Applied Probability
Concise Notes

MATH60045/70045

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Content from prior years assumed to be known.

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Contents

3 Discrete-time Markov Chains

3.1 Definition of discrete time Markov Chains

Definition 3.1.1. *A discrete-time stochastic process $X = \{X_n\}_{n\in\mathbb{N}_0}$ taking values in countable state space E a Markov chain if it satisfies the Markov condition*

$$P(X_n = j \mid X_{n-1} = i, X_{n-2} = x_{n-2}, \ldots, X_0 = x_0) = P(X_n = j \mid X_{n-1} = i), \forall n \in \mathbb{N}\ \forall x_0, \ldots, x_{n-2}, i, j \in E$$

Definition 3.1.2. *(Time Homogenous)*

1. *Markov Chain $\{X_n\}_{n\in\mathbb{N}_0}$ is time-homogenous if*

$$P(X_{n+1} = j \mid X_n = i) = P(X_1 = j \mid X_0 = i),\ \forall n \in \mathbb{N}_0, i, j \in E$$

2. *Transition matrix $P = (p_{ij})_{i,j\in E}$ is the $K \times K$ matrix of transition probabilities*

Definition 3.1.3. *(Stochastic Matrix)*
A square matrix P a stochastic matrix if

1. $p_{ij} \geq 0, \forall i, j$
2. $\sum_j p_{ij} = 1\ \forall i$

Theorem 3.1.4. *Transition matrix P is stochastic*

3.2 The n-step transition probabilities and Chapman-Kolmogorov equations

Definition 3.2.1. *$n \in \mathbb{N}$, we have*

$$P_n = (p_{ij}(n)) = P(X_{m+n} = j, X_m = i),\ m \in \mathbb{N}_0$$

The matrix of n-step transition probabilities.

Lemma 3.2.2. *For discrete markov chain $\{X_n\}_{n\geq 0}$ on state space E we have*

$$P(X_{n+m} = x_{n+m} | X_n = x_n, \ldots, X_0 = x_0) = P(X_{n+m} = x_{n+m} \mid X_n = x_n),\ m \in \mathbb{N}, \forall x_{n+m}, x_n, \ldots, x_0 \in E$$

Theorem 3.2.3. *Let $m \in \mathbb{N}_0, n \in \mathbb{N}$ Then we have $\forall i, j \in E$*

$$p_{ij}(m+n) = \sum_{l\in E} p_{il}(m) p_{lj}(n) \quad P_{m+n} = P_m P_n \quad P_n = P^n$$

Remark 3.2.4. *Extend definition for case $K = \infty$*
Let $\mathbf{x}$ a K-dimensional row vector, P a $K \times K$ matrix

$$(\mathbf{x}P)_j := \sum_{i\in E} x_i p_{ij}, \quad (P^2)_{ik} := \sum_{j\in E} p_{ij} p_{jk},\ i, j, k \in \mathbb{N}$$

Define P^n similarly and take $(P^0)_{ij} = \delta_{ij}$

3.3 Dynamics of a Markov Chain

Definition 3.3.1. *Denote probability mass function of X_n for $n \in \mathbb{N}_0$ by*

$$\nu_i^{(n)} = P(X_n = i), \ i \in E$$

Take $K = card(E)$, denote by $\nu^{(n)}$ the K-dimensional row vector with elements $\nu_i^n, i \in E$
*Call this the **marginal distribution** of chain at time $n \in \mathbb{N}_0$*

Theorem 3.3.3. *We have*

$$\nu^{(m+n)} = \nu^{(m)} P_n = \nu^{(m)} P^n, \ \forall n \in \mathbb{N}, m \in \mathbb{N}_0$$

So

$$\nu^{(n)} = \nu^{(0)} P_n = \nu^{(0)} P^n, \ \forall n \in \mathbb{N}$$

Theorem 3.3.4. *Let $X = \{X_n\}_{n \in \mathbb{N}_0}$ a Markov chain on countable state space E*
Then given initial distribution $\nu^{(0)}$ and transition matrix P, we determine all finite dimensional distributions of Markov chain.
$\forall 0 \le n_1 < n_2 < \cdots < n_{k-1} < n_k$ $(n_i \in \mathbb{N}_0, i = 1, \dots, k), k \in \mathbb{N}, x_1, \dots, x_k \in E$ *We have*

$$\begin{aligned} P(X_{n_1} = x_1, X_{n_2} = x_2, \dots, X_{n_k} = x_k) &= (\nu^{(0)} P^{n_1})_{x_1} (P^{n_2 - n_1})_{x_1 x_2} \cdots (P^{n_k - n_{k-1}}) x_{k-1} x_k \\ &= (\nu P^{n_1}) x_1 p_{x_1 x_2}(n_2 - n_1) \cdots p_{x_{k-1} x_k}(n_k - n_{k-1}) \end{aligned}$$

3.4 First passage/hitting times

Definition 3.4.1. *Define **first passage/hitting time** of X for state $j \in E$ as*

$$T_j = \min\{n \in N : X_n = j\}$$

If $X_n \neq j, \forall n \in \mathbb{N}$ then set $T_j = \infty$

Definition 3.4.2. *For $i, j \in E, n \in \mathbb{N}$ define **first passage probability***

$$f_{ij}(n) = P(T_j = n \mid X_0 = i) = P(X_n = j, X_{n-1} \neq j, \dots, X_1 \neq j \mid X_0 = i)$$

