

# Math40002 Analysis 1

# Problem Sheet 2

1. Fix  $S \subset \mathbb{R}$  with an upper bound, and suppose that  $S \neq \emptyset$  and  $S \neq \mathbb{R}$ . Give proofs or counterexamples to each of the following statements.

- (a) If  $S \subset \mathbb{Q}$  then  $\sup S \in \mathbb{Q}$ .
- (b) If  $S \subset \mathbb{R} \setminus \mathbb{Q}$  then  $\sup S \in \mathbb{R} \setminus \mathbb{Q}$ .
- (c) If  $S \subset \mathbb{Z}$  then  $\sup S \in \mathbb{Z}$ .
- (d)  $S \cap \left\{ \frac{n}{m} \in \mathbb{Q} : n, m \in \mathbb{N}, m \leq 10^{100} \right\}$  has a minimum if it is nonempty.
- (e) There exists a  $\max S$  if and only if  $\sup S \in S$ .
- (f)  $\sup S = \inf(\mathbb{R} \setminus S)$ .
- (g)  $\sup S = \inf(\mathbb{R} \setminus S) \iff S$  is an interval of the form  $(-\infty, a)$  or  $(-\infty, a]$ .

(a) **False**, eg  $S = \{x \in \mathbb{Q} : x < \sqrt{2}\}$  with  $\sup S = \sqrt{2}$ .

(b) **False**, eg  $S = \{x \in \mathbb{R} \setminus \mathbb{Q} : x < 0\}$  with  $\sup S = 0$ .

(c) **True**. Pick  $s_0 \in S$  and an integer  $N$  larger than a given upper bound for  $S$ . Then  $[s_0, N] \cap \mathbb{Z}$  is a finite set, so  $[s_0, N] \cap S$  is also finite and nonempty set of integers so has a maximum  $m \in \mathbb{Z}$ . By (e) this is also  $\sup S$ .

(d) **True**. Since it is  $\neq \emptyset$  pick an element  $s_0$ . Then  $S \cap \left\{ \frac{n}{m} \in \mathbb{Q} : n, m \in \mathbb{N}, m \leq 10^{100} \right\} \cap [0, s_0]$  is nonempty and *finite*! (Because  $m$  runs through the finite set  $\{1, 2, \dots, 10^{100}\}$  and  $n$  runs through the finite set  $\{1, 2, \dots, \lfloor s_0 10^{100} \rfloor\}$ .) It therefore has a minimum, which is clearly the minimum of  $S \cap \left\{ \frac{n}{m} \in \mathbb{Q} : n, m \in \mathbb{N}, m \leq 10^{100} \right\}$  because it is in the set and is  $\leq$  all elements  $s$  of the set that satisfy  $s \leq s_0$ . It is therefore  $\leq$  elements  $s$  of the set that satisfy  $s > s_0$  too!

(e) **True**. If  $\sup S \in S$  then it is the maximum element because it is an upper bound, so  $\sup S \geq s$  for all  $s \in S$ .

Conversely if there exists  $m = \max S$  then  $m$  is an upper bound, and given any other upper bound  $M$ ,  $M \geq m$  by definition of upper bound because  $m \in S$ . Therefore  $m$  is the least upper bound, i.e.  $\sup S = m \in S$ .

(f) **False**. E.g.  $S = \{0\}$  has  $\sup S = 0$  but  $\mathbb{R} \setminus S$  is not even bounded below so has no infimum.

(g) **True**. If  $S = (-\infty, a)$  then  $\sup S = a$  while  $\mathbb{R} \setminus S = [a, \infty)$  so  $\inf S^c = a$  also.

If  $S = (-\infty, a]$  then  $\sup S = a$  while  $\mathbb{R} \setminus S = (a, \infty)$  so  $\inf S^c = a$  also.

Conversely, if  $\sup S = \inf S^c$  then call this number  $a$ . Any  $x < a$  must be in  $S$ : if not then  $x \in S^c$  but  $x < \inf S^c$ , a contradiction. Similarly any  $x > a$  must be in  $S^c$ : if not then  $x \in S$  but  $x > \sup S$ , a contradiction.

Therefore  $(-\infty, a) \subseteq S$  and  $(a, \infty) \subseteq S^c$ . Finally either  $a \in S$  or  $a \in S^c$ , making  $S$  equal to  $(-\infty, a]$  or  $(-\infty, a)$  respectively.

2. Fix nonempty sets  $S_n \subset \mathbb{R}$ ,  $n = 1, 2, 3, \dots$ . Prove that

$$\sup \{\sup S_1, \sup S_2, \sup S_3, \dots\} = \sup \left( \bigcup_{n=1}^{\infty} S_n \right),$$

in the sense that if either exists then so does the other, and they are equal.

Upper bound is  $M_n \in \mathbb{R}$  such that  $M_n \geq s \forall s \in S_n$ .

$\sup S_n \in \mathbb{R}$  is an upper bound for  $S_n$  such that  $\sup S_n \leq M_n$  for all upper bounds  $M_n$ . It exists if and only if  $S_n$  is nonempty and has an upper bound.

