10.4.1.2.2

EE24BTECH11006 - Arnav Mahishi

Question:
The product of two consecutive integers is 306. We need to find the integers.

Variable	Description
x	The bigger integer out of the two we need to find
$g(x)$	The function we take to update x_n in the point iteration method
$x_{\alpha,n}$	The value of x after n iterations for the first root
$x_{\beta,n}$	The value of x after n iterations for the second root

TABLE 0: Caption

TABLE 0: Variables Used

Theoretical Solution:
Lets start by assuming the bigger integer as x and the smaller integer as $\frac{306}{x}$

$$\implies x - \frac{306}{x} = 1 \tag{0.1}$$

$$\implies x^2 - 306 = x \tag{0.2}$$

$$\implies x^2 - x - 306 = 0 \tag{0.3}$$

Using the quadratic formula:

$$x = \frac{1 \pm \sqrt{1^2 - (4 \cdot -306)}}{2} \tag{0.4}$$

$$x_1 = \frac{1 + \sqrt{1225}}{2} = 18 \tag{0.5}$$

$$x_2 = \frac{1 - \sqrt{1225}}{2} = -17 \tag{0.6}$$

If $x = 18$ the other integer will be 17 if $x = -17$ the other integer will be -18

$$p(2) = 41 - 72(-2) - 18(-2)^2 = 113 \tag{0.7}$$

$\therefore$ The integers can be $18, 17$ or $-18, -17$
Computational Solution:
Below are three methods to find the solutions of this quadratic equation,
Matrix-Based Method:

For a polynomial equation of form $x_n + b_{n-1}x^{n-1} + \cdots + b_2x^2 + b_1x + b_0 = 0$ we construct a matrix called companion matrix of form

$$\Lambda = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \vdots & 1 \\ -b_0 & -b_1 & -b_2 & \dots & -b_{n-1} \end{pmatrix} \tag{0.8}$$

The eigenvalues of this matrix are the roots of the given polynomial equation.
The solution given by the code is

$$x_1 = 18.000000 \tag{0.9}$$

$$x_2 = -17.00000 \tag{0.10}$$

Fixed Point Iterations: Take an initial guess x_0. The update difference equation will use the following function:

$$x = g(x) \tag{0.11}$$

For our problem,

$$g(x) = \sqrt{x^2 - 306} \tag{0.12}$$

To get both roots we do two iterations, now the update equations will be

$$x_{\alpha,n+1} = g(x_{\alpha,n}) \tag{0.13}$$

$$x_{\beta,n+1} = g\left(x_{\beta,n}\right) \tag{0.14}$$

$$\tag{0.15}$$

We take two iniital guesses $x_{\alpha,0}$ and $x_{\beta,0}$ close to each root. Then we continue calculating the each $x_{\alpha,n}$ and $x_{\beta,n}$ until

$$|x_{n+1} - x_n| < \epsilon \tag{0.16}$$

Where ϵ is the tolerance which we have taken as 1e-6. In each of the α series and β series we get a root.

This is proved by a theorem as follows:
Let $x = s$ be a solution of $x = g(x)$ and suppose that g has a continuous derivative in some interval J containing s. Then if $|g'| \leq K < 1$ in J, the iteration process defined above converges for any x_0 in J. The limit of the sequence $[x_n]$ is s.

This can also be solved by the Newton-Raphson Method,
Start with an initial guess x_0, and then run the following logical loop,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{0.17}$$

where ,

$$f(x) = x^2 - x - 306 \tag{0.18}$$
$$f'(x) = 2x - 1 \tag{0.19}$$

The problem with this method is if the roots are complex but the coeffcients are real, x_n either converges to an extrema or grows continuously without any bound. To get the complex solutions, however , we can just take the initial guess point to be a random complex number.
The output of a program written to find roots is shown below:

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Fixed-Point Iteration for Positive Root:
Iteration 1: x = 17.577532, f(x) = -14.607907
Iteration 2: x = 17.988261, f(x) = -0.410729
Iteration 3: x = 17.999674, f(x) = -0.011413
Iteration 4: x = 17.999996, f(x) = -0.000237
Iteration 5: x = 18.000000, f(x) = -0.000009
Converged to solution: x = 18.000000

Fixed-Point Iteration for Negative Root:
Iteration 1: x = -9.767519, f(x) = -200.828057
Iteration 2: x = -17.211406, f(x) = 7.443887
Iteration 3: x = -16.993781, f(x) = 0.217625
Iteration 4: x = -17.001183, f(x) = -0.006402
Iteration 5: x = -16.999995, f(x) = 0.000018
Iteration 6: x = -17.000000, f(x) = -0.000000
Converged to solution: x = -17.000000

Newton-Raphson Iteration for Positive Root:
Iteration 1: x = 19.448724, f(x) = 52.804159
Iteration 2: x = 18.055381, f(x) = 1.941406
Iteration 3: x = 18.000887, f(x) = 0.003057
Iteration 4: x = 18.000001, f(x) = 0.000007
Iteration 5: x = 18.000000, f(x) = 0.000000
Converged to solution: x = 18.000000

Newton-Raphson Iteration for Negative Root:
Iteration 1: x = -19.809749, f(x) = 106.235914
Iteration 2: x = -17.194357, f(x) = 6.840276
Iteration 3: x = -17.007687, f(x) = 0.073361
Iteration 4: x = -17.000017, f(x) = 0.000307
Iteration 5: x = -17.000000, f(x) = 0.000000
Converged to solution: x = -17.000000

Eigenvalues (Roots of the equation):
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Eigenvalue 1: -17.000000 +0.0000000i
Eigenvalue 2: 18.000000 + 0.0000000i
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