Probability that we visit state j at time n, given we start at i at time 0
Define $f_{ij}(0) = 0, f_{ij}(1) = p_{ij}, \forall i, j \in E$

Definition 3.4.4. *Define*

$$f_{ij} = P(T_j < \infty \mid X_0 = i)$$

For $i \neq j$, we have f_{ij} the probability that the chain ever visits state j, starting at i
*Call f_{ii} the **returning probability***

Proposition 3.4.5. $\forall i, j \in E$

$$f_{ij} = \sum_{n=1}^{\infty} f_{ij}(n)$$

Lemma 3.4.7. $\forall i, j \in E, n \in \mathbb{N}$, *we have*

$$\begin{aligned} p_{ij}(n) &= \sum_{l=0}^{n} f_{ij}(l) p_{jj}(n-l) \\ &= \sum_{l=1}^{n} f_{ij}(l) p_{jj}(n-l) \end{aligned}$$

3.5 Recurrence and transience

Definition 3.5.1. *Let $\{X_n\}_{n\in\mathbb{N}_0}$ be a markov chain on countable state space E.*

$$j \in E,\ P(X_n = j, \textit{for some } n \in \mathbb{N} \mid X_0 = j) = f_{jj} \begin{cases} 1, & \textit{recurrent}; \\ < 1, & \textit{transient} . \end{cases}$$

Theorem 3.5.2. $j \in E$

$$\sum_{n=1}^{\infty} p_{ij}(n) = \begin{cases} \infty, & \iff \quad \textit{recurrent} ; \\ < \infty, & \iff \quad \textit{transient} . \end{cases}$$

Define

$$N_j = \sum_{n=0}^{\infty} I_n^{(j)}, \quad I_n^{(j)} = I_{X_n=j} = \begin{cases} 1, & \text{if } X_n = j; \\ 0, & \text{if } X_n \neq j. \end{cases}$$

Theorem 3.5.3. *$j \in E$ transient*

1. $P(N_j = n \mid X_0 = j) = f_{jj}^{n-1}(1 - f_{jj})$ *for $n \in \mathbb{N}$ geometric distribution with param f_{jj}*
2. $i \neq j$

$$P(N_j = n \mid X_0 = i) = \begin{cases} 1 - f_{ij}, & \textit{if } n = 0; \\ f_{ij} f_{jj}^{n-1}(1 - f_{jj}), & \textit{if } n \in \mathbb{N}. \end{cases}$$

Corollary 3.5.4. *$j \in E$ transient*

1.

$$E(N_j \mid X_0 = j) = \frac{1}{1 - f_{jj}}$$

2. *$i \neq j$ we have*

$$E(N_j \mid X_0 = i) = \frac{f_{ij}}{1 - f_{jj}}$$

Theorem 3.5.5. *Given $X_0 = j$, we have*

$$E(N_j \mid X_0 = j) = \sum_{n=0}^{\infty} p_{jj}(n)$$

Sum may diverge to ∞

Corollary 3.5.6. *$j \in E$ transient then $p_{ij}(n) \xrightarrow[n\to\infty]{} 0, \forall i \in E$*

3.5.1 Mean recurrence time, null and positive recurrence

Definition 3.5.7. *The **mean recurrence time** μ_i of state $i \in E$ defined as $\mu_i = E[T_i \mid X_0 = i]$*

Theorem 3.5.8. *Let $i \in E$. We have $P(T_i = \infty \mid X_0 = i) > 0 \iff i$ transient, where we get*

$$\mu_i = E[T_i \mid X_0 = i = \infty]$$

Theorem 3.5.9. *For recurrent state $i \in E$ we have*

$$\mu_i = E[T_i \mid X_0 = i] = \sum_{n=1}^{\infty} n f_{ii}(n)$$

Can be finite or infinite.

Definition 3.5.10. *A recurrent state $i \in E$*

$$\mu_i = \begin{cases} \infty, & \textit{called } \textbf{\textit{null}}; \\ < \infty, & \textit{called } \textbf{\textit{positive}}. \end{cases}$$

Theorem 3.5.11. *Recurrent state $i \in E$ null $\iff p_{ii}(n) \xrightarrow[n\to\infty]{} 0$*
Further, if this holds, then $p_{ji}(n) \xrightarrow[n\to\infty]{} 0, \forall j \in E$

3.5.2 Generating functions for $p_{ij}(n), f_{ij}(n)$ (READING MATERIAL)

3.5.3 Example: Null recurrence/transience of a simple random walk (READING MATERIAL)

SEE FULL OFFICIAL NOTES

3.6 Aperiodicity and ergodicity

Definition 3.6.1. *Period of state i defined by*

$$d(i) = gcd\{n : p_{ii}(n) > 0\}$$

Definition 3.6.4. *A state ergodic if it is positive recurrent and aperiodic*

3.7 Communicating classes

Definition 3.7.1. *(Accessible and Communicating)*

1. *j accessible from i, $i \to j$, if $\exists m \in \mathbb{N}_0$ s.t $p_{ij}(m) > 0$*
2. *i, j communicate, if $i \to j$ and $j \to i$; write $i \leftrightarrow j$*

