1. Define a function $f : [a, b] \to \mathbb{R}$ by $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ -1, & x \notin \mathbb{Q}. \end{cases}$

   Prove that f is not integrable, but that f^2 is.

   Solution. Since $f(x)^2 = 1$ for all x, and constant functions are integrable, we know that f^2 is integrable. On the other hand, given any partition P of $[a, b]$ we have $\inf f(t) = -1$ and $\sup f(t) = 1$ on every interval, so that

$$L(f, P) = \sum_{i=0}^{n-1} (-1)\Delta x_i = -(b - a), \qquad U(f, P) = \sum_{i=0}^{n-1} (1)\Delta x_i = b - a$$

   independently of P. Thus $\underline{\int_a^b} f(x)\,dx = -(b - a)$ is not equal to $\overline{\int_a^b} f(x)\,dx = b - a$, and so f is not integrable.

2. Prove that any monotone increasing function $f : [a, b] \to \mathbb{R}$ is integrable, by considering its Darboux sums for partitions where every subinterval $[x_i, x_{i+1}]$ has the same length.

   Solution. Consider for all $n \in \mathbb{N}$ the partition

$$P_n = \left(a, a + \frac{b-a}{n}, a + 2\left(\frac{b-a}{n}\right), \dots, a + (n-1)\left(\frac{b-a}{n}\right), b\right),$$

   with $x_i = a + i(\frac{b-a}{n})$ for $0 \le i \le n$ and $\Delta x_i = \frac{b-a}{n}$ for $0 \le i < n$. Since f is monotone increasing, we have

$$m_i = \inf_{x_i \le t \le x_{i+1}} f(t) = f(x_i), \qquad M_i = \sup_{x_i \le t \le x_{i+1}} f(t) = f(x_{i+1}),$$

   and so

$$L(f, P_n) = \sum_{i=0}^{n-1} m_i \Delta x_i = (f(x_0) + f(x_1) + \cdots + f(x_{n-1}))\left(\frac{b-a}{n}\right)$$
$$U(f, P_n) = \sum_{i=0}^{n-1} M_i \Delta x_i = (f(x_1) + f(x_2) + \cdots + f(x_n))\left(\frac{b-a}{n}\right).$$

   from which we compute

$$U(f, P_n) - L(f, P_n) = (f(x_n) - f(x_0))\left(\frac{b-a}{n}\right) = (f(b) - f(a))\left(\frac{b-a}{n}\right).$$

   It follows that $\lim_{n\to\infty} (U(f, P_n) - L(f, P_n)) = 0$, and hence that f is integrable.

3. Define the *mesh* of a partition $P = (x_0, \dots, x_k)$ to be the maximal length of any subinterval:

$$\operatorname{mesh}(P) = \max_{0 \le i \le k-1} \Delta x_i = \max_{0 \le i \le k-1} (x_{i+1} - x_i).$$

Show that if $f : [a, b] \to \mathbb{R}$ is continuous and $\big(P_n\big)$ is any sequence of partitions of $[a, b]$ such that $\mathrm{mesh}(P_n) \to 0$ as $n \to \infty$, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} U(f, P_n).$$

The proof should follow the argument we used in lecture to show that continuous functions are integrable.

Solution. Fix $\epsilon > 0$. Since f is uniformly continuous on $[a, b]$, there is a $\delta > 0$ such that

$$\forall x, y \in [a, b],\ |x - y| < \delta \ \Rightarrow\ |f(x) - f(y)| < \frac{\epsilon}{b-a}.$$

Then $\lim\limits_{n\to\infty} \mathrm{mesh}(P_n) = 0$ implies that for this value of δ, there is an $N > 0$ such that $\mathrm{mesh}(P_n) < \delta$ for all $n \geq N$. Writing $P_n = (x_0, \dots, x_k)$, we compute that

$$U(f, P_n) - L(f, P_n) = \sum_{i=0}^{k-1} \left(\sup_{x_i \le t \le x_{i+1}} f(t) - \inf_{x_i \le t \le x_{i+1}} f(t) \right) \Delta x_i.$$

The extreme value theorem says that there are $y_i, z_i \in [x_i, x_{i+1}]$ such that

$$\sup_{x_i \le t \le x_{i+1}} f(t) = f(y_i), \qquad \inf_{x_i \le t \le x_{i+1}} f(t) = f(z_i),$$

and since $|z_i - y_i| \leq x_{i+1} - x_i \leq \mathrm{mesh}(P_n) < \delta$, we have $|f(z_i) - f(y_i)| < \frac{\epsilon}{b-a}$, so

$$\begin{aligned} U(f, P_n) - L(f, P_n) &= \sum_{i=0}^{k-1} (f(y_i) - f(z_i)) \\ &< \sum_{i=0}^{k-1} \frac{\epsilon}{b-a}(x_{i+1} - x_i) = \frac{\epsilon}{b-a}(b-a) = \epsilon. \end{aligned}$$

Since $U(f, P_n) - L(f, P_n) < \epsilon$ for all $n \geq N$, and we can find such an N for any $\epsilon > 0$, it follows that $\lim\limits_{n\to\infty} \big(U(f, P_n) - L(f, P_n)\big) = 0$, and so

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} U(f, P_n)$$

by Proposition 3.13 in the lecture notes.

