1. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with $f(\mathbb{R}) \subset \mathbb{Q}$. Prove that f is constant.

   Solution. Suppose that $f(x) \neq f(y)$ for some $x < y$. Then the interval $[f(x), f(y)]$ contains at least one irrational number r (in fact, uncountably many), say $r = f(x) + \frac{f(y)-f(x)}{\sqrt{2}}$ for concreteness. The intermediate value theorem says that there is some $c \in [x, y]$ such that $f(c) = r$, but $r \notin \mathbb{Q}$, contradiction.

2. Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions such that $f(x) = g(x)$ for all $x \in \mathbb{Q}$. Prove that $f(x) = g(x)$ for all $x \in \mathbb{R}$. Is this still true if we only assume that $f(x) = g(x)$ for $x \in \mathbb{Z}$?

   Solution. We fix $x \in \mathbb{R}$ and take a sequence of rational numbers $y_1, y_2, \dots \to y$. Since f and g are continuous we have $f(y_n) \to f(y)$ and $g(y_n) \to g(y)$, but the sequences $\big(f(y_n)\big)$ and $\big(g(y_n)\big)$ are identical, so their limits $f(y)$ and $g(y)$ must be the same.

   On the other hand, let $f(x) = \sin(\pi x)$ and $g(x) = 0$. Then $f(x) = g(x)$ for all $x \in \mathbb{Z}$, but $f(\frac{\pi}{2}) = 1 \neq g(\frac{\pi}{2})$.

3. Consider the function $f : [1, 2] \cap \mathbb{Q} \to \mathbb{R}$ defined by $f(x) = \left|x - \sqrt{2}\right|$. Prove that f does *not* have a minimum value. Why doesn't the extreme value theorem apply?

   Solution. We have $f(x) > 0$ for all x in the domain, since $\sqrt{2}$ is irrational. If we take a sequence of rational numbers $x_n \to \sqrt{2}$ then $f(x_n) \to 0$, so $\inf f(x) = 0$ and hence the infimum is not achieved anywhere on the domain. The extreme value theorem fails here because the domain is not closed, even though it's bounded and contains both a minimum and a maximum.

4. (*) Define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0, & x \text{ irrational} \\ 1/n, & x = m/n. \end{cases}$$

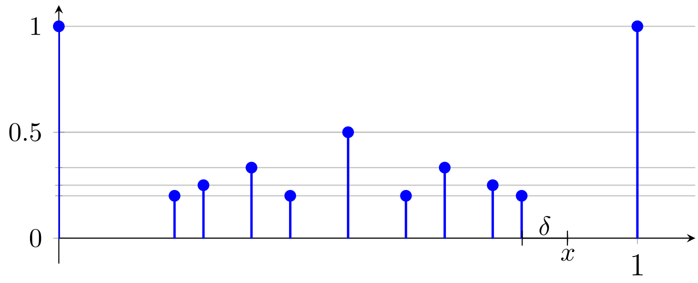
   Here all rational numbers $x = \frac{m}{n}$ are written in lowest terms, with $n > 0$.

   (a) Prove that if x is rational, then f is not continuous at x.

   (b) Prove that if x is irrational, then f is continuous at x.

   Solution. (a) If $x = \frac{m}{n}$, then we take $\epsilon = \frac{1}{2n}$ and find that for any $\delta > 0$ there are irrational y with $|y - x| < \delta$, and these satisfy $|f(y) - f(x)| = \frac{1}{n} > \epsilon$.

   (b) Suppose that $x \notin \mathbb{Q}$ and fix $\epsilon > 0$. There are only finitely many rational numbers $q_1, \dots, q_k$ with denominator at most $\frac{1}{\epsilon}$ between the integers $\lfloor x \rfloor$ and $\lceil x \rceil$, inclusive, because no more than $d+1$ of them can have a given denominator *d*. For example, if $\frac{1}{\epsilon} = 5$ and $0 < x < 1$, we might have the following picture, where $f(\frac{m}{n})$ is shown for all rational numbers in $[0,1]$ with $n \leq 5$:



We let

$$\delta = \min_j |x - q_j| > 0,$$

and then if $|y - x| < \delta$, it follows that y is either irrational ($\Rightarrow \ f(y) = 0$) or has denominator greater than $\frac{1}{\epsilon}$ ($\Rightarrow \ f(y) < \epsilon$). So for all such y we have $|f(y) - f(x)| = |f(y)| < \epsilon$, and this proves continuity at x.

