

Mathematics Year 1, Calculus and Applications I

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Solutions Problem Sheet 6

1. Let $\{r_n\}$ denote the rational numbers in the interval $(0, 1)$ arranged in the sequence whose first few terms are $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \dots$. Prove that the series $\sum_{n=1}^{\infty} r_n$ diverges.

Solution $\sum_{n=1}^{\infty} r_n$ diverges because r_n does not tend to zero as $n \rightarrow \infty$. In fact $r_n \rightarrow 1$ as $n \rightarrow \infty$.

2. Determine the convergence or divergence of the following infinite series:

$$\begin{aligned} (a) \quad & \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \quad (b) \quad \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} 5^n \quad (c) \quad \sum_{n=1}^{\infty} \left(\frac{n}{n+1} \right)^{n^2} \quad (d) \quad \sum_{n=1}^{\infty} \left(\frac{n}{n+1} \right)^{n^2} 4^n \\ (e) \quad & \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}} \quad (f) \quad \sum_{n=1}^{\infty} \frac{1}{n} (\sqrt{n+1} - \sqrt{n}) \quad (g) \quad \sum_{n=2}^{\infty} \frac{1}{(\log n)^{\log n}} \\ (h) \quad & \sum_{n=1}^{\infty} \frac{2^n}{(2n+1)!}, \quad (i) \quad \sum_{n=1}^{\infty} \frac{2^{n^2}}{n!}, \quad (j) \quad \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{\sqrt{n}} \right) \end{aligned}$$

Solution

- (a) Ratio test $\frac{a_{n+1}}{a_n} = \frac{((n+1)!)^2 (2n)!}{(2(n+1))! (n!)^2} = \frac{(n+1)^2}{(2n+2)(2n+1)} \rightarrow \frac{1}{4}$ as $n \rightarrow \infty$. Converges.
- (b) As above, only difference is the extra $\frac{5^{n+1}}{5^n}$ factor. Hence $\frac{a_{n+1}}{a_n} \rightarrow \frac{5}{4}$. Diverges.
- (c) Can do this in different ways: (i) Write $a_n = \left[\left(\frac{n}{n+1} \right)^n \right]^n$; defining $x_n = \left(\frac{n}{n+1} \right)^n$, calculate

$$x_n = \left(1 - \frac{1}{n+1} \right)^n = \left(1 - \frac{1}{n+1} \right)^{n+1} \frac{1}{\left(1 - \frac{1}{n+1} \right)} \rightarrow e^{-1} \quad \text{as } n \rightarrow \infty,$$

by definition of e . Hence $a_n \sim e^{-n}$ for large n and the series converges (geometric of ratio $e^{-1} < 1$).

(ii) Alternatively, note that

$$\log(a_n) = n^2 \log \left(\frac{n}{n+1} \right) = n^2 \log \left(1 - \frac{1}{n+1} \right) = n^2 \left(-\frac{1}{n+1} + \dots \right) \rightarrow -n,$$

as $n \rightarrow \infty$ hence $a_n \sim e^{-n}$ and the conclusion is the same as above. [I used Taylor's theorem to expand the log, only the first term is needed.]

- (d) As above, and on including the extra factor 4^n we have $a_n \rightarrow e^{-n} 4^n = (4 - e)^n$. Since $4 - e > 1$, the series diverges.
- (e) Series converges by the alternating series test since $1/\sqrt{n} \rightarrow 0$ as $n \rightarrow \infty$.
- (f) Here $a_n = \frac{1}{n} (\sqrt{n+1} - \sqrt{n}) = \frac{1}{n(\sqrt{n+1} + \sqrt{n})}$. Hence $a_n < \frac{1}{2n^{3/2}}$ for $n \geq 1$, and by comparison with $\int_1^{\infty} \frac{dx}{x^{3/2}}$ we have convergence.

- (g) Use the comparison test with the integral $I := \int_2^\infty \frac{dx}{(\log x)^{\log x}}$. Calculate I first making the substitution $y = \log x$

$$I = \int_{\log 2}^\infty \frac{e^y dy}{y^y} = \int_{\log 2}^\infty \frac{e^y dy}{e^{y \log y}} = \int_{\log 2}^\infty e^{-y(\log y - 1)} dy = \left(\int_{\log 2}^{\log M} + \int_{\log M}^\infty \right) e^{-y(\log y - 1)} dy$$

Now pick the constant M so that $\log M - 1 > 0$ (e.g. $M = 2e$ will do). The first integral is a finite number. The second integral can be estimated as follows

$$\int_{\log M}^\infty e^{-y(\log y - 1)} dy < \int_{\log M}^\infty e^{-y} dy = \frac{1}{M},$$

hence the integral and the series converge.

- (h) Ratio test $\frac{a_{n+1}}{a_n} = \frac{2}{(2n+3)(2n+2)} \rightarrow 0$ as $n \rightarrow \infty$. Converges.
 (i) Ratio test $\frac{a_{n+1}}{a_n} = \frac{1}{n+1} \frac{2^{n^2+2n+1}}{2^{n^2}} = \frac{2^{2n+1}}{n+1} \rightarrow \infty$ as $n \rightarrow \infty$.
 (j) Compare with

$$\lim_{M \rightarrow \infty} \int_1^M \left(\frac{1}{x} - \frac{1}{\sqrt{x}} \right) dx = \lim_{M \rightarrow \infty} (\log M - 2\sqrt{M} + 2) = \infty,$$

hence the series diverges.

3. (a) Prove that the series

$$\sum_{n=1}^\infty \frac{1}{n(n+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots = 1.$$

Use the result to prove that $\sum_{n=1}^\infty \frac{1}{n^2}$ converges, and obtain upper and lower bounds for this sum.

- (b) Find the sum of the series $\sum_{n=1}^\infty \frac{n}{(n+1)!}$.
 (c) Find the sum $\sum_{n=1}^\infty \frac{1+n}{2^n}$. [Hint: Differentiate a certain power series, justifying any operations.]

Solution

- (a) Consider the partial sum $S_N = \sum_{n=1}^N \frac{1}{n(n+1)} = \sum_{n=1}^N \left(\frac{1}{n} - \frac{1}{n+1} \right) = 1 - \frac{1}{N+1}$ (telescoping series). Hence $\sum_{n=1}^\infty \frac{1}{n(n+1)} = \lim_{N \rightarrow \infty} S_N = 1$.

Now

$$\sum_{n=1}^\infty \frac{1}{n(n+1)} > \sum_{n=1}^\infty \frac{1}{(n+1)^2} = \sum_{k=2}^\infty \frac{1}{k^2} = \sum_{n=1}^\infty \frac{1}{n^2} - 1,$$

hence $\sum_{n=1}^\infty \frac{1}{n^2} < 1 + \sum_{n=1}^\infty \frac{1}{n(n+1)} = 2$, i.e. converges with upper bound 2.

