Analysis 2 - Concise Notes

MATH50001

Term 1 Content

Arnav Singh

Colour Code - **Definitions** are **green** in these notes, **Consequences** are **red** and **Causes** are **blue**

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Mathematics
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United Kingdom

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Contents

1 Differentiation in Higher Dimensions

1.1 Euclidean Spaces

1.1.1 Preliminaries

Definition - Modulus Function

$$|x| := \begin{cases} x, & x \geq 0 \\ -x, & x<0 \end{cases}$$

Having the following properties:

(i) $\forall x \in \mathbb{R}, |x| \geq 0, |x| = 0 \iff x = 0$

(ii) $\forall x, y \in \mathbb{R}, |xy| = |x||y|$

(iii) $\forall x, y \in \mathbb{R}, |x + y| \leq |x| + |y|$ *(Triangle inequality)*

1.1.2 Euclidean space of dim. n

Define - Euclidean Space of dim. $n, \mathbb{R}^n$
Defined as the set of ordered n-tuples $(x^1, \ldots, x^n)$, s.t each $x^i \in \mathbb{R} \forall i$
$\mathbb{R}^n$ a vector space.

Define - Inner Product, $< \cdot, \cdot >, : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

$$< (x^1, x^2, \ldots, x^n), (y^1, y^2, \ldots, y^n) >= \sum_{i=1}^{n} x^i y^i$$

Define - Norm/Lengths, $|| \cdot || : \mathbb{R}^n \to \mathbb{R}$

$$||x|| = \sqrt{< x, x >}$$

Having the following properties:

(i) $\forall x \in \mathbb{R}^n, ||x|| \geq 0, ||x|| = 0 \iff x = \vec{0}$

(ii) $\forall \in \mathbb{R}, x \in \mathbb{R}^n ||\lambda x|| = |\lambda|||x||$

(iii) $\forall x, y \in \mathbb{R}^n, ||x + y|| \leq ||x|| + ||y||$ *(Triangle inequality)*

Definition - Cauchy-Schwartz Inequality

$$|\langle x, y\rangle| \leq ||x||||y||$$

1.1.3 Convergence of Sequences in Euclidean Spaces

Definition - Sequence in $\mathbb{R}^n$
An infinite ordered list, $x_0, x_1, \ldots,$ s.t $x_i \in \mathbb{R}^n\ \forall i$. Denoted $(x_i)_{i \geq 1}$ or $(x_i)_{i \in \mathbb{N}}$

Definition 1.1 - **Convergence**
A seq. $(x_i) \in \mathbb{R}^n$ **converges to** $x \in \mathbb{R}^n$ if $\forall \epsilon > 0, \exists N \in \mathbb{N}$ **s.t** $\forall i \geq \mathbb{N}, ||x_i - x|| < \epsilon$
Corollary
seq. $(x_i) \in \mathbb{R}^n$ converges to $x \in \mathbb{R}^n \iff$

$$\text{For } x_i = (x_i^1, \ldots, x_i^n) \text{ and } x = (x^1, \ldots, x^n)$$
$$x_i \to x \iff \forall k\ x_i^k \to x^k \text{ as } i \to \infty$$

1.2 Continuity

1.2.1 Open sets in Euclidean Spaces

Definition - Open Ball
Open ball of radius r is

$$B_r(x) = \{y \in \mathbb{R}^n : ||x - y|| < r\}$$

Definition 1.2 - Open sets
A set $U \subseteq \mathbb{R}^n$ is called **open**, if

$$\forall x \in U, \exists r > 0 \text{ such that} B_r(x) \subseteq U$$

1.2.2 Continuity at a point/on an open set

Definition 1.3 - Continuity at a point
Let $A \subset \mathbb{R}^n$ an open set, with $f : A \to R^n$
f **continuous at** $p \in A$ if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t } ||x - p|| < \delta \implies ||f(x) - f(p)|| < \epsilon$$

f is (pointwise) continuous on $A \subseteq \mathbb{R}^n \iff$ continuous $\forall p \in A$, we write f is continuous.
For small enough δ, we have $f(B_\delta(p)) \subseteq B_\epsilon(f(p))$

Theorem 1.2 - Composition of continuous functions
Let $A \subseteq \mathbb{R}^n$ open, $B \subseteq \mathbb{R}^m$ open and suppose $f : A \to B$ continuous at $p \in A$, and $g : B \to \mathbb{R}^l$ continuous at $f(p)$

$$\textbf{Then } g \circ f : A \to \mathbb{R}^l \textbf{ continuous at p}$$

Definition 1.4 - Limit of a function at a point
$A \subseteq \mathbb{R}^n$ an open set. f a function $f : A \to \mathbb{R}^m$, with $p \in A$ and $q \in \mathbb{R}^m$
Say $\lim_{x \to p} f(x) = q$ if $\forall \epsilon > 0, \exists \delta > 0$ **s.t** $\forall x \in A$ **with** $0 < ||x - p|| < \delta$ **we have** $||f(x) - p|| < \epsilon$

$$f \text{ continuous at } p \iff \lim_{x \to p} f(x) = q$$

Theorem 1.3 - Algebra of Limits
Suppose $A \subseteq \mathbb{R}^n$ open, with $p \in A$ and $f, g : A \to \mathbb{R}^n$

$$\lim_{x \to p} f(x) = F \text{ and } \lim_{x \to p} g(x) = G$$

Then:

