

1. Prove that if $f : [a, b] \rightarrow [0, \infty)$ is continuous and $f(c) \neq 0$ for some $c \in [a, b]$, then $\int_a^b f(x) dx > 0$.

Solution. We can take $c \in (a, b)$ without loss of generality, since if $f(x) = 0$ for all $x \in (a, b)$ then $f(a) = f(b) = 0$ as well by continuity. Since $\frac{f(c)}{2} > 0$, we also have

$$\exists \delta > 0 \text{ such that } |x - c| < \delta \Rightarrow |f(x) - f(c)| < \frac{f(c)}{2}$$

by the continuity of f at c . Then if $|x - c| \leq \delta$ we have $f(x) \geq \frac{f(c)}{2}$. Taking a smaller δ if needed to ensure that $[c - \delta, c + \delta] \subset [a, b]$, we can write

$$\begin{aligned} \int_a^b f(x) dx &= \int_a^{c-\delta} f(x) dx + \int_{c-\delta}^{c+\delta} f(x) dx + \int_{c+\delta}^b f(x) dx \\ &\geq \int_a^{c-\delta} 0 dx + \int_{c-\delta}^{c+\delta} \frac{f(c)}{2} dx + \int_{c+\delta}^b 0 dx \end{aligned}$$

since $f(x)$ is at least 0, $\frac{f(c)}{2}$, and 0 on the intervals $[a, c - \delta]$, $[c - \delta, c + \delta]$, and $[c + \delta, b]$ respectively. We evaluate these integrals one by one to get

$$\int_a^b f(x) dx \geq 0 + \delta \cdot f(c) + 0 > 0.$$

2. Suppose for some $f : [a, b] \rightarrow \mathbb{R}$ and integer $n \geq 1$ that the n th power f^n of f is integrable. Prove that if n is odd, then f is integrable. Why doesn't this work for n even, and can you find additional hypotheses on f that make it true in that case?

Solution. If n is odd then the n th root function $\sqrt[n]{\cdot} : \mathbb{R} \rightarrow \mathbb{R}$ is continuous, so if f^n is integrable then so is the composition $\sqrt[n]{f^n} = f$.

When n is even, we still have a continuous $\sqrt[n]{\cdot} : [0, \infty) \rightarrow [0, \infty)$, and $f(x)^n \geq 0$ for all x , so we can again conclude that $\sqrt[n]{f^n}$ is integrable. But in this case $\sqrt[n]{f^n} = |f|$, so we have only proved that $|f|$ is integrable. If $f(x) \geq 0$ for all x then $f = |f|$, so this proves that a nonnegative function f is integrable if f^n is.

3. Let $C[a, b]$ denote the set of continuous functions $f : [a, b] \rightarrow \mathbb{R}$, and define a function $d : C[a, b] \times C[a, b] \rightarrow \mathbb{R}$ by $d(f, g) = \int_a^b |f(x) - g(x)| dx$.

- (a) Prove that $d(f, g) = d(g, f)$ for all $f, g \in C[a, b]$.
- (b) Prove that $d(f, g) \geq 0$, with equality if and only if $f = g$.
- (c) Prove the triangle inequality $d(f, g) + d(g, h) \geq d(f, h)$.

These properties say that d is a *metric*, which is a notion of distance on $C[a, b]$.

- (d) Prove that if $f_n \rightarrow f$ uniformly on $[a, b]$, then $\lim_{n \rightarrow \infty} d(f_n, f) = 0$.

Solution. (a) Since $|f - g| = |g - f|$ on $[a, b]$, their integrals are the same.

(b) We have $|f(x) - g(x)| \geq 0$ for all x by the definition of absolute value, so $\int_a^b |f(x) - g(x)| dx \geq \int_a^b 0 dx = 0$. Since $|f(x) - g(x)|$ is continuous, problem 1 says that its integral is zero iff $|f(x) - g(x)| = 0$ for all x , or equivalently $f(x) = g(x)$ for all x .

