Sheet 0 Solutions

1. (i) $\mathbf{A}\cdot\mathbf{B} = (\mathbf{i}+3\mathbf{j}-2\mathbf{k})\cdot(4\mathbf{i}-2\mathbf{j}+4\mathbf{k}) = 4-6-8=-10$. (ii) $|\mathbf{A}| = \sqrt{1^2+3^2+2^2} = \sqrt{14}$.
(iii) $\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -2 \\ 4 & -2 & 4 \end{vmatrix} = 8\mathbf{i}-12\mathbf{j}-14\mathbf{k}$.
(iv) $(2\mathbf{A}+\mathbf{B})\cdot(\mathbf{A}-2\mathbf{B}) = (6\mathbf{i}+4\mathbf{j})\cdot(-7\mathbf{i}+7\mathbf{j}-10\mathbf{k}) = -42+28=-14$.

2. Consider a parallelogram with sides $\mathbf{B}$ and $\mathbf{C}$ and let $\widehat{\mathbf{n}}$ be the unit normal to this parallelogram in the direction $\mathbf{B}\times\mathbf{C}$, so that $\widehat{\mathbf{n}} = (\mathbf{B}\times\mathbf{C})/\left|\mathbf{B}\times\mathbf{C}\right|$.
The area of the parallelogram is $|\mathbf{B}\times\mathbf{C}|$ and it follows that the volume V of the parallelepiped is $|\mathbf{B}\times\mathbf{C}|\,h$ where $h = \mathbf{A}\cdot\widehat{\mathbf{n}}$ (>0) is the perpendicular height.
Therefore $V = (\mathbf{A}\cdot\widehat{\mathbf{n}})\,|\mathbf{B}\times\mathbf{C}| = \mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$ upon substituting for $\widehat{\mathbf{n}}$ from above.
If the system is not right-handed then $\mathbf{A}\cdot\widehat{\mathbf{n}}<0$ and so $V = \left|\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})\right|$.

3. (i) Differentiating, $\partial x/\partial u = 3u^2+v$ and so $\partial^2 x/\partial v\partial u = (\partial/\partial v)(\partial x/\partial u) = 1$.
We also have $\partial x/\partial v = u+3v^2$ and so $\partial^2 x/\partial u\partial v = 1$.
In a similar way $\partial y/\partial u = 2u, \partial y/\partial v = -2v$ and so $(\partial/\partial u)(\partial y/\partial v) = (\partial/\partial v)(\partial y/\partial u) = 0$.
(ii) Differentiating implicitly wrt x we have $1 = 3u^2u_x + vu_x + uv_x + 3v^2v_x$ and $0 = 2uu_x - 2vv_x$.
This can be written in matrix form as $\begin{pmatrix} 3u^2+v & u+3v^2 \\ 2u & -2v \end{pmatrix}\begin{pmatrix} u_x \\ v_x \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
Letting $u=1, v=0$ we have $\begin{pmatrix} u_x \\ v_x \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1/2 \\ 1 & -3/2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
and so $u_x = 0, v_x = 1$.
To get u_y and v_y we go back and differentiate implicitly wrt y. This gives $\begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} u_y \\ v_y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
After inverting the matrix we find $u_y = 1/2, v_y = -3/2$.

4. (i) First $(\partial^2/\partial x^2)(r) = (\partial/\partial x)(\partial r/\partial x) = (\partial/\partial x)(\partial/\partial x)(x^2+y^2+z^2)^{1/2} = (\partial/\partial x)(x/r) =$
$(1/r) - (x/r^2)(\partial r/\partial x) = 1/r - x^2/r^3$.
Similarly: $(\partial^2/\partial y^2)(r) = 1/r - y^2/r^3, (\partial^2/\partial z^2)(r) = 1/r - z^2/r^3$.
Hence $\nabla^2(r) = 3/r - r^2/r^3 = 2/r$.
(ii) This time consider $(\partial^2/\partial x^2)(r^{-1}) = (\partial/\partial x)(-r^{-2}\partial r/\partial x) = (\partial/\partial x)(-x/r^3) =$
$-1/r^3 + 3xr^{-4}\partial r/\partial x = -r^{-3} + 3x^2r^{-5}$.
Again we can obtain similar expressions for the y and z derivatives.
Putting them together: $\nabla^2(1/r) = -3r^{-3} + 3(x^2+y^2+z^2)r^{-5} = 0$.

