Game Theory

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February 24, 2024

Contents

1 Prelude

1.5 Strategies

Definition 1.1. A **move** refers to the action a player must make on their turn to progress from one game position to the next position

Definition 1.2. An **outcome** of a game refers to the final result of a game once the game has been played

Definition 1.3. A **strategy** for a player involves a complete description of all the moves that will be made in any game position, including responses to any random moves, and the opponent's moves. A strategy is a program which can be followed to play the game mechanically.

Definition 1.4. A **pure strategy** is a strategy that doesn't involve any self-imposed random chances of playing any moves.

Definition 1.5. Finite game - if all players in the game have a finite number of pure strategies. If at least one player has an infinite number of pure strategies, the game is called an **infinite game**.

2 Dominance, Best Response and Equilibria

Define the following notation to start with

Note. Player A will have pure strategies $A_s = \{a_1, a_2, \ldots\}$, the set may be finite or infinite. Similarly, player B will have pure strategies $B_s = \{b_1, b_2, \ldots\}$

Denote by $g_A(a_i, b_j)$ the payoff to player A when player A plays pure strategy a_i and player B plays pure strategy b_j.

Definition 2.6. Strategy $a \in A_s$ is **strictly dominated** by another strategy $a' \in A_s$ if

$$g_A(a, b) < g_A(a', b) \quad \forall b \in B_s$$

Definition 2.7. In an n-player game, a strategy $s_i \in S_i$ for player i is **strictly dominated** by another strategy $s_i' \in S_i$ if

$$g_i(s_i, s_{-i}) < g_i(s_i', s_{-i}) \quad \forall s_{-i} \in S_{-i}$$

s_{-i} denotes the strategies of all players other than i

Definition 2.8. $a \in A_s$ is weakly dominated by $a' \in A_s$ if

$$g_A(a, b) \leq g_A(a', b) \quad \forall b \in B_s$$

and there exists at least one $b \in B_s$ such that the inequality is strict

Definition 2.9. In an n-player game, a strategy $s_i \in S_i$ for player i is weakly dominated by another strategy $s_i' \in S_i$ if

$$g_i(s_i, s_{-i}) \leq g_i(s_i', s_{-i}) \quad \forall s_{-i} \in S_{-i}$$

and there exists at least one $s_{-i} \in S_{-i}$ such that the inequality is strict

Definition 2.10. In an n-player game, a strategy $s_i \in S_i$ for player i is **payoff equivalent** to another strategy $s_i' \in S_i$ if

$$g_i(s_i, s_{-i}) = g_i(s_i', s_{-i}) \quad \forall s_{-i} \in S_{-i}$$

Definition 2.11. In an n-player game, a strategy $s_i \in S_i$ for player i is a **best response** to a strategy profile $s_{-i} \in S_{-i}$ if

$$g_i(s_i, s_{-i}) \geq g_i(s_i', s_{-i}) \quad \forall s_i' \in S_i$$

Proposition 2.12. *A dominated strategy is never a best response*

2.6 Equilibria

Definition 2.13 (Nash Equilibrium)**.** An **equilibrium** of an n-player game is a strategy profile $s \in S$ such that

$$g_i(s_i, s_{-i}) \geq g_i(s_i', s_{-i}) \quad \forall s_i' \in S_i$$

for all players i.

2.8 Iterative Deletion of Dominated Strategies

Proposition 2.14. *In an N-player game, with strategy sets $S_1, S_2, \ldots, S_N$, let s_i, s_i' be two strategies for player i. Suppose s_i' weakly dominates or is payoff equivalent to s_i. Consider game G' with identical payoffs as G but where S_i is replaced by $S_i - \{s_i\}$, Then:*

1. *Any Nash equilibrium of G' is a Nash equilibrium of G*
2. *If s_i is dominated by s_i', then G and G' have the* ***same*** *equilibria*

Proposition 2.15. *Consider game G that upon performing iterative deletion of dominated strategies, results in game G' with a single strategy profile. Then, the single strategy profile is the unique equilibrium of G.*

3 Mixed Equilibria

3.1 Mixed Strategies

Definition 3.16. A **mixed** strategy for a player is a self-imposed randomization over the player's pure strategies. A mixed strategy is a probability distribution over the pure strategies. A mixed strategy α for player A is denoted as

$$\alpha = (p_1, p_2, \ldots, p_n), \quad \text{or}$$
$$\alpha = p_1 a_1 + p_2 a_2 + \ldots + p_n a_n, \quad \text{where} \quad \sum_{i=1}^{n} p_i = 1, \quad 0 \leq p_i \leq 1$$

We extend the pure strategy set A_s to the more general **mixed strategy set**, $\mathbb{A}_s$ - the infinite set of all possible α for player A.

