5 Topics: Continuous random variables, transformations of random variables, expectation

5.1 Prerequisites: Lecture 12

Exercise 5- 1: Let $X \sim \mathrm{Exp}(\lambda)$. Show that, if $x, y > 0$ then

$$\mathrm{P}(X > x + y | X > x) = \mathrm{P}(X > y).$$

This is called the *Lack of memory* property (for a continuous random variable).

Solution: Recall that, for $x, y > 0$, we have

$$f_X(x) = \lambda e^{-\lambda x},$$

and

$$\mathrm{P}(X > x) = \int_x^\infty \lambda e^{-\lambda u} du = e^{-\lambda x}.$$

Hence

$$\begin{aligned} \mathrm{P}(X > x + y | X > x) &= \frac{\mathrm{P}(X > x + y, X > x)}{\mathrm{P}(X > x)} = \frac{\mathrm{P}(X > x + y)}{\mathrm{P}(X > x)} \\ &= \frac{e^{-\lambda(x+y)}}{e^{-\lambda x}} = \mathrm{P}(X > y). \end{aligned}$$

Exercise 5- 2: (Suggested for personal/peer tutorial) The length of time (in hours) that a student takes to complete a one hour exam is a continuous random variable X with probability density function f_X defined by

$$f_X(x) = cx^2 + x, \qquad 0 < x \leq 1,$$

for some constant c, and zero otherwise.

(a) Find the value of c.

(b) By integration, find the cumulative distribution function F_X of X.

(c) Find the probability that a student completes the exam in less than half an hour.

(d) **Given** that a student takes longer than fifteen minutes to complete the exam, find the probability that they require at least half an hour, that is, find the conditional probability

$$\mathrm{P}\left(X > \frac{1}{2} \middle| X > \frac{1}{4}\right)$$

(e) In a class of two hundred students, find the probability that at most three students complete the exam in fewer than ten minutes.

Assume that the exam completion times for the two hundred students are independent random variables having the distribution specified above.

Hint: Consider discrete random variables $Y_1, ...Y_{200}$ where

$$Y_i = \begin{cases} 1 & \textit{student } i \textit{ completes the exam in fewer than ten minutes} \\ 0 & \textit{otherwise} \end{cases}$$

You need to calculate $\mathrm{P}(Y \leq 3)$ *where* $Y = \sum_{i=1}^{200} Y_i$.

Solution:

(a) Density function must integrate to 1 over $\mathrm{Im}X = [0,1]$, so

$$\int_0^1 f_X(x)\,dx = 1 \Longrightarrow \int_0^1 (cx^2 + x)\,dx = 1 \Longrightarrow \left[c\frac{x^3}{3} + \frac{x^2}{2}\right]_0^1 = 1 \Longrightarrow c = \frac{3}{2}$$

(b) Distribution function F_X given for $0 \leq x \leq 1$ by

$$F_X(x) = \int_0^x f_X(t)\,dt = \frac{x^3 + x^2}{2}$$

and $F_X(x) = 0$ for $x < 0$, and $F_X(x) = 1$ for $x > 1$.

(c) $\mathrm{P}(X < 1/2) = F_X(1/2) = \frac{3}{16}$.

(d) From the definition of the conditional probability, we have

$$\mathrm{P}(X > 1/2 | X > 1/4) = \frac{\mathrm{P}(X > 1/2, X > 1/4)}{\mathrm{P}(X > 1/4)} = \frac{\mathrm{P}(X > 1/2)}{\mathrm{P}(X > 1/4)} = \frac{1 - F_X(1/2)}{1 - F_X(1/4)} = \frac{104}{123}$$

(e) $Y \sim \mathrm{Bin}(200, \theta)$ where

$$\theta = \mathrm{P}(Y_i = 1) = \mathrm{P}(X < 1/6) = F_X(1/6) = \frac{7}{432}$$

Then

$$\mathrm{P}(Y \leq 3) = \mathrm{P}(Y = 0) + \mathrm{P}(Y = 1) + \mathrm{P}(Y = 2) + \mathrm{P}(Y = 3) = 0.593$$

Exercise 5- 3: The probability density function of continuous random variable X taking values in the range $\mathrm{Im}X = (0,2)$ is specified by

$$f_X(x) = \begin{cases} x & 0 < x < 1 \\ 2 - x & 1 \leq x < 2 \end{cases}$$

and zero otherwise. Find the cumulative distribution function of X, F_X, and hence find $\mathrm{P}(0.8 < X \leq 1.2)$.