To find a lower bound note that $\sum_{n=1}^\infty \frac{1}{n^2} > \sum_{n=1}^\infty \frac{1}{n(n+1)} = 1$.

- (b) Write $\frac{n}{(n+1)!} = \frac{1}{n!} - \frac{1}{(n+1)!}$, hence the partial sum

$$S_N := \sum_{n=1}^N \frac{n}{(n+1)!} = \sum_{n=1}^N \left(\frac{1}{n!} - \frac{1}{(n+1)!} \right) = 1 - \frac{1}{(N+1)!} \rightarrow 1 \quad \text{as } N \rightarrow \infty.$$

- (c) Consider the function $f(x) = \frac{1}{1-x}$ and in particular its Binomial expansion which converges absolutely for $|x| < 1$. The Binomial expansion is $f(x) = (1-x)^{-1} = 1+x+x^2+x^3+\dots$, and since the series converges absolutely we can differentiate to find $f'(x) = 1+2x+3x^2+\dots$. In particular

$$f'(1/2) = 1 + 2 \cdot \frac{1}{2} + 3 \cdot \frac{1}{2^2} + \dots = 1 + \sum_{n=1}^{\infty} \frac{n+1}{2^n} \quad (1)$$

But $f'(x) = \frac{1}{(1-x)^2}$ hence $f'(1/2) = 4$. Combining with (1) yields $\sum_{n=1}^{\infty} \frac{n+1}{2^n} = 3$.

4. Suppose that $\{a_n\}$ is a decreasing sequence of positive terms such that $\sum_{n=1}^{\infty} a_n$ converges. Prove that $na_n \rightarrow 0$ as $n \rightarrow \infty$. [Hint - consider the sum $a_{n+1} + a_{n+2} + \dots + a_{2n}$.]

Solution

Consider $a_{n+1} + a_{n+2} + \dots + a_{2n} = \sum_{n=1}^{2n} a_n - \sum_{n=1}^n a_n$. Since the series has positive terms and is decreasing we have the bound

$$a_{n+1} + a_{n+2} + \dots + a_{2n} \geq na_{2n}.$$

But $\left(\sum_{n=1}^{2n} a_n - \sum_{n=1}^n a_n\right) \rightarrow 0$ as $n \rightarrow \infty$ by Cauchy's test for convergence. Hence, by the squeeze theorem $na_{2n} \rightarrow 0$, equivalently $(2n)a_{(2n)} \rightarrow 0$ and hence $na_n \rightarrow 0$ as $n \rightarrow \infty$.

5. (a) For what values of α do the following series converge or diverge

$$(i) \sum_{n=2}^{\infty} \frac{1}{n(\log n)^{\alpha}} \quad (ii) \sum_{n=3}^{\infty} \frac{1}{n \log n (\log \log n)^{\alpha}}$$

- (b) Show that the following series converges

$$\sum_{n=2}^{\infty} \frac{\log(n+1) - \log n}{(\log n)^2}.$$

Solution

- (a) (i) Use the integral comparison test - we need to consider (for $\alpha \neq 1$ at the moment)

$$\int_2^M \frac{dx}{x(\log x)^{\alpha}} = \frac{(\log x)^{1-\alpha}}{1-\alpha} \Big|_2^M = \frac{(\log M)^{1-\alpha}}{1-\alpha} - \frac{\log 2}{1-\alpha},$$

and the improper integral converges as $M \rightarrow \infty$ only if $\alpha > 1$. Need to check $\alpha = 1$, i.e. the integral

$$\int_2^M \frac{dx}{x \log x} = \log(\log M) - \log(\log 2) \rightarrow \infty \quad \text{as } M \rightarrow \infty,$$

hence the integral diverges.

(ii) Again use the integral test hence consider

$$\int_3^M \frac{dx}{(x \log x)(\log \log x)^\alpha} = \frac{(\log \log x)^{1-\alpha}}{1-\alpha} \Big|_3^M,$$

hence convergence as $M \rightarrow \infty$ only if $\alpha > 1$. For $\alpha = 1$

$$\int_3^M \frac{dx}{x \log x (\log \log x)} = \log \log \log M - \log \log \log 3 \rightarrow \infty \quad \text{as } M \rightarrow \infty.$$

- (b) Combine the logarithms in the numerator to write the series as $\sum_2^\infty \frac{\log(1+\frac{1}{n})}{(\log n)^2}$.
Next we show that $\log(1+\frac{1}{n}) < \frac{1}{n}$. Can do this many ways (e.g. $(1+\frac{1}{n})^n < e$ from the definition of the exponential - see Chapter 1) but I will do it using Taylor's theorem that gives

$$\log\left(1 + \frac{1}{n}\right) = \frac{1}{n} - \frac{1}{2} \cdot \frac{1}{n^2} + \frac{1}{3} \cdot \frac{1}{n^3} - \dots \quad (2)$$

The power series converges absolutely for $|1/n| < 1$ and since the series starts with $n = 2$ we are fine. The series (2) is alternating with decreasing terms, hence it is smaller than its first term, i.e. $\log(1+\frac{1}{n}) < \frac{1}{n}$ for $n \geq 2$.

Now we are in a position to use the comparison theorem for series. We have

$$\sum_2^\infty \frac{\log(1+\frac{1}{n})}{(\log n)^2} < \sum_{n=2}^\infty \frac{(1/n)}{(\log n)^2},$$

and the latter sum is convergent by comparison with the integral

$$\int_2^M \frac{dx}{x(\log x)^2} = \frac{1}{\log 2} - \frac{1}{\log M} \rightarrow 0 \quad \text{as } M \rightarrow \infty.$$

6. For what values $p > 0$ does the series $\sum_{n=1}^\infty (1 - \frac{1}{n^p})^n$ converge.

Solution

Recall and use the result $\lim_{k \rightarrow \infty} (1 - \frac{1}{k})^k = e^{-1}$. Write $a_n = (1 - \frac{1}{n^p})^n$. If $p < 0$ then a_n does not tend to zero and the series diverges. If $p = 0$ then the series converges and is equal to 0. Hence $p \geq 0$ is a necessary condition for convergence. Take $p > 0$ and re-write

$$a_n = \left(\left(1 - \frac{1}{n^p}\right)^{n^p} \right)^{n/n^p}.$$

For large n , therefore, we have

$$a_n \sim \left(\frac{1}{e}\right)^{n/n^p} \rightarrow \begin{cases} e^{-1} & p = 1 \\ 1 & p > 1 \\ \exp(-n^{1-p}) & p < 1 \end{cases}$$

Clearly if $p \geq 1$ the series does not converge (the terms do not tend to zero). So the only possibility is $0 \leq p < 1$. We need to prove this. One way to do is to compare with the integral

$$\int_1^\infty \exp(-x^{1-p}) dx = \int_1^\infty \frac{y^{\frac{p}{1-p}}}{1-p} \exp(-y) dy,$$

where the substitution $y = x^{1-p}$ has been used. Now for any $0 < p < 1$ we can find an integer $N > \frac{p}{1-p}$, and so

$$\int_1^\infty \frac{y^{\frac{p}{1-p}}}{1-p} \exp(-y) dy \leq \int_1^\infty y^N \exp(-y) dy < \infty,$$

hence the series converges for $0 \leq p < 1$.