(c) The usual triangle inequality says that for all x we have

$$|f(x) - g(x)| + |g(x) - h(x)| \geq |f(x) - h(x)|,$$

and each of the three terms above is continuous and hence integrable, so by basic properties of integration we have

$$\begin{aligned} \int_a^b |f(x) - g(x)| dx + \int_a^b |g(x) - h(x)| dx \\ = \int_a^b (|f(x) - g(x)| + |g(x) - h(x)|) dx \\ \geq \int_a^b |f(x) - h(x)| dx \end{aligned}$$

which is equivalent to $d(f, g) + d(g, h) \geq d(f, h)$.

(d) For any $\epsilon > 0$, uniform convergence means that we can find an $N > 0$ such that

$$n \geq N \Rightarrow |f_n(x) - f(x)| < \frac{\epsilon}{b-a} \quad \forall x \in [a, b].$$

But then for all $n \geq N$ we have

$$d(f_n, f) = \int_a^b |f_n(x) - f(x)| dx < \int_a^b \frac{\epsilon}{b-a} dx = \epsilon.$$

(We have strict inequality because $\frac{\epsilon}{b-a} - |f_n(x) - f(x)|$ is continuous and strictly positive, hence its integral is also strictly positive.) Thus $0 \leq d(f_n, f) < \epsilon$ for all $n \geq N$, and we can find such an N for any $\epsilon > 0$, so this means that $d(f_n, f) \rightarrow 0$ as $n \rightarrow \infty$.

4. Evaluate $\int_1^x \frac{\sqrt{t^2 - 1}}{t} dt$ for $x \geq 1$. (Hint: what is the inverse of the integrand?)

Solution. We note that $f(t) = \frac{\sqrt{t^2 - 1}}{t} = \sqrt{1 - t^{-2}}$ is monotone increasing, and if $y = f(t)$ with $t \geq 1$ then

$$y^2 = 1 - \frac{1}{t^2} \Rightarrow f^{-1}(y) = t = \frac{1}{\sqrt{1 - y^2}}.$$

Thus we apply the formula for the integral of an inverse function to get

$$\begin{aligned} \int_0^{f(x)} \frac{1}{\sqrt{1 - t^2}} dt + \int_1^x f(t) dt &= xf(x) - 1f(1) \\ &= x\sqrt{1 - \frac{1}{x^2}} = \sqrt{x^2 - 1}. \end{aligned}$$

We know from lecture that $\sin^{-1}(t)$ is an antiderivative of $\frac{1}{\sqrt{1-t^2}}$, so then

$$\begin{aligned}\int_1^x f(t) dt &= \sqrt{x^2 - 1} - \int_0^{f(x)} \frac{1}{\sqrt{1-t^2}} dt \\ &= \sqrt{x^2 - 1} - \sin^{-1}(f(x)) \\ &= \sqrt{x^2 - 1} - \sin^{-1}\left(\frac{\sqrt{x^2 - 1}}{x}\right).\end{aligned}$$

We can optionally simplify the last term by writing $\theta = \sin^{-1}\left(\frac{\sqrt{x^2-1}}{x}\right)$, which lies between 0 and $\frac{\pi}{2}$ since $\frac{\sqrt{x^2-1}}{x} > 0$, and then noting that

$$\sin^2(\theta) + \cos^2(\theta) = 1 \implies \cos^2(\theta) = 1 - \left(\frac{x^2 - 1}{x^2}\right) = \frac{1}{x^2},$$

so $\cos(\theta) = \frac{1}{x}$. Thus $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \sqrt{x^2 - 1}$, or equivalently

$$\sin^{-1}\left(\frac{\sqrt{x^2-1}}{x}\right) = \cos^{-1}\left(\frac{1}{x}\right) = \tan^{-1}\left(\sqrt{x^2-1}\right),$$

and we can write for example that $\int_1^x f(t) dt$ is equal to

$$\int_1^x \frac{\sqrt{t^2-1}}{t} dt = \sqrt{x^2-1} - \tan^{-1}(\sqrt{x^2-1}).$$

5. In problem sheet 4 we constructed a smooth (i.e., infinitely differentiable) function $f : \mathbb{R} \rightarrow [0, \infty)$ such that $f(x) > 0$ if and only if $x \in (0, 1)$.

