12.8.3.8

EE24BTECH11006 - Arnav Mahishi

Question:
Find the area of the smaller region bounded by the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ and the line $\frac{x}{3}+\frac{y}{2}=1$

input	Description	value
a	Length of semi major axis of ellipse	3
b	Length of semi minor axis of ellipse	2
v	Quadratic form of matrix	$\begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix}$
u	Linear coefficient vector	0
f	Constant Term	$-(a^2b^2)$
h	One of the points the line passes through	$\begin{pmatrix} a \\ 0 \end{pmatrix}$
m	Slope of line	$\begin{pmatrix} \frac{1}{b} \\ \frac{-1}{a} \end{pmatrix}$
n	number of subintervals we are taking	1000
x_0	semi-major axis of ellipse	3
x_n	semi-minor axis of ellipse	2

TABLE 0: Variables Used

Theoretical Solution:
The point of intersection of the line with the ellipse is $x_i = h + k_i m$,
where,k_i is a constant and is calculated as follows:-

$$k_i = \frac{1}{m^\top Vm}\left(-m^\top (Vh+u) \pm \sqrt{\left[m^\top (Vh+u)\right]^2 - g(h)\left(m^\top Vm\right)}\right)$$

Substituting the input parameters in k_i,

$$k_i = \frac{1}{\begin{pmatrix} \frac{1}{b} & \frac{-1}{a} \end{pmatrix}\begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix}\begin{pmatrix} \frac{1}{b} \\ \frac{-1}{a} \end{pmatrix}}\left(-\begin{pmatrix} \frac{1}{b} & \frac{-1}{a} \end{pmatrix}\left(\begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix}\begin{pmatrix} a \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) \pm \right.$$
$$\left.\sqrt{\left[\begin{pmatrix} \frac{1}{b} & \frac{-1}{a} \end{pmatrix}\left(\begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix}\begin{pmatrix} a \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)\right]^2 - g(h)\left(\begin{pmatrix} \frac{1}{b} & \frac{-1}{a} \end{pmatrix}\begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix}\begin{pmatrix} \frac{1}{b} \\ \frac{-1}{a} \end{pmatrix}\right)}\right) \quad (0.1)$$

We get,

$$k_i = 0, -ab$$

Substituting k_i in $x_i = h + k_i m$ we get,

$$x_1 = \begin{pmatrix} a \\ 0 \end{pmatrix} + (0)\begin{pmatrix} \frac{1}{b} \\ \frac{-1}{a} \end{pmatrix} \tag{0.2}$$

$$\implies x_1 = \begin{pmatrix} a \\ 0 \end{pmatrix} \tag{0.3}$$

$$x_2 = \begin{pmatrix} a \\ 0 \end{pmatrix} + (-ab)\begin{pmatrix} \frac{1}{b} \\ \frac{-1}{a} \end{pmatrix} \tag{0.4}$$

$$\implies x_2 = \begin{pmatrix} a \\ 0 \end{pmatrix} + \begin{pmatrix} -a \\ b \end{pmatrix} \tag{0.5}$$

$$\implies x_2 = \begin{pmatrix} 0 \\ b \end{pmatrix} \tag{0.6}$$

The area of the smaller region bounded by the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and the line $\frac{x}{3} + \frac{y}{2} = 1$ is

$$= \int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx - \int_0^a \frac{b}{a}(a - x)\,dx \tag{0.7}$$

$$= \frac{b}{a}\left(\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} - ax + \frac{x^2}{2}\right)_0^a \tag{0.8}$$

$$= \frac{b}{a}\left(\frac{\pi a^2}{4} - \frac{a^2}{2}\right) = \frac{ab}{2}\left(\frac{\pi}{2} - 1\right) \tag{0.9}$$

The given area is $\frac{ab}{2}\left(\frac{\pi}{2} - 1\right)$ sq. units
$\therefore$ Upon substituting $a = 3, b = 2$ the given area is $3\left(\frac{\pi}{2} - 1\right)$sq. units ≈ 1.712 sq. units