Solution: We need to consider the ranges of integration carefully;

$$F_X(x) = \begin{cases} \displaystyle\int_0^x t\,dt & = \frac{x^2}{2} & 0 \leq x \leq 1 \\ \displaystyle\int_0^1 t\,dt + \int_1^x (2-t)\,dt & = 2x - \frac{x^2}{2} - 1 & 1 \leq x \leq 2 \end{cases}$$

and $F_X(x) = 0$ for $x < 0$, and $F_X(x) = 1$ for $x > 2$. Hence $\mathrm{P}(0.8 < X < 1.2) = F_X(1.2) - F_X(0.8) = 0.36$.

Exercise 5- 4: The *median* of a continuous random variable X is that value x such that $F_X(x) = 1/2$. Find the median of X when

(a) X has an *Exponential* distribution with parameter $\lambda > 0$, that is

$$f_X(x) = \lambda e^{-\lambda x}, \quad x > 0$$

and zero otherwise.

(b) $\log X$ has a normal distribution with parameters μ and σ^2, where $\mu \in \mathbb{R}, \sigma > 0$.

Solution:

(a) $X \sim \mathrm{Exp}(\lambda) \Rightarrow F_X(x) = 1 - e^{-\lambda x}$ for $x > 0$, so $F_X(x) = 1/2 \Leftrightarrow x = \frac{\log 2}{\lambda}$. Hence the median of X is given by $x = \frac{\log 2}{\lambda}$.

(b) We note that $\log X \sim \mathrm{N}(\mu, \sigma^2)$. We can define a new random variable by $Z = (\log X - \mu)/\sigma \sim \mathrm{N}(0, 1)$. The easiest way to see that Z is standard normally distributed is to use generating functions, which have not yet been used. Hence we derive the pdf of Z instead. For any $z \in \mathbb{R}$, we have, with $\mu \in \mathbb{R}, \sigma > 0$,

$$F_Z(z) = \mathrm{P}(Z \le z) = \mathrm{P}(\sigma^{-1}(\log X - \mu) \le z) = \mathrm{P}(\log X \le z\sigma + \mu).$$

Hence, the pdf of Z is given by

$$\begin{aligned} f_Z(z) &= \frac{d}{dz} F_z(z) = f_{\log X}(z\sigma + \mu)\sigma = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}((z\sigma + \mu) - \mu)^2 \right) \sigma \\ &= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2} \right), \end{aligned}$$

for any $z \in \mathbb{R}$, which is the density of a standard normally distributed random variable.

We note that, due to the symmetry of the standard normal distribution around 0, the median of the standard normal distribution satisfies $\Phi(z) = 1/2 \Leftrightarrow z = 0$. I.e. the median (and the mean) of the standard normal distribution is equal to 0.

Hence the median x of X satisfies:

$$\begin{aligned} F_X(x) &= 1/2 \\ \Leftrightarrow \mathrm{P}(X \le x) &= 1/2 \\ \Leftrightarrow \mathrm{P}((\log X - \mu)/\sigma \le (\log x - \mu)/\sigma) &= 1/2 \\ \Leftrightarrow F_Z((\log x - \mu)/\sigma) = \Phi((\log x - \mu)/\sigma) &= 1/2 \\ \Leftrightarrow (\log x - \mu)/\sigma &= 0 \\ \Leftrightarrow \log x &= \mu \\ \Leftrightarrow x &= e^{\mu}. \end{aligned}$$

Hence the median of X is given by $x = e^{\mu}$.

5.2 Prerequisites: Lecture 13

Exercise 5- 5: Suppose that X is a continuous random variable with range $\mathrm{Im}X = [0, 1]$, and probability density function f_X specified by

$$f_X(x) = 2(1 - x), \quad 0 \le x \le 1,$$

and zero otherwise. Find the probability distributions of random variables Y_1, Y_2 and Y_3 defined respectively by

(a) $Y_1 = 2X - 1$,

(b) $Y_2 = 1 - 2X$,

(c) $Y_3 = X^2$,

that is, in each case, find the range and the density function.

