Multivariable Calculus + Differential Equations Concise Notes

MATH50004

Term 1 Content

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Colour Code - **Definitions** are **green** in these notes, **Consequences** are **red** and **Causes** are **blue**

Content from MATH40004 assumed to be known.

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Contents

1 Vector Calculus

1.1 Prelim

Definition 1.1.1 - Einstein Summation Convention

$$a_i x_i = \sum_{i=1}^{3} x_i$$

Definition 1.1.2 - The Kronecker delta

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Definition 1.1.3 - The Permutation Symbol

$$\epsilon_{ijk} = \begin{cases} 0, & \text{if any 2 elements } i, j, k \text{ equal} \\ 1, & \text{if } i, j, k \text{ a cyclic permutation of } 1, 2, 3 \\ -1, & \text{if } i, j, k \text{ an acyclic permutation } 1, 3, 2 \end{cases}$$

Formula - Relation between Kroenecker Delta and Permutation Symbol

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$$

$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta kl$$

Definition 1.1.4 - Vector Products
Here are some identities:

- $\mathbf{a} \cdot \mathbf{b} = a_i b_i$
- $[\mathbf{a} \times \mathbf{b}]_i = \epsilon_{ijk} a_j b_k$
- $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \Rightarrow [a \times b]_i = \epsilon_{ijk} a_j b_k$
- $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \epsilon_{ijk} a_i b_j c_k$
- $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \Rightarrow [\mathbf{a} \times (\mathbf{b} \times \mathbf{c})]_i = (\mathbf{a} \cdot \mathbf{c}) b_i - (\mathbf{a} \cdot \mathbf{b}) c_i$

1.2 Gradient, Div, and Curl

Definition 1.2 - Gradient, Directional Derivatives
$\phi =$ constant, defines a surface in $3D$, varying the constant yields a family of surfaces.

$$\hat{\mathbf{n}}\frac{\partial \phi}{\partial n} = \nabla = (\frac{\delta}{\delta x}, \frac{\delta}{\delta y}, \frac{\delta}{\delta z}) \Rightarrow \nabla \phi = \frac{\delta \phi}{\delta x} + \frac{\delta \phi}{\delta y} + \frac{\delta \phi}{\delta z}$$

Thus, directional derivative towards $\mathbf{s}$ = $\frac{\delta \phi}{\delta s} = \nabla \phi \cdot \hat{\mathbf{s}}$
In **cylindrical coordinates** r, θ, z parametrized by $x = r\cos\theta$, $y = r\sin\theta$ yields $\nabla \phi = \hat{\mathbf{r}}\frac{\delta \phi}{\delta r} + \frac{\hat{\theta}}{r}\frac{\delta \phi}{\delta \theta} + \mathbf{k}\frac{\delta \phi}{\delta z}$

Definition 1.2.3 - Tangent Plane to $\phi(P)$

$$(\mathbf{r} - \mathbf{r}_p) \cdot (\nabla \phi)_P = 0$$

$$\left(\frac{\delta \phi}{\delta x}\right)_P (x - x_P) + \left(\frac{\delta \phi}{\delta y}\right)_P (y - y_P) + \left(\frac{\delta \phi}{\delta z}\right)_P (z - z_P) = 0$$

1.3 Divergence & Curl

Definition 1.3.1 - Divergence and Curl
$\mathbf{A}$ a vector function of position

$$\textbf{Div A} = \nabla \cdot \mathbf{A} = \frac{\delta A_1}{\delta x} + \frac{\delta A_2}{\delta y} + \frac{\delta A_3}{\delta z} \text{ where } A = A_1\hat{\mathbf{i}} + A_2\hat{\mathbf{j}} + A_3\hat{\mathbf{k}}$$

$$\textbf{Curl A} = \nabla \times \mathbf{A} = \hat{\mathbf{i}}\left(\frac{\delta A_3}{\delta y} - \frac{\delta A_2}{\delta z}\right) - \hat{\mathbf{j}}\left(\frac{\delta A_3}{\delta x} - \frac{\delta A_1}{\delta z}\right) + \hat{\mathbf{k}}\left(\frac{\delta A_2}{\delta x} - \frac{\delta A_1}{\delta y}\right)$$

Definition - Laplacian Operator

$$\nabla^2 \phi = \text{div}(\nabla \phi) = \frac{\delta^2 \phi}{\delta x^2} + \frac{\delta^2 \phi}{\delta y^2} + \frac{\delta^2 \phi}{\delta z^2}$$

1.4 Operations with Grad operator

Resulting Equalities

(i) $\nabla(\phi_1 + \phi_2) = \nabla \phi_1 + \nabla \phi_2$

(ii) $\text{div } (\mathbf{A} + \mathbf{B}) = \text{div } \mathbf{A} + \text{div } \mathbf{B}$

(iii) $\text{curl } (\mathbf{A} + \mathbf{B}) = \text{curl } \mathbf{A} + \text{curl } \mathbf{B}$

(iv) $\nabla(\phi \psi) = \phi \nabla \psi + \psi \nabla \phi$

(v) $\text{div}(\phi \mathbf{A}) = \phi \text{ div } \mathbf{A} + \nabla \phi \cdot \mathbf{A}$

(vi) $\text{curl}(\phi \mathbf{A}) = \phi \text{ curl } \mathbf{A} + \nabla \phi \times \mathbf{A}$

(vii) $\text{div}(\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \text{curl } \mathbf{A} - \mathbf{A} \cdot \text{curl } \mathbf{B}$

(viii) $\text{curl}(\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla) \ \mathbf{A} - \mathbf{B} \text{ div } \mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A} \text{ div } \mathbf{B}$

(ix) $\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} + (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{B} \times \text{curl } \mathbf{A} + \mathbf{A} \times \text{curl } \mathbf{B}$

