\xi^\mu}{\partial x^\rho}\frac{d^2x^\rho}{ds^2} = 0. \tag{113}$$

Let us multiply this equation by $\frac{\partial x^\alpha}{\partial\xi^\mu}$. Taking into account that $\frac{\partial\xi^\mu}{\partial x^\rho}\frac{\partial x^\alpha}{\partial\xi^\mu} = \frac{\partial x^\alpha}{\partial\xi^\rho} = \delta^\alpha_\rho$, we find

$$\frac{d^2x^\alpha}{ds^2} + \frac{d^2\xi^\mu}{dx^\rho dx^\beta}\frac{dx^\alpha}{d\xi^\mu}\frac{dx^\rho}{ds}\frac{dx^\beta}{ds} = 0. \tag{114}$$

Let us introduce the notation

$$\frac{d^2\xi^\mu}{dx^\rho dx^\beta}\frac{\partial x^\alpha}{\partial\xi^\mu} = \bar{\Gamma}^\alpha_{\rho\beta}, \tag{115}$$

We will show now that $\bar{\Gamma}^\alpha_{\rho\beta}$ is the Cristoffel symbol, determined by the relation (42).

In fact, taking into account that $\eta_{\mu\nu} = \eta_{\nu\mu}$, from Eq. (111) we find

$$\frac{\partial g_{\alpha\beta}}{\partial x^\rho} = \eta_{\mu\nu}\frac{\partial^2\xi^\mu}{\partial x^\rho\partial x^\alpha}\frac{\partial\xi^\nu}{\partial x^\beta} + \eta_{\mu\nu}\frac{\partial\xi^\mu}{\partial x^\alpha}\frac{\partial^2\xi^\nu}{\partial x^\rho\partial x^\beta} = \eta_{\mu\nu}\frac{\partial^2\xi^\mu}{\partial x^\rho\partial x^\alpha}\frac{\partial\xi^\nu}{\partial x^\beta} + (\alpha \rightleftarrows \beta). \tag{116}$$

From (111) and (115) it follows that

$$\bar{\Gamma}^\sigma_{\rho\alpha}g_{\sigma\beta} = \frac{\partial^2\xi^\mu}{\partial x^\rho\partial x^\alpha}\frac{\partial\xi^\sigma}{\partial x^\mu}\eta_{\tau\nu}\frac{\partial\xi^\tau}{\partial x^\sigma}\frac{\partial\xi^\nu}{\partial x^\beta} = \eta_{\mu\nu}\frac{\partial^2\xi^\mu}{\partial x^\rho\partial x^\alpha}\frac{\partial\xi^\nu}{\partial x^\beta}. \tag{117}$$

Thus, we find the relation

$$\frac{\partial g_{\alpha\beta}}{\partial x^\rho} = \bar{\Gamma}^\sigma_{\rho\alpha}g_{\sigma\beta} + \bar{\Gamma}^\sigma_{\rho\beta}g_{\sigma\alpha}. \tag{118}$$

If we change in (118) $\alpha \leftrightarrows \rho$ and $\beta \leftrightarrows \rho$ we obtain two additional relations

$$\frac{\partial g_{\rho\beta}}{\partial x^\alpha} = \bar{\Gamma}^\sigma_{\alpha\rho}g_{\sigma\beta} + \bar{\Gamma}^\sigma_{\alpha\beta}g_{\sigma\rho} \tag{119}$$

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and

$$\frac{\partial g_{\alpha\rho}}{\partial x^\beta} = \bar{\Gamma}^\sigma_{\beta\alpha}g_{\sigma\rho} + \bar{\Gamma}^\sigma_{\beta\rho}g_{\sigma\alpha}. \tag{120}$$

It is obvious from (115) that $\bar{\Gamma}^\sigma_{\alpha\rho} = \bar{\Gamma}^\sigma_{\rho\alpha}$. From (118), (119) and (120) we find that $\bar{\Gamma}^\sigma_{\alpha\rho}$ is given by the relation

$$\bar{\Gamma}^\sigma_{\alpha\rho}g_{\sigma\beta} = \bar{\Gamma}_{\beta\alpha\rho} = \frac{1}{2}(\frac{\partial g_{\alpha\beta}}{\partial x^\rho} + \frac{\partial g_{\rho\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\rho}}{\partial x^\beta}). \tag{121}$$

Comparing this relation with (42) we conclude that $\bar{\Gamma}_{\beta\alpha\rho} = \Gamma_{\beta\alpha\rho}$, where $\Gamma_{\beta\alpha\rho}$ is the standard Christoffell index. Thus, from (114) we have

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\rho\beta}\frac{dx^\rho}{ds}\frac{dx^\beta}{ds} = 0 \tag{122}$$

This equation is the equation for geodesics (99) which describes the motion of a body in curved Riemann space (gravitational field). We have demonstrated here that non inertial systems are equivalent to a gravitational field which is determined by the metric tensor.

