Mathematical Logic
Concise Notes

MATH60132

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Content from prior years assumed to be known.

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Contents

1 Propositional Logic

1.1 Propositional formula

Definition 1.1.1. *Proposition - a statement, either* ***True (T), (1)*** *or* ***False (F), (0)****, represented using* ***propositional variables***
Connectives and Truth Tables
For $\{p, \dots, q\}$ *a set of propositions, combine them using the following connectives*

1. ***Negation*** $(\neg p)$
2. ***Conjunction*** $(p \wedge q)$
3. ***Disjunction*** $(p \vee q)$
4. ***Implication*** $p \to q$
5. ***Biconditional*** $p \leftrightarrow q$

p	q	$p \wedge q$	$p \vee q$	$p \to q$	$p \leftrightarrow q$
0	0	0	0	0	1
0	1	0	1	1	0
1	0	0	0	1	0
1	1	1	0	1	1

Definition 1.1.2. *A* ***propositional formula*** *is obtained in the following way*

- *Any propositional variable a formula*
- *if* ϕ, ψ *are formulas then so are*

$$(\neg\phi),\ (\phi \vee \psi),\ (\phi \wedge \psi),\ (\phi \to \psi),\ (\phi \leftrightarrow \psi)$$

- *Any formula arises this way*

Definition 1.1.3. *Let* $n \in \mathbb{N}$

- *A* ***truth function*** *of* n *variables a function* $f : \{T, F\}^n \to \{T, F\}$
- *Suppose* ϕ *a formula with variables amongst* $p_1, \dots, p_n$

$$F_\phi : \{T, F\}^n \to \{T, F\}$$

whose values at $\big(x_1, \dots, x_n\big)$ *is the truth value of* ϕ *when* p_i *has value* x_i *for* $i = 1, \dots, n$
F_ϕ *the* ***truth function*** *of* ϕ

Definition 1.1.4. *We have the following*

- *A propositional formula* ϕ *a* ***tautology*** *if its truth function* F_ϕ *always has value* T
- *Say* ϕ, ψ *are* ***logically equivalent (LE)*** *if they have the same truth function,* $(F_\phi = F_\psi)$

Lemma 1.1.7. *There are* 2^{2^n} *truth functions of* n *variables*

Definition 1.1.8. *Say a set of connectives is* ***adequate*** *if for every* $n \geq 1$*, every truth function of* n *variables is the truth function of some formula involving only connectives from the set and variables* $p1, \dots, p_n$

Theorem 1.1.9. *Set* $\{\neg, \vee, \wedge\}$ *is adequate*

Corollary 1.1.10. *Suppose χ a formula whose truth function not always F. Then χ logically equivalent to formula in disjunctive normal form.*

Corollary 1.1.11. *The following set of connectives are adequate*

- $\{\neg, \vee\}$
- $\{\neg, \wedge\}$
- $\neg, \rightarrow$

We also have the ***(NOR)*** *connective $\{\downarrow\}$ is adequate*

p	q	$p \downarrow q$
T	T	F
T	F	F
F	T	F
F	F	T

1.2 A formal system for propositional logic

Definition 1.2.1.

- *A* ***formal deduction system*** *Σ has the following*
  - *An alphabet $A \neq \varnothing$ of symbols*
  - *A non-empty set $\mathcal{F}$ of the set of all finite sequence,* ***strings****, of elements of A the* ***formulas*** *of Σ*
  - *A subset $\mathcal{A} \subseteq \mathcal{F}$ called the* ***axioms*** *of Σ*
  - *A collection of* ***deduction rules***
- *A* ***proof*** *in Σ a finite sequence of formulas in $\mathcal{F}$ $\phi_1, \ldots, \phi_n$ such that each ϕ_i either an axiom, or obtained from $\phi_1, \ldots, \phi_{i-1}$ using one of the deduction rules.*
- *The last, or any, formula in a proof a* ***theorem*** *of Σ. Write $\vdash_\Sigma \phi$.*

Definition 1.2.2. *The formal system $\mathcal{L}$ for propositional logic has the following*

- *A* ***Alphabet*** *consisting of*
  - *variables, $p_1, \ldots, p_n$*
  - *connective, $\{\neg, \rightarrow\}$*
  - *punctuation, $), ($*
- *Formulas: finite strings of symbols from alphabet as follows*
  - *any variable p_i a formula*
  - *if ϕ, ψ formulas then so are $(\neg\pi)$ and $()\phi \rightarrow \psi)$*
  - *Any formula arises this way*
- *Axioms, suppose ϕ, ψ, χ are L-formulas. We have the following axioms for $\mathcal{L}$*

  (A1) $(\phi \rightarrow (\psi \rightarrow \phi))$

  (A2) $((\phi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \chi)))$

  (A3) $(((\neg\psi) \rightarrow (\neg\phi)) \rightarrow (\phi \rightarrow \psi))$
- *Deduction rule*

  (MP) ***Modus Ponens****. From ϕ and $(\phi \rightarrow \psi)$, deduce ψ*

Definition 1.2.4. *Suppose Γ a set of L-formulas*

- *A **deduction** from Γ a finite sequence of L-formulas $\phi_1, \ldots, \phi_n$ s.t each ϕ_i either an axiom, a formula in Γ or obtained from previous formulas via MP*
- *Write $\Gamma \vdash_L \phi$ if there is a deduction from Γ ending in ϕ. Say ϕ a **consequence** of Γ.* *$\varnothing \vdash_L \phi$ same as $\vdash_L \phi$*

Theorem 1.2.5. *(Deduction Theorem)*
Suppose Γ a set of L-formulas and ϕ, ψ L-formulas.
Suppose $\Gamma \cup \{\phi\} \vdash_L \psi$ then $\Gamma \vdash_L (\phi \to \psi)$

