Linear Algebra & Numerical Analysis Concise Notes

MATH50003

Term 1 Content

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Colour Code - **Definitions** are **green** in these notes, **Consequences** are **red** and **Causes** are **blue**

Content from MATH40003 assumed to be known.

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Contents

1 Prelim

Definition - Similair Matrices
$A, B \in M_n(F)$ similair $(A \sim B)$ if $\exists$ invertible $P \in M_n(F)$ s.t $P^{-1}AP = B$
$\sim$ is an equivalence relation.
Properties of Similair Matrices

- Same Determinant
- Same Char. Poly.
- Same eigenvalues
- Same rank Same Trace

Definition - Companion Matrix
Let $p(x)$ a monic polynomial of degree r; $p(x) = x^r + a_{r-1}x^{r-1} + \cdots + a_0$.
Companion matrix of $p(x)$;

$$C(p(x)) = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & 0 & \dots & 0 & -a_2 \\ & & & \dots & & \\ 0 & 0 & 0 & \dots & 1 & -a_{r-1} \end{pmatrix}$$

Geometry

Definition - Dot Product
$u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, vn)$

$$u \cdot v = \sum_{i=1}^{n} u_i v_i$$

Length of $u, ||u|| = \sqrt{u \cdot u}$
Distance between u and $v = ||u - v||$

- P orthogonal if $P^T P = I, (Pu \cdot Pv) = u \cdot v)$
- A symmetric if $A^T = A, (Au \cdot v = u \cdot Av)$

Properties of dot product

- linear in u, v
- symmetric; $u \cdot v = v \cdot u$
- $u \cdot v > 0, \forall u, v$

3 Algebraic and Geometric multiplicities of eigenvalues

Definition - Multiplicity of eigenvalues
For $T : V \to V$ a linear map with char. poly. $p(x)$ with roots λ, Then $\exists\ a(\lambda) \in \mathbb{N}$ the **algebraic multiplicity** of λ s.t

$$p(x) = (x - \lambda)^{a(\lambda)} q(x)$$

where λ not a root of $q(x)$
Geometric multiplicity $g(\lambda) = dim E_\lambda$, for E_λ the eigenspace of T

Theorem 3.2
$dimV = n$, Let $T : V \to V$ a linear map with finite distinct eigenvalues $\{\lambda_i\}_{i=1}^r$
Characteristic polynomial of T is

$$p(x) = \prod_{i=1}^{r} (x - \lambda_i)^{a(\lambda_i}$$

so $(\sum_{i=1}^r a(\lambda_i) = n$. Following are equivalent

- T diagonalisable
- $(\sum_{i=1}^r g(\lambda_i) = n$
- $g(\lambda_i) = a(\lambda_i) \forall i$ (This can be used to test for diagonalisability.)

4 Direct Sums

Define
For $\{U_i\}_{i=1,\dots,k}$ subspaces of vector space V. Sum of these subspaces is:

$$U_1 + \cdots + U_k = \{u_1 + \cdots + u_k : u_i \in U_i, \forall i\}$$

Definition - Direct Sums
V a vector space, $\{V_i\}_{i=1,\dots,k}$ subspaces of vector space V. **V a direct sum of $\{V_i\}$** if:

$$V = V_1 \oplus \cdots \oplus V_k$$

If $\forall v \in V$ can be expressed as $v = v_1 + \cdots + v_k$ for unique vectors $v_i \in V_i$
Corollary

$$V = V_1 \oplus \cdots \oplus V_k \iff dimV = \sum_{i=1}^{k} dimV_i \text{ and if } B_i \text{ a basis for } V_i, B = \bigcup_i B_i \text{ is a basis for } V$$

Definition - Invariant subspaces
$T : V \to V$ a linear map, W a subspace of V.

$$W \text{ is } T\text{-invariant if } T(W) \subseteq W, T(W) = \{T(w) : w \in W\}$$

Write $T_W : W \to W$ for the restriction of T to W

Notation - Direct sums of matrices

$$A_1 \oplus \cdots \oplus A_k = \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_k \end{pmatrix}$$

5 Quotient Spaces

Definition - Cosets V a vector space over F, with $W \leq V$ a subspace.

$$\text{Cosets } W + v \text{ for } v \in V \ W + v := \{w + v : w \in W\}$$

Quotient Space
Define V/W as a vector space of vectors $W + v$ over F

- Addition; $(W + v_1) + (W + v_2) = W + v_1 + v_2$
- Scalar Multiplication; $\lambda(W + v) = W + \lambda v$