4. (a) Prove for any $\theta \in \mathbb{R}$ and $n \in \mathbb{N}$ that if $\sin(\frac{\theta}{2}) \neq 0$, then

$$\sin(\theta) + \sin(2\theta) + \cdots + \sin(n\theta) = \frac{\sin(n\theta/2)\sin((n+1)\theta/2)}{\sin(\theta/2)}$$

using the formula $\sin(\alpha)\sin(\beta) = \frac{1}{2}(\cos(\alpha - \beta) - \cos(\alpha + \beta))$.

(b) Fix $t \in (0, \frac{\pi}{2}]$ so that $\sin(x)$ is monotone increasing on the interval $[0,t]$, and consider the partition $P_n = (0, \frac{t}{n}, \frac{2t}{n}, \dots, \frac{(n-1)t}{n}, t)$ of $[0,t]$. Compute the upper Darboux sum $U(\sin(x), P_n)$, and show that

$$\lim_{n\to\infty} U(\sin(x), P_n) = 2\sin^2(\tfrac{t}{2}).$$

Remark: This limit is equal to $1-\cos(t)$ by the double-angle formula $\cos(2\theta) = 1 - 2\sin^2(\theta)$, so problem 3 tells us that

$$\int_0^t \sin(x)\,dx = 2\sin^2\left(\frac{t}{2}\right) = 1 - \cos(t)$$

for all $t \in (0, \frac{\pi}{2}]$.

Solution. (a) If we call the sum S, then we have

$$\begin{aligned} S\sin\left(\frac{\theta}{2}\right) &= \sum_{k=1}^{n} \sin(k\theta)\sin\left(\frac{\theta}{2}\right) \\ &= \sum_{k=1}^{n} \frac{1}{2}\left[\cos\left(\left(k-\frac{1}{2}\right)\theta\right) - \cos\left(\left(k+\frac{1}{2}\right)\theta\right)\right] \\ &= \frac{1}{2}\left(\cos\left(\frac{\theta}{2}\right) - \cos\left(\frac{(2n+1)\theta}{2}\right)\right) \end{aligned}$$

because the sum in the second row telescopes. By one more application of the given identity, with $\alpha = \frac{(n+1)\theta}{2}$ and $\beta = \frac{n\theta}{2}$, we conclude that

$$S\sin\left(\frac{\theta}{2}\right) = \sin\left(\frac{(n+1)\theta}{2}\right)\sin\left(\frac{n\theta}{2}\right),$$

and we divide through by $\sin(\frac{\theta}{2})$ to solve for S.

(b) As in problem 2, the assumption that $\sin(x)$ is monotone increasing means that

$$U(\sin(x), P_n) = \sum_{i=0}^{n-1} \sin\left(\frac{(i+1)t}{n}\right)\frac{t}{n} = \frac{t}{n}\left(\sin(\theta) + \cdots + \sin(n\theta)\right)$$

with $\theta = \frac{t}{n}$, and so by part (a) we have

$$U(\sin(x), P_n) = \frac{t}{n}\cdot\frac{\sin(\frac{t}{2})\sin(\frac{(n+1)t}{2n})}{\sin(\frac{t}{2n})} = \frac{t/n}{\sin(t/2n)}\sin\left(\frac{t}{2}\right)\sin\left(\frac{t}{2}+\frac{t}{2n}\right).$$

We have $\lim_{x\to 0} \frac{tx}{\sin(tx/2)} = \lim_{x\to 0}\frac{t}{(t/2)\cos(tx/2)} = 2$ by l'Hôpital's rule, and $\frac{1}{n} \to 0$ as $x \to \infty$, so then

$$\lim_{n\to\infty} U(\sin(x), P_n) = 2\lim_{n\to\infty}\sin\left(\frac{t}{2}\right)\sin\left(\frac{t}{2}+\frac{t}{2n}\right) = 2\sin^2\left(\frac{t}{2}\right).$$

5. Let $f, g : [a, b] \to \mathbb{R}$ be bounded functions such that $f(x)$ and the product $f(x)g(x)$ are both integrable, and $f(x) \geq 0$ for all $x \in [a, b]$. If $c \leq g(x) \leq d$ for all $x \in [a, b]$, prove that

$$c \int_a^b f(x)\,dx \leq \int_a^b f(x)g(x)\,dx \leq d \int_a^b f(x)\,dx.$$

Solution. We claim that for any partition P of $[a, b]$, we have

$$cL(f, P) \leq L(fg, P) \leq U(fg, P) \leq dU(f, P).$$

To see this, if $P = (x_0, \ldots, x_n)$, then since $f(x)g(x) \geq cf(x)$ for all x, we have

$$\begin{aligned} L(fg, P) &= \sum_{i=0}^{n-1} \left(\inf_{t \in [x_i, x_{i+1}]} f(t)g(t) \right) \Delta x_i \\ &\geq \sum_{i=0}^{n-1} \left(\inf_{t \in [x_i, x_{i+1}]} cf(t) \right) \Delta x_i = cL(f, P) \end{aligned}$$

and the same argument with $f(x)g(x) \leq df(x)$ says that $U(fg, P) \leq dL(f, P)$.