Corollary 1.2.6. *(Hypothetical syllogism)*
Suppose ϕ, ψ, χ L-formulas, and $\vdash_L (\phi \to \psi)$ and $\vdash_L (\psi \to \chi)$ Then $\vdash_L (\phi \to \chi)$

Proposition 1.2.7. *Suppose ϕ, ψ are L-formulas. Then*

1. $\vdash_L ((\neg\phi) \to (\psi \to \phi))$
2. $\{(\neg\psi), \psi\} \vdash_L \phi$
3. $\vdash_L (((\neg\phi) \to \phi) \to \phi)$

1.3 Soundness and completeness of L

Theorem 1.3.1. *(Soundness of L)*
Suppose ϕ a theorem of L. Then ϕ a tautology

Definition 1.3.2. *A **propositional valuation** v an assignment of truth values to the propositional variables $p_1, \ldots, p_n$ So*

$$v(p_i) = T, F \quad i \in \mathbb{N}$$

Theorem 1.3.3. *(Generalisation of Soundness)*
Suppose Γ a set of formulas and ϕ a formula with $\Gamma \vdash_L \phi$ Suppose v a valuation with $v(\psi) = T, \forall \psi \in \Gamma$ Then $v(\phi) = T$

Theorem 1.3.4. *(Completeness (adequacy) of L)*
Suppose ϕ a tautology, i.e. $v(\phi) = T, \forall v$. Then $\vdash_L \phi$

Definition 1.3.6. *A set Γ of L-formulas is **consistent** if there is no L-formula ϕ such that $\Gamma \vdash_L \phi$ and $\Gamma \vdash_L (\neg\phi)$*

Proposition 1.3.7. *Suppose Γ a consistent set of L-formulas and $\Gamma \nvdash_L \phi$ Then $\Gamma \cup \{(\neg\phi)\}$ is consistent*

Proposition 1.3.8. *(Lindenbaum Lemma)*
*Suppose Γ a set of L-formulas. Then there is a consistent set of formulas $\Gamma^* \supseteq \Gamma$ s.t for every ϕ either $\Gamma^* \vdash_L \phi$ or $\Gamma^* \vdash_L (\neg\phi)$. Say Γ^* is **complete***

Lemma 1.3.9. *Let Γ^* as above. Then $\exists$ valuation v s.t for every L-formula ϕ, $v(\phi) = T$ iff $\Gamma^* \vdash_L \phi$*

Corollary 1.3.10. *Suppose Δ a consistent set of L-formulas, and $\Delta \nvdash_L \phi$ Then there is a valuation v s.t $v(\Delta) = T$ and $v(\phi) = F$*

Corollary 1.3.11. *Suppose Δ a set of L-formulas and ϕ an L-formula. Then*

1. *Δ consistent iff there is a valuation v with $v(\Delta) = T$, and*
2. *$\Delta \vdash_L \phi$ iff, for every valuation v with $v(\Delta) = T$, we have $v(\phi) = T$*

Theorem 1.3.12. *(Compactness theorem for L)*
Suppose Δ a set of L-formulas. The following are equivalent

1. *There is a valuation v s.t $v(\Delta) = T$*
2. *For every finite subset $\Delta_0 \subseteq \Delta$, there is a valuation w s.t $w(\Delta_0) = T$*

2 Predicate Logic

2.1 Structures

Definition 2.1.1. *Suppose A a set and $n \in \mathbb{N}_{\geq 1}$*

- *An n-**ary** relation on A a subset*
$$\overline{R} \subseteq A^n = \{(a_1, \ldots, a_n) \mid a_i \in A\}$$
- *An n-**ary** function on A a function*
$$\overline{f} : A^n \to A$$

Definition 2.1.2. *A **first-order structure** $\mathcal{A}$ consists of*

- *A non-empty set A, the domain of $\mathcal{A}$*
- *A set of relations on A*
$$\{\overline{R}_i \subseteq A^{n_i} \mid i \in I\}$$
- *A set of functions on A*
$$\{\overline{f}_j : A^{m_j} \to A \mid j \in J\}$$
- *A set of constants, elements of A*
$$\{\overline{c}_k \mid k \in K\}$$

  I, J, K simply indexing sets, which can be empty
$$(n_i \mid i \in I),\ (m_j \mid j \in J),\ K$$

*called the **signature** of $\mathcal{A}$*
Denote the structure by

$$\begin{aligned}\mathcal{A} &= \big\langle A; (\overline{R}_i \mid i \in I), (\overline{f}_j \mid j \in J), (\overline{c}_k \mid k \in K)\big\rangle \\ &= \langle \text{domain; relations, functions, constants}\rangle\end{aligned}$$

2.2 First-order languages

Definition 2.2.1. *A **first-order-language** $\mathcal{L}$ has an alphabet of symbols of the following types I, J, K*

Variables	x_0	x_1
Connectives	$\neg$	$\to$
Punctuation	( )	,
Quantifier	$\forall$	
Relation symbols	$R_i,\ i \in I$	
Function symbols	$f_j, j \in J$	
Constant symbols	$c_k, k \in K$	

indexing sets, which could have $J, K = \varnothing$

- *Each R_i comes equipped with arity n_i*
- *Each f_j comes equipped with arity m_j*

$$(n_i \mid i \in I) \quad (m_j \mid j \in J), \quad K$$

*Above called the **signature** of $\mathcal{L}$*

Definition 2.2.2. *A **term** of $\mathcal{L}$ defined as follows*

- *Any variable is a term*
- *Any constant symbol is a term*
- *If f an m-ary function symbol of $\mathcal{L}$ and $t_1, \ldots, t_m$ are terms then*

  $$f(t_1, \ldots, t_m)$$

  also a term
- *Any term arises this way*

Definition 2.2.3. *Use previous notation*

- *An **atomic formula** of $\mathcal{L}$ is of the form*

  $$R(t_1, \ldots, t_n)$$

  Where R an n-ary relation symbol of $\mathcal{L}$ and $t_1, \ldots, t_n$ are terms
- *Formulas of $\mathcal{L}$ are defined as follows*
  - *Any atomic formula is a formula*
  - *If ϕ, ψ are L-formulas, then*

    $$(\neg\phi),\ (\phi \to \psi),\ (\forall x)\phi$$

    are L-formulas, where x is any variable
  - *Every L-formula arises in this way*

Definition 2.2.4. *Suppose ϕ, ψ are L-formulas*

- *$(\exists x)\phi$ means $(\neg(\forall x)(\neg\phi))$*
- *$(\phi \vee \psi)$ means $((\neg\phi) \to \psi)$*