Suppose the left hand side exists, call it  $M$ . Then  $M \geq \sup S_i \geq s$  for each  $i$  and for each  $s \in S_i$ , so  $M$  is an upper bound for  $\bigcup_{n=1}^{\infty} S_n$ , so the right hand side  $N$  exists and  $M \geq N$ .

Similarly if the right hand side  $N$  exists then it is an upper bound for each  $S_i$ , so each  $\sup S_i$  exists and is  $\leq N$ . Therefore the set on the left hand side has an upper bound  $N$ , therefore its supremum  $M$  exists and  $M \leq N$ .

Therefore if either exists so does the other. In this case we have shown that  $M \geq N$  and  $M \leq N$ . Therefore  $M = N$ .

3. Take bounded, nonempty  $S, T \subset \mathbb{R}$ . Define  $S + T := \{s + t : s \in S, t \in T\}$ . Prove

$$\sup(S + T) = \sup S + \sup T.$$

Any element  $s + t$  of  $S + T$  is  $\leq \sup S + \sup T$ , so  $S + T$  is nonempty and has an upper bound  $\sup S + \sup T$ , so it has a supremum  $\leq \sup S + \sup T$ .

Given  $\epsilon > 0$  we know  $\sup S - \epsilon$  is not an upper bound for  $S$  because it is smaller than the least upper bound, so  $\exists s \in S$  such that  $s > \sup S - \epsilon$ . Similarly  $\exists t \in T$  such that  $t > \sup T - \epsilon$ .

Therefore  $\exists s + t \in S + T$  such that  $s + t > \sup S + \sup T - 2\epsilon$ , so we have proved that

$$\sup S + \sup T - 2\epsilon < \sup(S + T) \leq \sup S + \sup T \quad \forall \epsilon > 0,$$

which implies  $\sup(S + T) = \sup S + \sup T$ .

- 4.\* Fix  $a \in (0, \infty)$  and  $n \in \mathbb{N}$ . We will prove  $\exists x \in \mathbb{R}$  such that  $x^n = a$ . Set

$$S_a := \{s \in [0, \infty) : s^n < a\}$$

and show  $S$  is nonempty and bounded above, so we may define  $x := \sup S_a$ .

For  $\epsilon \in (0, 1)$  show  $(x + \epsilon)^n \leq x^n + \epsilon[(x + 1)^n - x^n]$ . (Hint: multiply out.)

Hence show that if  $x^n < a$  then  $\exists \epsilon \in (0, 1)$  such that  $(x + \epsilon)^n < a$ . (\*)

If  $x^n > a$  deduce from (\*) that  $\exists \epsilon \in (0, 1)$  such that  $(\frac{1}{x} + \epsilon)^n < \frac{1}{a}$ . (\*\*)

Deduce contradictions from (\*) and (\*\*) to show that  $x^n = a$ .

By the binomial theorem,

$$\begin{aligned} (x + \epsilon)^n &= x^n + n\epsilon x^{n-1} + \binom{n}{2}\epsilon^2 x^{n-2} + \dots + \epsilon^n \\ &< x^n + n\epsilon x^{n-1} + \binom{n}{2}\epsilon x^{n-2} + \dots + \epsilon \\ &= x^n + \epsilon((x + 1)^n - x^n), \end{aligned}$$

for  $\epsilon \in (0, 1)$  (so that  $\epsilon^k < \epsilon$ ) and  $x > 0$ .

Therefore, if  $x^n < a$  we can set  $\epsilon := \min\left(\frac{1}{2}, \frac{a - x^n}{(x+1)^n - x^n}\right) \in (0, 1)$  so that

$$(x + \epsilon)^n < x^n + \epsilon((x + 1)^n - x^n) \leq x^n + (a - x^n) = a,$$

as required. So  $x + \epsilon$  is both in  $S_a$  and  $> \sup S_a$  - a contradiction.

If  $x^n > a$  then  $(\frac{1}{x})^n < \frac{1}{a}$  so by (\*) applied to  $\frac{1}{x}$  and  $\frac{1}{a}$  we find  $\exists \epsilon \in (0, 1)$  such that  $(\frac{1}{x} + \epsilon)^n < \frac{1}{a}$ .

Thus  $(\frac{x}{1 + \epsilon x})^n > a \Rightarrow y^n > a$  for all  $y \geq \frac{x}{1 + \epsilon x}$ . That is,  $\frac{x}{1 + \epsilon x}$  is an upper bound for  $S_a$ , but is  $< x = \sup S_a$  - a contradiction.

5. Suppose  $0 < q \in \mathbb{Q}$  and  $a \in (0, \infty)$ . Write  $q = \frac{m}{n}$  with  $m, n \in \mathbb{N}$  and define

$$a^q := x^m,$$

where  $x =: a^{1/n}$  is defined in the last question. Show this is well defined, and make a definition of  $a^{-q}$ .

Show that  $(ab)^q = a^q b^q$  and  $(a^{q_1})^{q_2} = a^{q_1 q_2}$  for any  $a, b \in (0, \infty)$  and  $q, q_1, q_2 \in \mathbb{Q}$ .