5. (i)$\int_0^{2\pi}\sin^2\theta\;d\theta = (1/2)\int_0^{2\pi}(1-\cos 2\theta)\;d\theta = (1/2)\left[\theta - (1/2)\sin 2\theta\right]_0^{2\pi} = \pi$.
(ii) As above but write $\cos^2\theta = (1/2)(1+\cos 2\theta)$. Again the answer is equal to π.
(iii) Letting $u = \cos\theta$ the integral becomes $-\int_1^{-1} u^2 du = 2/3$.
(iv) Write $\cos^4\theta = (1/2)^2(1+\cos 2\theta)^2 = 1/4 + (1/2)\cos 2\theta + (1/4)(1/2)(1+\cos 4\theta) =$
$3/8 + (1/2)\cos 2\theta + (1/8)\cos 4\theta$.
After integrating, the trigonometric functions vanish, leaving an answer of $3\pi/4$.
(v) Write $\sin^4\theta = (1/2)^2(1-\cos 2\theta)^2$ and proceed as above. The answer is $3\pi/4$ once more.

6. In polar coordinates a small element of area $dxdy$ can be represented by $(rd\theta)(dr)$.
So, with $x = r\cos\theta, y = r\sin\theta$ we can write the integral as
$I = \int_0^{2\pi}\int_0^a (r^2\cos^2\theta - 2r^2\sin^2\theta)\;r\;dr\;d\theta = (\int_0^a r^3dr)(\int_0^{2\pi}\cos^2\theta - 2\sin^2\theta\;d\theta) = (a^4/4)(\pi - 2\pi) = -\pi a^4/4$.

7. (i) The integrating factor is $\exp(-3\int(x+1)^{-1}dx) = (x+1)^{-3}$.
Multiplying the ODE by this quantity we get $((x+1)^{-3}y)' = (x+1)$ and hence $(x+1)^{-3}y = x^2/2 + x + C$.
The solution is therefore $y = (x+1)^3(x^2/2 + x + C)$.
(ii) The homogeneous solution is $y_{HS} = A\cos x + B\sin x$.
For the particular solution we try $y_{PS} = \mu\cos 2x + \lambda\sin 2x$.
Then $y''_{PS} + y_{PS} = -3\mu\cos 2x - 3\lambda\sin 2x = \cos 2x$. Thus we need $\mu = -1/3, \lambda = 0$.
The general solution is therefore $y = A\cos x + B\sin x - (1/3)\cos 2x$.
(iii) The homogeneous solution is as in (ii).
The RHS is now contained in that solution so we need to modify y_{PS}.

We try $y_{PS} = Cx\cos x + Dx\sin x$ and find after some algebra that $y''_{PS} + y_{PS} = 2D\cos x - 2C\sin x = \cos x$. We therefore require $D = 1/2, C = 0$ and the solution is $y = A\cos x + B\sin x + (1/2)x\sin x$.

8. Using the substitution $x = e^t$ we see that $dy/dt = (dy/dx)(dx/dt) = xdy/dx$ and $d^2y/dt^2 = x(d/dx)(xdy/dx) = x^2d^2y/dx^2 + xdy/dx$.
Our ODE becomes $d^2y/dt^2 + dy/dt - 2y = e^{-t}$. This may now be solved in the standard way to yield $y = Ae^{-2t} + Be^t - (1/2)e^{-t}$. Writing back in terms of x we have $y = A/x^2 + Bx - 1/(2x)$.

9. If we let $\beta^2 = 1 - \lambda$ then the general solution of the ODE is $y = A\cos\beta x + B\sin\beta x$.
Applying the boundary conditions $y(0) = A = 0, y(\pi) = B\sin(\beta\pi) = 0$. Thus we need $\beta = \pm 1, \pm 2. \pm 3, \ldots$
The possible values of λ are therefore $0, -3, -8$ etc.