Math40003 Linear Algebra and Groups Term 2 Unseen 1

1. Prove that there is some matrix $A \in M_{n\times n}(\mathbb{R})$ such that $A^2 = -I_n$ if and only if n is even.

   - **If n is odd and such A exists, then $\det(A)^2 = \det(A^2) = -1$, contradicting $\det(A) \in \mathbb{R}$.**
   - **If n is even, we can take A the matrix with $1, -1$ alternating on the secondary diagonal, and 0 outside the secondary diagonal. (There were exercises on 2×2 matrices in the beginning of the module, as well as in Intro to Maths which hints to this general solution for n even.)**

2. A square matrix is a *block upper triangular matrix* if it is of the form

$$\begin{pmatrix} A_1 & * & \dots & * \\ 0 & A_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & * \\ 0 & \dots & 0 & A_k \end{pmatrix}$$

   Where $A_1, \dots A_k$ are square matrices, the zeros stand for blocks of square zero matrices, and $*$ can be anything.

   (a) Prove $\det \begin{pmatrix} A & * \\ 0 & B \end{pmatrix} = \det(A) \cdot \det(B)$.

   By induction of the size of A. Assume the equality holds if $A \in M_{n\times n}(F)$, and let $A \in M_{(n+1)\times(n+1)}(F)$. Notice that $\begin{pmatrix} A & * \\ 0 & B \end{pmatrix}_{i,j} = \begin{pmatrix} A_{i,j} \end{pmatrix}$ for all $1 \le i, j \le n+1$. So

$$\det \begin{pmatrix} A & * \\ 0 & B \end{pmatrix} = \sum_{i=1}^{n+1} (-1)^{i+1} [A]_{i,1} \det \left(\begin{pmatrix} A & * \\ 0 & B \end{pmatrix}_{i,1} \right) =$$

$$\sum_{i=1}^{n+1} (-1)^{i+1} [A]_{i,1} \det \left(\begin{pmatrix} A_{i,1} & * \\ 0 & B \end{pmatrix} \right) = \sum_{i=1}^{n+1} (-1)^{i+1} [A]_{i,1} \det(A_{i,1}) \cdot \det(B) =$$

$$\det(A) \cdot \det(B).$$

   (b) Deduce

$$\det \begin{pmatrix} A_1 & * & \dots & * \\ 0 & A_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & * \\ 0 & \dots & 0 & A_k \end{pmatrix} = \det(A_1) \cdot \det(A_2) \cdot \dots \cdot \det(A_k).$$

3. B is a *submatrix* of A if B is the result of removing any number of rows and columns from A. E.g., if $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$, then all the following matrices are examples of submatrices of A:

$$\begin{pmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 4 & 6 \\ 7 & 9 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 7 & 9 \end{pmatrix}, (4).$$

Prove that for an arbitrary matrix $A \neq 0$ (not necessarily square):
rank(A) is the maximal natural number n such that A has an $n \times n$ submatrix with non-zero determinant.

Let $A \in M_{m,l}(F)$. We will prove that $rank(A) \geq n$ iff A has an $n \times n$ submatrix with non-zero determinant.

$\Longleftarrow$ **If A has an $n \times n$ submatrix B with non-zero determinant, by equivalent condition on B, the rows of B are linearly independent, which in turn implies that A has n linearly independent rows.**

$\Longrightarrow$ **Step 1:**
If $\text{rank}(A) \geq n$, then A has at least n linearly independent rows. Remove the other rows from A. Now we have an $n \times l$ submatrix of A, call it A', with n linearly independent rows.
Step 2:
convince yourself that the existence of n l.i. rows implies $n \leq l$. Now since $\dim(CS(A')) = \dim(RS(A'))$, there are n l.i. columns in A'. by transposing the argument in Step 1, we get an $n \times n$ submatrix of A' with n l.i. columns, which in turn is also a submatrix of A. By equivalent conditions of invertability, its determinant is non-zero.