Definition 2.2.5. *(Interpretation)*
Suppose $\mathcal{L}$ a first-order-language with relation symbols, R_i of arity $n_i, i \in I$, functions symbols f_j of arity $m_j, j \in J$ and constant symbols $c_k, k \in K$
*An **$\mathcal{L}$-structure** is a structure*

$$\mathcal{A} = \left\langle A; (\overline{R}_i \mid i \in I), (\overline{f}_j \mid j \in J), (\overline{c}_k \mid k \in K) \right\rangle$$

of the same signature as $\mathcal{L}$
The correspondence

$$R_i \leftrightsquigarrow \overline{R_i}, \quad f_j \leftrightsquigarrow \overline{f_j}, \quad c_k \leftrightsquigarrow \overline{c_k}$$

*called an **interpretation** of $\mathcal{L}$*

Definition 2.2.6. *(Valuation)*
With the same notation, suppose $\mathcal{A}$ an $\mathcal{L}$-structure. A valuation in $\mathcal{A}$ is a function v from the set of terms of $\mathcal{L}$ to A satisfying

- *$v(c_k) = \overline{c_k}$*
- *if $t_1, \ldots, t_m$ are terms of $\mathcal{L}$ and f a m-ary function symbol then*

  $$v(f(t_1, \ldots, t_m)) = \overline{f}(v(t_1), \ldots, v(t_m)),$$

  where $\overline{f}$ an interpretation of f in $\mathcal{A}$

Lemma 2.2.7. *Suppose $\mathcal{A}$ an $\mathcal{L}$-structure and $a_0, a_1, \ldots \in A$. Then there is a unique valuation v in $\mathcal{A}$ with $v(x_l) = a_l, \forall l \in \mathbb{N}$ where variables of $\mathcal{L}$ are $x_0, x_1, \ldots$*

Definition 2.2.8. *Suppose $\mathcal{A}$ an $\mathcal{L}$-structure and x_l any variable. Suppose v, w are valuations in $\mathcal{A}$. Say v, w are x_l-**equivalent** if $v(x_m) = w(x_m)$, whenever $m \neq l$*

Definition 2.2.9. *Suppose $\mathcal{A}$ an $\mathcal{L}$-structure and v a valuation in $\mathcal{A}$*
Define for an $\mathcal{L}$-formula ϕ what is meant by v satisfies ϕ in $\mathcal{A}$

- *Atomic formulas. Suppose R an $n-$ary relation symbol and $t_1, \ldots, t_n$ are terms of $\mathcal{L}$ Then*

$$v \text{ satisfies the atomic formula } R(t_1, \ldots, t_n) \iff \overline{R}(v(t_1), \ldots, v(t_n)) \text{ holds in } \mathcal{A}$$

- *Suppose ϕ, ψ are $\mathcal{L}$-formulas*

$$\begin{aligned} v \text{ satisfies } (\neg A) \text{ in } \mathcal{A} &\iff v \text{ does not satisfy } \phi \text{ in } \mathcal{A} \\ v \text{ satisfies } (\phi \to \psi) \text{ in } \mathcal{A} &\iff \text{it is not the case that } v \text{ satisfies } \phi \text{ in } \mathcal{A} \text{ and } v \text{ does not satisfy } \psi \text{ in } \mathcal{A} \\ v \text{ satisfies } (\forall x_l)\phi \text{ in } \mathcal{A} &\iff \text{whenever } w \text{ a valuation in } \mathcal{A} \text{which is } x_l - \text{equivalent to } v, \text{ then } w \text{ satisfies } \phi \text{ in } \mathcal{A} \end{aligned}$$

Notation:
If v satisfies ϕ write $v[\phi] = T$ if not write $v[\phi] = F$
If every valuation in $\mathcal{A}$ satisfies ϕ say that ϕ is **true** in $\mathcal{A}$ or $\mathcal{A}$ a model of ϕ
Write $A \models \phi$, if $\mathcal{A} \models \phi$ for every $\mathcal{L}$-structure $\mathcal{A}$ say that ϕ is **logically valid**, and write $\models \phi$

Definition 2.2.13. *Suppose χ an $\mathcal{L}$-formula involving propositional variables $p_1, \ldots, p_n$. Suppose $\mathcal{L}$ a first-order language and $\phi_1, \ldots, \phi_n$ are $\mathcal{L}$-formulas.*
*A **substitution instance** of χ is obtained by replacing each p_i in χ by ϕ_i. Call the result θ*

Theorem 2.2.14. *We have*

- *θ an $\mathcal{L}$-formula, and*
- *if χ a tautology, then θ is logically valid*

Note: *not all logically valid formulas arise this way*

2.3 Bound and free variables in formula

Definition 2.3.1. *Suppose ϕ, ψ are $\mathcal{L}$-formulas, with $(\forall x_i)\phi$ occurring as a sub-formula of ψ*