To show this is well defined we have to show that if we replace  $\frac{m}{n}$  by  $\frac{pm}{pn}$  for some  $p \in \mathbb{N}$  then we get the same answer. (Why is this enough?) That is, we must show

$$\left(a^{\frac{1}{n}}\right)^m = \left(a^{\frac{1}{pn}}\right)^{pm}.$$

It would be sufficient to prove

$$a^{\frac{1}{n}} = \left(a^{\frac{1}{pn}}\right)^p.$$

The LHS is the unique (why?)  $x \in (0, \infty)$  such that  $x^n = a$ . So it is enough to show the RHS has this property, i.e. that

$$\left(\left(a^{\frac{1}{pn}}\right)^p\right)^n = a.$$

But the LHS is

$$\left(a^{\frac{1}{pn}}\right)^{np},$$

which is indeed  $a$  by the definition of  $a^{\frac{1}{pn}}$ .

Next, we'll prove that, for any  $a, b \in (0, \infty)$  and  $0 < q \in \mathbb{Q}$ , we have the identity  $(ab)^q = a^q b^q$ . Writing  $q = \frac{m}{n}$  with  $m, n \in \mathbb{N}$ , what we must show is that  $((ab)^{1/n})^m = (a^{1/n})^m (b^{1/n})^m = (a^{1/n} b^{1/n})^m$  - in the last inequality here we are allowed to play with the exponent  $m$  like this since  $m \in \mathbb{N}$ . Again, since  $m \in \mathbb{N}$ , it suffices to show that

$$(ab)^{1/n} = a^{1/n} b^{1/n}.$$

Now,  $(ab)^{1/n}$  is the unique positive real number that satisfies  $((ab)^{1/n})^n = ab$ , so we just need to show that  $a^{1/n} b^{1/n}$  satisfies this same property (since it is also a positive real number). We see that this is true since  $(a^{1/n} b^{1/n})^n = (a^{1/n})^n (b^{1/n})^n = ab$  where in the first equality we are using the fact that we can distribute the exponent  $n$  since  $n \in \mathbb{N}$ .

For showing  $(a^{q_1})^{q_2} = a^{q_1 q_2}$  we write  $q_1 = m_1/n_1$  and  $q_2 = m_2/n_2$  so the desired identity is

$$(((a^{1/n_1})^{m_1})^{1/n_2})^{m_2} = (a^{1/(n_1 n_2)})^{m_1 m_2}$$

Notice that we know the righthand side above is equal to  $((a^{1/(n_1 n_2)})^{m_1})^{m_2}$  (natural number exponents behave nicely) and so it suffices to show

$$((a^{1/n_1})^{m_1})^{1/n_2} = (a^{1/(n_1 n_2)})^{m_1}$$

Now, just like last time, by the uniqueness of  $1/n_2$  roots, to show the above identity it suffices to show that

$$((a^{1/(n_1 n_2)})^{m_1})^{n_2} = (a^{1/n_1})^{m_1}$$

Now, again we know that  $((a^{1/(n_1 n_2)})^{m_1})^{n_2} = (a^{1/(n_1 n_2)})^{m_1 n_2} = ((a^{1/(n_1 n_2)})^{n_2})^{m_1}$ . Again, we "cancel" the  $m_1$ 's so it suffices to show

$$(a^{1/(n_1 n_2)})^{n_2} = a^{1/n_1}$$

Now are are done since we know  $((a^{1/(n_1 n_2)})^{n_2})^{n_1} = a$  (skipping a few steps here).

For  $a^{-q}$ , you should define  $a^{-q} := 1/a^q$ .

6. For real numbers  $x, y, z$ , consider the following inequalities.

- |   |                                      |
|---|--------------------------------------|
| (a) $ x + y  \leq  x  +  y $                | (e) $ x  \leq  y  +  x - y $         |
| (b) $ x + y  \geq  x  -  y $                | (f) $ x  \geq  y  -  x - y $         |
| (c) $ x + y  \geq  y  -  x $                | (g) $ x - y  \leq  x - z  +  y - z $ |
| (d) $ x - y  \geq \left   x  -  y  \right $ |                                      |

**Prove** (a) from first principles. Why is it called the “triangle inequality”?

**Deduce** (b,c,d,e,f,g) from (a).

**First prove that  $x \leq |x|$  by considering the two cases  $x \leq 0, x > 0$ .**

**Then it follows that  $x + y \leq |x| + |y|$  and  $-(x + y) \leq |-x| + |-y| = |x| + |y|$ . Combining these two gives (a).**

**Called triangle rule because it says that if you make a triangle with vertices at  $0, x, x + y$  then the length of the side  $[0, x + y]$  is  $\leq$  the sum of the lengths of the other two sides  $[0, x]$  and  $[x, x + y]$ . It's a bit more convincing when  $0, x, x + y$  are all vectors in higher dimensions, eg.  $R^2$ , where the same result holds.**

**For (b) replace  $x, y$  in (a) by  $x + y, -y$  and rearrange. Similarly for all the others. For (g) write  $x - y = (x - z) - (y - z)$ .**

*Starred questions \* are good to prepare to discuss at your Problem Class.*