5. Let $f : [a, b] \to \mathbb{R}$ be continuous, and suppose that $f(a) \leq y \leq f(b)$.

   (a) Let $(a_0, b_0) = (a, b)$, and for all $n \geq 0$, define $m_n = \frac{a_n+b_n}{2}$ and

   $$(a_{n+1}, b_{n+1}) = \begin{cases} (a_n, m_n), & f(m_n) > y \\ (m_n, b_n), & f(m_n) \leq y. \end{cases}$$

   Prove that the sequences $\big(a_n\big)$ and $\big(b_n\big)$ converge to the same limit $L \in [a, b]$.

   (b) Prove that $f(L) = y$, concluding a new proof of the intermediate value theorem.

Solution. (a) We have $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ for all n, so the sequence $\big(a_n\big)$ is increasing and bounded above by b, while the sequence $\big(b_n\big)$ is decreasing and bounded below by a. Thus $\big(a_n\big) \to L_a$ and $\big(b_n\big) \to L_b$ for some $L_a \leq b$ and $L_b \geq a$. We also have

$$b_{n+1} - a_{n+1} = \frac{b_n - a_n}{2} \ \forall n \quad \Longrightarrow \quad \lim_{n\to\infty}(b_n - a_n) = 0,$$

but by the algebra of limits this means that $L_b - L_a = 0$, so $a \leq L_b = L_a \leq b$.

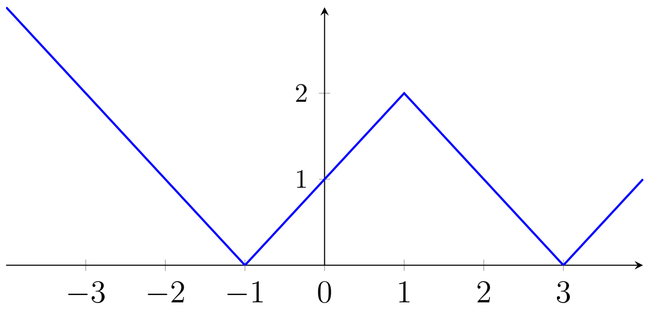
(b) By induction we see that $f(a_n) \leq y$ for all n and also $f(b_n) \geq y$ for all n. Since f is continuous, we have $f(a_n) \to f(L)$, which now implies that $f(L) \leq y$, and likewise $f(b_n) \to f(L)$ gives us $f(L) \geq y$. We combine these to conclude that $f(L) = y$.

6. For any nonempty set $S \subset \mathbb{R}$, define $d_S : \mathbb{R} \to \mathbb{R}$ by $d_S(x) = \inf_{s\in S} |x - s|$.

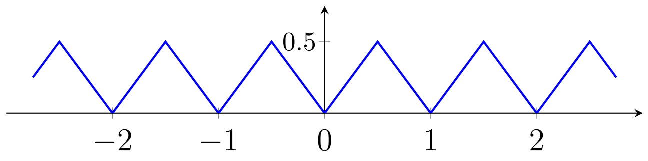
   (a) Describe or draw graphs of d_S when S is each of $\{0\}$, $\{-1, 3\}$, $\mathbb{Z}$, $\mathbb{Q}$.

(b) Prove that $|d_S(y) - d_S(x)| \leq |y - x|$ for all $x, y \in \mathbb{R}$, and conclude that d_S is continuous.

Solution. (a) We have $d_{\{0\}}(x) = |x|$. Here's a graph of $d_{\{-1,3\}}$:



And of $d_{\mathbb{Z}}$:



We have $d_{\mathbb{Q}}(x) = 0$, since there are rational numbers arbitrarily close to x.

(b) By the definition of $d_S(x)$, for all $n \geq 1$ there's an $s_n \in S$ such that $|x - s_n| < d_S(x) + \frac{1}{n}$, and the triangle inequality tells us that

$$|y - s_n| < |y - x| + |x - s_n| < |y - x| + d_S(x) + \frac{1}{n}.$$

Taking limits as $n \to \infty$ gives $d_S(y) \leq \inf_n |y - s_n| \leq |y - x| + d_S(x)$, hence

$$d_S(y) - d_S(x) \leq |y - x|.$$

We repeat this argument with x and y swapped to get $d_S(x) - d_S(y) \leq |y - x|$ as well, so $|d_S(y) - d_S(x)| \leq |y - x|$ as claimed.

Now for any $x \in \mathbb{R}$ and $\epsilon > 0$ we have $|y - x| < \epsilon \implies |d_S(y) - d_S(x)| \leq |y - x| < \epsilon$, so the definition of continuity at x is satisfied by taking $\delta = \epsilon$.

7. Let $f : \mathbb{R} \to \mathbb{R}$ be a monotonically increasing function, not necessarily continuous. Define $S(x) = \sup_{y<x} f(y)$ and $I(x) = \inf_{y>x} f(y)$.