(i) $\lim_{x \to p}(f(x) + g(x)) = F + G$

(ii) $\lim_{x \to p}(f(x)g(x)) = FG$

(iii) **If,** $G \neq 0$ **then** $\lim_{x \to p} \frac{f(x)}{g(x)} = \frac{F}{G}$

1.3 Derivative of a map of Euclidean Spaces

1.3.1 Derivative of a linear map

Lemma 1.5
The map $f : (a,b) \to \mathbb{R}$ differentiable at $p \in (a,b) \iff \exists$ map of the form $A_\lambda(x) = \lambda(x-p) + f(p)$ for some $\lambda \in \mathbb{R}$ s.t

$$\lim_{x \to p} \frac{|f(x) - A_\lambda(x)|}{|x-p|} = 0$$

Notation

$$h[v] \text{ for h a linear map, v a vector}$$

$$h(v) \text{ h a map, v a point in domain of h}$$

$$L(\mathbb{R}^n; \mathbb{R}^m) - \textbf{Set of linear maps from } \mathbb{R}^n \to \mathbb{R}^m$$

Definition 1.5 - Derivative in higher dimension
Suppose $\Omega \subset \mathbb{R}^n$ open. **The map $f : \Omega \to \mathbb{R}^m$ differentiable** at $p \in \Omega$ if **$\exists$ a linear map $\Lambda \in L(\mathbb{R}^n; R^m)$** such that

$$\lim_{x \to p} \frac{||f(x) - (\Lambda[x-p] + f(p))}{||x-p||} = 0$$

We write

$$Df(p) := \Lambda$$

Calling $Df(p)$ the derivative of f at p
Λ a $m \times n$ matrix called the **Jacobian**

Lemma 1.6 - Differentiable then continuous
$\Omega \subset \mathbb{R}^n$ open, $f : \Omega \to \mathbb{R}^m$ differentiable at $p \in \Omega \implies f$ continuous at p

Theorem 1.7 - Uniqueness of Derivative
*The derivative, **if it exists**, **is unique***

1.3.2 Chain Rule

Chain rule in $\mathbb{R}$
$f, g : \mathbb{R} \to \mathbb{R}, g$ differentiable at p, f differentiable at $g(p)$ Then $f \circ g$ differentiable at p with

$$(f \circ g)'(p) = f'(g(p))g'(p)$$

Theorem 1.8 - Chain rule in higher dim.
$\Omega \subset \mathbb{R}^n$ open, $\Omega' \subset \mathbb{R}^m$ open
With $g : \Omega \to \Omega'$ differentiable at $p \in \Omega$, $f : \Omega' \to \mathbb{R}^l$ differentiable at $g(p) \in \Omega'$
Then $h = f \circ g : \Omega \to \mathbb{R}^l$, differentiable at p, s.t

$$Dh(p) = D(f(g(p)) \circ Dg(p)$$

1.4 Directional Derivatives

1.4.1 Rates of change and Partial Derivatives

Definition - Directional Derivative
The **directional derivative** of f at p in the direction v is

$$\frac{\partial f}{\partial v}(p) := \lim_{t \to 0} \frac{1}{t}[f(p+vt) - f(p)] = Df(p)[v]$$

Definition - Partial derivatives
We can find any directional derivative at p, given we know the partial derivatives of f

$$D_i f(p) = \frac{\partial f}{\partial e_i}(p)$$

In $\mathbb{R}^3$ we have,

$$Df(p)[v] = \begin{pmatrix} D_1 f(p) & D_2 f(p) & D_3 f(p) \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix}$$

Definition - Gradient
Gradient of f at p

$$\nabla f(p) = \begin{pmatrix} D_1 f(p) \\ D_2 f(p) \\ D_3 f(p) \end{pmatrix} \qquad Df(p) = (\nabla f(p))^t$$

Theorem 1.9 - Jacobian
Suppose $\Omega \subset \mathbb{R}^n$ open and $f : \Omega \to \mathbb{R}^m$ of the form

$$f(x) = \left(f^1(x), f^2(x), \ldots, f^m(x)\right)$$

If f differentiable for some $p \in \Omega$ Then **Jacobian of f at p is:**

$$Df(p) = \begin{pmatrix} D_1 f^1(p) & \ldots & D_n f^1(p) \\ \vdots & \ddots & \vdots \\ D_1 f^m(p) & \ldots & D_n f^m(p) \end{pmatrix}$$

1.4.2 Relation between partial derivatives and differentiability

Theorem 1.12
Let $\Omega \subset \mathbb{R}^n$ open, $f : \Omega \to \mathbb{R}$. **Suppose the partial derivatives:**

$$D_i f(x) := \lim_{t \to 0} \frac{f(x + te_i - f(x)}{t}$$

exist $\forall x \in \Omega$, with each map $x \mapsto D_i f(x)$ continuous at $p, \forall i \implies$ f is differentiable at p