Theorem 3.7.2. *(Communication an equivalence relation)*
Satisfies, reflexivity, symmetry and transitivity

Theorem 3.7.4. *If $i \leftrightarrow j$ then*

1. *i, j have same period*
2. *i transient/recurrent $\iff$ j transient/recurrent*
3. *i null recurrent $\iff$ j null recurrent*

Definition 3.7.5. *Set of states C is*

1. ***closed** if $\forall i \in C, j \notin C, p_{ij} - 0$*
2. ***irreducible** if $i \leftrightarrow j, \forall i, j \in C$*

Theorem 3.7.6. *Let C a closed communicating class, transition matrix P restricted to C is stochastic*

3.7.1 The decomposition theorem

Theorem 3.7.8. *C a communicating class, consisting of recurrent states. Then C is closed*

Theorem 3.7.9. *State-space E can be partitioned uniquely into*

$$E = \underbrace{T}_{\text{transient states}} \cup \left(\bigcup_i \underbrace{C_i}_{\substack{\text{irreducible, closed} \\ \text{set of recurrent states}}} \right)$$

Theorem 3.7.11. *$K < \infty$ Then at least one state is recurrent and all recurrent states are positive.*

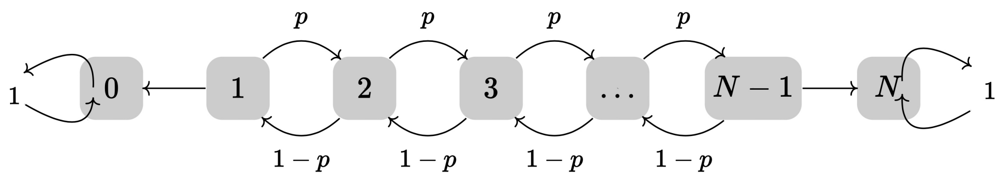
Theorem 3.7.12. *C a finite, closed communicating class $\implies$ all states in C positive recurrent*

3.7.2 Class properties

Type of Class	Finite	Infinite
Closed	positive recurrent	positive recurrent, null recurrent, transient
Not Closed	transient	transient

3.8 Application: The gambler's ruin problem

3.8.1 The problem and the results



Consider a gambler with initial fortune $i \in \{0, 1, \dots, N\}$. At each play of the game, the gambler has

- probability p of winning one unit
- probability q of losing one unit
- each successive game is independent

What is the probability, a gambler starting at i units, has their fortune reach N before 0 ?

Let X_n denote gamblers fortune at time n. Then $\{X_n\}_{n \in \mathbb{N}_0}$ is a Markov Chain with transition probabilities, shown in diagram above.
This yields 3 communicating classes.

$$\underbrace{C_1 = \{0\}, C_2 = \{N\}}_{\substack{\text{positive recurrent} \\ \text{since finite and closed}}}, T_1 = \{1, 2, \dots, N-1\}$$

Define the following for our problem:
Define first time X visits state i as

$$V_i = \min\{n \in \mathbb{N}_0 : X_n = i\}$$

$$h_i = h_i(N) = P(V_N < V_0 \mid X_0 = i)$$

This yields the following recurrence relation

$$h_i = h_{i+1}p + h_{i-1}q, \ i = 1, 2, \dots, N-1$$

Theorem 3.8.1. *From above we achieve*

$$h_i = h_i(N) = \begin{cases} \frac{1-(q/p)^i}{1-(q/p)^N}, & \text{if } p \neq \frac{1}{2}; \\ \frac{i}{N}, & \text{if } p = \frac{1}{2}. \end{cases}$$

Theorem 3.8.2. *We also have*

$$\lim_{N \to \infty} h_i(N) = h_i(\infty) = \begin{cases} 1 - (q/p)^i, & \text{if } p > \frac{1}{2}; \\ 0, & \text{if } p \leq \frac{1}{2}. \end{cases}$$

- $p > \frac{1}{2} \implies \frac{q}{p} < 1 \implies \lim_{N \to \infty} (\frac{q}{p})^N = 0$
- $p < \frac{1}{2} \implies \frac{q}{p} > 1 \implies \lim_{N \to \infty} = \infty$

3.9 Stationarity

Definition 3.9.1. *(Distributions)*

1. *row vector λ a* ***distribution*** *on E if*

$$\forall j \in E, \lambda_j \geq 0, \quad \text{and} \sum_{j \in E} = 1$$

2. *row vector λ with non-negative entries is called* ***invariant*** *for transition matrix P if*

$$\lambda P = \lambda$$

3. *row vector π is* ***invariant/stationary/equilibrium distribution*** *of Markov chain on E with transition matrix P if*

   (a) *π a distribution*

   (b) *it is invariant*

$$\pi P^n = \pi$$

3.9.1 Stationarity distribution for irreducible Markov Chains

Theorem 3.9.2. *An irreducible chain has stationary distribution $\pi \iff$ all states are positive recurrent. π unique stationary distribution, s.t $\pi_i = \mu_i^{-1} \forall i$*

