Mathematics Year 1, Calculus and Applications I

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Solutions Problem Sheet 3

1. $\underline{y = x\exp(-x)}$: $y = 0$ at $x = 0$; $y < 0$ for $x < 0$; $y > 0$ for $x > 0$; $y \to 0$ as $x \to \infty$, and $y \to -\infty$ as $x \to -\infty$. In addition $y' = (1-x)\exp(-x)$, hence there is a local maximum at $x = 1$. This is the only critical point. See Figure 1.

   $\underline{y = x^2\exp(-x^2)}$: The function is symmetric about $x = 0$ and $y \geq 0$ for all x. $y = 0$ at $x = 0$ and $y \to 0$ as $|x| \to \infty$. $y' = 2x(1-x^2)\exp(-x^2)$, hence $x = 0$ is a local minimum and $x = \pm 1$ are local maxima. Sketch in Figure 2.

   $\underline{y = e^x/x}$: $y \to \pm\infty$ as $x \to 0\pm$. $y \to \infty$ as $x \to +\infty$, and $y \to 0$ as $x \to -\infty$. Also, $y' = e^x(1/x - 1/x^2)$, so $x = 1$ is the only critical point - it must be a local minimum. $y > 0$ for $x > 0$ and $y < 0$ for $x < 0$. Sketch in Figure 3.

2. For the function $f(x) = \exp(1/x)$, $x \neq 0$.

   (a) What are the limits

   $$\lim_{x\to 0+} f(x) = +\infty, \qquad \lim_{x\to 0-} f(x) = 0, \qquad \lim_{x\to+\infty} f(x) = \lim_{x\to-\infty} f(x) = 1.$$

   (b) Defining $f(0) = 0$, the function is differentiable everywhere except possibly at $x = 0$, Here we consider

   $$\lim_{h\to 0} \frac{\exp(1/h) - 0}{h},$$

   which clearly does not exist if $h > 0$.

   (c) Calculate derivatives:

   $$\frac{df}{dx} = -\frac{1}{x^2}\exp(1/x),$$

   $$\frac{d^2f}{dx^2} = \frac{1}{x^4}\exp(1/x) + \frac{2}{x^3}\exp(1/x),$$

   $$\cdots$$

   $$\frac{d^nf}{dx^n} = (-1)^n\frac{1}{x^{2n}}\exp(1/x) + g_n(x)\exp(1/x),$$

   where the function $g_n(x)$ contains terms of size x^{-2n+1} at most for small negative x. Now, $\lim_{x\to 0-} |\frac{\exp(1/x)}{x^{2n}}| = \lim_{t\to+\infty} t^{2n}\exp(-t) = 0$, and hence

   $$\lim_{x\to 0-} g_n(x)\exp(1/x) = 0$$

   also by the comparison test (since it is x times something that already goes to 0).

   (d) From the result for d^2f/dx^2 we see that there is an inflection point at $x = -1/2$, $y = 1/e^2$. There are no critical points, and the asymptotes have been determined. The sketch is given in Figure 4.

3. The function $y = x\exp(1/x)$ is slightly different from that in problem 2. We have y behaving like x for large x and $\lim_{x\to 0-} x\exp(1/x) = 0$ but $\lim_{x\to 0+} x\exp(1/x) = +\infty$ as before. All derivatives are 0 at $x = 0-$ as before. Since $y' = (1 - 1/x)\exp(1/x)$ we must have a local minimum at $x = 1$, $y = e$. There are no other critical points. A sketch is given in Figure 5.

4. Need to show that the equation $\mathrm{e}^x = ax$ has at least one solution for any number a, except when $0 \le a < \mathrm{e}$.

   Lets do the easy cases first: (i) If $a = 0$ there is no root since $e^x > 0$. (ii) If $a < 0$ then $f(0) = 1$ and $\lim_{x\to -\infty}(e^x - ax) = -\infty$; by the intermediate value theorem there is at least one root (you can also see this graphically but that is not a proof).