1.5 Higher Derivatives

1.5.1 Higher derivatives as linear maps

Can think of the differential of f, $Df(p)$ as a map

$$Df : \Omega \to L(R^n; R^m) = \Omega \to \mathbb{R}^{mn}$$

$$p \mapsto Df(p)$$

if map Df is continuous $\implies$ $f : \Omega \to \mathbb{R}$ is **continuously differentiable**

Definition - Higher derivative
If $Df : \Omega \to \mathbb{R}^{mn}$ differentiable at p, denote derivative of Df as $DDf(p) : \mathbb{R}^n \to \mathbb{R}^{nm}$

$$DDf(p) \in L(\mathbb{R}^n; \mathbb{R}^{nm}) = L(\mathbb{R}^n; L(\mathbb{R}^n; \mathbb{R}^m))$$

Where $DDf(p)$ is a linear map $\mathcal{L} \in L(\mathbb{R}^n; L(\mathbb{R}^n; \mathbb{R}^m))$, satisfying:

$$\lim_{x \to p} \frac{||Df(x) - Df(p) - \mathcal{L}[x - p]||}{||x - p||} = 0$$

$DDf(p)$ takes an n-vector to a $m \times n$ matrix

Definition - Continuously differentiable
$f : \Omega \to \mathbb{R}^m$ is k-times dfferentiable with all continuous derivatives $\implies$ **f is k-times continuously differentiable**
Testing for k-times differentiability
For $f = \left(f^1(x), f^2(x), \ldots, f^m(x)\right)$
If f differentiable at $p \in \Omega \implies$ we have partial derivatives $D_i f^j : \Omega \to \mathbb{R}$.
If Df differentiable, then 2^{nd} partial derivatives exist

$$D_k D_i f^j(p) := \lim_{t \to 0} \frac{D_i f^j(p + te_k) - D_i f^j(p)}{t}$$

Easy to check these exist and are continuous $\implies$ k-times differentiability at p

1.5.2 Symmetry of mixed partial derivatives

Theorem 1.13 - Schwartz' Theorem
Suppose $\Omega \subset \mathbb{R}^n$ open and $f : \Omega \to \mathbb{R}$ differentiable $\forall p \in \Omega$
Suppose also, for $i, j \in \{1, \ldots, n\}$, 2^{nd} partial derivatives D_iD_jf and D_jD_if exist and are continuous $\forall p \in \Omega$

$$\forall p \in \Omega, D_iD_jf(p) = D_jD_if(p)$$

Definition - Hessian
The matrix of 2^{nd} partial derivatives at the point p

$$\text{Hess } f(p) = [D_iD_jf(p)]_{i,j=1,\ldots,n}$$

Schwartz' Theorem says Hess $f(p$ is a symmetric matrix

1.5.3 Taylor's Theorem

Definition - Multi-inidices
Multi-index $\alpha \in (\mathbb{N})^n, \alpha = \big(\alpha_1, \ldots, \alpha_n\big)$
We define $|a| = \sum_{i=1}^{n} \alpha_i$ and

$$D^\alpha f := (D_1)^{\alpha_1}(D_2)^{\alpha_2} \ldots (D_n)^{\alpha_n} f,$$

And for a vector $h = \big(h_1, \ldots, h_n\big)$

$$h^\alpha := (h^1)^{\alpha_1}(h^2)^{\alpha_2} \ldots (h^n)^{\alpha_n}$$

Also

$$\alpha! := \alpha_1!\alpha_2! \ldots \alpha_n!$$

helpful examples

$$D^{(0,3,0)}f(p) = D_2^3 f(p)$$
$$D^{(1,0,1)}f(p) = D_1D_3f(p)$$
$$(x, y, z)^{(2,1,5)} = x^2y^1z^5$$

Theorem 1.14 - Taylor's Theorem in higher dim.
Suppose $p \in \mathbb{R}^n$ and $f : B_r(p) \to \mathbb{R}$ a k-times continuously differentiable $\forall q \in B_r(p)$, for some $k \geq 1 \in \mathbb{N}$
Then $\forall h \in \mathbb{R}^n$ with $||h|| < r$ We have

$$f(p + h) = \sum_{|\alpha| \leq k-1} \frac{h^\alpha}{\alpha!} D^\alpha f(p) + R_k(p, h)$$

Sum over all $\alpha = \big(\alpha_1, \ldots, \alpha_n\big)$
with $|\alpha| \leq k - 1$ and remainder term

$$R_k(p, h) = \sum_{|\alpha|=k} \frac{h^\alpha}{\alpha!} D^a f(x)$$

for some x s.t $0 < ||x - p|| < ||h||$
Evidently

$$\lim_{h \to 0} \frac{|R_k(p, h)|}{||h||^{k-1}} = 0$$

1.6 Inverse & Implicit Function Theorem

1.6.1 Inverse Function Theorem

Theorem 1.15 - *(Inverse Function Theorem)*
Let Ω an open set in $\mathbb{R}^n$, $f : \Omega \to \mathbb{R}^n$ continuously differentiable on Ω, $\exists q \in \Omega$ s.t $Df(q)$ invertible
Then $\exists$ open sets $U \subset \Omega$ and $V \subset \mathbb{R}^n, q \in U, f(q) \in V$ s.t