9   Einstein Equations

In 1915 after many years of various attempts Einstein finally formulated equations of the General Relativity. His basic requirement was that equations of the General Theory of Relativity had to be invariant under a general transformation of coordinates in a curved Riemann space. He assumed that the conserved energy-momentum tensor of matter (and radiation) $T^{\alpha\beta}$ was proportional to a tensor $R^{\alpha\beta} + ag^{\alpha\beta}R$ formed by the Ricci tensor.

We have seen before that in a Riemann space the Bianci identity

$$(R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R)_{:\beta} = 0. \tag{123}$$

must be satisfied (see the section 5.). Thus, we have $a = -\frac{1}{2}$. The Einstein equations took the form

$$R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R = -8\pi GT^{\alpha\beta}. \tag{124}$$

Here G is the gravitational constant.[6] The coefficient $-8\pi G$ was chosen in order to ensure the correct Newtonian approximation.

From (124) follows that due to Bianci identity the equation

$$T^{\alpha\beta}_{:\beta} = 0, \tag{125}$$

which ensure conservation of energy and momentum, is *contained in the Einstein equations.*

In the empty space (no matter and no fields, except the gravitational field) we have

$$R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R = 0. \tag{126}$$

Contracting indexes in (129) we find $R - 2R = -R = 0$. Thus, the Einstein equation in the empty space can be written in the form[7]

$$R^{\alpha\beta} = 0. \tag{127}$$

In 1917 Einstein included in the equations of the General Relativity the tensor

$$\Lambda g^{\alpha\beta}, \tag{128}$$

where Λ is a constant, which is called the cosmological constant. The equations of General Relativity took the form

$$R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R + \Lambda g^{\alpha\beta} = -8\pi G T^{\alpha\beta}. \tag{129}$$

From the condition $g^{\alpha\beta}_{:\beta} = 0$ (see (67)) follows that the equation (125) is contained in the equations of the General Relativity also in the case of the cosmological constant.

From the Einstein equations (129) follows that evolution of the Universe is determined not only by attractive gravitational force but also by a proportional to Λ repulsive force. From analysis of existing cosmological data follows that about 70% of the density of the Universe at present time is due to the cosmological constant (or dark energy)

The Einstein equations are equations for the metric tensor $g^{\alpha\beta}$ which play a role of a potential of the gravitational field. From (91) follows that Einstein equations are the second order equations for the potential. Notice that ten components of $g_{\alpha\beta}$ describe not only potential but also coordinate system.

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[6]In the system $\hbar = c = 1$ G has dimension M^{-2} and the curvature tensor has dimension M^2. Thus, the tensor $T^{\alpha\beta}$ has dimension M^4.

[7]This does not mean that such a space is flat. Space is flat only if the Riemann tensor is equal to zero.

10 Energy-Momentum Tensor

Here we will consider an energy-momentum tensor for continuously distributed matter. The velocity of an infinitesimal element of the matter with coordinates x^α is given by

$$u^\alpha = \frac{dx^\alpha}{ds}. \tag{130}$$

Taking into account that

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta \tag{131}$$

we have

$$g_{\alpha\beta}u^\alpha u^\beta = 1. \tag{132}$$

In order to built the conserved energy-momentum tensor we will use the relation

$$\sqrt{-g}_{,\beta} = \sqrt{-g}\ \Gamma^\alpha_{\beta\alpha}. \tag{133}$$

Here g is the determinant of the metric tensor $g_{\alpha\beta}$ $(g < 0)$.

Using (133) for any vector A^α we find the following relation

$$\sqrt{-g}A^\alpha_{:\alpha} = \sqrt{-g}A^\alpha_{,\alpha} + \sqrt{-g}\Gamma^\alpha_{\sigma\alpha}A^\sigma = \sqrt{-g}A^\alpha_{,\alpha} + \sqrt{-g}_{,\alpha}A^\alpha = (\sqrt{-g}\ A^\alpha)_{,\alpha}. \tag{134}$$

Let us introduce now the scalar quantity $\rho(x)$ and assume that the vector $\sqrt{-g}\ \rho u^\alpha$ is conserved

$$(\sqrt{-g}\ \rho u^\alpha)_{,\alpha} = 0. \tag{135}$$

From this relation follows that $\sqrt{-g}\ \rho u^0$ is the density of matter and $\sqrt{-g}\ \rho u^i$ is the flux of matter.

