Topic: Discrete random variables and their distributions

In today's problem class we will be studying properties of discrete random variables.

1. Show that the function

$$p_X(x) = \frac{1}{1+\lambda}\left(\frac{\lambda}{1+\lambda}\right)^x$$

for parameter $\lambda > 0$ is a valid probability mass function for a discrete random variable X taking values on $\{0, 1, 2, ...\}$. Also, find $\mathrm{P}(X \leq x)$ for $x \in \mathbb{R}$.

Solution: Need to show p_X non-negative, and sums to one over the range of X. Clearly, p_X is nonnegative, and the sum of geometric progression gives result, that is,

$$\sum_{x=0}^{\infty} p_X(x) = \sum_{x=0}^{\infty} \frac{1}{1+\lambda}\left(\frac{\lambda}{1+\lambda}\right)^x = \frac{1}{1+\lambda}\left(1 - \frac{\lambda}{1+\lambda}\right)^{-1} = 1,$$

where we note that $\lambda/(1+\lambda) < 1$.

Let $x = 0, 1, ...$, then

$$\mathrm{P}(X \leq x) = \sum_{i=0}^{x} p_X(i) = \sum_{i=0}^{x} \frac{1}{1+\lambda}\left(\frac{\lambda}{1+\lambda}\right)^i = (1-\theta)\sum_{i=0}^{x}\theta^i = (1-\theta)\frac{1-\theta^{x+1}}{1-\theta} = 1 - \theta^{x+1},$$

where $\theta = \dfrac{\lambda}{1+\lambda}$. For general $x \in \mathbb{R}$, we have

$$\mathrm{P}(X \leq x) = \begin{cases} 0, & \text{if } x < 0, \\ 1 - \theta^{\lfloor x \rfloor + 1}, & \text{if } x \geq 0. \end{cases}$$

We note that $\lfloor x \rfloor$ denotes greatest integer less than or equal to x.

2. For what values of k is the following function a valid probability mass function?

$$p_X(x) = \begin{cases} \frac{k}{x(x+1)} & \text{if } x = n, n+1, n+2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

where n is a fixed positive integer.

Hint:

$$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}.$$

Solution:

$$\sum_{x=n}^{\infty} \frac{k}{x(x+1)} = k\sum_{x=n}^{\infty}\left(\frac{1}{x} - \frac{1}{x+1}\right) = k\left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+2} - \ldots\right)$$
$$= \frac{k}{n}.$$

Need $\sum p_X(x) = 1 \Rightarrow k = n$. Also, $p_X(x) \geq 0$ by construction (and also $p_X(x) \leq 1$).

3. If $X \sim \text{Poi}(\lambda)$ and we know that $\text{P}(X > 0) = 1 - \text{e}^{-0.5}$, determine $\text{P}(X \leq 1)$.

Solution: If $X \sim \text{Poi}(\lambda)$ so $p_X(x) = \text{P}(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, x = 0, 1, 2, \ldots$

$$\text{P}(X > 0) = 1 - \text{P}(X = 0) = 1 - e^{-0.5} \Rightarrow \text{P}(X = 0) = e^{-0.5}.$$

$$p_X(0) = \text{P}(X = 0) = e^{-\lambda} \Rightarrow \lambda = 0.5$$

$$\text{P}(X \leq 1) = p_X(0) + p_X(1) = e^{-0.5} + 0.5e^{-0.5} = 1.5e^{-0.5}.$$

4. If $X \sim \text{Poi}(\lambda)$ find the probability that X is odd.

Solution:

$$\sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^{2x+1}}{(2x+1)!} = e^{-\lambda}\left(\lambda + \frac{\lambda^3}{3!} + \frac{\lambda^5}{5!} + \ldots\right) = e^{-\lambda}\sinh(\lambda).$$

5. If $X \sim \text{Bin}(n, \theta)$, find $g(x)$ such that

$$p_X(x+1) = g(x)p_X(x), \quad x = 0, 1, \ldots, n-1.$$

Solution:

$$p_X(x) = \binom{n}{x}\theta^x(1-\theta)^{n-x},$$
$$p_X(x+1) = \binom{n}{x+1}\theta^{x+1}(1-\theta)^{n-x-1}.$$

Note that

$$\binom{n}{x+1} = \frac{n!}{(x+1)!(n-x-1)!} = \frac{n!(n-x)}{(x+1)x!(n-x)!} = \frac{(n-x)}{(x+1)}\binom{n}{x}.$$

So we can write

$$p_X(x+1) = \frac{(n-x)\theta}{(x+1)(1-\theta)}\binom{n}{x}\theta^x(1-\theta)^{n-x}.$$

I.e. when we choose

$$g(x) = \frac{\theta(n-x)}{(1-\theta)(x+1)}, x = 0, 1, 2, \ldots, n-1,$$

then the result follows.

