1. (a) Suppose that some function $f : (-R, R) \to \mathbb{R}$ is equal to the power series $\displaystyle\sum_{n=0}^{\infty} \frac{a_n x^n}{n!}$, which converges absolutely on $(-R, R)$. Prove that the Taylor series of f centered at $a = 0$ is precisely $\displaystyle\sum_{n=0}^{\infty} \frac{a_n x^n}{n!}$.

   (b) Compute the Taylor series of $f(x) = \frac{1}{1-x^2}$ centered at $a = 0$. What is $f^{(100)}(0)$?

Solution. (a) Since we can differentiate power series term by term inside their radius of convergence, it follows by induction that $f^{(k)}(x)$ exists and that

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n x^{n-k},$$

absolutely convergent on the interval $(-R, R)$, for all k. This gives us $f^{(k)}(0) = k!a_k$, and so $f(x)$ has Taylor series

$$P(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = \sum_{n=0}^{\infty} \frac{n!a_n \cdot x^n}{n!} = \sum_{n=0}^{\infty} a_n x^n.$$

In other words, if f is equal to some power series on $(-R, R)$ then that power series must be the Taylor series centered at $x = 0$, and so that power series is unique.

(b) Computing the derivatives of $f(x)$ gets messy very quickly, so instead we note that $f(x)$ is the sum of a geometric series

$$f(x) = 1 + x^2 + x^4 + x^6 + \cdots = \sum_{n=0}^{\infty} x^{2n}$$

on the interval $(-1, 1)$, and this is a power series, so it must be the Taylor series for $f(x)$. The coefficient of x^{100} is 1, and it's also supposed to be equal to $\frac{f^{(100)}(0)}{100!}$, so we must have $f^{(100)}(0) = 100!$.

2. (*) Let (a_n) denote the Fibonacci sequence, with $a_0 = 0$, $a_1 = 1$, and $a_{n+2} = a_{n+1} + a_n$ for all $n \geq 0$.

   (a) Prove by induction that $a_n < 2^n$ for all $n \geq 0$. What is the radius of convergence of the *exponential generating function*

$$F(x) = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!} = 0 + 1x + \frac{1x^2}{2} + \frac{2x^3}{6} + \frac{3x^4}{24} + \ldots?$$

   (b) Prove that $F''(x) = F'(x) + F(x)$, and that $F(0) = 0$ and $F'(0) = 1$.

   (c) Solve this differential equation for $F(x)$.

(d) Use the solution from part (c) to prove *Binet's formula*:

$$a_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

Solution. (a) We have $a_0 < 2^0$ and $a_1 < 2^1$, and if $a_n < 2^n$ and $a_{n+1} < 2^{n+1}$ then

$$a_{n+2} = a_{n+1} + a_n < 2^{n+1} + 2^n < 2 \cdot 2^{n+1} = 2^{n+2},$$

so it follows by induction that $a_k < 2^k$ for all $k \geq 0$.

We now have $\left|\frac{a_n x^n}{n!}\right| < \left|\frac{2^n x^n}{n!}\right| = \left|\frac{(2x)^n}{n!}\right|$, so the comparison test says that $F(x)$ converges absolutely whenever $\sum_{n=0}^{\infty} \frac{(2x)^n}{n!}$ does. The latter is equal to e^{2x} for all $x \in \mathbb{R}$, so $F(x)$ has infinite radius of convergence.