We will choose the symmetric energy-momentum tensor in the form

$$T^{\alpha\beta} = \rho u^\alpha u^\beta. \tag{136}$$

Let us calculate the covariant derivative of this tensor. We have

$$T^{\alpha\beta}_{:\beta} = u^\alpha(\rho u^\beta)_{:\beta} + u^\alpha_{:\beta}\rho u^\beta. \tag{137}$$

From (134) and (135) we find

$$\sqrt{-g}(\rho u^\beta)_{:\beta} = (\sqrt{-g}\rho u^\beta)_{,\beta} = 0. \tag{138}$$

Further, taking into account that $\frac{du^\alpha}{ds} = \frac{du^\alpha}{dx^\beta}u^\beta$, from the equation for the geodesic (99) we have

$$u^\alpha_{:\beta}u^\beta = 0. \tag{139}$$

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From (138) and (139) follows that the tensor $T^{\alpha\beta}$, given by the expression (136), is conserved

$$T^{\alpha\beta}_{:\beta} = 0. \tag{140}$$

11 Newtonian approximation

As we have seen in the section 7, in the gravitational field particles are moving along the geodesic. The equation of motion has the form

$$\frac{du^\alpha}{ds} = -\Gamma^\alpha_{\beta\rho} u^\beta u^\rho. \tag{141}$$

In this section we consider the movement of a non relativistic particle in a weak, stationary gravitational field. For the stationary field the metric tensor does not depend on time

$$g_{\alpha\beta,0} \simeq 0 \tag{142}$$

and

$$g_{i0} \simeq 0. \tag{143}$$

Taking into account that $g^{\alpha\beta} g_{\beta\rho} = \delta^\alpha_\rho$ we also have

$$g^{i0} \simeq 0, \quad g^{00} g_{00} \simeq 1, \quad g^{ik} g_{kl} \simeq \delta^i_l. \tag{144}$$

In the liner over velocity approximation from (143) we find

$$\frac{du^i}{ds} = -g^{ik} \Gamma_{k\beta\rho} u^\beta u^\rho \simeq -g^{ik} \Gamma_{k00} u^0 u^0. \tag{145}$$

From (42) and (142) we easily find that

$$\Gamma_{k00} \simeq -\frac{1}{2} g_{00,k}. \tag{146}$$

Further, in the linear approximation we have

$$\frac{du^i}{ds} = \frac{du^i}{dx^\alpha} u^\alpha \simeq \frac{du^i}{dx^0} u^0 \tag{147}$$

and

$$g_{\alpha\beta} u^\alpha u^\beta \simeq g_{00} (u^0)^2 = 1, \quad u^0 \simeq g_{00}^{-1/2}. \tag{148}$$

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From (145)-(148) we obtain the following equation of the motion of non relativistic particle in a weak gravitational field

$$\frac{du^i}{dx^0} = g^{ik}(g_{00}^{1/2})_{,k}.$$

(149)

From this equation we conclude that $\sqrt{g_{00}}$ plays a role of a potential of a particle in the gravitational field. In fact, the Einstein equation in the gravitational field has a form

$$R_{\alpha\beta} = 0.$$

(150)

We are considering a weak gravitational field. For such a field the metric tensor is approximately constant and Cristoffel symbols are small. Neglecting quadratic over Cristoffel symbols terms in the Ricci curvature tensor we have

$$\Gamma^\rho_{\alpha\rho,\beta} - \Gamma^\rho_{\alpha\beta,\rho} \simeq 0.$$

(151)

Using (42), from this equation we find

$$g^{\rho\tau}[g_{\tau\rho,\alpha\beta} - g_{\alpha\rho,\tau\beta} - g_{\tau\beta,\alpha\rho} + g_{\alpha\beta,\tau\rho}] = 0.$$

(152)

Let us put in this equation $\alpha = \beta = 0$. Taking into account (142), we find that g_{00} satisfies the Laplace equation

$$g^{ik}g_{00,ik} = 0.$$

(153)

If there is no gravitational field we have $g_{00} = 1$. In the case of the weak gravitational field we can present g_{00} in the form

$$g_{00} = 1 + 2V,$$

(154)

where V is a small quantity which satisfies the Laplace equation

$$\nabla^2 V = 0.$$

(155)

From (149) and (154) we find the equation

$$\frac{d\vec{u}}{dx^0} = -\nabla V$$

(156)

Comparing this equation with the equation of motion of a particle in the gravitational filed, we conclude that V can be identified with the Newton potential

$$V = -G\frac{M}{r},$$

(157)

where G is the gravitational constant and M is a mass at the origin.

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12 Spherically Symmetrical Gravitational Field. The Schwarzschild Solution.

In this section we will consider *the static, spherically symmetrical gravitational field* produced by a spherically symmetrical body at rest.