Solution: We can derive the densities from first principles:

(a) $Y_1 = 2X - 1 \implies \mathrm{Im}Y_1 = [-1, 1]$. Also,

$$F_{Y_1}(y) = \mathrm{P}(Y_1 \leq y) = \mathrm{P}(2X - 1 \leq y) = \mathrm{P}(\ X \leq (1+y)/2) = F_X((1+y)/2).$$

Hence

$$f_{Y_1}(y) = \frac{1}{2} f_X((1+y)/2) = \frac{1-y}{2}, \quad \text{for } -1 \leq y \leq 1,$$

and $f_{Y_1}(y) = 0$ otherwise.

(b) $Y_2 = 1 - 2X \implies \mathrm{Im}Y_2 = [-1, 1]$. Also,

$$F_{Y_2}(y) = \mathrm{P}(Y_2 \leq y) = \mathrm{P}(1 - 2X \leq y) = \mathrm{P}((X \geq (1-y)/2) = 1 - F_X((1-y)/2).$$

Hence

$$f_{Y_2}(y) = \frac{1}{2} f_X((1-y)/2) = (1+y)/2, \quad \text{for} -1 \leq y \leq 1,$$

and $f_{Y_2}(y) = 0$ otherwise.

(c) $Y_3 = X^2 \implies \mathrm{Im}Y_3 = [0, 1]$. Also,

$$F_{Y_3}(y) = \mathrm{P}(Y_3 \leq y) = \mathrm{P}(X^2 \leq y) = \mathrm{P}(-\sqrt{y} \leq X \leq \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y}).$$

Hence

$$f_{Y_3}(y) = \frac{1}{2\sqrt{y}} \left[f_X(\ \sqrt{y}\ ) + f_X(\ -\sqrt{y}\ ) \right] = \frac{1}{2\sqrt{y}} f_X(\ \sqrt{y}\ ) = \frac{(1-\sqrt{y})}{\sqrt{y}}, \quad \text{for } 0 < y \leq 1,$$

(as $f_X(x) = 0$ for $x < 0$) and $f_{Y_3}(y) = 0$ otherwise.

We could also use the general transformation formula for (a) and (b), as the transformations are 1-1.

(a) $g(t) = 2t - 1 \iff g^{-1}(t) = (1+t)/2$

(b) $g(t) = 1 - 2t \iff g^{-1}(t) = (1-t)/2$

Then use

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dt} \left\{ g^{-1}(t) \right\}_{t=y} \right|.$$

Exercise 5- 6: The continuous random variable X has a Uniform distribution on the interval $[-1, 1]$. Find the probability density function of random variables

(a) $Y = |X|$,

(b) $Z = X^2$.

Solution: We know that $f_X(x) = 1/2, \quad -1 \le x \le 1$ and zero otherwise. From first principles we get:

(a) $Y = |X| \implies \mathrm{Im}Y = [0,1]$ and

$$F_Y(y) = \mathrm{P}(Y \le y) = \mathrm{P}(|X| \le y) = \mathrm{P}(-y \le X \le y) = F_X(y) - F_X(-y).$$

Hence

$$f_Y(y) = \left[\, f_X(y) + f_X(-y) \,\right] = 1, \quad \text{for } 0 \le y \le 1,$$

and $f_Y(y) = 0$ otherwise.

(b) $Z = X^2 \implies \mathrm{Im}Z = [0,1]$ and

$$F_Z(z) = \mathrm{P}(Z \le z) = \mathrm{P}(X^2 \le z) = \mathrm{P}(-\sqrt{z} \le X \le \sqrt{z}) = F_X(\sqrt{z}) - F_X(-\sqrt{z}).$$

Hence,

$$f_Z(z) = \frac{1}{2\sqrt{z}}\left[\, f_X(\sqrt{z}) + f_X(-\sqrt{z}) \,\right] = \frac{1}{2\sqrt{z}}, \quad \text{for } 0 < z \le 1,$$

and $f_Z(z) = 0$ otherwise.

Exercise 5- 7: If X is **any** continuous random variable with distribution function F_X, show that

(a) the random variable $U = F_X(X)$ has a Uniform distribution on $(0,1)$; (*Hint:* You may first assume that F_X is invertible/strictly monotonically increasing. Then, as an extra challenge, can you find a proof which allows you to drop this additional assumption?)

(b) the random variables $Y_1 = -\log F_X(X)$ and $Y_2 = -\log(1-X)$ have an exponential distribution.

Solution: If X is continuous, then its cdf F_X is continuous and is monotonically increasing (but not necessarily strictly increasing). Also, $U = F_X(X)$ implies that $\mathrm{Im}U = (0,1)$.