- (a) Construct a smooth, monotone increasing function $g : \mathbb{R} \rightarrow [0, \infty)$ such that $g(x) = 0$ for all $x \leq 0$ and $g(x) = 1$ for all $x \geq 1$.
- (b) Given $a < b < c < d$, construct a smooth function $h : \mathbb{R} \rightarrow [0, \infty)$ satisfying

$$h(x) = 0 \text{ for all } x \notin [a, d], \quad h(x) = 1 \text{ for all } x \in [b, c],$$

and with h monotone increasing on $(-\infty, b]$ and decreasing on $[c, \infty)$.

Solution. (a) We construct g by integrating f . This gives us a constant function for $x \geq 1$ since $f(x) = 0$ there, but its value won't be 1, so we rescale it and define $g : \mathbb{R} \rightarrow \mathbb{R}$ by

$$g(x) = \begin{cases} 0, & x < -1 \\ \frac{1}{c} \int_0^x f(t) dt, & x \geq -1 \end{cases} \quad \text{where } c = \int_0^1 f(x) dx.$$

We claim that g is differentiable, with $g'(x) = \frac{1}{c}f(x)$. Indeed, for all $x \leq 0$ we have $g(x) = 0$ – when $-1 \leq x \leq 0$ this follows from the fact that $f(x) = 0$ for

$x \leq 0$ – so $g'(x) = 0 = \frac{1}{c}f(x)$ on $(-\infty, 0)$. And for $x > -1$ the fundamental theorem of calculus tells us that g is differentiable at x with $g'(x) = \frac{1}{c}f(x)$.

Since $g'(x) = f(x) \geq 0$, we know that g is monotone increasing (and hence nonnegative, since it is zero for all $x \leq 0$), and it is smooth since its derivative is infinitely differentiable. It only remains now to check that for $x \geq 1$ we have

$$\begin{aligned} g(x) &= \frac{1}{c} \int_0^x f(t) dt = \frac{1}{c} \left(\int_0^1 f(t) dt + \int_1^x f(t) dt \right) \\ &= \frac{1}{c} \left(\int_0^1 f(t) dt + \int_1^x 0 dt \right) = 1. \end{aligned}$$

- (b) The functions $h_1(x) = g\left(\frac{x-a}{b-a}\right)$ and $h_2(x) = g\left(\frac{d-x}{d-c}\right)$ are smooth since they are compositions of two smooth functions (one linear in x , and the other one g), and they satisfy

$$h_1(x) = \begin{cases} 0, & x \leq a \\ 1, & x \geq b \end{cases} \quad h_2(x) = \begin{cases} 0, & x \geq d \\ 1, & x \leq c \end{cases}$$

with h_1 and h_2 monotone increasing and constant, respectively, on $(-\infty, b]$ and constant and monotone decreasing, respectively, on $[c, \infty)$. It follows that $h(x) = h_1(x)h_2(x)$ has the desired properties: it is smooth, zero precisely outside (a, d) and one precisely on $[b, c]$, and increasing on $(-\infty, b]$ and decreasing on $[c, \infty)$.