Definition 3.17. Let player A have pure strategy set $A_s = \{a_1, \ldots, a_n\}$ and player B have pure strategy set $B_s = \{b_1, \ldots, b_m\}$.

If player A choses to play the mixed strategy $\alpha = (p_1, \ldots, p_n) \in \mathbb{A}_s$ and player B choses to play the mixed strategy $\beta = (q_1, \ldots, q_m) \in \mathbb{B}_s$, then the **expected payoff** to player A is

$$g_A(\alpha, \beta) = \sum_{i=1}^{n} \sum_{j=1}^{m} p_i q_j g_A(a_i, b_j)$$

If A_s, B_s are infinite sets then the summation is replaced by integration.

$$g_A(\alpha, \beta) = \int_x \int_y g_A(x, y) f_A(x) f_B(y) \, dx \, dy$$

where $f_A(x), f_B(y)$ are the probability density functions of the mixed strategies α, β respectively.

Definition 3.19. A pair of mixed strategies α^* for A and β^* for B, are said to be in mixed equilibrium if

$$g_A(\alpha^*, \beta^*) \geq g_A(\alpha, \beta^*) \quad \forall \alpha \in \mathbb{A}_s$$
$$\text{and} \quad g_B(\alpha^*, \beta^*) \geq g_B(\alpha^*, \beta) \quad \forall \beta \in \mathbb{B}_s$$

3.3 Finding mixed equilibria by considering Pure strategies

Proposition 3.20. *For any mixed strategies α^* of player A and β^* of player B, then*

$$\max_{\alpha \in \mathbb{A}_s} \{g_A(\alpha, \beta^*)\} = \max_{a \in A_s} \{g_A(a, \beta^*)\},$$
$$\max_{\beta \in \mathbb{B}_s} \{g_B(\alpha^*, \beta)\} = \max_{b \in B_s} \{g_B(\alpha^*, b)\}$$

Definition 3.21. Let c a constant. A mixed strategy α^*, for player A is an **equaliser strategy** if

$$g_A(\alpha^*, b) = c \quad \forall b \in \mathbb{B}_s$$

Similarly for player B

Proposition 3.22. *In a 2-player game, if α^* is an equaliser strategy for A using B's payoffs and β^* is an equaliser strategy for B using A's payoffs, then (α^*, β^*) is a mixed equilibrium*

3.4 Geometry of Games

Note. Define the convex hull of a set of points as the smallest convex set that contains all the points. For a set of points $\{x_1, \ldots, x_n\}$ with each $x_i \in \mathbb{R}^m$, form their convex hull as

$$C = \left\{ \sum_{i=1}^{n} \lambda_i x_i \mid \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1 \right\}$$

3.5 Existence of an equilibrium

Theorem 3.23 (Nash, 1951)**.** *Every finite game has at least one mixed equilibrium*

3.6 Finding equilibria by checking subgames

3.7 The upper envelope method

3.8 Degenerate games

Definition 3.24 (Degenerate game)**.** A 2-player game is said to be **degenerate** if some player has a mixed strategy that assigns positive probability to exactly k pure strategies so that the other player has more than k pure strategies.

4 Zero-sum games

4.3 Max-min and Min-max Strategies

Definition 4.25. A **max-min** strategy $\hat{\alpha} \in \mathbb{A}_s$ of player A is a strategy such that

$$\min_{\beta \in \mathbb{B}_s} \{g_A(\hat{\alpha}, \beta)\} = \max_{\alpha \in \mathbb{A}_s} \left\{ \min_{\beta \in \mathbb{B}_s} \{g_A(\alpha, \beta)\} \right\}$$

assuming that the maxima and minima exist. This also defines the **max-min payoff** to player A

Definition 4.26. A **min-max** strategy $\hat{\beta} \in \mathbb{B}_s$ of player B is a strategy such that

$$\max_{\alpha \in \mathbb{A}_s}\{g_B(\alpha, \hat{\beta})\} = \min_{\beta \in \mathbb{B}_s}\left\{\max_{\alpha \in \mathbb{A}_s}\{g_B(\alpha, \beta)\}\right\}$$

