NCERT-10.4.1.2.2

Arnav Mahishi
Dept. of Electrical Engg.
IIT Hyderabad.

January 23, 2025

Problem Statement

The product of two consecutive integers is 306. We need to find the integers.

Input Parameters

Variable	**Description**
x	The bigger integer out of the two we need to find
$g(x)$	The function we take to update x_n in the point iteration method
$x_{\alpha,n}$	The value of x after n iterations for the first root
$x_{\beta,n}$	The value of x after n iterations for the second root

Theoritical Soln

Lets start by assuming the bigger integer as x and the smaller integer as $\frac{306}{x}$

$$\implies x - \frac{306}{x} = 1 \tag{3.1}$$

$$\implies x^2 - 306 = x \tag{3.2}$$

$$\implies x^2 - x - 306 = 0 \tag{3.3}$$

Using the quadratic formula:

$$x = \frac{1 \pm \sqrt{1^2 - (4 \cdot -306)}}{2} \tag{3.4}$$

$$x_1 = \frac{1 + \sqrt{1225}}{2} = 18 \tag{3.5}$$

$$x_2 = \frac{1 - \sqrt{1225}}{2} = -17 \tag{3.6}$$

If $x = 18$ the other integer will be 17 if $x = -17$ the other integer will be -18

$$p(2) = 41 - 72(-2) - 18(-2)^2 = 113 \tag{3.7}$$

$\therefore$ The integers can be $18, 17$ or $-18, -17$

Qr Algorithm

The QR algorithm is as follows: QR decomposition

$$A = QR \tag{3.8}$$

Q is an $m \times n$ orthogonal matrix R is an $n \times n$ upper triangular matrix.
Given a matrix $A = [a_1, a_2, \ldots, a_n]$, where each a_i is a column vector of size $m \times 1$.
Normalize the first column of A:

$$q_1 = \frac{a_1}{\|a_1\|} \tag{3.9}$$

For each subsequent column a_i, subtract the projections of the previously obtained orthonormal vectors from a_i :

$$a'_i = a_i - \sum_{k=1}^{i-1} \langle a_i, q_k \rangle q_k \tag{3.10}$$

Normalize the result to obtain the next column of Q:

$$q_i = \frac{a'_i}{\|a'_i\|} \tag{3.11}$$

Repeat this process for all columns of A.
Finding R:-
After constructing the ortho-normal columns $q_1, q_2, \ldots, q_n$ of Q, we can compute the elements of R by taking the dot product of the original columns of A with the columns of Q:

$$r_{ij} = \langle a_j, q_i \rangle \text{ , for } i \leq j \tag{3.12}$$

Companion Matrix

For the given polynomial equation the companion matrix will be

$$\Lambda = \begin{pmatrix} 0 & 1 \\ 306 & 1 \end{pmatrix} \tag{3.13}$$

Initialization

Let $A_0 = A$, where A is the given matrix. QR Decomposition

For each iteration $k = 0, 1, 2, \ldots$:

Compute the QR decomposition of A_k, such that:

$$A_k = Q_k R_k \tag{3.14}$$

where:

Q_k is an orthogonal matrix ($Q_k^\top Q_k = I$). R_k is an upper triangular matrix.

The decomposition ensures $A_k = Q_k R_k$.

Form the next matrix A_{k+1} as:

$$A_{k+1} = R_k Q_k \tag{3.15}$$

Convergence
Repeat Step 2 until A_k converges to an upper triangular matrix T. The diagonal entries of T are the eigenvalues of A.
The eigenvalues of matrix will be the roots of the equation. We can intorduce a shift to make the convergence faster

$$\sigma = 1 \tag{3.16}$$

$$\Lambda_{\text{shifted}} = \begin{pmatrix} 0 & 1 \\ 306 & 1 \end{pmatrix} - \sigma I \tag{3.17}$$

$$\Lambda_{\text{shifted}} = \begin{pmatrix} -1 & 1 \\ 306 & 0 \end{pmatrix} \tag{3.18}$$

Where σ is the last diagonal element.

