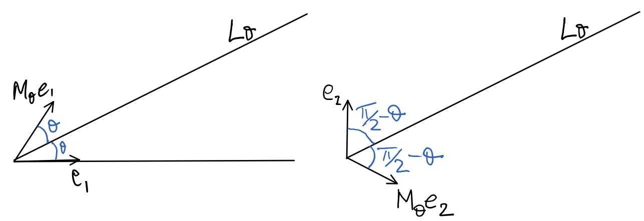
Math40003 Linear Algebra and Groups Problem Sheet 2

1. (a) Let M_θ be the reflection in the line $L_\theta = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 = x_1 \tan\theta\}$. Using any school geometry or trigonometry you like, show that the matrix representing M_θ is
$$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}.$$

Drawing it, you see that e_1 gets reflected to the unit vector making an angle 2θ with the x_1-axis, i.e. $\binom{\cos 2\theta}{\sin 2\theta}$.



Similarly e_2 makes an angle $\pi/2 - \theta$ anticlockwise from L_θ, so gets reflected to a unit vector whose angle is $\pi/2 - \theta$ *clockwise* from L_θ. Thus it makes an angle $\theta - (\pi/2 - \theta) = 2\theta - \pi/2$ with the x_1-axis, so is the unit vector $\binom{\cos(2\theta-\pi/2)}{\sin(2\theta-\pi/2}) = \binom{\sin 2\theta}{-\cos 2\theta}$.

Thus the matrix is $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$ as claimed.

(b) Let R_α be a rotation about the origin, and let M_β be the reflection in a line through the origin. Prove that $M_\beta R_\alpha$ is a reflection.

We compute the product

$$\begin{pmatrix} \cos 2\beta & \sin 2\beta \\ \sin 2\beta & -\cos 2\beta \end{pmatrix} \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} = \begin{pmatrix} \cos(2\beta-\alpha) & \sin(2\beta-\alpha) \\ \sin(2\beta-\alpha) & -\cos(2\beta-\alpha) \end{pmatrix}.$$

By (c) this is reflecton in $L_{\beta-\alpha/2}$

(c) Let M_α and M_β be reflections in straight lines through the origin. Prove that $M_\alpha M_\beta$ is a rotation.

We compute

$$\begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix} \begin{pmatrix} \cos 2\beta & \sin 2\beta \\ \sin 2\beta & -\cos 2\beta \end{pmatrix} = \begin{pmatrix} \cos 2(\alpha-\beta) & -\sin 2(\alpha-\beta) \\ \sin 2(\alpha-\beta) & \cos 2(\alpha-\beta) \end{pmatrix},$$

which is the rotation $R_{2(\alpha-\beta)}$.

2. * Let $\mathbb{R}[x]$ be the set of all polynomials with variable x and real coefficients, with standard addition and scalar multiplication. Show that this is a vector space over $\mathbb{R}$.

A1 **Follows from associativity of $\mathbb{R}$ and definition of polynomial addition. i.e. let $f(x), g(x), h(x) \in \mathbb{R}[x]$ then we have:**

$$\begin{array}{rcl} f(x) & = & \sum_{i=1}^{m} a_i x^i \\ g(x) & = & \sum_{i=1}^{n} b_i x^i \\ h(x) & = & \sum_{i=1}^{s} c_i x^i \end{array}$$

Let $t = max\{m, n, s\}$ then define $a_i = 0$ for $m \leq i \leq t$, similarly define $b_i = 0$ for $n \leq i \leq t$, and define $c_i = 0$ for $s \leq i \leq t$. So we get:

$$\begin{array}{rcl} f(x) & = & \sum_{i=1}^{t} a_i x^i \\ g(x) & = & \sum_{i=1}^{t} b_i x^i \\ h(x) & = & \sum_{i=1}^{t} c_i x^i \end{array}$$

Now

$$\begin{array}{rcl} f(x) + (g(x) + h(x)) & = & \sum_{i=1}^{t} a_i x^i + (\sum_{i=1}^{t} b_i x^i + \sum_{i=1}^{t} c_i x^i) \\ & = & \sum_{i=1}^{t} (a_i + (b_i + c_i)) x^i \\ & = & \sum_{i=1}^{t} ((a_i + b_i) + c_i) x^i \\ & = & (\sum_{i=1}^{t} a_i x^i + \sum_{i=1}^{t} b_i x^i) + \sum_{i=1}^{t} c_i x^i \\ & = & (f(x) + g(x)) + h(x) \end{array}$$