This also defines the **min-max payoff** to player B

Proposition 4.27. *In a zero-sum game, for $\alpha \in \mathbb{A}_s$, then*

$$\min_{\beta \in \mathbb{B}_s}\{g_A(\alpha, \beta)\} = \min_{b \in B_s}\{g_A(\alpha, b)\}$$

Similarly for $\beta \in \mathbb{B}_s$, then

$$\max_{\alpha \in \mathbb{A}_s}\{g_B(\alpha, \beta)\} = \max_{a \in A_s}\{g_B(a, \beta)\}$$

4.4 Relationship of Equilibria and Max-min/Min-max Strategies

Proposition 4.28. *In a finite zero-sum game with $\hat{\alpha} \in \mathbb{A}_s, \hat{\beta} \in \mathbb{B}_s$ then $(\hat{\alpha}, \hat{\beta})$ is a mixed equilibrium if and only if $\hat{\alpha}$ is a max-min strategy for A and $\hat{\beta}$ is a min-max strategy for B, and*

$$\max_{\alpha \in \mathbb{A}_s}\left\{\min_{\beta \in \mathbb{B}_s}\{g_A(\alpha, \beta)\}\right\} = \min_{\beta \in \mathbb{B}_s}\left\{\max_{\alpha \in \mathbb{A}_s}\{g_B(\alpha, \beta)\}\right\}$$

4.5 The Minimax theorem of Von Neumann

Theorem 4.29 (Von Neumann, 1928). *In a finite zero-sum game then*

$$\max_{\alpha \in \mathbb{A}_s}\left\{\min_{\beta \in \mathbb{B}_s}\{g_A(\alpha, \beta)\}\right\} = v = \min_{\beta \in \mathbb{B}_s}\left\{\max_{\alpha \in \mathbb{A}_s}\{g_B(\alpha, \beta)\}\right\}$$

*where v is the unique max-min payoff to A (and cost to B), called the **value** of the game.*

4.6 Finding solutions in small zero-sum games

Proposition 4.30. *Consider 2 zero-sum games G, G', where G' is obtained from G by deleting a weakly dominated strategy of one of the players. Then any equilibrium of G' is also an equilibrium of G, and G and G' have the **same value**.*

5 Cooperative Games

5.1 Bargaining sets

Definition 5.31. Bargaining (Negotiation) set, S, resulting from a 2-player game in strategic form is the convex hull of all payoff pairs, with the added constraint that

$$\forall (x, y) \in S, \quad x \geq t_A, \quad y \geq t_B$$

where t_A, t_B are the max-min payoff of player A and B respectively. Known as A and B's security level or **threat level**.

Call (t_A, t_B) the **threat point**

5.2 Bargaining Axioms

Definition 5.32 (Axioms for bargaining solution)**.** For a bargaining set S with threat point (t_A, t_B), a **Nash bargaining solution** $N(S) = (X, Y)$ is said to satisfy the following axioms:

(a) **Efficiency** - $(X, Y) \in S$

(b) **Pareto optimality** - (X, Y) are Pareto optimal, i.e. $\forall (x, y) \in S$ if $x \geq X$ and $y \geq Y$, then $(x, y) = (X, Y)$

(c) **Invariant under payoff scaling**, meaning if $a, c > 0$ and $b, d \in \mathbb{R}$ and we define S' to be the bargaining set

$$S' = \{(ax + b, cy + d) \mid (x, y) \in S\}$$

with threat point $(at_A + b, ct_B + d)$, then $N(S') = (aX + b, cY + d)$

(d) **Symmetry** - If $t_A = t_B$ and $(x, y) \in S$ implies $(y, x) \in S$ then we must have $X = Y$

(e) **Independence of irrelevant alternatives** - If S, T are bargaining sets with the same threat point and $S \subset T$, then either $N(S) = N(T)$ or $N(T) \notin S$

5.3 The Nash Bargaining Solution

Theorem 5.33. *Under the axioms of bargaining solution, (a)-(e) above. Every bargaining set S that contains a point (x, y) with $x > t_A, y > t_B$, has a unique Nash bargaining solution $N(S) = (X, Y)$*

*Obtained as the unique point $(x, y) \in S$ that maximises the **Nash product***

$$(x - t_A)(y - t_B)$$

6 Congestion Games

6.5 Components of a Congestion Game

Definition 6.34. A **congestion network** has the following components:

