Physics LATAM - Riemannian and Complex Geometry

Course: Physics LATAM - Riemannian and Complex Geometry
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This document contains detailed solutions to Assignment #2. The format follows the Homework 1 style: each exercise displays the problem followed by a clear solution. All LaTeX macros are placed inside math delimiters (... or

$$\ldots$$

) so the file compiles cleanly.

Section 2.1: Vector fields and differential forms

Exercise 2.1

Problem: Let $f : M \to N$ be a smooth map between smooth manifolds. Let X_1, X_2 be vector fields on M and Y_1, Y_2 be vector fields on N such that for $i = 1, 2$ the fields are f-related, i.e. $(df)(X_i) = Y_i \circ f$. Show that

$$(df)([X_1, X_2]) = [Y_1, Y_2] \circ f.$$

Solution:

Fix a smooth $g \in C^\infty(N)$. Since X_i and Y_i are f-related, for each i we have

$$X_i(g \circ f) = (Y_i g) \circ f.$$

Compute the Lie bracket acting on the pullback:

$$\begin{aligned}
[X_1, X_2](g \circ f) &= X_1\big(X_2(g \circ f)\big) - X_2\big(X_1(g \circ f)\big) \\
&= X_1\big((Y_2 g) \circ f\big) - X_2\big((Y_1 g) \circ f\big) \\
&= (Y_1(Y_2 g)) \circ f - (Y_2(Y_1 g)) \circ f \\
&= ([Y_1, Y_2] g) \circ f.
\end{aligned}$$

Since this holds for every g, the derivations agree, proving the claimed identity.

Exercise 2.2

Problem: In each case compute $d\omega$ and $F^*\omega$, and verify $F^*(d\omega) = d(F^*\omega)$.

(a) $M = N = \mathbb{R}^2$, $F(s,t) = (st, e^t)$, $\omega = x\,dy$.

Solution:

- $d\omega = dx \wedge dy$.
- Pullbacks: $x \circ F = st$, $y \circ F = e^t$. Hence

$$F^*\omega = (st)\,d(e^t) = ste^t\,dt.$$

- Differentiate:

$$d(F^*\omega) = d(ste^t) \wedge dt = te^t\,ds \wedge dt.$$

- Compute $F^*(d\omega) = F^*(dx \wedge dy) = F^*dx \wedge F^*dy$ with

$$F^*dx = d(st) = t\,ds + s\,dt, \qquad F^*dy = d(e^t) = e^t\,dt.$$

  Therefore

$$(t\,ds + s\,dt) \wedge (e^t\,dt) = te^t\,ds \wedge dt.$$

Thus $F^*(d\omega) = d(F^*\omega)$.

(b) $M = \mathbb{R}^2$, $N = \mathbb{R}^3$,

$$F(\theta, \phi) = ((\cos\phi + 2)\cos\theta, (\cos\phi + 2)\sin\theta, \sin\phi), \qquad \omega = y\,dz \wedge dx.$$

Solution: On $\mathbb{R}^3$ we have $d\omega = dy \wedge dz \wedge dx$. Since $\dim M = 2$, any 3-form on M vanishes, so $d(F^*\omega) = 0$ and $F^*(d\omega) = 0$, hence equality holds.

(c) $M = \{(u,v) \in \mathbb{R}^2 : u^2 + v^2 < 1\}$, $N = \mathbb{R}^3 \setminus \{0\}$,

$$F(u,v) = (u, v, 1 - u^2 - v^2), \qquad \omega = \frac{x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}.$$

Solution (sketch): The 2-form is closed on $\mathbb{R}^3 \setminus \{0\}$ (by homogeneity or direct computation), so $d\omega = 0$. Pullbacks to a 2-dimensional domain yield zero upon applying d, therefore $F^*(d\omega) = d(F^*\omega) = 0$.

Exercise 2.3

Problem: Let $\omega = x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy$ on $\mathbb{R}^3$. Compute ω in spherical coordinates, compute $d\omega$, and restrict to S^2.

Solution:

1. In spherical coordinates (ρ, ϕ, θ) with $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$, one computes

$$\omega = i_R(\text{vol}) = \rho^3 \sin\phi\,d\phi \wedge d\theta,$$

   where $R = \rho\partial_\rho$ and $\text{vol} = dx \wedge dy \wedge dz$.

2. Cartesian: $d\omega = 3\,dx \wedge dy \wedge dz$ (since $d(i_R \mathrm{vol}) = L_R \mathrm{vol} = (\mathrm{div} R)\mathrm{vol}$ and $\mathrm{div} R = 3$). Spherical: from $\omega = \rho^3 \sin\phi\, d\phi \wedge d\theta$ we get

$$d\omega = 3\rho^2 \sin\phi\, d\rho \wedge d\phi \wedge d\theta,$$

which agrees with the Cartesian expression.

3. Restricting to S^2 ($\rho = 1$) gives

$$\iota_{S^2}^* \omega = \sin\phi\, d\phi \wedge d\theta,$$

the standard area form (nowhere-vanishing).

Exercise 2.4

Problem: Exhibit a smooth vector field on S^2 that vanishes at exactly one point.

Solution (construction): Use stereographic projection from the North pole to identify $S^2 \setminus \{N\} \cong \mathbb{R}^2$. Push forward the constant field ∂_x on $\mathbb{R}^2$ to $S^2 \setminus \{N\}$ and set the value at N to be 0. One checks the extension is smooth and has exactly one zero.

Exercise 2.5 (Hairy-ball theorem)

Problem (sketch): Show S^n admits a nowhere-vanishing tangent vector field iff n is odd.

Solution (sketch): One standard chain of equivalences uses the antipodal map and degree theory: the antipodal map has degree $(-1)^{n+1}$, so it preserves orientation iff n is odd; this is equivalent to the existence of a nowhere-vanishing tangent field. See standard references for the full equivalence.

Section 2.2: Orientation

Exercise 2.6

Problem: Decide orientability of $\mathbb{RP}^n$.

Solution: $\mathbb{RP}^n$ is orientable iff n is odd (the antipodal map preserves orientation exactly when n is odd).

Exercise 2.7

Problem: Show $\mathbb{CP}^n$ is orientable.

Solution: Complex manifolds are canonically orientable; charts are holomorphic and transition maps have positive real Jacobian determinants when viewed as maps $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$, so $\mathbb{CP}^n$ is orientable.

Section 2.3: De Rham cohomology

Exercise 2.8

Problem: Let

$$\omega = |x|^{-n} \sum_{i=1}^{n} (-1)^{i-1} x_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$$

be the $(n-1)$-form on $\mathbb{R}^n \setminus \{0\}$. Show ω is closed but not exact.

Solution (outline): ω is the contraction of the Euclidean volume form with the radial vector field divided by $|x|^n$; by homogeneity one checks $d\omega = 0$. Restricting to the unit sphere gives a nonzero top-form with nonzero integral, so ω cannot be exact (Stokes).