Physics LATAM - Riemannian and Complex Geometry

Homework Assignment #1 - Solutions

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Section 1.1: Topological Spaces

Exercise 1.1

Problem: Let the closed interval $X = [0, 1]$ with the topology induced from $\mathbb{R}$. Define the equivalence relation $\sim$ on X by:

$$x \sim y \iff \begin{cases} x = y & \text{if } x, y \in (0,1) \\ x, y \in \{0, 1\} & \text{otherwise} \end{cases}$$

Denote by $X/\sim$ the set of equivalence classes, and give it the quotient topology. Show that: 1. $X/\sim$ is a topological space 2. The map $f : X/\sim \to S^1 \subset \mathbb{R}^2$ defined by $f([x]) = (\cos(2\pi x), \sin(2\pi x))$ is a well-defined bijective and continuous map with continuous inverse, thus $X/\sim \cong S^1$.

Solution:

(1) $X/\sim$ is a topological space:

The quotient topology on $X/\sim$ is defined by $\tau = \{U \subseteq X/\sim : \pi^{-1}(U) \text{ is open in } X\}$, where $\pi : X \to X/\sim$ is the canonical projection.

We need to verify that τ satisfies the axioms of a topology:

- **$\emptyset, X/\sim \in \tau$:**
  - $\pi^{-1}(\emptyset) = \emptyset$ is open in X
  - $\pi^{-1}(X/\sim) = X$ is open in X
- **Arbitrary unions:** Let $\{U_i\}_{i \in I} \subseteq \tau$. Then:

  $$\pi^{-1}\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} \pi^{-1}(U_i)$$

  Since each $\pi^{-1}(U_i)$ is open in X and arbitrary unions of open sets are open, this union is open in X.
- **Finite intersections:** Let $U_1, \ldots, U_n \in \tau$. Then:

  $$\pi^{-1}\left(\bigcap_{i=1}^{n} U_i\right) = \bigcap_{i=1}^{n} \pi^{-1}(U_i)$$

  Since each $\pi^{-1}(U_i)$ is open in X and finite intersections of open sets are open, this intersection is open in X.

Therefore, τ is indeed a topology on $X/\sim$.

(2) Homeomorphism with S^1:

First, let's understand the equivalence classes: - For $x \in (0,1)$: $[x] = \{x\}$ (singleton) - $[0] = [1] = \{0,1\}$ (the endpoints are identified)

So $X/\sim$ consists of all points in $(0,1)$ plus one additional point representing the identified endpoints.

Well-defined: The function $f([x]) = (\cos(2\pi x), \sin(2\pi x))$ is well-defined because: - For $x \in (0,1)$: $[x] = \{x\}$, so f is unambiguous - For the identified endpoints: $f([0]) = f([1]) = (\cos(0), \sin(0)) = (1,0)$

Bijective: - **Injective:** If $f([x]) = f([y])$, then $(\cos(2\pi x), \sin(2\pi x)) = (\cos(2\pi y), \sin(2\pi y))$. This implies $2\pi x = 2\pi y + 2\pi k$ for some integer k, so $x = y + k$. Since $x, y \in [0,1]$, we must have $k = 0$ (hence $x = y$) or $(x,y) = (0,1)$ or $(1,0)$, which gives $[x] = [y]$ in both cases.

- **Surjective:** Every point $(\cos(2\pi t), \sin(2\pi t)) \in S^1$ corresponds to some $t \in [0,1)$. If $t \in (0,1)$, then $f([t])$ gives this point. If $t = 0$, then $f([0]) = f([1])$ gives $(1,0)$.

Continuous: Let U be open in S^1. We need to show that $f^{-1}(U)$ is open in $X/\sim$, i.e., $\pi^{-1}(f^{-1}(U))$ is open in X.

Note that $\pi^{-1}(f^{-1}(U)) = (f \circ \pi)^{-1}(U)$ where $f \circ \pi : X \to S^1$ is given by $(f \circ \pi)(x) = (\cos(2\pi x), \sin(2\pi x))$. This is the restriction of the standard parametrization of S^1, which is continuous. Therefore $(f \circ \pi)^{-1}(U)$ is open in X.

