Imperial College London
Department of Mathematics

Solutions to Unseen Sheet 6 (week 9)

~~MATH40003 Linear Algebra and Groups Term 2, 2020/21~~

Unseen problem sheet for the tutorials in Week 9.

Question 1 In the following, X is a finite set and $G \leq \mathrm{Sym}(X)$.

For $x, y \in Y$, write $x \sim y$ if there exists $g \in G$ with $g(x) = y$.

(a) Prove that this is an equivalence relation on X.

The equivalence classes are of the form $\{g(x) : g \in G\}$ and are called the *orbits* of G on X.

The following formula gives a way of counting the number of orbits (it's sometimes referred to as Burnside's Lemma, or the Cauchy - Frobenius formula). If $g \in G$, let $\mathrm{Fix}(g) = \{x \in X : g(x) = x\}$, the set of fixed points of g in X.

Theorem: The number of orbits of G on X is $\frac{1}{|G|}\sum_{g\in G} |\mathrm{Fix}(g)|$.

(b) Check the formula in the case where $X = \{1, 2, 3\}$ and $G = S_3$.

Before proving the formula, let's give an application. This is a typical application where we want to count a number of objects 'up to symmetry'.

(c) Each of the 6 vertices of a regular hexagon is coloured with one of 3 possible colours (R,G,B), giving a total of 3^6 possible coloured hexagons. However, some of these colourings will look the same, up to (rotational) symmetries of the hexagon (of which there are 12). How many essentially different coloured hexagons are there? Here's the start of a solution:

Let G be the group of rotational symmetries of the hexagon. Let X be the set of coloured hexagons (so this has size 3^6). Each element of G gives a permutation of X and so we can think of G as a subgroup of $\mathrm{Sym}(X)$. We want to know how many orbits there are of G on X. So for each $g \in G$ we compute how many elements of X are fixed by g. ....

Complete the solution. Note that G is the dihedral group D_{12} with 12 elements, discussed in Section 4.3 of the notes. But you should be able to do the question without looking at this: just think about what the elements of G do to the hexagon.

(d) Prove the Theorem as follows. Call the orbits $X_1, \ldots, X_t$. If $x \in X_i$ then by Question 8 on Problem Sheet 7, $|X_i| = |G|/|G_x|$, where $G_x = \{g \in G : g(x) = x\}$. Now compute the size of the set

$$\{(g, x) : g \in G\,, x \in X\,, g(x) = x\}$$

in two different ways...

Complete the proof.

Solution: (a) This is standard.

(b) There is a single orbit of G on X. The identity element fixes 3 points; the three elements of order 2 in G each fix one point; the two elements of order 3 do not fix any points. So the average of the number of fixed points is:

$$(1.3 + 3.1 + 2.0)/6 = 1$$

which is indeed the number of orbits!

(c) Draw some pictures for the following! List the elements of G. First list the rotations in the plane of the hexagon:

(1) The identity;

(2) Rotations of order 6, so anticlockwise rotations through $2\pi k/6$ for $k = 1, 5$;

(3) rotations of order 3, so so anticlockwise rotations through $2\pi k/6$ for $k = 2, 4$;

(4) rotation of order 2, through angle π.

Then we have the symmetries which turn the hexagon over, of two types:

(5) ones about an axis through opposite vertices (3 of these);

(6) ones about an axis through the midpoints of opposite edges (3 of these).

For each of these six types, we count the number of elements of X fixed:

(1) 3^6

(2) 3

(3) 3^2

(4) 3^3

(5) 3^4

(6) 3^3.

Applying the formula, the number of orbits is therefore:

$$\frac{1}{12}(3^6 + 2.3 + 2.3^2 + 3^3 + 3.3^4 + 3.3^3) = 92.$$

(d) If $x \in X_i$ then $|G_x| = |G|/|X_i|$. So the size of the set is:

$$\sum_{x \in X} |G_x| = \sum_{i=1}^{t} \sum_{x \in X_i} |G_x| = \sum_{i=1}^{t} |X_i| \cdot |G|/|X_i| = t|G|.$$

On the other hand, the size of the set is:

$$\sum_{g \in G} |\mathrm{Fix}(g)|.$$

Hence the result.