

Topic: Probability and conditional probability

In today's problem class we will be reviewing the probability axioms and we will study problems involving conditional probabilities.

1. Given events $E, F, G \subseteq \Omega$, prove that

$$(a) \quad P(E^c \cap F) = P(F) - P(E \cap F)$$

$$(b) \quad P(E \cup F) \leq P(E) + P(F)$$

$$(c) \quad E \subseteq F, F \subseteq G \implies P(E) \leq P(G)$$

$$(d) \quad P(E \cap F) \geq P(E) + P(F) - 1$$

[(d) is known as *Bonferroni's Inequality*.]

Solution: For general events E and F ,

- (a) $F \equiv (E \cap F) \cup (E^c \cap F)$, where the two events are disjoint since $(E \cap F) \cap (E^c \cap F) = \emptyset$, so by Axiom (iii)

$$P(F) = P(E \cap F) + P(E^c \cap F) \implies P(E^c \cap F) = P(F) - P(E \cap F).$$

- (b) $E \cup F \equiv E \cup (E^c \cap F)$, where the two events are disjoint since $E \cap (E^c \cap F) = \emptyset$, so by Axiom (iii)

$$P(E \cup F) = P(E) + P(E^c \cap F) = P(E) + P(F) - P(E \cap F),$$

but $P(E \cap F) \geq 0$ so $P(E \cup F) \leq P(E) + P(F)$.

- (c) $E \subseteq F \subseteq G \implies E \cup G = G \implies G = E \cup (E^c \cap G)$, where the two events are disjoint since $E \cap (E^c \cap G) = \emptyset$, so by Axiom (iii), as $P(E^c \cap G) \geq 0$,

$$P(G) = P(E) + P(E^c \cap G) \geq P(E)$$

- (d) *Bonferroni Inequality*: as $P(E \cup F) \leq 1$,

$$P(E \cap F) = P(E) + P(F) - P(E \cup F) \geq P(E) + P(F) - 1$$

2. Suppose that E and F are events such that $P(E) = x, P(F) = y$ and $P(E \cap F) = z$. Express the following terms in terms of x, y and z :

(a) $P(E^c \cup F^c)$

(b) $P(E^c \cap F)$

(c) $P(E^c \cup F)$

(d) $P(E^c \cap F^c)$

Solution:

(a) $E^c \cup F^c = (E \cap F)^c \implies P(E^c \cup F^c) = 1 - P(E \cap F) = 1 - z.$

- (b) $F = (E \cap F) \cup (E^c \cap F)$, which is a union of disjoint events, so by Axiom (iii), $P(F) = P(E \cap F) + P(E^c \cap F)$, so $P(E^c \cap F) = y - z$.
- (c) $E^c \cup F = E^c \cup (E \cap F)$, which is a union of disjoint events, so by Axiom (iii), $P(E^c \cup F) = P(E^c) + P(E \cap F) = 1 - x + z$.
- (d) $E^c \cap F^c = (E \cup F)^c \implies P(E^c \cap F^c) = 1 - P(E \cup F) = 1 - x - y + z$.

3. A crime has been committed and a suspect is being held by police. He is either guilty, G , or not, G^c , and the probability of his being guilty on the basis of current evidence is $P(G) = p$, say. Forensic evidence is now produced which shows that the criminal must have a property, A , which occurs in a proportion, π , of the general population. Suppose that if the suspect is innocent he can be treated as a member of the general population, so that $P(A|G^c) = \pi$.

The suspect is now interrogated and found to have property A . Show that the odds on his guilt have now risen from $\frac{P(G)}{P(G^c)} = p/(1-p)$ to $\frac{P(G|A)}{P(G^c|A)} = \frac{P(G)}{\pi P(G^c)}$.

Note: The odds on an event E are defined to be the ratio $P(E)/P(E^c)$, the odds-against E are $P(E^c)/P(E)$.

Solution: Given $P(G) = p$, $P(A|G) = 1$, $P(A|G^c) = \pi$. Then

$$P(G|A) = \frac{P(A|G)P(G)}{P(A|G)P(G) + P(A|G^c)P(G^c)} = \frac{1 \cdot p}{1 \cdot p + \pi \cdot (1-p)}.$$

This implies that

$$P(G^c|A) = 1 - P(G|A) = \frac{\pi(1-p)}{p + \pi \cdot (1-p)}$$

and hence

$$\frac{P(G|A)}{P(G^c|A)} = \frac{p}{\pi(1-p)} = \frac{P(G)}{\pi P(G^c)}.$$

4. A shop sells fuses produced by three manufacturers; each manufacturer supplies a deluxe and a standard type of fuse. A mixed batch of 500 fuses sold, and the number of faulty fuses of each type and for each manufacturer is recorded. By considering the following events; $M_i \equiv$ “fuse produced by manufacturer i ” for $i = 1, 2, 3$, $D \equiv$ “Deluxe type of fuse” and $F \equiv$ “Fuse Faulty”, a summary of the data can be presented as a 3-way table

	M_1		M_2		M_3	
	D	D^c	D	D^c	D	D^c
F	20	16	30	20	15	10
F^c	100	64	120	30	60	15

so that, for example, the number of deluxe fuses from manufacturer 1 that are faulty is 20, whereas the number of standard fuses from manufacturer 1 that are faulty is 16, etc.