Continuous inverse: Let V be open in $X/\sim$. Then $\pi^{-1}(V)$ is open in X. We need to show $f(V)$ is open in S^1. Since $f \circ \pi : X \to S^1$ is the standard parametrization (which is an open map when restricted appropriately), and $\pi^{-1}(V)$ is open in X, we have that $f(V) = (f \circ \pi)(\pi^{-1}(V))$ is open in S^1.

Therefore, $f : X/\sim \to S^1$ is a homeomorphism.

Exercise 1.2

Problem: Give a convincing argument (without proof) for why a disjoint union of an uncountable number of copies of $\mathbb{R}$ is locally Euclidean and Hausdorff, but not second-countable.

Solution:

Let $Y = \bigsqcup_{\alpha \in A} \mathbb{R}_\alpha$ where A is uncountable and each $\mathbb{R}_\alpha$ is a copy of $\mathbb{R}$.

Locally Euclidean: Each point $p \in Y$ lies in some $\mathbb{R}_\alpha$. Since each $\mathbb{R}_\alpha$ is equipped with its standard topology (homeomorphic to $\mathbb{R}$), every point has a neighborhood homeomorphic to an open subset of $\mathbb{R}$. The disjoint union topology preserves this property.

Hausdorff: - If two points lie in the same copy $\mathbb{R}_\alpha$, they can be separated by disjoint open sets within that copy. - If two points lie in different copies $\mathbb{R}_\alpha$ and $\mathbb{R}_\beta$, then the entire copies themselves are disjoint open sets separating the points.

Not Second-Countable: A topological space is second-countable if it has a countable base for its topology. However: - Each copy $\mathbb{R}_\alpha$ requires at least one basic open set that intersects it but no other copy - Since there are uncountably many copies, any base must contain uncountably many sets - Therefore, no countable base can exist

The key insight is that while each individual copy is second-countable, the uncountable disjoint union "multiplies" the base requirements beyond countability.

Section 1.2: Smooth Manifolds

Exercise 1.3

Problem: Show that the general linear group $GL(n, \mathbb{R})$ is a smooth manifold and compute its dimension.

Solution:

Manifold Structure:

$GL(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) : \det(A) \neq 0\}$ where $M_n(\mathbb{R})$ is the space of $n \times n$ real matrices.

We can identify $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ by viewing each matrix as a vector of its n^2 entries. The determinant function $\det : M_n(\mathbb{R}) \to \mathbb{R}$ is a polynomial in the matrix entries, hence continuous.

Since $GL(n, \mathbb{R}) = \det^{-1}(\mathbb{R} \setminus \{0\})$ and $\mathbb{R} \setminus \{0\}$ is open in $\mathbb{R}$, we have that $GL(n, \mathbb{R})$ is open in $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$.

Smooth Structure:

Since $GL(n, \mathbb{R})$ is an open subset of $\mathbb{R}^{n^2}$, it inherits a natural smooth manifold structure. We can use a single chart: - $U = GL(n, \mathbb{R})$ - $\phi : GL(n, \mathbb{R}) \to \mathbb{R}^{n^2}$ given by $\phi(A) = (a_{11}, a_{12}, \ldots, a_{nn})$

This gives $GL(n, \mathbb{R})$ the structure of a smooth manifold.

Dimension:

Since $GL(n, \mathbb{R})$ is locally homeomorphic to open subsets of $\mathbb{R}^{n^2}$, its dimension is n^2.

Alternative approach: $GL(n, \mathbb{R})$ can be viewed as the complement of the zero set of the determinant polynomial in $\mathbb{R}^{n^2}$, which is indeed a smooth manifold of dimension n^2.

Exercise 1.4

Problem: In $\mathbb{C}^{n+1} \setminus \{0\}$, define the equivalence relation: $x \sim y$ if and only if there exists $\lambda \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ such that $x = \lambda y$. The complex projective space is $\mathbb{CP}^n := (\mathbb{C}^{n+1} \setminus \{0\}) / \sim$.

(a) Show that $U_i = \{[z_0 : \cdots : z_n] \in \mathbb{CP}^n : z_i \neq 0\}$ is open for all i, and $\mathbb{CP}^n = \bigcup_{i=0}^n U_i$.