Lemma 3.9.3. *For markov chain X we have $\forall j \in E, n, m \in \mathbb{N}$*

$$f_{jj}(m+n) = \sum_{i \in E, i \neq j} l_{ji}(m) f_{ij}(n)$$

For $l_{ji}(n) = P(X_n = i.T_j \geq n \mid X_0 = j)$

Corollary 3.9.4. *For Markov Chain X we have $\forall i, j \in E, i \neq j$ and $\forall n, m \in \mathbb{N}$*

$$f_{jj}(m+n) \geq l_{ji}(m) f_{ij}(n)$$

Lemma 3.9.5. *Let $i \neq j$ Then $l_{ji}(1) = p_{ji}$, and for integers $n \geq 2$*

$$l_{ji}(n) = \sum_{r \in E : r \neq j} p_{ri} l_{jr}(n-1)$$

Lemma 3.9.6. *$\forall j \in E$ of an irreducible, recurrent chain, the vector $\rho(j)$ satisfies $\rho_i(j) < \infty$ $\forall i$ and further $\rho(j) = \rho(j)P$*

Lemma 3.9.7. *Every irreducible, positive, recurrent chain has a stationary distribution*

Theorem 3.9.8. *If the chain is irreducible and recurrent, then $\exists \mathbf{x} > 0$ s.t $\mathbf{x} = \mathbf{xP}$ unique up to multiplicative constant.*

$$\text{Chain is} \begin{cases} \text{positive recurrent}, & \text{if } \sum_i x_i < \infty; \\ \text{null}, & \text{if } \sum_i x_i = \infty. \end{cases}$$

Lemma 3.9.9. *Let T a non-negative integer valued random variable on probability space $(\Omega, \mathcal{F}, P)$, with $A \in \mathcal{F}$ an event s.t $P(A) > 0$. Can show that*

$$E(T \mid A) = \sum_{n=1}^{\infty} P(T \geq n \mid A)$$

Theorem *(Dominated convergence theorem)*
Let $\mathcal{I}$ be a countable index set.
If $\sum_{i\in\mathcal{I}} a_i(n)$ is an absolutely convergent series $\forall n \in N$ s.t

1. $\forall i \in \mathcal{I}$ the limit $\lim\limits_{n\to\infty} a_i(n) = a_i$ exists
2. $\exists$ seq. $(b_i)_{i\in I}$ s.t $b_i \geq 0 \, \forall i$ and $\sum_{i\in\mathcal{I}} b_i < \infty$ s.t $\forall n, i : |a_i(n)| \leq b_i$

Then $\sum_{i\in\mathcal{I}} |a_i| < \infty$ and

$$\sum_{i\in I} a_i = \sum_{i\in I} \lim_{n\to\infty} a_i(n) = \lim_{n\to\infty} \sum_{i\in\mathcal{I}} a_i(n)$$

3.9.2 Limiting distribution

Definition 3.9.12. *A distribution π is the limiting distribution of a discrete-time Markov Chain if, $\forall i, j \in E$ we have*

$$\lim_{n\to\infty} p_{ij}(n) = \pi_j$$

Definition 3.9.14. *For irreducible aperiodic chain we have*

$$\lim_{n\to\infty} p_{ij}(n) = \frac{1}{\mu_j}$$

3.9.3 Ergodic Theorem

Theorem 3.9.16. *(Ergodic Theorem)*
Suppose we have irreducible Markov chain $\{X_n\}_{n\in\mathbb{N}_0}$ with state space E. Let μ_i the mean recurrence time to state $i \in E$

$$V_i(n) = \sum_{k=0}^{n-1} \mathbf{1}_{\{X_k=i\}}$$

The number of visits to i before n
So we have $V_i(n)/n$ the proportion of time before n spent at i

$$P\left(\frac{V_i(n)}{n} \to \frac{1}{\mu_i}, \text{ as } n \to \infty\right) = 1$$

Summary: Properties of irreducible Markov Chains
3 kinds of irreducible Markov Chains

1. **Positive recurrent**
   (a) Stationary distribution π exists
   (b) Stationary distribution is unique
   (c) All mean recurrence times are finite and $\mu_i = \frac{1}{\pi_i}$
   (d) $V_i(n)/n \xrightarrow[n\to\infty]{} \pi_i$
   (e) If chain aperiodic
   $$\lim_{n\to\infty} P(X_n = i) = \pi_i, \forall i \in E$$
2. **Null recurrent**
   (a) Recurrent, but all mean recurrence times are infinite
   (b) No stationary distribution exists
   (c) $V_i(n)/n \xrightarrow[n\to\infty]{} 0$

(d)

$$\lim_{n\to\infty} P(X_n = i) = 0, \forall i \in E$$

3. **Transient**

   (a) Any particular state is eventually never visited

   (b) No stationary distribution exists

   (c) $V_i(n)/n \xrightarrow[n\to\infty]{} 0$

   (d)

$$\lim_{n\to\infty} P(X_n = i) = 0, \forall i \in E$$

3.9.4 Properties of the elements of a stationary distribution associated with transient or null-recurrent states

Theorem 3.9.17. *Let X a time-homogeneous Markov Chain on countable state space E*
If π a stationary distribution of X, $i \in E$ either transient or null-recurrent, then $\pi_i = 0$

