Math40003 Linear Algebra and Groups Term 2 Unseen 2

1. Let F be a field, $A \in M_{n\times n}(F)$ and $\lambda \in F$. Prove that for all $\lambda \in F$, the sets $\{v \in F^n | Av = \lambda v\}$ and $\{v \in F^n | A^t v = \lambda v\}$ are subspaces of the same dimension. Conclude that λ is an eigenvalue of A if and only if λ is an eigenvalue of A^t.

   It is easy to see that these are subspaces: note that the first is equal to $\ker(A - \lambda I)$**.**

   To see that the dimensions are the same, it suffices to prove that the ranks of the matrices $(A - \lambda I)$ **and** $(A^T - \lambda I)$ **are equal. But the second is the transpose of the first, and so this follows from the fact that row rank is equal to column rank.**

   As $A = (A^t)^t$**, only one direction is required. If** λ **is an eigenvalue of** A**, by definition, there is some** $0 \neq v \in F^n$ **such that** $Av - \lambda v = 0$**. So** $\dim(\ker(A-\lambda I)) = n - \mathrm{rank}(A^t - \lambda I) = n - \mathrm{rank}\left((A - \lambda I)^t\right) = n - \mathrm{rank}(A^t - \lambda I) = \dim(\ker(A^t - \lambda I))$**. The conclusion follows from the observation that** λ **is an eigenvalue of** A **iff** $\dim\{v \in F^n | A^t v = \lambda v\} > 0$**.**

Definition 1. A matrix $A \in M_{n\times n}(\mathbb{R})$ is *column-stochastic* if all of its entries are non-negative and the sum of the entries in each column is equal to 1.

2. Prove that a column-stochastic matrix has $\lambda = 1$ as an eigenvalue.

   By Question 1, it suffices to show A^t **has** 1 **as an eigenvalue. The sum of every row in** A^t **is 1, so** $\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ **is a solution for** $A^t x = x$**.**

3. (a) Let $a_1, \dots, a_n \in \mathbb{R}$. Prove that $|\sum_{i=1}^n a_i| < \sum_{i=1}^n |a_i|$ if and only if there are i, j such that $a_i > 0$ and $a_j < 0$.

   This is more of an observation, nevertheless, it is an interesting questions in terms of "how to prove something that seems trivial"? Well, first, $\leq$ **is by the triangle inequality, and then it can be proceeded for** $n = 2$ **by dividing into cases, and then the claim follows by induction.**

   (b) Prove that if $v, w \in \mathbb{R}^n$ are linearly independent, then there is some $u \in \mathrm{span}\{v, w\}$ such with at least one positive entry and one negative entry.

   Put v, w **in rows of a matrix. Using row-echelon form, there are two leading 1's. Subtract one line from the other and we get 1 and -1 in two entries of the same row of a vector in the row space.**

(c) Let $A \in M_{n\times n}(\mathbb{R})$ be a column-stochastic matrix with all entries positive. Prove that $\{v \in \mathbb{R}^n | Av = v\}$ is a 1-dimensional vector space.

To prove it is 1-dimensional, in Question 2, we saw it is at least 1-dimensional. In Question 1, we saw that it has the same dimension as $\{v \in F^n | A^t v = v\}$. Assume $\{v \in F^n | A^t v = v\}$ contains two l.i. vectors. By Item 3b, there is some $\begin{pmatrix} v_1 \\ \dots \\ v_n \end{pmatrix} = v \in F^n$ such that $A^t v = v$ and v has both positive and negative entries. By Item 3a and since $[A]_{i,j} > 0$ for all i, j:

$$\sum_{i=1}^{n} |v_i| < \sum_{i=1}^{n}\sum_{j=1}^{n} A^t_{i,j} |v_j| = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{j,i} |v_j| = \sum_{j=1}^{n} |v_j| \sum_{i=1}^{n} A_{j,i} = \sum_{j=1}^{n} |v_j| .$$

Contradiction.

(d) With A as in (c), conclude there is a unique vector v such that $Av = v$, all of its entries are positive and the sum of its entries is 1.

4. Let $n \in \mathbb{N}$. Let $\mathcal{P} := \{1, \dots, n\}$ (elements of $\mathcal{P}$ are referred to as pages) and let $\mathcal{L} \subseteq \mathcal{P}^2$ be some set of ordered pairs of $\mathcal{L}$ (elements of $\mathcal{L}$ are called links). If $(a, b) \in \mathcal{L}$, we say there is a link from a to b. Fix some $0 < p < 1$. Consider the following random process :

- In stage 0: we pick a page with uniform distribution (i.e., the probability of every element of $\mathcal{P}$ to be picked is $1/n$).
- In stage $k+1$: Assuming in stage k we picked the page a, in stage $k+1$, with probability p we pick a new page out of $\mathcal{P}$ with uniform distribution, and with probability $1-p$ we pick, with uniform distribution, some page out of the pages with a link from a.

  Namely:
  - If $(a, b) \notin \mathcal{L}$, then the probability of picking b is $p \cdot 1/n$.
  - If $(a, b) \in \mathcal{L}$ and there are l many links from a, then the probability of picking b is $p \cdot 1/n + (1-p) \cdot 1/l$.

Find a matrix $A \in M_{n\times n}(\mathbb{R})$ and a vector $v \in \mathbb{R}^n$ such that the probability of picking page i in stage k is the i-th coordinate of $A^k v$. Prove A is column-stochastic.

$v = \begin{pmatrix} 1/n \\ \vdots \\ 1/n \end{pmatrix}$, $[A]_{i,j} = \begin{cases} p \cdot 1/n + (1-p) \cdot 1/l & \textbf{if } (i,j) \in \mathcal{L} \\ p \cdot (1/n) & \textbf{if } (i,j) \notin \mathcal{L} \end{cases}$. **$A$ is column-stochastic by definition and the fact that the probability of picking page i in stage k is the i-th coordinate of $A^k v$ follows by induction, definition of matrix multiplication and the total probability formula.**

In this sheet, you almost developed the basic PageRank, Google's ranking of web pages. For details, see the back of this page.

The process described in Question 4 describes a random surfer on the web. The surfer starts by surfing to a random web page picked at random, then with probability p (where p is quite small, e.g., 0.15) the surfer jumps to a completely random new page and with probability $(1 - p)$ the surfer follows one of the links on the page currently at.

It can be shown that the process $Av, A^2v, A^3v, \dots$ converges on each coordinate. Moreover, the limit vector, call it v^*, is precisely the unique vector found in Quesion 3d. However, this requires some more analysis. The i-th coordinate of v^* is the PageRank of page i.

Of course, there is more to Google's algorithm, e.g., what's the magic p, giving some links more weight than others etc. More importantly, how to compute v^*? Having said that, still, these are the basic foundations of Google's PageRank.

As for the question of computing (or at least, approximating) v^*, this can be solved by taking powers of matrices efficiently, a matter that will be addressed later in this module.

For further information, see:
`http://pi.math.cornell.edu/~mec/Winter2009/RalucaRemus/index.html`