Math40003 Linear Algebra and Groups   Problem Sheet 3

1. (a) Which of these sets of vectors are linearly independent? Which span $\mathbb{R}^3$?

   i. $(5,3,0)$, $(2,1,1)$
   ii. $(1,0,1)$, $(-1,1,0)$, $(0,1,1)$
   iii. $(1,3,1)$, $(2,1,1)$, $(-1,7,-5)$
   iv. $(1,-3,2)$, $(2,-1,1)$, $(2,-5,4)$, $(1,2,5)$

   (b) For which a,b,c are the vectors $(1,3,1)$, $(2,1,1)$, (a,b,c) linearly dependent?

   (a) **i. Linearly independent, does not span**
   ii. Linearly dependent since $v_1+v_2-v_3=0$; does not span
   iii. Linearly independent and spans $\mathbb{R}^3$
   iv. Linearly dependent since $-17v_1-v_2+10v_3-v_4=0$, but spans $\mathbb{R}^3$

   (b) Those satisfying $2a+b-5c=0$.

2. *Let V be a finite-dimensional vector space. For each of the following statements, say whether it is true or false. If it is true, give a justification; otherwise find a counterexample.

   (a) If $\{v_1,\dots,v_n\}$ is a basis, for V, and $\{x_1,\dots,x_r\}$ is a linearly independent subset of V with $r<n$, and if $v_i \notin Span\{x_1,\dots,x_r\}$ for all $i=1,\dots,n$, then $\{x_1,\dots,x_r,v_{r+1},\dots,v_n\}$ is a basis for V.

   (b) If U is a subspace of V, then $U+U=U$.

   (c) If U and W are subspaces of V, and $\dim U+\dim W=\dim V$, then $U\cap W=\{0_V\}$.

   (d) If $\dim V=n$ and $v_1\in V$, then there exist vectors $v_2,\dots,v_n$ in V such that $\{v_1,\dots,v_n\}$ spans V.

   (e) If W is a subspace of V , then $\dim W\le\dim V$ and $\dim W=\dim V$ if and only if $W=V$.

   (a) False: take $\{v_1,v_2,v_3\}$ to be the standard basis of $\mathbb{R}^3$, with $s=1$ and $x_1=(0,1,1)$.

   (b) True: since U is closed under addition, we have $U+U\subseteq U$; but since $0_V\in U$ we have $u=u+0_V\in U+U$ for all $u\in U$, and so $U\subseteq U+U$.

   (c) False: take $V=\mathbb{R}^2$, and $U=W=Span\{(1,0)\}$.

   (d) False: but only because v_1 might be 0_V. For any other v_1 it is true.

   (e) True: Let B be a basis for W. Now consider B as a subset of V. If $Span(B)=V$ then B is also a basis for V, so $\dim W=\dim V$. Otherwise, we have $v\in V\setminus Span(B)$, now $B'=\{v\}\cup B$ is LI. If this

does not span V we continue to add vectors until we get a spanning set B^*. Now B^* will be basis for V and $B \subseteq B^*$, so $|B| \leq |B^*|$.

It remains to show that if $\dim W = \dim V$ then $V = W$. Suppose $v \in V \setminus W$ then $v \notin Span(B)$, so $\{v\} \cup B$ is an LI subset of V, so $dim(V) \geq dim(W) + 1$ giving us a contradiction.

3. Which of the following sets of vectors in $\mathbb{R}^4$ are linearly independent? Extend those which are linearly independent to a basis of $\mathbb{R}^4$.

   (a) $(1, 2, 3, 0)$, $(-1, 2, 3, 0)$ (b) $(1, 2, 3, 0)$, $(-1, 2, 3, 0)$, $(0, 1, 2, 3)$

   (c) $(1, 1, -1, -1)$, $(1, -1, 1, -1)$, $(-1, 1, 1, -1)$, $(0, 1, 2, -3)$

   (a) Linearly independent An example of a basis is $\{v_1, v_2, e_3, e_4\}$, where v_1, v_2 are the vectors given.

   (b) Linearly independent Basis $\{v_1, v_2, v_3, e_1\}$.

   (c) Linearly dependent

4. Let $V = \mathbb{R}^{\mathbb{R}}$ (the vector space of functions from $\mathbb{R}$ to $\mathbb{R}$).

   (a) Show that the functions

   $$f_1(x) = 1, \quad f_2(x) = 1 + x + x^2, \quad f_3(x) = \sin x, \quad f_4(x) = \cos x$$

   are linearly independent.

