Riemannian and Complex Geometry

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— Exercises —

2.1 Vector fields and differential forms

Exercise 2.1. Let $f\colon M \to N$ be a smooth map between smooth manifolds. Let X_1, X_2 be vector fields on M and Y_1, Y_2 be vector fields on N such that $(\mathrm{d}f)(X_i) = Y_i$ for $i = 1, 2$. Show that

$$(\mathrm{d}f)([X_1, X_2]) = [Y_1, Y_2].$$

Exercise 2.2. In each of the following cases, compute $\mathrm{d}\omega$ and $F^*\omega$, and verify by direct computation that $F^*(\mathrm{d}\omega) = \mathrm{d}(F^*\omega)$.

1. $M = N = \mathbb{R}^2$; $F(s,t) = (st, e^t)$; $\omega = x\mathrm{d}y$.
2. $M = \mathbb{R}^2$, $N = \mathbb{R}^3$; $F(\theta,\varphi) = ((\cos\varphi + 2)\cos\theta, (\cos\varphi + 2)\sin\theta, \sin\varphi)$; $\omega = y\mathrm{d}z \wedge \mathrm{d}x$.
3. $M = \{(u,v) \in \mathbb{R}^2 \colon u^2 + v^2 < 1\}$, $N = \mathbb{R}^3 \setminus \{0\}$; $F(u,v) = \left(u, v, \sqrt{1 - u^2 - v^2}\right)$;

$$\omega = \frac{x\mathrm{d}y \wedge \mathrm{d}z + y\mathrm{d}z \wedge \mathrm{d}x + z\mathrm{d}x \wedge \mathrm{d}y}{(x^2 + y^2 + z^2)^{3/2}}.$$

Exercise 2.3. Define a 2-form ω on $\mathbb{R}^3$ by

$$\omega = x\mathrm{d}y \wedge \mathrm{d}z + y\mathrm{d}z \wedge \mathrm{d}x + z\mathrm{d}x \wedge \mathrm{d}y.$$

1. Compute ω in spherical coordinates (ρ, φ, θ) defined by

$$(x, y, z) = (\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi).$$

2. Compute $\mathrm{d}\omega$ in both Cartesian and spherical coordinates and verify that both expressions represent the same 3-form.

3. Let $\iota_{\mathbb{S}^2} : \mathbb{S}^2 \to \mathbb{R}^3$ be the natural inclusion. Compute the pullback $\iota_{\mathbb{S}^2}^* \omega$ to $\mathbb{S}^2$, using coordinates φ, θ on the open subset where these coordinates are defined.

4. Show that $\iota_{\mathbb{S}^2}^* \omega$ is a nowhere-vanishing 2-form on $\mathbb{S}^2$.

Exercise 2.4. Show that there is a smooth vector field on $\mathbb{S}^2$ that vanishes at exactly one point.

Exercise 2.5 (hairy ball theorem)**.** There exists a nowhere-vanishing vector field on $\mathbb{S}^n$ if and only if n is odd. Prove this by showing that the following are equivalent:

1. There exists a nowhere-vanishing vector field on $\mathbb{S}^n$.

2. There exists a continuous map $V\colon \mathbb{S}^n \to \mathbb{S}^n$ satisfying $V(x) \perp x$ (with respect to the Euclidean dot product on $\mathbb{R}^{n+1}$) for all $x \in \mathbb{S}^n$.

3. The antipodal map α is homotopic[1] to $\mathrm{id}_{\mathbb{S}^n}$.

4. The antipodal map α is orientation-preserving.

5. n is odd.

2.2 Orientation

Exercise 2.6. Prove that a real projective space $\mathbb{RP}^n$ is orientable if and only if n is odd.

Exercise 2.7. Prove that a complex projective space $\mathbb{CP}^n$ is always orientable.

2.3 De Rham cohomology

Exercise 2.8. Let ω be the $(n-1)$-form on $\mathbb{R}^n \setminus \{0\}$ defined by

$$\omega = |x|^{-n} \sum_{i=1}^{n} (-1)^{i-1} x^i \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n.$$

Show that ω is closed but not exact on $\mathbb{R}^n \setminus \{0\}$.

[1] Let $f, g\colon X \to Y$ be two continuous maps between topological spaces. Then f is *homotopic* to g if there exists a continuous map $F\colon X \times [0,1] \to Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$.