(b) Since the power series for F has infinite radius of convergence, we can differentiate term by term to get

$$F'(x) = \sum_{n=0}^{\infty} \frac{n a_n x^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{a_n x^{n-1}}{(n-1)!} = \sum_{m=0}^{\infty} \frac{a_{m+1} x^m}{m!},$$

where in the last step we substitute $m = 1$, and this also has infinite radius of convergence. We repeat this argument to get

$$F''(x) = \sum_{n=0}^{\infty} \frac{n a_{n+1} x^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{a_{n+1} x^{n-1}}{(n-1)!} = \sum_{m=0}^{\infty} \frac{a_{m+2} x^m}{m!}.$$

Since these power series all converge absolutely, we can rearrange them to get

$$\begin{aligned} F(x) + F'(x) &= \sum_{n=0}^{\infty} \frac{a_n x^n}{n!} + \sum_{n=0}^{\infty} \frac{a_{n+1} x^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(a_n + a_{n+1}) x^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{a_{n+2} x^n}{n!} = F''(x). \end{aligned}$$

We also have $F(0) = a_0 = 0$ and $F'(0) = a_1 = 1$ by inspection.

(c) The roots of $x^2 - x - 1 = 0$ are $r = \frac{1}{2}(1+\sqrt{5})$ and $s = \frac{1}{2}(1-\sqrt{5})$, so the general solution to $y'' - y' - y = 0$ is

$$y = c_1 e^{rx} + c_2 e^{sx}.$$

The initial conditions $y(0) = 0$ and $y'(0) = 1$ are equivalent to

$$\begin{aligned} c_1 + c_2 &= 0 \\ rc_1 + sc_2 &= 1, \end{aligned}$$

with solution $c_1 = \frac{1}{r-s} = \frac{1}{\sqrt{5}}$ and $c_2 = -c_1 = -\frac{1}{\sqrt{5}}$, so we have

$$F(x) = \frac{e^{rx} - e^{sx}}{\sqrt{5}}.$$

(d) From the above closed form for $F(x)$, we have

$$\begin{aligned} F(x) &= \frac{1}{\sqrt{5}}\left(\sum_{n=0}^{\infty}\frac{(rx)^n}{n!} - \sum_{n=0}^{\infty}\frac{(sx)^n}{n!}\right) \\ &= \sum_{n=0}^{\infty}\left(\frac{r^n - s^n}{\sqrt{5}}\right)\frac{x^n}{n!}. \end{aligned}$$

Since this power series is equal to $\displaystyle\sum_{n=0}^{\infty}\frac{a_n x^n}{n!}$, the coefficients of each x^n must be the same, so

$$a_n = \frac{r^n - s^n}{\sqrt{5}} = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

3. Recall that we defined $\pi = \inf\{y > 0 \mid \sin(y) = 0\}$.

   (a) Prove that $\sin(n\pi) = 0$ for all $n \in \mathbb{Z}$.

   (b) Prove that if $\sin(y) = 0$, then $y = n\pi$ for some $n \in \mathbb{Z}$. (Hint: write $y = q\pi + r$.)

   (c) Prove that $\cos(x) = 0$ if and only if $x = \frac{(2k+1)\pi}{2}$ for some $k \in \mathbb{Z}$.

Solution. (a) We already know that $\sin(0) = \sin(\pi) = 0$. We prove the claim for $n > 0$ by induction: if $n \geq 2$ then the angle addition formula says that

$$\begin{aligned} \sin(n\pi) &= \sin((n-1)\pi + \pi) \\ &= \underbrace{\sin((n-1)\pi)}_{=0}\cos(\pi) + \cos((n-1)\pi)\underbrace{\sin(\pi)}_{=0} = 0. \end{aligned}$$

And having solved the case where n is positive, we note that sin is an odd function, so if $n < 0$ then

$$\sin(n\pi) = -\underbrace{\sin((-n)\pi)}_{=0} = 0$$

since $-n > 0$. So $\sin(n\pi) = 0$ for all negative integers n as well.

(b) Suppose that $\sin(y) = 0$. Then we can write

$$y = q\pi + r, \qquad q \in \mathbb{Z},\ 0 \leq r < \pi,$$

and we compute using part (a) that

$$\sin(r) = \sin(y - q\pi) = \underbrace{\sin(y)}_{=0}\cos(q\pi) - \cos(y)\underbrace{\sin(q\pi)}_{=0} = 0.$$

If $r > 0$ then it belongs to the set $\inf\{y > 0 \mid \sin(y) = 0\}$, and then π could not be a lower bound since $r < \pi$. This would be a contradiction, so we must have $r = 0$ and hence $y = q\pi$.