7. This problem follows closely the derivation in class for the power series expansion for $\log(1+x)$.

- (a) Write down the sum of the geometric series $\sum_{k=0}^n r^k$.
- (b) Use (a) to show that

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - \dots + (-1)^{n-1} t^{2n-2} + (-1)^n \frac{t^{2n}}{1+t^2}.$$

- (c) Use (b) to show that

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots + (-1)^{n-1} \frac{x^{2n-1}}{2n-1} + R_n, \quad (3)$$

where R_n is the remainder which you should express as an integral involving x .

- (d) Show that the power series for $\tan^{-1} x$ converges absolutely for x in the closed interval $[-1, 1]$.
- (e) Use the power series to show that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$. How many terms do we have to keep in this series in order to estimate π with accuracy to 10 decimal places, i.e. with error less than 10^{-10} ?

Solution

- (a) $\sum_{k=0}^n r^k = 1 + r + r^2 + \dots + r^n = \frac{1-r^{n+1}}{1-r}$.
- (b) Use the result above but for n terms not $n+1$, i.e.

$$1 + r + \dots + r^{n-1} = \frac{1-r^n}{1-r} = \frac{1}{1-r} - \frac{r^n}{1-r} \quad \Rightarrow$$

$$\frac{1}{1-r} = 1 + r + \dots + r^{n-1} + \frac{r^n}{1-r}$$

and putting $r = -t^2$ gives

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - \dots + (-1)^{n-1} t^{2n-2} + (-1)^n \frac{t^{2n}}{1+t^2}, \quad (4)$$

as desired.

- (c) Now integrate (4) between 0 and x noting that $\int_0^x (1+t^2)^{-1} dt = \tan^{-1} x$ to find

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots + (-1)^{n-1} \frac{x^{2n-1}}{2n-1} + \int_0^x \frac{(-1)^n t^{2n}}{1+t^2} dt, \quad (5)$$

where the last integral is R_n .

- (d) For absolute convergence we can use the ratio test of successive terms in the series for $\tan^{-1} x$ above. Find

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|x|^{2n+1}}{2n+1} \cdot \frac{2n-1}{|x|^{2n-1}} \rightarrow |x|^2 \quad \text{as } n \rightarrow \infty,$$

hence we have absolute convergence as long as $|x| < 1$. The boundary points $|x| = 1$ must be considered separately since the ratio test gives no information for them. This comes from the requirement that for absolute convergence the remainder $R_n \rightarrow 0$ as $n \rightarrow \infty$. To prove this, estimate the integral form of R_n as follows

$$|R_n| \leq \left| \int_0^x t^{2n} dt \right| \leq \frac{|x|^{2n+1}}{2n+1} \rightarrow 0 \quad \text{as } n \rightarrow \infty \text{ if } |x| \leq 1.$$

Hence, we have absolute convergence of the power series for $\tan^{-1} x$ if $|x| \leq 1$.

- (e) Putting $x = 1$ in the power series (5) and noting that $\tan^{-1} 1 = \pi/4$ gives

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

An upper bound for the error in estimating π using this series after truncating an n terms, is given by $\frac{4}{2n+1}$ (why? because the series is alternating the error is less than the absolute value of the first term dropped). For accuracy 10^{-10} we need $2n+1 > 4 \times 10^{10}$, i.e. $n \gtrsim 2 \times 10^{10}$, a lot of terms!

8. Following up from the calculation of π above, here is a much more efficient way.

- (a) Starting from the addition formula for the tangent

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y},$$

introduce the inverse functions $x = \tan^{-1} u$ and $y = \tan^{-1} v$ to show that

$$\tan^{-1} u + \tan^{-1} v = \tan^{-1} \left(\frac{u+v}{1-uv} \right). \quad (6)$$

- (b) Show that choosing $(u+v)/(1-uv) = 1$ in expression (6), we have the following formula for π ,

$$\frac{\pi}{4} = \tan^{-1} u + \tan^{-1} v, \quad (7)$$

and that restricting u and v to be in the interval $(0, 1)$ we can express them as the one-parameter family

$$u = \frac{1-p}{1+p}, \quad v = p, \quad 0 < p < 1, \quad (8)$$

or equivalently

$$u = \frac{n-m}{n+m}, \quad v = \frac{m}{n}, \quad 0 < m < n, \quad (9)$$

where we picked p to be the rational number $p = m/n$.

Use your earlier findings regarding the power series for $\tan^{-1} x$ (equation (3)) to explain why the choices (8)-(9) are useful.

(c) Hence show that (first derived and used by Euler)

$$\frac{\pi}{4} = \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3}. \quad (10)$$

Noting that $\frac{\frac{1}{3} + \frac{1}{7}}{1 - \frac{1}{21}} = \frac{1}{2}$, show that $\tan^{-1} \frac{1}{2} = \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{7}$, which when combined with (10) gives the formula (used by Jurij Vega, 1754-1802, a Slovenian mathematician who got 140 digits accuracy to π using this formula)

$$\frac{\pi}{4} = 2 \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{7}, \quad (11)$$

and on use of $\frac{\frac{1}{5} + \frac{1}{8}}{1 - \frac{1}{40}} = \frac{1}{3}$ and previous results we also have

$$\frac{\pi}{4} = 2 \tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{7} + 2 \tan^{-1} \frac{1}{8}. \quad (12)$$

(d) If we use the expressions (10), (11) and (12), respectively, how many terms in the expansion (3) do we need to compute π to 10 decimals accuracy? Compare with your answer to question 8(e).

Solution

(a) From the definitions of u and v , the addition formula becomes

$$\begin{aligned} \tan(x+y) &= \frac{u+v}{1-uv} \quad \Rightarrow \\ x+y &= \tan^{-1} \left(\frac{u+v}{1-uv} \right) \quad \Rightarrow \\ \tan^{-1} u + \tan^{-1} v &= \tan^{-1} \left(\frac{u+v}{1-uv} \right), \end{aligned} \quad (13)$$

as required.

(b) Choosing $\frac{u+v}{1-uv} = 1$ in (13) immediately yields $\tan^{-1} u + \tan^{-1} v = \frac{\pi}{4}$. Continuing with $u, v \in (0, 1)$, write $v = p$, $0 < p < 1$, so that $u = \frac{1-p}{1+p}$. Restricting to rationals, let $p = m/n$, $0 < m < n$, so that $v = \frac{m}{n}$ and $u = \frac{n-m}{n+m}$.

