9-9.3-27

EE24BTECH11006 - Arnav Mahishi

Question: Using Integration, find the area of the smaller region bounded by the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and the line $\frac{x}{3} + \frac{y}{2} = 1$

input	value
a	3
b	2
v	$\begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix}$
u	0
f	$-(a^2 b^2)$
h	$\begin{pmatrix} a \\ 0 \end{pmatrix}$
m	$\begin{pmatrix} \frac{1}{b} \\ \frac{-1}{a} \end{pmatrix}$

TABLE 0: Input Parameters

Solution:

The point of intersection of the line with the ellipse is $x_i = h + k_i m$,

where,k_i is a constant and is calculated as follows:-

$$k_i = \frac{1}{m^\top V m} \left(-m^\top (Vh + u) \pm \sqrt{[m^\top (Vh + u)]^2 - g(h)(m^\top V m)} \right)$$

Substituting the input parameters in k_i,

$$k_i = \frac{1}{\left(\frac{1}{b} \quad \frac{-1}{a} \right) \begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix} \begin{pmatrix} \frac{1}{b} \\ \frac{-1}{a} \end{pmatrix}} \left(-\left(\frac{1}{b} \quad \frac{-1}{a} \right) \left(\begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix} \begin{pmatrix} a \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) \pm \right.$$

$$\left. \sqrt{\left[\left(\frac{1}{b} \quad \frac{-1}{a} \right) \left(\begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix} \begin{pmatrix} a \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) \right]^2 - g(h) \left(\left(\frac{1}{b} \quad \frac{-1}{a} \right) \begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix} \begin{pmatrix} \frac{1}{b} \\ \frac{-1}{a} \end{pmatrix} \right)} \quad (0.1)$$

We get,

$$k_i = 0, -ab$$

Substituting k_i in $x_i = h + k_i m$ we get,

$$x_1 = \begin{pmatrix} a \\ 0 \end{pmatrix} + (0) \begin{pmatrix} \frac{1}{b} \\ \frac{-1}{a} \end{pmatrix} \tag{0.2}$$

$$\implies x_1 = \begin{pmatrix} a \\ 0 \end{pmatrix} \tag{0.3}$$

$$x_2 = \begin{pmatrix} a \\ 0 \end{pmatrix} + (-ab) \begin{pmatrix} \frac{1}{b} \\ \frac{-1}{a} \end{pmatrix} \tag{0.4}$$

$$\implies x_2 = \begin{pmatrix} a \\ 0 \end{pmatrix} + \begin{pmatrix} -a \\ b \end{pmatrix} \tag{0.5}$$

$$\implies x_2 = \begin{pmatrix} 0 \\ b \end{pmatrix} \tag{0.6}$$

The area of the smaller region bounded by the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and the line $\frac{x}{3} + \frac{y}{2} = 1$ is

$$= \int_0^a \frac{b}{a} \sqrt{a^2 - x^2} \, dx - \int_0^a \frac{b}{a} (a - x) \, dx \tag{0.7}$$

$$= \frac{b}{a} \left(\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} - ax + \frac{x^2}{2} \right) \Bigg|_0^a \tag{0.8}$$

$$= \frac{b}{a} \left(\frac{\pi a^2}{4} - \frac{a^2}{2} \right) = \frac{ab}{2} \left(\frac{\pi}{2} - 1 \right) \tag{0.9}$$

The given area is $\frac{ab}{2} \left(\frac{\pi}{2} - 1 \right)$ sq. units

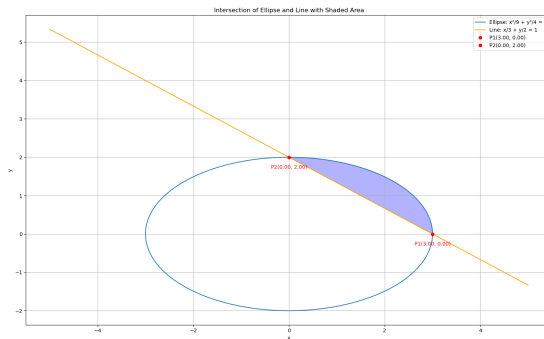$\therefore$ Upon substituting $a = 3, b = 2$ the given area is $3 \left(\frac{\pi}{2} - 1 \right)$ sq. units ≈ 1.712 sq. units

Fig. 0.1: Plot of Ellipse and Line