Can verify this using vector space axioms.
Dimension of V/W

$$dimV/W = dimV - dimW$$

Definition - Quotient Map
$T : V \to V$ a linear map, W a T-invariant subspace of V. Quotient map: $\bar{T} : V/W :\to V/W$ such that

$$\bar{T}(W + v) = W + T(v), \qquad \forall v \in V$$

6 Triangularisation

Lemma - Diagonal Matrices

$$A = \begin{pmatrix} \lambda_1 & & & & \\ 0 & \lambda_2 & & * & \\ & & \cdot & & \\ 0 & & & \cdot & \\ 0 & 0 & & & \lambda_n \end{pmatrix}, B = \begin{pmatrix} \mu_1 & & & & \\ 0 & \mu_2 & & * & \\ & & \cdot & & \\ 0 & & & \cdot & \\ 0 & 0 & & & \mu_n \end{pmatrix}$$

- Characteristic polynomial of $A = \prod_{i=1}^{n}(x - \lambda_i)$, eigenvalues $= \{\lambda_i\}$
- $detA = \prod_{i=1}^{n} \lambda_i$
- AB also upper triangular, with $diag(AB) = \lambda_1\mu_1, \dots, \lambda_n\mu_n$

Theorem 6.2 - Triangularisation Theorem
V an n dimensional vector space over F, $T : V \to V$ a linear map,
Where $\chi(T) = \prod_{i=1}^{n}(x - \lambda_i)$, where $\lambda_i \in F \ \forall i \implies \exists$ basis B of V s.t $[T]_B$ upper triangular

7 The Cayley-Hamilton Theorem

Theorem. 7.1 - *(Cayley-Hamilton Theorem)*
V a finite dimensional vector space over F. $T : V \to V$ a linear map with char. poly. $p(x)$

$$p(T) = 0$$

8 Polynomials

Definition - Polynomials over a field
F a field,$p(x)$ over F, for $p(x) = \sum_i a_i x^i, F[x] = \{p(x) : a_i \in F\}$
Degree of polynomial
$deg(p(x)) =$ the highest power of x in $p(x)$
Euclidean Algorithm
$f, g \in F[x]$ with $deg(g) \geq 1$, Then $\exists q, r \in F[x] s.t$

$$f = gq + r$$

for either $r = 0$ or $deg(r) < deg(g)$

Definition - Greatest Common Divisor (GCD) of polynomials
$f, g \in F[x] \backslash \{0\}$, **Say $d \in F[x]$ the gcd of f, g if:**

(i) $d|f$ **and** $d|g$

(ii) **if** $e(x) \in F[x]$ **and** $e|f$ **and** $e|g$ **Then** $e|d$

Say f, g are co-prime if $gcd(f, g) = 1$
Corollary
$d = gcd(f, g) \implies \exists r, s \in F[x]$ s.t $d = rf + sg$

Definiton - Irreducible polynomials
$p(x) \in F[x]$ irreducible over F if $deg(p) \geq 1$ and p not factorisable over F as a product of $\{f_i\} \in F$ s.t $deg(f_i \leq deg(p)$
Corollary
$p(x) \in F[x]$ irreducible, $\{g_i\} \in F[x]$, if $p|g_1 \dots g_r \implies p|g_i$ for some i

Theorem 8.7 - *(Unique Factorization Theorem)*
$f(x) \in F[x]$ s.t $deg(f) \geq 1$

$$f = p_1 \dots p_r$$

where each $p_i \in F[x]$ irreducible. **Factorisation of f is unique up to scalar multiplication**

9 The minimal polynomial of a linear map

Definition -Minimal polynomial
Say $m(x) \in F[x]$ a minimal polynomial for $T : V \to V$ if

(i) $m(T) = 0$

(ii) $m(x)$ monic

(iii) $deg(m)$ is as small as possible s.t (i) and (ii)

Properties of the minimal polynomial

- For T a linear map, its minimal polynomial $m_T(x)$ is unique
- $p(x) \in F[x], p(T) = 0 \iff m_T(x)|p(x)$
- $m_T(x)|c_T(x)$ the char. poly. of T
- $\lambda \in F$ a root of $c_T(x) \implies \lambda$ a root of $m_T(x)$
- $A, B \in M_n(F)$ s.t $A \sim B \implies m_A(x) = m_B(x)$

Theorem 9.3
$p(x) \in F[x]$ an irreducible factor of $c_T(x) \implies p(x)|m_T(x)$ *Corollaries*

- $c_T(x) = c_{T_W}(x)c_{\bar{T}}(x)$
- $m_{T_W}(x)$ and $m_{\bar{T}}(x)$ both divide $m_T(x)$

10 Primary Decomposition

Theorem 10.1 - ***(Primary Decomposition Theorem)***
V a finite dimensional vector space over F, $T : V \to V$ a linear map with $m_T(x)$
Let factorisation of $m_T(x)$ into irreducible polynomials be:

$$m_T(x) = \prod_{i=1}^{k} f_i(x)^{n_i}$$

Where $\{f_i(x)\}$ all distinct irreducible polynomials in $F[x]$
For $1 \le i \le k$, define:

$$V_i = ker(f_i(T)^{n_i})$$

Then

1. $V = V_1 \oplus \dots \oplus V_k$ *(Call this the* ***primary decomposition*** *of V w.r.t T)*
2. each V_i is T-invariant
3. each restriction T_{V_i} has minimal polynomial $f_i(x)^{n_i}$