These choices for u, v ensure that we are within the radius of convergence of the power series for $\tan^{-1} u$ and $\tan^{-1} v$ found in problem 7.

(c) Pick $m = 1$, $n = 3$, so that $u = \frac{1}{2}$, $v = \frac{1}{3}$, and (13) gives

$$\frac{\pi}{4} = \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} \quad (14)$$

Picking $u = 1/3$, $v = 1/7$ makes $(u+v)/(1-uv) = 1/2$ as given, and hence (13) becomes

$$\tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{7} = \tan^{-1} \frac{1}{2} \quad (15)$$

Now eliminate $\tan^{-1} \frac{1}{2}$ between expressions (15) and (14) to find the required formula

$$\frac{\pi}{4} = 2 \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{7}. \quad (16)$$

Next picking $u = 1/5$, $v = 1/8$ gives $(u + v)/(1 - uv) = 1/3$, hence

$$\tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{8} = \tan^{-1} \frac{1}{3},$$

which when combined with (16) gives

$$\frac{\pi}{4} = 2 \tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{7} + 2 \tan^{-1} \frac{1}{8}. \quad (17)$$

- (d) To estimate how many terms we need for an error of 10^{-10} for each of formulas (14), (16) and (17), first note that it is sufficient to consider the power series for the largest argument of \tan^{-1} appearing in each of these formulas, i.e. $1/2$, $1/3$ and $1/5$, respectively. (Any smaller arguments present will contribute smaller errors automatically, so no need to bother with them further.) For (14) and on use of formula (5) we have

$$\tan^{-1} \frac{1}{2} = \frac{1}{2} - \frac{1}{3 \cdot 2^3} + \frac{1}{5 \cdot 2^5} - \dots + (-1)^{n-1} \frac{1}{(2n-1)2^{2n-1}}.$$

Now, $2^{34} \approx 1.7 \times 10^{10}$, so if $n = 17$ are used the error is guaranteed to be smaller than 10^{-10} .

For the Vega expression (16) the n th term has size $\frac{1}{(2n-1)3^{2n-1}}$. Since $3^{21} \approx 1.05 \times 10^{10}$, $n = 10$ will do.

Finally, for formula (17) the n th term has size $\frac{1}{(2n-1)5^{2n-1}}$ and since $13 \times 5^{13} \approx 1.6 \times 10^{10}$ we see that $2n - 1 = 13$, i.e. $n = 6$ terms are enough.

The differences with the number of terms needed in question 6 are striking, i.e. 6 terms instead of 2×10^{10} terms! The moral of the exercise is - do analysis before you compute!

9. (a) *Binomial Theorem.* Let $f(x) = (1+x)^s$ where s is a real number. Use induction arguments to show that $f^{(n)}(x) = s(s-1)\dots(s-n+1)(1+x)^{s-n}$ and hence write down the Taylor series for $f(x)$ including the remainder term. Hence show that $(1+x)^s$ converges uniformly (i.e. it is analytic) for $|x| < 1$.
- (b) Use the Binomial Theorem to compute $(126)^{1/3}$ and $\sqrt{96}$ to 4 decimals.
- (c) Write out the Maclaurin series for $1/\sqrt{1+x^2}$ using the binomial series. What is $\left. \frac{d^{20}}{dx^{20}} \left(\frac{1}{\sqrt{1+x^2}} \right) \right|_{x=0}$?
- (d) Find the Maclaurin series for $g(x) = \sqrt{1+x} + \sqrt{1-x}$, and hence calculate $g^{(20)}(0)$ and $g^{(2001)}(0)$.

Solution

- (a) *Induction.* Start with $f(x) = (1+x)^s$ with s real. Differentiate once, $f'(x) = s(1+x)^{s-1}$, hence the formula is true for $n = 1$. Assume it is true for n , i.e. $f^{(n)} = s(s-1)\dots(s-(n-1))(1+x)^{s-n}$, differentiating once more we find $f^{(n+1)}(x) = s(s-1)\dots(s-(n-1))(s-n)(1+x)^{s-(n+1)}$, hence the formula holds for $n+1$ also and the induction proof is complete.

We can now write down the Taylor series of $f(x)$ about $x = 0$ (i.e. its Maclaurin series). This is, with remainder where ξ is a number between 0 and x ,

$$\begin{aligned} f(x) &= f(0) + xf'(0) + \frac{x^2}{2!}f^{(2)}(0) + \dots + \frac{x^n}{n!}f^{(n)}(0) + \frac{x^{n+1}}{(n+1)!}f^{(n+1)}(\xi), \Rightarrow \\ &= 1 + sx + s(s-1)\frac{x^2}{2!} + \dots + \frac{s(s-1)\dots(s-n+1)}{n!}x^n + \frac{s(s-1)\dots(s-n)}{(n+1)!}(1+\xi)^{s-n-1}. \end{aligned}$$

Using the ratio test for convergence we find

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} &= \lim_{n \rightarrow \infty} \frac{|s(s-1)\dots(s-n)|}{(n+1)!} \frac{|x|^{n+1}}{|x|^n} \frac{n!}{|s(s-1)\dots(s-n+1)|} \\ &= \lim_{n \rightarrow \infty} \frac{|x||s-n|}{n} = |x|, \end{aligned}$$

hence we need $|x| < 1$ for absolute convergence.

(b)

$$\begin{aligned} (126)^{1/3} &= (125+1)^{1/3} = 5 \left(1 + \frac{1}{5}\right)^{1/3} = 5 \left[1 + \frac{1}{3} \cdot \frac{1}{5} + \frac{(\frac{1}{3})(-\frac{2}{3})}{2!} \frac{1}{5^2} \right. \\ &\quad \left. + \frac{(\frac{1}{3})(-\frac{2}{3})(-\frac{5}{3})}{3!} \frac{1}{5^3} + \frac{(\frac{1}{3})(-\frac{2}{3})(-\frac{5}{3})(-\frac{8}{3})}{4!} \frac{1}{5^4} + \dots \right] \end{aligned}$$

The last term has absolute value $\frac{5 \times 80}{81 \times 24 \times 725} \approx 3 \times 10^{-4}$, hence it needs to be kept. The next term can be neglected.

$$\begin{aligned} \sqrt{96} &= \sqrt{100-4} = 10 \left(1 - \frac{1}{25}\right)^{1/2} = 10 \left[1 - \frac{1}{2} \cdot \frac{1}{25} + \frac{(1/2)(-1/2)}{2!} \frac{1}{25^2} \right. \\ &\quad \left. + \frac{(1/2)(-1/2)(-3/2)}{3!} \frac{1}{25^3} + \dots \right] \end{aligned}$$

- (c) Use $x \rightarrow x^2$ and $s = -1/2$ to find

$$(1+x^2)^{-1/2} = 1 - \frac{1}{2}x^2 + \dots + \frac{(-1/2)(-3/2)\dots(-1/2-n+1)}{n!}x^{2n} + \dots$$

The only term that survives in $\frac{d^{20}}{dx^{20}} \left(\frac{1}{\sqrt{1+x^2}} \right) \Big|_{x=0}$ comes from the x^{20} term, i.e. $n = 10$. Hence

$$\frac{d^{20}}{dx^{20}} \left(\frac{1}{\sqrt{1+x^2}} \right) \Big|_{x=0} = \frac{(-1/2)(-3/2)\dots(-1/2-10+1)}{10!} 20!$$

Note that the $20!$ factor comes from differentiating x^{20} twenty times.