A2 **Follows from commutitivity of $\mathbb{R}$ and definition of polynomial addition.**

A3 **0_V here is the polynomial 0.**

A4 **The inverse of $f(x) = \sum_{i=1}^{m} a_i x^i$ is $-f(x) = \sum_{i=1}^{m} -a_i x^i$ clearly we get $f(x) + (-f(x)) = 0_V$.**

A5 **Follows from distributivity of $\times$ over $+$ in $\mathbb{R}$ and definition of polynomial addition.**

A6 **Follows from distributivity of $+$ over $\times$ in $\mathbb{R}$ and definition of multiplying a polynomial by a constant/scalar.**

A7 **Follows from commutativity of $\times$ in $\mathbb{R}$ and definition of multiplying a polynomial by a constant/scalar.**

A8 **Follows from definition of multiplying a polynomial by a constant/scalar.**

3. Decide for each of the following sets, whether it is a vector space with the indicated operations of addition and scalar multiplication:

   (a) The set $\mathbb{R}^2$, with the usual addition but with scalar multiplication defined by

$$r \odot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ry \\ rx \end{pmatrix}.$$

Consider A7, and let $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$.

$$\begin{aligned} r \odot (s \odot \mathbf{v}) &= r \odot \left(s \odot \begin{pmatrix} x \\ y \end{pmatrix} \right) \\ &= r \odot \begin{pmatrix} sy \\ sx \end{pmatrix} \\ &= \begin{pmatrix} rsx \\ rsy \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} (rs) \odot \mathbf{v} &= (rs) \odot \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} rsy \\ rsx \end{pmatrix} \end{aligned}$$

Because $r \odot (s \odot \mathbf{v}) \neq (rs) \odot \mathbf{v}$, A7 is not satisfied.

(b) The set $\mathbb{R}^2$, with the usual scalar multiplication but with addition defined by

$$\begin{pmatrix} x \\ y \end{pmatrix} \oplus \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} y+s \\ x+r \end{pmatrix}.$$

All of the axioms A5 - A8 are satisfied since the usual definition of scalar multiplication is used.

Consider the axiom A1, and let $\mathbf{u} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} x_3 \\ y_3 \end{pmatrix}$, then

$$\begin{aligned} (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w} &= \left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \oplus \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right) \oplus \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} \\ &= \begin{pmatrix} y_1 + y_2 \\ x_1 + x_2 \end{pmatrix} \oplus \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} \\ &= \begin{pmatrix} x_1 + x_2 + y_3 \\ y_1 + y_2 + x_3 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} \mathbf{u} \oplus (\mathbf{v} + \mathbf{w}) &= \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \oplus \left(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \oplus \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} \right) \\ &= \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \oplus \begin{pmatrix} y_2 + y_3 \\ x_2 + x_3 \end{pmatrix} \\ &= \begin{pmatrix} y_1 + x_2 + x_3 \\ x_1 + y_2 + y_3 \end{pmatrix} \end{aligned}$$

Since $(\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w} \neq \mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w})$ then A1 is not satisfied.

(c) The set $\mathbb{R}^2$, with addition and scalar multiplication defined by

$$\begin{pmatrix} x \\ y \end{pmatrix} \oplus \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x+a+1 \\ y+b \end{pmatrix} \quad \text{and} \quad r \odot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} rx+r-1 \\ ry \end{pmatrix}.$$

Consider A1, and let $\mathbf{u} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} x_3 \\ y_3 \end{pmatrix}$, then

$$\begin{aligned}
(\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w} &= \left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \oplus \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right) \oplus \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} \\
&= \begin{pmatrix} x_1+x_2+1 \\ y_1+y_2 \end{pmatrix} \oplus \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} \\
&= \begin{pmatrix} x_1+x_2+x_3+2 \\ y_1+y_2+y_3 \end{pmatrix}
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) &= \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \oplus \left(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \oplus \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} \right) \\
&= \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \oplus \begin{pmatrix} x_2+x_3+1 \\ y_2+y_3 \end{pmatrix} \\
&= \begin{pmatrix} x_1+x_2+x_3+2 \\ y_1+y_2+y_3 \end{pmatrix}
\end{aligned}$$

Because $(\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w} = \mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w})$, A1 is satisfied.