(c) We use the double-angle formula $\sin(2x) = 2\sin(x)\cos(x)$ to see that

$$\cos(x) = 0 \implies \sin(2x) = 0 \implies 2x = n\pi \text{ for some } n \in \mathbb{Z}$$

by part (b), so if $\cos(x) = 0$ then $x = \frac{n\pi}{2}$ for some integer n. If n is odd then $\sin(\frac{n\pi}{2}) \neq 0$ since $\frac{n\pi}{2}$ isn't an integer multiple of π, so

$$n = 2k+1 \implies \cos\left(\frac{n\pi}{2}\right) = \frac{\sin(n\pi)}{2\sin\left(\frac{n\pi}{2}\right)} = 0.$$

On the other hand, if n is even then

$$n = 2k \implies \cos^2\left(\frac{n\pi}{2}\right) + \underbrace{\sin^2\left(\frac{n\pi}{2}\right)}_{=\sin^2(k\pi)=0} = 1 \implies \cos^2\left(\frac{n\pi}{2}\right) = 1,$$

so $\cos(\frac{n\pi}{2}) = \pm 1$ is nonzero. Thus $\cos(\frac{n\pi}{2}) = 0$ if and only if n is odd, completing the proof.

4. In this problem we will show that the mysterious constant π lies strictly between $2\sqrt{2} \cong 2.828\ldots$ and 3.2. (Can you use these same ideas to do better?)

(a) Use the third-order Taylor polynomial for $\cos(x)$, centered at $x = 0$, to prove that if $0 < x \leq \frac{\pi}{2}$ then

$$1 - \frac{x^2}{2} < \cos(x) \leq 1 - \frac{x^2}{2} + \frac{x^4}{24}.$$

(b) Evaluate one or both of these inequalities at $x = \frac{\pi}{2}$ and conclude that $\pi > 2\sqrt{2}$.

(c) Show that $\cos(2) < 0$ and hence that $\frac{\pi}{2} < 2$. Once you've done this, use a calculator to do the same for $\cos(1.6)$ and deduce that $\pi < 3.2$.

Solution. (a) We apply Taylor's theorem as follows: for any $0 < x \leq \frac{\pi}{2}$, there exists $t \in (0, x)$ such that

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{\cos(t)}{4!}x^4,$$

since $1 - \frac{x^2}{2!}$ is the third-order Taylor polynomial for $\cos(x)$ and $\frac{d^4}{dx^4}\cos(x) = \cos(x)$. Then $0 < t < \frac{\pi}{2}$, and cos is nonzero on this interval by part (c) of the previous problem while $\cos(0) = 1$ is positive, so we must have $\cos(t) > 0$. Thus the error term $\frac{\cos(t)}{24}x^4$ is positive for x in this interval, and it is at most $\frac{x^4}{24}$ since $\cos(t) \leq 1$ (which follows from $\sin^2(t) + \cos^2(t) = 1$), implying both of the desired inequalities.

(b) Using the lower bound $1 - \frac{x^2}{2} < \cos(x)$ for $x \in (0, \frac{\pi}{2}]$, we set $x = \frac{\pi}{2}$ to conclude that

$$0 = \cos\left(\frac{\pi}{2}\right) > 1 - \frac{(\pi/2)^2}{2},$$

which we rearrange to get $\pi^2 > 8$, or $\pi > 2\sqrt{2}$.

(c) Supposing that $2 \leq \frac{\pi}{2}$, we must have $\cos(2) \geq 0$ exactly as in part (b). We take the upper bound from part (a) to get

$$0 \leq \cos(2) \leq 1 - \frac{2^2}{2} + \frac{2^4}{24} = -\frac{1}{3}$$

which is absurd, so we must have $\frac{\pi}{2} < 2$, or $\pi < 4$.
The same argument works if we instead suppose that $1.6 \leq \frac{\pi}{2}$: then

$$0 \leq \cos(1.6) \leq 1 - \frac{1.6^2}{2} + \frac{1.6^4}{24} = -\frac{13}{1875}$$

again gives us a contradiction, so then $\frac{\pi}{2} < 1.6$, or $\pi < 3.2$.