6. (a) Given $a < b < 0$, evaluate $\int_a^b \frac{1}{x} dx$. Be careful not to take the logarithm of a negative number along the way!
- (b) Check that $\tan(x) = \frac{\sin(x)}{\cos(x)}$ is strictly monotone increasing on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$, with

$$\lim_{x \downarrow -\frac{\pi}{2}} \tan(x) = -\infty \quad \text{and} \quad \lim_{x \uparrow \frac{\pi}{2}} \tan(x) = +\infty.$$

- (c) Let $\tan^{-1} : \mathbb{R} \rightarrow (-\frac{\pi}{2}, \frac{\pi}{2})$ be the inverse function to $\tan(x)$. Prove for all $x \in \mathbb{R}$ that

$$\cos(\tan^{-1}(x)) = \frac{1}{\sqrt{1+x^2}}.$$

- (d) Fix $\theta \in (0, \frac{\pi}{2})$. Find a convenient substitution which proves that

$$\int_0^\theta \tan(x) dx = -\log(\cos(\theta)).$$

- (e) Prove for $x > 0$ that $\int_0^x \tan^{-1}(t) dt = x \tan^{-1}(x) - \frac{1}{2} \log(1+x^2)$.

Solution. (a) We'd like to use the fundamental theorem of calculus, because we've seen that $\frac{d}{dx} \log(x) = \frac{1}{x}$, but this fact only makes sense if $x > 0$! One way to fix it is to compose \log and the absolute value function, and then apply the chain rule: we have

$$\frac{d}{dx} \log |x| = \frac{1}{|x|} \cdot \frac{d}{dx} |x| = \begin{cases} +1/|x|, & x > 0 \\ -1/|x|, & x < 0. \end{cases}$$

In either case we have $\frac{d}{dx} \log |x| = \frac{1}{x}$ for all $x \neq 0$, so now we can apply the fundamental theorem of calculus:

$$\int_a^b \frac{1}{x} dx = (\log |x|)|_a^b = \log |b| - \log |a| = \log \left(\frac{|b|}{|a|} \right) = \log \left(\frac{b}{a} \right).$$

Alternatively, we can substitute $y = -x$ to get

$$\int_a^b \frac{1}{x} dx = \int_{-a}^{-b} -\frac{1}{y}(-1) dy = \int_{-a}^{-b} \frac{1}{y} dy = - \int_{-b}^{-a} \frac{1}{y} dy.$$

Since $0 < -b < -a$, this last term is equal to

$$- (\log(y)|_{-b}^{-a}) = -(\log(-a) - \log(-b)) = \log |b| - \log |a| = \log \left(\frac{b}{a} \right).$$

(b) We use the quotient rule to compute that as long as $\cos(x) \neq 0$, we have

$$\frac{d}{dx} \tan(x) = \frac{\cos(x) \cdot \cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)} = \frac{1}{\cos^2(x)} > 0,$$

and so $\tan(x)$ is strictly increasing on $(-\frac{\pi}{2}, \frac{\pi}{2})$. As $x \rightarrow \pm \frac{\pi}{2}$ we have $\cos(x) \rightarrow 0$ while $\sin(x) \rightarrow \pm 1$, so then $\tan(x) \rightarrow \pm \infty$, and since \tan is increasing on $(-\frac{\pi}{2}, \frac{\pi}{2})$ it must follow that $\tan(x) \rightarrow -\infty$ as $x \downarrow -\frac{\pi}{2}$ while $\tan(x) \rightarrow +\infty$ as $x \uparrow \frac{\pi}{2}$.

(c) We wish to compute $\cos(\theta)$, where $\tan(\theta) = x$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. We square both sides of the latter equation:

$$x^2 = \frac{\sin^2(\theta)}{\cos^2(\theta)} = \frac{1 - \cos^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)} - 1 \implies \cos^2(\theta) = \frac{1}{1 + x^2}.$$

Now $\cos(\theta)$ is strictly positive in the given range, so we can take square roots to conclude that

$$\cos(\tan^{-1}(x)) = \cos(\theta) = \left(\frac{1}{1 + x^2} \right)^{1/2}$$

as claimed.

