1. (a) Show that $f(x) = x^{1/2}$ is differentiable on $(0, \infty)$, and compute its derivative.
   (b) Do the same for $f(x) = x^{1/n}$, where n is any positive integer.
   (c) Now do the same for $f(x) = x^{m/n}$, where m and n are positive integers.

Solution. (a) We observe for $x, a > 0$ that $x - a = (x^{1/2} - a^{1/2})(x^{1/2} + a^{1/2})$, so

$$\lim_{x\to a} \frac{x^{1/2} - a^{1/2}}{x - a} = \lim_{x\to a} \frac{1}{x^{1/2} + a^{1/2}} = \frac{1}{2a^{1/2}}.$$

The last step implicitly uses the fact that $x^{1/2}$ is continuous for $x > 0$, but this follows from it being the inverse of the continuous function x^2 on the same interval. Therefore f is differentiable for all $x > 0$, with $f'(x) = \frac{1}{2}x^{-1/2}$.

(b) Each $x^{1/n}$ is continuous for $x > 0$, since it is the inverse of the continuous function x^n. We apply the identity

$$\begin{aligned} x - a &= (x^{1/n})^n - (a^{1/n})^n \\ &= (x^{1/n} - a^{1/n})(x^{(n-1)/n} + x^{(n-2)/n}a^{1/n} + \cdots + x^{1/n}a^{(n-2)/n} + a^{(n-1)/n}) \end{aligned}$$

to write

$$\lim_{x\to a} \frac{x^{1/n} - a^{1/n}}{x - a} = \lim_{x\to a} \frac{1}{\sum_{i=0}^{n-1} x^{(n-1-i)/n}a^{i/n}} = \frac{1}{na^{(n-1)/n}} = \frac{a^{(1-n)/n}}{n}.$$

So $f(x) = x^{1/n}$ has derivative $f'(x) = \frac{1}{n}x^{1/n-1}$.

(c) At this point we might well guess that $x^{m/n}$ should have derivative $\frac{m}{n}x^{m/n-1}$, and we can prove it by induction. We've already done the case $m = 1$. If the claim holds for exponent $\frac{m-1}{n}$ then we let $f(x) = x^{(m-1)/n}$ and $g(x) = x^{1/n}$, and apply the product rule to $h(x) = f(x)g(x) = x^{m/n}$ to see that $x^{m/n}$ is differentiable on $(0, \infty)$ with derivative

$$\begin{aligned} h'(x) &= f'(x)g(x) + f(x)g'(x) \\ &= \left(\frac{m-1}{n}x^{(m-1)/n-1}\right) x^{1/n} + x^{\frac{m-1}{n}} \left(\frac{1}{n}x^{1/n-1}\right) \\ &= \frac{m-1}{n}x^{m/n-1} + \frac{1}{n}x^{m/n-1} \\ &= \frac{m}{n}x^{m/n-1}. \end{aligned}$$

This proves the claim by induction on m.

2. A function $f : \mathbb{R} \to \mathbb{R}$ is called *Hölder continuous* with exponent $\alpha > 0$ if there is a constant $C \geq 0$ such that

$$|f(x) - f(y)| \leq C|x - y|^\alpha$$

for all $x, y \in \mathbb{R}$. Show that if $\alpha > 1$ then f is differentiable, and $f'(x) = 0$.

Remark: We will see in lecture soon that if $f' \equiv 0$ then f must be constant.

Solution. For $x \neq y$ we can write

$$\left|\frac{f(x)-f(y)}{x-y}\right| \leq C|x-y|^{\beta},$$

where $\beta = \alpha - 1$ is strictly positive. But then $\lim_{x\to y} |x-y|^{\beta} = 0$ for any fixed y, so on the left side we must have

$$\lim_{x\to y}\left|\frac{f(x)-f(y)}{x-y}\right| = 0$$

and this imeans that $f'(y)$ exists and is zero.

3. Find all $x \in \mathbb{R}$ where $f(x) = \begin{cases} 0, & x \notin \mathbb{Q} \\ x^2, & x \in \mathbb{Q} \end{cases}$ is differentiable and compute its derivative.