(a)
- Suppose F_X is strictly montonically increasing. In this case, F_X is invertible, i.e.

  $$F_U(u) = \mathrm{P}(U \le u) = \mathrm{P}(F_X(X) \le u) = \mathrm{P}(X \le F_X^{-1}(u)) = F_X(F_X^{-1}(u)) = u, \quad 0 < u < 1,$$

  which implies that $U \sim \mathrm{U}(0,1)$.
- The general case (when we drop the strict monotonicity assumption) can be proven as follows:

  Step 1: For **any** random variable X with cdf F_X, we show that for any $x \in \mathbb{R}$:

  $$\mathrm{P}(F_X(X) \le F_X(x)) = F_X(x).$$

  We can decompose the event $\{\omega \in \Omega : F_X(X(\omega)) \le F_X(x)\}$ as follows (where we drop the ω notation as usual):

  $$\{F_X(X) \le F_X(x)\} = [\{F_X(X) \le F_X(x)\} \cap \{X \le x\}] \cup [\{F_X(X) \le F_X(x)\} \cap \{X > x\}].$$

  Let us look at the two events in the square brackets separately. For the first one, we note that, since F_X is monotonically increasing (not necessarily strictly), we have that

  $$\{X \le x\} \subseteq \{F_X(X) \le F_X(x)\}.$$

Hence

$$[\{F_X(X) \leq F_X(x)\} \cap \{X \leq x\}] = \{X \leq x\}.$$

Next, let us look at the event in the second square bracket:

$$\{F_X(X) \leq F_X(x)\} \cap \{X > x\}.$$

Due to the monotonicity of F_X, we have that

$$\{F_X(X) < F_X(x)\} \cap \{X > x\} = \emptyset.$$

Hence,

$$\{F_X(X) \leq F_X(x)\} \cap \{X > x\} = \{F_X(X) = F_X(x)\} \cap \{X > x\}.$$

So, by the law of total probability, we have

$$\begin{aligned} \mathrm{P}(F_X(X) \leq F_X(x)) &= \mathrm{P}(\{F_X(X) \leq F_X(x)\} \cap \{X \leq x\}) + \mathrm{P}(\{F_X(X) \leq F_X(x)\} \cap \{X > x\}) \\ &= \mathrm{P}(X \leq x) + \mathrm{P}(\{F_X(X) = F_X(x)\} \cap \{X > x\}) = F_X(x), \end{aligned}$$

since $\mathrm{P}(\{F_X(X) = F_X(x)\} \cap \{X > x\}) = 0$. To see this, note that either we have

$$\{F_X(X) = F_X(x)\} \cap \{X > x\} = \{F_X(X) = F_X(x), X > x\} = \emptyset,$$

which has probability zero, or

$$\{F_X(X) = F_X(x)\} \cap \{X > x\} = \{F_X(X) = F_X(x), X > x\} \neq \emptyset,$$

which implies that X needs to lie in the interior of an interval where F_X is constant, i.e. there exists an interval $I = [a, b]$ (or $I = [a, b)$) (with $a < b \leq \infty$) such that $x \in I$ and $F_X(u) = F_X(x)$ for all $u \in I$. Then, we have

$$\{F_X(X) = F_X(x)\} \cap \{X > x\} = \{F_X(X) = F_X(x), X > x\} = \{x < X, X \in I\}.$$

Hence

$$\mathrm{P}(\{F_X(X) = F_X(x)\} \cap \{X > x\}) = \mathrm{P}(\{X > x, X \in I\}) = F_X(b-) - F_X(x) = 0,$$

where $F_X(b-)$ denotes the left limit of the distribution function F_X in b.
Step 2: If X has a continuous cdf F_X, then the random variable $U = F_X(X)$ has the uniform distribution $U(0, 1)$.
Suppose $u \in (0, 1)$. Since F_X is continuous, there exists an $x \in \mathbb{R}$ such that $F_X(x) = u$. Applying the result from Step 1 leads to

$$\mathrm{P}(U \leq u) = \mathrm{P}(F_X(X) \leq F_X(x)) = u,$$

which implies that $U \sim \mathrm{U}(0, 1)$.
Note: In this question we assumed that X is continuous, which implies that its cumulative distribution function F_X is continuous.