(d) Writing $\tan(x) = \frac{\sin(x)}{\cos(x)}$, we notice that the numerator is (up to sign) the derivative of the denominator, so we write $y = -\cos(x)$ and then

$$\int_0^\theta \tan(x) dx = \int_0^\theta \left(-\frac{1}{y(x)} \right) y'(x) dx = \int_{-1}^{-\cos(\theta)} -\frac{1}{y} dy = (-\log |y|)|_{-1}^{-\cos(\theta)}$$

which simplifies to $-\log |\cos(\theta)|$. By hypothesis we have $0 < \cos(\theta) < 1$, so we can simplify using $|\cos(\theta)| = \cos(\theta)$ and we are done.

- (e) We use the theorem from lecture about integrals of inverse functions, since \tan is strictly monotone on $(0, \frac{\pi}{2})$: if $y \in (0, \frac{\pi}{2})$ then

$$\int_0^y \tan(t) dt + \int_0^{\tan(y)} \tan^{-1}(t) dt = y \tan(y) - 0 \tan(0).$$

We will eventually set $y = \tan^{-1}(x)$, which is in the interval $(0, \frac{\pi}{2})$ since $x > 0$. The first term on the left is equal to $-\log(\cos(y))$ by the previous part, so we move it to the right:

$$\int_0^{\tan(y)} \tan^{-1}(t) dt = y \tan(y) + \log(\cos(y))$$

and then set $y = \tan^{-1}(x)$ as promised to get

$$\int_0^x \tan^{-1}(t) dt = x \tan^{-1}(x) + \log(\cos(\tan^{-1}(x))).$$

We use $\cos(\tan^{-1}(x)) = (1 + x^2)^{-1/2}$ to simplify the right hand side further:

$$\int_0^x \tan^{-1}(t) dt = x \tan^{-1}(x) - \frac{1}{2} \log(1 + x^2).$$

7. (a) Check that the derivative of $x \log(x) - x$ is $\log(x)$.
 (b) Use Darboux sums to prove for all integers $n \geq 1$ that

$$\log((n-1)!) \leq \int_1^n \log(x) dx \leq \log(n!).$$

- (c) Evaluate the integral in (b) and deduce that

$$\frac{1}{n} \leq \frac{\log(n!)}{n} - \log\left(\frac{n}{e}\right) \leq \log\left(1 + \frac{1}{n}\right) + \frac{\log(n+1)}{n}$$

for all $n \geq 1$.

- (d) Conclude that $\lim_{n \rightarrow \infty} \frac{n}{\sqrt[n]{n!}} = e$.

Remark: this is a weak version of *Stirling's formula* $n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$.

Solution. (a) We have $\frac{d}{dx}(x \log x) = \log(x) + x \frac{1}{x} = \log(x) + 1$ by the product rule, so $\frac{d}{dx}(x \log(x) - x) = (\log(x) + 1) - 1 = \log(x)$.

- (b) We know that $\log(x)$ is integrable on $[1, n]$ since it is continuous, so if we take the partition $P = (1, 2, \dots, n)$ of $[1, n]$ then we have

$$L(\log(x), P) \leq \int_1^n \log(x) dx \leq U(\log(x), P).$$

Since $\log(x)$ is monotone increasing, we compute that

$$L(\log(x), P) = \sum_{i=1}^{n-1} \log(i) \cdot 1 = \log(1 \cdot 2 \cdots (n-1)) = \log((n-1)!)$$

$$U(\log(x), P) = \sum_{i=1}^{n-1} \log(i+1) \cdot 1 = \log(2 \cdot 3 \cdots n) = \log(n!),$$

so we put these together to get $\log((n-1)!) \leq \int_1^n \log(x) dx \leq \log(n!)$.