(x) $\text{curl } (\nabla \phi) = 0$

(xi) $\text{curl } (\text{curl } \mathbf{A}) = \nabla(\text{div } \mathbf{A}) - \nabla^2 \mathbf{A}$

(xii) $\text{div } (\text{curl } \mathbf{A}) = 0$

1 Integration

Definition 1.4.6 - Scalar and Vector Fields
If at each point of region V, scalar function ϕ defined - **ϕ a scalar field over V**
Similarly if vector function A defined $\forall v \in V$, **A a vector field.**
If curl $A = 0$, A is an **irrotational vector field**. If div $A = 0$, A a **solenoidal vector field**

1.5 Path Integrals

Definition 1.5.1 - Definition of a Path Integral

$$\lim_{n\to\infty}\sum_{n=1}^{N} f_n \delta s_n = \int_\gamma f ds \Rightarrow \int_\gamma \mathbf{F}\cdot d\mathbf{r} \int_\gamma \mathbf{F}\cdot\hat{\mathbf{t}}\ ds \text{ where } \hat{\mathbf{t}} \text{ is the normalized vector tangent to the path}$$

Definition 1.5.3 - Conservative forces

If $F = \nabla\phi$ for a **differentiable scalar function** ϕ, **F is said to be a conservative field**, which has the following properties:

$$\int_\gamma \mathbf{F}\cdot d\mathbf{r} = \phi(B) - \phi(A)$$

Result independent of path joining **A** and **B**, in particular for γ a closed curve ($B \equiv A$) We have:

$$\oint_\gamma \mathbf{F}\cdot d\mathbf{r} = 0$$

Call this a **circulation of F** around γ
If a vector field **F** s.t $\oint_\gamma F\cdot dr = 0$, for any closed curve γ say **F a conservative field**, if $\mathbf{F} = \nabla\phi \implies \mathbf{F}$ conservative.
If **F** conservative $\implies$ can always find differentiable scalar function ϕ s.t $\mathbf{F} = \nabla\phi$, call ϕ the **potential of field F**

Definition 1.5.4 - Calculation of Path Integrals

When $\mathbf{F} = \mathbf{F}(x, y, z)$ and the path γ can be parametrized by $(x(t), y(t), z(t))$, then:

$$\mathbf{r} = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}} \Rightarrow d\mathbf{r} = \frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}} + \frac{dz}{dt}\hat{\mathbf{k}}$$

$$\implies \int_\gamma \mathbf{F}\cdot d\mathbf{r} = \int_{t_0}^{t_1}\left(\mathbf{F}_1\frac{dx}{dt} + \mathbf{F}_2\frac{dy}{dt} + \mathbf{F}_3\frac{dz}{dt}\right)dt$$

1.6 Surface Integrals

Definition 1.6.1 - Surface Integral

Consider a surface S,where we find the surface integral of $f = f(P)$ over S.
Dividing S into small elements of area δS_i, with f_i the values of f at typical points P_i of δS_i
The **surface integral of f over S** is

$$\int_S f dS = \lim_{\substack{N\to\infty \\ max(\delta S_n)\to 0}} \sum_{n=1}^{N} f_n \delta S_n$$

f may be a vector or a scalar.

1.6.2 Types of Surfaces

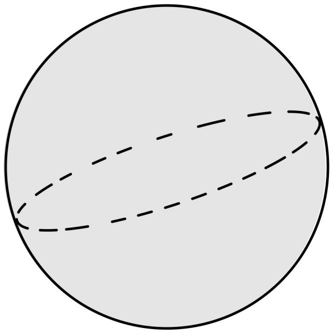


Figure 1: Closed Surface

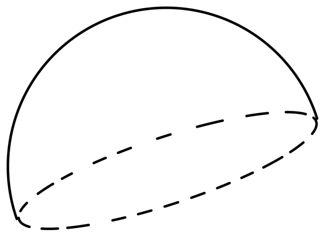


Figure 2: Open Surface

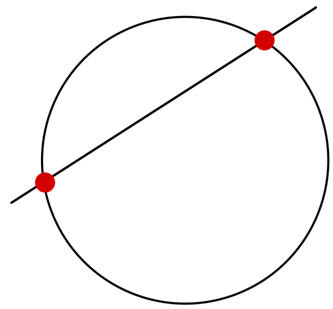


Figure 3: Convex Surface

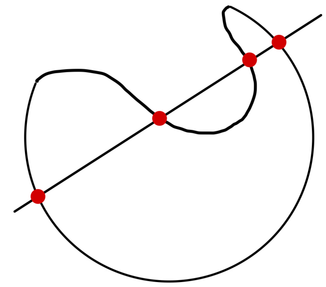
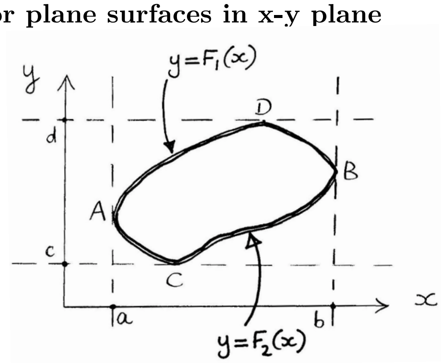


Figure 4: Non-Convex Surface

Definitions

1. **Closed Surface** - Divides $3D$ space into 2 non-connected regions; interior and exterior.
2. **Open Surface** - Does not divide $3D$ space into 2 non-connected regions - has a rim which can be represented by closed curve.
   Can think of closed surfaces as sum of 2 open surfaces.
3. **Convex Surface** - A surface which is crossed by a straight line at most twice

