Riemannian and Complex Geometry

Lecturers Daniel Galviz & Bowen Liu
Teaching Assistants: Anis Bousclet & Julian Alzate
PhysicsLatam.com/RC

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— Exercises —

1.1 Topological spaces

Exercise 1.1. Let the closed interval $X = [0, 1]$ with the topology induced from $\mathbb{R}$. Define the equivalence relation $\sim$ on X by:

$$x \sim y \quad \text{iff} \quad \begin{cases} x = y, & \text{if } x, y \in (0,1), \\ x, y \in \{0,1\}. \end{cases}$$

Denote by $X/_\sim$ the set of equivalence classes, and give it the quotient topology:

$$\tau = \{U \subseteq X/_\sim \mid \pi^{-1}(U) \text{ is open in } X\},$$

where $\pi \colon X \to X/_\sim$ is the continuous projection sending each point to its equivalence class. Show that

(1) $X/_\sim$ is a topological space;

(2) The map $f \colon X/_\sim \to S^1 \subset \mathbb{R}^2$ defined by

$$f([x]) = (\cos(2\pi x), \sin(2\pi x))$$

is a well-defined bijective and continuous map, and the inverse is also continuous.

Thus $X/_\sim$ is homeomorphic to S^1.

Exercise 1.2. Give a convincing argument (without proof) for why a disjoint union of an uncountable number of copies of $\mathbb{R}$ is locally Euclidean and Hausdorff, but not second-countable.

1.2 Smooth manifolds

Exercise 1.3. Show that the general linear group $\mathrm{GL}(n, \mathbb{R})$ is a smooth manifold and compute its dimension.

Exercise 1.4. In $\mathbb{C}^{n+1} \setminus \{0\}$, we define the following equivalence relation: for $x, y \in \mathbb{C}^{n+1} \setminus \{0\}$, $x \sim y$ if and only if there exists $\lambda \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ such that $x = \lambda y$. The *complex projective space* is the quotient space

$$\mathbb{CP}^n := (\mathbb{C}^{n+1} \setminus \{0\})/\sim.$$

We write $[z_0 : \cdots : z_n] \in \mathbb{CP}^n$ for the equivalence class of $(z_0, \ldots, z_n)$ in $\mathbb{CP}^n$.

(a) For $i = 0, 1, \ldots, n$, let $U_i = \{[z_0, \ldots, z_n] \in \mathbb{CP}^n : z_i \neq 0\}$. Show that U_i is open in $\mathbb{CP}^n$ for all $i = 0, \ldots, n$, and $\mathbb{CP}^n = \bigcup_{i=0}^n U_i$.

(b) For $i = 0, \ldots, n$, define $\phi_i : U_i \to \mathbb{C}^n$ by

$$\phi_i([z_0 : \cdots : z_n]) = \left(\frac{z_0}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \ldots, \frac{z_n}{z_i} \right).$$

Show that $\phi_i : U_i \to \mathbb{C}^n$ is a homeomorphism and that $\phi_i \circ \phi_j^{-1} : \mathbb{C}^n \to \mathbb{C}^n$ is of class C^∞.

(c) Making the identification $\mathbb{C}^n \cong \mathbb{R}^{2n}$ by $(x_1 + \sqrt{-1}y_1, \ldots, x_n + \sqrt{-1}y_n) \mapsto (x_1, y_1, \ldots, x_n, y_n)$, and conclude that $\mathbb{CP}^n$ is a differentiable manifold of real dimension $2n$ which is compact and connected.

Remark 1.1. The *real projective space* $\mathbb{RP}^n$ is defined in the same way by replacing $\mathbb{C}$ by $\mathbb{R}$.

Exercise 1.5. Which of the following subsets of $\mathbb{R}^2$ are topological manifolds when endowed with the subset topology from $\mathbb{R}^2$? Argue without precise proof.

- $M_1 := \{(x, |x|) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$
- $M_2 := \{(x, \pm|x|) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$
- $M_3 := \{(x, y) \in \mathbb{R}^2 \mid y = 0 \text{ or } y = \frac{1}{m} \text{ for some } m \in \mathbb{N}\}$
- $M_4 := \{(x, y) \in \mathbb{R}^2 \mid y = \frac{1}{m} \text{ for some } m \in \mathbb{N}\}$

Exercise 1.6. Let M be the boundary of the standard cube $[-1, 1]^2$ in $\mathbb{R}^2$, i.e.

$$M := \{(x, \pm 1) \in \mathbb{R}^2 \mid |x| \leq 1\} \cup \{(\pm 1, y) \in \mathbb{R}^2 \mid |y| \leq 1\},$$

equipped with the subset topology of $\mathbb{R}^2$.

1. Show that M is a topological manifold.
2. Can one find a smooth atlas on M containing the four charts below? Prove your answer.

$$\begin{aligned} \varphi_+ &: \{(1, y) \mid |y| < 1\} \to \mathbb{R}, \quad (1, y) \mapsto y \\ \varphi_- &: \{(-1, y) \mid |y| < 1\} \to \mathbb{R}, \quad (-1, y) \mapsto y \\ \psi_+ &: \{(x, 1) \mid |x| < 1\} \to \mathbb{R}, \quad (x, 1) \mapsto x \\ \psi_- &: \{(x, -1) \mid |x| < 1\} \to \mathbb{R}, \quad (x, -1) \mapsto x \end{aligned}$$

Exercise 1.7. Give a topological manifold with two different smooth structures. Can you find one with uncountably many different smooth structures?

1.3 Smooth maps

Exercise 1.8. Let M, N be smooth manifolds and let $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ be smooth atlases for M and N, respectively. Show that a map $f : M \to N$ is smooth if, for each α and β, the map $\psi_\beta \circ f \circ \varphi_\alpha^{-1}$ is smooth on its domain of definition.

Exercise 1.9. Show that being diffeomorphic defines an equivalence relation on smooth manifolds.

Exercise 1.10. Let $\mathbb{T} = \mathbb{S}^1 \times \mathbb{S}^1$ be the torus and α be any irrational number. Show that the map $f \colon \mathbb{R} \to \mathbb{T}$ by

$$f(t) = (e^{2\pi\sqrt{-1}t}, e^{2\pi\sqrt{-1}t\alpha}) \tag{1.1}$$

is a smooth injective immersion, but not an embedding.

Exercise 1.11. Considering the map

$$\begin{aligned} f \colon \mathrm{GL}(n, \mathbb{R}) &\to \mathbb{R} \\ A &\mapsto \det A. \end{aligned}$$

Show that f is a smooth map and the special linear group $\mathrm{SL}(n, \mathbb{R}) = f^{-1}(1)$ is a submanifold of $\mathrm{GL}(n, \mathbb{R})$. Compute the dimension of $\mathrm{SL}(n, \mathbb{R})$.

Exercise 1.12. Show that $\mathbb{CP}^1$ is diffeomorphic to $\mathbb{S}^2$.

Exercise 1.13. Define

$$\begin{aligned} f \colon \mathbb{RP}^2 &\to \mathbb{R}^4 \\ [x : y : z] &\mapsto \frac{1}{x^2 + y^2 + z^2}(x^2 - y^2, xy, xz, yz). \end{aligned}$$

Show that f is an embedding of $\mathbb{RP}^2$ into $\mathbb{R}^4$.

1.4 Tangent bundle and cotangent bundle

Exercise 1.14. Compute the transition function for $T\mathbb{S}^2$ associated with the two local trivializations determined by stereographic coordinates.

Exercise 1.15. Show that $T\mathbb{S}^1 \cong \mathbb{S}^1 \times \mathbb{R}$.