Imperial College London
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Solutions to Question Sheet 6

MATH40003 Linear Algebra and Groups — Term 2, 2020/21

Problem sheet released on Friday of week 7. All questions can be attempted before the tutorials at the end of week 8. Solutions will be released following the tutorials.

Question 1 Suppose $(G, \cdot)$ is a group and H is a subgroup of G. Prove that each of the following is an equivalence relation on G (where g, h are elements of G):

(i) $g \sim_1 h$ if and only if there is $k \in G$ with $h = kgk^{-1}$;

(ii) $g \sim_2 h$ if and only if $h^{-1}g \in H$.

In the case where $(G, .)$ is the group $(\mathbb{R}^2, +)$ and H is the subgroup $\{(x, x) \in \mathbb{R}^2 : x \in \mathbb{R}\}$, describe geometrically the $\sim_2$-equivalence classes. What are the $\sim_1$-equivalence classes?

Solution: (i) Clearly $g \sim_1 g$ (take $k = e$). If $g \sim_1 h$ take k with $kgk^{-1} = h$. Then $g = k^{-1}hk = k^{-1}h(k^{-1})^{-1}$. So $h \sim_1 g$. Finally if $g \sim_1 h$ and $h \sim_1 f$ take $k, j \in G$ with $h = kgk^{-1}$ and $f = jhj^{-1}$. So $f = jkgk^{-1}j^{-1} = (jk)g(jk)^{-1}$, so $g \sim_1 f$, as required.

(ii) $g \sim_2 g$ as $g^{-1}g = e \in H$. If $g \sim_2 h$ then $h^{-1}g \in H$, so $g^{-1}h = (h^{-1}g)^{-1} \in H$, whence $h \sim_2 g$. If $g \sim_2 h$ and $h \sim_2 f$ then $g^{-1}h, h^{-1}f \in H$. So taking the product, $g^{-1}f \in H$ and $g \sim_2 f$. (Note that each of the three things to be verified corresponds to one of the conditions in the test for a subgroup.)

In the example the equivalence class C containing a point $(a, b) \in \mathbb{R}^2$ has the property that $(c, d) \in C$ iff there is $(x, x) \in H$ with $(c, d) = (a, b) + (x, x)$. So we might write $C = (a, b) + H$. In other words, C is the line through (a, b) which is parallel to the line H.

This group is abelian (and written additively), so $g \sim_1 h$ iff there is k with $h = k + g - k = g$. So the $\sim_1$-classes are just sets of size 1 (i.e. $\sim_1$ is the equality relation!).

Question 2 Suppose $(G, \cdot)$ is a group and H, K are subgroups of G.

(i) Show that $H \cap K$ is a subgroup of G.

(ii) Show that if $H \cup K$ is a subgroup of G then either $H \subseteq K$ or $K \subseteq H$.

Solution: (i) Use the test from the notes. As $e \in H \cap K$ we have $H \cap K \neq \emptyset$. If $g, h \in H \cap K$ then $g, h \in H$, so $gh \in H$ as H is a subgroup. Similarly $gh \in K$, so $gh \in H \cap K$. Also $g^{-1} \in H$ as H is a subgroup and $g \in H$; similarly $g^{-1} \in K$. So $g^{-1} \in H \cap K$.

(ii) If not, there exist $h \in H \setminus K$ and $k \in K \setminus H$. We have $hk \in H \cup K$, so $hk \in H$ or $hk \in K$. In the first case we have $hk = h'$ for some $h' \in H$. Rearranging, we obtain $k = h'h^{-1}$. As $h, h' \in H$ and H is a subgroup, this means $k \in H$ contradicting how it was chosen. But also the case $hk \in K$ leads to a similar contradiction. Thus no such choice of h, k is possible: we have either $H \subseteq K$ or $K \subseteq H$.

Question 3 Which of the following groups are cyclic?

(a) S_2.

(b) $\mathrm{GL}(2, \mathbb{R})$.