5. Fix an integer $r \geq 0$ and define $f : [1, b] \to \mathbb{R}$ by $f(x) = x^r$, where $b > 1$.

(a) Let $P_n = (1, b^{1/n}, b^{2/n}, \dots, b^{(n-1)/n}, b)$ be a partition of $[1, b]$. Compute the lower Darboux sum $L(f, P_n)$, and show that $U(f, P_n) = b^{r/n} L(f, P_n)$.

(b) Prove that $\lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} U(f, P_n)$, and compute their common value.

Solution. (a) Since $f(x)$ is monotone increasing, we compute that

$$m_i = \inf_{t\in[b^{i/n}, b^{(i+1)/n}]} t^r = b^{ir/n}, \qquad M_i = \sup_{t\in[b^{i/n}, b^{(i+1)/n}]} t^r = b^{(i+1)r/n}.$$

On each interval $[b^{i/n}, b^{(i+1)/n}]$ we have $\Delta x_i = b^{i/n}(b^{1/n} - 1)$, so

$$\begin{aligned} L(f, P_n) &= \sum_{i=0}^{n-1} b^{ir/n} \cdot b^{i/n}(b^{1/n} - 1) = (b^{1/n} - 1) \sum_{i=0}^{n-1} (b^{(r+1)/n})^i \\ &= (b^{1/n} - 1) \frac{b^{r+1} - 1}{b^{(r+1)/n} - 1} \\ &= \frac{b^{r+1} - 1}{b^{r/n} + b^{(r-1)/n} + \cdots + b^{1/n} + 1}. \end{aligned}$$

Similarly, we compute that

$$\begin{aligned} U(f, P_n) &= \sum_{i=0}^{n-1} b^{(i+1)r/n} \cdot b^{i/n}(b^{1/n} - 1) \\ &= b^{r/n} \cdot \sum_{i=0}^{n-1} b^{ir/n} \cdot b^{i/n}(b^{1/n} - 1) = b^{r/n} L(f, P_n). \end{aligned}$$

(b) We note that $\lim_{n\to\infty} L(f, P_n) = \dfrac{b^{r+1} - 1}{r + 1}$. In particular, the sequence $\big(L(f, P_n)\big)$ is bounded above by any single value $U(f, P_n)$, so we write $L(f, P_n) < C$ for some constant $C > 0$, and then we have

$$0 \leq U(f, P_n) - L(f, P_n) = (b^{r/n} - 1)L(f, P_n) < C(b^{r/n} - 1)$$

for all $n \geq 0$ by part (a). The right side approaches 0 as $n \to \infty$, hence so does the middle part by the squeeze theorem, and this means that

$$\lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} (U(f, P_n) - L(f, P_n)) + \lim_{n\to\infty} L(f, P_n)$$

exists and is equal to $0 + \lim_{n\to\infty} L(f, P_n) = \dfrac{b^{r+1} - 1}{r + 1}$, by the algebra of limits.
Remark: we don't really need r to be an integer, since we can still evaluate

$$\begin{aligned}
\lim_{n\to\infty} \frac{b^{r+1} - 1}{b^{r/n} + b^{(r-1)/n} + \cdots + b^{1/n} + 1} &= (b^{r+1} - 1) \lim_{n\to\infty} \frac{b^{1/n} - 1}{b^{(r+1)/n} - 1} \\
&= (b^{r+1} - 1) \lim_{x\downarrow 0} \frac{b^{x} - 1}{b^{(r+1)x} - 1} \\
&= \frac{b^{r+1} - 1}{r + 1}
\end{aligned}$$

using l'Hôpital's rule.