1.6.3 Evaluating surface integrals for plane surfaces in x-y plane



dS infinitesimal area $\implies$ think of as approx. plane.
Vector areal element $d\mathbf{S}$ is the vector $\hat{\mathbf{n}}dS$ for $\hat{\mathbf{n}}$ the unit normal vector to dS.
For a plane lying in $z = 0$, we can say $dS = dxdy$

For a rectangle, $x = a, b$ and $y = c, d$ circumscribing convex S. We let

$$y = \begin{cases} F_1(x) & \text{upper half ADB} \\ F_2(x) & \text{lower half ACB} \end{cases}$$

1.6.5 Projection of an area onto a plane

$$dS = \frac{d\Sigma}{|\hat{\mathbf{n}} \cdot \hat{\mathbf{k}}|}$$

1.6.6 The Projection Theorem

P a point on surface S, which at no point is orthogonal to $\mathbf{k}$

$$\int_S f(P)dS = \int_\Sigma f(P)\frac{dx\ dy}{|\hat{\mathbf{n}} \cdot \hat{\mathbf{k}}|}$$

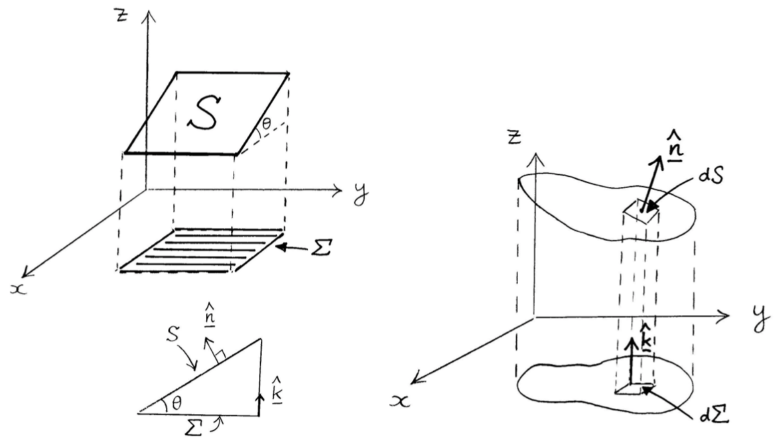


Figure 9: Left; Projection of plane area S onto $x-y$ plane
Figure 9: Right; Projection of curved surface S onto $x-y$ plane

For a projection of S onto $z=0$, with $\hat{\mathbf{n}}$ normal to S
For S given by $z=\phi(x,y)$

$$\int_S f(x,y,z)dS = \int_{\Sigma_z} f(x,y,\phi(x,y))\frac{dx\ dy}{|\hat{\mathbf{n}}\cdot\hat{\mathbf{k}}|}$$

Projecting onto $x=0$ or $y=0$

$$\int_S f(P)dS = \int_{\Sigma_x} f(x,y,\phi(x,y))\frac{dy\ dz}{|\hat{\mathbf{n}}\cdot\hat{\mathbf{i}}|} = \int_{\Sigma_y} f(x,y,\phi(x,y))\frac{dx\ dz}{|\hat{\mathbf{n}}\cdot\hat{\mathbf{j}}|}$$

Σ_x, projection onto $x=0$, Σ_y, projection onto $y=0$

1.7 Volume Integrals

Definition 1.7.1 - Volume Integral
Considering a volume τ, split into N subregions, $\{\delta\tau_i\}$, with $\{P_i\}$ typical points of $\{\delta\tau_i\}$.

$$\int_\tau f d\tau = \lim_{\substack{N\to\infty \\ max(\delta\tau_i)\to 0}} \sum_{i=1}^{N} f(P_i)\delta\tau_i$$

In Cartesian coordinates, the volume element $d\tau = dxdydz$

1.8 Results relating line,surface and volume integrals

1.8.1 Green's Theorem in the plane

R a closed plane region bounded by a simple plane closed convex curve in $x-y$ plane.
L, M continuous functions of x, y with continuous derivatives throughout R. Then:

$$\oint_C (L\ dx + M\ dy) = \int_R (\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y})dxdy,$$

For C the boundary of R described in the counter-clockwise sense.

1.8.2 Vector forms of Green's Theorem

(i) 2D Stokes Theorem
Let $F = L\mathbf{i} + M\mathbf{j}$ and $d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j}$. Then

$$\text{curl } \mathbf{F} = \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} \right) \mathbf{k}$$

Over region R write $dxdy = dS$.

$$\begin{aligned} \oint_C F \cdot dr &= \int_R k \cdot \text{curl } F dS \\ &= \int_R \text{curl } F \cdot d\mathbf{S}, \qquad d\mathbf{S} = \hat{\mathbf{k}} dS \end{aligned} \tag{1}$$

(ii) Divergence Theorem in 2D
Let $\mathbf{F} = M\mathbf{i} - L\mathbf{j}$. Then

$$\text{div } \mathbf{F} = \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}$$

So we can rewrite Green's Theorem as

$$\int_R \text{div } \mathbf{F} dxdy = \oint_C F \cdot \hat{\mathbf{n}} ds$$

Green's Theorem holds for more complicated geometries too, if C not convex we can see it as the composition of 2 or more simple convex closed curves.
Joining A, A' form C_1, C_2 enclosing R_1, R_2 s.t $R_1 + R_2 = R$

