

NCERT-12.9.1.7

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January 16, 2025

1 Problem

2 Solution

- Input Parameters
- Laplace Transforms
- Simplification and Substitution
- Radius of Convergence
- Computational Solution
- Trapezoidal Rule
- Bilinear Transform

Problem Statement

Solve the differential equation $y''' + 2y'' + y' = 0$ with initial conditions $y(0) = 1, y'(0) = -1$, and $y''(0) = 1$

Input Parameters

Variable	Description
n	Order of given differential equation
y_i	i th derivative of the function in the equation
c	constant in the equation
a_i	coefficient of i th derivative of the function in the equation
$\mathbf{y}(t)$	Vector containing all 1 and y_i from $i = 0$ to $i = n - 1$
$\mathbf{y}'(t)$	Vector containing 1 and y'_i from $i = 0$ to $i = n - 1$
A	the coefficient matrix that transforms each y_i to its derivative
h	the stepsize between each t we are taking
t_o	The start time from which we are plotting
t_f	The end time at which we stop plotting

Laplace Transforms

We apply the Laplace transform to each term in the equation. The Laplace transforms for the derivatives of $y(t)$ are:

$$\mathcal{L}y'(t) = sY(s) - y(0) \quad (3.1)$$

$$\mathcal{L}y''(t) = s^2Y(s) - sy(0) - y'(0) \quad (3.2)$$

$$\mathcal{L}y'''(t) = s^3Y(s) - s^2y(0) - sy'(0) - y''(0) \quad (3.3)$$

Simplification and Substitution

$$(s^3 + 2s^2 + s) Y(s) = s^2 + s \quad (3.4)$$

$$Y(s) = \frac{s^2 + s}{s(s+1)^2} \quad (3.5)$$

$$\Rightarrow Y(s) = \frac{1}{s+1} \quad (3.6)$$

Now, take the inverse Laplace transform:

$$\mathcal{L}^{-1} \left(\frac{1}{s+1} \right) = e^{-t} \quad (3.7)$$

Radius of Convergence

Radius of Convergence:

The denominator indicates a pole at $s = -1$. To ensure convergence of the Laplace transform integral, the real part of s must satisfy:

$$\operatorname{Re}(s) > -1 \quad (3.8)$$

Since the ROC extends infinitely to the right in the s -plane, the radius of convergence is:

$$R = \infty \quad (3.9)$$

Computational Solution

y_i is the i th derivative of the function then

$$\begin{pmatrix} y'_0 \\ y'_1 \\ y'_2 \\ \vdots \\ y'_{n-1} \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ -\left(\sum_{i=0}^{n-1} a_i y_i\right) - c \\ a_n \end{pmatrix} \quad (3.10)$$

$$\Rightarrow \begin{pmatrix} 1 \\ y'_0 \\ y'_1 \\ y'_2 \\ \vdots \\ y'_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 1 & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \frac{-c}{a_n} & \frac{-a_0}{a_n} & \frac{-a_1}{a_n} & \frac{-a_2}{a_n} & \cdots & \cdots & \frac{-a_{n-1}}{a_n} \end{pmatrix} \begin{pmatrix} 1 \\ y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{pmatrix} \quad (3.11)$$

$$\Rightarrow \mathbf{y}'_k = \mathbf{A} \mathbf{y}_k \quad (3.12)$$

Trapezoidal Rule

Using the trapezoidal rule,

$$J = \int_a^b f(x) dx \quad (3.13)$$

$$\approx h \left(\frac{1}{2} f(x) + f(x_1) + f(x_2) \cdots + f(x_{n-1}) + \frac{1}{2} f(b) \right) \quad (3.14)$$

$$\frac{\mathbf{y}_{k+1} - \mathbf{y}_k}{h} = A \cdot \frac{\mathbf{y}_{k+1} + \mathbf{y}_k}{2} \quad (3.15)$$

$$\rightarrow \mathbf{y}_{k+1} = \left(I - \frac{h}{2} A \right)^{-1} \cdot \left(I + \frac{h}{2} A \right) \cdot \mathbf{y}_k \quad (3.16)$$

Bilinear Transform

Apply Bilinear Transform with $T = h$ on laplace transform above

$$s = \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}} \quad (3.17)$$

$$Y(z) = \frac{1}{\frac{2}{T} \cdot \frac{1 - z^{-1}}{1 + z^{-1}} + 1} \quad (3.18)$$

$$Y(z) = \frac{1 + z^{-1}}{\frac{2}{h} (1 - z^{-1}) + (1 + z^{-1})} \quad (3.19)$$

$$\Rightarrow Y(z) = \frac{1 + z^{-1}}{\left(1 + \frac{2}{h}\right) - \left(1 - \frac{2}{h}\right) z^{-1}} \quad (3.20)$$

$$\Rightarrow \text{Taking } \alpha = \frac{1 - \frac{2}{h}}{1 + \frac{2}{h}} \quad (3.21)$$

$$\Rightarrow Y(z) = \frac{(1 - \alpha)(1 + z^{-1})}{2(1 + \alpha z^{-1})} \quad (3.22)$$

$$(1 + \alpha z^{-1}) Y(z) = \frac{1 - \alpha}{2} (1 + z^{-1}) \quad (3.23)$$

Applying Inverse Z transform, we get the following difference equation

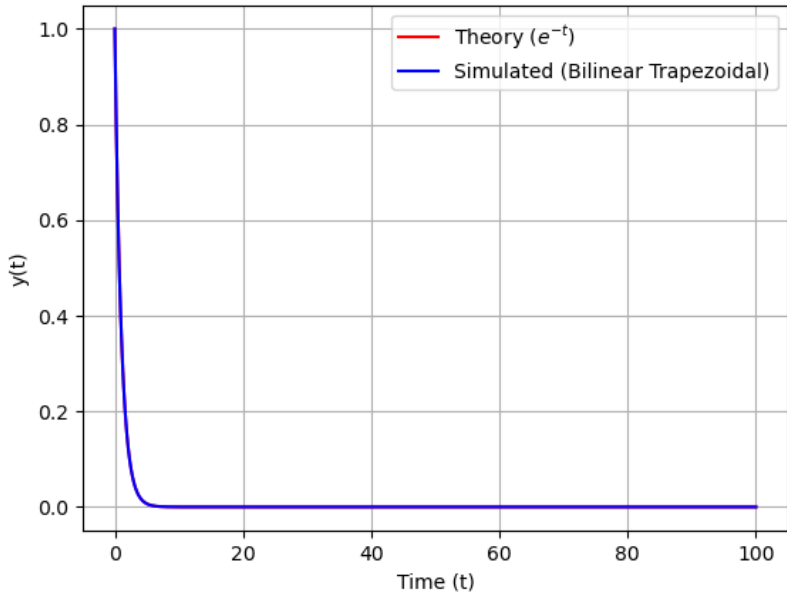
$$y_n = \frac{1-\alpha}{2} (1 + y_{n-1}) \quad (3.25)$$

$$\Rightarrow y_n = \frac{2}{1-\alpha} \delta(n) + \frac{2}{1-3\alpha} y_{n-1}, |\alpha| < 1 \quad (3.26)$$

The ROC of the z transform is $|Z| > |\alpha|$

Plot

When k ranges from 0 to $\frac{t_o - t_f}{h}$ in increments of 1, discretizing the steps gives us all \mathbf{y}_k . Record the $y_{0,k}$ for each k we got and then plot the graph. The result will be as given below.



As you can see the graph of bilnear transform is much more accurate compared to the one formed using euler