(i) $f : U \to V$, a bijection

(ii) $f^{-1} : V \to U$, continuously differentiable

(iii) $\forall y \in V$,

$$Df^{-1}(y) = [Df(f^{-1}(y))]^{-1}$$

1.6.2 Implicit Function Theorem

Theorem 1.16 - ***(Implicit Function Theorem - Simple version)***
$\Omega \subset \mathbb{R}^2$ open
$F : \Omega \to \mathbb{R}$ continuously differentiable and $\exists (x', y') \in \Omega$ s.t

(i) $F(x', y') = 0$, and

(ii) $D_2 F(x', y') \neq 0$

$\implies \exists$ open sets $A, B \subset \mathbb{R}$ with $x' \in A, y' \in B$ with a map $f : A \to B$ s.t

$$(x, y) \in A \times B \text{ satisfies } F(x, y) = 0 \iff y = f(x) \text{for some } x \in A$$

with $f : A \to B$ continuously differentiable.

Definition - **C^1-diffeomorphism**
$\Omega, \Omega' \subset \mathbb{R}^n$ open.
Say $f : \Omega \to \Omega'$ a C^1-diffeormorphism, if $f : \Omega \to \Omega'$ a bijection, continuously differentiable, and $\forall x \in \Omega, Df(x)$ **invertible**
$\mathcal{D}$ the set of all C^1-diffeomorphisms from $\Omega \to \Omega$, a group under group law; composition.

1.6.4 Implicit Function Theorem - General Form

Theorem 1.17 - ***(Implicit Function Theorem)***
$\Omega \subset \mathbb{R}^n, \Omega' \subset \mathbb{R}^m$ open sets
$F : \Omega \times \Omega' \to \mathbb{R}^m$ continuously differentiable on $\Omega \times \Omega'$ and sps $\exists (a, b) \in \Omega \times \Omega'$ s.t

(i) $f(p) = 0$ and,

(ii) $m \times n$ matrix

$$(D_{n+j} f^i(p)), \qquad 1 \leq i, j \leq m$$

invertible

$\implies \exists$ open sets $A \subset \Omega, B \subset \Omega'$ with $a \in A, b \in B$ with a map $g : A \to B$ s.t

$$g(x, y) = 0 \text{ for some } (x, y) \in A \times B \iff y = g(x) \text{for some } x \in A$$

with $g : A \to B$ continuously differentiable.

2 Metric and Topological Spaces

2.1 Metric Spaces

2.1.1 Motivation + Definition

Definition 2.1 - Metric
X an arbitrary set
Metric a function $d : X \times X \to \mathbb{R}$ satisfying:

(M1) $\forall x, y \in X;\ d(x,y) \geq 0, d(x,y) = 0 \iff x = y$ ***(positivity)***

(M2) $\forall x, y \in X;\ d(x,y) = d(y,x)$ ***(symmetry)***

(M3) $\forall x, y, z \in X d(x,y) \leq d(x,z) + d(z,y)$ ***(triangle inequality)***

Definition 2.2 - Metric space
Pair of a set and metric; $M = (X, d)$
Call elements of X points,with $d(x,y)$ distance between x, y w.r.t d

Definition

$$C([a,b]) = \{f : [a,b] \to \mathbb{R} | f : [a,b] \to \mathbb{R}\text{continuous}\}$$

2.1.2 Examples of metrics

Examples

- $d_2(x,y) = ||x - y||$; Euclidean metric on $\mathbb{R}^n$
- $d_{\text{disc}}(x,y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \end{cases}$
- $d_\infty(x,y) = \sup_{k \geq 1} |x^k - y^k|$
- $d_\infty(f,g) = \max_{a \leq t \leq b} |f(t) - g(t)|$ where $f, g \in C([a,b])$ ***(supremum/uniform metric)***

Definition 2.3. Induced metrics

(X, d) a metric space
$Y \subseteq X$, define $d|_Y : Y \times Y \to \mathbb{R}$ as $d|_Y(x,y) = d(x,y)\ \forall x, y \in Y$

Definition 2.3. Metric Subspace

Say $(Y, d|_Y)$ a metric subspace of (X, d)

Definition 2.4. Product metric space

(X_1, d_1) and (X_2, d_2) metric spaces.
define metric using d_1, d_2 $d : (X_1 \times X_2) \times (X_1 \times X_2) \to \mathbb{R}$.
$(X_1 \times X_2, d)$ a product metric space.

2.1.3 Normed Vector Spaces

Definition 2.5. Norm in Metric Spaces

V a vector space on $\mathbb{R}$. Say $|| \cdot || : V \to \mathbb{R}$ a **norm** on V if

(N1) $\forall v \in V,\ ||v|| \geq 0$ and $||v|| = 0 \iff v = 0$

(N2) $\forall v \in V, \forall \lambda \in \mathbb{R},\ ||\lambda v|| = |\lambda| \cdot ||v||$

(N3) $\forall u, v \in V,\ ||u + v|| \leq ||u|| + ||v||$

Definition - Normed vector space
A pair of a vector space $(V, || \cdot ||)$
note $|| \cdot ||$ is a metric on $V \implies$ normed vector space a metric space.