- *Say ϕ the **scope** of a that quantifier $(\forall x_i)$ here in ψ*
  *An occurrence of a variable x_j in ψ is **bound** if it is in the scope of a quantifier $(\forall x_j)$ in ψ or it is the x_j here in $(\forall x_j)$*
- *Otherwise, it is a **free** occurrence of x_j. Variables having a free occurrence in ψ are called **free variables** of ψ*
- *A formula with no free variables called a **closed formula** or a **sentence**, of $\mathcal{L}$*

Definition 2.3.2. *If ψ an $\mathcal{L}$-formula with free variables amongst $x_1, \ldots, x_n$, might write*

$$\psi(x_1, \ldots, x_n)$$

instead of ψ. If $t_1, \ldots, t_n$ are terms, by

$$\psi(t_1, \ldots, t_n)$$

we denote the $\mathcal{L}$-formula obtained by replacing each free occurrence of x_i in ψ by t_i

Theorem 2.3.3. *Suppose ϕ closed $\mathcal{L}$-formula and $\mathcal{A}$ an $\mathcal{L}$-structure. Then either $A \models \phi$ or $A \models (\neg\phi)$. More generally if ϕ has free variables amongst $x_1, \dots, x_n$ and v, w valuations in $\mathcal{A}$ with*

$$v(x_i) = w(x_i), \quad i = 1, \dots, n$$

Then $v[\phi] = T \iff w[\phi] = T$. Allow $n = 0$, for no free variables

Remark 2.3.4. *If $\mathcal{A}$ an $\mathcal{L}$-structure and $\psi(x_1, \dots, x_n)$ an $\mathcal{L}$-formula, whose free variables are amongst $x_1, \dots, x_n$ and $a_1, \dots, a_n \in A$ for domain A,then we write*

$$A \models \psi(a_1, \dots, a_n)$$

to mean $v[\psi] = T$ for every valuation v in $\mathcal{A}$ with

$$v(x_i) = a_i, \quad i = 1, \dots, n$$

Definition 2.3.5. *Let ϕ an $\mathcal{L}$-formula, x_i a variable, t an $\mathcal{L}$-term.*
*Say that t is **free for** x_i in ϕ if there is no variable x_j in t s.t x_i has a free occurrence within the scope of a quantifier $(\forall x_j)$ in ϕ*

Theorem 2.3.6. *Suppose $\phi(x_1)$ an $\mathcal{L}$-formula, possibly with other free variables. Let t be a term free for x_1 in ϕ, then*

$$\models ((\forall x_1)\phi(x_1) \to \phi(t))$$

In particular, if $\mathcal{A}$ an $\mathcal{L}$-structure, with $\mathcal{A} \models (\forall x_1 \phi(x_1),)$ then $A \models \phi(t)$

Lemma 2.3.7. *Suppose v a valuation in $\mathcal{A}$. Let v' be the valuation in $\mathcal{A}$ which is x_1-equivalent to v with $v'(x_1) = v(t)$. Then $v'[\phi(x_1)] = T \iff v[\phi(t)] = T$*

2.4 The formal system $K_{\mathcal{L}}$

Definition 2.4.1. *Suppose $\mathcal{L}$ a first-order language. The formal system $K_{\mathcal{L}}$ has, as formulas, $\mathcal{L}$-formulas, and the following*

- *Axioms. For L-formulas, ϕ, ψ, χ*

(A1) $(\phi \to (\psi \to \phi))$

(A2) $((\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi)))$

(A3) $(((\neg\psi) \to (\neg\phi)) \to (\phi \to \psi))$

(K1) $((\forall x_i)\phi(x_i) \to \phi(t))$*, where t a term free for x_i in ϕ and ϕ can have other free variables*

(K2) $((\forall x_i)(\phi \to \psi) \to (\phi \to (\forall x_i)\psi))$*, if x_i is not free in ϕ*

- *Deduction rules:*

(MP) Modus Ponens. From ϕ and $\phi \to \psi$, deduce ψ

*(Gen) **Generalisation**. From ϕ, deduce $(\forall x_i)\phi$*

*A **proof** in $K_{\mathcal{L}}$ a finite sequence of $\mathcal{L}$-formulas, each of which an axiom or deduced from previous formulas in proof using a deduction rule.*
*A **theorem** of $K_{\mathcal{L}}$ the last (or any) formula in some proof.*
Write: $\vdash_{K_{\mathcal{L}}}$ if ϕ a theorem in $K_{\mathcal{L}}$

Definition 2.4.2. *Suppose Σ a set of $\mathcal{L}$-formulas and ψ an $\mathcal{L}$-formula. A* ***deduction*** *of ψ from Σ a finite sequence of formulas, ending with ψ,each of which is one of*

- *an axiom*
- *a formula in Σ*
- *obtained from earlier formulas using a deduction rule, with restriction that when Gen applied, it does not involve a variable occurring freely in Σ*

Write $\Sigma \vdash_{K_{\mathcal{L}}} \psi$ if there is a deduction from Σ to ψ

Remark 2.4.3. *We have*

- *if Σ consists of closed formulas, do not need to worry about the restriction on Gen*
- *$\phi \vdash_{K_{\mathcal{L}}} \psi$ if there is a deduction from Σ to ψ*
- *Without the restriction would have*

$$\{\phi\} \vdash (\forall x_i)\phi, \text{ not sensible}$$

- *Should have, if $\Sigma' \subseteq \Sigma$ and $\Sigma' \vdash \phi \implies \Sigma \vdash \phi$, So we modify the definition accordingly*

Theorem 2.4.4. *Suppose ϕ an $\mathcal{L}$-formula, which is a substitution instance of a propositional tautology χ, then*

$$\vdash_{K_{\mathcal{L}}} \phi$$

Theorem 2.4.5. *(Soundness of $K_{\mathcal{L}}$)*
If $\vdash_{K_{\mathcal{L}}} \phi$, then $\models \phi$, that is it is logically valid