   It remains to consider $a > 0$. There is probably another solution but I did it this way: Take the difference defined by $f(x) = e^x - ax$. Find the local minima for this (there is no local maximum since $f \to \infty$ as $x \to \infty$) by setting $f'(x_m) = 0$, i.e. $e^x_m - a = 0$, giving $x_m = \log a$. Hence $f(x_m) = a(1 - \log a)$ which immediately shows that $a = e$ gives a solution. If $a < e$ we have $a(1 - \log a) > 0$ hence $f(x) > 0$ and there cannot be a solution. If $a > e$ we have $f(x_m) < 0$, and since $f(0) = 1$, the intermediate value theorem guarantees a root.

5. We are given the function

$$f(x) = \begin{cases} \exp(-1/x^2) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

   (a) Using the definition of the derivative we have

$$f'(0) = \lim_{h\to 0} \frac{\exp(-1/h^2) - 0}{h} = \lim_{t\to\infty} t\exp(-t^2) = 0,$$

   hence the derivative exists and $f'(0) = 0$.

   (b) Use the chain rule, $\frac{d}{dx}(e^{-1/x^2}) = \frac{2}{x^3}e^{-1/x^2}$ and combined with (a) above we have

$$f'(x) = \begin{cases} \frac{2}{x^3}\exp(-1/x^2) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

   (c) We can see that $f^{(n)}(x)$ will contain a term proportional to $x^{-3n}e^{-1/x^2}$ along with smaller inverse powers of x (the x^{-3n} is the most singular as $x \to 0$). Since

$$\lim_{x\to 0} \frac{e^{-1/x^2}}{x^{3n}} = 0, \qquad (why?)$$

   we also define $f^{(n)}(0) = 0$ and hence all higher derivatives exist.

   (d) To sketch the function we note that $f(x) \ge 0$, it is symmetric about $x = 0$, and $\lim_{|x|\to\infty} f(x) = e^0 = 1$. All derivatives are zero at $x = 0$ and there are inflection points at $x = \pm\sqrt{2/3}$, $y = e^{-3/2}$. A plot is provided in Figure 6.

6. Can write $f(x) = e^{x\log x}$, hence $f'(x) = x^x(1 + \log x)$. Considering $\lim_{x\to 0+} x^x(1 + \log x)$, we note that $\lim_{x\to 0+} x^x = 1$ (why?), and hence $\lim_{x\to 0+} f'(x)$ does not exist and in fact tends to $-\infty$. This means that the tangent at $x = 0$ is vertical. In addition, there is a local minimum at $x = e^{-1}$, $y = e^{-1/e}$, and clearly f is positive and becomes large for x large.

   A plot is given in Figure 7.

7. We can use the result $\frac{d}{dx}x^x = x^x(1+\log x)$ in problem 7 also. Compute

$$\frac{d}{dx}\left(x^{x^x}\right) = \frac{d}{dx}e^{x^x \log x} = x^{x^x}\left(x^{x-1} + x^x(1+\log x)\log x\right)$$

8. Yes. One example is $\log_2 \sqrt{2} = 1/2$.

9. (a) Need to find $\lim_{a\to 0} \frac{1}{a}\log\left(\frac{e^a-1}{a}\right)$. The function $\frac{e^a-1}{a}$ has the form $'0/0'$ and so L'Hôpital's rule can be used to see that $\lim_{a\to 0}\frac{e^a-1}{a} = 1$. Hence, $\frac{1}{a}\log\left(\frac{e^a-1}{a}\right)$ is of the form $'0/0'$ and what we have shown is that L'Hôpital's rule can be applied directly to find

$$\lim_{a\to 0}\frac{1}{a}\log\left(\frac{e^a-1}{a}\right) = \lim_{a\to 0}\frac{\frac{e^a}{e^a-1} - \frac{1}{a}}{1} = \lim_{a\to 0}\frac{ae^a - (e^a-1)}{a(e^a-1)}$$

$$= \lim_{a\to 0}\frac{ae^a}{ae^a + e^a - 1} = \lim_{a\to 0}\frac{ae^a + e^a}{ae^a + 2e^a} = \frac{1}{2}.$$

[We can do this much more easily using Taylor's Theorem that is coming a bit later.]

