Math40003 Linear Algebra and Groups Unseen 2

Here is a modified version of Question 4 from last week.

4'. Let $\alpha_1, \dots, \alpha_k \in \mathbb{R}$. Show that the following are equivalent:

i. $\alpha_1 + \cdots + \alpha_k = 1$.

ii. For every system of linear equations (in n variables over $\mathbb{R}$) $A'x = b'$, if $v_1, \dots, v_k$ are all solutions to $A'x = b'$, then $\alpha_1 v_1 + \cdots + \alpha_k v_k$ is also a solution to $A'x = b'$.

iii. There is some system of linear equations (in n variables) $A'x = b'$ **such that** $b' \neq 0$ and $v_1, \dots, v_k$ such that $v_1, \dots, v_k, \alpha_1 v_1 + \cdots + \alpha_k v_k$ are all solutions to $A'x = b'$.

Can you see how this solves Question 4 from last week?

1. Let $A \in M_{m\times n}(\mathbb{R}), B \in M_{n\times p}(\mathbb{R})$. Show that the following are equivalent:

   i. $AB = \mathcal{O}_{m,p}$, where $\mathcal{O}_{m,p}$ is the $m \times p$ matrix whose entries are all 0.

   ii. $B^t A^t = \mathcal{O}_{p,m}$, where $\mathcal{O}_{m,p}$ is the $m \times p$ matrix whose entries are all 0.

   iii. Every column of B is a solution to the system $Ax = 0$.

   iv. Every row of A is a solution to the system $B^t x = 0$.

2. Let $Ax = 0$ be a system of linear equations in n variables over $\mathbb{R}$ and let $v_1, \dots, v_k \in \mathbb{R}^n$. Show that $v_1, \dots, v_k$ are all solutions to $Ax = 0$ if and only if for all $\alpha_1, \dots, \alpha_k \in \mathbb{R}$, $\alpha_1 v_1 + \cdots + \alpha_k v_k$ is a solution to $Ax = 0$.

3. For every matrix $A \in M_{m\times n}(\mathbb{R})$ and $\vec{b} = (b_1, \dots, b_n) \in \mathbb{R}^n$, denote by $\begin{pmatrix} A \\ \vec{b} \end{pmatrix}$ the $(m+1) \times n$ matrix which has the rows of A as its first m rows and $\vec{b}$ as its $(m+1)$-th row.

   Let $\vec{0} = (0, \dots, 0) \in \mathbb{R}^n$ and let $A \in M_{m\times n}(\mathbb{R})$ such that the rows of A are $\vec{a}_1, \dots \vec{a}_m \in \mathbb{R}$. Prove that $\begin{pmatrix} A \\ \vec{b} \end{pmatrix}$ is row equivalent to $\begin{pmatrix} A \\ \vec{0} \end{pmatrix}$ if and only if there are $\alpha_1, \dots, \alpha_m \in \mathbb{R}$ such that $\alpha_1 \vec{a}_1 + \cdots + \alpha_m \vec{a}_m = \vec{b}$.

4. (a) Find a matrix $A \in M_{4\times 4}(\mathbb{R})$, such that $(1, 0, 0, 0)$ and $(0, 1, 0, 0)$ are both solutions to $Ax = 0$ but $(0, 0, 1, 0)$ and $(0, 0, 0, 1)$ are not.

   (b) Find a matrix $A \in M_{4\times 4}(\mathbb{R})$, such that $(2, 4, 2, 14)$ and $(1, 2, 2, 11)$ are both solutions to $Ax = 0$ but $(2, 3, 4, 5)$ and $(2, 4, 2, 4)$ are not.

   (c) Find a matrix $A \in M_{5\times 5}(\mathbb{R})$, such that the solution set to $Ax = 0$ is precisely the set

$$S = \{\, a(1, 1, 0, -1, 0) + b(1, 1, 1, -3, 1) + c(1, 1, 2, -5, 3) \mid a, b, c \in \mathbb{R} \,\}.$$

Hint: use Questions 1,2,3.

5. Let $A = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 4 & 6 \end{pmatrix}$.

(a) Find all matrices X such that $AX = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

(b) Find all matrices X such that $XA = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$.