For the static, spherically symmetrical field we have

$$g_{\alpha\beta,0} = 0, \quad g_{0i} = 0. \tag{158}$$

Let us choose the spherical coordinates r, θ, ϕ. The most general expression for ds^2 in spherically symmetrical static case has the form

$$ds^2 = U(r)dt^2 - V(r)dr^2 - W(r)(d\theta^2 + \sin^2\theta d\phi^2), \tag{159}$$

where U, V and W are functions of r. The coordinate r can be arbitrary chosen. We can determine r in such a way that $W(r) = r^2$.[8] It is convenient to present functions U and V in the form $U(r) = e^{2\nu(r)}$ and $V(r) = e^{2\lambda(r)}$. The interval ds^2 takes the form

$$ds^2 = e^{2\nu}dt^2 - e^{2\lambda}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{160}$$

We have $x^0 = t$, $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$. From (160) follows that nonzero components of the metric tensor $g_{\alpha\beta}$ are given by

$$g_{00} = e^{2\nu}, \quad g_{11} = -e^{2\lambda}, \quad g_{22} = -r^2, \quad g_{33} = -r^2\sin^2\theta. \tag{161}$$

Taking into account that $g^{\alpha\beta}g_{\beta\rho} = \delta^\alpha_\rho$ we find

$$g^{00} = e^{-2\nu}, \quad g^{11} = -e^{-2\lambda}, \quad g^{22} = -r^{-2}, \quad g^{33} = -r^{-2}\sin^{-2}\theta. \tag{162}$$

Let us now calculate the Christoffel symbols $\Gamma^\alpha_{\beta\gamma}$. From (42), (161) and (162) we find

$$\Gamma^0_{10} = \nu', \quad \Gamma^1_{00} = e^{2(\nu-\lambda)}\nu', \quad \Gamma^1_{11} = \lambda', \quad \Gamma^2_{12} = \Gamma^3_{13} = r^{-1}, \quad \Gamma^1_{22} = -re^{-2\lambda},$$

$$\Gamma^3_{23} = \frac{\cos\theta}{\sin\theta}, \quad \Gamma^1_{33} = -r\sin^2\theta e^{-2\lambda}, \quad \Gamma^2_{33} = -\sin\theta\cos\theta. \tag{163}$$

Other components of the the Christoffel symbols are equal to zero.

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[8]In this case the length of the circle with the center at the center of coordinates is equal to $2\pi r$.

From (70), (86) and (163) for nonzero components of the Ricci curvature tensor we find the following expressions

$$R_{00} = \left(\lambda'\nu' - \frac{2\nu'}{r} - \nu'' - \nu'^2 \right) e^{2(\nu-\lambda)}, \qquad (164)$$

$$R_{11} = \nu'' - \frac{2\lambda'}{r} - \lambda'\nu' + \nu'^2, \qquad (165)$$

$$R_{22} = \left(+1 + r\nu' - r\lambda' \right) e^{-2\lambda} - 1 \qquad (166)$$

and

$$R_{33} = R_{22} \sin^2\theta. \qquad (167)$$

The Einstein equations for the gravitational field outside of a body which produce the field have a form

$$R_{\alpha\beta} = 0. \qquad (168)$$

From (164), (165) and (168) we obtain the following equation

$$\nu' + \lambda' = 0. \qquad (169)$$

From this equation we find

$$\nu(r) + \lambda(r) = C, \qquad (170)$$

where C is a constant. At $r \to \infty$ the space is flat and $\nu = \lambda = 0$. Thus, we have $C = 0$ and

$$\lambda(r) = -\nu(r). \qquad (171)$$

From (166) and (168) we have

$$\left(1 + 2r\nu' \right) e^{2\nu} = \left(re^{2\nu} \right)' = 1. \qquad (172)$$

Thus, we find

$$e^{2\nu} = 1 + \frac{C_1}{r}, \qquad (173)$$

where C_1 is a constant. At large r the Newton approximation is valid and g_{00} is given by

$$g_{00} = 1 + 2V, \quad V = -G\frac{M}{r}, \qquad (174)$$

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where M is the mass of the body which produce the gravitational field. Comparing (173) and (174) we conclude that $C_1 = -2GM$ and

$$g_{00} = e^{2\nu} = 1 - G\frac{2M}{r}. \qquad (175)$$

We also have $g_{11} = e^{-2\nu} = (1 - G\frac{2M}{r})^{-1}$. Thus for the static, spherically symmetrical gravitational field we find the following Schwarzschield solution of the Einstein equations

$$ds^2 = (1 - G\frac{2M}{r})dt^2 - (1 - G\frac{2M}{r})^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \qquad (176)$$

The Schwarzschield metric modify the Newton theory of the motion of planets around the sun. It perfectly explains precession of the perihelion of Mercury.

Let us notice that from (176) it follows that the distance between points r_2 and r_1 at the same radius is larger than $(r_2 - r_1)$. In fact, we have

$$\int_{r_1}^{r_2} \frac{dr}{(1 - G\frac{2M}{r})} > (r_2 - r_1). \qquad (177)$$

We also have $g_{00} \leq 1$. For the proper time we find

$$d\tau = \sqrt{g_{00}}\, dt \leq dt \qquad (178)$$

From (175) and (178) we conclude that at infinity t and τ are the same but at a finite distance there is a slowdown of time with respect to a time at infinity.