   (b) Which of the following functions lie in $Span(f_1, f_2, f_3, f_4)$?

   $$5 - 3x - 3x^2, \quad \tan x, \quad 10 - x - x^2 + \sin(x + \pi/3).$$

   (a) If $\exists \lambda_i \in \mathbb{R}$ such that $\lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3 + \lambda_4 f_4 = \mathbf{0}$ (the zero function), then

   $$\lambda_1 f_1(x) + \lambda_2 f_2(x) + \lambda_3 f_3(x) + \lambda_4 f_4(x) = 0 \text{ for all } x \in \mathbb{R}.$$

   Putting $x = 0, \pi, 2\pi$ gives the equations $\lambda_1 + \lambda_2 + \lambda_4 = 0$, $\lambda_1 + (1 + \pi + \pi^2)\lambda_2 - \lambda_4 = 0$, $\lambda_1 + (1 + 2\pi + 4\pi^2)\lambda_2 + \lambda_4 = 0$, from which it easily follows that $\lambda_1 = \lambda_2 = \lambda_4 = 0$. Now put $x = \pi/2$ to get $\lambda_3 = 0$ also. So f_1, f_2, f_3, f_4 are linearly independent.

   (b) The function $\tan x$ is not in $Span(f_1, \ldots, f_4)$ — otherwise $\exists \lambda_i \in \mathbb{R}$ such that $\tan x = \lambda_1 f_1(x) + \lambda_2 f_2(x) + \lambda_3 f_3(x) + \lambda_4 f_4(x)$ for all $x \in \mathbb{R}$, which we check to be impossible by putting $x = 0, \pi, 2\pi$ etc. as above.

   The other two functions are in $Span(f_1, \ldots, f_4)$: use the addition formula to show that for the last one.

5. (a) Write down an infinite number of different bases of $\mathbb{R}^2$ (in finite time).
   (b) Find a basis for $W = Span(x^2 - 1,\ x^2 + 1,\ 4,\ 2x - 1,\ 2x + 1) \leq \mathbb{R}[x]$.

   Recall: $\mathbb{R}[x]$ *is the set of real polynomials in the variable* x

   (a) E.g. $\{(1,0),(0,t)\}$ **are bases for any** $t \in \mathbb{R} \setminus \{0\}$**.**

   (b) Call these polynomials $p_1, \ldots, p_5$**. Since** $p_1 \neq 0$ **we keep it. Since** p_2 **is not a scalar multiple of** p_1**, we keep it. Next,** $p_3 = 2(p_2 - p_1) \in Span(p_1, p_2)$**, so we throw it away. Now** p_4 **has an** x **term, so cannot be in** $\langle p_1, p_2 \rangle$**, so keep it. Finally,** $p_5 = p_4 + \frac{1}{2}p_3 \in Span(p_1, p_2, p_4)$ **so discard it. So a basis is** $\{p_1, p_2, p_4\}$**.**

6. Let V be the vector space of all 3×3 matrices over $\mathbb{R}$.

   (i) Find a basis of V consisting of invertible matrices.

   (ii) Let $W = \{A \in V : A^t = A\}$. Show $W \leq V$ and compute $\dim W$.

   (iii) Let $W \subset V$ be the set of matrices whose columns, rows, and both diagonals add to 0. Show $W \leq V$ and find a basis for W.

   (a) E.g. $I, I - 2E_{33}, I - 2E_{22}$ **together with** $I + E_{ij}$ **for all** $i \neq j$**, where** E_{ij} **has a 1 in the** ij**th position only.**

   (b) The general symmetric matrix

$$\begin{pmatrix} a & x & y \\ x & b & z \\ y & z & c \end{pmatrix}$$

   is $aE_{11} + bE_{22} + cE_{33} + x(E_{12} + E_{21}) + y(E_{13} + E_{31}) + z(E_{23} + E_{32})$**. So a basis for this subspace is** $E_{11}, E_{22}, E_{33}, E_{12} + E_{21}, E_{13} + E_{31}, E_{23} + E_{32}$**, and its dimension is 6.**

   (c) First show the middle entry must be 0 — e.g. add each diagonal and the middle row to get that $0 =$ **the sum of the 1st and third columns plus twice the middle entry.**

   So the magic matrices have the form $\begin{pmatrix} a & b & -a-b \\ c & 0 & -c \\ -a-c & -b & -a \end{pmatrix}$**, with** $2a + b + c = 0$ **to get the last row to add to 0.**

   Get a basis by taking first $a = 1, b = 0$ **then** $a = 0, b = 1$**. So dimension** $= 2$**.**