2.1.4 Open sets in metric spaces

Definition 2.6. **Open ball in metric spaces**

(X, d), with $x \in X, \epsilon \in \mathbb{R}; \epsilon > 0$

$$\text{Ball radius } \epsilon;\ B_\epsilon(x) = \{x' \in X | d(x, x') < \epsilon\}$$

notation; $B_\epsilon(x, X, d)$

Definition 2.7. **Open set in metric space**

(X, d) a metric space. $U \subseteq X$ open in (X, d) if:

$$\forall u \in U,\ \exists \delta > 0 \in \mathbb{R} \text{ s.t } B_\delta(u) \subset U$$

Definition 2.8. **Topologically equivalent**

d_1, d_2 metrics on a set X **topologically equivalent** if:

$$\forall\ U \subseteq X,\ U \text{ open in } (X, d_1) \iff U \text{ open in } (X, d_2)$$

2.1.5 Convergence in Metric Spaces

Definition 2.9. **Convergence in Metric Spaces**

(X, d) a metric space. $(x_n)_{n \geq 1}$ a sequence in X.
Say $(x_n) \to x \in (X, d)$ if

$$\forall\ \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t } \forall\ n \geq N, d(x, x_n) < \epsilon$$

Lemma 2.7. - if (x_n) converges in $(X, d) \implies$ limit is unique
Corollary - d_1, d_2 topologically equivalent $\iff$ (x_n) converges in (X, d_1) and (X, d_2)

2.1.6 Closed sets in metric spaces

Definition 2.10. **Closed set in Metric Spaces**

(X, d) a metric space. $V \subseteq X$ a set.
V **closed** in (X, d) if $\forall\ (x_n) \in V$ s.t $(x_n) \to x$ convergent in $(X, d) \implies x \in V$

Theorem 2.9.

(X, d) a metric space. $V \subseteq X$

$$V \text{ closed in } (X, d) \iff X \backslash V \text{ open in } (X, d)$$

Lemma 2.10

(i) Intersection of closed sets in (X, d) is a closed set in (X, d)

(ii) Finite union of closed sets in (X, d) a closed set in (X, d)

2.1.7 Interior, isolated, limit, and boundary points in metric spaces

Definition 2.11. - 2.12.

(X, d) a metric space, $V \subset X,\ x \in X$

(i) x an **interior/inner point** of V if
$$\exists \delta > 0, \quad \text{s.t } B_\delta(x) \subset V$$

(a) **Interior of V; V°** - $\{v \in V : v \text{ an interior point of } V\}$

(ii) x a **limit/accumulation point** of V if
$$\forall \delta > 0, (B_\delta(x) \cap V)\backslash\{x\} \neq \emptyset$$

Note: *not all limit points of V are in V*

(b) **Closure of V; $\bar{V}$** - $V \cup \{v \text{ a limit point of } V\}$

(iii) x a **boundary point of V** if
$$\forall \delta > 0, B_\delta \cap V \neq \emptyset \text{ and } B_\delta(x)\backslash V \neq \emptyset$$

(c) **Boundary of V; ∂V -** $\{v \in X : v \text{ a boundary point of } V\}$

(iv) x an **isolated point** of V if
$$\exists \delta > 0, \text{ s.t } V \cap B_\delta(x) = \{x\}$$

Lemma 2.11 (X, d) a metric space, $V \subseteq X$
$x \in X$ a limit point of $V \iff \exists$ sequence in $V\backslash\{x\}$ converging to x.

Definition 2.13. **Dense and Seperable subsets**

(X, d) a metric space

- $V \subseteq X$ **dense** in X if $\bar{V} = X$
- (X, d) seperable if, $\exists$ dense countable subset of X

2.1.8 Continuous maps of metric spaces

Definition 2.14. **Continuity in metric spaces**

$(X, d_X), (Y, d_Y)$ metric spaces.
$f : X \to Y$ a map

(i) f **continuous** at $x \in X$ if
$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t } \forall x' \in X \text{ s.t } d_X(x', x) < \delta, d_Y(f(x), f(x')) < \epsilon$$

(ii) $f : X \to Y$ continuous if f continuous $\forall x \in X$

(iii) $f : X \to Y$ uniformly continuous if f continuous $\forall x \in X$ with $\delta = \delta(\epsilon)$ not depending on x

Theorem 2.12.

$(A_1, d_1), (A_2, d_2)$ metric spaces
$f : A_1 \to A_2$ continuous $\iff$ pre-image of any open set in A_2 is an open set in A_1
$f : A_1 \to A_2$ continuous $\iff$ pre-image of any closed set in A_2 is a closed set in A_1

Theorem 2.13.

$(X, d_X), (Y, d_Y)$ metric spaces
$f : X \to Y$ a map;

$$f \text{ continuous at } x \in X \iff \text{ for any sequence } (x_n) \to x;\ f(x_n) \to f(x) \text{ in } (Y, d_Y)$$

Definition 2.15. **Homeomorphism**

$(X_1, d_1), (X_2, d_2)$ metric spaces.

(i) $f : X_1 \to X_2$ a **homeomorphism** if

- $f : X_1 \to X_2$ a bijection
- $f : X_1 \to X_2$ and $f^{-1} : X_2 \to X_1$ continuous

(ii) Say $(X_1, d_1), (X_2, d_2)$ **homeomorphic** if $\exists$ homeomorphism from X_1 to X_2

Definition 2.16.