(c) $\left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in \{1, -1\} \right\}$ under matrix multiplication.

(d) $(\mathbb{Q}, +)$.

Solution:

(a) Yes. It is $\langle g \rangle$, where $g = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

(b) No. $\mathrm{GL}(2, \mathbb{R})$ is not abelian, so it cannot be cyclic.

(c) No. Every element has order 1 or 2, so they all generate proper cyclic subgroups.

(d) No. Suppose $\frac{p}{q}$ is a generator, in lowest terms. All of the powers of this generator have the form $\frac{np}{q}$ for $n \in \mathbb{Z}$. But such an element has denominator at most q, and this is a contradiction (since $\mathbb{Q}$ has elements with denominators greater than q).

Question 4 Let G and H be finite groups. Let $G \times H$ be the set $\{(g, h) \mid g \in G, h \in H\}$ with the binary operation $(g_1, h_1) * (g_2, h_2) = (g_1 g_2, h_1 h_2)$.

(a) Show that $(G \times H, *)$ is a group.

(b) Show that if $g \in G$ and $h \in H$ have orders a, b respectively, then the order of (g, h) in $G \times H$ is the lowest common multiple of a and b.

(c) Show that if G and H are both cyclic, and $\gcd(|G|, |H|) = 1$, then $G \times H$ is cyclic. Is the converse true?

Solution:

(a) Easy; just check the group axioms. The identity is (e_G, e_H).

(b) We have $(g, h)^t = (g^t, h^t)$. Now

$$\begin{aligned} (g^t, h^t) = (e_G, e_H) &\iff a \text{ divides } t \text{ and } b \text{ divides } t \\ &\iff \mathrm{lcm}(a, b) \text{ divides } t. \end{aligned}$$

So $\mathrm{ord}(g, h)$ is $\mathrm{lcm}(a, b)$.

(c) Let $|G| = m$ and $|H| = n$. Since $G \times H$ has order mn, it is cyclic if and only if there exists an element (g, h) with order mn. Let $g \in G$ have order m and $h \in H$ have order n. By (b), (g, h) has order mn, so $G \times H$ is cyclic. The converse is also true. Let $(g, h) \in G \times H$ have order mn and suppose g has order a and h has order b. Then a divides m and b divides m and $lcm(a, b)$ is equal to mn (by (b)). It follows that $a = m$ and $b = n$ and m, n are coprime.

Question 5 Find an example of each of the following:

(a) an element of order 3 in the group $\mathrm{GL}(2, \mathbb{C})$.

(b) an element of order 3 in the group $\mathrm{GL}(2, \mathbb{R})$.

(c) an element of infinite order in the group $\mathrm{GL}(2, \mathbb{R})$.

(d) an element of order 12 in the group S_7.

Solution:

(a) E.g. $\begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix}$, where $\omega = \mathrm{e}^{2\pi\mathrm{i}/3}$, or as in (b).

(b) E.g. $\begin{pmatrix} \cos 2\pi/3 & \sin 2\pi/3 \\ -\sin 2\pi/3 & \cos 2\pi/3 \end{pmatrix}$.

(c) E.g. $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

(d) E.g. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 1 & 6 & 7 & 5 \end{pmatrix}$, or $(1234)(567)$ in cycle notation.

Question 6 Prove that if $\{x_1, \ldots, x_n\}$ is any finite subset of $(\mathbb{Q}, +)$, then the subgroup $\langle x_1, \ldots, x_n \rangle$ is cyclic.

Solution: Let $d_1, \ldots, d_n$ be the denominators when $x_1, \ldots, x_n$ are expressed in lowest terms. Then each of $x_1, \ldots, x_n$ is in the cyclic subgroup generated by $1/\ell$, where ℓ is $\mathrm{lcm}(d_1, \ldots, d_n)$. So $\langle x_1, \ldots, x_n \rangle$ is a subgroup of the cyclic group $\langle 1/\ell \rangle$ and is therefore cyclic (theorem in notes).