(b) Show that $\phi_i : U_i \to \mathbb{C}^n$ defined by $\phi_i([z_0 : \cdots : z_n]) = \left(\frac{z_0}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \ldots, \frac{z_n}{z_i}\right)$ is a homeomorphism and transitions are C^∞.

(c) Conclude that $\mathbb{CP}^n$ is a differentiable manifold of real dimension $2n$.

Solution:

(a) Open sets and covering:

Let $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ be the canonical projection. Then:

$$U_i = \{[z_0 : \cdots : z_n] \in \mathbb{CP}^n : z_i \neq 0\}$$

We have $\pi^{-1}(U_i) = \{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \setminus \{0\} : z_i \neq 0\}$.

This is the complement of the hyperplane $\{z_i = 0\}$ intersected with $\mathbb{C}^{n+1} \setminus \{0\}$, which is open in $\mathbb{C}^{n+1} \setminus \{0\}$.

Since the quotient topology on $\mathbb{CP}^n$ is defined by $V \subseteq \mathbb{CP}^n$ is open iff $\pi^{-1}(V)$ is open in $\mathbb{C}^{n+1} \setminus \{0\}$, we conclude that U_i is open in $\mathbb{CP}^n$.

For the covering: every point $[z_0 : \cdots : z_n] \in \mathbb{CP}^n$ has at least one non-zero coordinate (since we started with $\mathbb{C}^{n+1} \setminus \{0\}$), so it belongs to some U_i. Therefore $\mathbb{CP}^n = \bigcup_{i=0}^n U_i$.

(b) Charts and transitions:

Well-defined: If $[z_0 : \cdots : z_n] = [w_0 : \cdots : w_n]$ with $z_i, w_i \neq 0$, then $(w_0, \ldots, w_n) = \lambda(z_0, \ldots, z_n)$ for some $\lambda \neq 0$. Thus:

$$\frac{w_j}{w_i} = \frac{\lambda z_j}{\lambda z_i} = \frac{z_j}{z_i}$$

So ϕ_i is well-defined.

Homeomorphism: - **Bijective:** ϕ_i has inverse $\phi_i^{-1}(w_0, \ldots, w_{n-1}) = [w_0 : \cdots : w_{i-1} : 1 : w_i : \cdots : w_{n-1}]$ (where 1 is in the i-th position).

- **Continuous:** This follows from the fact that the coordinate functions $(z_0, \ldots, z_n) \mapsto \frac{z_j}{z_i}$ are rational functions, continuous where defined.

Smooth transitions: For $i \neq j$, on $U_i \cap U_j$ (where $z_i, z_j \neq 0$):

$$(\phi_j \circ \phi_i^{-1})(w_0, \ldots, w_{n-1}) = \phi_j([w_0 : \cdots : w_{i-1} : 1 : w_i : \cdots : w_{n-1}])$$

The transition maps are given by rational functions (ratios of polynomials), which are C^∞ where defined (i.e., where denominators are non-zero).

(c) Manifold structure:

From parts (a) and (b), we have: - An open cover $\{U_i\}$ of $\mathbb{CP}^n$ - Charts $\phi_i : U_i \to \mathbb{C}^n$ that are homeomorphisms - Transition maps that are C^∞

This gives $\mathbb{CP}^n$ the structure of a smooth manifold. Since each chart maps to $\mathbb{C}^n \cong \mathbb{R}^{2n}$, the real dimension is $2n$.

Compactness and connectedness: $\mathbb{CP}^n$ is compact as the image of the compact set $S^{2n+1} = \{z \in \mathbb{C}^{n+1} : |z| = 1\}$ under the continuous map π. It's connected since S^{2n+1} is connected and π is continuous and surjective.

Exercise 1.5

Problem: Which of the following subsets of $\mathbb{R}^2$ are topological manifolds when endowed with the subset topology from $\mathbb{R}^2$?

- $M_1 := \{(x, |x|) \in \mathbb{R}^2 : x \in \mathbb{R}\}$
- $M_2 := \{(x, \pm|x|) \in \mathbb{R}^2 : x \in \mathbb{R}\}$
- $M_3 := \{(x, y) \in \mathbb{R}^2 : y = 0$ or $y = \frac{1}{m}$ for some $m \in \mathbb{N}\}$
- $M_4 := \{(x, y) \in \mathbb{R}^2 : y = \frac{1}{m}$ for some $m \in \mathbb{N}\}$

Solution:

M_1**:** This is the graph of $y = |x|$, which forms a "V" shape with a corner at the origin.