- (d) We know from part (a) that $\sqrt{1+x} + \sqrt{1-x}$ will give a series that only contains even powers of x . Hence we conclude immediately that $g^{(2001)}(0) = 0$ (in fact any odd derivative will be zero at $x = 0$).

For $g^{(20)}(0)$ the only non-zero term comes from the coefficient of x^{20} . Now

$$(1 \pm x)^{1/2} = 1 + \dots + (1/2)(1/2 - 1) \dots (1/2 - 20 + 1) \frac{x^{20}}{20!} + \dots,$$

hence

$$g^{(20)}(0) = 2(1/2)(1/2 - 1) \dots (1/2 - 20 + 1).$$

10. Find the radius of convergence of the following series:

$$\begin{aligned} (1) \sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} x^n \quad (2) \sum_{n=1}^{\infty} \frac{n^n}{(n!)} x^n \quad (3) \sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!} x^n \quad (4) \sum_{n=1}^{\infty} \frac{n^{5n}}{(2n)! n^{3n}} x^n \\ (5) \sum_{n=1}^{\infty} \frac{(3n)!}{(n!)^2} x^n \quad (6) \sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{2^n} x^n \quad (7) \sum_{n=1}^{\infty} \frac{\log n}{2^n} x^n \quad (8) \sum_{n=1}^{\infty} \frac{1 + \cos 2\pi n}{3^n} x^n \\ (9) \sum_{n=1}^{\infty} n x^n \quad (10) \sum_{n=1}^{\infty} \frac{\sin(2\pi n)}{n!} x^n \quad (11) \sum_{n=1}^{\infty} n^2 x^n \quad (12) \sum_{n=1}^{\infty} \frac{\cos n^2}{n^n} x^n \\ (13) \sum_{n=1}^{\infty} \frac{n}{\log n} x^n \quad (14) \sum_{n=1}^{\infty} \frac{(-1)^n}{n! - 1} x^n \quad (15) \sum_{n=1}^{\infty} \frac{n!}{n^n} x^n \quad (16) \sum_{n=1}^{\infty} \frac{(-1)^n + 1}{n!} x^n \end{aligned}$$

You may use Stirling's formula

$$n! = (2\pi n)^{1/2} n^n e^{-n} e^{\theta/12n}, \quad 0 \leq \theta \leq 1,$$

in its appropriate form for large n .

[**Answers:** (1) $1/4$, (2) $1/e$, (3) 27 , (4) $4/e^2$, (5) 0 , (6) 2 , (7) 2 , (8) 3 , (9) 1 , (10) ∞ , (11) 1 , (12) ∞ , (13) 1 , (14) ∞ , (15) e , (16) ∞ .]

Solution

These problems are solved using the ratio test and the given Stirling formula if you wish. I didn't bother with it, but the way you could use it is to replace $n!$ in any of the sums by the formula - this is valid for large n and it would get rid of the n^n terms. I kept it as a ratio and that needs the familiar result $e = \lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n$.

Will denote by a_n the n th term in a series and by R the radius of convergence.

(1)

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(2n+2)!(n!)^2}{((n+1)!)^2(2n)!} |x| = \frac{(2n+2)(2n+1)}{(n+1)^2} |x| \rightarrow 4|x| \Rightarrow R = \frac{1}{4}$$

(2)

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)^{n+1} n!}{(n+1)! n^n} |x| = \left(1 + \frac{1}{n}\right)^n |x| \rightarrow e|x| \Rightarrow R = \frac{1}{e}$$

(3)

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)^3}{(3n+3)(3n+2)(3n+1)}|x| \rightarrow \frac{|x|}{27} \Rightarrow R = 27$$

(4)

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\left(1 + \frac{1}{n}\right)^{5n}}{\left(1 + \frac{1}{n}\right)^{3n}} \frac{(n+1)^5}{(2n+2)(2n+1)(n+1)^3} |x| \rightarrow \frac{e^5}{e^3} \frac{|x|}{4} \Rightarrow R = \frac{4}{e^2}$$

(5)

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(3n+3)(3n+2)(3n+1)}{(n+1)^2} |x| \sim 9n|x| \Rightarrow R = 0$$

(6) Note first that $a_n = 0$ if n is even, so need to consider the new series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2^{2n-1}}$,
so

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|x|^{2n+1} 2^{2n+1}}{|x|^{2n-1} 2^{2n+1}} \rightarrow \frac{|x|^2}{4} \Rightarrow R = 2$$

(7)

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\log(n+1) 2^n}{2^{n+1} \log(n)} |x| \rightarrow \frac{|x|}{2} \Rightarrow R = 2$$

(8) Note that $\cos 2n\pi = 1$ for all $n \geq 1$, hence

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|x|}{3} \Rightarrow R = 3$$

(9)

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)|x|^{n+1}}{n|x|^n} \rightarrow |x| \Rightarrow R = 1$$

(10) Since $\sin(2\pi n) = 0$ for all positive integers, the sum is 0 for all x . Hence $R = \infty$.

(11)

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)^2 |x|^{n+1}}{n^2 |x|^n} \rightarrow |x| \Rightarrow R = 1$$

(12)

$$\begin{aligned} \frac{|a_{n+1}|}{|a_n|} &= \frac{|\cos(n+1)|^2 n^n}{|\cos n^2| (n+1)^{n+1}} |x| = \frac{|\cos(n+1)|^2}{|\cos n^2|} \frac{1}{(n+1)} \frac{1}{\left(1 + \frac{1}{n}\right)^n} |x| \\ &\leq M \frac{|x|}{(n+1)} \rightarrow 0 \Rightarrow R = \infty \end{aligned}$$

The bound M follows since $|\cos n^2|$ with n an integer is always bounded away from 0.

(13)

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1) \log n}{n \log(n+1)} |x| \rightarrow 1 \Rightarrow R = 1$$

(14)

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n! - 1)}{[(n+1)! - 1]} |x| \rightarrow \frac{|x|}{n+1} \Rightarrow R = \infty$$

(15) The coefficients of x^n in a_n are the reciprocals of those in problem (2). Hence $R = e$.