6. Let $X \sim \text{Bin}(n, p)$ and let $q = 1 - p$. Show that $Y = n - X \sim \text{Bin}(n, q)$.

Solution: For $y \in \{0, 1, \ldots, n\}$, we have

$$p_Y(y) = \mathrm{P}(Y = y) = \mathrm{P}(n - X = y) = \mathrm{P}(X = n - y) = \binom{n}{n-y} p^{n-y}(1-p)^{n-(n-y)}$$
$$= \binom{n}{y} q^y (1-q)^{n-y},$$

since

$$\binom{n}{n-y} = \frac{n!}{(n-y)!(n-(n-y))!} = \frac{n!}{(n-y)!y!} = \binom{n}{y}.$$

So the p.m.f. of Y has the functional form of the p.m.f. of a $\mathrm{Bin}(n, q)$ random variable.

7. Let $X \sim \mathrm{Bin}(n, p)$ for $n \in \mathbb{N}, 0 < p < 1$, and set $q = 1 - p$. Show that $Y := n - X \sim \mathrm{Bin}(n, q)$.

Solution: Let $y \in \{0, 1, \ldots, n\}$, then

$$\mathrm{P}(Y = y) = \mathrm{P}(n - X = y) = \mathrm{P}(X = n - y) = \binom{n}{n-y} p^{n-y}(1-p)^{n-(n-y)}$$
$$= \binom{n}{y} q^y (1-p)^{n-y},$$

and 0 otherwise, which is the pmf of a $\mathrm{Bin}(n, q)$ random variable.

8. Let $X \sim \mathrm{Bin}(n, p)$ for an even $n \in \mathbb{N}$, and $p = 1/2$. Show that the distribution of X is symmetric about $n/2$, i.e.

$$\mathrm{P}(X = \frac{n}{2} + j) = \mathrm{P}(X = \frac{n}{2} - j),$$

for all nonnegative integers j.

Solution: From the previous question, we deduce that $X \sim \mathrm{Bin}(n, 1/2)$ implies that $Y := n - X = \mathrm{Bin}(n, 1/2)$. Hence

$$\mathrm{P}(X = k) = \mathrm{P}(Y = k) = \mathrm{P}(n - X = k) = \mathrm{P}(X = n - k),$$

for all $k \in \mathbb{N}_0$. Since n is even, we can set $k = \frac{n}{2} + j$, for any $j \in \mathbb{N}_0$, and obtain

$$\mathrm{P}\left(X = \frac{n}{2} + j\right) = \mathrm{P}\left(X = \frac{n}{2} - j\right).$$

9. A fair coin is tossed n times. Let H, T denote the discrete random variables corresponding to the number of heads and the number of tails, respectively, in n tosses of the coin. Define the discrete random variable $X = H - T$. Find the image/range and probability mass function of X.

Solution: Let $H =$"Number of Heads", $T =$"Number of Tails". Then $T = n - H$ and $X = H - T = H - (n - H) = 2H - n$. Thus

$$\mathrm{Im}X = \{-n, -n+2, -n+4, ..., n-4, n-2, n\}.$$

We note that $H, T \sim \mathrm{Bin}(n, 1/2)$. Hence,

$$\begin{aligned} p_X(x) &= \mathrm{P}(X = x) = \mathrm{P}(2H - n = x) = \mathrm{P}(H = (x+n)/2) = p_H((x+n)/2)) \\ &= \binom{n}{\frac{x+n}{2}} \left(\frac{1}{2}\right)^n, \end{aligned}$$

for $x \in \mathrm{Im}X$ and $p_X(x) = 0$ otherwise.