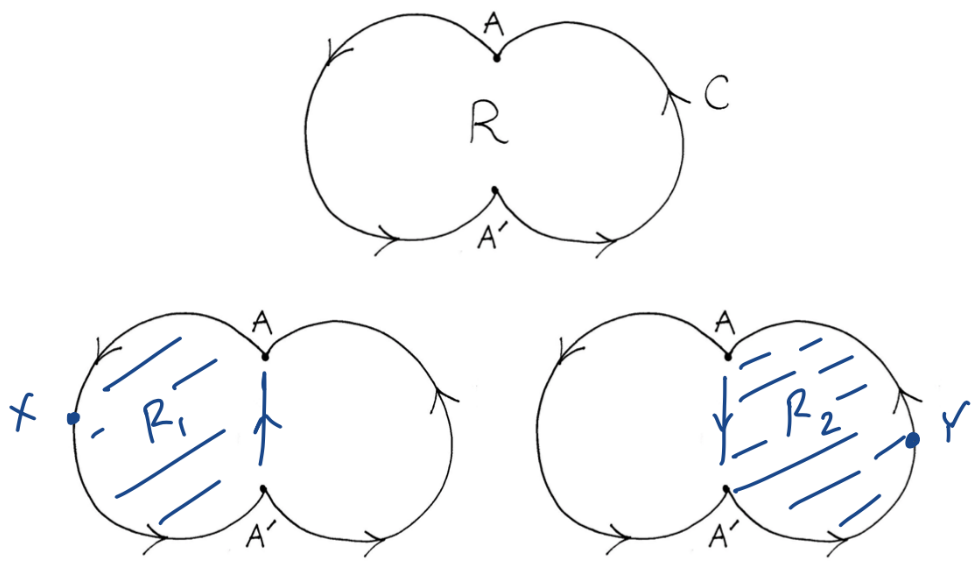


Figure 13: A non-convex boundary

$$\begin{gathered} \oint_C \mathbf{F} \cdot dr = \oint_{C_1} \mathbf{F} \cdot d\mathbf{r} + \oint_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_R \text{curl } \mathbf{F} \cdot d\mathbf{S} \\ \oint_{C_1} = \int_{AXA'} + \int_{A'}^{A} \\ \oint_{C_2} = \int_{A'YA} + \int_{A}^{A'} \end{gathered} \tag{2}$$

1.8.4 Green's Theorem in multiply-connected regions

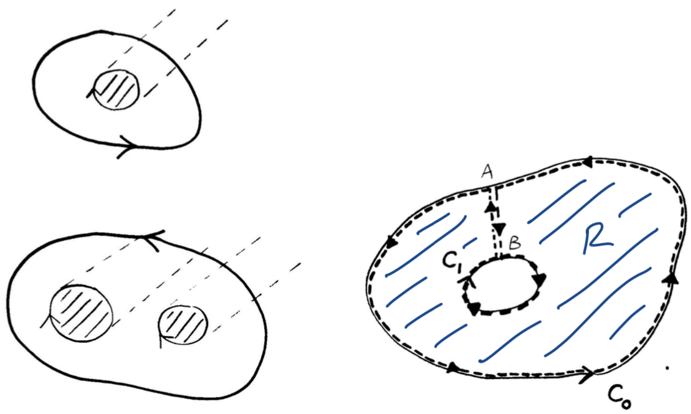


Figure 14: Left; Doubly- and triply- connected regions

Figure 14: Right; Green's Theorem in multiply-connected regions

R **simply-connected** if any closed curve in R can be shrunk to a point without leaving R.
For $2D$ any region with a hole in it; **not simply connected**, we say it is **multiply-connected**
Green's theorem still holds in multiply-connected regions. C interpreted as the entire inner and outer boundary.

For doubly-connected region, describe outer C_0 anti-clockwise, C_1 clockwise, and join them via A on C_0 and B on C_1
R now a simply connected region bounded by $(C_0 + AB + C_1 + BA)$

$$\int_R \text{curl } \mathbf{F} \cdot d\mathbf{S} = \left(\oint_{C_0} + \int_A^B + \oint_{C_0} + \int_B^A \right) (\mathbf{F} \cdot d\mathbf{r})$$

$$\int_R \text{curl } \mathbf{F} \cdot d\mathbf{S} = \left(\oint_{C_0} + \oint_{C_1} \right) (\mathbf{F} \cdot d\mathbf{r}) = \left(\oint_C \mathbf{F} \cdot d\mathbf{r} \right)$$

Where $C = C_0 + C_1$

1.8.5 Flux

If S is a surface then the flux of A across S is defined as

$$\int_S \mathbf{A} \cdot \hat{\mathbf{n}} dS$$

If S a closed surface then by convention draw unit normal $\hat{\mathbf{n}}$ **out** of S.

1.8.6 The divergence theorem

If τ the volume enclosed by a closed surface S with unit outward normal $\hat{\mathbf{n}}$ and $\mathbf{A}$ is a vector field with continuous derivatives throughout τ, then:

$$\int_S \mathbf{A} \cdot \hat{\mathbf{n}} dS = \int_\tau div \mathbf{A} d\tau$$

1.8.7 The Divergence theorem in more complicated geometries

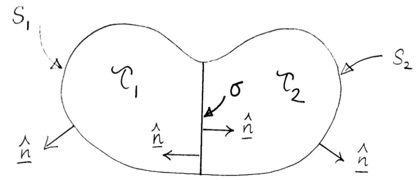


Figure 17: The divergence theorem for a non-convex surface

(i) **Non-convex surfaces** *non-convex surface S can be divided by surfaces(s) σ into 2 (or more) parts S_1 and S_2 which together with σ form convex surfaces $S_1+\sigma, S_2+\sigma/$*
Applying divergence theorem to the convex parts, upon addition yields the same result as before.

(ii) **A region with internal boundaries**
(a) *Simply-connected regions* - e.g space between concentric spheres..