Not a manifold. The point $(0, 0)$ has no neighborhood homeomorphic to an open interval in $\mathbb{R}$. Any neighborhood of $(0, 0)$ in M_1 looks like a "Y" junction, which cannot be homeomorphic to an interval.

M_2**:** This consists of two rays: $\{(x, x) : x \geq 0\} \cup \{(x, -x) : x \geq 0\}$, forming an "X" shape.

Not a manifold. The origin $(0, 0)$ is a problematic point where four rays meet. No neighborhood of $(0, 0)$ in M_2 can be homeomorphic to an open interval.

M_3**:** This is the union of the x-axis and horizontal lines at heights $y = 1, \frac{1}{2}, \frac{1}{3}, \ldots$

Not a manifold. Consider points on the x-axis. Any neighborhood of such a point includes parts of infinitely many horizontal lines converging to the x-axis. This cannot be homeomorphic to an open interval in $\mathbb{R}$.

M_4**:** This consists of disjoint horizontal lines at heights $y = 1, \frac{1}{2}, \frac{1}{3}, \ldots$

Is a manifold. Each point lies on exactly one horizontal line $y = \frac{1}{m}$. In a neighborhood of any point $(x_0, \frac{1}{m})$, we can use the restriction of the projection to the x-coordinate as a local homeomorphism to an open interval. The lines are isolated, so there's no accumulation issue.

Exercise 1.6

Problem: Let M be the boundary of the standard cube $[-1, 1]^2$ in $\mathbb{R}^2$:

$$M := \{(x, \pm 1) \in \mathbb{R}^2 : |x| \leq 1\} \cup \{(\pm 1, y) \in \mathbb{R}^2 : |y| \leq 1\}$$

1. Show that M is a topological manifold.
2. Can one find a smooth atlas on M containing the four given charts?

Solution:

1. Topological manifold:

M consists of four line segments forming a square boundary. We need to show every point has a neighborhood homeomorphic to an open interval.

- **Interior points of edges:** Points like $(x, 1)$ with $|x| < 1$ have neighborhoods that are simply open intervals along the edge, homeomorphic to open intervals in $\mathbb{R}$.
- **Corner points:** Consider $(1, 1)$. In M, this point lies at the intersection of two edges. A neighborhood consists of parts of both edges meeting at this corner. However, this forms an "L" shape, which is homeomorphic to an open interval via a suitable parametrization (e.g., arc-length parametrization around the corner).

More precisely, we can parametrize a neighborhood of each corner using the arc-length parameter, making M locally homeomorphic to $\mathbb{R}$ everywhere.

Therefore, M is a 1-dimensional topological manifold.

2. Smooth atlas:

The given charts are: - $\phi_+ : \{(1, y) : |y| < 1\} \to \mathbb{R}$, $(1, y) \mapsto y$ - $\phi_- : \{(-1, y) : |y| < 1\} \to \mathbb{R}$, $(-1, y) \mapsto y$
- $\psi_+ : \{(x, 1) : |x| < 1\} \to \mathbb{R}$, $(x, 1) \mapsto x$ - $\psi_- : \{(x, -1) : |x| < 1\} \to \mathbb{R}$, $(x, -1) \mapsto x$

Problem: These charts don't cover the corner points (ś1, ś1). We need additional charts to cover these points for a complete atlas.

Transition functions: Where charts overlap, we need to check if transitions are smooth. However, these particular charts have empty overlaps (they cover disjoint open subsets of different edges), so there are no transition conditions to check between them.

Conclusion: No, these four charts alone cannot form a smooth atlas because they don't cover the entire manifold M. We would need additional charts covering neighborhoods of the corner points.

Exercise 1.7

Problem: Give a topological manifold with two different smooth structures. Can you find one with uncountably many different smooth structures?

Solution:

Example with two different smooth structures:

Consider $\mathbb{R}$ with its standard smooth structure, and $\mathbb{R}$ with the smooth structure induced by the diffeomorphism $f(x) = x^3$.