(b) We have $Y_1 = -\log F_X(X) = -\log U \Longrightarrow \mathrm{Im}Y_1 = (0, \infty)$ and

$$F_{Y_1}(y_1) = \mathrm{P}(Y_1 \leq y_1) = \mathrm{P}(-\log U \leq y_1) = \mathrm{P}(U \geq e^{-y_1}) = 1 - F_U(e^{-y_1}) = 1 - e^{-y_1}, \quad y_1 > 0,$$

and 0 otherwise, which implies that $Y_1 \sim \mathrm{Exp}(1)$.
Also, we have $Y_2 = -\log(1 - F_X(X)) = -\log(1 - U) \Longrightarrow \mathrm{Im}Y_2 = (0, \infty)$, and

$$\begin{aligned} F_{Y_2}(y_2) &= \mathrm{P}(Y_2 \leq y_2) = \mathrm{P}(-\log(1 - U) \leq y_2) = \mathrm{P}(\log((1 - U)^{-1} \leq y_2) \\ &= \mathrm{P}((1 - U)^{-1}) \leq e^{y_2}) = \mathrm{P}(e^{-y_2} \leq (1 - U)) = \mathrm{P}(U \leq 1 - e^{-y_2}) = 1 - e^{-y_2}, \end{aligned}$$

for $y_2 > 0$ and 0 otherwise, which implies that $Y_2 \sim \mathrm{Exp}(1)$.

Exercise 5- 8: Suppose that random variable X has a standard normal distribution.

(a) Find the cumulative distribution function (cdf) of $Y = X^2$ in terms of the standard normal c.d.f. Φ.

Hint: For the c.d.f. of Y, we have

$$\mathrm{P}(Y \leq y) = \mathrm{P}(X^2 \leq y) = \mathrm{P}(|X| \leq \sqrt{y}).$$

(b) Find the probability density function of Y, f_Y.

(c) Identify (by name) the probability distribution of Y.

Solution:

(a) $X \sim \mathrm{N}(0,1)$, and thus if Φ and ϕ are the standard normal c.d.f. and p.d.f. respectively. We have immediately that $Y = X^2 \implies \mathrm{Im}Y = [0, \infty)$, and

$$F_Y(y) = \mathrm{P}(Y \leq y) = \mathrm{P}(X^2 \leq y) = \mathrm{P}(-\sqrt{y} \leq X \leq \sqrt{y}) = \Phi(\sqrt{y}) - \Phi(-\sqrt{y}).$$

(b) By differentiation, we have

$$f_Y(y) = \frac{1}{2\sqrt{y}}\,[\,\phi(\,\sqrt{y}\,) + \phi(\,-\sqrt{y}\,)\,] = \left(\frac{1}{2\pi}\right)^{1/2} y^{-1/2}\, e^{-y/2}, \quad \text{for } 0 < y < \infty,$$

and $f_Y(y) = 0$ otherwise.

(c) By inspection, we have that $Y \sim \mathrm{Gamma}\left(\frac{1}{2}, \frac{1}{2}\right) \equiv \chi^2_1$.

Exercise 5- 9: [From the January test in 2021] Let X be a continuous random variable with probability density given by

$$f_X(x) = \begin{cases} 2x, & \text{for } 0 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Define the random variable $Y = g(X)$, where

$$Y = g(X) = \begin{cases} X, & \text{for } 0 \leq X \leq 1/2, \\ \frac{1}{2}, & \text{for } X > \frac{1}{2}. \end{cases}$$

a) Find the image of Y.

b) Find the cumulative distribution function of Y.

c) Sketch the cumulative distribution function. Is Y a continuous or discrete random variable or is it neither discrete nor continuous?

Solution:

a) We note that $\mathrm{Im}X = [0, 1]$ and $\mathrm{Im}Y = [0, 1/2]$.

b) From $\mathrm{Im}Y = [0, 1/2]$ we can deduce that $F_Y(y) = 0$ for $y < 0$ and $F_Y(y) = 1$ for $y > 1/2$. For $0 \le y < 1/2$, we have

$$F_Y(y) = \mathrm{P}(Y \le y) = \int_0^y 2x dx = y^2.$$

Also,

$$\mathrm{P}(Y = 1/2) = \mathrm{P}(X \ge 1/2) = \int_{1/2}^1 2x dx = 1 - \frac{1}{4} = \frac{3}{4}.$$

