12.6.5.6

EE24BTECH11006 - Arnav Mahishi

Question:
Find the maximum profit that a company can make, if the profit function is given by $p(x) = 41 - 72x - 18x^2$

input	Description	value
μ	Size of step	0.01
$p(x)$	The profit function	$41 - 72x - 18x^2$
Tolerance	Maximum gradient until we can say x_n has converged	1e-5
x_0	Inital Guess of maxima	0

TABLE 0: Variables Used

Theoretical Solution:
To find critcal points we equate $\frac{dp(x)}{dx} = 0$. Let $y = p(x)$

$$\frac{dy}{dx} = -72 - 36x \tag{0.1}$$

$$\implies -72 - 36x = 0 \tag{0.2}$$

$$\implies x = -2 \tag{0.3}$$

To find whether $x = -2$ is a maxima or minima we need to take double derivative

$$\frac{d^2y}{dx^2} = -36 \tag{0.4}$$

As the double derivative is negative for all x so the point at $x = -2$ is a maxima

$$p(2) = 41 - 72(-2) - 18(-2)^2 = 113 \tag{0.5}$$

$\therefore$ The maximum profit the company can make is 113 at $x = -2$
Computational Solution:
We use the method of gradient descent to find the local maximum of the given function. Since the coefficient of $\left(x^2 < 0\right)$, the function is concave down, and we expect to find a local maximum. Hence we apply gradient ascent. The iterative formula (Difference Equation) is as follows:

$$x_{n+1} = x_n + \mu f'(x_n) \tag{0.6}$$

$$f'(x_n) = -72 - 36x_n \tag{0.7}$$

$$\implies x_{n+1} = x_n + \mu(-72 - 36x_n) \tag{0.8}$$

$$= (1 - 36\mu)x_n - 72\mu \tag{0.9}$$

Applying Unilateral Z-transform,

$$zX(z) - zx_0 = (1-36\mu)X(z) - 72\mu\frac{z}{z-1} \tag{0.10}$$

$$(z-(1-36\mu))X(z) = zx_0 - 72\mu\frac{z}{z-1} \tag{0.11}$$

$$X(z) = \frac{zx_0}{z-(1-36\mu)} - \frac{72\mu z}{(z-1)(z-(1-36\mu))} \tag{0.12}$$

The ROC is determined by the stability condition:

$$|1-36\mu| < 1 \tag{0.13}$$

$$\implies -1 < 1-36\mu < 1 \tag{0.14}$$

$$\implies 0 < \mu < \frac{1}{18} \tag{0.15}$$

If μ satisfies the previous condition,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0 \tag{0.16}$$

$$\implies \lim_{n\to\infty} \|\mu(-72-36x_n)\| = 0 \tag{0.17}$$

$$\implies -72\mu - 36\mu \lim_{x\to\infty} \|x_n\| = 0 \tag{0.18}$$

$$\implies \lim_{x\to\infty} \|x_n\| = -2 \tag{0.19}$$

Choosing a step-size in the ROC ($\mu = 0.01$), initial guess $x_0 = 0$ and toleance 1e-5, we perform $x_{n+1} = (1-36\mu)x_n - 72\mu$ until $f'(x)$ is less than the tolerance and we get x_n to be the local maxima. After convergence we get:

$$x_n = -2 \tag{0.20}$$

$$\implies p(x_n) = p(-2) = 41 - 72(-2) - 18(-2)^2 = 113 \tag{0.21}$$

$\therefore$ The maximum profit the company can make is 113

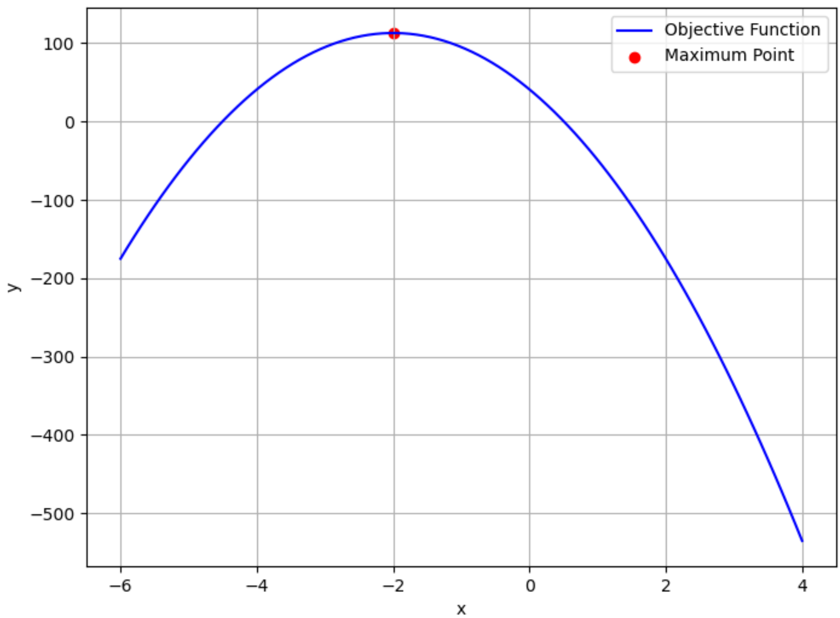

Fig. 0.1: Maximum Value of Objective Function