Corollary 2.4.6. *(Consistency of $K_{\mathcal{L}}$)*
There is no formula, ϕ, with $\vdash_{K_{\mathcal{L}}} \phi$ and $\vdash_{K_{\mathcal{L}}} (\neg\phi)$

Theorem 2.4.8. *(Deduction theorem)*
Supposed $\mathcal{L}$ a first-order language, Σ a set of $\mathcal{L}$-formulas, and ϕ, ψ are $\mathcal{L}$-formulas.
Then if $\Sigma \cup \{\phi\} \vdash_{K_{\mathcal{L}}} \psi \implies \Sigma \vdash_{K_{\mathcal{L}}} (\phi \to \psi)$

2.5 Gödel's completeness theorem

Definition 2.5.1. *A set Σ of $\mathcal{L}$-formulas is* ***consistent*** *if there is no formula ϕ with*

$$\Sigma \vdash_{K_{\mathcal{L}}} \phi, \quad \Sigma \vdash_{K_{\mathcal{L}}} (\neg\phi)$$

By Soundness/ 2.4.7, $\varnothing$ is consistent so $K_{\mathcal{L}}$ is consistent

Remark 2.5.2. *If Σ inconsistent, then*

$$\Sigma \vdash_{K_{\mathcal{L}}} \chi, \quad \forall \ L\text{-formula } \chi$$

Proposition 2.5.2. *Suppose Σ a consistent set of closed L-formulas and ϕ a closed L-formula.*

1. *Comparing 1.3.7, if $\Sigma \nvdash_{K_{\mathcal{L}}} \phi$, then $\Sigma \cup \{(\neg\phi)\}$ is consistent*
2. *Comparing the Lindenbaum lemma (1.3.8), there is a consistent set $\Sigma^* \supseteq \Sigma$ of closed L-formulas such that for every closed L-formula ϕ, either $\Sigma^* \vdash_{K_{\mathcal{L}}} \psi$ or $\Sigma^* \vdash_{K_{\mathcal{L}}} (\neg\phi)$*

Theorem 2.5.3. *(Model existence theorem)*
Suppose Σ a consistent set of closed L-formulas. Then there is a countable L-structure $\mathcal{A}$ such that

$$A \models \Sigma, \ i.e \ A \models \sigma, \forall \sigma \in \Sigma$$

Theorem 2.5.4. *Let Σ a set of closed L-formulas, ϕ a closed L-formula.*
If every model Σ is a model of ϕ, then $\Sigma \vdash_{K_{\mathcal{L}}} \phi$. That is

$$\text{if } \mathcal{A} \models \Sigma, \text{ or } \mathcal{A} \models \sigma, \forall \sigma \in \Sigma \implies \mathcal{A} \models \phi, \text{ then } \Sigma \vdash_{K_{\mathcal{L}}} \phi$$

Theorem 2.5.5. *(Gödel's completeness theorem for $K_{\mathcal{L}}$)*
If ϕ an L-formula with $\models \phi$, then ϕ a theorem of $K_{\mathcal{L}}$ i.e. $\vdash_{K_{\mathcal{L}}} \phi$

Corollary 2.5.6. *(Compactness theorem for $K_{\mathcal{L}}$)*
Suppose Σ a set of closed L-formulas and every finite subset of Σ has a model. Then Σ has a model.

2.6 Equality

Definition 2.6.1. *Suppose $\mathcal{L}^E$ a first-order language with a distinguished binary relation symbol E*

- *An $\mathcal{L}^E$-structure in which E is interpreted as equality $=$ is a **normal** $\mathcal{L}^E$-structure*
- *The following are **axioms of equality**, Σ_E*
  - $(\forall x_1)E(x_1, x_1)$
  - $(\forall x_1)(\forall x_2)(E(x_1, x_2) \to E(x_2, x_1))$
  - $(\forall x_1)(\forall x_2)(\forall x_3)(E(x_1, x_2) \to (E(x_2, x_3) \to E(x_1, x_3)))$
  - *For each n-ary relation symbol R of $\mathcal{L}^E$*

    $$(\forall x_1, \ldots, x_n)(\forall y_1, \ldots, y_n)((R(x_1, \ldots, x_n) \wedge E(x_1, y_1) \wedge \ldots \wedge E(x_n, y_n)) \to R(y_1, \ldots, y_n))$$

  - *For each m-ary function symbol f of $\mathcal{L}^E$*

    $$(\forall x_1, \ldots, x_m)(\forall y_1, \ldots, y_m)((E(x_1, y_1)) \wedge \ldots \wedge E(x_m, y_m)) \to E(f(x_1, , \ldots, x_m), f(y_1, \ldots, y_m))$$

Remark 2.6.2. *Some remarks/defs*

- *If $\mathcal{A}$ a normal $\mathcal{L}^E$-structure, then $A \models \Sigma_E$*
- *Suppose $\mathcal{A} = \langle A; \overline{E}, \ldots \rangle$ an $\mathcal{L}^E$-structure and $A \models \Sigma_E$. Then $\overline{E}$ an equivalence relation on A*
  Denote for $a \in A$
  $$\hat{a} = \{b \in A \mid \overline{E}(a, b) \text{ holds}\}$$
  the equivalence class of a. Let
  $$\hat{A} = \{\hat{a} \mid a \in A\}$$
  Make $\hat{A}$ into an $\mathcal{L}^E$-structure $\hat{\mathcal{A}}$
  - *if R an n-ary relation symbol, $\hat{a}_1, \ldots, \hat{a}_n \in \hat{A}$*
    Say $\overline{R}(\hat{a}_1, \ldots, \hat{a}_n)$ holds in $\hat{\mathcal{A}} \iff \overline{R}(a_1, \ldots, a_n)$holds in $\mathcal{A}$, this is well defined by Σ_E
  - *Similarly, if f an $m-$ary function symbol and $\hat{a}_1, \ldots, \hat{a}_m \in \hat{A}$ let*

    $$\overline{f}(\hat{a}_1, \ldots, \hat{a}_m) = \overline{f}(a_1 \widehat{, \ldots ,} a_m)$$

