# Math40003 Linear Algebra and Groups,
# Term 2 Unseen 5 (week 8)

1. Suppose $q, n \in \mathbb{N}$ and $\mathbb{F}_q$ is a field with $q$ elements.
   (a) Find a formula (depending on $n$ and $q$) for the number of bases $v_1, \dots, v_n$ of $\mathbb{F}_q^n$ (consider choosing $v_1, v_2, \dots$ in turn).
   (b) What is the order of $GL_n(\mathbb{F}_q)$, the group of all invertible $n \times n$ matrices over $\mathbb{F}_q$.
   (c) Suppose $0 \neq \alpha \in \mathbb{F}_q$. Show that the number of matrices in $GL_n(\mathbb{F}_q)$ with determinant $\alpha$ does not depend on the choice of $\alpha$. When do these matrices form a subgroup of $GL_n(\mathbb{F}_q)$?
   (d) Find a formula (depending on $n$ and $q$) for the order of $SL_n(\mathbb{F}_q)$, the group of all $n \times n$ matrices over $\mathbb{F}_q$ with determinant 1.

2. Let $(G, .)$ be a finite group and let $A, B \subseteq G$ be subsets. Prove that if $|A| + |B| > |G|$ then $G = AB$ where $AB = \{ a.b | a \in A, b \in B \}$

3. Let $F$ be a finite field. Prove that every element of $F$ is a sum of two squares, i.e., for every $a \in F$, there are $b_1, b_2 \in F$ such that $a = b_1^2 + b_2^2$. Is it true that every $n \in \mathbb{N}$ is a sum of two squares of $\mathbb{N}$?

4. Let $X$ be a set and let $G \leq Sym(X)$ be a subgroup. $G$ *acts freely* if $\forall x \in X$, $g, h \in G$: $g(x) = h(x) \implies g = h$. $G$ *is transitive* if $\forall x, y \in X$, $\exists g \in G$: $g(x) = y$. Prove that if $G$ is transitive and acts freely, then $|G| = |X|$.