1. A finite set of nodes

2. A finite set of directed edges, each edge, e, an ordered pair written AB from node A to node B

3. Each edge e has an associated cost function $c_e(x)$ giving value when there are x users on edge e, with $c_e(x)$ weakly increasing in x

$$x \leq y \implies c_e(x) \leq c_e(y)$$

Definition 6.35. To form a **congestion game**, we need the following components:

1. A congestion network

2. N users of network with each user having a origin node, O_i and a destination node D_i

3. A strategy of user i is a path P_i from $O_i \to D_i$. Given strategy P_i for each user i, the **flow** on edge e is the number of users using edge e

$$f_e = \|\{i : e \in P_i\}\|$$

4. The **cost** to user i of using path P_i is the sum of the costs of the edges in P_i

$$\text{Cost}_i(P_i) = \sum_{e \in P_i} c_e(f_e)$$

Definition 6.36. Say P_i a **best response** for user i if against strategies P_j, $j \neq i$, then

$$\sum_{e \in P_i} c_e(f_e) \leq \sum_{e \in P_i \cap Q_i} c_e(f_e) + \sum_{e \in P_i / Q_i} c_e(f_e + 1)$$

holds for every possible alternative path Q_i for user i

Definition 6.37. In a congestion game with N users strategies $P_1, P_2, \ldots, P_N$ of all N users define an **equilibrium** if each strategy is a best response to the other strategies. i.e if the above inequality holds for all i

6.6 Existence of Equilibrium in Congestion Games

Theorem 6.38. *Every congestion game has at least one equilibrium*

6.7 Price of Anarchy

Definition 6.39. The **price of anarchy** of a congestion game is the ratio of the cost of the worst equilibrium to the cost of the best possible solution

$$\text{PoA} = \frac{\text{Worst average cost per user in any equilibrium}}{\text{Average cost per user in social optimum}} = \frac{\max_P \sum_i \text{Cost}_i(P_i)}{\min_P \sum_i \text{Cost}_i(P_i)}$$

Proposition 6.40. *For atomic flow congestion games, the price of anarchy is at most* 5/2

Proposition 6.41. *For split-able flow congestion games, the price of anarchy is at most* 4/3

7 Combinatorial Games

These are 2-player, perfect information games with no chance moves. They come in 2 types:

- **Partizan games** - where the players have different sets of moves
- **Impartial games** - where the players have the same set of moves

7.0.1 The Ending Condition

A combinatorial game ends when there are no legal moves left for any player. The game is then said to be in a **terminal position**. This is a necessary condition for a game to be a combinatorial game.

7.0.2 The Normal Play Convention

The normal play convention is that the player who cannot move loses the game. This is a necessary condition for a game to be a combinatorial game.

7.1 Nim and Impartial Games

Definition 7.42. An **option** of a game position in a combinatorial game is a position that can be reached in one move from the player to move.

7.1.1 Winning and Losing Positions

Impartial games, game positions belong to one of 2 classes:

- **Winning positions** - the player to move has a winning move
- **Losing positions** - the player to move has no winning move

Proposition 7.43. *In an impartial game, a game position is losing if and only if all its options are winning positions. A game is winning if and only if at least one of its options is a losing position; moving to that position is a winning move.*

Proposition 7.44. *A Nim position is losing if and only if the Nim sum equals zero for all columns in the binary representation of the position; such a position is called a* ***zero position****. A Nim position is winning if and only if the Nim sum is not zero.*

7.2 Top-down induction

7.2.1 Partial and Total Orders

Definition 7.45. A binary relation $\simeq$ on a set S is a **partial order** if, for all $x, y, z \in S$, we have:

- **Reflexivity** - $x \simeq x$
- **Antisymmetry** - $x \simeq y$ and $y \simeq x$ implies $x = y$
- **Transitivity** - $x \simeq y$ and $y \simeq z$ implies $x \simeq z$

If in addition to the above, for all $x, y \in S$, we have:

- **Comparability** - $x \simeq y$ or $y \simeq x$

then $\simeq$ is a **total order**

Definition 7.46. For a given partial order $\simeq$ on a set S, we define the **strict order** $\sim$ corresponding to $\simeq$ by; for all $x, y \in S$:

$$x \sim y \iff x \simeq y \text{ and } x \neq y$$

Definition 7.47. An element $x \in S$ is **maximal** if there is no $y \in S$ such that $x \sim y$

7.2.2 Back to Top-Down Induction

Definition 7.48. Consider a set S of games, defined by a starting game and all the games that can be reached from it via any sequence of moves of the players. For two games; $G, H \in S$, we call H **simpler** than G, denoted with the binary relation $H \leq G$, if there is a sequence of moves that leads from G to H. We allow for $G = H$ where this sequence is empty.

