1. Give an example of a compact set $S \subset R$ and a continuous function $f : S \to \mathbb{R}$ which does *not* satisfy the intermediate value theorem: in other words, there are points $a < b$ in S and some x between $f(a)$ and $f(b)$ such that $f(c) \neq x$ for all $c \in S$.

   Solution. Let $S = [0, 1] \cup [3, 4]$. This is closed (as a union of two closed intervals) and bounded, so it is compact. The function $f : S \to \mathbb{R}$ given by $f(x) = x$ is continuous, and it satisfies $f(1) = 1$ and $f(3) = 3$, but there is no $c \in S$ such that $f(c) = 2$.

2. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is continuous, then $f^{-1}(c) = \{x \in \mathbb{R} \mid f(x) = c\}$ is closed.

   Solution. Let $(x_n) \subset f^{-1}(c)$ be a sequence which converges to a limit $x \in \mathbb{R}$. By sequential continuity we have $f(x_n) \to f(x)$, but $f(x_n) = c$ for all n, so $f(x) = c$ as well and thus $x \in f^{-1}(c)$. It follows that the limit of any convergent sequence in $f^{-1}(c)$ also lies in $f^{-1}(c)$, so $f^{-1}(c)$ is closed.

3. (*) Let $(S_n)_{n\in\mathbb{N}}$ denote a decreasing sequence of nonempty subsets of $\mathbb{R}$, meaning that
$$S_1 \supset S_2 \supset S_3 \supset \dots .$$
   Let $S = \bigcap_{n=1}^{\infty} S_n$ be their intersection.

   (a) Give an example where all of the S_n are open and S is empty.

   (b) Prove that if all of the S_n are compact, then S is nonempty. (Hint: consider the sequence $x_n = \inf(S_n)$.)

   Solution. (a) Take $S_n = (0, \frac{1}{n})$ for all $n \geq 1$.

   (b) Let $x_n = \inf(S_n)$ for all $n \geq 1$. This exists since S_n is bounded, and in fact $x_n \in S_n$ since S_n is closed. Moreover, we have
$$S_n \supset S_{n+1} \quad \Longrightarrow \quad x_n = \inf(S_n) \leq \inf(S_{n+1}) = x_{n+1},$$
   so the sequence (x_n) is monotone increasing; and
$$S_n \subset S_1 \quad \Longrightarrow \quad \sup(S_1) \geq \sup(S_n) \geq \inf(S_n) = x_n,$$
   so (x_n) is bounded above by $\sup(S_1)$. Since (x_n) is monotone increasing and bounded above, it converges, say $x_n \to y$.
   We claim that $y \in S$. If not, then there is some $N \geq 1$ such that $y \notin S_N$, and yet the inclusion $S_n \subset S_N$ for all $n \geq N$ implies that
$$x_n \in S_N \text{ for all } n \geq N.$$
   The set S_N is closed and contains the convergent sequence $(x_n)_{n\geq N}$, so it must also contain the limit y, and this is a contradiction. Thus $y \in S$ after all.

4. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and $S \subset \mathbb{R}$ is compact, then the image $f(S)$ is also compact.

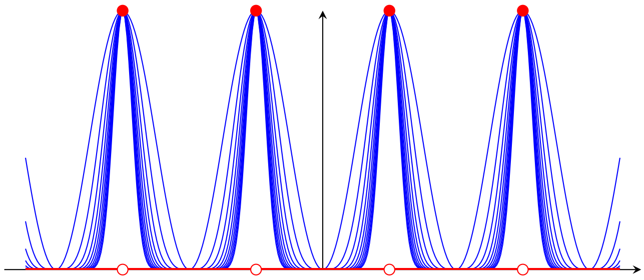
Solution. Let $(y_n) \subset f(S)$ be an arbitrary sequence, and write $y_n = f(x_n)$ for $x_n \in S$. Since S is compact, there is a convergent subsequence (x_{n_i}), with $x_{n_i} \to x \in S$. But then by continuity we have $f(x_{n_i}) \to f(x)$, so the subsequence y_{n_i} converges to $f(x) \in f(S)$. Since every sequence in $f(S)$ has a convergent subsequence with limit in $f(S)$, we conclude that $f(S)$ is compact.

5. Give a family of continuous functions $f_n : \mathbb{R} \to \mathbb{R}$ for all $n \in \mathbb{N}$ such that the f_n converge pointwise to a function $f : \mathbb{R} \to \mathbb{R}$ with infinitely many discontinuities.

Solution. Let $f_n(x) = (\sin(x))^{2n}$. Then we define $f(x)$ by

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \left(\sin^2(x)\right)^n = \begin{cases} 1, & \sin^2(x) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

The f_n are graphed below in blue for $1 \leq n \leq 10$, and the limit f is shown in red.



This is discontinuous at every point of the form $x = (2k+1)\frac{\pi}{2}$, $k \in \mathbb{Z}$.

6. Recall that $\cos(x) = \text{Re}(E(ix))$ and $\sin(x) = \text{Im}(E(ix))$ have power series

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \qquad \sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

(a) Use the identity $E(ix)E(-ix) = E(0) = 1$ to prove that $\cos^2(x) + \sin^2(x) = 1$ for all $x \in \mathbb{R}$.

(b) Prove that $|\sin(x)| \leq |x|$ for all $x \in \mathbb{R}$. (Hint: reduce to the case $0 \leq x \leq 1$.)