$(X, d_X), (Y, d_Y)$ metric spaces with $f : X \to Y$

(i) f is **Lipschitz** if $\exists$ constant $M > 0$ s.t $\forall x_1, x_2 \in X, d_Y(f(x_1), f(x_2)) \leq M \cdot d_X(x_1, x_2)$

(ii) f is **bi-Lipschitz** if $\exists$ constants $M_1, M_2 > 0$ s.t $\forall x_1, x_2 \in X$

$$M_2 \cdot d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2)) \leq M_1 \cdot d_X(x_1, x_2)$$

Corollary; *any bi-Lipschitz map is injective*

(iii) f an **isometry/distance preserving** if $\forall x_1, x_2 \in X$;

$$d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2)$$

2.2 Topological Spaces

2.2.2 Topology on a set

Definition 2.17. **Topology**

A an arbitrary set. τ a collection of subsets of A
τ a **topology** on A if:

(T1) $\emptyset \in \tau$ and $A \in \tau$

(T2) $G_\alpha \in \tau$ for α in a (finite) set $I \implies \bigcup_{\alpha \in I} G_\alpha \in \tau$

(T3) $G_1, G_2, \ldots, G_m \in \tau \implies \bigcap_{i=1}^m G_i \in \tau$

A **topological space**; (A, τ) a pair of a set A and topology τ on A. Each element in τ an open set in (A, τ)
U a neighbourhood of a if $U \in \tau$ and $a \in U$

Example 2.25. Some Topologies

1. **Coarse topology -** A arbitrary set, $\tau = \{\emptyset, A\}$
2. **Induced topology -** (X, d) a metric space, with τ the collection of all open sets in (X, d)
3. **Order Topology -** $A = \mathbb{R}$ with τ collection of subsets of $\mathbb{R}$ of form $(a, +\infty)$, $a \in \mathbb{R} \cup \{-\infty, +\infty\}, (infty, +\infty) := \emptyset$
4. **Discrete Topology** - A arbitrary, $\tau = \mathcal{P}(A)$
5. **Product topology -**

Definition. **Metrisable topological space**
Say topological space (X, τ) **metrisable** if $\exists$ metric on X which induces a topology τ.
Definition. **Induced and Subspace topology**
(X, τ) a topological space. $Y \subset X$

$$\tau_Y = \{U \cap Y | U \in \tau\}$$

τ_Y the **induced topology** on Y from (X, τ)
(Y, τ_Y) has the **subspace topology** induced from (X, τ)

Definition 2.18. Stronger topology

A a set, with τ_1, τ_2
Say τ_1 stronger (or finer) than τ_2 if $\tau_2 \subset \tau_1$
Lemma 2.14.
(A, τ)
A set $G \subset A$ open $\iff \forall\ x \in G,\ \exists$ neighbourhood of x contained in G

Definition 2.19. Interior in Topological space

(A, τ) a topological space. $\Omega \subseteq A$
$z \in \Omega$ an interior point of Ω if

$$\exists U \in \tau \text{ s.t } z \in U \text{ and } U \subset \Omega$$

interior of Ω; Ω° $= \{z \in \Omega | z \text{ an interior point of } \Omega\}$
Properties of interior

- $S \subset T \implies S^\circ \subset T^\circ$
- S open in $A \iff S = S^\circ$
- S° largest open set contained in S

2.2.3 Convergence, and Hausdorff property

Definition 2.20. Convergence in Topological Spaces

(A, τ) a topological space. $(x_n)_{n \geq 1}$ a sequence in A
(x_n) **converges** in (A, τ) if

$$\exists x \in A \text{ s.t } \forall\ G \in \tau \text{ with } x \in G,\ \exists N \in \mathbb{N},\ \text{s.t } \forall n \geq N, x_n \in G$$

Definition 2.21. Hausdorff

(A, τ) called **Hausdorff** if:

$$\forall x, y \in A\ x \neq y,\ \exists \text{ open set } U, V \text{ s.t } x \in U, y \in V \text{ and } U \cap V = \emptyset$$

Say U and V seperate x and y

Theorem 2.14.

(A, τ) a Hausdorff topological space. (x_n) a sequence in A.
if (x_n) convergent in $(A, \tau) \implies$ limit is unique.

2.2.4 Closed sets in topological spaces

Definition 2.22. Closed set in Topological space

(A, τ) a topological space.
$V \subseteq A$. Say V closed in $(A, \tau) \iff A \backslash V \in \tau$

Lemma 2.17.
(A, τ) a topological space $\implies \emptyset$ and A closed in (A, τ)

(i) intersection of closed sets in (A, τ) is a closed set in (A, τ)

(ii) union of a finite number of closed sets in (A, τ) is a closed set in (A, τ)

Definition 2.23. Limit/Accumulation point in Topological Spaces

(A, τ), a topological space, $S \subseteq A$
$x \in A$ a **limit/accumulation point** of S if

$$\forall\ U \text{ a neighbourhood of } x,\ (S \cap U) \backslash \{x\} \neq \emptyset$$

x not necessarily in S
Closure of S, $\bar{S}$ $= S \cup \{x \in A | x \text{ a limit point of } S\}$
Lemma
S closed in $(A, \tau) \iff S = \bar{S}$

2.2.5 Continuous maps on topological spaces

Definition 2.24. Continuity in topological space

$(X, \tau_X), (Y, \tau_Y)$ with $f : X \to Y$
f continuous on X if:

$$\forall \text{open sets } U \in Y,\ f^{-1}(U) \text{ open in } X$$

Theorem 2.20.

