Math40003 Linear Algebra and Groups — Unseen 4

1. Consider $\mathbb{R}$ as a $\mathbb{Q}$-vector space. Show $\{1, \sqrt{2}, \sqrt{3}\}$ is linearly independent.

2. Given a field F and an F-vector space V we denote the dimension of V as an F-vector space by $\dim_F(V)$.

   (a) Find $\dim_{\mathbb{R}}(\mathbb{C})$.

   (b) Let $V = M_{n\times m}(\mathbb{C})$. Find $\dim_{\mathbb{C}}(V)$ and $\dim_{\mathbb{R}}(V)$.

   (c) Let E, F be fields such that $F \subseteq E$ and E is a finite-dimensional F-vector space. Let V be a finite dimensional E-vector space. Prove that V is an F-vector space and $\dim_F(V) = \dim_E(V) \cdot \dim_F(E)$.

3. Consider $\mathbb{R}$ as a $\mathbb{Q}$-vector space. Let $p_1, \ldots, p_n$ be distinct prime numbers, and let a be any positive real number. Show $\{\log_a(p_1), \ldots, \log_a(p_n)\}$ is linearly independent.

   You may want to use the Fundamental Theorem of Arithmetic:
   Every positive integer $n > 1$ can be represented in exactly one way as a product of prime powers: $n = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k} = \prod_{i=1}^{k} p_i^{n_i}$
   where $p_1 < p_2 < \cdots < p_k$ are primes and the n_i are positive integers.

4. Prove $\{\sin(nx) \mid n \in \mathbb{N}\}$ is linearly independent in $\mathbb{R}^{\mathbb{R}}$.

   Notice that an infinite set is linearly independent if and only if every finite subset of it is linearly independent.

   Hint: Find a linear combination of 0 with a minimal number of elements of the set. Use derivatives to contradict minimality.