Physics LATAM - Riemannian and Complex Geometry — Homework 3 Solutions

Course: Physics LATAM - Riemannian and Complex Geometry
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Date: October 29, 2025
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Exercise 3.1 — Equivalence of two definitions of vector bundle

Problem. Show the two standard definitions of a rank-r real vector bundle are equivalent: (A) a total space with local trivializations $phi : \pi^{-1}(U) \to U \times \mathbb{R}^r$; and (B) an open cover with smooth transition maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r, \mathbb{R})$ satisfying the cocycle conditions.

Solution (construction in both directions).

Direction (A) $\to$ (B).

1. Let (U, ϕ) and (V, ψ) be two trivializations. On the overlap $U \cap V$ both give vector-space isomorphisms of each fibre with $\mathbb{R}^r$. Define $g_{UV} : U \cap V \to GL(r, \mathbb{R})$ by the relation

   $$g_{UV}(x) = (p_2 \circ \psi) \circ (p_2 \circ \phi)^{-1} \in GL(r, \mathbb{R}),$$

   where $p_2 : U \times \mathbb{R}^r \to \mathbb{R}^r$ is projection to the second factor. Concretely, if $\phi(v) = (x, v_\phi)$ and $\psi(v) = (x, v_\psi)$ for $v \in E_x$, then $v_\psi = g_{UV}(x) v_\phi$.

2. The functions g_{UV} are smooth because they are compositions of smooth maps coming from the trivializations. They satisfy the cocycle conditions: $g_{UU} = \mathrm{id}$ and on triple overlaps $g_{UV} g_{VW} g_{WU} = \mathrm{id}$ by straightforward composition.

Direction (B) $\to$ (A).

1. Start from the disjoint union $\bigsqcup_\alpha (U_\alpha \times \mathbb{R}^r)$. Impose the equivalence relation: $(x, v, \alpha) \sim (x, g_{\alpha\beta}(x)v, \beta)$ for $x \in U_\alpha \cap U_\beta$. Denote the quotient by E and write $\pi : E \to M$ for the map induced by $(x, \cdot, \alpha) \mapsto x$.
2. The cocycle condition ensures this relation is transitive, so E is well-defined. Local charts $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^r$ are obtained by mapping the equivalence class of (x, v, α) to (x, v). The transition functions between these canonical charts are exactly the $g_{\alpha\beta}$, so E is a smooth manifold with the required local trivializations and fibrewise vector-space structure.
3. One checks the local charts give a smooth atlas (compatibility of charts follows from smoothness of $g_{\alpha\beta}$). Thus the two definitions agree.

Exercise 3.2 — Line bundle triviality from a nowhere-vanishing section

Problem. Show that a real line bundle $L \to M$ which admits a smooth section s with $s(x) \neq 0$ for all x is trivial.

Solution.

1. Define a map $\Phi : M \times \mathbb{R} \to L$ by

$$\Phi(x, t) = t\, s(x).$$

   This map is smooth and fibre-preserving: $\Phi(x, t) \in L_x$ for all (x, t). It is linear in the second factor and so a bundle map.
2. Show Φ is a vector-bundle isomorphism. Surjectivity: for any $v \in L_x$ the fibre is one-dimensional, so there exists a scalar t with $v = ts(x)$; hence $v = \Phi(x, t)$. Injectivity: if $\Phi(x, t) = 0$ then $ts(x) = 0$ and since $s(x) \neq 0$ we get $t = 0$.
3. The inverse map is explicitly $\Phi^{-1}(v) = (\pi(v), \lambda(v))$, where $\lambda(v)$ is the unique real number with $v = \lambda(v)s(\pi(v))$; the dependence on v is smooth because in any local trivialization we represent s by a nonvanishing function and dividing by it is a smooth operation. Thus Φ is a diffeomorphism and linear on each fibre. Therefore $L \cong M \times \mathbb{R}$ is trivial.

Conclusion. A nowhere-vanishing global section provides a global frame for a line bundle, so the bundle is trivial.

Exercise 3.3 — Poincaré disk isometric to hyperbolic upper half-plane

Problem. Show the unit disk model (Poincaré disk) with metric

$$ds^2 = \frac{4|dz|^2}{(1-|z|^2)^2}, \qquad |z| < 1,$$

is isometric to the upper half-plane model $\mathbb{H} = \{w \in \mathbb{C} : \operatorname{Im} w > 0\}$ with metric

$$ds^2 = \frac{|dw|^2}{(\operatorname{Im} w)^2}.$$

Solution (explicit Möbius map and pullback check).