3.9.5 Existence of a stationary distribution on a finite state space

Theorem 3.9.19. *If state space finite $\implies \exists$ at least one positive recurrent communicating class*

Theorem 3.9.20. *Suppose finite state space. The stationary distribution π for transition matrix P unique $\iff$ there is a unique closed communicating class*

Corollary 3.9.21. *Markov chain on finite state space, and $N \geq 2$ closed classes.*
C_i the closed classes of Markov chain and $\pi^(i)$ the stationary distribution associated with class C_i using construction

$$\pi_j^{(i)} = \begin{cases} \pi_j^{C_i}, & \text{if } j \in C_i; \\ 0, & \text{if } j \notin C_i. \end{cases}$$

Then every stationary distribution of Markov Chain represented as

$$\sum_{i=1}^{N} \omega_i \pi^{(i)}$$

For weights $\omega_i \geq 0, \sum_{i=1}^{n} \omega_i = 1$

3.9.6 Limiting distributions on a finite state space

Theorem 3.9.23. *Let $K = |E| < \infty$ Suppose for some $i \in E$ that*

$$\lim_{n\to\infty} p_{ij}(n) = \pi_j, \quad \forall j \in E$$

Then π a stationary distribution

3.10 Time reversibility

Theorem 3.10.1. *For irreducible, positive recurrent Markov chain $\{X_n\}_{n\in 0,1,\ldots,N}, N \in \mathbb{N}$ assume π a stationary distribution, and P a transition matrix, and $\forall n \in \{0, 1, \ldots, N\}$ the marginal distribution $\nu^{(n)} = \pi$*

$$Y_n = X_{N-n}, \quad \textit{The reversed chain defined for } n \in \{0, 1, \ldots, N\}$$

We have Y a Markov chain, satisfying

$$P(Y_{n+1} = j \mid Y_n = i) = \frac{\pi_j}{\pi_i} p_{ji}$$

Definition 3.10.2. *$X = \{X_n : n \in \{0, 1, \dots, N\}\}$ an irreducible Markov chain with stationary distribution π and marginal distributions $\nu^{(n)} = \pi,\ \forall n \in \{0, 1, \dots, N\}$*
*Markov chain X **time-reversible** if transition matrices of X and its reversal Y are the same.*

Theorem 3.10.3. *$\{X_n\}_{n\in\{0,1,\dots,N\}}$ time-reversible $\iff, \forall i, j \in E$*

$$\pi_i p_{ij} = \pi_j p_{ji}$$

Theorem 3.10.4. *For irreducible chain, if $\exists\pi$ s.t 3.10.1 holds $\forall i, j \in E$. Then the chain is time-reversible (once in its stationary regime) and positive recurrent with stationary distribution π*

4 Properties of the Exponential Distribution

4.1 Definition and basic properties

Definition 4.1.1. *(Exponential distribution)*
A continuous random variable X is $X \sim Exp(\lambda)$ if it has density function

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x > 0; \\ 0, & \text{if otherwise.} \end{cases}$$

Cumulative distribution function

$$F_X(x) = \begin{cases} 0, & \text{if } x \leq 0; \\ 1 - e^{-\lambda x}, & \text{if } x > 0. \end{cases}$$

Survival function of the exponential distribution is given by

$$P(X > x) = \begin{cases} 1, & \text{if } x \leq 0; \\ e^{-\lambda x}, & \text{if } x > 0. \end{cases}$$

Theorem 4.1.2. *$X \sim Exp(\lambda)$ for $\lambda > 0$ Then*

1. *$E(X) = \frac{1}{\lambda}$*
2. *$\lambda X \sim Exp(1)$*

Theorem 4.1.3. *Let $n \in \mathbb{N}$ and $\lambda > 0$. Consider independent and identically distributed random variables $H_i \sim Exp(\lambda)$, for $i = 1, \dots, n$*
Let $J_n := \sum_{i=1}^{n} H_i$ Then J_n follows the Gamma(n, λ) distribution, i.e

$$f_{J_n}(t) = \frac{\lambda^n}{\Gamma(n)} t^{n-1} e^{-\lambda t}$$

Theorem 4.1.4. *Let $n \in \mathbb{N}$ and $\lambda_1, \dots, \lambda_n$. Consider independent random variables $H_i \sim Exp(\lambda_i)$ for $i = 1, \dots, n$. Let $H := \min\{H_1, \dots, H_n\}$ Then*

1. *$H \sim Exp(\sum_{i=1}^{n} \lambda i)$*
2. *For any $k = 1, \dots, n, P(H = H_k) = \lambda_k / \sum_{i=1}^{n} \lambda_i$*