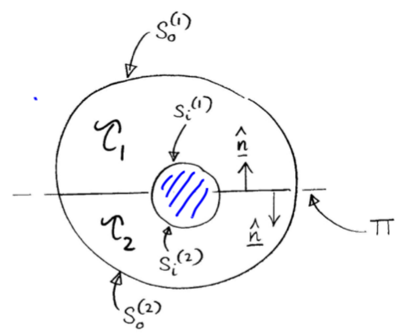


Figure 18: Simply-connected regions

Given interior surface S_i and outer surface S_o. A plane Π cutting both S_o, S_i, divides S_o, S_i into open $S_o^{(1)}, S_o^{(2)}$ and $S_i^{(1)}, S_i^{(2)}$ respectively.
Apply divergence theorem to τ_1, τ_2 bounded by closed $S_o^{(1)}+S_i^{(1)}+\Pi$ and $S_o^{(2)}+S_i^{(2)}+\Pi$. Upon addition contribution from Π cancels.

$$\int_{S_o+S_i}\mathbf{A}\cdot\hat{\mathbf{n}}dS=\int_S\mathbf{A}\cdot\hat{\mathbf{n}}dS=\int_{\tau_1}div\mathbf{A}d\tau+\int_{\tau_2}div\mathbf{A}d\tau=\int_\tau div\mathbf{A}d\tau$$

(b) *Multiply-connected regions*
e.g. region between 2 cyclinders.

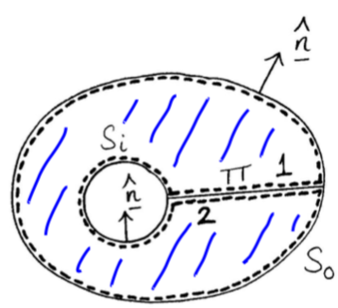


Figure 18: Multiply-connected regions

Given interior surface S_i and outer surface S_o, linked by plane Π.
Consider the closed surface, enclosing simply connected region τ

$$S_i + \text{ side 1 of } \Pi + S_o + \text{ side 2 of } \Pi$$

Applying divergence theorem to τ. Once again gives

$$\int_{S_0+S_i}\mathbf{A}\cdot\hat{\mathbf{n}}dS=\int_\tau div\mathbf{A}d\tau$$

1.8.8 Green's identity in 3D

For ϕ and ψ 2 scalar fields with continuous derivatives. We consider $\mathbf{A} = \phi\nabla\psi$, for which we have

$$div\mathbf{A} = \phi\nabla^2\psi + (\underline{\nabla}\phi)\cdot(\underline{\nabla}\psi)$$

$$\hat{\mathbf{n}}\cdot\mathbf{A} = \phi(\underline{\nabla}\psi)\cdot\hat{\mathbf{n}} = \phi\frac{\partial\psi}{\partial n}$$

Green's first identity

$$\int_S \left\{\phi\frac{\partial\psi}{\partial n}\right\} dS = \int_\tau \phi\nabla^2\psi + (\underline{\nabla}\phi)\cdot(\underline{\nabla}\psi)d\tau$$

Green's Second identity

$$\int_S \left\{\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right\} dS = \int_\tau \phi\nabla^2\psi - \psi\nabla^2\phi d\tau$$

1.8.9 Green's identities in 2D

Divergence theorem in 2D: $\int_F div\mathbf{F}dxdy = \oint_C \mathbf{F}\cdot\hat{\mathbf{n}}ds$
Giving the following Green's identities:

$$\oint_C \phi\frac{\partial\psi}{\partial n}ds = \int_R [\phi\nabla^2\psi + (\nabla\psi)\cdot(\nabla\phi)dxdy$$

and

$$\oint_C \left[\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right] ds = \int_R \left[\phi\nabla^2\psi - \psi\nabla^2\phi\right] dxdy$$

$\int_R \phi\nabla^2\psi\ dxdy = \oint_C \phi\frac{\partial\psi}{\partial n}ds - \int_R(\nabla\psi)\cdot(\nabla\phi)dxdy$ - **Looks like Integration by parts**

1.8.10 Gauss' Flux Theorem

Let S a closed surface with outward unit normal $\hat{\mathbf{n}}$ and let O the origin of the coordinate system.
$\mathbf{A} = \frac{\mathbf{r}}{r^3}$ Then:

$$\int_S \frac{\hat{\mathbf{n}}\cdot\mathbf{r}}{r^3} = \begin{cases} 0, & \text{if } O \text{ is exterior to } S \\ 4\pi, & \text{if } O \text{ interior to } S \end{cases}$$

1.8.11 Stokes Theorem

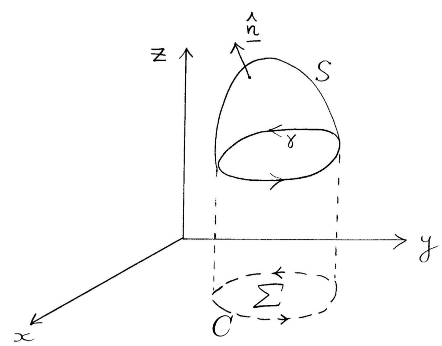


Figure 20: Diagram for proof of Stokes' Theorem

Suppose S is **open** surface with simple closed curve γ forming its boundary.
A a vector field with continuous partial derivatives, Then:

$$\oint_\gamma \mathbf{A}\cdot d\mathbf{r} = \int_S curl\mathbf{A}\cdot\hat{\mathbf{n}}dS$$

This holds for **any** open surface with γ as a boundary.
Theorem
For **A** continuously differentiable and simply connected region:

$$\underbrace{\oint_\gamma \mathbf{A}\cdot d\mathbf{r} = 0}_{\mathbf{A}\text{ conservative}} \iff curl\mathbf{A} = 0,\ \text{throughout region for which } \gamma \text{ is drawn}$$

1.9 Curvilinear Coordinates

1.9.1 Intro + Definition

Consider generally cartesian coordinates: (x_1, x_2, x_3) with each expressible as single-valued differentiable functions of the new coorinates (u_1, u_2, u_3)

$$x_i = x_i(u_1, u_2, u_3)$$

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij} = \frac{\partial x_i}{\partial u_1}\frac{\partial u_1}{\partial x_j} + \frac{\partial x_i}{\partial u_2}\frac{\partial u_2}{\partial x_j} + \frac{\partial x_i}{\partial u_3}\frac{\partial u_3}{\partial x_j}$$

