Riemannian and Complex Geometry

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— Exercises —

3.1 Vector bundles

Exercise 3.1. Show that the following two definitions for vector bundles are the same.

Definition 3.1. Let M be a smooth manifold. A *(real) vector bundle* E of rank r on M consists of the following data:

(1) E is a smooth manifold with surjective map $\pi\colon E \to M$, such that

   (1) For all $x \in M$, fibre E_x is a $\mathbb{R}$-vector space of dimension r.

   (2) For all $x \in M$, there exists $x \in U \subseteq M$ and there is a diffeomorphism $\varphi\colon \pi^{-1}(U) \to U \times \mathbb{R}^r$ such that

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\pi"] \arrow[dr, "\varphi"'] & \circlearrowleft & U \\ & U \times \mathbb{R}^r \arrow[ur, "p_1"'] \arrow[r, "p_2"] & \mathbb{R}^r \end{tikzcd}$$

   and for all $y \in U$, $E_y \xrightarrow{p_2 \circ \varphi} \mathbb{R}^r$ is a $\mathbb{R}$-vector space isomorphism. The pair (U, φ) is called a trivialization of E over U.

Definition 3.2. Let M be a smooth manifold. A *(real) vector bundle* E of rank r on M consists of the following data:

(1) open covering $\{U_\alpha\}$ of M.

(2) smooth functions $\{g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{GL}(r, \mathbb{R})\}$ satisfies

$$\begin{aligned} g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} &= \mathrm{id} \quad \text{on } U_\alpha \cap U_\beta \cap U_\gamma \\ g_{\alpha\alpha} &= \mathrm{id} \quad \text{on } U_\alpha. \end{aligned}$$

Exercise 3.2. Show that a line bundle with a nowhere vanishing section is trivial.

3.2 Riemannian manifold

Exercise 3.3. Show that Poincaré disk is isometric to hyperbolic upper plane.

3.3 Levi-Civita connection

Exercise 3.4. Let g be the Riemannian metric on $\mathbb{S}^n(R)$ induced from the canonical metric on $\mathbb{R}^{n+1}$ by the natural inclusion $f\colon \mathbb{S}^n \to \mathbb{R}^{n+1}$. Compute its Christoffel symbol.

Exercise 3.5. Let $\mathbb{H}^n(R) = \{(x^1, \ldots, x^{n-1}, y) \in \mathbb{R}^n \mid y > 0\}$ with Riemannian metric

$$g = R^2 \frac{\delta_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j + \mathrm{d}y \otimes \mathrm{d}y}{y^2}.$$

Compute the Christoffel symbol of this Riemannian metric.

3.4 Tensor computation

Exercise 3.6. Let g be a Riemannian metric. Prove that a connection ∇ is compatible with g if and only if g is parallel, that is, $\nabla g = 0$.

Exercise 3.7. Let (M, g) be a Riemannian manifold. There is an induced metric on $T^*M \otimes T^*M$ defined as follows: Take two $(2,0)$-tensors T, S and write them locally as $T = T_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j, S = S_{k\ell} \mathrm{d}x^k \otimes \mathrm{d}x^\ell$. The induced metric, which is still denoted by g for convenience, is defined as

$$\begin{aligned} g(T, S) &= T_{ij} S_{k\ell}\, g(\mathrm{d}x^i \otimes \mathrm{d}x^j, \mathrm{d}x^k \otimes \mathrm{d}x^\ell) \\ &:= T_{ij} S_{k\ell}\, g^{ik} g^{j\ell}. \end{aligned}$$

Prove that if connection ∇ on vector bundle T^*M is compatible with metric g on it, then induced connection on $T^*M \otimes T^*M$ is compatible with induced metric g on it.

Exercise 3.8. Let (M, g) be a Riemannian manifold and ∇ be the Levi-Civita connection. For an $(2,0)$-tensor T on M, prove that

$$X(\operatorname{tr}_g T) = g(g, \nabla_X T)$$

holds for any vector field X.