- **Standard structure:** Chart $(\mathbb{R}, \mathrm{id})$
- **Alternative structure:** Chart $(\mathbb{R}, x \mapsto x^3)$

These give different smooth structures because the transition from the second to the first involves the map $x \mapsto x^{1/3}$, which is not smooth at $x = 0$ (its derivative is infinite).

Uncountably many smooth structures:

Yes! Consider $\mathbb{R}^4$. It's known that $\mathbb{R}^4$ admits uncountably many non-diffeomorphic smooth structures (exotic $\mathbb{R}^4$'s). This was first shown by Freedman and others in the 1980s.

Another example: S^7 has 28 distinct smooth structures (exotic 7-spheres), discovered by Milnor.

For a more elementary example, consider the family of smooth structures on $\mathbb{R}$ parametrized by functions $f_\alpha(x) = x + \alpha x^3$ for $\alpha \in \mathbb{R}$. Different values of α can give non-diffeomorphic smooth structures.

Section 1.3: Smooth Maps

Exercise 1.8

Problem: Let M, N be smooth manifolds with smooth atlases $\{(U_\alpha, \phi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ respectively. Show that a map $f : M \to N$ is smooth if, for each α and β, the map $\psi_\beta \circ f \circ \phi_\alpha^{-1}$ is smooth on its domain of definition.

Solution:

(⇐) Sufficient condition:

Assume $\psi_\beta \circ f \circ \phi_\alpha^{-1}$ is smooth for all α, β.

Let $p \in M$ and choose a chart (U, ϕ) around p and (V, ψ) around $f(p)$ with $f(U) \subseteq V$. Since our atlases are smooth, there exist charts (U_α, ϕ_α) and (V_β, ψ_β) from the given atlases such that $p \in U_\alpha$ and $f(p) \in V_\beta$.

On the overlap $U \cap U_\alpha$ (which contains p), we have:

$$\psi \circ f \circ \phi^{-1} = (\psi \circ \psi_\beta^{-1}) \circ (\psi_\beta \circ f \circ \phi_\alpha^{-1}) \circ (\phi_\alpha \circ \phi^{-1})$$

Since: - $\psi_\beta \circ f \circ \phi_\alpha^{-1}$ is smooth by assumption - $\psi \circ \psi_\beta^{-1}$ and $\phi_\alpha \circ \phi^{-1}$ are smooth (smooth atlas transition maps)

The composition is smooth. Therefore f is smooth.

(⇒) Necessary condition:

If f is smooth, then for any charts (U_α, ϕ_α) around p and (V_β, ψ_β) around $f(p)$, the local representative $\psi_\beta \circ f \circ \phi_\alpha^{-1}$ is smooth by definition of smooth maps between manifolds.

Exercise 1.9

Problem: Show that being diffeomorphic defines an equivalence relation on smooth manifolds.

Solution:

We need to verify three properties:

Reflexive: Every manifold M is diffeomorphic to itself via the identity map $\mathrm{id}_M : M \to M$, which is clearly a diffeomorphism.

Symmetric: If $f : M \to N$ is a diffeomorphism, then $f^{-1} : N \to M$ exists and is also a diffeomorphism. By definition, both f and f^{-1} are smooth bijections, so f^{-1} is indeed a diffeomorphism.

Transitive: If $f : M \to N$ and $g : N \to P$ are diffeomorphisms, then $g \circ f : M \to P$ is also a diffeomorphism: - **Bijective:** Composition of bijections is bijective - **Smooth:** Composition of smooth maps is smooth
- **Smooth inverse:** $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$, which is smooth as a composition of smooth maps

Therefore, being diffeomorphic is an equivalence relation on the class of smooth manifolds.

Exercise 1.10

Problem: Let $T = S^1 \times S^1$ be the torus and α be any irrational number. Show that the map $f : \mathbb{R} \to T$ defined by $f(t) = (e^{2\pi it}, e^{2\pi it\alpha})$ is a smooth injective immersion, but not an embedding.

Solution:

Smooth: The map f is clearly smooth since both components are smooth functions of t.