   Solution. We know that $f(x)$ is not continuous at any nonzero $x = a$, because we can find a sequence of rationals $r_n \to a$ with $f(r_n) = r_n^2 \to a^2$ and a sequence of irrationals $s_n \to a$ with $f(s_n) = 0 \to 0$, and these limits are not equal, whereas if f were continuous at a then they would have both been equal to $f(a)$. Since f is continuous at all points where it is differentiable, it cannot be differentiable anywhere except possibly at $x = 0$.

   On the other hand, for nonzero x we have

$$\frac{f(x)-f(0)}{x-0} = \frac{f(x)}{x} = \begin{cases} 0, & x \notin \mathbb{Q} \\ x, & x \in \mathbb{Q} \end{cases}.$$

   Regardless of whether $x \neq 0$ is rational or irrational, it follows that

$$\left|\frac{f(x)-f(0)}{x-0}\right| \leq |x| \text{ for all } x \neq 0 \implies \lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = 0.$$

   So f is differentiable at x if and only if $x = 0$, and $f'(0) = 0$.

4. (a) Show, using $\sin(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$ and $\cos(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$, that

$$\lim_{x\to 0}\frac{\sin(x)}{x} = 1 \quad \text{and} \quad \lim_{x\to 0}\frac{1-\cos(x)}{x} = 0.$$

   (b) Use the angle addition formulas to prove that $\sin(x)$ and $\cos(x)$ are differentiable and determine their derivatives. (Note: you may *not* just differentiate the power series term by term, because we have not yet proved that this gives the right answer.)

Solution. (a) We observe for $x \neq 0$ that

$$\frac{\sin(x)}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots = 1 - x^2 \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+3)!} \right).$$

The power series in parentheses has infinite radius of convergence (why?), so it defines a continuous function with value $\frac{1}{6}$ at $x = 0$ and hence

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1 - 0^2 \cdot \frac{1}{6} = 1$$

by the algebra of limits. Similarly for the cosine expression, when $x \neq 0$ we have

$$\frac{1 - \cos(x)}{x} = \frac{x}{2!} - \frac{x^3}{4!} + \frac{x^5}{6!} - \cdots = x \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+2)!} \right),$$

and again the sum in parentheses is continuous, with value $\frac{1}{2}$ at $x = 0$, so

$$\lim_{x \to 0} \frac{1 - \cos(x)}{x} = 0 \cdot \frac{1}{2} = 0.$$

(b) We compute the derivative of $\sin(x)$ by

$$\begin{aligned}
\lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h} &= \lim_{h \to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} \\
&= \lim_{h \to 0} \left(\sin(x) \left(\frac{\cos(h) - 1}{h} \right) + \cos(x) \left(\frac{\sin(h)}{h} \right) \right) \\
&= \sin(x) \cdot 0 + \cos(x) \cdot 1,
\end{aligned}$$

and so $\sin(x)$ has derivative $\cos(x)$. Similarly for the derivative of $\cos(x)$:

$$\begin{aligned}
\lim_{h \to 0} \frac{\cos(x+h) - \cos(x)}{h} &= \lim_{h \to 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h} \\
&= \lim_{h \to 0} \left(\cos(x) \left(\frac{\cos(h) - 1}{h} \right) - \sin(x) \left(\frac{\sin(h)}{h} \right) \right) \\
&= \cos(x) \cdot 0 - \sin(x) \cdot 1,
\end{aligned}$$

so $\cos(x)$ is differentiable, with derivative $-\sin(x)$.

5. Recall that we defined $\log : (0, \infty) \to \mathbb{R}$ as the inverse function of e^x.

(a) Using only this and formal properties of e^x, prove for $x > 0$ and $0 < |h| < x$ that

$$\frac{\log(x+h) - \log(x)}{h} = \frac{1}{x} \frac{\log\left(1 + \frac{h}{x}\right)}{h/x}.$$

(b) Prove by a substitution that $\lim_{y\to 0} \frac{\log(1+y)}{y} = \lim_{x\to 0} \frac{x}{e^x - 1}$, and that the latter limit is 1. (Hint: use the power series definition of e^x to evaluate $\lim_{x\to 0} \frac{e^x-1}{x}$.)