In our case:

$$\Lambda = \begin{pmatrix} 0 & 1 \\ 306 & 1 \end{pmatrix} \tag{3.19}$$

$$\implies B = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 306 & 1 \end{pmatrix} \tag{3.20}$$

$$\implies d = \frac{a_{11} - a_{22}}{2} = -0.5 \tag{3.21}$$

$$\implies shift = a_{22} - sgn\,(d)\,\sqrt{d^2 + a_{12}a_{21}} = 18.5 \tag{3.22}$$

$$\hat{A}_0 = A_0 - shift \cdot I = \begin{pmatrix} -18.5 & 1 \\ 306 & -17.5 \end{pmatrix} \tag{3.23}$$

$$\implies Q_0 = \begin{pmatrix} -0.0603 & 0.9982 \\ 0.9982 & 0.0603 \end{pmatrix} \tag{3.24}$$

$$R_0 = \begin{pmatrix} 306.5586 & -17.5285 \\ 0 & -0.0579 \end{pmatrix} \tag{3.25}$$

$$\implies A_1 = R_0 Q_0 + shift \cdot I = \begin{pmatrix} -17.4965 & -304.9422 \\ -0.0578 & 18.4965 \end{pmatrix} \tag{3.26}$$

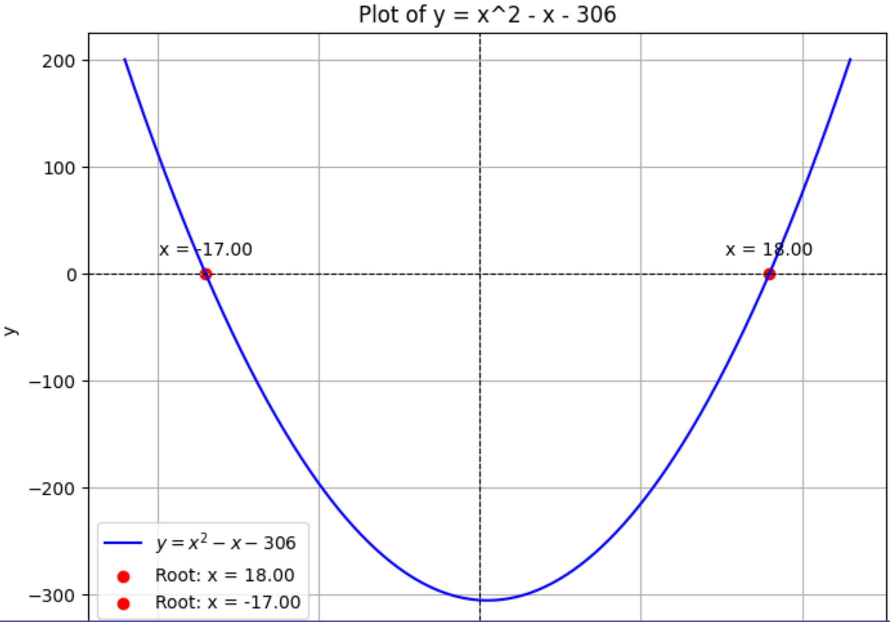
After 19 iterations the matrix converges to

$$A_{19} = \begin{pmatrix} -17 & -305 \\ 4.97 \times 10^{-11} & 18 \end{pmatrix} \tag{3.27}$$

The sub-diagonal elements are nearly zero confirming convergence and the diagonal elements are the eigenvalues. The eigen values given by the code are

$$x_1 = 18.000000000433293, x_2 = -17.000000000043303 \tag{3.28}$$

Plot



Plot of y = x^2 - x - 306
200
100
0
−100
−200
−300
y
x = -17.00
x = 18.00
$y = x^2 - x - 306$
Root: x = 18.00
Root: x = -17.00