Math40003 Linear Algebra and Groups — Unseen 3

1. Let $(F, \oplus, \otimes, 0_F, 1_F)$ be a field.
   (a) Prove that if $a, b \in F$ such that $a \otimes b = 0_F$, then either $a = 0_F$ or $b = 0_F$.
   (b) Prove that if $a, b, c \in F$ such that $c \neq 0_F$ and $a \otimes c = b \otimes c$, then $a = b$.
   (c) Prove that if$a, b, c \in F$ such that $a \oplus c = b \oplus c$, then $a = b$.

2. Let $(F, \oplus, \otimes, 0_F, 1_F)$ be a field and let $S \subseteq F$ be a *finite non-empty* subset of F closed under addition and multiplication, that is, such that

$$\forall s, t \in S : s \oplus t \in S \text{ and } s \otimes t \in S.$$

   (a) Prove that for all $s, t \in S$ there is $u \in S$ such that $t \oplus u = s$.
   Hint: use the pigeon-hole principle: if A is a finite set and $f : A \to A$ is an injective function, then f is surjective.
   (b) For $a \in F$, let $\oplus^n a = a \oplus a \oplus \cdots \oplus a$ (n times). Prove there is an $s \in S$ and and $n \geq 1$ such that $\oplus^n a = 0_F$.
   (c) Prove that $(S, \oplus, \otimes, 0_F, 1_F)$ is a field.
   Notice that we are not assuming that $0_F, 1_F \in S$, so this is something you need prove as well.

3. Consider $\mathbb{C}^2$ as a vector space over $\mathbb{C}$ in the usual way. Which of the following subsets of $\mathbb{C}^2$ are subspaces?
   (a) $U_1 = \{(2a + 3b, 4a - c) \mid a, b, c \in \mathbb{C}\}$.
   (b) $U_2 = \{(a - 1, a - b) \mid a, b \in \mathbb{C}\}$.
   (c) $U_3 = \{(2a - 1, 4a - 2) \mid a \in \mathbb{R}\}$.
   (d) $U_4 = \{(2a - 1, 4a - 2) \mid a \in \mathbb{C}\}$.

4. Consider $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ with their usual field structure.
   (a) Prove or disprove the following statements.
      i. $\mathbb{Q}$ with usual addition and scalar multiplication given by the multiplication in $\mathbb{R}$ is an $\mathbb{R}$-vector space.
      ii. $\mathbb{R}$ with the usual addition and scalar multiplication given by the product in $\mathbb{C}$ is a $\mathbb{C}$-vector space.
      iii. $\mathbb{C}$ is a $\mathbb{Q}$-vector space (with the usual additive structure and scalar multiplication given by the product in $\mathbb{C}$).
      iv. $\mathbb{R}$ is an $\mathbb{R}$-vector space (with the usual additive structure and scalar multiplication given by the product in $\mathbb{R}$).
   (b) Fill in the blanks and prove the following statement. Let E, F be fields such that $E \boxed{} F$. Then $\boxed{}$ is an $\boxed{}$-vector space. In particular, F is an F-vector space.