    This also well defined by Σ_E
  - *if c a constant symbol, then interpret c as $\hat{\overline{c}}$ in $\hat{\mathcal{A}}$, where $\overline{c}$ the interpretation in $\mathcal{A}$*

Lemma 2.6.3. *Suppose $\mathcal{A}$ an $\mathcal{L}^E$-structure with $\mathcal{A} \models \Sigma_E$. Let v a valuation in $\mathcal{A}$. Let $\hat{\mathcal{A}}$ be as given above. Let $\hat{v}$ be the valuation in $\hat{\mathcal{A}}$ with*
$$\hat{v}(x_i) = \widehat{v(x_i)}$$
Then for every $\mathcal{L}^E$-formula, $\phi, \hat{v}$ satisfies ϕ in $\hat{\mathcal{A}} \iff v$ satisfies ϕ in $\mathcal{A}$
In particular, if ϕ is closed, then $\mathcal{A} \models \phi \iff \hat{\mathcal{A}} \models \phi$

Lemma 2.6.4. *Suppose Δ a set of closed $\mathcal{L}^E$-formulas*
*Then Δ has a **normal model**, that is a normal $\mathcal{L}^E$-structure, $\mathcal{B}$ with $\mathcal{B} \models \sigma, \forall \sigma \in \Delta \iff \Delta \cup \Sigma_E$ has a model.*

Theorem 2.6.5. *(Compactness theorem for normal models)*
Suppose $\mathcal{L}^E$ a countable language with equality, and Δ a set of closed $\mathcal{L}^E$-formulas such that every finite subset of Δ has a normal model. Then Δ has a normal model

Notation: Write $\mathcal{L}^=$ instead of $\mathcal{L}^E$ and $x_1 = x_2$ instead of $E(x_1, x_2)$

Theorem 2.6.6. *(Countable downward Löwnenheim-Skolem theorem)*
Suppose $\mathcal{L}^=$ a countable first-order language, with equality and $\mathcal{B}$ a normal $\mathcal{L}^=$-structure
Then there is a countable normal $\mathcal{L}^=$-structure $\mathcal{A}$ such that, for every closed $\mathcal{L}^=$-formula, ϕ, $\mathcal{B} \models \phi \iff \mathcal{A} \models \phi$

2.7 Examples and applications

We let $\mathcal{L}^=$ be a first-order language with equality and binary relation symbol $\leq$

Definition 2.7.1. *We have*

- *A **linear order** $\mathcal{A} = \langle A; \leq_A \rangle$ a normal model of*

$$\begin{aligned}
&\phi_1 : (\forall x_1)(\forall x_2)(((x_1 \leq x_2) \wedge (x_2 \leq x_1)) \leftrightarrow (x_1 = x_2)) \\
&\phi_2 : (\forall x_1)(\forall x_2)(\forall x_3)(((x_1 \leq x_2) \wedge (x_2 \leq x_3)) \to (x_1 \leq x_3)) \\
&\phi_3 : (\forall x_1)(\forall x_2)((x_1 \leq x_2) \vee (x_2 \leq x_1))
\end{aligned}$$

- *it is **dense** if also*

$$\phi_4 : (\forall x_1)(\forall x_2)(\exists x_3)(\underbrace{(x_1 < x_2)}_{((x_1 \leq x_2) \wedge (x_1 \neq x_2))} \to ((x_1 < x_3) \wedge (x_3 < x_2)))$$

- *it is **without endpoints if***

$$\begin{aligned}
&\phi_5 : (\forall x_1)(\exists x_2)(x_1 < x_2) \\
&\phi_6 : (\forall x_1)(\exists x_2)(x_2 < x_1)
\end{aligned}$$

Let $\Delta = \{\phi_1, \ldots, \phi_6\}$

- *$\mathcal{Q} = \langle \mathbb{Q}; \leq \rangle$ a normal model of Δ*
- *$\mathcal{R} = \langle \mathbb{R}; \leq \rangle$ also a model of Δ*

Theorem 2.7.2. *We have*

1. *For every closed $\mathcal{L}^=$-formula ϕ $\mathcal{Q} \models \phi \iff \mathcal{R} \models \phi$*
2. *There is an algorithm which decides, given a closed $\mathcal{L}^=$-formula ϕ, whether $\mathcal{Q} \models \phi$ or $\mathcal{Q} \not\models \phi$, that is $\mathcal{Q} \models (\neg\phi)$ (by 2.3.3)*

Definition 2.7.3. *We have*

1. *Linear orders $\mathcal{A} = \langle A; \leq_A \rangle$ and $\mathcal{B} = \langle B; \leq_B \rangle$ are **isomorphic** if there is a bijection $\alpha : A \to B$ such that $\forall a, a' \in A, a \leq_A a' \iff \alpha(a) \leq_B \alpha(a')$*
2. *if $\mathcal{A}, \mathcal{B}$ isomorphic and ϕ closed, then $\mathcal{A} \models \phi \iff \mathcal{B} \models \phi$*

Theorem 2.7.4. *(Cantor)*
If $\mathcal{A}, \mathcal{B}$ countable dense linear orders without endpoints, then $\mathcal{A}, \mathcal{B}$ are isomorphic

Lemma 2.7.5. *(Los-Vaught test)*
*Let $\Sigma = \Sigma_E \cup \Delta$. Then for every closed $\mathcal{L}^=$-formula ϕ we have either $\Sigma \vdash_{K_{\mathcal{L}^=}} \phi$ or $\Sigma \vdash_{K_{\mathcal{L}^=}} (\neg\phi)$. Say that Σ is **complete***