13   Gravitational Waves

In this section we will discuss gravitational waves, one of the very important recently observed consequence of the General Relativity. For the weak gravitational field in the empty space the Einstein equations (in the linear approximation) have the form

$$\Gamma^{\rho}_{\alpha\rho,\beta} - \Gamma^{\rho}_{\alpha\beta,\rho} \simeq 0. \qquad (179)$$

Using (42), from this equation we have

$$g^{\tau\rho}[g_{\tau\rho,\alpha\beta} - g_{\tau\alpha,\rho\beta} - g_{\tau\beta,\alpha\rho} + g_{\alpha\beta,\tau\rho}] = 0. \qquad (180)$$

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It is convenient to use harmonic coordinates which provide the closest approximation to the flat case. In these coordinates the following condition holds

$$g^{\tau\rho}\Gamma^{\alpha}_{\tau\rho} = 0 \tag{181}$$

From (181) we find

$$g^{\tau\rho}(g_{\alpha\tau,\rho} - \frac{1}{2}g_{\tau\rho,\alpha}) = 0. \tag{182}$$

Let us differentiate this equation over x^{β}. In the linear approximation we have

$$g^{\tau\rho}(g_{\tau\alpha,\rho\beta} - \frac{1}{2}g_{\tau\rho,\alpha\beta}) = 0. \tag{183}$$

If we perform in (183) the change $\alpha \leftrightarrows \beta$, we find

$$g^{\tau\rho}(g_{\tau\beta,\rho\alpha} - \frac{1}{2}g_{\tau\rho,\alpha\beta}) = 0. \tag{184}$$

Adding (180), (183) and (184) we obtain the d'Alambert equation for each $g_{\alpha\beta}$:

$$g^{\tau\rho}g_{\alpha\beta,\tau\rho} = 0. \tag{185}$$

The solution of the equation (185) are gravitational waves traveling with the speed of light.

Let us consider gravitational waves moving in one direction, determined by a vector l_{α}, which satisfies the equation $g^{\alpha\beta}l_{\alpha}l_{\beta} = 0$. In this case the metric tensor depends on $lx = l_{\alpha}x^{\alpha}$. We have

$$g_{\tau\rho,\alpha} = u_{\tau\rho}l_{\alpha}, \tag{186}$$

where $u_{\tau\rho} = \frac{dg_{\tau\rho}}{dlx}$. For harmonic coordinates from (183) we find

$$g^{\tau\rho}u_{\tau\alpha}l_{\rho} = \frac{1}{2}g^{\tau\rho}u_{\tau\rho}l_{\alpha} = \frac{1}{2}ul_{\alpha}, \tag{187}$$

where $u = g^{\tau\rho}u_{\tau\rho}$. This equation can be rewritten in the form

$$u^{\rho}_{\alpha}l_{\rho} = \frac{1}{2}ul_{\alpha} \tag{188}$$

or

$$(u^{\tau\rho} - \frac{1}{2}g^{\tau\rho}u)l_{\rho} = 0. \tag{189}$$

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Further from (42) and (186) we find

$$\Gamma^\alpha_{\tau\beta} = \frac{1}{2}(u^\alpha_\tau l_\beta + u^\alpha_\beta l_\tau - u_{\tau\beta}l^\alpha). \tag{190}$$

We will consider now the energy-momentum tensor of the gravitational waves. The invariant action is given by

$$I = \int \mathcal{L}\sqrt{-g}d^4x, \tag{191}$$

where the action density $\mathcal{L}$ is equal to

$$\mathcal{L} = g^{\tau\rho}(\Gamma^\beta_{\tau\rho}\Gamma^\alpha_{\beta\alpha} - \Gamma^\alpha_{\tau\beta}\Gamma^\beta_{\rho\alpha}). \tag{192}$$

In the harmonic coordinates we have

$$\mathcal{L} = -g^{\tau\rho}\Gamma^\alpha_{\tau\beta}\Gamma^\beta_{\rho\alpha} = -\frac{1}{4}(u^\alpha_\tau l_\beta + u^\alpha_\beta l_\tau - u_{\tau\beta}l^\alpha)(u^\beta_\rho l_\alpha + u^\beta_\alpha l_\rho - u_{\rho\alpha}l^\beta). \tag{193}$$

Taking into account that $g^{\alpha\beta}l_\alpha l_\beta = 0$ we conclude that for the gravitational waves moving in one direction $\mathcal{L} = 0$.

The (pseudo) energy-momentum tensor t^β_α is determined by the relation

$$16\pi t^\rho_\tau\sqrt{-g} = (\Gamma^\rho_{\alpha\beta} - g^\rho_\beta\Gamma^\sigma_{\alpha\sigma})(g^{\alpha\beta}\sqrt{-g})_{,\tau} - g^\rho_\tau\mathcal{L}. \tag{194}$$

We have
$$g^{\alpha\beta}_{,\tau} = -g^{\alpha\rho}g^{\beta\sigma}g_{\rho\sigma,\tau} = u^{\alpha\beta}l_\tau \tag{195}$$

and

$$\sqrt{-g}_{,\tau} = \frac{1}{2}\sqrt{-g}g^{\alpha\beta}g_{\alpha\beta,\tau} = \frac{1}{2}\sqrt{-g}ul_\tau \tag{196}$$

