Math40002 Analysis 1 — Problem Sheet 3

1. * Which of the following sequences are convergent and which are not? What is the limit of the convergent ones? Give proofs for each.

   (a) $\frac{n+7}{n}$ (b) $\frac{n}{n+7}$ (c) $\frac{n^2+5n+6}{n^3-2}$ (d) $\frac{n^3-2}{n^2+5n+6}$ (e) $\frac{1-n(-1)^n}{n}$

(a) This tends to 1. For any $\epsilon > 0$ pick $N \in \mathbb{N}$ such that $N > \frac{7}{\epsilon}$. Then for $n \geq N$, $|a_n - 1| = \frac{7}{n} \leq \frac{7}{N} < \epsilon$.

(b) This tends to 1. For any $\epsilon > 0$ pick $N \in \mathbb{N}$ such that $N > \frac{7}{\epsilon}$. Then for $n \geq N$, $|a_n - 1| = \frac{7}{n+7} < \frac{7}{N} < \epsilon$.

(c) This tends to 0.

Notice that for $n \geq 5$, $5n \leq n^2$ and $6 < n^2$, so $n^2 + 5n + 6 < 3n^2$. And also $2 < \frac{1}{2}n^3$ so $n^3 - 2 > \frac{1}{2}n^3$. Therefore $\frac{n^2+5n+6}{n^3-2} < \frac{3n^2}{\frac{1}{2}n^3} = \frac{6}{n}$.

For any $\epsilon > 0$ pick $N \in \mathbb{N}$ such that $N > \frac{6}{\epsilon}$ and $N \geq 5$. Then for $n \geq N$, $|a_n| < \frac{6}{n} \leq \frac{6}{N} < \epsilon$.

(d) This does not converge to any real number. Suppose for a contradiction that it converged to $a \in \mathbb{R}$. Then taking $\epsilon = 1$ we find $N \in \mathbb{N}$ such that $n \geq N \;\Rightarrow\; |a_n - a| < 1 \;\Rightarrow\; a_n < a + 1$.

But for $n \geq 2$ (so that $n^3/2 > 2$) we have $a_n > \frac{n^3 - n^3/2}{n^2+5n^2+6n^2} = n/24$. So for $n > 24(a+1)$ we find that $a_n > a + 1$, which contradicts the line above.

(e) This does not converge. Suppose for a contradiction that it converged to $a \in \mathbb{R}$. Then taking $\epsilon = \frac{1}{2}$ we find $N \in \mathbb{N}$ such that $n \geq N \;\Rightarrow\; |a_n - a| < \frac{1}{2} \;\Rightarrow\; a_n - \frac{1}{2} < a < a_n + \frac{1}{2}$.

For even $n \geq N$ this gives $a < \frac{1-n}{n} + \frac{1}{2} = \frac{1}{n} - \frac{1}{2} \leq 0$ $(*)$ while for odd $n \geq N$ it gives $a > \frac{1+n}{n} - \frac{1}{2} = \frac{1}{n} + \frac{1}{2} > 0$, contradicting $(*)$.

2. We've defined what it means for (a_n) to converge to a real number $a \in \mathbb{R}$ as $n \to \infty$. Professor Lee Beck thinks infinity is cool, so he comes up with some definitions of $a_n \to +\infty$ as $n \to \infty$. Which are right and which are wrong? For any wrong ones, illustrate its wrongness with an example.

   (a) $\forall a \in \mathbb{R},\ a_n \not\to a$.

   (b) $\forall \epsilon > 0\ \exists N \in \mathbb{N}$ such that $n \geq N \;\Rightarrow\; |a_n - \infty| < \epsilon$.

   (c) $\forall R > 0\ \exists N \in \mathbb{N}$ such that $n \geq N \;\Rightarrow\; a_n > R$.

   (d) $\forall a \in \mathbb{R}\ \exists \epsilon > 0$ such that $\forall N \in \mathbb{N}\ \exists n \geq N$ such that $|a_n - a| \geq \epsilon$.