In the case where each $f_i(x) = (x - \lambda_i)$

$$\implies m_T(x) = \prod_{i=1}^{k} (x - \lambda_i)^{n_i}$$

With λ_i distinct eigenvalues of T and $V_i = ker(T - \lambda_i I)^{n_i}$
We call V_i the **generalised λ_i-eigenspace of T**

Corollary
A linear map $T : V \to V$ diagonalisable $\iff m_T(x) = \prod_{i=1}^{k}(x - \lambda_i)$ a product of distinct linear factors

Corollary
For $T : V \to V$ a linear map, with $g_1(x), g_2(x) \in F[x]$ coprime polynomials s.t $g_1(T)g_2(T) = 0$

1. Then $V = V_1 \oplus V_2$, where $V_i = ker g_i(T), i = 1, 2$ with each V_i being T-invariant
2. Suppose $m_T(x) = g_1(x)g_2(x) \implies m_{T_{V_i}}(x) = g_i(x), i = 1, 2$

11 Jordan Canonical Form

Definition - Jordan Block
F a field and let $\lambda \in F$. Define $n \times n$ matrix:

$$J_n(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda & 1 & \dots & 0 & 0 \\ 0 & 0 & \lambda & \dots & 0 & 0 \\ & & & \dots & & \\ 0 & 0 & 0 & \dots & \lambda & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda \end{pmatrix}$$

Properties of the Jordan Blocks

1. characteristic and minimal polynomials of $J, = (x - \lambda)^n)$
2. λ the only eigenvalue of J, with $a(\lambda) = n, g(\lambda) = 1$
3. $J - \lambda I = J_n(0)$, multiplication by $J - \lambda I$ sends basis vectors as such:

$$e_n \to e_{n-1} \to \dots \to e_2 \to e_1 \to 0$$

4. $(J - \lambda I)^n = 0$, and for $i < n$, $rank((J - \lambda I)^i) = n - i$. And under multiplication:

$$e_n \to e_{n-i}, e_{n-1} \to e_{n-i-1} \dots$$

Lemma
Let $A = A_1 \oplus \cdots \oplus A_k$ for each i let A_i have char. poly $c_i(x)$ and min. poly. $m_i(x)$.

- $c_A(x) = \prod_{i=1}^{k} c_i(x)$
- $m_A(x) = lcm(m_1(x), \ldots, m_k(x))$
- $\forall \lambda$ eigenvalues of A, $dim E_\lambda(A) = \sum_{i=1}^{k} dim E_\lambda(A_i)$
- $\forall q(x) \in F[x]$, $q(A) = q(A_1) \oplus \cdots \oplus q(A_k)$

Theorem 11.3 - *(Jordan Canonical Form)*
$A \in M_n(F)$, suppose $c_A(x)$ a product of linear factors over F.
Then

1. A similair to matrix of form
$$J = J_{n_1}(\lambda_1) \oplus \cdots \oplus J_{n_k}(\lambda_k)$$
This is the Jordan Canonical Form (JCF) of A
2. Matrix J from above, is uniquely determined by A up to order of Jordan blocks

Computing the JCF
JCF theorem says $A \sim J$, a JCF matrix.
$A \sim J \implies$ same characteristic polynomial, eigenvalues, geometric multiplicities, minimal polynomial and $q(A) \sim q(J)$ for any polynomial q.

For each eigenvalue λ, collect all Jordan blocks as such;

$$J = \underbrace{(J_{n_1}(\lambda) \oplus \cdots \oplus J_{n_a}(\lambda))}_{\lambda-\text{blocks of J}} \oplus \underbrace{(J_{m_1}(\mu) \oplus \cdots \oplus J_{m_b}(\mu))}_{\mu-\text{blocks of J}} \oplus \ldots$$

Properties of JCF
J as above, λ an eigenvalue;

1. $n_1 + \cdots + n_a = a(\lambda)$
2. $a =$ number of λ-blocks $= g(\lambda)$
3. $\max(n_1, \ldots, n_a) = r$, where $(x-\lambda)^r$ the highest power of $(x-\lambda)$ dividing $m_A(x)$

Theorem 11.6.

$T : V \to V$ a linear map s.t $c_T(x)$ a product of linear factors $\implies \exists$ basis B of V s.t $[T]_B$ a JCF matrix
Definition.- Nilpotent Matrix
$A^k = 0$ for some $k \in \mathbb{N}$

Theorem 11.7.