Theorem 4.1.5. *Consider a countable index set E and $\{H_i : i \in E\}$ independent random variables with $H_i \sim Exp(\lambda_i), \forall i \in E$. Suppose that $\sum_{i\in E} \lambda_i < \infty$ and set $H := \inf_{i\in E} H_i$*
Then the infimum is attained at a unique random value I of E with probability 1
H, I are independent, with $H \sim Exp(\sum_{i\in E} \lambda_i < \infty)$ and $P(I = i) = \lambda_i / \sum_{k\in E} \lambda_k$

Remark 4.1.6. *Suppose we have $X \sim Exp(\lambda_X), Y \sim Exp(\lambda_Y)$, Then*

$$P(X < Y) = P(\min\{X, Y\} = X) = \frac{\lambda_X}{\lambda_X + \lambda_Y}$$

4.2 Lack of memory property

Theorem 4.2.1. *(Lack of memory property)*
A continuous random variable $X : \Omega \to (0,\infty)$ has an exponential distribution $\iff$ has the lack of memory property

$$P(X > x + y \mid X > x) = P(X > y), \quad \forall x, y > 0$$

Remark 4.2.2. *A random variable $X : \Omega \to (0,\infty)$ has an exponential distribution $\iff$ has lack of memory property:*

$$P(X > x + y \mid X > x) = P(X > y), \quad \forall x, y > 0$$

4.3 Criterion for the convergence/divergence of an infinite sum of independent exponentially distributed random variables

Theorem 4.3.1. *Consider sequence of independent random variables $H_i \sim Exp(\lambda_i)$ for $0 < \lambda_i < \infty$ for all $i \in \mathbb{N}$ and let $J_\infty = \sum_{i=1}^{\infty} H_i$, Then:*

1. *If $\sum_{i=1}^{\infty} \frac{1}{\lambda_i} < \infty \implies P(J_\infty < \infty) = 1$*
2. *If $\sum_{i=1}^{\infty} \frac{1}{\lambda_i} = \infty \implies P(J_\infty = \infty) = 1$*

Lemma 4.3.2. *For $x \geq 1$, we have*

$$\log\left(1 + \frac{1}{x}\right) \geq \log(2)\frac{1}{x}$$

$$\log(1 + x) > \frac{x}{x+1}, \quad \text{for } x > -1$$

5 Poisson Process

5.1 Remarks on continuous-time stochastic processes on a countable state space

5.3 Some Definitions

Definition 5.3.0. *A stochastic process $\{N_t\}_{t\geq 0}$ a **counting process** if N_t represents the total number of 'events' that have occurred up to time t*
Having the following properties:

1. *$N_0 = 0$*
2. *$\forall t \geq 0, N_t \in \mathbb{N}_0$*
3. *If $0 \leq s \leq t, N_s \leq N_t$*
4. *For $s < t, N_t - N_s =$ the number of events in interval $(s, t]$*
5. *Process is piecewise constant and has upward jumps of size 1 i.e $N_t - N_{t-} \in \{0, 1\}$*

Definition 5.3.1. *Let $(J_n)_{n\in\mathbb{N}_0}$ a strictly increasing sequence of positive random variables s.t $J_0 = 0$ almost surely.*
Define process $\{N_t\}_{t\geq 0}$ as

$$N_t = \sum_{n=1}^{\infty} \mathbf{1}_{\{J_n \leq t\}},$$

Interpret J_n as the (random) time at which the nth event occurs.
The nth jump time.

5.3.1 Poisson Process: First Definition

Definition 5.3.0. *Define $o(\cdot)$ notation.*
A function f is $o(\delta)$ if

$$\lim_{\delta \downarrow 0} \frac{f(\delta)}{\delta} = 0$$

With the following properties

- *if f, g are $o(\delta)$ then so is $f + g$*
- *if f is $o(\delta)$ and $c \in \mathbb{R}$ then cf is $o(\delta)$*

Definition 5.3.3. *A **Poisson process** $\{N_t\}_{t\geq 0}$ of rate $\lambda > 0$ is a non-decreasing stochastic process with values in $\mathbb{N}_0$ satisfying:*

1. $N_0 = 0$[1]
2. *Increments are independent, that is given any $n \in \mathbb{N}$ and $0 \leq t_0 < t_1 < t_2 < \ldots < t_n$ random variables $N_{t_0}, N_{t_1} - N_{t_0}, N_{t_2} - N_{t_1}, N_{t_3} - N_{t_2}, \ldots, N_{t_n} - N_{t_{n-1}}$ are independent*
3. *The increments are stationary, Given any 2 distinct times $0 \leq s < t, \forall k \in \mathbb{N}_0$*

$$P(N_t - N_s = k) = P(N_{t-s} = k)$$

4. *There is a 'single arrival', i.e $\forall t \geq 0, \delta > 0, d \to 0$:*

$$\begin{aligned} P(N_{t+\delta} - N_t = 1) &= \lambda\delta + o(\delta) \\ P(N_{t+\delta} - N_t \geq 2) &= o(\delta) \end{aligned}$$

5.3.2 Poisson Process: Second definition

Definition 5.3.4. *A **Poisson Process** $\{N_t\}_{t\geq 0}$ of rate $\lambda > 0$ is a stochastic process with values in $\mathbb{N}_0$ satisfying*

