# 9.1.7

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**Question**:
Solve the differential equation $y'''+2y''+y' = 0$ with initial conditions $y(0) = 1$,$y'(0) = -1$ , and $y''(0) = 1$
**Solution:**

| Variable | Description |
|---|---|
| $n$ | Order of given differential equation |
| $y_i$ | $i$th derivative of the function in the equation |
| $c$ | constant in the equation |
| $a_i$ | coefficient of $i$th derivative of the function in the equation |
| $\mathbf{y}(t)$ | Vector containing all 1 and $y_i$ from $i = 0$ to $i = n-1$ |
| $\mathbf{y}'(t)$ | Vector containing 1 and $y_i'$ from $i = 0$ to $i = n-1$ |
| $A$ | the coefficient matrix that transforms each $y_i$ to its derivative |
| $h$ | the stepsize between each t we are taking |
| $t_o$ | The start time from which we are plotting |
| $t_f$ | The end time at which we stop plotting |

TABLE 0: Variables Used

Theoretical Solution: We apply the Laplace transform to each term in the equation. The Laplace transforms for the derivatives of $y(t)$ are:

$$\mathcal{L}y'(t) = sY(s) - y(0) \tag{0.1}$$
$$\mathcal{L}y''(t) = s^2Y(s) - sy(0) - y'(0) \tag{0.2}$$
$$\mathcal{L}y'''(t) = s^3Y(s) - s^2y(0) - sy'(0) - y''(0) \tag{0.3}$$

Now, applying the Laplace transform to the entire differential equation:

$$\mathcal{L}\{y''' + 2y'' + y'\} = 0 \tag{0.4}$$
$$\mathcal{L}\{y'''(t)\} + 2\mathcal{L}\{y''(t)\} + \mathcal{L}\{y'(t)\} = 0 \tag{0.5}$$
$$(s^3Y(s) - s^2y(0) - sy'(0) - y''(0)) + 2(s^2Y(s) - sy(0) - y'(0)) + (sY(s) - y(0)) = 0 \tag{0.6}$$

Substitute the initial conditions $y(0) = 1$, $y'(0) = -1$, and $y''(0) = 1$:

$$\left(s^3Y(s) - s^2 \cdot 1 - s \cdot (-1) - 1\right) + 2\left(s^2Y(s) - s \cdot 1 - (-1)\right) + (sY(s) - 1) = 0 \tag{0.7}$$
$$s^3Y(s) - s^2 + s - 1 + 2s^2Y(s) - 2s + 2 + sY(s) - 1 = 0 \tag{0.8}$$

Simplify the equation:

$$
\begin{aligned}
(s^3 + 2s^2 + s)Y(s) - (s^2 - s + 1) - (2s - 2) - 1 &= 0 && (0.9)\\
(s^3 + 2s^2 + s)Y(s) - s^2 - s + 1 - 2s + 2 - 1 &= 0 && (0.10)\\
(s^3 + 2s^2 + s)Y(s) - (s^2 + s) &= 0 && (0.11)
\end{aligned}
$$

Now, solve for $Y(s)$:

$$\left(s^3 + 2s^2 + s\right)Y(s) = s^2 + s \tag{0.12}$$

$$Y(s) = \frac{s^2 + s}{s(s+1)^2} \tag{0.13}$$

$$\implies Y(s) = \frac{1}{s+1} \tag{0.14}$$

Now, take the inverse Laplace transform:

$$\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) = e^{-t} \tag{0.15}$$

Thus, the solution to the differential equation is:

$$y(t) = e^{-t} \tag{0.16}$$

Radius of Convergence:
The denominator indicates a pole at $s = -1$.To ensure convergence of the Laplace transform integral, the real part of $s$ must satsify:

$$Re(s) > -1 \tag{0.17}$$

Since the ROC extends infinitely to the right in the s-plane, the radius of convergence is:

$$R = \infty \tag{0.18}$$

Computational Solution:
Consider the given linear differential equation

$$a_n y_n + a_{n-1} y_{n-1} + \cdots + a_1 y_1 + a_0 y_0 + c = 0 \tag{0.19}$$

Where $y_i$ is the $i$th derivative of the function then

$$\begin{pmatrix} y_0' \\ y_1' \\ y_2' \\ \vdots \\ y_{n-1}' \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ \frac{-(\sum_{i=0}^{i=n-1} a_i y_i)-c}{a_n} \end{pmatrix} \tag{0.20}$$