$S : V \to V$ a nilpotent linear map $\implies \exists$ basis B of V s.t

$$[S]_B = J_{n_1}(0) \oplus \cdots \oplus J_{n_k}(0)$$

Computing a Jordan Basis

Finding the Jordan Basis B as above.
We have $V = V_1 \oplus \cdots \oplus V_k$ by Primary Decomposition Theorem.
Take each restriction T_{V_i} each with 1 eigenvalue.
Let $S_i = T_{V_i} - \lambda_i I$ so each S_i nilpotent.

Step 1 - Compute subspaces

$$V \supset S(V) \supset S^2(V) \supset \cdots \supset S^r(V) \supset 0$$

$S^{r+1}(V) = 0$

Step 2 - Find basis of $S^r(V)$, Using the following rules extend to basis of $S^{r-1}(V)$:

Given basis $u_1, S(u_1), \ldots, S^{m_1-1}(u_1), \ldots u_r, S(u_r), \ldots, S^{m_r-1}(u_r)$

(1) for each i add vector $v_i \in V$ s.t $u_i = S(v_i)$

(2) note $ker(S)$ contains linearly independent vectors

$$S^{m_1-1}(u_1), \ldots, S^{m_r-1}(u_r)$$

extend to basis of $ker(S)$ by adding vectors $w_1, \ldots, w_s$ with dim $ker(S) = r + s$
Yielding

$$v_1, S(v_1), \ldots, S^{m_1}(v_1), \ldots, v_r, S(v_r), \ldots, S^{m_r}(v_r), w_1, \ldots, w_s$$

Step 3 - Repeat successively finding Jordan bases of $S^{r-2}, \ldots, S(V), V$

12 Cyclic Decomposition & Rational Canonical Form

Definition - Cyclic Subspaces
V a finite dimensional vector space over F, and $T : V \to V$ a linear map.
Let $0 \neq v \in V$ and define

$$\begin{aligned} Z(v, T) &= \{f(T)(v) \ : \ f(x) \in F[x]\} \\ &= \mathrm{Sp}(v, T(v), T^2(v), \ldots) \end{aligned}$$

Say $Z(v, T)$ the T-cyclic subspace of V generated by v.
$Z(v, T)$ is $T-$invariant. Write T_v
Definition - T-annihilator of v and $Z(v, T)$

Considering, $v, T(v), T^2(V), \ldots$ with $T^k(v)$ first vector in span of previous ones

$$\implies T^k(v) = -a_0 v - a_1 T(v) - \cdots - a_{k-1} T(v)$$

$T-$annihilator of v and $Z(v, T)$ is

$$m_v(x) = x^k + a_{k-1} x^k + \cdots + a_0 \in F[x]$$

This is monic polynomial of smallest degree s.t $m_v(T)(v) = 0$ also with $m_v(T)(w) = 0 \ \forall w \in Z(v, T)$

Theorem 12.2. (Cyclic Decomposition Theorem)

V a finite dimensional vector space over F
$T : V \to V$ a linear map. Suppose $m_T(x) = f(x)^k$ for irreducible $f(x) \in F[x]$
$\implies \exists v_1, \ldots, v_r \in V$ s.t

$$V + Z(v_1, T) \oplus \cdots \oplus Z(v_r, T)$$

where

(1) each $Z(v_i, T)$ has T-annihilator $f(x)^{k_i}$ for $1 \leq i \leq r, \ k = k_1 \geq k_2 \geq \cdots \geq k_r$

(2) r and $k_1, \ldots, k_r$ uniquely determined by T

Corollary 12.3
T a finite dimensional vector space over F
$\implies \exists$ basis B of V s.t

$$[T]_B = C(f(x)^{k_1}) \oplus \cdots \oplus C(f(x)^{k_r})$$

Corollary 12.3
$A \in M_n(F)$, with $m_A(x) = x^k$

$$\implies A \sim C(x^{k_1} \oplus \cdots \oplus C(x^{k_r})$$

Theorem 12.5. (Rational Canonical Form Theorem)

V be finite dimensional over field F with $T : V \to V$ a linear map with

$$m_T(x) = \prod_{i=1}^{t} f_i(x)^{k_i}$$

with $\{f_i(x)\}_{i=1}^t \in F[x]$ set of distinct irreducible polynomials $\implies \exists$ basis B of V s.t

$$[T]_B = C(f_1(x)^{k_{11}}) \oplus \cdots \oplus C(f_1(x)^{k_{1r_1}}) \oplus \dots$$
$$\oplus C(f_t(x)^{k_{t1}}) \oplus \cdots \oplus C(f_t(x)^{k_{tr_t}})$$

where for each i

$$k_i = k_{i1} \geq \cdots \geq k_{ir_i}$$

with r_i and $k_{i1}, \dots, k_{ir_i}$ uniquely determined by T

Corollary 12.6

$A \in M_n(F)$ s.t $m_A(x) = \prod_{i=1}^t f_i(x)^{k_i}$ distinct irreducible polynomials.
$\implies A \sim C(f_1(x)^{k_{11}}) \oplus \cdots \oplus C(f_1(x)^{k_{1r_1}}) \oplus \cdots \oplus C(f_t(x)^{k_{t1}}) \oplus \cdots \oplus C(f_t(x)^{k_{tr_t}})$
Computing the RCF $T : V \to V$ we have

$$c_T(x) = \prod_{i=1}^{t} f_i(x)^{n_i}, \quad m_T(x) = \prod -i = 1^t f_i(x)^{k_i}$$