(c) We use part (a) and the fundamental theorem of calculus to evaluate

$$\int_1^n \log(x) dx = x \log(x) - x \Big|_{x=1}^{x=n} = n \log(n) - (n-1).$$

Then part (b) says that $n \log(n) - (n-1) \leq \log(n!)$, or

$$\frac{1}{n} \leq \frac{\log(n!)}{n} - \log(n) + 1 = \frac{\log(n!)}{n} - \log\left(\frac{n}{e}\right)$$

after some rearranging. Similarly, if we let $m = n-1$ then the leftmost inequality from (b) tells us that

$$\log(m!) \leq (m+1) \log(m+1) - m,$$

and we divide through by m and rearrange to get

$$\begin{aligned} \frac{\log(m!)}{m} - \log(m) + 1 &\leq \left(1 + \frac{1}{m}\right) \log(m+1) - \log(m) \\ &= \log\left(\frac{m+1}{m}\right) + \frac{\log(m+1)}{m}. \end{aligned}$$

Relabeling the variable n gives $\frac{\log(n!)}{n} - \log\left(\frac{n}{e}\right) \leq \log\left(1 + \frac{1}{n}\right) + \frac{\log(n+1)}{n}$.

(d) Applying the squeeze theorem to part (c) shows us that

$$\lim_{n \rightarrow \infty} \left(\frac{\log(n!)}{n} - \log\left(\frac{n}{e}\right) \right) = 0,$$

or equivalently

$$\lim_{n \rightarrow \infty} \log\left(\frac{\sqrt[n]{n!}}{n/e}\right) = 0 \Rightarrow \lim_{n \rightarrow \infty} \frac{\sqrt[n]{n!}}{n/e} = 1.$$

We apply the algebra of limits to conclude that $\lim_{n \rightarrow \infty} \frac{n}{\sqrt[n]{n!}} = e$.

8. (*) Let $f : [N, \infty) \rightarrow [0, \infty)$ be a nonnegative, monotone decreasing function.

- (a) Let $S_n = \sum_{k=N}^n f(k)$ for all integers $n \geq N$. Use Darboux sums to prove that

$$S_n - f(N) \leq \int_N^n f(x) dx \leq S_{n-1}.$$

- (b) Prove that the series $\sum_{k=N}^{\infty} f(k)$ converges if and only if the limit

$$\int_N^{\infty} f(x) dx \stackrel{\text{def}}{=} \lim_{x \rightarrow \infty} \int_N^x f(t) dt$$

(called an *improper integral*) exists. This is the *integral test* for convergence.

- (c) Prove that if the series $S = \sum_{k=N}^{\infty} f(k)$ converges, so $I = \int_N^{\infty} f(x) dx$ exists, then
- $$I \leq S \leq I + f(N).$$

Solution. (a) Fix an integer $n \geq N$ and consider the partition

$$P_n = (N, N+1, N+2, \dots, n)$$

of $[N, n]$. Since f is monotone decreasing we can compute the Darboux sums of f with respect to P :

$$L(f, P_n) = \sum_{k=N}^{n-1} f(k+1), \quad U(f, P_n) = \sum_{k=N}^{n-1} f(k).$$

We proved on the last problem sheet that f is integrable on $[N, n]$ since it is monotone, so we have

$$\sum_{k=N+1}^n f(k) = L(f, P_n) \leq \int_N^n f(x) dx \leq U(f, P_n) = \sum_{k=N}^{n-1} f(k).$$

The left and right sides are $S_n - f(N)$ and S_{n-1} respectively.

- (b) (\Leftarrow) If $I = \int_N^{\infty} f(x) dx$ exists then for any $\epsilon > 0$, there is an $M \geq 0$ such that

$$n \geq M \Rightarrow \left| \int_N^n f(x) dx - I \right| < \epsilon.$$

We use this together with part (a) to deduce that

$$n \geq M \Rightarrow S_n \leq \int_N^n f(x) dx + f(N) < I + f(N) + \epsilon.$$

The sequence (S_n) of partial sums is therefore bounded above, and it is increasing since $f(n) \geq 0$ for all n , so it converges.

