Math40003 Linear Algebra and Groups **Unseen 1**

1. Let A be a matrix over $\mathbb{R}$ with 3 rows and m columns and let $x \in \mathbb{R}^m$, so that Ax is in $\mathbb{R}^3$. Prove or disprove the following statements:

   (a) If $Ax = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ has a solution, then $Ax = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ has a solution.

   (b) If $Ax = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ has a *unique* solution, then $Ax = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ has a *unique* solution.

   (c) If $Ax = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ has *no* solution, then $Ax = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ has *no* solution.

   (d) If $Ax = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $Bx = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ both have unique solutions,

   then $(A + B)x = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$ has a unique solution.

2. For which of the items in Question 1 the answer will change if we require A to be a 3×3 matrix? Prove.

3. Let $Ax = b$ be a system of linear equations in 4 variables over $\mathbb{R}$. Assume $(4, -2, -2, 4)$ and $(-2, 4, 4, -2)$ are solutions and $(2, 2, 2, 2)$ is not a solution.

   (a) Prove $b \neq \vec{0}$.

   (b) Prove $(0, 2, 2, 0)$ is a solution.

   (c) Prove $(0, 4, 4, 0)$ is not a solution.

4. Find a system of linear equations $Ax = b$ in variables $x_1, \ldots, x_k$ (over an arbitrary field), such that the following are equivalent:

   i. $(\alpha_1, \ldots, \alpha_k)$ is a solution to $Ax = b$.

   ii. For every system of linear equations (in n variables) $A'x = b'$ over the same field of definition as A, if $v_1, \ldots, v_k$ are all solutions to $A'x = b'$, then $\alpha_1 v_1 + \cdots + \alpha_k v_k$ is also a solution to $A'x = b'$.

   iii. There is some system of linear equations (in n variables) $A'x = b'$ over the same field of definition as A, **such that** $b' \neq 0$ and $v_1, \ldots, v_k$ such that $v_1, \ldots, v_k, \alpha_1 v_1 + \cdots + \alpha_k v_k$ are all solutions to $A'x = b'$.

5. Prove that for any $A \in M_{m \times n}(\mathbb{R})$, the following are equivalent:

i. There is some $B \in M_{n\times m}(\mathbb{R})$ such that $BA = I_n$

ii. The system of linear equations $Ax = 0$ has a unique solution.

iii. A is row-equivalent to $\begin{pmatrix} I_n \\ O \end{pmatrix}$, i.e. it is row-equivalent to the matrix whose rows are the rows of I_n for the first n rows, followed by $m - n$ rows of zeros.

Note that the question does not assume $m \geq n$.

6. Prove that for any $A \in M_{m\times n}(\mathbb{R})$, if there are $B_1, B_2 \in M_{n\times m}$ such that $B_1 A = I_n$ and $AB_2 = I_m$ then $B_1 = B_2$.