   (e) $\forall \epsilon > 0\ \exists N \in \mathbb{N}$ such that $\forall n \geq N,\ a_n > \frac{1}{\epsilon}$.

   (f) $\forall n \in \mathbb{N},\ a_{n+1} > a_n$.

   (g) $\forall R \in \mathbb{R},\ \exists n \in N$ such that $a_n > R$.

   (h) $1/\max(1, a_n) \to 0$.

(a) Wrong: eg $(-1)^n$.

(b) Wrong: ∞ not a real number, so $|a_n - \infty|$ doesn't mean anything.

(c) Correct! However big a number (R) you give me, once I go sufficiently far ($\geq N$) down the sequence, it is always bigger than R.

(d) Wrong: eg $(-1)^n$.

(e) Correct! This is equivalent to (c), with $R = \frac{1}{\epsilon}$.

(f) Wrong: eg $1 - \frac{1}{n}$.

(g) Wrong: eg $(-1)^n n$.

(h) Correct! The max is just there to make sure we don't divide by 0. So this definition says that $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $n \geq N \;\Rightarrow\; |1/\max(1, a_n)| < \epsilon$, which implies that $\max(1, a_n) > \epsilon^{-1}$.

So for all $R > 1$, setting $\epsilon = 1/R$ we see that $\exists N \in \mathbb{N}$ such that $n \geq N \;\Rightarrow\; \max(1, a_n) > R$ which implies that $a_n > R$ (since $R > 1$). Therefore this gives definition (c).

3. Let (a_n) be a sequence converging to $a \in \mathbb{R}$. Suppose (b_n) is another sequence which is different than (a_n) but only differs from (a_n) in finitely many terms, that is the set $\{n \in \mathbb{N} : a_n \neq b_n\}$ is non-empty and finite. Prove (b_n) converges to a.

Since $\{n \in \mathbb{N} : a_n \neq b_n\}$ is a finite non-empty set it has a maximum element which we call M. Now let $\epsilon > 0$. Since (a_n) converges to a there exists $M_\epsilon \in \mathbb{N}$ such that $n \geq M_\epsilon \Rightarrow |a_n - a| < \epsilon$.

Now, we take $N_\epsilon = \max(M_\epsilon, M)$ and so it follows that $n \geq N_\epsilon \Rightarrow |b_n - a| = |a_n - a| < \epsilon$ where the first equality holds because $n \geq M$ and the second holds because $n \geq M_\epsilon$. Therefore (b_n) converges to a.

4. Let $S \subset \mathbb{R}$ be nonempty and bounded above. Show that there exists a sequence of numbers $s_n \in S$, $n = 1, 2, 3, \ldots$, such that $s_n \to \sup S$.

Given any $n \in \mathbb{N}$, $\sup S - \frac{1}{n}$ is not an upper bound for S, because it is less than the smallest upper bound $\sup S$. Therefore there exists an element $s_n \in S$ such that $s_n > \sup S - \frac{1}{n}$.

Of course we also have $s_n \leq \sup S$ by definition of $\sup$, so $|s_n - \sup S| < \frac{1}{n}$.

Given any $\epsilon > 0$, fix $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon}$. Then $n \geq N \;\Rightarrow\; |s_n - \sup S| < \frac{1}{n} \leq \frac{1}{N} < \epsilon$. So $s_n \to \sup S$.

5. Give *without proof* examples of sequences (a_n), (b_n) with the following properties.

   (i) Neither of a_n, b_n is convergent, but $a_n + b_n$, $a_n b_n$ and a_n / b_n all converge.

   Eg. $a_n = (-1)^n, \; b_n = (-1)^{n+1}$.

   (ii) a_n converges, b_n is *un*bounded, but $a_n b_n$ converges.

   Eg. $a_n = 0, \; b_n = n$. Or $a_n = n^{-2}, \; b_n = n$.

   (iii) a_n converges, b_n bounded, but $a_n b_n$ diverges.

   Eg. $a_n = 1, \; b_n = (-1)^n$.