Math40002 Analysis 1 — Problem Sheet 6

1. Let $a_1 = 1$ and $a_{n+1} = \sqrt{2a_n}$. Prove that (a_n) converges and compute the limit.

A graph, or a few examples and a little experimentation (if you didn't do any, hang your head in shame!) seems to show that a_n is monotonic increasing. So we try to prove that:

Since $a_n > 0$ we have $a_{n+1} > a_n \iff \sqrt{2}a_n > a_n \iff 2a_n > a_n^2 \iff 2 > a_n$ so we want to show inductively that $a_n < 2$. True for $n = 1$, so assume true for n. Then $a_{n+1} = \sqrt{2a_n} < \sqrt{2 \times 2} = 2$ so true for $n + 1$, so true for all n.

Therefore a_n is indeed a monotonic increasing sequence, bounded above by 2. It therefore converges to a limit $a = \sup\{a_n : n \in \mathbb{N}\}$. By the algebra of limits, the identity

$$a_{n+1}^2 = 2a_n$$

converges to the identity

$$a^2 = 2a.$$

But $a > 0$ so we see that $a = 2$. Notice we did NOT take limits in $a_{n+1} = \sqrt{2a_n}$ to give $a = \sqrt{2a}$ because we have not proved this!!

2. Fix $r > 1$. By the ratio test prove that $n/r^n \to 0$ as $n \to \infty$.

$\frac{(n+1)/r^{n+1}}{n/r^n} = \frac{1+1/n}{r} \to 1/r < 1$. **So by the ratio test, $n/r^n \to 0$.**

Conclude that $n^{1/n} < r$ for sufficiently large n. Hence prove $n^{1/n} \to 1$ as $n \to \infty$.

Taking $\epsilon = 1$ we find $\exists N \in \mathbb{N}$ such that $n \geq N \;\Rightarrow\; n/r^n < 1 \;\Rightarrow\; n^{1/n} < r$.

Fix $\epsilon > 0$. Then putting $r = 1+\epsilon$ in the above we find $N \in \mathbb{N}$ such that $n \geq N \;\Rightarrow\; 1 < n^{1/n} < 1+\epsilon \;\Rightarrow$ $|n^{1/n} - 1| < \epsilon$. Therefore $n^{1/n} \to 0$.

3. Fix $M \in \mathbb{R}$. Prove $M^n/n! \to 0$. Hence show the sequence $(n!)^{1/n}$ is unbounded.

Ratio test: $\frac{M^{n+1}/(n+1)!}{M^n/n!} = \frac{M}{n+1} \to 0$ as $n \to \infty$. Therefore $M^n/n! \to 0$.

In particular $\exists N_M \in \mathbb{N}$ such that $M^n/n! < 1$ for all $n \geq N$. Thus $(n!)^{1/n} > M$. Since M was arbitrary we find that $(n!)^{1/n}$ is unbounded.

4.* Which of the statements (a)–(d) imply $(*)$ and which are implied by $(*)$?

$$\exists\, a \in \mathbb{R} \text{ such that } \forall \epsilon > 0\ \forall N \in \mathbb{N}\ \exists\, n \geq N,\ |a_n - a| < \epsilon. \qquad (*)$$

(a) $\exists\, a \in \mathbb{R}$ such that $\forall \epsilon > 0\ \exists\, N \in \mathbb{N}$ such that $\forall n \geq N,\ |a_n - a| < \epsilon$.

(b) $\exists\, a \in \mathbb{R}$ and $\exists\, \epsilon > 0$ such that $\forall N \in \mathbb{N}\ \forall n \geq N,\ |a_n - a| < \epsilon$.

(c) $\forall a \in \mathbb{R}\ \exists\, \epsilon > 0$ such that $\forall N \in \mathbb{N}\ \forall n \geq N,\ |a_n - a| < \epsilon$.

(d) $\exists\, a \in \mathbb{R}$ such that $\exists\, N \in \mathbb{N}$ such that $\forall \epsilon > 0,\ \forall n \geq N,\ |a_n - a| < \epsilon$.

$(*)$ is equivalent to the existence of a convergent subsequennce of (a_n) (exercise!) whereas the listed statements are

(a) a_n **is convergent**

(b) a_n **is bounded**

(c) a_n **is bounded**

(d) a_n **is a constant sequence beyond some point** a_N

By the Bolzano-Weierstrass theorem, all 4 imply $(*)$, but none are implied by it.

5. We saw in lectures that the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ diverges.
What about $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots$? Prove your answer.

Diverges: call it $\sum_{n=1}^{\infty} a_n$ where $a_n = \frac{1}{2n-1} \geq \frac{1}{2n} =: b_n$.