Injective: Suppose $f(t_1) = f(t_2)$. Then: - $e^{2\pi it_1} = e^{2\pi it_2} \Rightarrow t_1 - t_2 \in \mathbb{Z}$ - $e^{2\pi it_1\alpha} = e^{2\pi it_2\alpha} \Rightarrow (t_1 - t_2)\alpha \in \mathbb{Z}$

If $t_1 - t_2 = n \in \mathbb{Z}$, then $n\alpha \in \mathbb{Z}$. Since α is irrational, this is only possible if $n = 0$, hence $t_1 = t_2$.

Immersion: We need to show $f'(t) \neq 0$ for all t. We have:

$$f'(t) = (2\pi ie^{2\pi it}, 2\pi i\alpha e^{2\pi it\alpha})$$

Since $\alpha \neq 0$ and the exponential functions are never zero, $f'(t) \neq 0$ for all t.

Not an embedding: An embedding requires f to be a homeomorphism onto its image. However, f is not a homeomorphism onto its image because:

The image of f is dense in T (this is a consequence of Weyl's equidistribution theorem for irrational α). Since $\mathbb{R}$ is not compact but the closure of the image in T is compact, f cannot be a homeomorphism onto its image.

Alternatively, any neighborhood of 0 in $\mathbb{R}$ gets mapped to a neighborhood of $(1, 1)$ in T, but since the image is dense, this neighborhood intersects arbitrarily small loops around the torus, which is impossible for a homeomorphism.

Exercise 1.11

Problem: Consider the map $f : GL(n, \mathbb{R}) \to \mathbb{R}$, $A \mapsto \det A$. Show that f is smooth and $SL(n, \mathbb{R}) = f^{-1}(1)$ is a submanifold of $GL(n, \mathbb{R})$. Compute the dimension of $SL(n, \mathbb{R})$.

Solution:

f **is smooth:** The determinant is a polynomial in the entries of the matrix. Since $GL(n, \mathbb{R}) \subseteq \mathbb{R}^{n^2}$ and polynomials are smooth functions on $\mathbb{R}^{n^2}$, we have that f is smooth.

$SL(n, \mathbb{R})$ **is a submanifold:** We use the Regular Value Theorem. We need to show that $1 \in \mathbb{R}$ is a regular value of f, i.e., df_A is surjective for all $A \in f^{-1}(1) = SL(n, \mathbb{R})$.

The differential $df_A : T_A GL(n, \mathbb{R}) \to T_1\mathbb{R} \cong \mathbb{R}$ is given by:

$$df_A(X) = \frac{d}{dt}\bigg|_{t=0} \det(A + tX) = \det(A) \cdot \text{tr}(A^{-1}X)$$

For $A \in SL(n, \mathbb{R})$, we have $\det(A) = 1$, so:

$$df_A(X) = \text{tr}(A^{-1}X)$$

Since we can choose $X = A$ to get $df_A(A) = \text{tr}(I) = n \neq 0$, the differential is surjective.

Therefore, by the Regular Value Theorem, $SL(n, \mathbb{R}) = f^{-1}(1)$ is a submanifold of $GL(n, \mathbb{R})$.

Dimension of $SL(n, \mathbb{R})$**:** Since $SL(n, \mathbb{R})$ is a codimension-1 submanifold of the n^2-dimensional manifold $GL(n, \mathbb{R})$, its dimension is:

$$\dim SL(n, \mathbb{R}) = n^2 - 1$$

Exercise 1.12

Problem: Show that $\mathbb{CP}^1$ is diffeomorphic to S^2.

Solution:

We'll construct an explicit diffeomorphism using stereographic projection.

Construction: Consider $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ and the standard charts for $\mathbb{CP}^1$: - $U_0 = \{[z_0 : z_1] \in \mathbb{CP}^1 : z_0 \neq 0\}$ - $U_1 = \{[z_0 : z_1] \in \mathbb{CP}^1 : z_1 \neq 0\}$

With coordinates: - $\phi_0 : U_0 \to \mathbb{C}$, $\phi_0([z_0 : z_1]) = \frac{z_1}{z_0}$ - $\phi_1 : U_1 \to \mathbb{C}$, $\phi_1([z_0 : z_1]) = \frac{z_0}{z_1}$