With the following matrix equation

$$\begin{pmatrix} \partial x_1/\partial u_1 & \partial x_1/\partial u_2 & \partial x_1/\partial u_3 \\ \partial x_2/\partial u_1 & \partial x_2/\partial u_2 & \partial x_2/\partial u_3 \\ \partial x_3/\partial u_1 & \partial x_3/\partial u_2 & \partial x_3/\partial u_3 \end{pmatrix} \begin{pmatrix} \partial u_1/\partial x_1 & \partial u_1/\partial x_2 & \partial u_1/\partial x_3 \\ \partial u_2/\partial x_1 & \partial u_2/\partial x_2 & \partial u_2/\partial x_3 \\ \partial u_3/\partial x_1 & \partial u_3/\partial x_2 & \partial u_3/\partial x_3 \end{pmatrix} = I$$

Or more succinctly

$$J(x_u) \cdot J(u_x) = I$$

We say $J(x_u)$ the **Jacobian matrix** for the (x_1, x_2, x_3) system.

$$det\,(J(x_u)) \neq 0 \implies J(u_x) \text{ exists}$$
$$det(J(x_u)) = \tfrac{1}{det(J(u_x))}$$

We say (u_1, u_2, u_3) define a curvilinear coordinate system.
With each u_i = constant, defining a family of surfaces, with a member of each family passing through each $P(x, y, z)$
Let $(\hat{\mathbf{a_1}}, \hat{\mathbf{a_2}}, \hat{\mathbf{a_3}})$ unit vectors at P in the direction normal to $u_i = u_i(P)$, s.t u_i increasing in the direction $\hat{\mathbf{a_i}}$

$$\hat{\mathbf{a_i}} = \frac{\nabla \mathbf{u_i}}{|\nabla \mathbf{u_i}|}$$

if we have that $(\hat{\mathbf{a_1}}, \hat{\mathbf{a_2}}, \hat{\mathbf{a_3}})$ mutually orthogonal $\implies$ **orthogonal curvilinear coordinate system.**

$$\frac{\partial \mathbf{r}}{\partial u_i} = \hat{\mathbf{e_i}} h_i$$

For which we define $h_i = |\partial \mathbf{r}/\partial u_i|$. We call these the **length scales**

1.9.2 Path element

$\mathbf{r} = \mathbf{r}(u_1, u_2, u_3)$ **path element** $d\mathbf{r}$ given by

$$\begin{aligned} d\mathbf{r} &= \frac{\partial \mathbf{r}}{\partial u_1} du_1 + \frac{\partial \mathbf{r}}{\partial u_2} du_2 + \frac{\partial \mathbf{r}}{\partial u_3} du_3 \\ &= h_1 du_1 \hat{e_1} + h_2 du_2 \hat{e_2} + h_3 du_3 \hat{e_3} \end{aligned}$$

For an orthongal system

$$(ds)^2 = (d\mathbf{r}) \cdot (d\mathbf{r}) = h_1 (du_1)^2 + h_2 (du_2)^2 + h_3 (du_3)^2$$

$$\hat{e}_i = \hat{a}_i = \frac{\nabla \mathbf{u_i}}{|\nabla \mathbf{u_i}|}$$

1.9.3 Volume Element

$$\begin{aligned} d\tau &= (h_1 du_1)(h_2 du_2)(h_3 du_3) \\ &= h_1 h_2 h_3 du_1 du_2 du_3 \end{aligned}$$

1.9.4 Surface element

For u_1 constant.

$$dS = h_2 h_3 du_2 du_3$$

similarly for u_2, u_3

1.9.5 Properties of various orthogonal coordinates

(i) **Cartesisan coordinates** (x, y, z)

$$d\tau = dxdydz \qquad d\mathbf{r} = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}}$$
$$(ds)^2 = (d\mathbf{r}) \cdot (d\mathbf{r}) = (dx)^2 + (dy)^2 + (dz)^2$$

We have $h_1 = h_2 = h_3$

(ii) **Cylindrical polar coordinates** (r, ϕ, z)
Related to cartesian by

$$x = r\cos\theta \quad y = r\sin\phi \quad z = z$$

$$\begin{aligned}
\frac{\partial \mathbf{r}}{\partial r} &= (\frac{\partial x}{\partial r})\hat{\mathbf{i}} + (\frac{\partial y}{\partial r})\hat{\mathbf{j}} + (\frac{\partial z}{\partial r})\hat{\mathbf{k}} = (\cos\phi)\hat{\mathbf{i}} + (\sin\phi)\hat{\mathbf{j}} & (\frac{\partial \mathbf{r}}{\partial r}) \cdot (\frac{\partial \mathbf{r}}{\partial \phi}) &= 0 & h_1 &= |\frac{\partial \mathbf{r}}{\partial r}| = 1 \\
\frac{\partial \mathbf{r}}{\partial \phi} &= (\frac{\partial x}{\partial \phi})\hat{\mathbf{i}} + (\frac{\partial y}{\partial \phi})\hat{\mathbf{j}} + (\frac{\partial z}{\partial \phi})\hat{\mathbf{k}} = -(r\sin\phi)\hat{\mathbf{i}} + (r\cos\phi)\hat{\mathbf{j}} & (\frac{\partial \mathbf{r}}{\partial r}) \cdot (\frac{\partial z}{\partial \phi}) &= 0 & h_2 &= |\frac{\partial \mathbf{r}}{\partial \phi}| = r \\
\frac{\partial \mathbf{r}}{\partial z} &= \hat{\mathbf{k}} & (\frac{\partial \mathbf{r}}{\partial \phi}) \cdot (\frac{\partial \mathbf{r}}{\partial z}) &= 0 & h_3 &= |\frac{\partial \mathbf{r}}{\partial z}| = 1
\end{aligned}$$