1. $N_0 = 0$
2. *Increments are independent, that is given any $n \in \mathbb{N}$ and $0 \leq t_0 < t_1 < t_2 < \ldots < t_n$ random variables $N_{t_0}, N_{t_1} - N_{t_0}, N_{t_2} - N_{t_1}, N_{t_3} - N_{t_2}, \ldots, N_{t_n} - N_{t_{n-1}}$ are independent*
3. *The increments are stationary, Given any 2 distinct times $0 \leq s < t, \forall k \in \mathbb{N}_0$*

$$P(N_t - N_s = k) = P(N_{t-s} = k)$$

4. $\forall t \geq 0, N_t \sim Poi(\lambda t)$

$$\forall k \in \mathbb{N}_0, P(N_t = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

5.3.3 Right-continuous modification

Definition 5.3.0. *For 2 stochastic processes $\{X_t\}_{t\geq 0}, \{Y_t\}_{t\geq 0}$, say X a modification of Y if*

$$X_t = Y_t, \text{ almost surely for each} t \geq 0$$

$$P(X_t = Y_t) = 1, \forall t \geq 0$$

Can show that for each Poisson process, $\exists!$ modification which is càdlàg, (right continuous with left limits).

Remark 5.3.5. *Note that the jump chain of the Poisson Process given by $Z = (Z_n)_{n\in\mathbb{N}_0}$, where $Z_n = n, n \in \mathbb{N}_0$*

5.3.4 Equivalence of definitions

Theorem 5.3.6. *Definition 5.3.3, 5.3.4 are equivalent*

Lemma 5.3.7. *Laplace transform of a Poisson random variable of mean* $\lambda t, X \sim Poi(\lambda t)$ *for* $\lambda > 0, t > 0$ *is given by*

$$\mathcal{L}_X(u) = \exp\{\lambda t[e^{-u} - 1]\}, \quad \forall u > 0$$

5.4 Some properties of Poisson processes

5.4.1 Inter-arrival time distribution

Definition 5.4.1. *Let* $\{N_t\}_{t\geq 0}$ *a Poisson process of rate* $\lambda > 0$
Then the inter-arrival times are independently and identically distributed exponential random variables with parameter λ

5.4.2 Time to the n^{th} event

Theorem 5.4.2. *We have* $\forall n \in \mathbb{N}$*, the time to the* n^{th} *event* J_n *follows a Gamman,* λ *distribution, with density*

$$f_{J_n}(t) = \frac{\lambda^n}{\Gamma(n)} t^{n-1} e^{-\lambda t}, \ t > 0$$

5.4.3 Poisson process: Third definition

Definition 5.4.4. *A* ***Poisson process*** $\{N_t\}_{t\geq 0}$ *of rate* $\lambda > 0$ *is a stochastic process with values in* $\mathbb{N}_0$ *s.t*

1. $H_1, H_2, \ldots$ *denote independently and identically exponentially distributed random variables with parameter* $\lambda > 0$
2. *Let* $J_0 = 0$ *and* $J_n = \sum_{i=1}^n H_i$
3. *Define*

$$N_t = \sup\{n \in \mathbb{N}_0 : J_n \leq t\}, \quad \forall t \geq 0$$

Theorem 5.4.5. *Definitions 5.3.3, 5.3.4, 5.4.4 are equivalent*

5.4.4 Conditional distribution of the arrival times

Theorem 5.4.6. *Let* $\{N_t\}_{t\geq 0}$ *be a Poisson process of rate* $l > 0$*. Then* $\forall n \in \mathbb{N}, t > 0$*, the conditional density of* $\big(J_1, \ldots, J_n\big)$ *given by* $N_t = n$ *is given by*

$$f_{\big(J_1, \ldots, J_n\big)}\big(t_1, \ldots, t_n | N_t = n\big) = \begin{cases} \frac{n!}{t^n}, & \text{if } 0 < t_1 < \ldots < t_n \leq t; \\ 0, & \text{otherwise .} \end{cases}$$

Remark 5.4.7. *The above theorem says, conditional on the fact n events have occured in* $[0, t]$*, the times* $\big(J_1, \ldots, J_n\big)$ *at which the events occur, when considered as unordered random variables are independently and uniformly distributed on* $[0, t]$

5.5 Some extensions to Poisson processes

5.5.1 Superposition

Theorem 5.5.2. *Given n independent Poisson processes* $\{N_t^(1)\}_{t\geq 0}, \ldots, \{N_t^{(n)}\}_{t\geq 0}$ *with respective rates,* $\lambda_1, \ldots, \lambda_n > 0$ *define*

$$N_t = \sum_{i=1}^n N_t^{(i)}, \quad t \geq 0$$

Then $\{N_t\}_{t\geq 0}$ *a Poisson process with rate* $\lambda = \sum_{i=1}^n \lambda_i$ *and is called a* ***superposition of Poisson processes***