(\implies) If $\sum_{k=N}^{\infty} f(k)$ converges to some S , then given $\epsilon > 0$, there is an $M \geq 0$ such that

$$n \geq M \implies |S_n - S| = \left| \sum_{k=N}^n f(k) - S \right| < \epsilon.$$

Using part (a), we have

$$n \geq M + 1 \implies \int_N^n f(x) dx \leq S_{n-1} < S + \epsilon.$$

Now the function $F(x) = \int_N^x f(t) dt$ is increasing, since for any $x < y$ we have

$$\begin{aligned} F(y) &= \int_N^y f(t) dt = \int_N^x f(t) dt + \int_x^y f(t) dt \\ &\geq \int_N^x f(t) dt + \int_x^y 0 dt = F(x), \end{aligned}$$

and it is bounded above since for any $x \geq N$ we have $F(x) \leq F(m) < S + \epsilon$ for some integer $m \geq \max(x, M + 1)$. Thus the limit $\lim_{x \rightarrow \infty} F(x) = \int_N^{\infty} f(t) dt$ exists, as desired.

Remark: we can't use the fundamental theorem of calculus to assert that $F'(x) = f(x) \geq 0$ and thus prove that $F(x)$ is increasing, because we do not know that f is continuous.

- (c) In part (b) we proved for any $\epsilon > 0$ and all large enough x and n that $\int_N^x f(t) dt < S + \epsilon$ and $S_n < I + f(N) + \epsilon$ respectively, so

$$I = \lim_{x \rightarrow \infty} \int_N^x f(t) dt \leq S + \epsilon, \quad S = \lim_{n \rightarrow \infty} S_n \leq I + f(N) + \epsilon.$$

These hold for any $\epsilon > 0$, so we must actually have $I \leq S$ and $S \leq I + f(N)$.

9. Consider for any real s the series $\sum_{n=1}^{\infty} \frac{1}{n^s}$.

- Prove that this series is not convergent if $s \leq 0$.
- Use the integral test to prove that for $s > 0$, the series converges if and only if $s > 1$. If $s > 1$, show that $\frac{1}{s-1} < \sum_{n=1}^{\infty} \frac{1}{n^s} < \frac{s}{s-1}$.
- Prove for any $a > 1$ that the series converges uniformly to a continuous function on $[a, \infty)$, and hence it defines a continuous function $\zeta : (1, \infty) \rightarrow \mathbb{R}$ called the *Riemann zeta function*. Can it be extended continuously to $[1, \infty)$?
- (Harder!) Prove that $\zeta(s)$ is continuously differentiable, and compute its derivative. It may help to first show that $\lim_{x \rightarrow \infty} \frac{\log(x)}{x^\epsilon} = 0$ for any $\epsilon > 0$.

Solution. (a) If $s = 0$ or $s < 0$ then $\lim_{n \rightarrow \infty} \frac{1}{n^s}$ is 1 or ∞ , and since it is not 0 the series does not converge in either case.

(b) Since $f(x) = \frac{1}{x^s}$ is nonnegative and monotone decreasing on $[1, \infty)$, the series $\sum_{n=1}^{\infty} \frac{1}{n^s}$ converges if and only if the limit

$$\int_1^{\infty} \frac{1}{t^s} dt = \lim_{x \rightarrow \infty} \int_1^x \frac{1}{t^s} dt$$

exists. We apply the fundamental theorem of calculus: if $s \neq 1$ then

$$\int_1^x \frac{1}{t^s} dt = \left. \frac{t^{1-s}}{1-s} \right|_{t=1}^{t=x} = \frac{x^{1-s} - 1}{1-s}.$$

As $x \rightarrow \infty$, this diverges if $s < 1$ and converges to $\frac{1}{s-1}$ if $s > 1$. If instead $s = 1$ then we have

$$\lim_{x \rightarrow \infty} \int_1^x \frac{1}{t} dt = \lim_{x \rightarrow \infty} \log(t) \Big|_{t=1}^{t=x} = \lim_{x \rightarrow \infty} \log(x) = \infty.$$

So the improper integral $\int_1^{\infty} \frac{1}{t^s} dt$ exists if and only if $s > 1$, and if it exists then it is equal to $\frac{1}{s-1}$, so by part (c) we have

$$\frac{1}{s-1} \leq \sum_{n=1}^{\infty} \frac{1}{n^s} \leq \frac{1}{s-1} + 1,$$

or $\frac{1}{s-1} \leq \zeta(s) \leq \frac{s}{s-1}$.