(b) For $\lim_{a\to\infty}\frac{1}{a}\log\left(\frac{e^a-1}{a}\right)$ I can save myself all the differentiations by noting that $\log x$ is a strictly increasing function and hence $\log\left(\frac{e^a-1}{a}\right) < \log\left(\frac{e^a}{a}\right) = a - \log a$. Hence

$$\lim_{a\to\infty}\frac{1}{a}\log\left(\frac{e^a-1}{a}\right) < \lim_{a\to 0}\left(\frac{a-\log a}{a}\right) = \lim_{a\to 0}\left(1 - \frac{\log a}{a}\right) = 1,$$

since $\lim_{a\to\infty}\frac{\log a}{a} = 0$, and by use of the squeezing theorem.

10. $\underline{\lim_{x\to 1} x^{1/(1-x^2)}}$ of form '1^∞'.

$$x^{1/(1-x^2)} = \exp\left(\frac{1}{1-x^2}\log x\right); \quad \lim_{x\to 1}\frac{\log x}{1-x^2} = \lim_{x\to 1}\frac{1/x}{-2x} = -1/2$$

so $\lim_{x\to 1} x^{1/(1-x^2)} = e^{-1/2}$ since $\exp(x)$ is a continuous function.

$\underline{\lim_{x\to 0}(\tan x)^x}$, $x > 0$, is of form '0^0'.

$$(\tan x)^x = \exp\left(x\log(\tan x)\right); \quad x\log(\tan x) = \frac{\log(\tan x)}{(1/x)},$$

which is of the form '∞/∞' so can use L'H rule to find

$$\lim_{x\to 0}\frac{\log(\tan x)}{(1/x)} = \lim_{x\to 0}\frac{\frac{\sec^2 x}{\tan x}}{-\frac{1}{x^2}} = -\lim x \to 0\frac{x^2}{\sin x} = 0.$$

Hence $\lim_{x\to 0}(\tan x)^x = 1$.

$\underline{\lim_{x\to\infty}[\log x - \log(x-1)]} = \lim_{x\to\infty}\log\left(\frac{1}{1-1/x}\right) = 0.$

$\underline{\lim_{x\to 1}\frac{\log x}{e^x-1}} = \frac{\lim_{x\to 1}\log x}{e-1} = 0.$

$\underline{\lim_{x\to 0}\frac{\cos x - 1 + x^2/2}{x^4}} = \lim_{x\to 0}\frac{-\sin x + x}{4x^3} = \lim_{x\to 0}\frac{-\cos x + 1}{12x^2} = \lim_{x\to 0}\frac{-\sin x}{24x} = -1/24.$

11. *Suppose that f is continuous at $x = x_0$, that $f'(x)$ exists for x in an interval about x_0, $x \neq x_0$, and that $\lim_{x\to x_0} f'(x) = m$. Prove that $f'(x_0)$ exists and equals m. [Hint. Use the mean value theorem.]*

    We are given $\lim_{x\to x_0} f(x) = f(x_0)$. Also, $\lim_{x\to x_0} f'(x) = m$, hence I can write this as

$$\lim_{x\to x_0}\left[\frac{f(x)-f(x_0)}{x-x_0} - m\right] = 0,$$

    and since f' exists near x_0 except possibly at x_0, we can use the MVT to find a c between x and x_0 such that the above limit has the form

$$\lim_{x\to x_0}\left[f'(c) - m\right] = 0.$$

    Now as $x \to x_0$, the number c is squeezed between x and x_0, tends to x_0 in the limit, and the result follows.

12. (a) Clearly $u(0) = 0$ and $u(h) = U$, so BC satisfied. The rest, differentiate and substitute in to verify it is a solution.

    (b) With $q = du/dy$ we have

$$\nu\frac{dq}{dy} = (Vq - P/\rho) \quad \Rightarrow \quad \frac{dq}{(Vq - P/\rho)} = \frac{1}{\nu}dy.$$

    Integrate to find

$$\frac{1}{V}\log(Vq - P/\rho) = \frac{1}{\nu}y + K_1 \quad \Rightarrow \quad q = \frac{du}{dy} = K_2\,\mathrm{e}^{Vy/\nu} + \frac{P}{\rho V},$$

    where K_2 is a constant. One more integration gives

$$u(y) = K_3\,\mathrm{e}^{Vy/\nu} + \frac{P}{\rho V}y + K_4, \tag{1}$$

    where K_3, K_4 are constants. Using the boundary conditions gives

$$\begin{aligned} &\underline{u(0) = 0} \qquad K_3 + K_4 = 0,\\ &\underline{u(h) = U} \qquad K_3\,\mathrm{e}^{Vh/\nu} + K_4 = -\frac{P}{\rho V}h, \end{aligned}$$

    hence

$$K_3 = \frac{Ph}{\rho V(1-\mathrm{e}^R)} = -K_4,$$

    with $R = Vh/\nu$. Substitution into (1) gives the desired solution.