3 Set theory

3.0 Basic set theory

- Extensionality - Sets A, B are **equal** $\iff \forall x, x \in A \iff x \in B$
- **Natural numbers** ; $\mathbb{N} = \{0, 1, \ldots\}$

$$0 = \varnothing \quad \ldots, n+1 = \{0, \ldots, n\}, \quad \ldots$$

  - Note that, for $m, n \in \mathbb{N}$

$$m < n \iff m \in n \iff m \subsetneq n$$

- Ordered pairs. The **ordered pair** (x, y) is the set $\{\{x\}, \{x, y\}\}$
  - For example, for any $x, y, z, w, (x, y) = (z, w) \iff x = z$ and $y = w$
  - If A, B sets then

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

$$A^0 = \{\varnothing\}, \quad A^1 = A \quad A^2 = A \times A, \quad \ldots \quad A^{n+1} = A^n \times A, \quad \ldots$$

$$\bigcup_{n \in \mathbb{N}} A^n = \{\text{finite sequences of elements of } A\}$$

- Functions. Think of $f : A \to B$ as a subset of $A \times B$

$$f : \underbrace{A}_{domf} \to \underbrace{B}_{ranf}$$

  $X \subseteq A$ define $f[X] = \{f(a) \mid a \in X\} \subseteq B$

- Set of functions from A to B is

$$B^A \subseteq \mathcal{P}(A \times B)$$

  where $\mathcal{P}$ is the powerset.

3.1 Cardinality

Definition 3.1.1. *Sets A, B are **equinumerous**, or of the **same cardinality**, if there is a bijection $f : A \to B$*
Write $A \approx B$ or $\mid A \mid = \mid B \mid$

Definition 3.1.2. *We have*

- *A set is **finite** if it is equinumerous with some element $n = \{0, \ldots, n-1\}$ of $\mathbb{N}$*
- *A set A is **countably infinite** if it is equinumerous with $\mathbb{N}$*
- ***Countable** is finite or countably finite*

Remark 3.1.3. *(Basic facts)*

- *Every subset of countable set is countable*
- *A set A is countable $\iff$ there is an injective function $f : A \to \mathbb{N}$*
- *if A, B countable then $A \times B$ countable*
- *$A_0, A_1, \ldots$ countable, then $\bigcup_{i \in \mathbb{N}} A_i$ countable. (requires axiom of choice)*

Theorem 3.1.4. *(Cantor)*
There is no surjective function

$$f : X \to \mathcal{P}(X)$$

Definition 3.1.5. *For sets A, B write $\mid A \mid \leq \mid B \mid$ or $A \leq B$, if there is injective function $f : A \to B$*

Theorem 3.1.6. *(Schröder-Bernstein)*
Suppose A, B are sets, and $f : A \to B, g : B \to A$ are injective functions. Then $A \approx B$ i.e if $\mid A \mid \leq \mid B \mid, \mid B \mid \leq \mid A \mid \implies \mid A \mid = \mid B \mid$

3.2 Axioms for set theory

Zermelo-Fraenkel axioms (ZF)
Axioms, that denote how we are allowed to build sets, expressed in a first-order language, with equality, using a single binary relation $\in$
Avoid the **Russell Paradox**

$$S = \{x \mid x \text{ a set and } x \notin x\}$$

If S a set, is $S \in S$?

$$(\exists S)(\forall x)((x \in S) \leftrightarrow (x \notin x))$$

leads to inconsistency!

Axiom 1 (Axiom of Extensionality).
Two sets are equal $\iff$ they have the same elements

$$(\forall x)(\forall y)((x = y) \leftrightarrow (\forall z)((z \in x) \leftrightarrow (z \in y)))$$

Axiom 2 (Empty set axiom)

$$(\exists x)(\forall y)(y \notin x)$$

There is a unique set x with this property, $\varnothing$

Axiom 3 (Pairing axiom)
Given sets x, y we can form $z = \{x, y\}$

$$(\forall x)(\forall y)(\exists z)(\forall w)((w \in z) \leftrightarrow ((w = x) \vee (w = y)))$$

Axiom 4 (Union Axiom)
For any set A there is a set $B = \bigcup A$

$$(\forall A)(\exists B)(\forall x)((x \in B) \leftrightarrow (\exists z)((z \in A) \wedge (x \in z)))$$

So

$$B = \bigcup \{z \mid z \in A\}$$

Axiom 5 (Power set axiom)
For any set A , there is a set $\mathcal{P}(A)$ whose elements are the subsets of A

$$(\forall A)(\exists B)(\forall z)((z \in B) \leftrightarrow \underbrace{(z \subseteq A)}_{(\forall y)((y \in z) \to (y \in A))})$$

Axiom 6 (Axiom scheme of specfication)
Suppose $P(x, y_1, \ldots, y_r)$ a formula in our language then we have axiom

$$(\forall A)(\forall y_1) \ldots (\forall y_r)(\exists B)(\forall x)((x \in B) \leftrightarrow ((x \in A) \wedge P(x, y_1, \ldots, y_r)))$$

This guarantees, we can form subset of A

$$B = \{x \in A \mid P(x, y_1, \ldots, y_r) \text{ holds}\}$$

for all given sets $A, y_1, \ldots, y_r$

Definition 3.2.1. *For set a define* ***successor*** *of a as*

$$a^\dagger = a \cup \{a\}$$

A set A ***inductive*** *if*

$$((\varnothing \in A) \wedge (\forall x)((x \in A) \to (x^\dagger \in A)))$$

Axiom 7 (Axiom of infinity)

$$(\exists A)((\varnothing \in A) \wedge (\forall x)((x \in A) \to (x^\dagger \in A)))$$