$\{f_i(x)\}$ all distinct irreducible polynomials in $F[x]$
enough to find; $rank(f_i(T)^r) \forall i \in \{1, \dots, t\}, 1 \leq r \leq k_i$

13 The Dual Space

Definition - Linear functional
V a vector space over F
A **linear functional** on V a linear map $\phi : V \to F$ s.t

$$\phi(\alpha v_1 + \beta v_2) = \alpha\phi(v_1) + \beta\phi(v_2) \qquad \forall v_i \in V, \forall \alpha, \beta \in F$$

Operations of linear functionals

(i) $(\phi_1 + \phi_2)(v) = \phi_1(v) + \phi_2(v), \qquad \forall v \in V$

(ii) $(\lambda\phi)(v) = \lambda\phi(v), \qquad \forall \lambda \in F, \forall v \in V$

Definition - **The dual space**

$$V^* = \{\phi | \phi : V \ toF \text{ a linear functional } \}$$

V^* a vector space over F w.r.t above multiplication and addition.
Dimension
$\{v_i\}_i$ a basis of V with eigenvalues $\{\lambda\}_i$
$\exists! \phi \in V^*$ sending $v_i \to \lambda_i$

$$\phi(\sum \alpha_i v_i) = \sum \alpha_i \lambda_i$$

Proposition 13.1
Let $n = dimV$ with $\{v_1, \ldots, v_n\}$ a basis of V
$\forall i$ define $\phi_i \in V^*$ by

$$\phi_i(v_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$\implies \phi_i(\sum \alpha_j v_j) = \alpha_i \implies \{\phi_1, \ldots, \phi_n\}$ a basis of V^* the **dual basis** of B
$dimV^* = n = dimV$
Definition - Annihilators
V a finite dimensional vector space over F and V^* the dual space. $X \subset V$. Say annihilator $X^0 of X$:

$$X^0 = \{\phi \in V^* : \phi(x) = 0 \forall x \in X\}$$

X^0 a subspace of V^*
Proposition 13.2.
W subspace of $V \implies dimW^0 = dimV - dimW$

14 Inner Product Spaces

Definition - Inner Product
$F = \mathbb{R}$ or $\mathbb{C}$. V a vector space over F
Inner product on V a map $(u, v) : V \times V \to F$ satisfying

(i) $({}_1 v_1 +_2 v_2, w) =_1 (v_1, w) + (v_2, w)$

(ii) $(w, v) = (\bar{w, v})$

(iii) $(v, v) > 0$ if $v \neq 0$

$\forall v_i, v, w \in V$ and ${}_i \in F$. Call such a vector space V with inner product $(,)$ an **inner product space**.

Properties of Inner Product Space

- right-linear for $F = \mathbb{R}$; $(v,_1 w_1 +_2 w_2) = \bar{}_1(v, w_1) + \bar{}_2(v, w_2)$
- $(v, v) \in \mathbb{R}$
- $(0, v) = 0 \forall v \in V$
- symmetry; $F = \mathbb{R} \implies (w, v) = (v, w)$
- $(v, w) = (v, x) \forall v \in V \implies w = x$

Matrix of an inner product V a finite dimensional inner product space. $B = \{v_1, \ldots, v_n\}$ a basis.
Defining $a_{ij} = (v_i, v_j)$. So we have $a_{ji} = \bar{a_{ij}}$

F $\mathbb{R} \implies A$ symmetric

F $\mathbb{C} \implies A$ hermitian

$v, w \in V \implies (v, w) = [v]_B^T A [\bar{w}]_B$

Definition - Positive definite
Hermitian matrix A positive-definite if $x^T A \bar{x} > 0$ $\forall$ non-zero $x \in F^n$

Proposition 14.1
For $u, v, w \in V$ we have

(i) $|(u, v)| \leq ||u|| ||v||$ *(Cauchy-Schwarz Inequality)*

(ii) $||u + v|| \leq ||u|| + ||v||$

(iii) $||u - v|| \leq ||u - w|| + ||w - v||$ *(Triangle inequalities)*

Dual Space
Let V an inner product space over $F = \mathbb{R}$ or $\mathbb{C}$
$v \in V$ define

$$f_v : V \to F$$

$$f_v(w) = (w, v)$$

$\implies f_v$ linear functional $\in V*$

Definition - $\bar{V}$
$\bar{V}$ has same vectors as V

- Addition in $\bar{V}$ same as V
- Scalar multiplication; $\lambda * v = \bar{}v$

