9.1.7

EE24BTECH11006 - Arnav Mahishi

Question:
Solve the differential equation $y''' + 2y'' + y' = 0$ with initial conditions $y(0) = 1, y'(0) = -1$, and $y''(0) = 1$
Solution:

Variable	Description
n	Order of given differential equation
y_i	ith derivative of the function in the equation
c	constant in the equation
a_i	coefficient of ith derivative of the function in the equation
$\overrightarrow{V}(t)$	Vector containing all 1 and y_i from $i = 0$ to $i = n-1$
$\overrightarrow{V}'(t)$	Vector containing 1 and y_i' from $i = 0$ to $i = n-1$
A	the coefficient matrix that transforms each y_i to its derivative
h	the stepsize between each t we are taking

TABLE 0: Variables Used

Theoretical Solution: We apply the Laplace transform to each term in the equation. The Laplace transforms for the derivatives of $y(t)$ are:

$$\mathcal{L}y'(t) = sY(s) - y(0) \tag{0.1}$$

$$\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0) \tag{0.2}$$

$$\mathcal{L}\{y'''(t)\} = s^3Y(s) - s^2y(0) - sy'(0) - y''(0) \tag{0.3}$$

Now, applying the Laplace transform to the entire differential equation:

$$\mathcal{L}\{y''' + 2y'' + y'\} = 0 \tag{0.4}$$

$$\mathcal{L}\{y'''(t)\} + 2\mathcal{L}\{y''(t)\} + \mathcal{L}\{y'(t)\} = 0 \tag{0.5}$$

$$(s^3Y(s) - s^2y(0) - sy'(0) - y''(0)) + 2(s^2Y(s) - sy(0) - y'(0)) + (sY(s) - y(0)) = 0 \tag{0.6}$$

Substitute the initial conditions $y(0) = 1$, $y'(0) = -1$, and $y''(0) = 1$:

$$\left(s^3Y(s) - s^2 \cdot 1 - s \cdot (-1) - 1\right) + 2\left(s^2Y(s) - s \cdot 1 - (-1)\right) + (sY(s) - 1) = 0 \tag{0.7}$$

$$s^3Y(s) - s^2 + s - 1 + 2s^2Y(s) - 2s + 2 + sY(s) - 1 = 0 \tag{0.8}$$

Simplify the equation:

$$(s^3 + 2s^2 + s)Y(s) - (s^2 - s + 1) - (2s - 2) - 1 = 0 \quad (0.9)$$
$$(s^3 + 2s^2 + s)Y(s) - s^2 - s + 1 - 2s + 2 - 1 = 0 \quad (0.10)$$
$$(s^3 + 2s^2 + s)Y(s) - (s^2 + s) = 0 \quad (0.11)$$

Now, solve for $Y(s)$:

$$\left(s^3 + 2s^2 + s\right) Y(s) = s^2 + s \quad (0.12)$$
$$Y(s) = \frac{s^2 + s}{s(s+1)^2} \quad (0.13)$$
$$\implies Y(s) = \frac{1}{s+1} \quad (0.14)$$

Now, take the inverse Laplace transform:

$$\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) = e^{-t} \quad (0.15)$$

Thus, the solution to the differential equation is:

$$y(t) = e^{-t} \quad (0.16)$$

Computational Solution:
Consider the given linear differential equation

$$a_n y_n + a_{n-1} y_{n-1} + \cdots + a_1 y_1 + a_0 y_0 + c = 0 \quad (0.17)$$

Where y_i is the ith derivative of the function then

$$\begin{pmatrix} y_0' \\ y_1' \\ y_2' \\ \vdots \\ y_{n-1}' \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ \frac{-\left(\sum_{i=0}^{i=n-1} a_i y_i\right) - c}{a_n} \end{pmatrix} \quad (0.18)$$

$$\implies \begin{pmatrix} 1 \\ y_0' \\ y_1' \\ y_2' \\ \vdots \\ y_{n-1}' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 1 & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \frac{-c}{a_n} & \frac{-a_0}{a_n} & \frac{-a_1}{a_n} & \frac{-a_2}{a_n} & \cdots & \cdots & \frac{-a_{n-1}}{a_n} \end{pmatrix} \begin{pmatrix} 1 \\ y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{pmatrix} \quad (0.19)$$

$$\implies \overrightarrow{V}'(t) = A\overrightarrow{V}(t) \quad (0.20)$$

Where $\overrightarrow{V}(t)$ is the vector $\begin{pmatrix} 1 \\ y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{pmatrix}$ at a t. Using the definition of a derivative we get

$$y'(t) = \lim_{h \to 0} \frac{y(t+h) - y(t)}{h} \tag{0.21}$$

$$y(t+h) = y(t) + hy'(t) \tag{0.22}$$

$$\implies \overrightarrow{V}(t+h) = \overrightarrow{V}(t) + h\left(A\overrightarrow{V}(t)\right) \tag{0.23}$$

When t ranges from t_o to t_f in increments of h, discretizing the steps gives us all $\overrightarrow{V}(x)$ from t_o to t_f in increments of h Record the y_0 for each $\overrightarrow{V}(x)$ we got and then plot the graph. The result will be as given below.

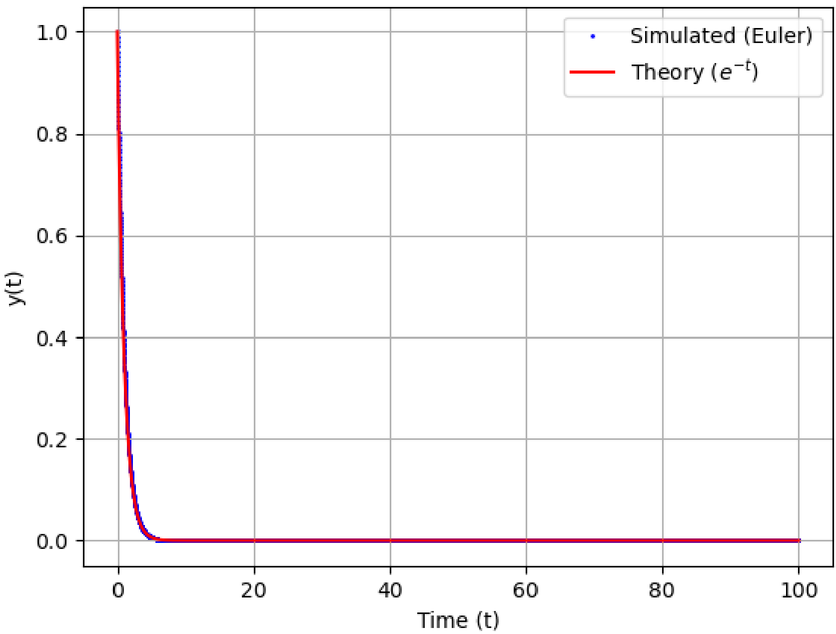

Fig. 0.1: Comparison between the Theoretical solution and Computational solution