    (c) Here we need to use L'Hôpital's rule (later you will see how to do this using Taylor's Theorem also). Rewrite the solution and calculate

$$u = \frac{(Py/\rho)(1-\mathrm{e}^{Vh/\nu}) + U(V - Ph/\rho U)(1-\mathrm{e}^{Vy/\nu})}{V(1-\mathrm{e}^{Vh/\nu})}$$

    As $V \to 0$ this is of the form $0/0$, hence

$$\lim_{V\to 0} u = \lim_{V\to 0}\frac{-(Phy/\rho\nu)\mathrm{e}^{Vh/\nu} + U(1-\mathrm{e}^{Vy/\nu}) - U(V - Ph/\rho U)(y/\nu)\mathrm{e}^{Vy/\nu}}{1 - \mathrm{e}^{Vh/\nu} - (Vh/\nu)\mathrm{e}^{Vh/\nu}}$$

This also has the form 0/0 hence need one more differentiation

$$\lim_{V\to 0} u =$$

$$\lim_{V\to 0} \frac{-(Pyh^2/\rho\nu^2)\mathrm{e}^{Vh/\nu} - (Uy/\nu)\mathrm{e}^{Vy/\nu} - (Uy/\nu)\mathrm{e}^{Vy/\nu} + U(Ph/\rho U - V)(y^2/\nu^2)\mathrm{e}^{Vy/\nu}}{-(h/\nu)\mathrm{e}^{Vh/\nu} - (h/\nu)\mathrm{e}^{Vh/\nu} - V(h^2/\nu^2)\mathrm{e}^{Vh/\nu}}$$

$$= \frac{-\frac{Ph^2y}{\rho\nu^2} - \frac{2Uy}{\nu} + \frac{UPhy^2}{\rho U\nu^2}}{-\frac{2h}{\nu}}$$

$$= U\frac{y}{h} + \frac{1}{2}\frac{Ph^2}{\rho\nu}\left(\frac{y}{h} - \left(\frac{y}{h}\right)^2\right)$$

which is the required result.

(d) Straightforward calculation gives

$$Q = \frac{1}{2}\rho Uh\left(1 + \frac{Ph^2}{6\rho\nu U}\right).$$

If $U = 0$ we get $Q = \frac{\rho Ph^3}{12\nu}$, hence for fixed P, μ we have that Q is proportional to h^3. Hence halving h will result in Q going down by a factor of 8.

(e) If $V \gg 1$ and also $V \gg \nu/h$, $V \gg Ph/\rho U$, then we can see that the largest possible term is

$$u(y) \approx U\mathrm{e}^{V(y-h)/\nu}. \tag{2}$$

Since $y - h \le 0$ we see immediately that unless $|y - h|$ is small then $u = 0$. In fact see from (2) that the solution will be zero everywhere except when $(h-y)V$ is an order one quantity, i.e. there is a layer near the upper wall of thickness $1/V$ where the solution goes from 0 to its wall value U.