Proposition 14.2.
V finite-dimensional. Define $\pi : \bar{V} \to V*$ as

$$\pi(v) = f_v \quad \forall v \in V$$

$\implies \pi$ a vector space isomorphism

Definition - Orthogonality
$\{v_1, \ldots, v_k\}$ orthogonal if $(v_i, v_j) = 0$ $\forall i, j$ $i \neq j$
Orthonormal if also $||v_i|| = 1$ $\forall i$

Definition - $W^\perp$
$W \subseteq V$ define

$$W^\perp = \{u \in V : (u, w) = 0 \ \forall w \in W\}$$

Proposition
V a finite dimensional inner product space. $W \leq V$

$$\implies V = W \oplus W^\perp$$

Theorem 14.5
V a finite dimensional inner product space

(i) V has orthonormal basis

(ii) Any orthonormal set of vectors $\{w_1, \ldots, w_r\}$ can be extended to orthonormal basis of V

Gram-Schmidt Process

***Step 1* -** Start with basis $\{v_1, \ldots, v_n\}$ of V

***Step 2* -** let $u_1 = \frac{v_1}{||v_1||}$ define $w_2 = v_2 - (v_2, u_1)u_1$
$\implies (w_2, u1) = 0, \quad$ let $u_2 = \frac{w_2}{||w_2||}$
$\implies \{u_1, u_2\}$ orthonormal

***Step 3* -** Let

$$w_3 = v_3 - (v_3, u_1)u_1 - (v_3, u_2)u_2$$

With $u_3 = \frac{w_3}{||w_3||} \implies \{u_1, u_2, u_3\}$

***Step 4* -** Continue, for i^{th} step

$$u_i = \frac{w_i}{||w_i||} \quad w_i = v_i - (v_i, u_1)u_1 - \cdots - (v_i, u_{i-1})u_{i-1}$$

Yielding after n steps an orthonormal basis $\{u_1, \ldots, u_n\}$ with

$$\text{Sp}(u_1, \ldots, u_i) = \text{Sp}(v_1, \ldots, v_i) \quad \forall i \in \{1, \ldots, n\}$$

Projections
V an inner product space. $v, w \in V \backslash 0$
Projection of v along w defined to be w for $\frac{(v,w)}{(w,w}$.
For $W \leq V, v \in V$
define projection of V along W as follows:

$$V = W \oplus W^{\perp}$$
$$v = w + w' \quad \text{for unique } w \in W, w' \in W^{\perp}$$

Define **orthogonal projection** map along W.

$$\pi_W : V \to W$$
$$\pi_W(v) = w$$

Proposition 14.7.
V an inner product space. $W \leq V$ with π_W orthogonal projection map along W.

(i) $v \in V \implies \pi_W$ vector in W closest to V
i.e for $w \in W$, $||w - v||$ minimum for $w = \pi_W(v)$

(ii) $\text{dist}(v, w)$ denotes shortest distance from v to any vector in W
$\implies \text{dist}(v, w) = ||v - \pi_W(v)$

(iii) $\{v_1, \ldots, v_r\}$ orthonormal basis of W
$\implies \pi_W(v) = \sum_{j=1}^{r}(v, v_j)v_j$

Change of orthonormal basis

Proposition 14.8
V an inner product space. $E = \{e_1, \ldots, e_n\}$, $F = \{f_1, \ldots, f_n\}$ orthonormal basis of V
$P = (p_{ij})$ change of basis matrix.

$$f_i = \sum_{j=1}^{n} p_{ji}e_j \implies P^T\bar{P} = I$$

Definition

- $P \in M_n(\mathbb{R}) : P^TP = I \implies$ orthogonal matrix
- $P \in M_n(\mathbb{C}) : P^T\bar{P} = I \implies$ unitary matrix

Properties of the above matrices

(i) length-preserving maps of $\mathbb{R}^n, \mathbb{C}^n$ (isometries)
i.e $||Pv|| = ||v|| \quad \forall v$

(ii) Set of all isometries form a group - *classical group*
orthogonal group; $O(n, \mathbb{R}) = \{P \in M_n(\mathbb{R}) : P^TP = I\}$
Unitary Group; $U(n, \mathbb{C}) = \{P \in M_n(\mathbb{C}) : P^T\bar{P} = I$

15 Linear maps on inner product spaces

Proposition 15.1.
V a finite dimensional inner product space. $T : V \to V$ a linear map
$\implies \exists!$ linear map $T^* : V \to V$ s.t $\forall u, v \in V$

$$(T(u), v) = (u, T^*(v))$$

Say T^* - **adjoint of T**
T **self-adjoint** if $T = T^*$

Proposition 15.2.
V an inner product space with orthonormal basis $E = \{v_1, \ldots, v_n\}$
$T : V \to V$ a linear map, $A = [T]_E$
$\implies [T^*]_E = \bar{A}^T$ if field $\mathbb{R} \implies A$ symmetric, if field $\mathbb{C} \implies A$ hermitian

Theorem 15.3. Spectral Theorem
V an inner product space. $T: V \to V$ a self-adjoint linear map $\implies V$ has orthonormal basis of T-eigenvectors.