(c) We apply the Weierstrass M-test. Setting $b = \frac{1+a}{2}$, so that $1 < b < a$, we define $M_n = \frac{1}{n^b}$ for all n , and then we have

$$\left| \frac{1}{n^s} \right| \leq \frac{1}{n^b} = M_n \text{ for all } s \in [a, \infty).$$

The series $\sum_{n=1}^{\infty} M_n$ converges to $\zeta(b)$, so the M-test says that $\sum_{n=1}^{\infty} \frac{1}{n^s}$ converges uniformly on $[a, \infty)$ to a continuous function $\zeta(s)$.

Since $\zeta(s)$ is continuous on any interval $[a, \infty)$, if we are given $x > 1$ then we know that ζ is continuous on $[\frac{1+x}{2}, \infty)$, and this interval contains x , so in particular $\zeta(s)$ is continuous at $s = x$. Thus it is continuous on all of $(1, \infty)$.

From part (b) we have $1 < (s-1)\zeta(s) < s$ for all $s > 1$, and so $\lim_{s \downarrow 1} (s-1)\zeta(s) = 1$

by the squeeze theorem. If it were possible to extend $\zeta(s)$ continuously to $s = 1$, then the algebra of limits would also tell us that $\lim_{s \downarrow 1} (s-1)\zeta(s) = 0\zeta(1) \neq 1$, and this is a contradiction.

(d) We write $f_n(x) = \sum_{k=1}^n \frac{1}{k^x}$ for all $n \geq 1$. Since $\frac{1}{n^x} = e^{-x \log(n)}$ has derivative $(-\log(n))e^{-x \log(n)} = \frac{-\log(n)}{n^x}$, we compute that

$$f'_n(x) = - \sum_{k=1}^n \frac{\log(k)}{k^x}.$$

Given $x > 1$, we set $a = \frac{1+x}{2}$ and $b = x + 1$, so that $x \in (a, b)$ and $f_n \rightarrow f$ on $[a, b]$. Since we already know that $f_n \rightarrow \zeta$ uniformly on $[a, b]$, if we can show that f'_n converges uniformly on $[a, b]$ then it will follow that ζ is differentiable on $[a, b]$ (and in particular at $s = x$), with

$$\zeta'(s) = \lim_{n \rightarrow \infty} f'_n(s) = - \sum_{n=1}^{\infty} \frac{\log(n)}{n^s}.$$

The f'_n are continuous on $[a, b]$, so their uniform limit ζ' will be continuous on $[a, b]$ as well and in particular at $s = x$.

To prove that f'_n converges uniformly on $[a, b]$, we again apply the Weierstrass M-test. Let $\epsilon = \frac{a-1}{2} > 0$. We have $\log(n) < n^\epsilon$ for all sufficiently large n , because l'Hôpital's rule says that

$$\lim_{x \rightarrow \infty} \frac{\log(x)}{x^\epsilon} = \lim_{x \rightarrow \infty} \frac{1/x}{\epsilon x^{\epsilon-1}} = \lim_{x \rightarrow \infty} \frac{1}{\epsilon x^\epsilon} = 0.$$

Thus if we set $M_n = \frac{1}{n^{1+\epsilon}}$, then for all $s \geq a$ and large enough n we have

$$\left| -\frac{\log(n)}{n^s} \right| < \frac{n^\epsilon}{n^a} = \frac{1}{n^{a-\epsilon}} = \frac{1}{n^{1+\epsilon}} = M_n.$$

The series $\sum_{n=1}^{\infty} M_n$ converges to $\zeta(1 + \epsilon)$, so the M-test says that $f'_n(s) =$

$-\sum_{k=1}^n \frac{\log(k)}{k^s}$ converges uniformly on $[a, b]$, as desired.