Math40003 Linear Algebra and Groups — Problem Sheet 6

1. Let $\mathcal{B} = \{(1,3),(1,2)\} \subseteq \mathbb{R}^2$.

   (a) Show that $\mathcal{B}$ is a basis of $\mathbb{R}^2$.

   (This works only because we are in dimension 2. Why?) $(1,2)$ is not a scalar multiple of $(1,3)$. (There are other ways to solve this that work in general.)

   (b) Compute the basis change matrix form $\mathcal{B}$ to the canonical basis of $\mathbb{R}^2$ $(\{(1,0)^t,(0,1)^t\})$.

   The matrix is

   $$P = \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}.$$

   (c) Compute the basis change matrix from the canonical basis to $\mathcal{B}$.

   The desired matrix is found by inverting P. This can be done by elementary row operations. (notice that here I haven't written out the operation I am performing for each step. However, this is something you should always do in your solutions, here I left the operations unspecified because understanding what is going on at each step is a good review exercise for you.)

   $$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -1 & -3 & 1 \end{array}\right) \sim$$
   $$\left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & -1 & -3 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 3 & -1 \end{array}\right)$$

   The desired matrix is $\begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix}$.

2. Let

   $$\mathcal{B} = \left\{ \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} \right\} \subseteq \mathbb{R}^3.$$

   (a) Show that $\mathcal{B}$ is a basis of $\mathbb{R}^3$.

   The matrix

   $$P = \begin{pmatrix} -1 & 0 & 0 \\ 2 & 1 & 1 \\ 1 & 3 & 4 \end{pmatrix}$$

   has rank 3 as it can be shown by bringing it in Reduced Row-Echelon Form. (We do this below as part of the solution to the next point.)

(b) Compute the basis change matrix from the canonical basis of $\mathbb{R}^3$ to $\mathcal{B}$.
The basis change matrix we are after is computed by inverting P. This can be done as usual by reducing P with row operations. We get

$$\left(\begin{array}{ccc|ccc} -1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 3 & 4 & 1 & 0 & 1 \end{array}\right) \sim$$
$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & -5 & -3 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 7 & 4 & -1 \\ 0 & 0 & 1 & -5 & -3 & 1 \end{array}\right).$$

Thus $P^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 7 & 4 & -1 \\ -5 & -3 & 1 \end{pmatrix}$**.**

3. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map given by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y \\ x+z \\ x+z \end{pmatrix}$$

Write the matrix ${}_{\mathcal{B}}[T]_{\mathcal{B}}$ where $\mathcal{B}$ is the basis

$$\mathcal{B} = \left\{ \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\} \subseteq \mathbb{R}^3.$$

Let $\mathcal{C}$ be the canonical basis of $\mathbb{R}^3$. We have

$$A = {}_{\mathcal{C}}[T]_{\mathcal{C}} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

The basis change from $\mathcal{B}$ to $\mathcal{C}$ is

$$B = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$$

The matrix ${}_{\mathcal{B}}[T]_{\mathcal{B}} = B^{-1}AB$, so it remains to compute B^{-1} and the product of the three matrices. We compute B^{-1} using row operations:

$$\left(\begin{array}{ccc|ccc} 2 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -3 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 2 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 2 \end{array}\right) \sim$$
$$\left(\begin{array}{ccc|ccc} 2 & 0 & 0 & -2 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & -2 \end{array}\right).$$

Thus

$$_{\mathcal{B}}[T]_{\mathcal{B}} = B^{-1}AB = \begin{pmatrix} -1 & -1 & 1 \\ -1 & 0 & 0 \\ -4 & -3 & 3 \end{pmatrix}.$$

4. Let V be a vector space of dimension 2 (over an arbitrary field containing $\mathbb{Q}$) and let

$$\mathcal{B} = \{v_1, v_2\} \qquad \mathcal{B}' = \{v_1', v_2'\}$$

be two bases of V such that

$$v_1 = 6v_1' - 2v_2' \qquad v_2 = 9v_1' - 4v_2'.$$

(a) Compute the basis change matrix from $\mathcal{B}$ to $\mathcal{B}'$.

The question gives us the coordinates of v_1 with respect to $\mathcal{B}'$. They are

$$\begin{pmatrix} 6 \\ -2 \end{pmatrix}$$

The coordinates of v_2 are also given by the question. The basis change matrix from $\mathcal{B}$ to $\mathcal{B}'$ is the matrix whose columns are the coordinates of the vectors of $\mathcal{B}$ written with respect to the basis $\mathcal{B}'$. Thus the desired basis change matrix is

$$\begin{pmatrix} 6 & 9 \\ -2 & -4 \end{pmatrix}$$

If in doubt ask yourself what happens to the coordinates of the vector v_1 when fed to the matrix $_{\mathcal{B}'}[Id]_{\mathcal{B}}$: the coordinates of v_1 with respect to $\mathcal{B}$ are $(1,0)^t$ and the coordinates of v_1 with respect to $\mathcal{B}'$ are $(6,-2)^t$. When multiplying a matrix by $(1,0)^t$ we get its first column, so the first column of the matrix we are after has to be $(6,-2)^t$. The same argument applies to the second column.

(b) Compute the coordinates of $-3v_1 + 3v_2$ with respect to $\mathcal{B}'$.

This can be done by substituting the expression of v_2 and v_2 in $-3v_1+3v_2$. Since we already have the basis change matrix, however it suffices to compute

$$\begin{pmatrix} 6 & 9 \\ -2 & -4 \end{pmatrix} \begin{pmatrix} -3 \\ 3 \end{pmatrix} = \begin{pmatrix} 9 \\ -6 \end{pmatrix}.$$

Check the result using the substitution method.

(c) Why did we assume that the base field contained $\mathbb{Q}$? What happens if we try and answer the same questions with base field $\mathbb{Z}/2\mathbb{Z}$?) **If the base field is $\mathbb{Z}/2\mathbb{Z}$ then $\mathcal{B}$ is not a basis because $v_1 = 0$.**

5. Let A be a square matrix of dimension n over an arbitrary field. Assume that there exists $m \in \mathbb{N}$ such that A^m is the zero matrix. Show that $\mathrm{I}_n + A$ is invertible.

We write

$$\mathrm{I}_n = \mathrm{I}_n + A^m = (\mathrm{I}_n + A)(\mathrm{I}_n - A + A^2 \cdots + (-1)^{m-1} A^{m-1}).$$