$(X, \tau_X), (Y, \tau_Y)$ with $f : X \to Y$
f continuous $\iff$ pre-image of closed set in Y is closed in X

Theorem 2.21.

$(X, \tau_X), (Y, \tau_Y), (Z, \tau_Z)$
$f : X \to Y, g : Y \to Z$ continuous $\implies$ $g \circ f : X \to Z$ continuous

Definition 2.25. Homeomorphisms in Topological space

$f : X \to Y$ a homeomorphism is $f : X \to Y$ bijective with f and f^{-1} continuous

Definition 2.25. Topologically equivalent in Topological space

$(X, \tau_X), (Y, \tau_Y)$ **topologically equivalent/homeomorphic** if $\exists$ homeomorphism from $X \to Y$

2.3 Connectedness

2.3.1 Connected sets

Definition 2.26. Disconnected sets

For (X, d) a metric space, consider $T \subseteq X$. T **disconnected**,if $\exists$ open sets $U, V \in X$ s.t:

(i) $U \cap V = \emptyset$

(ii) $T \subseteq U \cup V$

(iii) $T \cap U \neq \emptyset$ and $T \cap V \neq \emptyset$

Set connected if not disconnected. i.e for any 2 of the properties that hold from above the 3rd cannot.
Lemma 2.23.
(X, d) a metric space. $T \subseteq X$

$$T \text{ disconnected} \iff \exists \text{ continuous } f : T \to \mathbb{R} \text{ s.t } f(T) = \{0, 1\}$$

Theorem 2.22.

Consider $(\mathbb{R}, d)$, $S \subseteq \mathbb{R}$

$$S \text{ connected} \iff S \text{ an interval}$$

2.3.2 Continuous maps + Connected sets

Theorem 2.27.

(A, d_1) and (A, d_2) metric spaces. $f : A_1 \to A_2$ continuous map
$S \subset A$ connected $\implies$ $f(S)$ connected
Corollary 2.28.
$f : (X, d_X) \to (Y, d_Y)$ a homeomorphism

$$X \text{ connected} \iff Y \text{ connected}$$

Theorem 2.29.

(X, d) connected metric space, $f : X \to \mathbb{R}$ continuous. Assume $\exists a, b \in X$ s.t $f(a) < 0, f(b) > 0 \implies \exists c \in X$ s.t $f(c) = 0$

2.3.3 Path Connected Sets

Definition 2.28. Path

Under (X,d) given $a,b \in X$
Path from $a \to b$ a continuous map $f:[0,1] \to X$ s.t $f(0)=a, f(1)=b$

Definition 2.29. Path Connected

(X,d) path connected if $\forall a,b \in X, \exists$ path from $a \to b$ in X

Theorem 2.30.

if (X,d) path connected $\implies$ connected

2.4 Compactness

2.4.1 Compactness by covers

Definition 2.30. Covers

(X,d) a metric space. $Y \subseteq X$

(i) collection R of open subsets of X an **open cover** for Y if

$$Y \subseteq \bigcup_{v \in R} v$$

(ii) Given open cover R for Y
Say C a **sub-cover** of R for Y if $C \subseteq R$ and $Y \subseteq \bigcup_{v \in R} v$

(iii) Open cover R for Y is a **finite cover** if R has finitely many elements.

Definition 2.31. Compact

(X,d) a metric space
$Y \subseteq X$ compact in (X,d) if every open cover for Y has a finite sub-cover.

Proposition 2.32.
$a,b \in \mathbb{R},\ a \leq b$ in (R,d_1) we have $[a,b]$ compact

Proposition 2.33.
(X,d) a metric space, $Y \subseteq X$
X compact, Y closed $\implies$ Y compact.

Theorem 2.34.

(X,d) a metric space $Y \subset X$

$$Y \text{ compact} \implies Y \text{ closed}$$

Theorem 2.35.

$(X,d_X),(Y,d_Y)$ metric spaces. Considering $(X \times Y, d)$
$d((x_1,y_1),(x_2,y_2)) = d_1(x_1,x_2) + d_2(y_1,y_2)$
X,Y compact $\implies$ $(X \times Y, d)$ compact
Corollary.
$[a_1,b_1] \times [a_2,b_2] \cdots \times [a_{n-1},b_{n-1}] \times [a_n,b_n]$ compact in $\mathbb{R}^n$

Definition 2.32. Bounded

(X,d) non-empty metric space, $Z \subseteq X$
Z **bounded** in (X,d) if $\exists M \in \mathbb{R}$ s.t $\forall x,y \in Z; d(x,y) \leq M$
S arbitrary set. $f:S \to X$ bounded if $f(S)$ bounded in X
Lemma 2.37.
(X,d) compact metric space $\implies$ X bounded

Theorem 2.36. Heine-Borel

Consider $(\mathbb{R}^n, d_2)$, $X \subseteq \mathbb{R}^n$
X compact $\iff$ X closed and bounded