Now $\sum_{n=1}^{\infty} b_n = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$ diverges to ∞, so since all terms are positive and $a_n \geq b_n$ then by comparison $\sum_{n=1}^{\infty} a_n$ diverges to ∞ also.

6.† Let $\sum_{n\geq 1} a_n$ be the series obtained from $\sum_{n\geq 1} \frac{1}{n}$ deleting all the terms $\frac{1}{n}$ such that the base 10 expansion of n contains the digit 4. Prove this series converges.

Consider the positive integers n with exactly k digits in their base 10 expansion. We have 9 choices for their first digit (which cannot be 0) and 10 for each of the others, i.e. 9.10^{k-1} overall.

But for the numbers without a 4 in their base 10 expansion, we have 8.9^{k-1} choices, by the same reasoning.

Each of these numbers n is $\geq 10^{k-1}$ (the smallest number with k digits). So the sum of $1/n$ over all n with k digits, none of them 4, is

$$< 8.9^{k-1}/10^{k-1} = 8(0.9)^{k-1}.$$

Summing over all $k = 1, 2, \ldots$ we find that any partial sum of the series in the question is bounded above by $8/(1-0.9) = 80$. Since these partial sums are monotonically increasing they converge.

7. Prove *from first principles* that you can multiply a series by a constant $c \in \mathbb{R}$ term by term, i.e. if $\sum_{n=1}^{\infty} a_n$ is convergent then $\sum_{n=1}^{\infty} ca_n$ is convergent to $c\sum_{n=1}^{\infty} a_n$.

Let $s_n = \sum_{i=1}^{n} a_i$ be the nth partial sum of $\sum a_n$. Then by definition, saying it converges to $A := \sum_{n=1}^{\infty} a_n$ says that if we fix any $\epsilon > 0$,

$$\exists N \in N \text{ such that } \forall n \geq N,\ |s_n - A| < \epsilon.$$

Applying this to $\frac{\epsilon}{c} > 0$ gives

$$\exists N \in N \text{ such that } \forall n \geq N,\ |s_n - A| < \epsilon/c \ \Rightarrow\ |cs_n - cA| < \epsilon.$$

But cs_n is the nth partial sum of $\sum ca_n$, so this says that $cs_n \to cA$, i.e. $\sum ca_n$ converges to $cA = c\sum a_n$.

8. Given a real sequence (a_n), define a new sequence $b_n := \frac{1}{n}\sum_{i=1}^{n} a_i$ by averaging.

   (a) For any $a \in \mathbb{R}$, $N > 1$ and $n \geq N$, let $A(N) := \sum_{i=1}^{N-1} |a_i - a|$. Show that

$$|b_n - a| \ \leq\ \frac{A(N)}{n} + \frac{\sum_{i=N}^{n} |a_i - a|}{n}.$$

$$\begin{aligned} |b_n - a| \ = \ \left|\frac{1}{n}\left(\sum_{i=1}^{n} a_i - na\right)\right| \ &= \ \frac{1}{n}\left|\sum_{i=1}^{n}(a_i - a)\right| \\ &\leq \ \frac{1}{n}\sum_{i=1}^{N-1} |a_i - a| + \frac{1}{n}\sum_{i=N}^{n} |a_i - a| \qquad (*) \end{aligned}$$

by the triangle inequality.

   (b) Suppose that $a_n \to a$. Prove carefully that $b_n \to a$.

Proving $b_n \to a$ is now easy *if we get the order of the argument right.* Fixing ϵ first, we can make the second term $< \epsilon$ for *fixed* $N = N_\epsilon$. Then we tend $n \to \infty$ in the first term with N fixed to make that term $\to 0$. If you do it in another order, it won't work...

So: suppose that $a_n \to a$. That is, fixing any $\epsilon > 0$,

$$\exists N \in \mathbb{N} \text{ such that } \Big(n \geq N \;\Rightarrow\; |a_n - a| < \epsilon \Big).$$

This controls the right hand term in (1):

$$\frac{1}{n}\sum_{i=N}^{n} |a_i - a| \;<\; \frac{1}{n}\sum_{i=N}^{n} \epsilon \;=\; \frac{1}{n}(n - N + 1)\epsilon \;<\; \epsilon \qquad (**)$$

for all $n \geq N$.

Now fixing N we know that $A(N)/n \to 0$ as $n \to \infty$. In fact take $M \in \mathbb{N}$ such that $M > A(N)/\epsilon$. Then for all $n \geq M$ we have

$$n > \frac{A(N)}{\epsilon} \;\Rightarrow\; \frac{A(N)}{n} < \epsilon. \qquad (***)$$

Plugging $(**)$, $(***)$ into $(*)$ gives

$$|b_n - a| \;<\; \epsilon + \epsilon \;=\; 2\epsilon$$

for $n \geq \max(N, M)$. Therefore $b_n \to a$, as required.

