9.1.7

EE24BTECH11006 - Arnav Mahishi

Question:
Solve the differential equation $y'''+2y''+y' = 0$ with initial conditions $y(0) = 1$,$y'(0) = -1$, and $y''(0) = 1$
Solution:

Variable	Description
n	Order of given differential equation
y_i	ith derivative of the function in the equation
c	constant in the equation
a_i	coefficient of ith derivative of the function in the equation
$\mathbf{V}(t)$	Vector containing all 1 and y_i from $i = 0$ to $i = n-1$
$\mathbf{V}'(t)$	Vector containing 1 and y_i' from $i = 0$ to $i = n-1$
A	the coefficient matrix that transforms each y_i to its derivative
h	the stepsize between each t we are taking
t_o	The start time from which we are plotting
t_f	The end time at which we stop plotting

TABLE 0: Variables Used

Theoretical Solution: We apply the Laplace transform to each term in the equation. The Laplace transforms for the derivatives of $y(t)$ are:

$$\mathcal{L}y'(t) = sY(s) - y(0) \tag{0.1}$$

$$\mathcal{L}y''(t) = s^2Y(s) - sy(0) - y'(0) \tag{0.2}$$

$$\mathcal{L}y'''(t) = s^3Y(s) - s^2y(0) - sy'(0) - y''(0) \tag{0.3}$$

Now, applying the Laplace transform to the entire differential equation:

$$\mathcal{L}\{y''' + 2y'' + y'\} = 0 \tag{0.4}$$

$$\mathcal{L}\{y'''(t)\} + 2\mathcal{L}\{y''(t)\} + \mathcal{L}\{y'(t)\} = 0 \tag{0.5}$$

$$(s^3Y(s) - s^2y(0) - sy'(0) - y''(0)) + 2(s^2Y(s) - sy(0) - y'(0)) + (sY(s) - y(0)) = 0 \tag{0.6}$$

Substitute the initial conditions $y(0) = 1$, $y'(0) = -1$, and $y''(0) = 1$:

$$\left(s^3Y(s) - s^2 \cdot 1 - s \cdot (-1) - 1\right) + 2\left(s^2Y(s) - s \cdot 1 - (-1)\right) + (sY(s) - 1) = 0 \tag{0.7}$$

$$s^3Y(s) - s^2 + s - 1 + 2s^2Y(s) - 2s + 2 + sY(s) - 1 = 0 \tag{0.8}$$

Simplify the equation:

$$
\begin{aligned}
(s^3+2s^2+s)Y(s)-(s^2-s+1)-(2s-2)-1 &= 0 &(0.9)\\
(s^3+2s^2+s)Y(s)-s^2-s+1-2s+2-1 &= 0 &(0.10)\\
(s^3+2s^2+s)Y(s)-(s^2+s) &= 0 &(0.11)
\end{aligned}
$$

Now, solve for $Y(s)$:

$$\left(s^3+2s^2+s\right)Y(s)=s^2+s \tag{0.12}$$

$$Y(s)=\frac{s^2+s}{s(s+1)^2} \tag{0.13}$$

$$\implies Y(s)=\frac{1}{s+1} \tag{0.14}$$

Now, take the inverse Laplace transform:

$$\mathcal{L}^{-1}\left(\frac{1}{s+1}\right)=e^{-t} \tag{0.15}$$

Thus, the solution to the differential equation is:

$$y(t)=e^{-t} \tag{0.16}$$

Computational Solution:
Consider the given linear differential equation

$$a_n y_n + a_{n-1}y_{n-1}+\cdots+a_1y_1+a_0y_0+c=0 \tag{0.17}$$

Where y_i is the ith derivative of the function then

$$\begin{pmatrix} y_0' \\ y_1' \\ y_2' \\ \vdots \\ y_{n-1}' \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ \frac{-\left(\sum_{i=0}^{i=n-1} a_i y_i\right)-c}{a_n} \end{pmatrix} \tag{0.18}$$

$$\implies \begin{pmatrix} 1 \\ y_0' \\ y_1' \\ y_2' \\ \vdots \\ y_{n-1}' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 1 & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \frac{-c}{a_n} & \frac{-a_0}{a_n} & \frac{-a_1}{a_n} & \frac{-a_2}{a_n} & \cdots & \cdots & \frac{-a_{n-1}}{a_n} \end{pmatrix} \begin{pmatrix} 1 \\ y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{pmatrix} \tag{0.19}$$

$$\implies \mathbf{V}'(t)=A\mathbf{V}(t) \tag{0.20}$$

Where $\mathbf{V}(t)$ is the vector $\begin{pmatrix} 1 \\ y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{pmatrix}$ at a t. At any N, by defining derivative:

$$y'_N(t) = \lim_{h \to 0} \frac{y_N(t+h) - y_N(t)}{h} \tag{0.21}$$

$$y_N(t+h) = y_N(t) + h y'_N(t) \tag{0.22}$$

$$\implies y_0(t+h) = y_0(t) + h y'_0(t) \tag{0.23}$$

$$\implies y_1(t+h) = y_1(t) + h y'_1(t) \tag{0.24}$$

$$\implies y_2(t+h) = y_2(t) + h y'_2(t) \tag{0.25}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \tag{0.26}$$

$$\implies y_{n-1}(t+h) = y_{n-1}(t) + h y'_{n-1}(t) \tag{0.27}$$

$$\implies \mathbf{V}(t+h) = \mathbf{V}(t) + h(A\mathbf{V}(t)) \tag{0.28}$$

When t ranges from t_o to t_f in increments of h, discretizing the steps gives us all $\mathbf{V}(x)$ from t_o to t_f in increments of h Record the y_0 for each $\mathbf{V}(x)$ we got and then plot the graph. The result will be as given below.

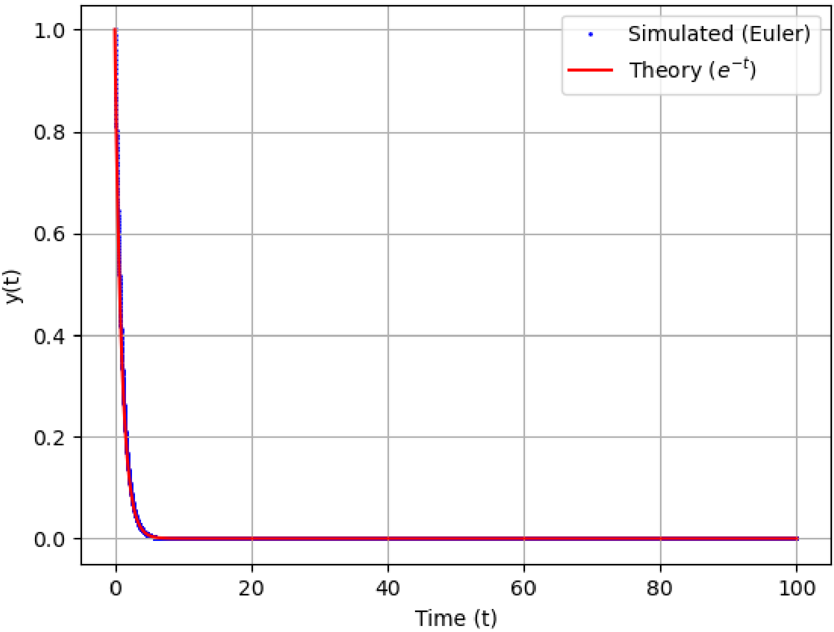

Fig. 0.1: Comparison between the Theoretical solution and Computational solution