Definition 3.2.2. *Let A an inductive set, can form using specification the set*

$$\mathbb{N} = \{x \in A \mid \text{ if } B \text{ is an inductive set then } x \in B\}$$

Theorem 3.2.3. *We have*

1. *$\mathbb{N}$ an inductive set, if B an inductive set then $\mathbb{N} \subseteq B$*
2. *Proof by induction works. Suppose $P(x)$ a property of sets, that is a formula, such that*
   (a) *$P(\varnothing)$ holds, and*
   (b) *for every $k \in \mathbb{N}$, if $P(k)$ holds, then $P(k^\dagger)$ holds*

Then $P(n)$ holds for all $n \in \mathbb{N}$

3.3 Well orderings

Definition 3.3.1. *A* ***linear ordering*** *$\langle A; \leq \rangle$ a* ***well orderings*** *or* ***woset*** *of every non-empty subset of A has a least element*

$$(\forall X)(((X \subseteq A) \wedge (X \neq \varnothing)) \to (\exists x)((x \in X) \wedge (\forall y)((y \in X) \to (x \in y))))$$

Definition 3.3.2. *Suppose $\mathcal{A}_1, \mathcal{A}_2$ are* ***similar*** *or* ***isomorphic*** *if there is a bijection*

$$\alpha : A_1 \to A_2 \text{ s.t. } \forall a, b \in A_1, \text{ if } a <_1 b \iff \alpha(a) <_2 \alpha(b)$$

Write $\mathcal{A}_1 \simeq \mathcal{A}_2$
Call α a similarity between $\mathcal{A}_1, \mathcal{A}_2$

Definition 3.3.3. *Define the following*

- *The* ***reverse-lexicographic product*** *is*

  $$\mathcal{A}_1 \times \mathcal{A}_2 = \langle A_1 \times A_2; \leq \rangle$$

  where $(a_1, a_2) \leq (a_1', a_2') \iff a_2 <_2 a_2'$ and $a_1 \leq_1 a_1'$
  In $\mathcal{A}_2$ replace each element by a copy of $\mathcal{A}_1$

- *Regard A_1, A_2 as disjoint, by replacing them by similar orderings on disjoint sets, such as*

  $$A_1 \times \{0\} = \{(a, 0) : a \in A_1\}$$
  $$A_2 \times \{1\} = \{(a, 1) : a \in A_2\}$$

  Define sum

  $$\mathcal{A}_1 + \mathcal{A}_2 = \langle A_1 \cup A_2; \leq \rangle$$

  Where $\leq$ the union of $\leq_1, \leq_2$ and $a_1 \leq a_2, a_1 \in A_1, a_2 \in A_2$

Lemma 3.3.4. *With this notation*

1. *$\mathcal{A}_1 + \mathcal{A}_2, \mathcal{A}_1 \times \mathcal{A}_2$ are linearly ordered sets*
2. *If $\mathcal{A}_1, \mathcal{A}_2$ are wosets then so are $\mathcal{A}_1 + \mathcal{A}_2, \mathcal{A}_1 \times \mathcal{A}_2$*

3.4 Ordinals

Definition 3.4.1. *Define the following*

1. *A set X a* ***transitive set*** *if every element of X is also a subset of X.*
   That is if $y \in x \in X \implies y \in X$

2. *A set α is an* ***ordinal*** *If*

   - *α a transitive set*
   - *the relation $<$ on α given by, for $x, y \in \alpha$, we have that $x < y \iff x \in y$ a strict well ordering on α*

Lemma 3.4.2. *If α an ordinal then so is $a^{\dagger} = a \cup \{a\}$*

Proposition 3.4.3. *We have*

1. *if $n \in \omega$ then n an ordinal*

2. *ω a transitive set*

Proposition 3.4.4. *We have*

1. *If α an ordinal then $\alpha \notin \alpha$*

2. *If α an ordinal and $\beta \in \alpha$ then β an ordinal*

3. *If α, β ordinals and $\alpha \subsetneq \beta$ then $\alpha \in \beta$*

4. *If α an ordinal, then $\alpha = \{\beta \mid \beta \text{ an ordinal and } \beta \in \alpha\}$*

Definition 3.4.5. *α, β ordinals, write $\alpha < \beta$ to mean $\alpha \in \beta$ and $\alpha \leq \beta \iff \alpha \subseteq \beta$*

Theorem 3.4.6. *Suppose α, β, γ are ordinals*

1. *If $\alpha < \beta, \beta < \gamma \implies \alpha < \gamma$*

2. *If $\alpha \leq \beta, \beta \leq \alpha \implies \alpha = \beta$*

3. *Exactly one of the following hold*
$$\alpha < \beta, \quad \alpha = \beta, \quad \beta < \alpha$$

4. *if X a non-empty set of ordinals, then X has a least element δ, and moreover*
$$\delta = \bigcap X$$

Corollary 3.4.7. *We have*

1. *If X a non-empty set of ordinals, then $\bigcup X$ is an ordinal*

2. *ω is an ordinal*

Theorem 3.4.8. *If $\langle A; \leq \rangle$ a well ordered set, then there is a unique ordinal α which is similar to $\langle A; \leq \rangle$*

Definition 3.4.9. *Suppose $\langle A; \leq \rangle$ a woset. Say $X \subseteq A$ an* ***initial segment*** *of A if whenever $y < x <\in X$ then $y \in X$, it is proper if $X \neq A$*

Lemma 3.4.10. *Suppose $\langle A, \leq \rangle$ a woset. If $X \subset A$ is a proper initial segment of A there is $z \in A$ with $X = A[z]$*

Proposition 3.4.11. *Suppose $(A; \leq)$ a woset and and $f : A \to A$ which is order preserving and $f[A]$ is an initial segment of A, then $f(x) = x$ for all $x \in A$*