Math40003 Linear Algebra and Groups
Term 2 Unseen 4A, Linear Algebra (Week 7)

In this exercise, we generalize the definition of the inner product you saw in the lectures. To avoid confusion we refer to this new definition as a "general inner product"

Definition 1. Let V be an $\mathbb{R}$-vector space. A *general inner product* on V is a binary function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ such that the following properties hold for all $v, u, w \in V$, $\alpha \in \mathbb{R}$:

(i) $\langle v, u \rangle = \langle u, v \rangle$.

(ii) $\langle \alpha v + u, w \rangle = \alpha \langle v, w \rangle + \langle u, w \rangle$.

(iii) $\langle v, v \rangle \geq 0$.

(iv) if $\langle v, v \rangle = 0$, then $v = 0$.

Similarly, we define a general norm on V: $\|v\| := \sqrt{\langle v, v \rangle}$.

1. For each of the following, determine whether $\langle \cdot, \cdot \rangle$ is a general inner product:

   (a) Let $V = M_{n \times m}(\mathbb{R})$ and let $\langle A, B \rangle := \operatorname{trace}(AB^\top)$.
   True. $\operatorname{trace}(AB^\top) = \operatorname{trace}((AB^\top)^\top) = \operatorname{trace}(BA^\top)$**, so we have Item (i). Item (ii) is obvious from linearity of the trace. For Item (iii) and Item (iv), open the definition and see that** $\operatorname{trace}(AA^\top) = \sum_{i=1}^{n} \sum_{j=1}^{m} (a_{i,j})^2$**.**

   (b) Let V be the space of all continuous functions on the interval $[a, b]$, with point-wise addition and scalar multiplication. Let $\langle g, f \rangle := \int_a^b f(x) g(x) dx$.
   True. Item (i) is by commutativity, Item (ii) is by linearity of the integral and distributivity in $\mathbb{R}$**. Item (iii) is obvious since** $(f(x))^2 \geq 0$**.**
   For Item (iv), the requirement of continuity is needed: if f **is not identically** 0**, there is some** $x_0 \in [a, b]$ **such that** $f(x_0) = y_0 > 0$**. By continuity, there is some interval** $[c, d]$ **of length** $\delta > 0$ **such that** $f(x) > y_0/2$ **for all** $x \in [c, d]$**. By positivity of** $(f(x))^2$**,**

$$\langle f, f \rangle = \int_a^b (f(x))^2 dx \geq \int_c^d (f(x))^2 dx \geq \int_c^d y_0/2 \, dx = \delta \cdot y_0/2 > 0.$$

   (c) Let V be the set of random variables on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$. Let $\langle X, Y \rangle := \mathbb{E}[X \cdot Y]$.
   False. Positivity fails as there are random variables which are not identically 0 **but have** $\mathbb{P}[X = 0] = 1$**.**

2. Prove that every n-dimensional inner product space is isomorphic to $\mathbb{R}^n$ with the inner product from class. Namely: Let $\langle (a_1, \ldots, a_n), (b_1, \ldots, b_n) \rangle_s := \sum_{i=1}^n a_i b_i$ (we call this the *standard inner product*, a.k.a the dot product). Let V be an $\mathbb{R}$-vector space with general inner product $\langle \cdot, \cdot \rangle$. Prove that there is an invertible linear transformation $T : \mathbb{R}^n \to B$ such that $\forall v, u \in \mathbb{R}^n : \langle v, u \rangle_s = \langle T(v), T(u) \rangle$.

   This is a bit complicated. First, one should ask oneself "where should the standard basis of $\mathbb{R}^n$ be mapped to?". Since T must be invertible, it needs to be mapped to a basis. Since T must preserve the inner product, it should be mapped to an *orthonormal basis*. So the first step would be figuring out that V has an orthonormal basis. This can be done by tracing GramSchmidt and observing that it works in general. Now if $\mathcal{B} := (b_1, \ldots, b_n)$ is an orthonormal basis for V and $\mathcal{E} = (e_1, \ldots, e_n)$ is the standard basis for $\mathbb{R}^n$, then there is a unique linear transformation $T : \mathbb{R}^n \to V$ such that $T(e_i) = b_i$ for all $i \leq n$. Invertability is easy to see, for the inner product we can now just observe that for every $v = \sum_{i=1}^n \alpha_i e_i, u = \sum_{i=1}^n \beta_i e_i$:

$$\langle v, u \rangle = \sum_{i=1}^n \sum_{j=1}^n \alpha_i \beta_j \langle e_i, e_j \rangle = \sum_{i=1}^n \alpha_i \beta_i = \sum_{i=1}^n \sum_{j=1}^n \alpha_i \beta_j \langle b_i, b_j \rangle = \langle T(v), T(u) \rangle .$$

3. Prove the following for general inner products: Let V be an $\mathbb{R}$-vector space and let $\langle \cdot, \cdot \rangle$ be a general inner product on V. Prove:

   (a) Cauchy-Schwarz inequality:

      i. $\forall v, u \in V : \langle v, u \rangle \leq \|v\| \cdot \|u\|$

      ii. $\forall v, u \in V : \langle v, u \rangle = \|v\| \cdot \|u\| \iff \{ v, u \}$ are linearly dependent.

   (b) The triangle inequality: $\forall v, u \in V : \|v + w\| \leq \|v\| + \|w\|$.

      For both Cauchy-Schwarz and the triangle inequality – one can either trace back the proofs in the lecture notes that clearly hold in the general case, or observe that it suffices to prove them for vector spaces of dimension at most 2 (as if the inequalities hold in subspace containing u, v then it holds in V in general). Then apply the isomorphism from Question item 2 to $\mathbb{R}^2$ and deduce the inequalities.

   (c) The Pythagorean theorem:
      $\forall v, u \in V : \langle v, u \rangle = 0 \iff \|v\|^2 + \|u\|^2 = \|u + v\|^2$.
      LHS: $\langle v, v \rangle + \langle u, u \rangle$**.**
      RHS: $\langle u, u \rangle + \langle v, v \rangle + 2 \langle u, v \rangle$**.**

   (d) The Parallelogram law: $\forall v, u \in V : \|v + u\|^2 + \|v - u\|^2 = 2 \|v\|^2 + 2 \|u\|^2$.
      LHS: $\langle u, u \rangle + \langle v, v \rangle + 2 \langle u, v \rangle + \langle u, u \rangle + \langle v, v \rangle - 2 \langle u, v \rangle$**.**
      RHS: $2 \langle u, u \rangle + 2 \langle v, v \rangle$**.**