Consider A2 and let $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$, then

$$\begin{aligned}
\mathbf{v} \oplus \mathbf{w} &= \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \oplus \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \\
&= \begin{pmatrix} x_1+x_2+1 \\ y_1+y_2 \end{pmatrix}
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{w} \oplus \mathbf{v} &= \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \oplus \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \\
&= \begin{pmatrix} x_1+x_2+1 \\ y_1+y_2 \end{pmatrix}
\end{aligned}$$

Because $\mathbf{v} \oplus \mathbf{w} = \mathbf{w} \oplus \mathbf{v}$, A2 is satisfied.

Consider A3, and let $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{e} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, then

$$\begin{aligned}
\mathbf{e} \oplus \mathbf{v} &= \begin{pmatrix} -1 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} x \\ y \end{pmatrix} \\
&= \begin{pmatrix} x - 1 + 1 \\ y \end{pmatrix} \\
&= \begin{pmatrix} x \\ y \end{pmatrix} \\
&= \mathbf{v}
\end{aligned}$$

Therefore, A3 is satisfied.

Consider A4, and let $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$, then

$$\begin{aligned}
\mathbf{v} \oplus (-1 \odot \mathbf{v}) &= \begin{pmatrix} x \\ y \end{pmatrix} \oplus \left(-1 \odot \begin{pmatrix} x \\ y \end{pmatrix}\right) \\
&= \begin{pmatrix} x \\ y \end{pmatrix} \oplus \begin{pmatrix} -x - 1 - 1 \\ -y \end{pmatrix} \\
&= \begin{pmatrix} x \\ y \end{pmatrix} \oplus \begin{pmatrix} -x - 2 \\ -y \end{pmatrix} \\
&= \begin{pmatrix} -1 \\ 0 \end{pmatrix} \\
&= \mathbf{e}
\end{aligned}$$

Therefore, A4 is satisfied.

Consider A5 and let $\mathbf{v} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$, then

$$\begin{aligned}
r \odot (\mathbf{v} \oplus \mathbf{w}) &= r \odot \begin{pmatrix} x_1 + x_2 + 1 \\ y_1 + y_2 \end{pmatrix} \\
&= \begin{pmatrix} r(x_1 + x_2 + 1) + r - 1 \\ r(y_1 + y_2) \end{pmatrix}
\end{aligned}$$

and

$$\begin{aligned}
(r \odot \mathbf{v}) \oplus (r \odot \mathbf{w}) &= r \odot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \oplus r \odot \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \\
&= \begin{pmatrix} rx_1 + r - 1 \\ ry_1 \end{pmatrix} \oplus \begin{pmatrix} rx_2 + r - 1 \\ ry_2 \end{pmatrix} \\
&= \begin{pmatrix} rx_1 + r - 1 + rx_2 + r - 1 + 1 \\ ry_1 + ry_2 \end{pmatrix} \\
&= \begin{pmatrix} r(x_1 + x_2 + 1) + r - 1 \\ r(y_1 + y_2) \end{pmatrix}
\end{aligned}$$

Because $r \odot (\mathbf{v} \oplus \mathbf{w}) = (r \odot \mathbf{v}) \oplus (r \odot \mathbf{w})$, A5 is satisfied.

Consider A6, and let $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$, then

$$\begin{aligned} (r+s) \odot \mathbf{v} &= (r+s) \odot \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} (r+s)x + (r+s) - 1 \\ (r+s)y \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} (r \odot \mathbf{v}) \oplus (s \odot \mathbf{v}) &= r \odot \begin{pmatrix} x \\ y \end{pmatrix} \oplus s \odot \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} rx + r - 1 \\ ry \end{pmatrix} \oplus \begin{pmatrix} sx + s - 1 \\ sy \end{pmatrix} \\ &= \begin{pmatrix} rx + r - 1 + sx + s - 1 + 1 \\ ry + sy \end{pmatrix} \\ &= \begin{pmatrix} (r+s)x + (r+s) - 1 \\ (r+s)y \end{pmatrix} \end{aligned}$$

Because $(r+s) \odot \mathbf{v} = (r \odot \mathbf{v}) \oplus (s \odot \mathbf{v})$, A6 is satisfied.