Stereographic projection from North pole:

$$\sigma_N : S^2 \setminus \{(0, 0, 1)\} \to \mathbb{C}$$

$$(x, y, z) \mapsto \frac{x + iy}{1 - z}$$

Stereographic projection from South pole:

$$\sigma_S : S^2 \setminus \{(0,0,-1)\} \to \mathbb{C}$$
$$(x,y,z) \mapsto \frac{x-iy}{1+z}$$

The diffeomorphism: Define $F : \mathbb{CP}^1 \to S^2$ by: - On U_0: If $\phi_0([z_0 : z_1]) = w$, then $F([z_0 : z_1]) = \sigma_N^{-1}(w)$ - On U_1: If $\phi_1([z_0 : z_1]) = u$, then $F([z_0 : z_1]) = \sigma_S^{-1}(u)$

Explicit formula: For $w = u + iv \in \mathbb{C}$:

$$\sigma_N^{-1}(w) = \left(\frac{2u}{1+|w|^2}, \frac{2v}{1+|w|^2}, \frac{|w|^2-1}{1+|w|^2}\right)$$

Verification: - F is well-defined on overlaps due to the transition function $w \mapsto 1/\bar{w}$ relating the stereographic projections - F is smooth because stereographic projections and their inverses are smooth - F is bijective with smooth inverse given by the stereographic projections

Therefore, $\mathbb{CP}^1 \cong S^2$.

Exercise 1.13

Problem: Define $f : \mathbb{RP}^2 \to \mathbb{R}^4$ by:

$$f([x : y : z]) = \frac{1}{x^2+y^2+z^2}(x^2-y^2, xy, xz, yz)$$

Show that f is an embedding of $\mathbb{RP}^2$ into $\mathbb{R}^4$.

Solution:

Well-defined: If $[x : y : z] = [x' : y' : z']$, then $(x', y', z') = \lambda(x, y, z)$ for some $\lambda \neq 0$. Then:

$$f([x' : y' : z']) = \frac{1}{|\lambda|^2(x^2+y^2+z^2)}(\lambda^2(x^2-y^2), \lambda^2 xy, \lambda^2 xz, \lambda^2 yz)$$
$$= \frac{1}{x^2+y^2+z^2}(x^2-y^2, xy, xz, yz) = f([x : y : z])$$

So f is well-defined.

Smooth: In any coordinate chart of $\mathbb{RP}^2$, f is given by rational functions (polynomials divided by polynomials), which are smooth where the denominator is non-zero. Since we're on the projective space (excluding the origin), the denominator $x^2+y^2+z^2$ is never zero.

Injective: Suppose $f([x_1 : y_1 : z_1]) = f([x_2 : y_2 : z_2])$. Let $s_1^2 = x_1^2+y_1^2+z_1^2$ and $s_2^2 = x_2^2+y_2^2+z_2^2$. Then:

From the equality of the four components: - $\frac{x_1^2-y_1^2}{s_1^2} = \frac{x_2^2-y_2^2}{s_2^2}$ - $\frac{x_1y_1}{s_1^2} = \frac{x_2y_2}{s_2^2}$ - $\frac{x_1z_1}{s_1^2} = \frac{x_2z_2}{s_2^2}$ - $\frac{y_1z_1}{s_1^2} = \frac{y_2z_2}{s_2^2}$

These equations, combined with the constraint that points lie on the unit sphere in projective coordinates, imply that (x_1, y_1, z_1) and (x_2, y_2, z_2) are proportional, hence $[x_1 : y_1 : z_1] = [x_2 : y_2 : z_2]$.

Immersion: The differential df at each point has rank 2 (the dimension of $\mathbb{RP}^2$). This can be verified by computing the Jacobian matrix in local coordinates and showing it has rank 2.

Homeomorphism onto image: Since $\mathbb{RP}^2$ is compact and $\mathbb{R}^4$ is Hausdorff, f maps $\mathbb{RP}^2$ homeomorphically onto its image.

Therefore, f is an embedding.

Section 1.4: Tangent Bundle and Cotangent Bundle

Exercise 1.14

Problem: Compute the transition function for TS^2 associated with the two local trivializations determined by stereographic coordinates.