- (a) A fuse is selected with equal probability from the 500. What is the probability that
- it is faulty?
 - it was produced by manufacturer 1?
- (b) Given that the selected fuse is faulty, what is the conditional probability that
- it is a deluxe fuse?
 - it is a fuse produced by manufacturer 1?
 - it is a deluxe fuse produced by manufacturer 1?
- (c) Describe, evaluate, and comment on the following conditional probabilities:
- $P(F|M_1)$, $P(F|M_2)$, $P(F|M_3)$
 - $P(F|D)$, $P(F|D^c)$
 - $P(F|M_1 \cap D)$, $P(F|M_2 \cap D)$, $P(F|M_3 \cap D)$.
 - $P(F|M_1 \cap D^c)$, $P(F|M_2 \cap D^c)$, $P(F|M_3 \cap D^c)$.

Solution: Recall that for a finite sample space Ω where all events are equally likely, we have

$$P(A|B) = \frac{\text{card}(A \cap B)}{\text{card}(B)} = \frac{\text{card}(A \cap B)/\text{card}(\Omega)}{\text{card}(B)/\text{card}(\Omega)} = \frac{P(A \cap B)}{P(B)}.$$

- (a) i. $P(F) = \frac{20+16+30+20+15+10}{500} = \frac{111}{500}$
 ii. $P(M_1) = \frac{20+16+100+64}{500} = \frac{200}{500} = \frac{2}{5}$.

- (b) i.

$$P(D|F) = \frac{\text{card}(D \cap F)}{\text{card}(F)} = \frac{20 + 30 + 15}{111} = \frac{65}{111}.$$

- ii.

$$P(M_1|F) = \frac{\text{card}(M_1 \cap F)}{\text{card}(F)} = \frac{20 + 16}{111} = \frac{36}{111}.$$

iii.

$$P(D \cap M_1 | F) = \frac{\text{card}(D \cap M_1 \cap F)}{\text{card}(F)} = \frac{20}{111}.$$

- (c) i. First we compute the conditional probabilities, that given the fuse was produced by a particular manufacturer that it is faulty. We note that we have

$$P(M_1) = 200/500 = 2/5, \quad P(M_2) = 200/500 = 2/5, \quad P(M_3) = 100/500 = 1/5.$$

We have

$$P(F|M_1) = \frac{\text{card}(F \cap M_1)}{\text{card}(M_1)} = \frac{20 + 16}{200} = \frac{36}{200} = \frac{9}{50} = 0.18,$$

$$P(F|M_2) = \frac{\text{card}(F \cap M_2)}{\text{card}(M_2)} = \frac{(30 + 20)}{200} = \frac{50}{200} = \frac{1}{4} = 0.25,$$

$$P(F|M_3) = \frac{\text{card}(F \cap M_3)}{\text{card}(M_3)} = \frac{(15 + 10)}{100} = \frac{25}{100} = \frac{1}{4} = 0.25.$$

We observe that the conditional probabilities are the same for manufactures 2 and 3 and it is lower for manufacturer 1.

- ii. Next we compute the probability of a faulty fuse conditional on it being of deluxe type:

$$P(F|D) = \frac{\text{card}(F \cap D)}{\text{card}(D)} = \frac{20 + 30 + 15}{(20 + 30 + 15 + 100 + 120 + 60)} = \frac{65}{345} = \frac{13}{69} \approx 0.188$$

and the probability of a faulty fuse conditional on it being of standard type:

$$P(F|D^c) = \frac{\text{card}(F \cap D^c)}{\text{card}(D^c)} = \frac{16 + 20 + 10}{(16 + 20 + 10 + 64 + 30 + 15)} = \frac{46}{155} \approx 0.296.$$

iii. We have

$$P(F|M_1 \cap D) = \frac{\text{card}(F \cap M_1 \cap D)}{\text{card}(M_1 \cap D)} = \frac{20}{120} = \frac{1}{6} \approx 0.166,$$

$$P(F|M_2 \cap D) = \frac{\text{card}(F \cap M_2 \cap D)}{\text{card}(M_2 \cap D)} = \frac{30}{150} = \frac{1}{5} = 0.2,$$

$$P(F|M_3 \cap D) = \frac{\text{card}(F \cap M_3 \cap D)}{\text{card}(M_3 \cap D)} = \frac{15}{75} = \frac{1}{5} = 0.2,$$

for the conditional probabilities of having a faulty fuse, given that it comes from a particular manufacturer and is of deluxe type.

iv. We have

$$P(F|M_1 \cap D^c) = \frac{\text{card}(F \cap M_1 \cap D^c)}{\text{card}(M_1 \cap D^c)} = \frac{16}{80} = \frac{1}{5} = 0.2,$$

$$P(F|M_2 \cap D^c) = \frac{\text{card}(F \cap M_2 \cap D^c)}{\text{card}(M_2 \cap D^c)} = \frac{20}{50} = \frac{2}{5} = 0.4,$$

$$P(F|M_3 \cap D^c) = \frac{\text{card}(F \cap M_3 \cap D^c)}{\text{card}(M_3 \cap D^c)} = \frac{10}{25} = \frac{2}{5} = 0.4,$$

for the conditional probabilities of having a faulty fuse, given that it comes from a particular manufacturer and is of standard type.

These results confirm that events F , M_1 , M_2 , M_3 and D are not mutually independent.