(c) Show that $\log(x)$ is differentiable, and find its derivative.

Solution. (a) We have the identity $\log(xy) = \log(x) + \log(y)$, which follows from

$$e^{\log(xy)} = xy = e^{\log(x)}e^{\log(y)} = e^{\log(x)+\log(y)},$$

and similarly for all $c \in \mathbb{R}$ we have

$$e^{c\log(x)} = \left(e^{\log(x)}\right)^c = x^c = e^{\log(x^c)} \implies c\log(x) = \log(x^c).$$

Using both of these identities together, we conclude that

$$\frac{\log(x+h) - \log(x)}{h} = \frac{1}{x} \cdot \frac{x}{h} \log\left(\frac{x+h}{x}\right) = \frac{1}{x} \frac{\log\left(1+\frac{h}{x}\right)}{h/x}.$$

(b) Letting $x = \log(1+y)$, we have $e^x = 1+y$ and so $\frac{\log(1+y)}{y} = \frac{x}{e^x-1}$. Since log is continuous and injective and $\log(1) = 0$, the condition $y \to 0$ is equivalent to $\log(1+y) \to 0$, so that $\lim_{y\to 0} \frac{\log(1+y)}{y} = \lim_{x\to 0} \frac{x}{e^x-1}$ as claimed.

To evaluate the limit, we write

$$e^x - 1 = \sum_{n=1}^{\infty} \frac{x^n}{n!} = x\left(\frac{1}{1!} + \frac{x}{2!} + \frac{x^2}{3!} + \dots\right),$$

so after some rearranging we have

$$\frac{e^x-1}{x} = 1 + x\left(\sum_{n=0}^{\infty} \frac{x^n}{(n+2)!}\right)$$

for all $x \neq 0$. The power series in parentheses has infinite radius of convergence, so it defines a continuous function, and as $x \to 0$ we have

$$\lim_{x\to 0} \frac{e^x-1}{x} = 1 + 0 \cdot \frac{1}{2} = 1.$$

Now the algebra of limits tells us that $\lim_{x\to 0} \frac{x}{e^x-1} = \frac{1}{1} = 1$.

(c) From part (a), the derivative of $\log(x)$ is given by

$$\lim_{h\to 0} \frac{\log(x+h)-\log(x)}{h} = \lim_{h\to 0} \frac{1}{x} \frac{\log\left(1+\frac{h}{x}\right)}{h/x}.$$

For fixed $x > 0$, we can substitute $y = \frac{h}{x}$, and then $h \to 0$ is equivalent to $y \to 0$, so we have

$$\lim_{h\to 0} \frac{\log(x+h)-\log(x)}{h} = \lim_{y\to 0} \frac{1}{x} \frac{\log(1+y)}{y} = \frac{1}{x}.$$

Thus $\log(x)$ has derivative $\frac{1}{x}$.

6. ($*$) Let $f : [a, b] \to \mathbb{R}$ be a differentiable function. We will prove that $f'(x)$ has the *intermediate value property* even though it may not be continuous. In both parts we will suppose that $f'(a) < f'(b)$ and fix some t such that $f'(a) < t < f'(b)$.

   (a) Let $g(x) = f(x) - tx$. Prove that there is some $c \in (a, b)$ such that $g(c) < g(a)$. (Hint: what is $g'(a)$?) Similarly, prove that there is some $d \in (a, b)$ such that $g(d) < g(b)$. In other words, $g(x)$ is not minimized at $x = a$ or at $x = b$.

   (b) Show that $g'(y) = 0$ for some $y \in (a, b)$, and deduce that $f'(y) = t$.