2.4.2 Sequential Compactness

Definition 2.33. Sequentially compact

(X, d) sequentially compact, if for every sequence in X has convergent subsequence in (X, d)

$$\forall (x_n)_{n\geq 1} \in X,\ \exists (x_{n_k})_{k\geq 1},\ x \in X \text{ s.t } x_{n_k} \to x$$

Lemma 2.39.
(X, d) a metric space. with sequence $(x_n)_{n\geq 1}$ s.t $\exists (x_{n_k})_{k\geq 1},\ x \in X$ s.t $x_{n_k} \to x$.

$$\iff \exists x \in X \text{ s.t } \forall \epsilon > 0 \text{ there are infinitely many } i \text{ s.t } x_i \in B_\epsilon(x)$$

Theorem 2.41. Bolzanno-Weierstrass

Any bounded sequence in $\mathbb{R}^n$ has convergent subsequence.

Theorem 2.40. + 2.42.

(X, d) metric space.

$$X \text{ Compact } \iff X \text{ Sequentially Compact}$$

2.4.3 Continuous maps + Compact Sets

Theorem 2.41.

$(X, d_X), (Y, d_Y)$ metric spaces.
$f : X \to Y$ a continuous map if

$$Z \text{ compact in } X \implies f(Z) \text{ compact in } Y$$

Corollary 2.44.
$(X, d_X), (Y, d_Y)$ metric spaces, $f : X \to Y$ a homeomorphism

$$\implies X \text{ compact } \iff Y \text{ compact}$$

Theorem 2.45.

Every continuous map from compact metric space to a metric space is uniformly continuous.
Corollary 2.46. $f : [a, b] \to \mathbb{R}$ continuous $\implies f$ uniformly continuous

Theorem 2.47.

(X, d_X) compact, $f : X \to \mathbb{R}$ continuous $\implies f$ bounded above and below attaining its upper & lower bounds

Theorem 2.48.

$f : \mathbb{R} \to \mathbb{R}$ continuous w.r.t Euclidean metrics on domain and range.
$\forall\ [a, b]$ we have $f([a, b])$ of the form $[m, M]$ for $m, M \in \mathbb{R}$

2.5 Completeness

2.5.1 Complete metric spaces Banach space

Definition 2.34. Cauchy Sequence

(X, d) a metric $(x_n)_{n\geq 1}$ sequence in X
Say $(x_n)_{n\geq 1}$ a **Cauchy sequence** in (X, d) if

$$\forall \epsilon > 0, \exists N_\epsilon \in \mathbb{N} \text{ s.t } \forall n, m \geq N_\epsilon \text{ we have } d(x_n, x_m) < \epsilon$$

Definition 2.35. Complete & Banach

(i) metric space (X, d) **complete** if every Cauchy sequence in X converges to a limit in X

(ii) Normed vector space $(V, ||\cdot||)$ a **Banach space** if V with induced metric space $d_{||||}$ a complete metric space.

Theorem 2.51.

Assume $(f_n : [a, b] \to \mathbb{R})_{n\geq 1}$ sequence of continuous functions converging uniformly to $f : [a, b] \to \mathbb{R} \implies f : [a, b] \to \mathbb{R}$ continuous

Theorem 2.52.

Metric space $(C([a, b]), d_\infty)$ is complete or equivalently $(C([a, b]), ||\cdot||_\infty)$ a Banach space

Theorem 2.53.

(X, d) a compact metric space $\implies (X, d)$ complete

2.5.2 Arzelà-Ascoli

Definition 2.36. **Uniformly bounded & Uniformly equi-continuous**

Let $\mathcal{C}$ a collection of functions $f : [a, b] \to \mathbb{R}$

1. Say collection $\mathcal{C}$ **uniformly bounded** if $\exists M$ s.t $\forall f \in \mathcal{C}$ and $\forall x \in [a, b] \implies |f(x)| < M$
2. Say collection $\mathcal{C}$ **uniformly equi-continuous** if $\forall \epsilon > 0, \exists \delta > 0$ s.t $\forall f \in \mathcal{C}$ and $\forall x_1, x_2 \in [a, b]$ s.t $|x_1 - x_2| < \delta$ we have $|f(x_1) - f(x_2)| < \epsilon$

Theorem 2.54. **Arzelà-Ascoli**

Assume $\mathcal{C}$ collection of continuous functions $f : [a, b] \to \mathbb{R}$ if $\mathcal{C}$ uniformly bounded and uniformly equi-continuous $\implies$ every sequence in $\mathcal{C}$ has convergent subsequence in $(C([a, b], d_\infty)$

2.5.3 Fixed point theorem

Definition 2.37. **Contracting**

(X_1, d_1) and (X_2, d_2), with $f : X_1 \to X_2$
Say f **contracting** if $\exists K \in (0, 1)$ s.t $\forall a, b \in X$ we have

$$d_2(f(a), f(b)) \leq K \cdot d_1(a, b)$$

Every contracting map is continuous.

Definition 2.37. **Fixed point**

$f : X \to X$ say $x \in X$ a **fixed point** of f if $f(x) = x$

Theorem 2.55. **Banach fixed point theorem**

(X, d) a non-empty complete metric space.
$f : X \to X$ a contracting map $\implies$ f has unique fixed point in X