$$\implies \begin{pmatrix} 1 \\ y_0' \\ y_1' \\ y_2' \\ \vdots \\ y_{n-1}' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 1 & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \frac{-c}{a_n} & \frac{-a_0}{a_n} & \frac{-a_1}{a_n} & \frac{-a_2}{a_n} & \cdots & \cdots & \frac{-a_{n-1}}{a_n} \end{pmatrix} \begin{pmatrix} 1 \\ y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{pmatrix} \tag{0.21}$$

$$\implies \mathbf{y}_k' = A\mathbf{y}_k \tag{0.22}$$

Where $\mathbf{y}_k$ is the vector $\begin{pmatrix} 1 \\ y_{0,k} \\ y_{1,k} \\ y_{2,k} \\ \vdots \\ y_{n-1,k} \end{pmatrix}$ at a $k$. At any $n$, by defining derivative:

$$y_{n,k}' = \lim_{h\to 0} \frac{y_{n,k+1} - y_{n,k}}{h} \tag{0.23}$$

$$y_{n,k+1} = y_{n,k} + hy_{n,k}' \tag{0.24}$$

$$\implies y_{0,k+1} = y_{0,k} + hy_{0,k}' \tag{0.25}$$

$$\implies y_{1,k+1} = y_{1,k} + hy_{1,k}' \tag{0.26}$$

$$\implies y_{2,k+1} = y_{2,k} + hy_{2,k}' \tag{0.27}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \tag{0.28}$$

$$\implies y_{n-1,k+1} = y_{n-1,k} + h\left(\frac{-\left(\sum_{i=0}^{i=n-1} a_i y_i\right) - c}{a_n}\right) \tag{0.29}$$

$$\implies \mathbf{y}_{k+1} = \mathbf{y}_k + h\left(A\mathbf{y}_k\right) \tag{0.30}$$

When $k$ ranges from 0 to $\frac{t_o - t_f}{h}$ in increments of 1, discretizing the steps gives us all $\mathbf{y}_k$, Record the $y_{0,k}$ for each $k$ we got and then plot the graph. The result will be as given below.

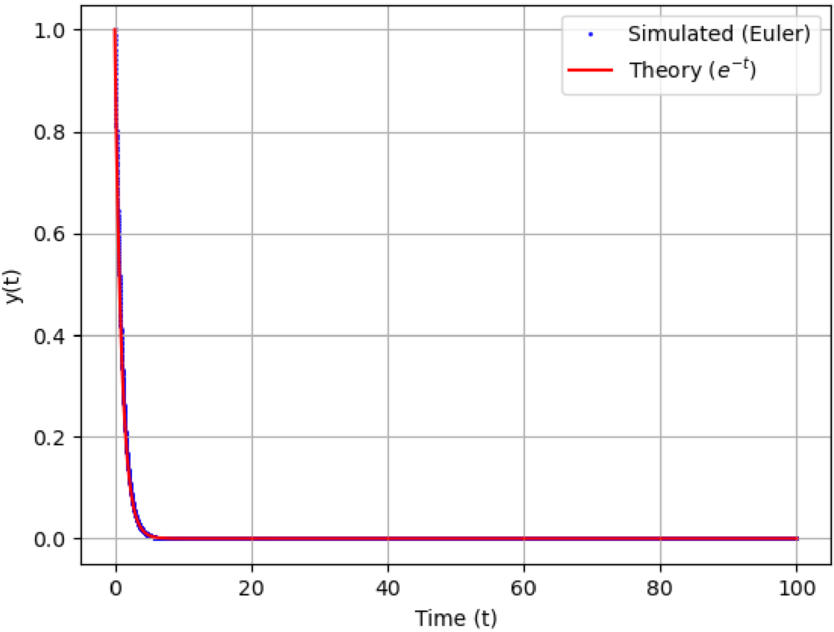


Fig. 0.1: Comparison between the Theoretical solution and Computational solution