Solution. (a) We know that $g(x)$ is differentiable, with $g'(x) = f'(x) - t$, so in particular $g'(a) < 0$. Fixing $\epsilon = |g'(a)| > 0$, there is $\delta > 0$ such that

$$a < x < a + \delta \;\Rightarrow\; \left| \frac{g(x) - g(a)}{x - a} - g'(a) \right| < \epsilon = -g'(a),$$

and since $x - a$ is positive, this implies that

$$\frac{g(x) - g(a)}{x - a} - g'(a) < -g'(a) \;\Rightarrow\; g(x) - g(a) < 0.$$

Thus $g(x) < g(a)$ for all $x \in (a, a + \delta)$ and we can take $c = \min(a + \frac{\delta}{2}, \frac{a+b}{2})$. (The point of taking this minimum is just to make sure that $c \in (a, b)$.)
Similarly, we have $g'(b) > 0$, so for $\epsilon = g'(b)$ we can find $\delta > 0$ such that

$$b - \delta < x < b \;\Rightarrow\; \left| \frac{g(x) - g(b)}{x - b} - g'(b) \right| < \epsilon = g'(b).$$

We deduce from this and the fact that $x - b < 0$ that

$$\frac{g(x) - g(b)}{x - b} - g'(b) \geq -g'(b) \;\Rightarrow\; g(x) - g(b) \leq 0,$$

so $g(x) < g(b)$ for $x \in (b - \delta, b)$ and we can take $d = \max(b - \frac{\delta}{2}, \frac{a+b}{2})$.

(b) We know that g is continuous since it is differentiable, so the extreme value theorem says that $g(x)$ achieves a minimum at some $y \in [a, b]$. By part (a) we know that $y \neq a$ and $y \neq b$, so $y \in (a, b)$, and since y is a local minimum of g it follows that $g'(y) = 0$. Since $g'(x) = f'(x) - t$, we must have $f'(y) = t$.

7. The goal of this problem is to construct a continuous function which is not differentiable anywhere. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = |x|$ for $-1 \leq x \leq 1$ and $f(x + 2) = f(x)$ for all $x \in \mathbb{R}$. Then define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \sum_{i=0}^{\infty} \left(\frac{3}{4}\right)^i f(4^i x).$$

   (a) Draw a graph of $f(x)$, and convince yourself that it is continuous.

   (b) Prove that g is continuous.

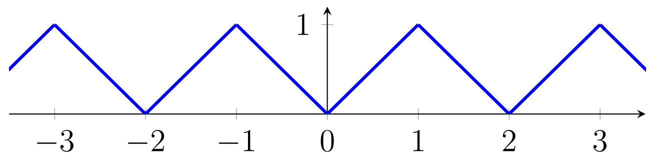
(c) Fix $x \in \mathbb{R}$ and an integer $n \in \mathbb{N}$. Let ϵ_n be $+\frac{1}{2}$ if there is no integer in the interval $(4^n x, 4^n x + \frac{1}{2})$, or $-\frac{1}{2}$ if there is no integer in $(4^n x - \frac{1}{2}, 4^n x)$. Check that one of these is always possible, and then define

$$d_i(x) = \frac{f(4^i(x + \frac{\epsilon_n}{4^n})) - f(4^i x)}{\epsilon_n / 4^n}.$$

Show that $|d_i(x)| = 4^i$ for all $i \le n$, and that $d_i(x) = 0$ for all $i > n$.

(d) Prove that $\left|\frac{g(x+\frac{\epsilon_n}{4^n}) - g(x)}{\epsilon_n/4^n}\right| \ge 3^n - (3^{n-1} + 3^{n-2} + \cdots + 1) = \frac{3^n + 1}{2}$. Conclude that g is not differentiable at x.

Solution. (a)



(b) We let $M_i = \left(\frac{3}{4}\right)^i$ for all i. Then each summand $(\frac{3}{4})^i f(4^i x)$ of g is continuous and satisfies the bound $|(\frac{3}{4})^i f(4^i x)| \le |(\frac{3}{4})^i| = M_i$, and the sum $\sum_{i=0}^{\infty} M_i$ converges, so the Weierstrass M-test proves that g is continuous.