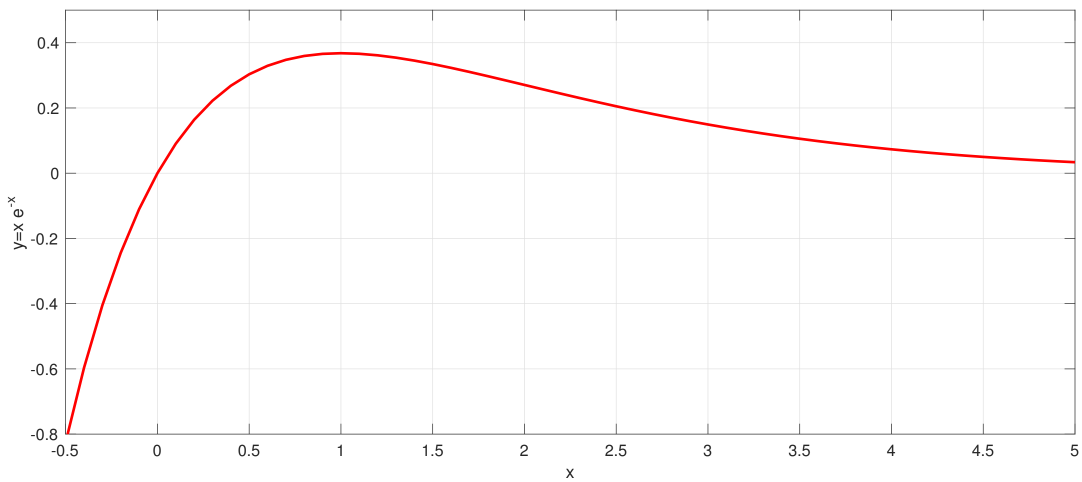

Figure 1: The function $y = x\exp(-x)$.

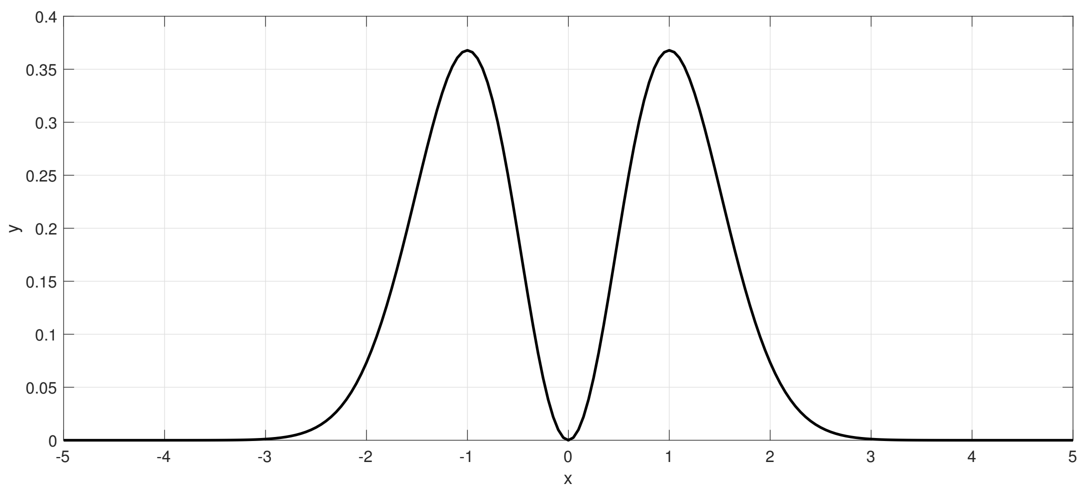

Figure 2: The function $y = x^2 \exp(-x^2)$.