Now we apply this claim to show that

$$c \underline{\int_a^b} f(x)\,dx = \sup_P cL(f, P) \leq \sup_P L(fg, P) = \underline{\int_a^b} f(x)g(x)\,dx,$$

so $c \int_a^b f(x)\,dx \leq \int_a^b f(x)g(x)\,dx$ since f and fg are both integrable, and likewise

$$\overline{\int_a^b} f(x)g(x)\,dx = \inf_P U(fg, P) \leq \inf_P dU(f, P) = d\overline{\int_a^b} f(x)\,dx$$

implies that $\int_a^b f(x)g(x)\,dx \leq d \int_a^b f(x)\,dx$.

6. (*) Define $f : [0, 1] \to \mathbb{R}$ by $f(x) = \begin{cases} 0, & x \notin \mathbb{Q} \\ 1/|q|, & x = \frac{p}{q} \in \mathbb{Q}. \end{cases}$

We proved in problem sheet 1 that f is discontinuous at all rational numbers.

(a) Compute the lower Darboux integral $\underline{\int_0^1} f(x)\,dx$.

(b) Consider the partition $P_n = (0, \frac{1}{n^3}, \frac{2}{n^3}, \ldots, \frac{n^3-1}{n^3}, 1)$ of $[0, 1]$. Show for n large that there are at most n^2 subintervals $[\frac{i}{n^3}, \frac{i+1}{n^3}]$ on which

$$M_i = \sup_{\frac{i}{n^3} \leq t \leq \frac{i+1}{n^3}} f(t)$$

is at least $\frac{1}{n}$.

(c) Prove that $U(f, P_n) \leq \frac{2}{n}$ for n large. (Hint: break the sum into terms where $M_i \geq \frac{1}{n}$ and terms where $M_i < \frac{1}{n}$.)

(d) Conclude that f is integrable, and compute $\int_0^1 f(x)\,dx$.

Solution. (a) We have $\inf_{t\in[x_i,x_{i+1}]} f(t) = 0$ on any interval, so the lower Darboux sum for any partition $P = (x_0, \dots, x_k)$ of $[0, 1]$ is

$$L(f, P) = \sum_{i=0}^{k-1} 0 \cdot \Delta x_i = 0,$$

and thus $\underline{\int_0^1} f(x)\,dx = \sup_P L(f, P) = 0$.

(b) If $f(t) \geq \frac{1}{n}$ then t must be a rational number of the form $\frac{p}{q}$ with $|q| \leq n$. On the interval $[0, 1]$ there are at most

$$2 + 1 + 2 + 3 + \cdots + (n - 1) = \frac{n(n - 1)}{2} + 2$$

of these: the first two counts 0 and 1, and then for each $q \geq 2$ we count at most $q-1$ additional values $\frac{1}{q}, \frac{2}{q}, \dots, \frac{q-1}{q}$ (though possibly fewer, because some of these may not be in lowest terms). And each such value of t belongs to at most two intervals, with equality iff $t = \frac{i}{n^3}$ and $0 < i < n^3$, so at most

$$2\left(\frac{n(n-1)}{2} + 2\right) = n^2 - n + 4 \leq n^2 \quad \text{(for } n \geq 4\text{)}$$

intervals $[\frac{i}{n^3}, \frac{i+1}{n^3}]$ contain a point t with $f(t) \geq \frac{1}{n}$. Then $M_i \geq \frac{1}{n}$ on these intervals, and $M_i \leq \frac{1}{n+1}$ on all other subintervals of $[0, 1]$.

(c) Since $M_i \leq 1$ for all i, we can write

$$\begin{aligned}
U(f, P_n) &= \sum_{M_i \geq \frac{1}{n}} M_i \Delta x_i + \sum_{M_i < \frac{1}{n}} M_i \Delta x_i \\
&= \frac{1}{n^3}\left(\sum_{M_i \geq \frac{1}{n}} M_i + \sum_{M_i < \frac{1}{n}} M_i\right) \\
&\leq \frac{1}{n^3}\left(\sum_{M_i \geq \frac{1}{n}} 1 + \sum_{M_i < \frac{1}{n}} \frac{1}{n}\right).
\end{aligned}$$

The first sum has at most n^2 terms, and the second sum has at most n^3 terms, so

$$U(f, P_n) \leq \frac{1}{n^3}\left(n^2(1) + n^3\left(\frac{1}{n}\right)\right) = \frac{2n^2}{n^3} = \frac{2}{n}.$$

(d) From part (c), we have

$$\overline{\int_0^1} f(x)\,dx = \inf_P U(f, P) \leq \inf_n U(f, P_n) \leq \inf_n \frac{2}{n} = 0.$$

But the upper Darboux integral is also as at least as big as $\underline{\int_0^1} f(x)\,dx = 0$, so the two are equal and we have $\int_0^1 f(x)\,dx = 0$.