Topic: Expectation of random variables

In today's problem class we will be computing expectations of various discrete and continuous random variables.

1. Compute the mean of the following random variables.
   (a) $X \sim \text{Poi}(\lambda)$,
   (b) $X \sim \text{Exp}(\lambda)$,

Solution: Recall that we have

$$\mathrm{E}(X) = \begin{cases} \sum_n n\mathrm{P}(X = n), & \text{if } X \text{ is discrete,} \\ \int x f_X(x)dx, & \text{if } X \text{ is continuous.} \end{cases}$$

(a) In the case when $X \sim \text{Poi}(\lambda)$, we have

$$\begin{aligned} \mathrm{E}(X) &= \sum_{n=0}^{\infty} n\mathrm{P}(X = n) = \sum_{n=0}^{\infty} n\frac{\lambda^n}{n!}e^{-\lambda} = e^{-\lambda}\sum_{n=1}^{\infty}\frac{\lambda^n}{(n-1)!} \\ &= e^{-\lambda}\lambda\sum_{n=0}^{\infty}\frac{\lambda^n}{n!} = e^{-\lambda}e^{\lambda}\lambda = \lambda. \end{aligned}$$

(b) In the case when $X \sim \text{Exp}(\lambda)$, we have

$$\mathrm{E}(X) = \int_0^{\infty} x\lambda e^{-\lambda x}dx = \frac{1}{\lambda}\int_0^{\infty} ze^{-z}dz = \Gamma(2)\frac{1}{\lambda} = \frac{1}{\lambda}.$$

2. Let X be a continuous random variable with the following p.d.f.:

$$f_X(x) = \begin{cases} \theta\lambda e^{-\lambda x}, & x \geq 0; \\ (1-\theta)\lambda e^{\lambda x}, & x < 0. \end{cases}$$

where λ and θ are constants such that $\lambda > 0$ and $0 \leq \theta \leq 1$.

   (a) Show that $f_X(x)$ is a valid p.d.f..
   (b) Find $\mathrm{E}(X)$. *Hint: You might find it useful to look up the definition of the Gamma function.*
   (c) Find $\text{Var}(X)$

Solution:

(a) $f_X(x) \geq 0$ for all $x \in \mathbb{R}$, since $\lambda > 0, \theta \in (0, 1)$ and the exponential function is always positive, and

$$\begin{aligned} \int_{-\infty}^{\infty} f_X(x)dx &= \int_{-\infty}^{0}(1-\theta)\lambda e^{\lambda x}dx + \int_0^{\infty}\theta\lambda e^{-\lambda x}dx \\ &= (1-\theta)e^{\lambda x}\Big|_{-\infty}^{0} - \theta e^{-\lambda x}\Big|_0^{\infty} = 1 - \theta + \theta = 1. \end{aligned}$$

(b) For the expectation, we have

$$\mathrm{E}(X) = \int_{-\infty}^{\infty} xf_X(x)dx = \int_{-\infty}^{0}(1-\theta)\lambda xe^{\lambda x}dx + \int_0^{\infty}\theta\lambda xe^{-\lambda x}dx.$$

Recall that the Gamma function is defined as

$$\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}dx, \qquad t > 0.$$

In the case, when $t \in \mathbb{N}$, we have that $\Gamma(t) = (t-1)!$.

We have, using the transformation $z = -\lambda x$, $dz = -\lambda dx$:

$$\int_{-\infty}^0 (1-\theta)\lambda x e^{\lambda x}dx = (1-\theta)\int_\infty^0 (-z)e^{-z}(-1)dz\frac{1}{\lambda} = (1-\theta)(-1)\int_0^\infty ze^{-z}dz\frac{1}{\lambda}$$
$$= -\frac{(1-\theta)}{\lambda}\Gamma(2) = \frac{(\theta-1)}{\lambda}.$$

Similarly, using the transformation $z = \lambda x$, $dz = \lambda dx$:

$$\int_0^\infty \theta\lambda x e^{-\lambda x}dx = \int_0^\infty \theta z e^{-z}dz\frac{1}{\lambda}$$
$$= \frac{\theta}{\lambda}\Gamma(2) = \frac{\theta}{\lambda}.$$

Hence,

$$\mathrm{E}(X) = \frac{(\theta-1)}{\lambda} + \frac{\theta}{\lambda} = \frac{(2\theta-1)}{\lambda}.$$

(c) We compute the second moment first:

$$\mathrm{E}(X^2) = \int_{-\infty}^\infty x^2 f_X(x)dx = \int_{-\infty}^0 (1-\theta)\lambda x^2 e^{\lambda x}dx + \int_0^\infty \theta\lambda x^2 e^{-\lambda x}dx.$$

For the first integral, using the same transformation as above $z = -\lambda x$, $dz = -\lambda dx$, we get:

$$\int_{-\infty}^0 (1-\theta)\frac{1}{\lambda}\lambda^2 x^2 e^{\lambda x}dx = (1-\theta)\frac{1}{\lambda}\int_\infty^0 z^2 e^{-z}(-1)dz\frac{1}{\lambda} = \frac{(1-\theta)}{\lambda^2}\int_0^\infty z^2e^{-z}dz$$
$$= \frac{(1-\theta)}{\lambda^2}\Gamma(3) = \frac{2(1-\theta)}{\lambda^2}.$$

Similarly, for the second integral, using the transformation $z = \lambda x$, $dz = \lambda dx$:

$$\int_0^\infty \theta\lambda x^2 e^{-\lambda x}dx = \frac{\theta}{\lambda^2}\int_0^\infty z^2e^{-z}dz$$
$$= \frac{\theta}{\lambda^2}\Gamma(3) = \frac{2\theta}{\lambda^2}.$$

So altogether we have

$$\mathrm{E}(X^2) = \frac{2}{\lambda^2}$$

and hence

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2 = \frac{2}{\lambda^2} - \frac{(2\theta-1)^2}{\lambda^2} = \frac{1-4\theta^2+4\theta}{\lambda^2}.$$