Yielding length and volume elements:

$$(ds)^2 = (dr)^2 + r^2(d\phi)^2 + (dz)^2 \qquad d\tau = rdrd\phi dz$$

(iii) **Spherical polar coordinates** (r, θ, ϕ)
Related to cartesian by:

$$x = r\sin\theta\cos\phi \quad y = r\sin\theta\sin\phi \quad z = r\cos\theta$$

$$\begin{aligned}
\frac{\partial \mathbf{r}}{\partial r} &= (\sin\theta\cos\phi)\hat{\mathbf{i}} + (\sin\theta\sin\phi)\hat{\mathbf{j}} + (\cos\theta)\hat{\mathbf{k}} & (\frac{\partial \mathbf{r}}{\partial r}) \cdot (\frac{\partial \mathbf{r}}{\partial \theta}) &= 0 & h_1 &= |\frac{\partial \mathbf{r}}{\partial r}| = 1 \\
\frac{\partial \mathbf{r}}{\partial \theta} &= (r\cos\theta\cos\phi)\hat{\mathbf{i}} + (r\cos\theta\sin\phi)\hat{\mathbf{j}} + (-r\sin\theta)\hat{\mathbf{k}} & (\frac{\partial \mathbf{r}}{\partial r}) \cdot (\frac{\partial \mathbf{r}}{\partial \phi}) &= 0 & h_2 &= |\frac{\partial \mathbf{r}}{\partial \theta}| = r \\
\frac{\partial \mathbf{r}}{\partial \phi} &= (-r\sin\theta\sin\phi)\hat{\mathbf{i}} + (r\sin\theta\cos\phi)\hat{\mathbf{j}} + (0)\hat{\mathbf{k}} & (\frac{\partial \mathbf{r}}{\partial \phi}) \cdot (\frac{\partial \mathbf{r}}{\partial \theta}) &= 0 & h_3 &= |\frac{\partial \mathbf{r}}{\partial \phi}| = r\sin\theta
\end{aligned}$$

Volume element:

$$d\tau = r^2 \sin\theta drd\theta d\phi$$

1.9.6 Gradient in orthogonal curvilinear coordinates

Let $\nabla\Phi = \lambda_1\hat{\mathbf{e}}_\mathbf{1} + \lambda_2\hat{\mathbf{e}}_\mathbf{2} + \lambda_3\hat{\mathbf{e}}_\mathbf{3}$.
In a general coordinate system for λ_is to be found.

$$d\mathbf{r} = h_1 du_1 \hat{e}_1 + h_2 du_2 \hat{e}_2 + h_3 du_3 \hat{e}_3$$

$$\begin{aligned}
d\mathbf{\Phi} &= (\frac{\partial \phi}{\partial u_1})du_1 + (\frac{\partial \phi}{\partial u_2})du_2 + (\frac{\partial \phi}{\partial u_3})du_3 \\
&= (\frac{\partial \phi}{\partial x})dx + (\frac{\partial \phi}{\partial y})dy + (\frac{\partial \phi}{\partial z})dz \\
&= \boxed{(\nabla\Phi) \cdot d\mathbf{r} = \lambda_1 h_1 du_1 + \lambda_2 h_2 du_2 + \lambda_3 h_3 du_3}
\end{aligned}$$

$$h_i\lambda_i = \frac{\partial \Phi}{\partial u_i}$$

$$\implies \nabla\Phi = \frac{\hat{\mathbf{e}}_1}{h_1}\frac{\partial \Phi}{\partial u_1} + \frac{\hat{\mathbf{e}}_2}{h_2}\frac{\partial \Phi}{\partial u_2} + \frac{\hat{\mathbf{e}}_3}{h_3}\frac{\partial \Phi}{\partial u_3}$$

(i) **Cylindrical polars** (r, ϕ, z)

We have: $\begin{array}{l} h_1 = 1 \\ h_2 = r \\ h_3 = 1 \end{array} \implies \nabla = \hat{r}\frac{\partial}{\partial r} + \frac{\hat{\phi}}{r}\frac{\partial}{\partial \phi} + \hat{z}\frac{\partial}{\partial z}$

(ii) **Spherical polars** (r, θ, ϕ)

We have: $\begin{array}{l} h_1 = 1 \\ h_2 = r \\ h_3 = r\sin\theta \end{array} \implies \nabla = \hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial \phi}$

1.9.7 Expressions for unit vectors

$$\hat{\mathbf{e}}_i = h_i \nabla u_i$$

Alternatively, unit vectors orthogonal $\implies$ if we know 2 already then

$$\hat{\mathbf{e}}_1 = (\hat{\mathbf{e}}_2 \times \hat{\mathbf{e}}_3) = h_2 h_3 (\nabla u_2 \times \nabla u_3)$$

1.9.8 Divergence in orthogonal curvilinear coordinates

Suppose we have vector field

$$\mathbf{A} = A_1\hat{\mathbf{e}}_1 + A_2\hat{\mathbf{e}}_2 + A_3\hat{\mathbf{e}}_3$$

$$\implies \nabla \cdot \mathbf{A} = \frac{1}{h_1 h_2 h_3}\left\{\frac{\partial}{\partial u_1}(A_1 h_2 h_3) + \frac{\partial}{\partial u_2}(A_2 h_3 h_1) + \frac{\partial}{\partial u_3}(A_3 h_1 h_2)\right\}$$

So we have divergence in other coordinate systems as follows:

(i) **Cylindrical polars** (r, ϕ, z)

We have: $\begin{array}{l} h_1 = 1 \\ h_2 = r \\ h_3 = 1 \end{array} \implies \nabla \cdot A = \frac{\partial A_1}{\partial r} + \frac{A_1}{r} + \frac{1}{r}\frac{\partial A_2}{\partial \phi} + \frac{\partial A_3}{\partial z}$

(ii) **Spherical polars** (r, θ, ϕ)

We have: $\begin{array}{l} h_1 = 1 \\ h_2 = r \\ h_3 = r\sin\theta \end{array} \implies \nabla \cdot A = \frac{1}{r^2\sin\theta}\left\{\frac{\partial}{\partial r}(r^2\sin\theta A_1) + \frac{\partial}{\partial \theta}(r\sin\theta A_2) + \frac{\partial}{\partial \phi}(rA_3)\right\}$

1.9.9 Curl in orthogonal curvilinear coordinates

$$curl\mathbf{A} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

(i) **Cylindrical polars**

$$curl\mathbf{A} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\phi} & \hat{\mathbf{k}} \\ \partial/\partial r & \partial/\partial \phi & \partial/\partial z \\ A_1 & A2 & A_3 \end{vmatrix}$$

(ii) **Spherical polars**

$$curl\mathbf{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\phi} & r\sin\theta\hat{\phi} \\ \partial/\partial r & \partial/\partial \phi & \partial/\partial z \\ A_1 & rA2 & r\sin\theta A_3 \end{vmatrix}$$

1.9.10 The Laplacian in orthogonal curvilinear coordinates

From the above grad and div;

$$\nabla^2\Phi = \nabla\cdot(\nabla\Phi)$$
$$= \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\Phi}{\partial u_1}\right)+\frac{\partial}{\partial u_2}\left(\frac{h_1h_3}{h_2}\frac{\partial\Phi}{\partial u_2}\right)+\frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\Phi}{\partial u_3}\right)\right\}$$

(i) **Cylindrical polars** (r,ϕ,z)

$$\nabla^2\Phi = \frac{1}{r}\left\{\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right)+\frac{\partial}{\partial\phi}\left(\frac{1}{r}\frac{\partial\Phi}{\partial\phi}\right)+\frac{\partial}{\partial z}\left(r\frac{\partial\Phi}{\partial z}\right)\right\}$$
$$= \frac{\partial^2\Phi}{\partial r^2}+\frac{1}{r}\frac{\partial\Phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\phi^2}+\frac{\partial^2\Phi}{\partial z^2}$$

(ii) **Spherical polars** (r,θ,ϕ)

$$\nabla^2\Phi = \frac{1}{r^2\sin\theta}\left\{\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial\Phi}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right)+\frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}\frac{\partial\Phi}{\partial\phi}\right)\right\}$$
$$= \frac{\partial^2\Phi}{\partial r^2}+\frac{2}{r}\frac{\partial\Phi}{\partial r}+\frac{\cot\theta}{r^2}\frac{\partial\Phi}{\partial\theta}+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}$$

1.10 Changes of variables in surface integration

Suppose we have surface S, parametrized by quantities u_1, u_2. We can write:

$$x = x(u_1,u_2),\quad y = y(u_1,u_2),\quad z = z(u_1,u_2)$$

Consider surface to be comprised of arbitrarily small parallelograms, its sides given by keeping either u_1 or u_2

$$dS = \text{ Area of parallelogram with sides } \frac{\partial\mathbf{r}}{\partial u_1}du_1 \text{ and } \frac{\partial\mathbf{r}}{\partial u_2}du_2$$
$$= |\mathbf{J}|du_1du_2$$

Vector Jacobian J given by $\mathbf{J} = \frac{d\mathbf{r}}{du_1}\times\frac{d\mathbf{r}}{du_2}$.
Useful in substitution of surface integrals:

$$\int_S f(x,y,z)dS = \int_S F(u_1,u_2)|\mathbf{J}|du_1du_{2]}$$

$F(u_1,u_2) = f(x(u_1,u_2),y(u_1,u_2),z(u_1,u_2))$

For S a region R in the $x-y$ plane we can write:

$$\int_R f(x,y)dxdy = \int_R F(u_1,u_2)|det(J(x_u))|du_1du_2$$
$$|\mathbf{J}| = |\frac{d\mathbf{r}}{du_1}\times\frac{d\mathbf{r}}{du_2}| = det(J(x_u)) = \begin{vmatrix}\partial x/\partial u_1 & \partial x/\partial u_2\\ \partial y/\partial u_1 & \partial y/\partial u_2\end{vmatrix}$$

For a surface described by $z = f(x,y)$. We have $x = u_1, y = u_2$ and $\mathbf{r} = (x,y,f(x,y))$
We have:

$$\frac{\partial\mathbf{r}}{\partial u_1} = \frac{\partial\mathbf{r}}{\partial x} = \hat{\mathbf{i}}+\frac{\partial f}{\partial x}\hat{\mathbf{k}}$$
$$\frac{\partial\mathbf{r}}{\partial u_2} = \frac{\partial\mathbf{r}}{\partial y} = \hat{\mathbf{j}}+\frac{\partial f}{\partial y}\hat{\mathbf{k}}$$

$$|\frac{\partial\mathbf{r}}{\partial u_1}\times\frac{\partial\mathbf{r}}{\partial u_2}| = \left\|\begin{matrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 1 & 0 & \partial f/\partial x\\ 0 & 1 & \partial f/\partial y\end{matrix}\right\|$$
$$= \sqrt{1+|\nabla f|^2}$$

So we have area of surface given by

$$\int_\Sigma \sqrt{1+|\nabla f|^2}dxdy$$

for the projection of S onto the $x-y$ plane.