Thus, we find
$$(g^{\alpha\beta}\sqrt{-g})_{,\tau} = -(u^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}u)\sqrt{-g}l_\tau \tag{197}$$

From (196) and (197) we have

$$\Gamma^\sigma_{\alpha\sigma}(g^{\alpha\beta}\sqrt{-g})_{,\tau} = \sqrt{-g}_{,\alpha}(-u^{\alpha\beta} + \frac{1}{2}g^{\alpha\beta}u)l_\tau = \frac{1}{2}\sqrt{-g}ul_\alpha(-u^{\alpha\beta} + \frac{1}{2}g^{\alpha\beta}u)l_\tau. \tag{198}$$

Taking into account (189) we conclude that

$$\Gamma^\sigma_{\alpha\sigma}(g^{\alpha\beta}\sqrt{-g})_{,\tau} = 0. \tag{199}$$

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We have

$$16\pi t^\rho_\tau \sqrt{-g} = -\Gamma^\rho_{\alpha\beta}(u^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}u)l_\tau = -\frac{1}{2}(u^\rho_\alpha l_\beta + u^\rho_\beta l_\alpha - u_{\alpha\beta}l^\rho)(u^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}u)l_\tau \tag{200}$$

From (189) it is easy to see that only the term

$$16\pi t^\rho_\tau \sqrt{-g} = \frac{1}{2}(u_{\alpha\beta}u^{\alpha\beta} - \frac{1}{2}u^2)l^\rho l_\tau. \tag{201}$$

is different from zero and transforms as tensor. Thus in the coordinate system, which is moving in the direction of the vector l_α, $t^\rho_\tau \sqrt{-g}$ is the energy-momentum tensor.

Discovery of predicted by the GTR gravitational wave signals by LIGO and later Virgo collaborations, awarded by the Nobel Prize in 2017, is a great success of this theory.

14   Friedman Equations

The modern cosmology is based on *the Cosmological Principle* which states that at large scales the distribution of Galaxies in the Universe is isotropic and homogeneous. The Cosmological Principle was formulated by Einstein in 1917. From the requirement of the isotropy and homogeneity it follows that the metrics of the Universe is the Friedman-Robertson-Walker metrics:

$$ds^2 = dt^2 - a^2(t)(\frac{dr^2}{1 - kr^2} + r^2\,(d\theta^2 + \sin^2\theta d\phi^2)). \tag{202}$$

Here t is a cosmological universal time, r is the comoving distance, $a(t)$ is the scale factor. The distance between any two Galaxies is given by the relation

$$d(t) = a(t)\,r. \tag{203}$$

From (203) follows
$$v(t) = \dot{a}(t)\,r, \tag{204}$$

where $v(t) = \dot{a}(t)$ is the relative velocity of a Galaxy. From (203) and (204) we find the famous Hubble law

$$v(t) = H(t)\,d(t), \tag{205}$$

where

$$H(t) = \frac{\dot{a}(t)}{a(t)} \tag{206}$$

is the Hubble parameter.

The parameter k in (202) is the curvature of the space. For the isotropic and homogeneous Universe k takes three values: $k = 1$ (closed Universe), $k = 0$ (flat Universe), ($k = -1$ open Universe).

Comparing (202) with the general expression for the interval $ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$ we conclude that

$$x^0 = t, \quad x^1 = r, \quad x^2 = \theta, \quad x^3 = \phi$$

and nonzero components of the metric tensor are equal to

$$g_{00} = 1, \quad g_{11} = -a^2\frac{1}{1-kr^2}, \quad g_{22} = -a^2r^2, \quad g_{33} = -a^2r^2\sin^2\theta. \tag{207}$$

Taking into account relation $g^{\alpha\beta}g_{\beta\rho} = \delta^\alpha_\rho$ we find that $g^{00} = 1$ and $g^{ii} = g_{ii}^{-1}$.

In order to find components of the Ricci tensor

$$R_{\alpha\beta} = R^\rho_{\alpha\beta\rho} = -\Gamma^\rho_{\alpha\beta,\rho} + \Gamma^\rho_{\alpha\rho,\beta} - \Gamma^\tau_{\alpha\beta}\Gamma^\rho_{\tau\rho} + \Gamma^\tau_{\alpha\rho}\Gamma^\rho_{\tau\beta} \tag{208}$$

we need to calculate the Christoffel symbols

$$\Gamma_{\sigma\beta\gamma} = \frac{1}{2}(g_{\sigma\beta,\gamma} + g_{\sigma\gamma,\beta} - g_{\beta\gamma,\sigma}). \tag{209}$$

