NCERT-12.9.1.7

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Problem Statement

Solve the differential equation $y''' + 2y'' + y' = 0$ with initial conditions $y\left(0\right) = 1$,$y'\left(0\right) = -1$, and $y''\left(0\right) = 1$

Input Parameters

Variable	**Description**
n	Order of given differential equation
y_i	ith derivative of the function in the equation
c	constant in the equation
a_i	coefficient of ith derivative of the function in the equation
$\mathbf{V}(t)$	Vector containing all 1 and y_i from $i = 0$ to $i = n - 1$
$\mathbf{V}'(t)$	Vector containing 1 and y_i' from $i = 0$ to $i = n - 1$
A	the coefficient matrix that transforms each y_i to its derivative
h	the stepsize between each t we are taking
t_o	The start time from which we are plotting
t_f	The end time at which we stop plotting

Laplace Transforms

We apply the Laplace transform to each term in the equation. The Laplace transforms for the derivatives of $y\left(t\right)$ are:

$$\mathcal{L}y'\left(t\right) = sY\left(s\right) - y\left(0\right) \tag{3.1}$$

$$\mathcal{L}y''\left(t\right) = s^2Y\left(s\right) - sy\left(0\right) - y'\left(0\right) \tag{3.2}$$

$$\mathcal{L}y'''\left(t\right) = s^3Y\left(s\right) - s^2y\left(0\right) - sy'\left(0\right) - y''\left(0\right) \tag{3.3}$$

Simplification and Substitution

$$\left(s^3 + 2s^2 + s\right) Y(s) = s^2 + s \tag{3.4}$$

$$Y(s) = \frac{s^2 + s}{s(s+1)^2} \tag{3.5}$$

$$\implies Y(s) = \frac{1}{s+1} \tag{3.6}$$

Now, take the inverse Laplace transform:

$$\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) = e^{-t} \tag{3.7}$$

Computational Solution

y_i is the ith derivative of the function then

$$\begin{pmatrix} y_0' \\ y_1' \\ y_2' \\ \vdots \\ y_{n-1}' \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ \frac{-\left(\sum_{i=0}^{i=n-1} a_i y_i\right) - c}{a_n} \end{pmatrix} \tag{3.8}$$

$$\implies \begin{pmatrix} 1 \\ y_0' \\ y_1' \\ y_2' \\ \vdots \\ y_{n-1}' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 1 & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \frac{-c}{a_n} & \frac{-a_0}{a_n} & \frac{-a_1}{a_n} & \frac{-a_2}{a_n} & \cdots & \cdots & \frac{-a_{n-1}}{a_n} \end{pmatrix} \begin{pmatrix} 1 \\ y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{pmatrix} \tag{3.9}$$

$$\implies \mathbf{y}_k' = A\mathbf{y}_k \tag{3.10}$$

Difference Equation

At any n, by defining derivative:

$$y'_{n,k} = \lim_{h \to 0} \frac{y_{n,k+1} - y_{n,k}}{h} \tag{3.11}$$

$$y_{n,k+1} = y_{n,k} + hy'_{n,k} \tag{3.12}$$

$$\implies y_{0,k+1} = y_{0,k} + hy'_{0,k} \tag{3.13}$$

$$\implies y_{1,k+1} = y_{1,k} + hy'_{1,k} \tag{3.14}$$

$$\implies y_{2,k+1} = y_{2,k} + hy'_{2,k} \tag{3.15}$$

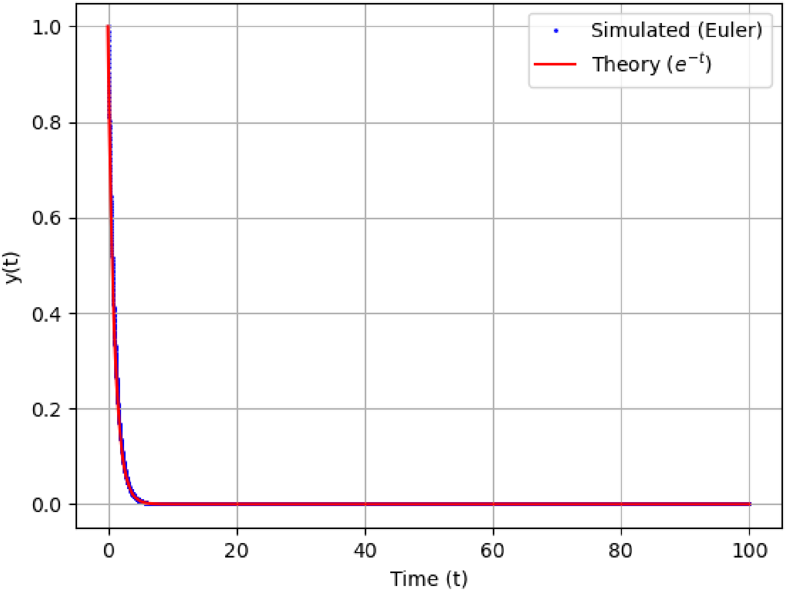
$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \tag{3.16}$$

$$\implies y_{n-1,k+1} = y_{n-1,k} + h\left(\frac{-\left(\sum_{i=0}^{i=n-1} a_i y_i\right) - c}{a_n}\right) \tag{3.17}$$

$$\implies \mathbf{y}_{k+1} = \mathbf{y}_k + h\left(A\mathbf{y}_k\right) \tag{3.18}$$

Plot

When k ranges from 0 to $\frac{t_o - t_f}{h}$ in increments of 1, discretizing the steps gives us all $\mathbf{y}_k$, Record the $y_{0,k}$ for each k we got and then plot the graph. The result will be as given below.


1.0
0.8
0.6
0.4
0.2
0.0
y(t)
0
20
40
60
80
100
Time (t)
Simulated (Euler)
Theory (e^{-t})

Codes

Code: https://github.com/arnavmahishi
/EE1003/tree/main/assignments/Problem%201/codes