(a) Prove for all $x \in \mathbb{R}$ that $S(x) \leq f(x) \leq I(x)$.

(b) Prove for all $x \in \mathbb{R}$ that $S(x) = I(x)$ if and only if f is continuous at x.

(c) Find an injective mapping

$$\{x \in \mathbb{R} \mid f \text{ is not continuous at } x\} \to \mathbb{Q}.$$

Conclude that f is continuous at all but at most countably many real numbers.

Solution. (a) By monotonicity, we have $f(y) \leq f(x)$ for all $y < x$, so $f(x)$ is an upper bound for $\{f(y) \mid y < x\}$ and hence $f(x) \geq S(x)$. The proof that $f(x) \leq I(x)$ is the same.

(b) ($\Longrightarrow$) Suppose that $S = f(x) = I$, and fix $\epsilon > 0$. We can find $z < x$ such that $f(z) > f(x) - \epsilon$, since $f(x) = \sup_{z<x} f(z)$, and likewise $w > x$ such that $f(w) < f(x) + \epsilon$. If we let $\delta = \min(x - z, w - x) > 0$, then $|y - x| < \delta$ implies $z < y < w$, and hence implies

$$f(y) \in [f(z), f(w)] \subset (f(x) - \epsilon, f(x) + \epsilon)$$

by the monotonicity of f. So $|f(y) - f(x)| < \epsilon$ whenever $|y - x| < \delta$ and the continuity of f at x follows.

($\Longleftarrow$) Suppose that f is continuous at x. Then for any $\epsilon > 0$ we can find $\delta > 0$ such that $|y - x| < \delta$ implies $|f(y) - f(x)| < \epsilon$, and if we take $y = x \pm \frac{\delta}{2}$ then

$$S \geq f(x - \tfrac{\delta}{2}) > f(x) - \epsilon \qquad \text{and} \qquad I \leq f(x + \tfrac{\delta}{2}) < f(x) + \epsilon.$$

Now $f(x)$ is an upper bound for $\{f(y) \mid y < x\}$ by monotonicity, so we have

$$f(x) \geq S > f(x) - \epsilon \text{ for all } \epsilon > 0 \quad \Longrightarrow S = f(x)$$

and similarly $f(x) \leq I < f(x) + \epsilon$ leads to $f(x) = I$.

(c) Parts (a) and (b) say that if f is discontinuous at x then the open interval $(S(x), I(x))$ is nonempty, so we can pick a rational number q_x in this interval. If $x < y$ are two points of discontinuity, then we have $I(x) \leq f(\frac{x+y}{2}) \leq S(y)$, so the intervals $(S(x), I(x))$ and $(S(y), I(y))$ are disjoint and thus $q_x \neq q_y$. Therefore the mapping $x \mapsto q_x$ is injective.

8. Prove that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if for every open set $U \subset \mathbb{R}$, the preimage
$$f^{-1}(U) = \{x \in \mathbb{R} \mid f(x) \in U\}$$
is open.

Solution. ($\Longrightarrow$): Suppose that f is continuous, and fix an open set $U \subset \mathbb{R}$. Let x be a point of $f^{-1}(U)$; then $f(x) \in U$ by definition, and since U is open, there is some $\epsilon > 0$ such that the whole open interval $(f(x) - \epsilon, f(x) + \epsilon)$ is a subset of U. Since f is continuous at x, there is $\delta > 0$ such that $|y - x| < \delta$ implies $|f(y) - f(x)| < \epsilon$, hence
$$f(y) \in (f(x) - \epsilon, f(x) + \epsilon) \subset U.$$
But then $y \in f^{-1}(U)$ for all such y, so $(x - \delta, x + \delta) \subset U$. Since we can find such a neighborhood for any $x \in f^{-1}(U)$, it follows that $f^{-1}(U)$ is open.

($\Longleftarrow$): We will show that f is continuous at any $x \in \mathbb{R}$. Fix $\epsilon > 0$ and let $U = (f(x) - \epsilon, f(x) + \epsilon)$. Then $f^{-1}(U)$ contains x by definition, and since U is open, so is $f^{-1}(U)$. This means that $f^{-1}(U)$ contains an open neighborhood $(x-\delta, x+\delta)$ of x for some $\delta > 0$. Now if $|y - x| < \delta$ then

$$y \in f^{-1}(U) \implies f(y) \in U = (f(x) - \epsilon, f(x) + \epsilon) \implies |f(y) - f(x)| < \epsilon,$$

and we can do this for any $\epsilon > 0$, so f is continuous at x.