Corollary 15.4.

- $A \in M_n(\mathbb{R}) \implies \exists$ orthogonal P s.t $P^{-1}AP$ diagonal
- $A \in M_n(\mathbb{C}) \implies \exists$ unitary P s.t $P^{-1}AP$ diagonal

Lemma 15.5.
$T: V \to V$ self-adjoint

(i) eigenvalues of T real

(ii) eigenvectors for distinct eigenvalues, orthogonal to each other

(iii) If $W \subseteq V$, $T-$invariant $\implies W^{\perp}$ is also $T-$invariant

16 Bilinear & Quadratic Forms

Definition. - Bi-linear form
V a vector space over F
Bi-linear form on V a map; $(,): V \times V \to F$ which is both right and left-linear.
i.e $\forall \alpha, \beta \in F$

- $(\alpha v_1 + \beta v_2, w) = \alpha(v_1, w) + \beta(v_2, w)$
- $(v, \alpha w_1 + \beta w_2) = \alpha(v, w_1) + \beta(v, w_2)$

General example
F a field, $V = F^n$ with $A \in M_n(F)$
$\implies (u, v) = u^T A v \quad \forall u, v \in V$ a bilinear form on V
Matrices
$(,)$ a bilinear form on finited dimensional vector space V. With $B = \{v_1, \ldots, v_n\}$
A matrix of $(,)$ w.r.t B, So $(a_{ij}) = (v_i, v_j) \implies \forall u, v \in V \ (u, v) = [u]_B^T A[v]_B$

Definition - Symmetric & Skew-symmetric
Bilinear form $(,)$ on V is

- **Symmetric** if $(u, v) = (v, u) \ \forall u, v \in V$
- **Skew symmetric** if $(v, u) = -(u, v) \ \forall u, v \in V$

Definition - Characteristic of Field F
char of field F is the smallest $n \in \mathbb{N}_+$ s.t $n = 0$. if no such n exists say $char(F) = 0$

Lemma 16.1.
V a vector space over F with $char(F) \neq 2$
$(,)$ skew-symmetric bilinear form on $V \implies (v, v) = 0 \ \forall v \in V$

$$(v, v) = -(v, v) \implies 2(v, v) = 0 \iff 2 = 0 \text{ or } (v, v) = 0$$

Orthogonality
Theorem 16.2
bilinear form $(,)$ has property that

$$(v, w) = 0 \iff (w, v) = 0$$
$$\iff$$
$$(,) \text{ skew-symmetric or symmetric}$$

Definition - Non-degenerate
$(,)$ on V **non-degenerate** if $V^{\perp} = \{0\}$. Where $V^{\perp}$ defined analogously w.r.t bilinear forms.

$$\forall u \in V, \ (u, v) = 0 \forall v \in V \implies u = 0$$

$V^\perp = \{0\} \iff$ matrix of $(,)$ w.r.t a basis is invertible.
Dual Space

Proposition 16.3.
Suppose $(,)$ non-degenerate bilinear form on a finite dimensional vector space V.

(i) $v \in V$ define $f_v \in V^*$
$f_v(u) = (v, u) \quad \forall u \in V$
$\implies \phi : V \to V^*$ mapping $v \mapsto f_v$ $(v \in V)$ an isomorphism

(ii) $\forall W \leq V$ we have $dim(W^\perp) = dim(V) - dim(W)$

Bases
Definition
$A, B \in M_n(F)$ **congruent** if $\exists$ invertible $P \in M_n(F)$ s.t

$$B = P^T AP$$

A, B congruent $\implies$ bilinear forms $(u, v)_1 = u^T Av$ and $(u, v)_2 = u^T Bv$ are **equivalent**

Skew-symmetric bilinear forms
Theorem 16.4.
V a finite dimensional vector space over F where $char(F) \neq 2$
$(,)$ non-degenerate skew-symmetric bilinear form on V. Then

(i) $dim(V)$ even

(ii) $\exists$ basis $B = \{e_1, f_1, \dots, e_m, f_m\}$ of V
s.t matrix of $(,)$ w.r.t B is a block-diagonal matrix

$$J_m = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{m \text{ blocks}}$$

So that $(e_i, f_i) = -(f_i, e_i) = 1$
$(e_i, e_j) = (f_i, f_j) = (e_i, f_j) = (f_j, e_i) = 0 \quad \forall i \neq j$