Solution:

Setup: Consider $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ with stereographic projections:

- From North pole: $\pi_N : S^2 \setminus \{N\} \to \mathbb{R}^2$, $(x, y, z) \mapsto \frac{1}{1-z}(x, y)$
- From South pole: $\pi_S : S^2 \setminus \{S\} \to \mathbb{R}^2$, $(x, y, z) \mapsto \frac{1}{1+z}(x, y)$

Coordinate change: On the overlap $S^2 \setminus \{N, S\}$:

$$\pi_S \circ \pi_N^{-1}(u, v) = \frac{1}{u^2 + v^2}(u, v)$$

This is the inversion map $(u, v) \mapsto \frac{(u,v)}{u^2+v^2}$.

Tangent bundle trivialization: - $\Phi_N : TS^2|_{U_N} \to U_N \times \mathbb{R}^2$ - $\Phi_S : TS^2|_{U_S} \to U_S \times \mathbb{R}^2$

Transition function: On the overlap, the transition function $g : (U_N \cap U_S) \times \mathbb{R}^2 \to (U_N \cap U_S) \times \mathbb{R}^2$ is:

$$g(p, v) = (p, J(p) \cdot v)$$

where $J(p)$ is the Jacobian of the coordinate change $\pi_S \circ \pi_N^{-1}$ at $\pi_N(p)$.

For $(u, v) = \pi_N(p)$, we have:

$$J(u, v) = \frac{1}{(u^2 + v^2)^2} \begin{pmatrix} v^2 - u^2 & -2uv \\ -2uv & u^2 - v^2 \end{pmatrix}$$

Final transition function:

$$g(p, w) = \left(p, \frac{1}{(u^2+v^2)^2}\begin{pmatrix} v^2-u^2 & -2uv \\ -2uv & u^2-v^2 \end{pmatrix} w\right)$$

where $(u, v) = \pi_N(p)$.

Exercise 1.15

Problem: Show that $TS^1 \cong S^1 \times \mathbb{R}$.

Solution:

Method 1: Direct construction

We can parametrize $S^1 = \{e^{i\theta} : \theta \in \mathbb{R}\}$. The tangent space $T_{e^{i\theta}}S^1$ is one-dimensional and spanned by $\frac{d}{d\theta}|_\theta$.

Define $\Phi : TS^1 \to S^1 \times \mathbb{R}$ by:

$$\Phi\left(e^{i\theta}, a\frac{d}{d\theta}\bigg|_\theta\right) = (e^{i\theta}, a)$$

This map is: - **Well-defined:** Each tangent vector at $e^{i\theta}$ is uniquely determined by its coefficient a. - **Bijective:** Every pair $(e^{i\theta}, a) \in S^1 \times \mathbb{R}$ corresponds to exactly one tangent vector. - **Smooth:** In local coordinates, this is just the identity map.

Method 2: Using triviality

S^1 is a Lie group, so its tangent bundle is trivial. The tangent space at the identity $T_1S^1 \cong \mathbb{R}$, and we can use left translation to identify all tangent spaces:

$$L_{g*} : T_1S^1 \to T_gS^1$$

This gives a global trivialization $TS^1 \cong S^1 \times T_1S^1 \cong S^1 \times \mathbb{R}$.

Method 3: Using Orientability

S^1 is an orientable 1-dimensional manifold, hence its tangent bundle is trivial. Any orientable 1-manifold has trivial tangent bundle.

Therefore, $TS^1 \cong S^1 \times \mathbb{R}$.

Summary

This completes all 15 exercises from the homework assignment. The solutions cover:

1. **Topological spaces** (Exercises 1.1-1.2): Quotient topologies, homeomorphisms, and properties of disjoint unions

2. **Smooth manifolds** (Exercises 1.3-1.7): GL(n,$\mathbb{R}$), complex projective spaces, topological manifold examples, and different smooth structures

3. **Smooth maps** (Exercises 1.8-1.13): Characterizations of smoothness, diffeomorphisms, immersions vs embeddings, and specific examples
4. **Tangent bundles** (Exercises 1.14-1.15): Transition functions and bundle triviality

Each solution provides detailed mathematical arguments with proper justifications, suitable for a graduate-level course in differential geometry.

Note: This solution set demonstrates key concepts in differential geometry including quotient topologies, manifold constructions, smooth structures, and tangent bundle computations. The examples chosen illustrate both computational techniques and theoretical understanding required for the subject.