(c) Prove that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sin(x)$ is uniformly continuous. (Hint: use the identity $\sin(\alpha) - \sin(\beta) = 2\cos(\frac{\alpha+\beta}{2})\sin(\frac{\alpha-\beta}{2})$.)

Solution. (a) We have $E(-ix) = \cos(-x) + i\sin(-x) = \cos(x) - i\sin(x)$, since $\cos(-x) = \cos(x)$ and $\sin(-x) = -\sin(x)$ by inspecting their power series. So

$$\begin{aligned} 1 = E(ix)E(-ix) &= (\cos(x) + i\sin(x))\,(\cos(x) - i\sin(x)) \\ &= (\cos(x))^2 + (\sin(x))^2. \end{aligned}$$

(b) By part (a) we have $|\sin(x)| \leq 1$ for all $x \in \mathbb{R}$, so it suffices to prove that $|\sin(x)| \leq |x|$ for $|x| \leq 1$, since if $|x| > 1$ then $|\sin(x)| \leq 1 < |x|$ anyway. Moreover, since $|\sin(-x)| = |\sin(x)|$ and $|-x| = |x|$, we have $|\sin(-x)| \leq |-x|$ if and only if $|\sin(x)| \leq |x|$. So it suffices to consider $x \geq 0$, leaving only the case $0 \leq x \leq 1$ to be proved.

Restricting our attention to $[0, 1]$ now, we pair consecutive terms in the power series as follows:

$$\begin{aligned}\sin(x) &= x - \left(\frac{x^3}{3!} - \frac{x^5}{5!}\right) - \left(\frac{x^7}{7!} - \frac{x^9}{9!}\right) - \cdots - \left(\frac{x^{4n+3}}{(4n+3)!} - \frac{x^{4n+5}}{(4n+5)!}\right) - \dots \\ &\leq x - 0 - 0 - \cdots - 0 - \cdots = x,\end{aligned}$$

where each term in parentheses is positive because $\frac{x^{4n+3}}{(4n+3)!} \geq \frac{x^{4n+5}}{(4n+5)!}$ on the interval $0 \leq x \leq 1$. So $\sin(x) \leq x$, and for a lower bound we group terms differently:

$$\begin{aligned}\sin(x) &= \left(x - \frac{x^3}{3!}\right) + \left(\frac{x^5}{5!} - \frac{x^7}{7!}\right) + \cdots + \left(\frac{x^{4n+1}}{(4n+1)!} - \frac{x^{4n+3}}{(4n+3)!}\right) + \dots \\ &\geq 0 + 0 + \cdots + 0 + \cdots = 0,\end{aligned}$$

because $\frac{x^{4n+1}}{(4n+1)!} \geq \frac{x^{4n+3}}{(4n+3)!}$ on the interval $0 \leq x \leq 1$ for each $n \geq 0$. Combining these inequalities, we have $0 \leq \sin(x) \leq x$, which implies that $|\sin(x)| \leq |x|$ on the interval $[0, 1]$, as claimed.

(c) The identity can be proved by writing

$$\begin{aligned}\sin(\alpha) - \sin(\beta) &= \sin\left(\frac{\alpha+\beta}{2} + \frac{\alpha-\beta}{2}\right) - \sin\left(\frac{\alpha+\beta}{2} - \frac{\alpha-\beta}{2}\right) \\ &= \left(\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) + \cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)\right) \\ &\quad - \left(\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) - \cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)\right) \\ &= 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right).\end{aligned}$$

With it in hand, we have for any $x, y \in \mathbb{R}$ an inequality

$$\begin{aligned}|f(x) - f(y)| &= \left|2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)\right| \\ &\leq 2\left|\sin\left(\frac{x-y}{2}\right)\right|,\end{aligned}$$

since $|\cos(\theta)| \leq 1$ for all θ by part (a). Now we apply $|\sin(\theta)| \leq |\theta|$ from part (b) to get

$$|f(x) - f(y)| \leq 2\left|\frac{x-y}{2}\right| = |x - y|$$

for all $x, y \in \mathbb{R}$. Thus if we are given any $\epsilon > 0$, we can set $\delta = \epsilon > 0$, and we have

$$|x - y| < \delta \implies |f(x) - f(y)| \le |x - y| < \delta = \epsilon$$

for all $x, y \in \mathbb{R}$, proving that f is indeed uniformly continuous.

7. Give an example of a sequence of functions $f_1, f_2, f_3, \cdots : \mathbb{R} \to \mathbb{R}$ and constants $M_1, M_2, M_3, \cdots \in \mathbb{R}$ such that $|f_i(x)| \le M_i$ for all $x \in \mathbb{R}$ and the sum $\displaystyle\sum_{i=1}^{\infty} M_i$ converges, but $\displaystyle\sum_{i=1}^{\infty} f_i(x)$ is *not* continuous.

Solution. Take $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$ and let $f_i(x) = \dfrac{f(x)}{2^i}$ for all i. Then $|f_i(x)| \le \frac{1}{2^i}$, and certainly $\displaystyle\sum_{i=1}^{\infty} \frac{1}{2^i} = 1$ converges, but we have

$$\sum_{i=1}^{\infty} f_i(x) = \sum_{i=1}^{\infty} \frac{f(x)}{2^i} = f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

and this sum is not continuous.