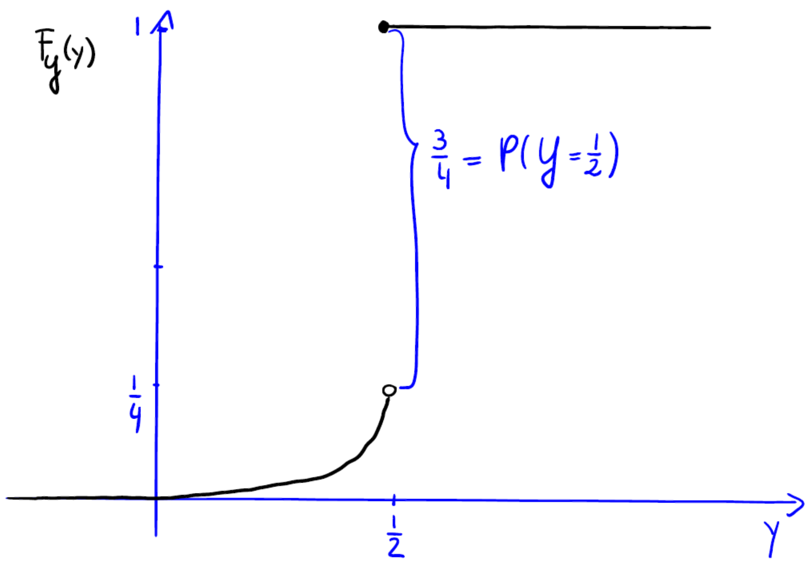
Hence

$$\mathrm{P}(Y \le 1/2) = \mathrm{P}(Y < 1/2) + \mathrm{P}(Y = 1/2) = \frac{1}{4} + \frac{3}{4} = 1.$$

Altogether, we get

$$F_Y(y) = \begin{cases} 0, & y < 0, \\ y^2, & 0 \le y < 1/2, \\ 1, & y \ge 1/2. \end{cases}$$

c) We sketch the cdf of Y as follows.



We note that the cdf of Y is continuous except for the point $y = 1/2$, where it is only right-continuous. The random variable Y is neither discrete nor continuous, but can be viewed as a mixture of a continuous and a discrete random variable.

5.3 Prerequisites: Lecture 14

Exercise 5- 10: The annual profit (in millions of pounds) of a manufacturing company is a function of product demand. If X is the continuous random variable corresponding to the demand in a given year, then the annual profit is also a continuous random variable, Y say, where

$$Y = 2(1 - e^{-2X})$$

If X has an Exponential distribution with parameter $\lambda = 6$, find the expected annual profit.

Solution: Using the general result for expectations of functions (LOTUS) with $g(X) = 2\left(1 - e^{-2X}\right)$

$$\begin{aligned} \mathrm{E}[g(X)] &= \int_{-\infty}^{\infty} g(x) f_X(x)\, dx = \int_0^{\infty} 2\left(1 - e^{-2x}\right) \lambda e^{-\lambda x}\, dx \\ &= 2\lambda \int_0^{\infty} \left(e^{-\lambda x} - e^{-(2+\lambda)x}\right) dx \\ &= 2\lambda \left[\frac{1}{\lambda} - \frac{1}{2+\lambda}\right] = \frac{4}{2+\lambda} = \frac{1}{2}. \end{aligned}$$

Exercise 5- 11: Consider the random variable Y as defined in Exercise 5- 9. How could you define the expectation for this random variable?

Solution: Since Y is a mixture of a continuous and discrete random variable, it has a ”density” function only on a certain subset of its image and a ”probability mass function” on the remaining discrete points. More precisely, we can define $f_Y(y) := 2y$ for $0 \leq y < 1/2$, i.e. we differentiate the cdf of Y wherever it is differentiable. The function f_Y is not a true density since it does not integrate to one! Moreover, for the points where the cdf of Y jumps, we collect the point probabilities (similar to a pmf). In our case, there exists only one point, where the cdf jumps: at $y = 1/2$, with $\mathrm{P}(Y = 1/2) = 3/4$. Then

$$\int_0^{1/2} 2y dy + \mathrm{P}(Y = 1/2) = \frac{1}{4} + \frac{3}{4}.$$

We can define the expectation of X as follows:

$$\begin{aligned} \mathrm{E}(Y) &= \int y f_Y(y) dy + \sum_y y \mathrm{P}(Y = y) \\ &= \int_0^{1/2} 2y^2 dy + \frac{1}{2}\mathrm{P}(Y = 1/2) \\ &= \frac{1}{12} + \frac{3}{8} = \frac{11}{24}. \end{aligned}$$