(16) Odd terms are zero, hence $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} x^n = \sum_{k=1}^{\infty} \frac{2}{(2k)!} x^{2k}$, hence

$$\frac{|a_{k+1}|}{|a_k|} = \frac{|x|^2}{(2k+2)(2k+1)} \rightarrow 0 \Rightarrow R = \infty$$

-
11. Find the Taylor series of the function $f(x) = \int_1^x \log t \, dt$ for x near 1. Do the same for the function $x \log x$ and compare the two. What do you conclude?

Solution

Generally, $f(x) = f(1 + x - 1) = f(1) + (x - 1)f^{(1)}(1) + \frac{(x-1)^2}{2!}f^{(2)}(1) + \dots$ is the Taylor expansion near $x = 1$. For $f(x) = \int_1^x \log t \, dt$ calculate

$$f'(x) = \log x \quad f^{(2)}(x) = \frac{1}{x}, \quad f^{(3)}(x) = -\frac{1}{x^2}, \quad f^{(4)}(x) = \frac{2}{x^3}, \quad \dots$$

Hence

$$f(x) = \frac{(x-1)^2}{2!} - \frac{(x-1)^3}{3!} + \frac{(x-1)^4}{4!} 2! - \frac{(x-1)^5}{5!} 3! + \dots, \quad (18)$$

is the Taylor series for the given integral.

Now define $g(x) = x \log x$ and again calculate derivatives

$$g'(x) = 1 + \log x, \quad g^{(2)}(x) = \frac{1}{x}, \quad g^{(3)}(x) = -\frac{1}{x^2}, \dots,$$

i.e. $f^{(k)}(x) = g^{(k)}(x)$ for all $k \geq 2$. Hence the Taylor expansion of $x \log x$ is

$$\begin{aligned} x \log x &= (x-1) + \frac{(x-1)^2}{2!} - \frac{(x-1)^3}{3!} + \frac{(x-1)^4}{4!} 2! - \frac{(x-1)^5}{5!} 3! + \dots, \\ &= (x-1) + f(x), \end{aligned} \quad (19)$$

on use of (18). This of course is expected since

$$f(x) = \int_1^x \log t \, dt = [t \log t]_1^x - \int_1^x dt = x \log x - (x-1),$$

in complete agreement with (19).

12. Find the first four non-vanishing terms of the Maclaurin series for the following functions:

$$(a) \ x \cot x \quad (b) \ e^{\sin x}, \quad (c) \frac{\sqrt{\sin x}}{\sqrt{x}}$$

$$(d) \ e^{e^x}, \quad (e) \ \sec x, \quad (f) \ \log \sin x - \log x$$

Solution

- (a) I avoided differentiating $x \cot x$ three times and used Maclaurin combined with Binomial - good to see it done this way also - can be much quicker.

$$\begin{aligned} x \cot x &= x \frac{\cos x}{\sin x} = x \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \right] \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \right]^{-1} \\ &= \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \right] \left[1 - \left(\frac{x^2}{3!} - \frac{x^4}{5!} + \frac{x^6}{7!} + \dots \right) \right]^{-1} \\ &= \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \right] \left[1 + \left(\frac{x^2}{3!} - \frac{x^4}{5!} + \frac{x^6}{7!} + \dots \right) + \left(\frac{x^2}{3!} - \frac{x^4}{5!} + \frac{x^6}{7!} + \dots \right)^2 \right. \\ &\quad \left. + \left(\frac{x^2}{3!} - \frac{x^4}{5!} + \frac{x^6}{7!} + \dots \right)^3 + \dots \right] \end{aligned}$$

and since we only need terms up to and including x^6 we only keep relevant terms in the second bracket, i.e.

$$\begin{aligned} &= \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \right] \left[1 + \left(\frac{x^2}{3!} - \frac{x^4}{5!} + \frac{x^6}{7!} + \dots \right) + \left(\frac{x^4}{(3!)^2} - 2 \frac{x^6}{(3!)(5!)} + \dots \right) \right. \\ &\quad \left. + \left(\frac{x^6}{(3!)^3} + \dots \right) \right] = \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \right] \left[1 + \frac{x^2}{3!} + \left(\frac{1}{(3!)^2} - \frac{1}{5!} \right) x^4 \right. \\ &\quad \left. + \left(\frac{1}{(3!)^2} - \frac{2}{(3!)(5!)} \right) x^6 + \dots \right] = 1 + \left(\frac{1}{3!} - \frac{1}{2!} \right) x^2 + \left(\frac{1}{(3!)^2} - \frac{1}{5!} - \frac{1}{(2!)(3!)} + \frac{1}{4!} \right) x^4 \\ &\quad + \left(\frac{1}{(3!)^2} - \frac{2}{(3!)(5!)} - \frac{1}{(2!)(3!)^2} + \frac{1}{(2!)(5!)} + \frac{1}{(3!)(4!)} - \frac{1}{6!} \right) x^6 + \dots \end{aligned}$$

- (b) Need four derivatives of $f(x) = e^{\sin x}$, since $f^{(3)}(0) = 0$ as we see below:

$$\begin{aligned} f'(x) &= (\cos x)e^{\sin x}, & f^{(2)}(x) &= (\cos^2 x)e^{\sin x} - (\sin x)e^{\sin x}, \\ f^{(3)}(x) &= (\cos^3 x)e^{\sin x} - 3(\cos x \sin x)e^{\sin x} - (\cos x)e^{\sin x} \\ f^{(4)}(x) &= (\cos^4 x)e^{\sin x} - 3(\cos^2 x \sin x)e^{\sin x} - 3(\cos^2 x \sin x)e^{\sin x} \\ &\quad - 3(\cos^2 x - \sin^2 x)e^{\sin x} - (\cos^2 x)e^{\sin x} + (\sin x)e^{\sin x} \end{aligned}$$

Hence the first 4 non-zero terms are

$$e^{\sin x} = 1 + x + \frac{x^2}{2!} - 3 \frac{x^3}{3!}.$$

(c) Use the Binomial expansion after canceling x :

$$\begin{aligned}\frac{\sqrt{\sin x}}{\sqrt{x}} &= \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots\right)^{1/2} = 1 - \frac{1}{2} \left(\frac{x^2}{3!} - \frac{x^4}{5!} + \frac{x^6}{7!} + \dots\right) \\ &+ \frac{1}{2!} \left(\frac{1}{2}\right) \left(-\frac{1}{2}\right) \left(\frac{x^2}{3!} - \frac{x^4}{5!} + \frac{x^6}{7!} + \dots\right)^2 \\ &+ \frac{1}{3!} \left(\frac{1}{2}\right) \left(-\frac{1}{2}\right) \left(-\frac{3}{2}\right) \left(\frac{x^2}{3!} - \frac{x^4}{5!} + \frac{x^6}{7!} + \dots\right)^3 + \dots \\ &= 1 - \frac{x^2}{2 \cdot 3!} + \left(\frac{1}{2 \cdot 5!} - \frac{1}{4 \cdot 2! \cdot 3!}\right) x^4 + \left(-\frac{1}{2 \cdot 7!} + \frac{1}{2 \cdot 2! \cdot 3! \cdot 5!} + \frac{1}{8(3!)^2}\right) x^6 + \dots\end{aligned}$$

(d) Need 3 derivatives of $f(x) = e^{e^x}$:

$$f' = e^x e^{e^x}, \quad f^{(2)} = e^x e^{e^x} + (e^x)^2 e^{e^x}, \quad f^{(3)} = e^x e^{e^x} + (e^x)^2 e^{e^x} + 2(e^x)^2 e^{e^x} + (e^x)^3 e^{e^x},$$

hence

$$e^{e^x} = e + ex + 2e \frac{x^2}{2!} + 5e \frac{x^3}{3!} + \dots$$

(e) Write

$$\sec x = (\cos x)^{-1} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right)^{-1}$$

and use the Binomial expansion as in parts (a) and (c). The calculations are almost identical, so I will omit them.