From (207) and (209) we find

$$\begin{aligned}
\Gamma^0_{11} &= \frac{a\dot{a}}{1-kr^2}, \quad \Gamma^0_{22} = a\dot{a}r^2, \quad \Gamma^0_{33} = a\dot{a}r^2\sin^2\theta, \\
\Gamma^1_{01} &= \Gamma^1_{10} = \Gamma^2_{02} = \Gamma^2_{20} = \Gamma^3_{03} = \Gamma^3_{30} = \frac{\dot{a}}{a}, \\
\Gamma^1_{11} &= \frac{rk^2}{1-kr^2}, \quad \Gamma^1_{22} = -r(1-kr^2), \quad \Gamma^1_{33} = -(1-kr^2)r\sin^2\theta, \\
\Gamma^2_{12} &= \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}, \\
\Gamma^2_{33} &= -\sin\theta\cos\theta, \quad \Gamma^3_{23} = \Gamma^3_{32} = \frac{1}{\tan\theta}. \tag{210}
\end{aligned}$$

For the components of the Ricci tensor from (208) and (210) we have

$$R_{00} = 3\frac{\ddot{a}}{a}, \quad R_{ik} = \left[\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 2\frac{k}{a^2}\right]g_{ik}. \tag{211}$$

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The Ricci curvature is equal to

$$R = 6 \left[\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right]. \tag{212}$$

Let turn now to the Einstein equation (129). Cosmology is based on the assumption that matter can be considered as an perfect fluid and the energy-momentum tensor of the matter is given by the expression

$$T_{\alpha\beta} = (\rho + p)u_\alpha u^\beta - p \ g_{\alpha\beta}. \tag{213}$$

Here ρ and p are the density and pressure of matter and $u^\alpha = \frac{dx^\alpha}{ds}$ is the velocity. From the Cosmological Principle (isotropy) it follows that $u = (1,0,0,0)$. Thus, for 00 component we find

$$3\frac{\ddot{a}}{a} - 3 \left[\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right] + \Lambda = -8\pi G\rho \tag{214}$$

From (214) we obtain the first Friedman equation

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi}{3}G\rho + \frac{1}{3}\Lambda - \frac{k}{a^2}. \tag{215}$$

Let us consider now (ii) component of the Einstein equation. We have

$$\left[\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 2\frac{k}{a^2} \right] g_{ii} - 3 \left[\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right] g_{ii} + \Lambda g_{ii} = 8\pi G p g_{ii}. \tag{216}$$

From (215) and (216) we find the second Freedman equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}G(\rho + 3p) + \frac{1}{3}\Lambda. \tag{217}$$

Notice that terms in the right-hand side of (217) have *different signs*: the first gravitational term is negative (attraction) and the second Λ-term is positive (repulsion). If the second term dominates, the expansion rate of the Universe will be accelerating. As it is well known, we are living now in the accelerating Universe.

A   Brief History of the Discovery of the Einstein Equations

Here I will discuss a brief history of the A.Einstein equation (for details see, for example, [5]). Einstein formulated the Special Relativity in its final form in 1905 in the very first paper on the subject. He wrote more than ten different papers on the General Relativity before he came to the ultimate formulation of the Equations of the General Relativity in 1915. Einstein moved forward by "trial and error method" during eight years of very intensive work.[9]

In 1907 Einstein came to an idea of a free-falling cabin which he called "the happiest thought of my life".[10] For an observer in a free-falling cabin there is no gravitational field (more exactly, because of the equality of inertial and gravitational masses, in the free-falling cabin the gravitational force and the force of inertia cancel each other.) When several observers are falling in the gravitational field they have no possibility to decide that the gravitational field exist. Einstein thought that this is a powerful argument in favor of necessity of a generalization of the postulate of relativity on non inertial coordinate systems.

The road to the General Theory of Relativity started in 1907 in a review article [6] which Einstein wrote by the invitation of the editor of the Jahrbuch. The first part of the review was devoted to the Special Theory of Relativity. In the second part of the review Einstein tried to generalize the principle of relativity on non inertial systems. He considered uniform gravitational field. In the Newton theory an inertial system in which there is such a field is equivalent (in terms of mechanical motion) to the non inertial system with a constant acceleration (and without gravitational field). Einstein postulated that all processes flow in both systems in the same way. He called this postulate *the equivalence principle*. On the basis of the equivalence principle Einstein calculated in [6] such effects of the gravitational field as the red-shift of spectral lines and bending of the light.

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[9]In the letter to Ehrenfest in 1913, apologizing for a long silence, Einstein wrote "My excuse lies in the literally superhuman efforts which I have devoted to the gravitational problem" (Pais p.223)

[10]Einstein remembered : "I was sitting in a chair in the patent office at Bern when all of a sudden a thought occurred to me: If a person falls freely he will not feel his own weight. I was startled. This simple thought made a deep impression on me. It impelled me to the theory of gravitation" (Pais, p.179).

In 1908 Einstein started his academic career. He got a teaching position at the Bern University. In 1909 he received the position of associate professor of theoretical physics at the University of Zürich. In 1911 Einstein took the position of full professor in the German University at Prague. From 1907 till 1911 he published many papers mainly on quantum theory of light and even took part in some experiment. In Prague he returned to the gravitation.