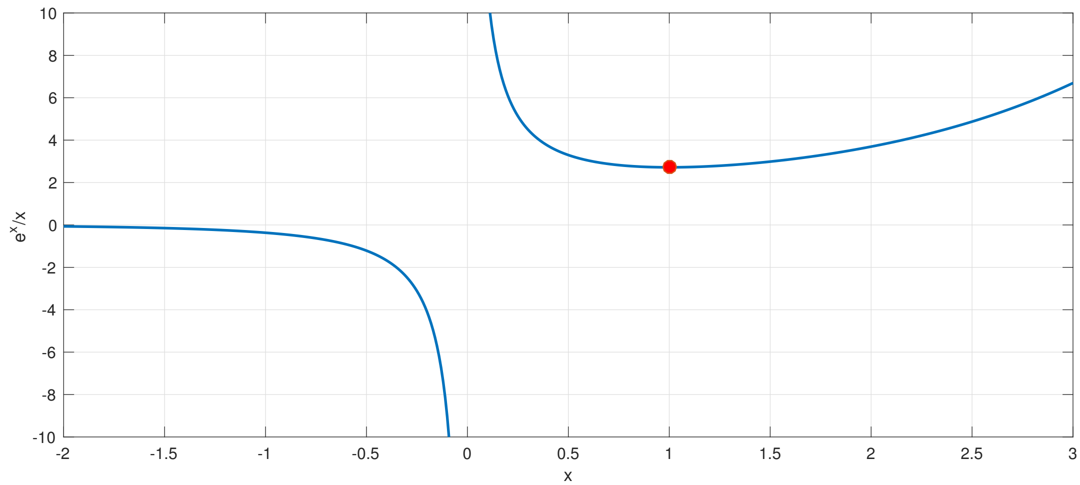

Figure 3: The function $y = \exp(x)/x$. The red dot denotes the point $(1, e)$ where the local minimum is attained.

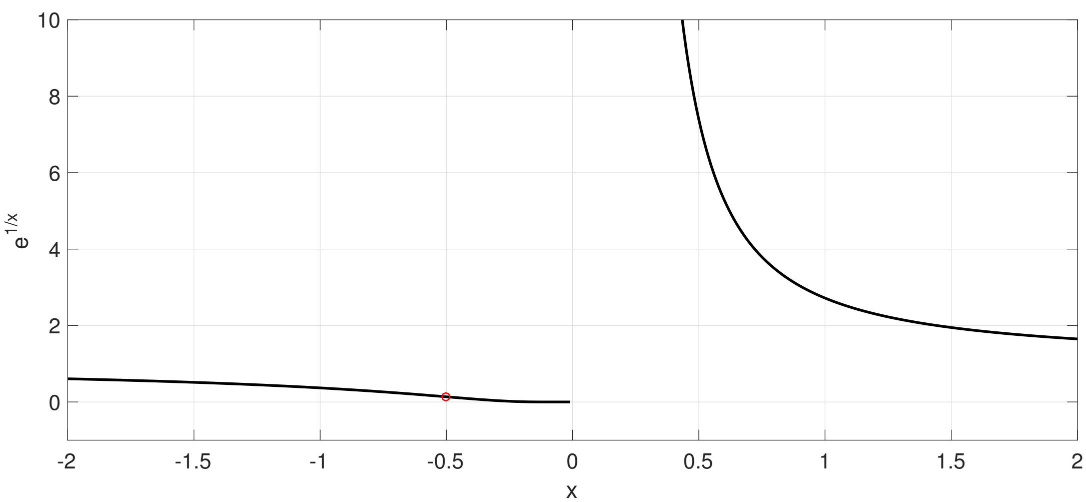

Figure 4: The function $y = \exp(1/x)$. The red dot denotes the point $(1/2, 1/e^2)$ where there is an inflection point

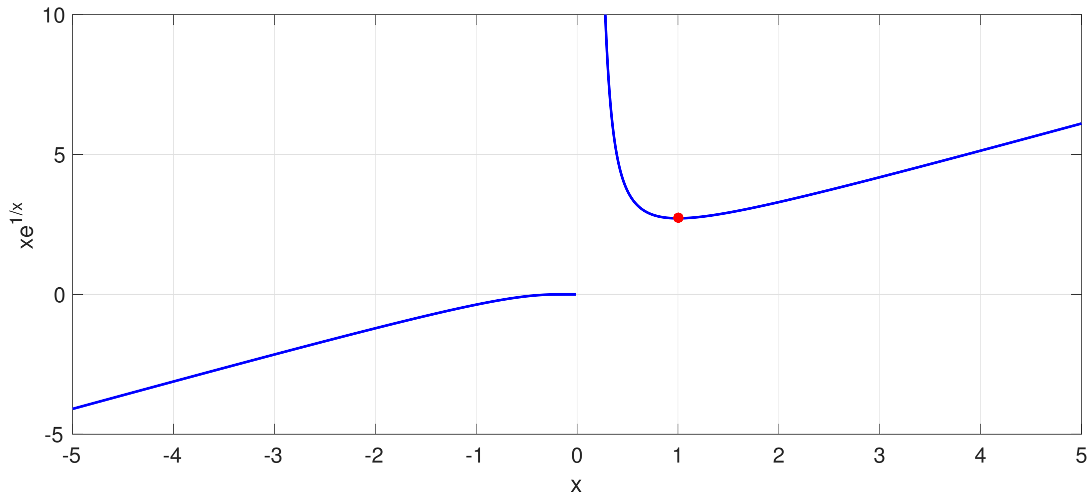

Figure 5: The function $y = x\exp(1/x)$. The red dot denotes the local minimum point $(1, e)$.