If you try differentiating $f(x) = \sec x$ to find the Maclaurin series, you will need 6 derivatives since $f^{(1)}(0) = f^{(3)}(0) = f^{(5)}(0) = 0$ by virtue of the function being even. The terms increase substantially - try it that way and the Binomial way to appreciate the differences.

(f) Here we expand $\sin x$ and cancel the log:

$$\begin{aligned}f(x) &:= \log \sin x - \log x = \log x \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots\right) - \log x \\ &= \cancel{\log x} + \log \left[1 - \left(\frac{x^2}{3!} - \frac{x^4}{5!} + \frac{x^6}{7!} - \frac{x^8}{9!} + \dots\right)\right] - \cancel{\log x}\end{aligned}\tag{20}$$

If I define $y = \left(\frac{x^2}{3!} - \frac{x^4}{5!} + \frac{x^6}{7!} - \frac{x^8}{9!} + \dots\right)$ then the Taylor series we need is

$$\log(1 - y) = -y - \frac{y^2}{2} - \frac{y^3}{3} - \frac{y^4}{4} + \dots,$$

and the first 4 non-zero terms follow by keeping terms up to and including x^8 in each y -term

$$\begin{aligned}f(x) &= -\left(\frac{x^2}{3!} - \frac{x^4}{5!} + \frac{x^6}{7!} - \frac{x^8}{9!} + \dots\right) - \frac{1}{2} \left(\frac{x^4}{(3!)^2} - \frac{2}{(3!)(5!)} x^6 + \frac{1}{(5!)^2} x^8 + \frac{2}{(3!)(7!)} x^8\right) \\ &- \frac{1}{3} \left(\frac{x^6}{(3!)^2} - \frac{3x^8}{(3!)^2(5!)}\right) - \frac{1}{4} \frac{x^8}{(3!)^4} + \dots\end{aligned}$$

-
13. Consider the function $h(x)$ defined on the interval $[-\pi, \pi]$ and given by

$$h(x) = \begin{cases} \frac{1}{x} - \frac{1}{2\sin(x/2)} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Use a Maclaurin expansion to show that $h(x)$ is continuous and has a continuous first derivative at $x = 0$.

Solution

Will expand $\sin(x/2)$ and then use a Binomial expansion:

$$\begin{aligned} \frac{1}{x} - \frac{1}{2\sin(x/2)} &= \frac{1}{x} - \frac{1}{2\left(\frac{x}{2} - \frac{(x/2)^3}{3!} + \frac{(x/2)^5}{5!} + \dots\right)} \\ &= \frac{1}{x} - \frac{1}{x} \left(1 - \frac{(x/2)^2}{3!} + \frac{(x/2)^4}{5!} + \dots\right)^{-1} = \frac{1}{x} - \frac{1}{x} \left(1 + \frac{(x/2)^2}{3!} - \frac{(x/2)^4}{5!} + \frac{(x/2)^4}{(3!)^2} + \dots\right) \\ &= -\frac{x}{24} + Kx^3 + \dots, \end{aligned}$$

where K is a known constant. It follows that the function is both continuous and has a continuous first derivative at $x = 0$. [In fact all derivatives exist at $x = 0$ and are continuous.]

14. Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = f(x)/(1-x)$.

- By multiplying the power series of $f(x)$ and $1/(1-x)$, show that $g(x) = \sum_{n=0}^{\infty} b_n x^n$, where $b_n = a_0 + \dots + a_n$ is the n th partial sum of the series $\sum_{n=0}^{\infty} a_n$.
- Suppose that the radius of convergence of $f(x)$ is greater than 1 and that $f(1) \neq 0$. Show that $\lim_{n \rightarrow \infty} b_n$ exists and is not equal to zero. What does this tell you about the radius of convergence of $g(x)$?
- Let $\frac{e^x}{1-x} = \sum_{n=0}^{\infty} b_n x^n$. What is $\lim_{n \rightarrow \infty} b_n$?

Solution

- The power series of $1/(1-x)$ follows from the identity $\frac{1-x^{n+1}}{1-x} = 1+x+\dots+x^n$ which we have seen often, by having $|x| < 1$ and sending $n \rightarrow \infty$. Thus

$$\begin{aligned} g(x) &= (a_0 + a_1x + a_2x^2 + a_3x^3 + \dots)(1 + x + x^2 + x^3 + \dots) \\ &= a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + \dots + (a_0 + a_1 + \dots + a_n)x^n + \dots \end{aligned}$$

and so $b_n = a_0 + a_1 + \dots + a_n$ as required.

- If $f(x)$ has radius of convergence greater than 1 and $f(1) \neq 0$, we can evaluate the power series for f at $x = 1$ to obtain

$$f(1) = \sum_{n=0}^{\infty} a_n \neq 0.$$

Now $\lim_{n \rightarrow \infty} b_n = \sum_{n=0}^{\infty} a_n = f(1) \neq 0$. This means that the radius of convergence of $g(x)$ is 1 since on use of the ratio test for $\sum_{n=0}^{\infty} b_n x^n$ we find

$$\lim_{n \rightarrow \infty} \frac{|b_{n+1}|}{|b_n|} |x| = \frac{|f(1)|}{|f(1)|} |x|.$$

- (c) You can do the expansion directly but that is not the way. Using (a) and (b) above, we identify $f(x) = e^x$. Clearly the radius of convergence is greater than 1 and $f(1) = e \neq 0$. Hence

$$\lim_{n \rightarrow \infty} b_n = e.$$

15. (a) Write the Maclaurin series for the functions $1/\sqrt{1-x^2}$ and $\sin^{-1} x$. For what values of x do they converge?
- (b) Find the terms up to and including x^3 in the series for $\sin^{-1}(\sin x)$ by substituting the series for $\sin x$ into the series for $\sin^{-1} x$.
- (c) Use the substitution method from part (b) to obtain the first five terms of the series for $\sin^{-1} x$ by using the relation $\sin^{-1}(\sin x) = x$ and solving for a_0 to a_5 .
- (d) Find the terms up to and including x^5 of the Maclaurin series for the inverse function $g(s)$ of $f(x) = x^3 + x$. [Hint: Use the relation $g(f(x)) = x$ and solve for the coefficients in the series for g .]