(c) The open interval $(4^n x - \frac{1}{2}, 4^n x + \frac{1}{2})$ cannot contain two integers because their difference would be strictly less than one, so in particular the disjoint subintervals $(4^n x - \frac{1}{2}, 4^n x)$ and $(4^n x, 4^n x + \frac{1}{2})$ cannot both contain integers. This justifies the claim that we can choose ϵ_n.

Since there are no integers between $4^n(x + \frac{\epsilon_n}{4^n}) = 4^n x + \epsilon_n$ and $4^n x$, the graph of f between these is a straight line segment of slope ± 1. It follows that

$$\left| f\left(4^n\left(x + \frac{\epsilon_n}{4^n}\right)\right) - f(4^n x)\right| = \left|4^n\left(x + \frac{\epsilon_n}{4^n}\right) - 4^n x\right| = |\epsilon_n|,$$

and so $|d_n(x)| = \frac{|\epsilon_n|}{|\epsilon_n/4^n|} = 4^n$.

In fact, for $0 \le i < n$ the interval between $4^i(x + \frac{\epsilon}{4^n})$ and $4^i x$ cannot contain an integer either, because if it did have some $m \in \mathbb{Z}$ then $4^{n-i}m$ would have been an integer in the interval from $4^n(x + \frac{\epsilon}{4^n})$ to $4^n x$. So the same argument says that

$$|d_i(x)| = \frac{\left|f\left(4^i\left(x + \frac{\epsilon_n}{4^n}\right)\right) - f(4^i x)\right|}{|\epsilon_n|/4^n} = \frac{\left|4^i\left(x + \frac{\epsilon_n}{4^n}\right) - 4^i x\right|}{|\epsilon_n|/4^n} = 4^i.$$

For all $i > n$, however, the difference $4^i(x + \frac{\epsilon_n}{4^n}) - 4^i x = 4^{i-n}\epsilon_n = 4^{i-n}(\pm\frac{1}{2})$ is an even integer, and f is periodic with period 2, so $f(4^i(x + \frac{\epsilon_n}{4^n})) = f(4^i x)$ and hence $d_i(x) = 0$.

(d) Recalling the definition of g, we have

$$\begin{aligned}
\left|\frac{g(x+\frac{\epsilon_n}{4^n})-g(x)}{\epsilon_n/4^n}\right| &= \left|\sum_{i=0}^{\infty}\left(\frac{3}{4}\right)^i \frac{f(4^i(x+\frac{\epsilon_n}{4^n}))-f(4^i x)}{\epsilon_n/4^n}\right| \\
&= \left|\sum_{i=0}^{\infty}\left(\frac{3}{4}\right)^i d_i(x)\right| \\
&= \left|\sum_{i=0}^{n}\left(\frac{3}{4}\right)^i d_i(x)\right| \\
&\geq \left|\left(\frac{3}{4}\right)^n d_n(x)\right| - \sum_{i=0}^{n-1}\left|\left(\frac{3}{4}\right)^i d_i(x)\right|.
\end{aligned}$$

In the last two steps we have used part (c) to throw away all terms where $i > n$, followed by the triangle inequality. Using part (c) again, we can simplify to

$$\begin{aligned}
\left|\frac{g(x+\frac{\epsilon_n}{4^n})-g(x)}{\epsilon_n/4^n}\right| &\geq 3^n - \sum_{i=0}^{n-1} 3^i \\
&= 3^n - \frac{3^n-1}{2} = \frac{3^n+1}{2}.
\end{aligned}$$

Since $x+\frac{\epsilon_n}{4^n} \to x$ as $n \to \infty$, if g were differentiable at x then we would have

$$\lim_{n\to\infty} \frac{g(x+\frac{\epsilon_n}{4^n})-g(x)}{\epsilon_n/4^n} = g'(x).$$

But this limit does not exist because the nth term has absolute value at least $\frac{3^n+1}{2} \to \infty$, so g is not differentiable at x.