Corollary 16.5.
If A invertible skew-symmetric $n \times n$ matrix over F where $char(F) \neq 2 \implies n$ even and A congruent to J_m

Symmetric bilinear forms

Theorem 16.6.
V a finite dimensional vector space over F where $char(F) \neq 2$
$(,)$ a non-degenerate symmetric bilinear form on V
$\implies V$ has orthogonal basis $B = \{v_1, \dots, v_n\}$

$$(v_i, v_j) = 0 \quad \text{for } i \neq j$$
$$(v_i, v_i) = \alpha_i \neq 0 \quad \forall i$$

Matrix of $(,)$ w.r.t $B = diag(\alpha_1, \dots, \alpha_n)$

Corollary 16.7.
A invertible symmetric matrix over $F, char(F) \neq 2$
$\implies A$ congruent to diagonal matrix
Computing orthogonal basis for 16.6

1. find v_1 s.t $(v_1, v_1) \neq 0$
2. Compute $v_1^\perp$ and find $v_2 \in v_1^\perp$ s.t $(v_2, v_2) \neq 0$
3. Compute $Sp(v_1, v_2)^\perp$ and find $v_3 \in Sp(v_1, v_2)^\perp$ s.t $(v_3, v_3) \neq 0$
4. Continue until you get orthogonal basis

Quadratic Form

Assume from now F s.t $char(F) \neq 2$, V a finite dimensional vector space over F

Definition - Quadratic form
Quadratic form on V a map $Q : V \to F$ of form

$$Q(v) = (v, v) \qquad \forall v \in V$$

$(,)$ a symmetric bilinear form on V
Q non-degenerate if $(,)$ non-degenerate.

Remarks

(i) given Q we find $(u, v) = \frac{1}{2}[Q(u + v) - Q(u) - Q(v)]$

(ii) $V = F^n$ every symmetric bilinear forms s.t

$$(x, y) = x^T Ay \qquad \text{for } A = A^T, (x, y \in V)$$

For $\mathbf{x} = (x_1, \dots, x_n)^T$

$$\begin{aligned} Q(x) &= x^T Ax \\ &= \sum_{i,j} a_{ij} x_i x_j \\ &= \sum_{i=1}^{n} a_{ii} x_i^2 + 2 \sum i < j a_{ij} x_i x_j \end{aligned}$$

A general homogeneous quadratic polynomial in $x_1, \dots, x_n$ (all terms of degree 2)

Change of variables

Definition - Equivalent Quadratic Forms

$V = F^n,\ Q : V \to F$
$Q(x) = x^T Ax\ \forall x \in V, A$ symmetric
Take $y = (y_1, \dots, y_n)^T$ s.t $x = Py$ for P invertible
$\implies Q(x) = y^T P^T APy = Q'(y)$
If such a P exists we say Q, Q' **equivalent**
note:
Congruent matrices $A, P^T AP$
$A \sim P^T AP \iff P$ orthogonal

Theorem 16.8.
$V = F^n$, $Q : V \to F$ non-degenerate quadratic form

(i) if $F = \mathbb{C} \implies Q$ equivalent to form

$$Q_0(x) = x_1^2 + \cdots + x_n^2 \quad (x \in \mathbb{C}^n)$$

Has matrix I_n

(ii) if $F = \mathbb{R} \implies Q$ equivalent to unique $Q_{p,q}; p + q = n$

$$Q_{p,q}(x) = x_1^2 + \cdots + x_p^2 - (x_{p+1}^2 + \cdots + x_{p+q}^2) \quad (x \in \mathbb{C}^n)$$

Has matrix $I_{p,q} = \begin{pmatrix} \mathrm{I}_p & 0 \\ 0 & \text{-}\mathrm{I}_q \end{pmatrix}$

(iii) if $F = \mathbb{Q} \implies \exists$ infinitely many inequivalent non-degenerate quadratic forms on $\mathbb{Q}^n$

Definition - isometry
$f = (,)$ a non-degenerate symmetric/skew-symmetric bilinear form on finite dimensional vector space V
Isometry of f a linear map $T : V \to V$ s.t

$$(T(u), T(v)) = (u, v) \quad \forall u, v \in V$$

T invertible since f non-degenerate.
Definition - Isometry Group

$$I(V, f) = \{T : T \text{ an isometry }\}$$

forms a subgroup of general linear group $GL(V)$
Equivalently;
fix basis B of V, A matrix of f w.r.t B if $[T]_B = X \implies T \in I(V, f) \iff X^T AX = A$

$$\implies I(v, f) \cong \{X \in GL(n, F) : X^T AX = A\}$$

- f skew-symmetric $\implies$ there is only one form (up to equivalence) so we get one isometry group; Classical ***symplectic group*** $\mathrm{Sp}(V, f)$
- f symmetric $\implies$ there are many forms, forming the isometry groups; the classical ***orthogonal groups*** $O(V, f)$