Proposition 7.49. *The binary relation $\leq$ ('simpler than') on a set S of games is a partial order*

Proposition 7.50. *Every non-empty subset, T, of S has a minimal element*

Theorem 7.51 (Top-down induction)**.** *Consider a set S with a partial order $\simeq$ such that every non-empty subset of S has a minimal element. Let $P(x)$ be a statement about an element $x \in S$ that may be true or false. Assume that $P(x)$ holds whenever $P(y)$ holds for all $y \in S$ such that $y \sim x$. Then $P(x)$ is true for all $x \in S$. That is*

$$(\forall x : \ (\forall y \sim x : P(y)) \implies P(x)) \implies (\forall x : P(x))$$

7.3 Game Sums

Definition 7.52. Suppose that G and H are games with options $G_1, \ldots, G_n$ and $H_1, \ldots, H_m$ respectively. Then the **game sum** $G+H$ is the game with options $G_1+H, \ldots, G_n+H, G+H_1, \ldots, G+H_m$

Proposition 7.53. *Denoting the losing game with* ***no options*** *by* 0*, then for any games G, H and J we have*

- ***Commutativity of +:***
$$G + H = H + G$$
- ***Associativity of +:***
$$(G + H) + J = G + (H + J)$$
- ***Identity of +:***
$$G + 0 = G$$

7.4 Equivalence of Games

Definition 7.54. Two games G and H are called **equivalent**, written $G \equiv H$, if and only if for any other game J, the game sum $G + J$ is losing if and only if $H + J$ is losing

Lemma 7.55. *The binary relation of equivalence, $\equiv$, is an equivalence relation between games, this means that it is:*

- ***Reflexive*** - $G \equiv G$
- ***Symmetric*** - $G \equiv H$ *implies* $H \equiv G$
- ***Transitive*** - $G \equiv H$ *and* $H \equiv J$ *implies* $G \equiv J$

Proposition 7.56. *Two Nim piles are equivalent if and only if they have the same size*

Proposition 7.57. *G is a losing game if and only if $G \equiv 0$*

Corollary 7.58. *Any two losing games are equivalent*

Lemma 7.59. *For all games G, H and K we have:*

$$G \equiv H \implies G + K \equiv H + K$$

Lemma 7.60. *Let J be a losing game. Then $G + J \equiv G$ for any game G*

Proposition 7.61 (The Copycat Principle)**.** *$G + G \equiv 0$ for any impartial game G*

Lemma 7.62. *For impartial games G and H, then $G \equiv H$ if and only if $G + H \equiv 0$*

7.5 Notation for Nim Piles

Definition 7.63. If G is a **single** Nim pile with $n \geq 0$ tokens in it, then we denote this game by $*n$. This game is specified by its n options, defined recursively as

$$*0, *1, \dots, *(n-1)$$

Definition 7.64. If $G \equiv *m$ for an impartial game G, then m is called the **Nim value** of G

7.6 The Mex Rule

Definition 7.65. For a finite set of natural numbers S, the **minimum excluded number** of S, written $mex(S)$, is defined as

$$mex(S) = \min\{n \in \mathbb{N} \mid n \notin S\}$$

In other words, $mex(S)$ is the smallest non-negative integer not contained in S e.g. $mex(\{0, 1, 3, 4, 6\}) = 2$

Theorem 7.66 (The Mex Rule)**.** *Any impartial game G has **Nim value** m, where m is uniquely determined as follows; for each option H of G, let H have Nim value s_H, and let $S = \{s_H : H \text{ is an option of } G\}$. Then $m = mex(S)$, that is, $G \equiv *(mex(S))$*

7.7 Sums of Nim Piles

Definition 7.67. If $*k \equiv *m + *n$, then we call k the **Nim sum** of m and n, and write $k = m \oplus n$

Theorem 7.68. *Let $n \in \mathbb{Z}^+$, and represent n as a unique sum of powers of 2, i.e. write $n = 2^a + 2^b + 2^c + \dots$, where $a > b > c > \dots \geq 0$. Then*

$$*n \equiv *2^a \oplus *2^b \oplus *2^c \oplus \dots$$