Computational Solution:
Using the Trapezoidal rule which approximates the integral of a function $f(x)$ over an interval $[a, b]$ by dividing the interval into n subintervals and approximating the area under the curve as a series of trapezoids

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right] \tag{0.10}$$

Where x_0 is semi-major axis of ellipse and x_n is semi-minor axis of the ellipse and h is the width of each subinterval.

$$x_i = x_0 + i \cdot h \tag{0.11}$$

$$h = \frac{b - a}{n} \tag{0.12}$$

In the case of our problem of the area between the line and ellipse the area is computed by:

$$A = \int_{x_{start}}^{x_{end}} \left(f_{ellipse}(x) - f_{line}(x)\right) dx \tag{0.13}$$

$$f_{ellipse}(x) = \sqrt{4\left(1 - \frac{x^2}{9}\right)} \tag{0.14}$$

$$f_{line}(x) = 2 - \frac{2x}{3} \tag{0.15}$$

Where $[x_{start}, x_{end}]$ are the intersection points. Using the difference equation which iteratively computes the integral by adding the contributions of each subinterval. Let A_k represent the area approximation after k subintervals:

$$A_k = A_{k-1} + \frac{h}{2}\left[f(x_k) + f(x_{k-1})\right] \tag{0.16}$$

$$x_k = x_{start} + k \cdot h \tag{0.17}$$

$$\implies A_k = A_{k-1} + \frac{h}{2}\left[\left(f_{ellipse}(x_n) - f_{line}(x_k)\right) + \left(f_{ellipse}(x_{k-1}) - f_{line}(x_{k-1})\right)\right] \tag{0.18}$$

Start with $A_0 = 0$ then iteratively calculate $f_{ellipse}(x_i)$ and $f_{line}(x_i)$ for each i from x_{start} to x_{end} and calculate each A_k from 0 to n. The value of A_n will be the approximate area of the ellipse. As n tends to infinity A_n will be the exact area of the ellipse.

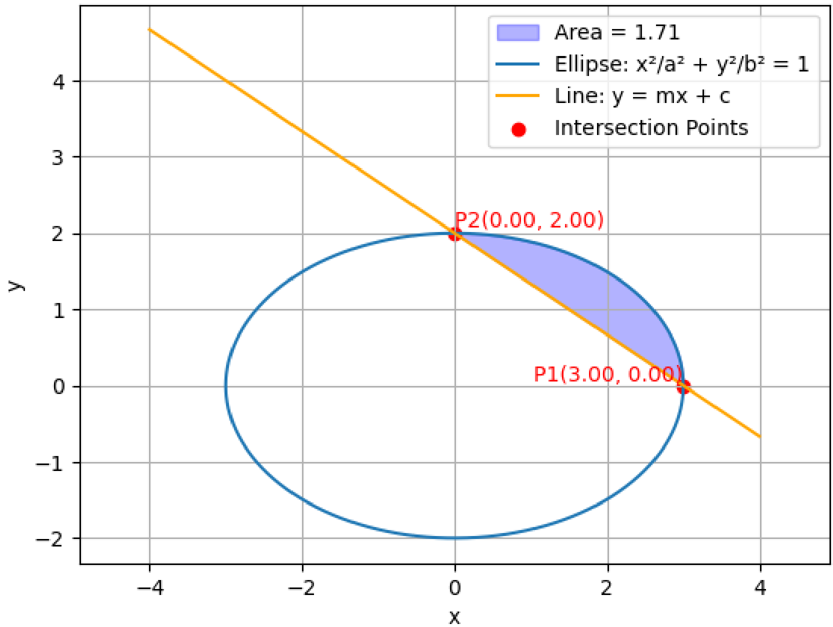

Fig. 0.1: Plot of area enclosed between the line and ellipse