1. Consider the Möbius transformation

$$w = \phi(z) = i\frac{1+z}{1-z}, \qquad |z| < 1.$$

   This map sends the unit disk to the upper half-plane. Two elementary algebraic identities are key:

   (i) the derivative

$$\phi'(z) = \frac{2i}{(1-z)^2}, \qquad |\phi'(z)| = \frac{2}{|1-z|^2};$$

   (ii) the imaginary part of w:

$$\operatorname{Im} \phi(z) = \frac{1-|z|^2}{|1-z|^2}.$$

2. Pull back the upper-half-plane metric via ϕ. One gets

$$\begin{aligned}\phi^*\left(\frac{|dw|^2}{(\operatorname{Im} w)^2}\right) &= \frac{|\phi'(z)|^2|dz|^2}{\left(\operatorname{Im}\phi(z)\right)^2} \\ &= \frac{(2/|1-z|^2)^2|dz|^2}{((1-|z|^2)/|1-z|^2)^2} \\ &= \frac{4|dz|^2}{(1-|z|^2)^2},\end{aligned}$$

   which is exactly the Poincaré disk metric. Therefore ϕ is an isometry between the two models.

Exercise 3.4 — Christoffel symbols for the round sphere metric

Problem. Let $S^n \subset \mathbb{R}^{n+1}$ be the unit sphere with the metric induced from the Euclidean metric on $\mathbb{R}^{n+1}$. Compute the Christoffel symbols of the Levi-Civita connection in stereographic coordinates (the standard local coordinates on $S^n \setminus \{N\}$).

Solution (use conformal metric from stereographic projection).

1. Stereographic projection from the North pole gives coordinates $u = (u^1, \dots, u^n) \in \mathbb{R}^n$ on the sphere minus the pole. The pullback metric is conformally flat with conformal factor

$$g_{ij}(u) = \lambda(u)^2 \delta_{ij}, \qquad \lambda(u) = \frac{2}{1 + |u|^2},$$

   where $|u|^2 = \sum_k (u^k)^2$. (This is standard; derive by the projection formula or recall the textbook expression.)

2. For a conformal metric $g_{ij} = \lambda^2 \delta_{ij}$ the Christoffel symbols (with respect to the flat coordinate basis) are

$$\Gamma^k_{ij} = \delta_{jk}\partial_i(\ln \lambda) + \delta_{ik}\partial_j(\ln \lambda) - \delta_{ij}\partial_k(\ln \lambda).$$

   This formula follows from the general expression

$$\Gamma^k_{ij} = \frac{1}{2} g^{k\ell}(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}),$$

   substituting $g_{ij} = \lambda^2 \delta_{ij}$ and simplifying.

3. Compute the derivatives of $\ln \lambda$ for the stereographic factor. We have

$$\ln \lambda = \ln 2 - \ln(1 + |u|^2), \qquad \partial_i \ln \lambda = -\frac{2u^i}{1 + |u|^2}.$$

4. Substitute into the conformal formula to obtain an explicit expression:

$$\boxed{\Gamma^k_{ij}(u) = -\frac{2}{1 + |u|^2}(\delta_{jk}u^i + \delta_{ik}u^j - \delta_{ij}u^k).}$$

   This is the Christoffel symbol in these coordinates. It is symmetric in i, j as required, and one may check it agrees with the geometric fact that geodesics are great circles when transformed back to the sphere.

Exercise 3.5 — Christoffel symbols for the hyperbolic upper-half metric

Problem. For $H^n(\mathbb{R}) = \{(x^1, \dots, x^{n-1}, y) : y > 0\}$ with metric

$$g = \frac{R^2}{y^2} \sum_{i=1}^{n-1} dx^i \otimes dx^i + \frac{R^2}{y^2} dy \otimes dy = \frac{R^2}{y^2}\, \delta_{ab}\, dx^a dx^b,$$

compute the Christoffel symbols.

Solution (conformal-flat calculation).

1. This metric is conformal to the Euclidean metric with conformal factor $\lambda = R/y$. So we may use the same conformal formula as in the previous exercise:

$$\Gamma^k_{ij} = \delta_{jk}\partial_i(\ln\lambda) + \delta_{ik}\partial_j(\ln\lambda) - \delta_{ij}\partial_k(\ln\lambda).$$

2. Compute derivatives of $\ln\lambda$:

$$\ln\lambda = \ln R - \ln y, \qquad \partial_{x^i}\ln\lambda = 0\ (i = 1, \dots, n-1), \quad \partial_y \ln\lambda = -\frac{1}{y}.$$

3. Using indices a, b, c ranging over $(x^1, \dots, x^{n-1}, y)$, the only nonzero partial derivatives above involve y. Plugging into the formula gives the nonzero Christoffel components (writing the last coordinate index as n to denote y):

   - For k corresponding to an x-direction ($k \le n-1$):

   $$\Gamma^k_{in} = \Gamma^k_{ni} = -\frac{1}{y}\,\delta_{ik}, \qquad (1 \le i, k \le n-1).$$

   - For $k = n$ (the y-direction):

   $$\Gamma^n_{ij} = \frac{1}{y}\,\delta_{ij}, \qquad (1 \le i, j \le n-1), \Gamma^n_{nn} = -\frac{1}{y}.$$

4. Collecting the nonzero components succinctly (where indices $i, j, k \le n-1$ run over the x-coordinates and index n denotes y):

$$\Gamma^k_{in} = \Gamma^k_{ni} = -\frac{1}{y}\,\delta_{ik}, \qquad (1 \le i, k \le n-1),$$

$$\Gamma^n_{ij} = \frac{1}{y}\,\delta_{ij}, \qquad (1 \le i, j \le n-1),$$

$$\Gamma^n_{nn} = -\frac{1}{y}.$$

For the $n = 2$ case (the familiar upper-half plane), this reduces to the components given in the worked example above.