Consider A7, and let $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$, then

$$\begin{aligned} r \odot (s \odot \mathbf{v}) &= r \odot \left(s \odot \begin{pmatrix} x \\ y \end{pmatrix} \right) \\ &= r \odot \begin{pmatrix} sx + s - 1 \\ sy \end{pmatrix} \\ &= \begin{pmatrix} r(sx + s - 1) + r - 1 \\ rsy \end{pmatrix} \\ &= \begin{pmatrix} r(sx + s) - 1 \\ rsy \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} (rs) \odot \mathbf{v} &= rs \odot \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} rsx + rs - 1 \\ rsy \end{pmatrix} \end{aligned}$$

Because $r \odot (s \odot \mathbf{v}) = (rs) \odot \mathbf{v}$, then A7 is satisfied.

Consider A8, and let $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$**, then**

$$\begin{aligned} 1 \odot \mathbf{v} &= 1 \odot \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} x+1-1 \\ y \end{pmatrix} \\ &= \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \mathbf{v} \end{aligned}$$

Therefore, A8 is satisfied.

Therefore, the set $\mathbb{R}^2$ **with addition described by** $\begin{pmatrix} x \\ y \end{pmatrix} \oplus \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x+a+1 \\ y+b \end{pmatrix}$ **and scalar multiplication described by** $r \odot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} rx+r-1 \\ ry \end{pmatrix}$ **is a vector space.**

4. Let F be a field. Show every F-vector space V with additive identity 0_V has the following properties:

   (a) The vector 0_V is the unique vector satisfying the equation $0_V \oplus v = v$ for all vectors v in V.

   (b) Let 0 be the additive identity in F. Then $0 \odot v = 0_V$ for all vectors v in V.

(a) Suppose that there are two vectors 0_v **and** $0'_V$ **that satisfy**

$$0_V \oplus \mathbf{v} = \mathbf{v} \quad 0'_V \oplus \mathbf{v} = \mathbf{v} \quad \forall \mathbf{v} \in V.$$

Let $\mathbf{v} = 0'_V$ **in the first equation and** $\mathbf{v} = 0_V$ **in the second equation gives**

$$0_V \oplus 0'_V = 0'_V \quad 0'_V \oplus 0_V = 0_V$$

By the commutative law, A2, $0_V \oplus 0'_V = 0'_V \oplus 0_V$**, therefore,** $0_V = 0'_V$ **and hence the zero vector is unique.**

(b) Using the distributive law, A6, for any $\mathbf{v} \in V$**,**

$$0 \odot \mathbf{v} = (0+0) \odot \mathbf{v} = (0 \odot \mathbf{v}) \oplus (0 \odot \mathbf{v})$$

By the additive identity axiom, A3, and the commutative law, A2,

$$0 \odot \mathbf{v} = 0 \odot \mathbf{v} \oplus (0 \odot \mathbf{v}) = (0 \odot \mathbf{v}) \oplus 0_V$$

Therefore, $(0 \odot \mathbf{v}) \oplus (0 \odot \mathbf{v}) = (0 \odot \mathbf{v}) \oplus 0_V$**, and therefore,** $0 \odot \mathbf{v} = 0_V$**.**

5. Describe all subspaces of $\mathbb{R}^3$.

   (a) The set containing the zero vector, $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$**, i.e. the zero subspace,**

   (b) the set describing any straight line going through the origin (including the x, y, and z axes),

   (c) the set describing any plane that goes through the origin,

   (d) and $\mathbb{R}^3$**.**

6. Let U, W be subspaces of a vector space V over F. Show that $U \cup W$ is a subspace of V iff either $U \subseteq W$ or $W \subseteq U$.

   (⇒) Suppose $U \cup W$ **is a subspace. Suppose, for contradiction, we have** $w \in W \setminus U$ **and** $u \in U \setminus W$**. Then** $u + w \notin U \cup W$**:**

   Suppose $u + w \in U$ **then** $w = (u + w) + (-u) \in U$ **which contradicts** $w \in W \setminus U$**, so** $u + w \notin U$**. Similarly we get** $u + w \notin W$**.**

   This contradicts SS2 for $U \cup W$**, so either** $W \setminus U = \emptyset$ **(so** $W \subseteq U$**) or** $U \setminus W = \emptyset$ **(so** $U \subseteq W$**).**

   (⇐) Clear as either $U \cup W = U$ **or** $U \cup W = W$