Math40003 Linear Algebra and Groups — Unseen 5

1. Let $A \in M_{n\times n}(F)$ for some field F. Prove there is some non-zero polynomial $p(x)$ such that $p(A) = 0$.
   Do not use Cayley-Hamilton or any other advanced material you might know. Only use what we learned in this module until now.

2. For a matrix A, we denote by $\mathrm{RSp}(A)$ and $\mathrm{CSp}(A)$ the row space and column space of A, respectively.

   (a) Let $A, A' \in M_{m\times n}(F)$, $B \in M_{n\times p}(F)$ for some field F. Prove that if $\mathrm{RSp}(A) = \mathrm{RSp}(A')$, then $\mathrm{RSp}(AB) = \mathrm{RSp}(A'B)$.

   (b) Let $A, A' \in M_{n\times p}(F)$, $B \in M_{m\times n}(F)$ for some field F. Prove that if $\mathrm{CSp}(A) = \mathrm{CSp}(A')$, then $\mathrm{CSp}(BA) = \mathrm{CSp}(BA')$.
   Try not to duplicate the same proof as in **??**. Rather, try finding a clever and short argument that uses **??**.

3. Let F be a field, and $A \in M_{n\times n}(F)$ such that $A^{n+1} = 0$. Prove $A^n = 0$.