Exercise 3.6 — Metric compatibility equivalent to parallel metric

Problem. Let g be a Riemannian metric and ∇ a connection. Prove that ∇ is compatible with g (i.e. preserves inner products under parallel transport) iff $\nabla g = 0$ (the metric is covariantly constant).

Solution.

1. Recall the coordinate formula for the covariant derivative of a $(0,2)$-tensor g is

   $$(\nabla_X g)(Y, Z) = X(g(Y, Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z).$$

2. If ∇ is compatible with g, parallel transport along any curve preserves the inner product. Differentiating the inner product of two parallel vector fields along the curve gives zero; writing that derivative in the form above yields $(\nabla_X g)(Y, Z) = 0$ for all vector fields X, Y, Z. Hence $\nabla g = 0$.

3. Conversely, if $\nabla g = 0$, let $Y(t), Z(t)$ be parallel vector fields along a curve $\gamma(t)$ (so $\nabla_{\dot\gamma} Y = 0$ and similarly for Z). Then

   $$\frac{d}{dt} g(Y, Z) = (\nabla_{\dot\gamma} g)(Y, Z) + g(\nabla_{\dot\gamma} Y, Z) + g(Y, \nabla_{\dot\gamma} Z) = 0 + 0 + 0 = 0.$$

   Hence parallel transport preserves inner products; this is exactly metric compatibility.

Therefore the two formulations are equivalent.

Exercise 3.7 — Induced connection on tensor product preserves induced metric

Problem. Let (M,g) be Riemannian and consider the induced metric on $T^*M\otimes T^*M$ defined by $g(T,S):=T_{ij}S_{k\ell}g^{ik}g^{j\ell}$. If a connection ∇ on T^*M is compatible with g, show the induced connection on $T^*M\otimes T^*M$ is compatible with the induced metric.

Solution.

1. The induced connection on the tensor product is the Leibniz (product) connection: for T,S tensor fields,

$$\nabla_X(T\otimes S)=(\nabla_X T)\otimes S+T\otimes(\nabla_X S).$$

2. Compute the derivative of the inner product:

$$\begin{aligned} X(g(T,S)) &= X(g^{ik}g^{j\ell}T_{ij}S_{k\ell}) \\ &= (\nabla_X g^{ik})g^{j\ell}T_{ij}S_{k\ell}+g^{ik}(\nabla_X g^{j\ell})T_{ij}S_{k\ell} \\ &\quad + g^{ik}g^{j\ell}(\nabla_X T_{ij})S_{k\ell}+g^{ik}g^{j\ell}T_{ij}(\nabla_X S_{k\ell}). \end{aligned}$$

3. Because ∇ is compatible with g on T^*M, we have $\nabla_X g^{ik}=0$ (equivalently $\nabla_X g_{ik}=0$ and raising indices commutes with ∇). Therefore the first two terms vanish and we get

$$X(g(T,S))=g(\nabla_X T,S)+g(T,\nabla_X S),$$

which is exactly the condition that the induced connection is metric-compatible on $T^*M\otimes T^*M$.

This completes the proof.

Exercise 3.8 — Trace and covariant derivative identity

Problem. Let (M,g) be a Riemannian manifold with Levi-Civita connection ∇. For a $(2,0)$-tensor T prove that for any vector field X,

$$X(\operatorname{tr}_g T)=g^{ij}(\nabla_X T)_{ij},$$

where $\operatorname{tr}_g T=g^{ij}T_{ij}$ is the trace using the inverse metric g^{ij}.

Solution.

1. Compute the derivative of the trace using the product rule:

$$\begin{aligned} X(\operatorname{tr}_g T) &= X(g^{ij}T_{ij}) \\ &= (\nabla_X g^{ij})T_{ij} + g^{ij}X(T_{ij}). \end{aligned}$$

2. Express $X(T_{ij})$ in terms of covariant derivatives:

$$(\nabla_X T)_{ij} = X(T_{ij}) - T_{kj}\Gamma^k_i(X) - T_{ik}\Gamma^k_j(X),$$

   where the last two terms are the Christoffel contributions. Contracting this with g^{ij} and using $\nabla g = 0$ (so $\nabla_X g^{ij} = 0$) removes the connection correction terms, leaving exactly

$$X(\operatorname{tr}_g T) = g^{ij}(\nabla_X T)_{ij}.$$

3. This identity is a direct and useful consequence of metric compatibility of the Levi-Civita connection.