NCERT-12.9.1.7

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Problem Statement

Solve the differential equation $y''' + 2y'' + y' = 0$ with initial conditions $y\left(0\right) = 1$,$y'\left(0\right) = -1$, and $y''\left(0\right) = 1$

Input Parameters

Variable	Description
n	Order of given differential equation
y_i	ith derivative of the function in the equation
c	constant in the equation
a_i	coefficient of ith derivative of the function in the equation
$\mathbf{y}(t)$	Vector containing all 1 and y_i from $i = 0$ to $i = n - 1$
$\mathbf{y}'(t)$	Vector containing 1 and y_i' from $i = 0$ to $i = n - 1$
A	the coefficient matrix that transforms each y_i to its derivative
h	the stepsize between each t we are taking
t_o	The start time from which we are plotting
t_f	The end time at which we stop plotting

Laplace Transforms

We apply the Laplace transform to each term in the equation. The Laplace transforms for the derivatives of $y\left(t\right)$ are:

$$\mathcal{L}y'\left(t\right) = sY\left(s\right) - y\left(0\right) \tag{3.1}$$

$$\mathcal{L}y''\left(t\right) = s^2Y\left(s\right) - sy\left(0\right) - y'\left(0\right) \tag{3.2}$$

$$\mathcal{L}y'''\left(t\right) = s^3Y\left(s\right) - s^2y\left(0\right) - sy'\left(0\right) - y''\left(0\right) \tag{3.3}$$

Simplification and Substitution

$$\left(s^3 + 2s^2 + s\right) Y(s) = s^2 + s \tag{3.4}$$

$$Y(s) = \frac{s^2 + s}{s(s+1)^2} \tag{3.5}$$

$$\implies Y(s) = \frac{1}{s+1} \tag{3.6}$$

Now, take the inverse Laplace transform:

$$\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) = e^{-t} \tag{3.7}$$

Radius of Convergence

Radius of Convergence:
The denominator indicates a pole at $s = -1$.To ensure convergence of the Laplace transform integral, the real part of s must satsify:

$$Re\,(s) > -1 \tag{3.8}$$

Since the ROC extends infinitely to the right in the s-plane, the radius of convergence is:

$$R = \infty \tag{3.9}$$

Computational Solution

y_i is the ith derivative of the function then

$$\begin{pmatrix} y_0' \\ y_1' \\ y_2' \\ \vdots \\ y_{n-1}' \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ \frac{-\left(\sum_{i=0}^{i=n-1} a_i y_i\right) - c}{a_n} \end{pmatrix} \tag{3.10}$$

$$\implies \begin{pmatrix} 1 \\ y_0' \\ y_1' \\ y_2' \\ \vdots \\ y_{n-1}' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 1 & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \frac{-c}{a_n} & \frac{-a_0}{a_n} & \frac{-a_1}{a_n} & \frac{-a_2}{a_n} & \cdots & \cdots & \frac{-a_{n-1}}{a_n} \end{pmatrix} \begin{pmatrix} 1 \\ y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{pmatrix} \tag{3.11}$$

$$\implies \mathbf{y}_k' = A\mathbf{y}_k \tag{3.12}$$

Trapezoidal Rule

Using the trapezoidal rule,

$$J = \int_a^b f(x)\, dx \tag{3.13}$$

$$\approx h\left(\frac{1}{2}f(x) + f(x_1) + f(x_2)\cdots + f(x_{n-1}) + \frac{1}{2}f(b)\right) \tag{3.14}$$

$$\frac{\mathbf{y}_{k+1} - \mathbf{y}_k}{h} = A \cdot \frac{\mathbf{y}_{k+1} + \mathbf{y}_k}{2} \tag{3.15}$$

$$\rightarrow \mathbf{y}_{k+1} = \left(I - \frac{h}{2}A\right)^{-1} \cdot \left(I + \frac{h}{2}A\right) \cdot \mathbf{y}_k \tag{3.16}$$

Bilnear Transform

Apply Bilinear Transform with $T = h$ on laplace transform above

$$s = \frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}} \tag{3.17}$$

$$Y(z) = \frac{1}{\frac{2}{T} \cdot \frac{1-z^{-1}}{1+z^{-1}} + 1} \tag{3.18}$$

$$Y(z) = \frac{1 + z^{-1}}{\frac{2}{h}\left(1 - z^{-1}\right) + \left(1 + z^{-1}\right)} \tag{3.19}$$

$$\implies Y(z) = \frac{1 + z^{-1}}{\left(1 + \frac{2}{h}\right) - \left(1 - \frac{2}{h}\right) z^{-1}} \tag{3.20}$$

$$\implies \text{Taking } \alpha = \frac{1 - \frac{2}{h}}{1 + \frac{2}{h}} \tag{3.21}$$

$$\implies Y(z) = \frac{(1 - \alpha)\left(1 + z^{-1}\right)}{2\left(1 + \alpha z^{-1}\right)} \tag{3.22}$$

$$\left(1 + \alpha z^{-1}\right) Y(z) = \frac{1 - \alpha}{2}\left(1 + z^{-1}\right) \tag{3.23}$$

Applying Inverse Z transform, we get the following difference equation

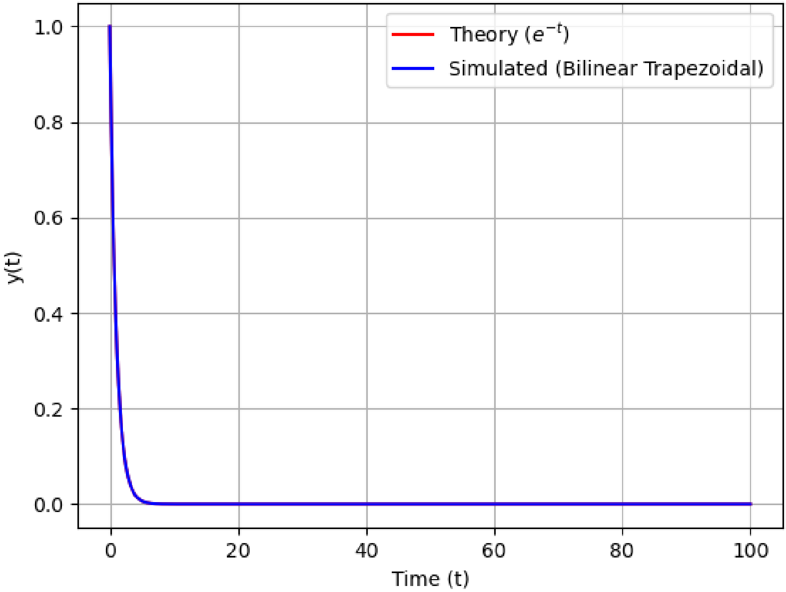
$$y_n = \frac{1-\alpha}{2}\left(1 + y_{n-1}\right) \tag{3.25}$$

$$\implies y_n = \frac{2}{1-\alpha}\delta\left(n\right) + \frac{2}{1-3\alpha}y_{n-1}, |\alpha| < 1 \tag{3.26}$$

The ROC of the z transform is 1 as $|\alpha| < 1$

Plot

When k ranges from 0 to $\frac{t_o - t_f}{h}$ in increments of 1, discretizing the steps gives us all $\mathbf{y}_k$, Record the $y_{0,k}$ for each k we got and then plot the graph. The result will be as given below.


Theory (e^{-t})
Simulated (Bilinear Trapezoidal)
1.0
0.8
0.6
0.4
0.2
0.0
y(t)
0
20
40
60
80
100
Time (t)

As you can see the graph of bilnear transform is much more accurate compared to the one formed using euler