Solution

- (a) Use the Binomial expansion:

$$(1-x^2)^{-1/2} = 1 + (1/2)x^2 + (1/2)(3/2)\frac{x^4}{2!} + (1/2)(3/2)(5/2)\frac{x^6}{3!} + \dots \quad (21)$$

The radius of convergence is $|x| < 1$.

Since $\sin^{-1} x = \int_0^x \frac{dt}{\sqrt{1-t^2}}$, and as long as $|x| < 1$ we can integrate (21) to establish the power series for $\sin^{-1} x$ (the radius of convergence is 1 of course):

$$\sin^{-1} x = x + \frac{(1/2)}{3}x^3 + \frac{(1/2)(3/2)}{5 \cdot 2!}x^5 + \frac{(1/2)(3/2)(5/2)}{7 \cdot 3!}x^7 + \dots \quad (22)$$

- (b) A direct substitution of $\sin x = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right)$ into (22) gives, up to and including x^3 ,

$$\begin{aligned} \sin^{-1}(\sin x) &= \left(x - \frac{x^3}{3!} + \dots\right) + \frac{(1/2)}{3} \left(x - \frac{x^3}{3!} + \dots\right)^3 + \dots \\ &= x - \cancel{\frac{x^3}{3!}} + \cancel{\frac{(1/2)}{3}}x^3 + \dots \\ &= x + \dots \end{aligned}$$

- (c) Start by assuming a power series expansion for $\sin^{-1} y$ with the first 5 terms to be found, i.e. write

$$\sin^{-1} y = a_0 + a_1 y + a_2 y^2 + a_3 y^3 + a_4 y^4 + a_5 y^5 + \dots, \quad (23)$$

and now put $y = \sin x$ where $|x| < 1$. The left hand side of (23) becomes $\sin^{-1}(\sin x)$ which equals x . Hence we have

$$\begin{aligned} x &= a_0 + a_1 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right) + a_2 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right)^2 + a_3 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right)^3 \\ &\quad + a_4 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right)^4 + a_5 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right)^5 + \dots \end{aligned}$$

The only way this can be satisfied is for the coefficients to balance, i.e.

$$\begin{aligned}
 x^0 : \quad & a_0 = 0 \\
 x^1 : \quad & a_1 = 1 \\
 x^2 : \quad & a_2 = 0 \\
 x^3 : \quad & -\frac{a_1}{3!} + a_3 = 0 \quad \Rightarrow \quad a_3 = \frac{1}{3!} \\
 x^4 : \quad & a_4 = 0 \\
 x^5 : \quad & \frac{a_1}{5!} - \frac{3a_3}{3!} + a_5 = 0 \quad \Rightarrow \quad a_5 = \frac{1}{2 \cdot 3!} - \frac{1}{5!} = \frac{(1/2)(3/2)}{10}
 \end{aligned}$$

[Could have anticipated $a_0 = a_2 = a_4 = 0$ since the function is odd.]

These are exactly the first three non-zero coefficients found in (22).

(d) To find the expansion for the inverse function, assume

$$g(s) = a_0 + a_1 s + a_2 s^2 + a_3 s^3 + a_4 s^4 + a_5 s^5 + \dots \quad (24)$$

Since $f(x) = x + x^3$ is an odd function of x , then $g(s)$ is also an odd function of s , hence $a_0 = a_2 = a_4 = \dots = 0$. If you didn't use this it will come out of the calculations that I include for completeness (similar to part (c)). Using the identity $g(f(x)) = x$ in (24), we have

$$x = a_0 + a_1(x + x^3) + a_2(x + x^3)^2 + a_3(x + x^3)^3 + a_4(x + x^3)^4 + a_5(x + x^3)^5 + \dots$$

Equating powers of x gives the coefficients

$$\begin{aligned}
 x^0 : \quad & a_0 = 0 \\
 x^1 : \quad & a_1 = 1 \\
 x^2 : \quad & a_2 = 0 \\
 x^3 : \quad & a_1 + a_3 = 0 \quad \Rightarrow \quad a_3 = -1 \\
 x^4 : \quad & a_4 = 0 \\
 x^5 : \quad & 3a_3 + a_5 = 0 \quad \Rightarrow \quad a_5 = 3
 \end{aligned}$$

hence

$$g(s) = s - s^3 + 3s^5 + \dots$$

16. * (This problem will guide you through an example of the use of power series to solve differential equations.)

Consider the differential equation

$$\frac{d^2 y}{dx^2} + y = 0. \quad (25)$$

- (i) Verify that $y = A \sin x + B \cos x$ where A, B are arbitrary constants, is the general solution of (25).

- (ii) Look for a solution of (25) in the form of a power series

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

and by equating different powers of x determine all possible values of a_n .

- (iii) Use your results to (i) and (ii) to find power series expansions for $\sin x$ and $\cos x$.

Solution

- (i) Straightforward, just substitute in and verify that LHS=0.

- (ii) If $y(x) = \sum_{n=0}^{\infty} a_n x^n$, then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2},$$

hence the differential equation becomes

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0,$$

and writing $n = k + 2$ in the first sum and after that $k = n$ for the new dummy variable, we have

$$\sum_{n=0}^{\infty} [(n+1)(n+2) a_{n+2} + a_n] x^n = 0.$$

Now equate to zero powers of x (reason: the functions x^n form a linearly independent basis in the vector space of polynomials). I will do even powers first:

$$\begin{aligned} (2 \cdot 1) a_2 &= -a_0, & (4 \cdot 3) a_4 &= -a_2, & (6 \cdot 5) a_6 &= -a_4, & \dots & \Rightarrow \\ a_2 &= -\frac{1}{2!} a_0, & a_4 &= -\frac{1}{4!} a_0, & a_6 &= -\frac{1}{6!} a_0, & \dots \end{aligned}$$

Now equate odd powers and we see that everything can be expressed in terms of a_1 :

$$a_3 = -\frac{1}{3!} a_1, \quad a_5 = -\frac{1}{5!} a_1, \quad a_7 = -\frac{1}{7!} a_1, \quad \dots$$

Hence, we can write the power series solution as

$$y(x) = a_0 \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \right] + a_1 \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \right]. \quad (26)$$

- (iii) It is seen from (26) that the solutions in the square brackets are linearly independent - they come with arbitrary constants a_0 and a_1 , respectively. As shown in part (i) the linearly independent solutions are $\cos(x)$ and $\sin(x)$. Clearly the bracket multiplied by a_0 is the power series for $\cos(x)$ since it is an even function of x , and the bracket multiplied by a_1 corresponds to the power series for $\sin(x)$ since it is an odd function of x .