12.6.5.6

EE24BTECH11006 - Arnav Mahishi

Question:
Find the maximum profit that a company can make, if the profit function is given by $p(x) = 41 - 72x - 18x^2$

input	Description	value
μ	Size of step	0.01
$p(x)$	The profit function	$41 - 72x - 18x^2$
Tolerance	Maximum gradient until we can say x_n has converged	1e-5
x_0	Inital Guess of maxima	0

TABLE 0: Variables Used

Theoretical Solution:
To find critcal points we equate $\frac{dp(x)}{dx} = 0$. Let $y = p(x)$

$$\frac{dy}{dx} = -72 - 36x \tag{0.1}$$
$$\implies -72 - 36x = 0 \tag{0.2}$$
$$\implies x = -2 \tag{0.3}$$

To find whether $x = -2$ is a maxima or minima we need to take double derivative

$$\frac{d^2y}{dx^2} = -36 \tag{0.4}$$

As the double derivative is negative for all x so the point at $x = -2$ is a maxima

$$p(2) = 41 - 72(-2) - 18(-2)^2 = 113 \tag{0.5}$$

$\therefore$ The maximum profit the company can make is 113 at $x = -2$

Computational Solution:
We use the method of gradient descent to find the local maximum of the given function. Since the coefficient of $\left(x^2 < 0\right)$, the function is concave down, and we expect to find a local maximum. Hence we apply gradient ascent. The iterative formula (Difference Equation) is as follows:

$$x_{n+1} = x_n + \mu f'(x_n) \tag{0.6}$$
$$f'(x_n) = -72 - 36x_n \tag{0.7}$$
$$\implies x_{n+1} = x_n + \mu(-72 - 36x_n) \tag{0.8}$$
$$= (1 - 36\mu)x_n - 72\mu \tag{0.9}$$

Applying Unilateral Z-transform,

$$zX(z) - zx_0 = (1 - 36\mu) X(z) - 72\mu \frac{z}{z-1} \tag{0.10}$$

$$(z - (1 - 36\mu)) X(z) = zx_0 - 72\mu \frac{z}{z-1} \tag{0.11}$$

$$X(z) = \frac{zx_0}{z - (1 - 36\mu)} - \frac{72\mu z}{(z-1)(z - (1 - 36\mu))} \tag{0.12}$$

The ROC is determined by the stability condition:

$$|1 - 36\mu| < 1 \tag{0.13}$$

$$\implies -1 < 1 - 36\mu < 1 \tag{0.14}$$

$$\implies 0 < \mu < \frac{1}{18} \tag{0.15}$$

If μ satisfies the previous condition,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0 \tag{0.16}$$

$$\implies \lim_{n\to\infty} \|\mu(-72 - 36x_n)\| = 0 \tag{0.17}$$

$$\implies -72\mu - 36\mu \lim_{x\to\infty} \|x_n\| = 0 \tag{0.18}$$

$$\implies \lim_{x\to\infty} \|x_n\| = -2 \tag{0.19}$$

Choosing a step-size in the ROC ($\mu = 0.01$), initial guess $x_0 = 0$ and tolerance 1e-5, we perform $x_{n+1} = (1 - 36\mu) x_n - 72\mu$ until $f'(x)$ is less than the tolerance and we get x_n to be the local maxima. After convergence we get:

$$x_{min} = -1.999997 \tag{0.20}$$

We can pose the question as the following quadratic programming question. Find the point lying on the line $y = 1$, which is nearest to the origin
We can formulate the problem as follows:

$$\min_{\mathbf{x}} \left\| e_2^{\top} \mathbf{x} \right\|^2 \tag{0.21}$$

$$\text{s.t.} \tag{0.22}$$

$$\mathbf{x}^{\top} V \mathbf{x} + 2\mathbf{u}^{\top} \mathbf{x} + f = 0 \tag{0.23}$$

$$V = \begin{pmatrix} -18 & 0 \\ 0 & 0 \end{pmatrix} \tag{0.24}$$

$$\mathbf{u} = \begin{pmatrix} -36 \\ -0.5 \end{pmatrix} \tag{0.25}$$

$$f = 0 \tag{0.26}$$

In the current form, the constraint is non-convex since the constraint defines a set which is not convex, since points on the line joining any 2 points on the curve don't belong to the set. However, if we become lenient and make the constraint

$$\mathbf{x}^\top V\mathbf{x} + 2\mathbf{u}^\top\mathbf{x} + f \leq 0 \tag{0.27}$$

The constraint becomes convex. Using scipy.optimize to solve this convex optimization problem, we get

$$\text{Optimal x} : [[-2.00000002] \tag{0.28}$$
$$[113.000000006047071]] \tag{0.29}$$

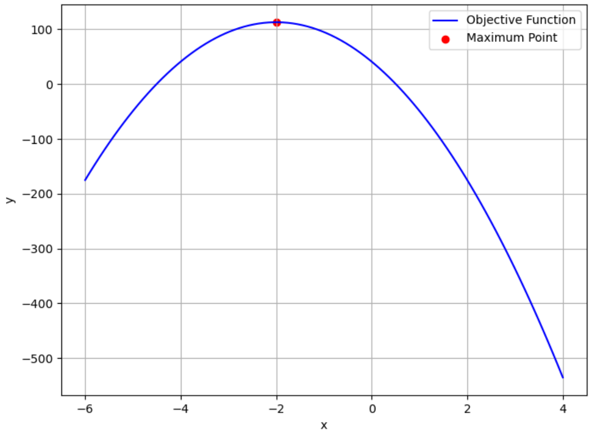

Fig. 0.1: Solution to set of linear equations