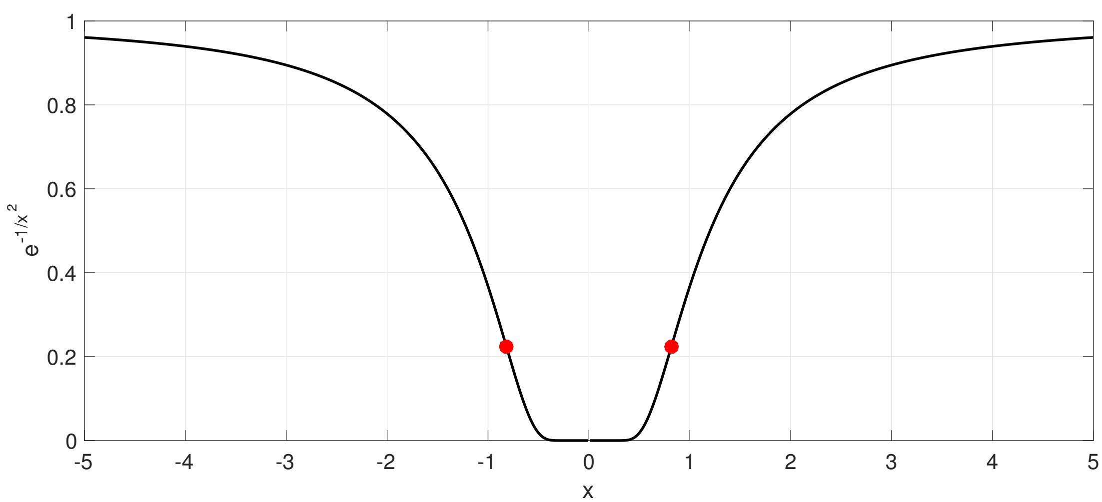

Figure 6: The function $y = \exp(-1/x^2)$. The red dots denote the inflection points $(\sqrt{2/3}, e^{-3/2})$.

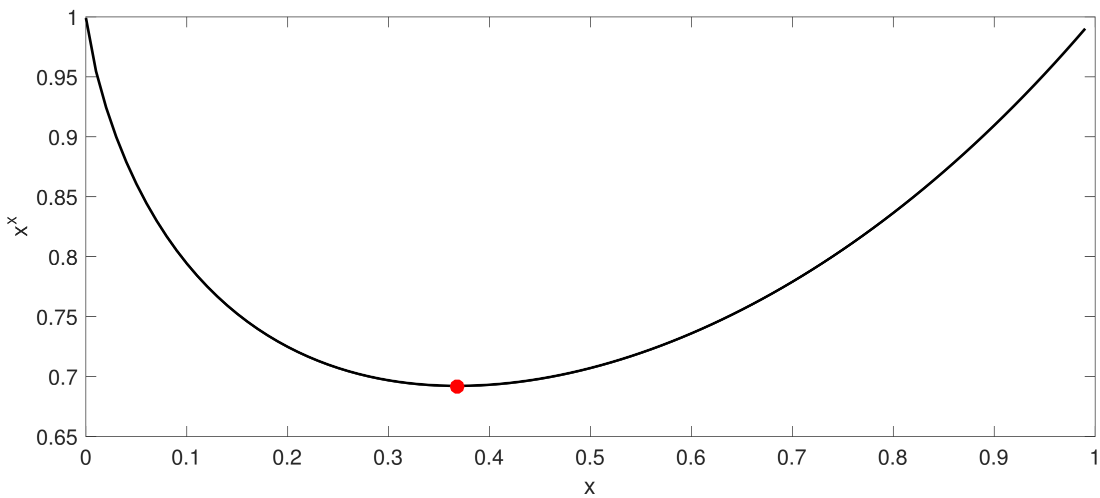

Figure 7: The function $y = x^x$. The red dot denotes the local minimum $(1/e, e^{-e})$.