(c) Give (without proof) an example with a_n divergent but b_n convergent.

Eg take $a_n = (-1)^n$. Then $a_n \not\to 0$, but $b_{2n} = 0$ and $b_{2n+1} = -(2n+1)^{-1}$, so $b_n \to 0$.

(d) Suppose $\sum_{n=1}^{\infty} a_n$ is convergent, does it follow that $\sum_{n=1}^{\infty} b_n$ is also convergent, and to the same value ? *Hint: consider the sequence* $a_n = \begin{cases} 1 & n = 1, \\ 0 & n > 1. \end{cases}$

For the example given in the hint we have $\sum_{n=1}^{\infty} a_n = 1$ but $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges to infinity (lectures). So the answer is "no".

9. For which values of $a, b \in \mathbb{R}$ does $\sum_{n=1}^{\infty} n^a/b^n$ converge or diverge ? (*Give a proof in the MATH40004 sense, and a proof in the proof sense when $a \in \mathbb{Z}$, $b \in \mathbb{R}$.*)

Ratio test: $\frac{(n+1)^a/b^{n+1}}{n^a/b^n} = \frac{(1+1/n)^a}{b} \to 1/b$ as $n \to \infty$. So it converges for $|b| > 1$ and diverges for $|b| < 1$.

[Note: we used $(1 + 1/n)^a \to 1$ as $n \to \infty$, which we haven't proved for arbitrary $a \in \mathbb{R}$. So in this sense it's all a bit MATH40004. But we have proved this for $a \in \mathbb{Z}$, by the algebra of limits.]

When $b = 1$ we have $\sum_{n=1}^{\infty} n^a$ which we have seen in lectures is convergent for $a < -1$ and divergent for $a \geq -1$.

When $b = -1$ we have $\sum_{n=1}^{\infty} (-1)^n n^a$ which we have seen in lectures is absolutely convergent for $a < -1$. For $a \geq 0$ it is divergent because $(-1)^n n^a \not\to 0$. For $a = -1$ we have seen in lectures that it converges (but not absolutely).

This just leaves $a \in (-1, 0)$, $b = -1$, i.e. $\sum_{n=1}^{\infty} (-1)^n n^a$ for $a \in (-1, 0)$. By the alternating series test these all converge, but not absolutely.

10. **MATH40004 question for fun.** Write down the unique degree $d+1$ polynomial $p(x)$ with roots $0, \lambda_1, \lambda_2, \ldots, \lambda_d$ and $p'(0) = 1$.

It is $p(x) = x \prod_{n=1}^{d} \left(1 - \frac{x}{\lambda_n}\right)$.

"Apply" your formula to $d = \infty$ and $p(x) = \sin x$, and compare coefficients of x^3 or x^5 on both sides to evaluate

(a) $$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$ (b)† $$\sum_{n=1}^{\infty} \frac{1}{n^4} = ?$$

We get

$$\sin x \;=\; x\prod_{n=1}^{\infty}\left(1-\frac{x}{n\pi}\right)\left(1+\frac{x}{n\pi}\right) \;=\; x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2\pi^2}\right).$$

(a) Taking coefficients of x^3 on both sides gives

$$-\frac{1}{3!} \;=\; \sum_{n=1}^{\infty} -\frac{1}{n^2\pi^2}\,.$$

Multiplying both sides by $-\pi^2$ gives the result.

(b) Taking coefficients of x^5 on both sides instead gives

$$\frac{1}{5!} \;=\; \sum_{m>n}\left(\frac{-1}{n^2\pi^2}\right)\left(\frac{-1}{m^2\pi^2}\right) \;=\; \frac{1}{2}\sum_{n=1}^{\infty}\sum_{m\neq n}\frac{1}{m^2n^2\pi^4} \;=\; \sum_{n=1}^{\infty}\frac{1}{2n^2\pi^4}\left(\frac{\pi^2}{6}-\frac{1}{n^2}\right) \;=\; \frac{1}{72}-\sum_{n=1}^{\infty}\frac{1}{2n^4\pi^4}.$$

Here for the second $=$ we have used the symmetry of $\frac{1}{m^2n^2}$ under $m \leftrightarrow n$, while in the final $=$ we have used $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$. Rearranging gives $\sum_{n=1}^{\infty}\frac{1}{n^4}=2\pi^4\left(\frac{1}{72}-\frac{1}{120}\right)=\frac{\pi^4}{90}$.