5.5.2 Thinning

Theorem 5.5.5. *Let $\{N_t\}_{t\geq 0}$ a Poisson process with rate $\lambda > 0$. Assume that each arrival, independent of other arrivals, is marked as a type k event with probability p_k for $k = 1, \dots, n$ where $\sum_{i=1}^{n} p_i = 1$.*
Let $N_t^{(k)}$ denote the number of type k events in $[0, t]$. Then $\{N_t^{(k)}\}_{t\geq 0}$ a Poisson process with rate λp_k and the processes

$$\{N_t^(1)\}_{t\geq 0}, \dots, \{N_t^{(n)}\}_{t\geq 0}$$

*are independent. Each process called a **thinned Poisson process***

5.5.3 Non-homogeneous Poisson processes

Definition 5.5.6. *Let $\lambda : [0, \infty) \mapsto (0, \infty)$ denote a non-negative and locally integrable function, called the **intensity function***
*A non-decreasing stochastic process $N = \{N_t\}_{t\geq 0}$ with values in $\mathbb{N}_0$ called a **non-homogeneous Poisson process** with intensity function $(\lambda(t))_{t\geq 0}$ if it satifies the following:*

1. *$N_0 = 0$*
2. *N has independent increments*
3. *'Single arrival' property, For $t \geq 0, \delta > 0$*

$$P(N_{t+\delta} - N_t = 1) = \lambda(t)\delta + o(\delta)$$
$$P(N_{t+\delta} - N_t \geq 2) = o(\delta)$$

Note that (3) also implies that

$$P(N_{t+\delta} - N_t = 0) = 1 - \lambda(t) + o(\delta)$$

Theorem 5.5.7. *Let $N = \{N_t\}_{t\geq 0}$ denote a non-homogeneous Poisson process with continuous intensity function $(\lambda(t))_{t\geq 0}$ Then*

$$N_t \sim Poi(m(t)), \quad \textit{where} \quad m(t) = \int_0^t \lambda(s)ds$$

i.e. $\forall t \geq 0, n \in \mathbb{N}_0$

$$P(N_t = n) = \frac{[m(t)]^n}{n!} e^{-m(t)}$$

5.5.4 Compound Poisson processes

Definition 5.5.12. *Let $\{N_t\}_{t\geq 0}$ be a Poisson process of rate $\lambda > 0$.*
$Y_1, Y_2, \dots$ be a sequence of independent and identically distributed random variables, that are independent of $\{N_t\}_{t\geq 0}$. Then the process $\{S_t\}_{t\geq 0}$ with

$$S_t = \sum_{i=1}^{N_i} Y_i, \quad t \geq 0$$

*is a **compound Poisson process***

Theorem 5.5.13. *Let $\{S_t\}_{t\geq 0}$ a compound Poisson process. Then for $t \geq 0$*

$$E(S_t) = \lambda t E(Y_1), \quad Var(S_t) = \lambda t E(Y_1^2)$$

as defined in Definition 5.5.12

5.6 The Cramér-Lundberg model in insurance mathematics

Definition 5.6.1. *The **Cramér-Lundberg model** is given by the following five conditions.*

1. *Claim size process is denoted by $Y = (Y_k)_{k\in\mathbb{N}}$, for Y_k denoting the positive i.i.d random variables with finite mean $\mu = E(Y)1$ and variance $\sigma^2 = Var(Y_1) \leq \infty$*
2. *Claim times occur at the random instants of time*
$$0 < J_1 < J_2 < \dots a.s..$$
3. *The claim arrival process is denoted by*
$$N_t = \sup\{n \in \mathbb{N} : J_n \leq t\}, t \geq 0$$
which is the number of claims in the interval $[0, t]$.
4. *The inter-arrival times are denoted by*
$$H_1 = J_1, H_k = J_k - J_{k-1}, k = 2, 3, \dots$$
and are independent and exponentially distributed with parameter λ
5. *sequences $(Y_k, (H_k))$ are independent of each other*

Definition 5.6.3. *The **Total claim amount** is defined as the process $(S_t)_{t\geq 0}$ satisfying*

$$S_t = \begin{cases} \sum_{i=1}^{N_t} Y_i, & \text{if } N_t > 0; \\ 0, & \text{if } N_t = 0. \end{cases}$$

Observe that the total claim amount is modelled as a compound Poisson process.

Theorem 5.6.4. *The total claim amount distribution given by*

$$P(S_t \leq x) = \sum_{n=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!} P\left(\sum_{i=1}^{n} Y_i \leq x\right), \quad x \geq 0, t \geq 0$$

and $P(S_t \leq x) = 0$ for $x < 0$

Definition 5.6.5. *The **risk process** $\{U_t\}_{t\geq 0}$ is defined as*

$$U_t = u + ct - S_t, \quad t \geq 0$$

*where $u \geq 0$, the **initial capital** and $c > 0$ denotes the **premium income rate***

Definition 5.6.7. *We have the following definitions*

1. *The **ruin probability in finite time** is given by*
$$\psi(u, T) = P(U_t < 0 \ \text{ for some } t \leq T),\ 0 < T < \infty, u \geq 0$$
2. *The **ruin probability in infinite time** is given by*
$$\psi(u) := \psi(u, \infty), u \geq 0$$

Theorem 5.6.8.

$$E(U_t) = u + ct - \lambda t\mu + (c - \lambda\mu)t$$

A minimal requirement for choosing the premium could be

$$c > \lambda\mu$$

*referred to as the **net profit condition***

5.7 The coalescent process