9-9.3-27

EE24BTECH11006 - Arnav Mahishi

Question: Using Integration, find the area of the smaller region bounded by the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ and the line $\frac{x}{3}+\frac{y}{2}=1$

Input	**Description**	**Value**
a	Length of semi major axis of ellipse	3
b	Length of semi minor axis of ellipse	2
v	Quadratic form of matrix	$\begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix}$
u	Linear coefficient vector	0
f	Constant Term	$-(a^2b^2)$
h	One of the points the line passes through	$\begin{pmatrix} a \\ 0 \end{pmatrix}$
m	Slope of line	$\begin{pmatrix} \frac{1}{b} \\ \frac{-1}{a} \end{pmatrix}$

TABLE 0: Input Parameters

Solution:
The point of intersection of the line with the ellipse is $x_i = h + k_i m$,
where,k_i is a constant and is calculated as follows:-

$$k_i = \frac{1}{m^\top V m}\left(-m^\top (Vh+u) \pm \sqrt{[m^\top (Vh+u)]^2 - g(h)(m^\top V m)}\right)$$

Substituting the input parameters in k_i,

$$k_i = \frac{1}{\begin{pmatrix} \frac{1}{b} & \frac{-1}{a} \end{pmatrix}\begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix}\begin{pmatrix} \frac{1}{b} \\ \frac{-1}{a} \end{pmatrix}}\left(-\begin{pmatrix} \frac{1}{b} & \frac{-1}{a} \end{pmatrix}\left(\begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix}\begin{pmatrix} a \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) \pm \right.$$
$$\left.\sqrt{\left[\begin{pmatrix} \frac{1}{b} & \frac{-1}{a} \end{pmatrix}\left(\begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix}\begin{pmatrix} a \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)\right]^2 - g(h)\left(\begin{pmatrix} \frac{1}{b} & \frac{-1}{a} \end{pmatrix}\begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix}\begin{pmatrix} \frac{1}{b} \\ \frac{-1}{a} \end{pmatrix}\right)}\right) \quad (0.1)$$

We get,

$$k_i = 0, -ab$$

Substituting k_i in $x_i = h + k_i m$ we get,

$$x_1 = \begin{pmatrix} a \\ 0 \end{pmatrix} + (0) \begin{pmatrix} \frac{1}{b} \\ \frac{-1}{a} \end{pmatrix} \tag{0.2}$$

$$\implies x_1 = \begin{pmatrix} a \\ 0 \end{pmatrix} \tag{0.3}$$

$$x_2 = \begin{pmatrix} a \\ 0 \end{pmatrix} + (-ab) \begin{pmatrix} \frac{1}{b} \\ \frac{-1}{a} \end{pmatrix} \tag{0.4}$$

$$\implies x_2 = \begin{pmatrix} a \\ 0 \end{pmatrix} + \begin{pmatrix} -a \\ b \end{pmatrix} \tag{0.5}$$

$$\implies x_2 = \begin{pmatrix} 0 \\ b \end{pmatrix} \tag{0.6}$$

The area of the smaller region bounded by the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and the line $\frac{x}{3} + \frac{y}{2} = 1$ is

$$= \int_0^a \frac{b}{a} \sqrt{a^2 - x^2}\, dx - \int_0^a \frac{b}{a} (a - x)\, dx \tag{0.7}$$

$$= \frac{b}{a} \left(\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} - ax + \frac{x^2}{2} \right)_0^a \tag{0.8}$$

$$= \frac{b}{a} \left(\frac{\pi a^2}{4} - \frac{a^2}{2} \right) = \frac{ab}{2} \left(\frac{\pi}{2} - 1 \right) \tag{0.9}$$

The given area is $\frac{ab}{2} \left(\frac{\pi}{2} - 1 \right)$ sq. units

$\therefore$ Upon substituting $a = 3, b = 2$ the given area is $3 \left(\frac{\pi}{2} - 1 \right)$ sq. units ≈ 1.712 sq. units

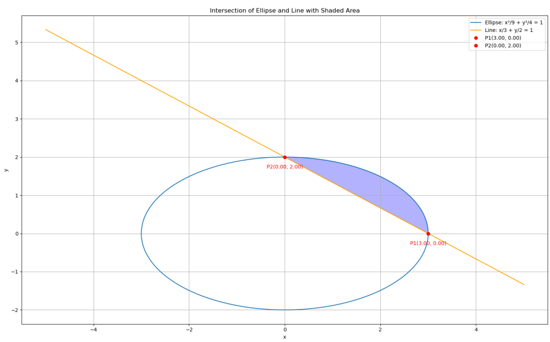

Fig. 0.1: Plot of Ellipse and Line