Einstein realized that deflection of light in the gravitational field leads to the detectable effect of displacement of the position of stars, observed near the edge of the solar disc. From the equivalence principle he came to fundamental conclusion that the light velocity depend on the gravitational potential. In 1912 Einstein returned back to Zürich. He took the professor position at ETH (Swiss Federal Institute of Technology).

Toward the end of his Prague time Einstein suggested that the Riemann geometry is a correct mathematical tool for General Theory of Relativity. ("Euclidean geometry must be abandoned if non inertial frames are admitted on the equal footing"). In Zurich Einstein started very fruitful collaboration with mathematician Marcel Grossmann, his friend from the student years . In the paper "Outline of a Generalized Theory of Relativity and of a Theory of Gravitation" [7] Einstein and Grossmann made a significant progress in the development of a Theory of General Relativity. For the first time they used the Riemann geometry as a mathematical apparatus of a Theory of Gravitation with the basic idea that gravitational field is determined by the metric tensor $g_{\alpha\beta}$. They suggested that $g_{\alpha\beta}$ is determined by a energy-momentum tensor and the equation of the General Relativity has the following tensor form

$$X_{\alpha\beta} = kT_{\alpha\beta}, \tag{218}$$

where $T_{\alpha\beta}$ is a conserved energy-momentum tensor and $X_{\alpha\beta}$ is a second rank tensor determined by quantities which are generated by the metric tensor.

Einstein and Grossmann naturally looked for such $X_{\alpha\beta}$ which in the Newtonian limit of a weak gravitational field gave the Poisson equation

$$\triangle\phi = 4\pi G\rho. \tag{219}$$

The covariant derivative of the metric tensor is equal to zero

$$g_{\alpha\beta:\sigma} = 0. \tag{220}$$

On the basis of this equation it was concluded in [7] "that the sought for equations will be covariant only with respect to a certain group of transformations..."

Einstein and Grossmann considered the Ricci curvature tensor $R_{\alpha\beta}$ as a possible candidate for $X_{\alpha\beta}$. However, they did not managed in [7] to get Newtonian limit from the Ricci tensor.

Einstein finally came to the correct equation of the General Relativity in 2015 when he was in Berlin.[11] In November 1915 Einstein published four papers (every Thursday, November 4, 11, 18 and 25 [8, 9, 10, 11]). He came to the conclusion that in the left-hand side of the equation (218) the Ricci curvature tensor must enter. However, in the first paper Einstein required invariance of the equation for the gravitation under the unimodular transformations. In the second paper he put even more severe limitation: he required unimodular invariance with $\sqrt{-g} = 1$ (g is the determinant of the metric tensor). In the third paper Einstein obtained very important result. From the theory, proposed in the previous paper, for the first time he obtained quantitatively correct value of the angle of the rotation of the perithelium of Mercury. This gave him confidence of the correctness of the approach he was pursuing .[12]

Finally, in the November 25 paper Einstein returned back to the requirement of the invariance under general transformations and obtained the correct equation of the General relativity

$$R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R = -8\pi G T^{\alpha\beta}. \tag{221}$$

At that time Einstein did not know the Biancci identities (see [5]). He assumed the energy-momentum conservation

$$T^{\alpha\beta}_{:\rho} = 0 \tag{222}$$

and the term $\frac{1}{2}g^{\alpha\beta}R$ in (221) was obtained by Einstein from the requirement (223). [13]

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[11]On November 28 1915 Einstein wrote to Sommerfeld : " During the past month I had one of the most exciting and strenuous times of my life, but also one of the most successful ones".

[12]Einstein used the field equation in empty space and condition $\sqrt{-g} = 1$. In modern calculations the same equation is used in the empty space. The condition $\sqrt{-g} = 1$ fixes the coordinate system.

[13]Simultaneously with Einstein the equation (221) was obtained by Hilbert from the variational principle [12]. There exist a lot of publications in which Einstein and Gilbert papers are compared and discussed (see, for example, [5, 13]). I will cite only the famous Pauli's encyclopedia paper [14]: " Simultaneously and independently on Einstein covariant

As we mention before, Einstein was confused with the equation (220) in 1913 when he started to apply the Riemann geometry to the gravitation. Later, after the fundamental equation (221) was formulated, Einstein was the first who understood that due to (220) it is possible to include in the equation for the metric tensor the cosmological constant Λ [16]. The equation took its final form

$$R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R + \Lambda g^{\alpha\beta} = -8\pi G T^{\alpha\beta}. \tag{223}$$

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equation of the (gravitational) field were established by Gilbert. The Gilbert's presentation was, however, a little consonant to physicists because, firstly, Gilbert introduced variational principle axiomatically and, secondly, and more importantly, his equation was obtained not for arbitrary material system but for special theory of matter proposed by Mie [15]

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