Math40003 Linear Algebra and Groups **Unseen 6**

1. Let $D : \mathbb{C}[x] \to \mathbb{C}[x]$ be the function sending a polynomial to its first derivative, that is, $D : f(x) \mapsto f'(x)$.

   (a) Show that D is a linear transformation.

   (b) Find $\operatorname{im} D$ and $\ker D$.

   (c) Is D injective, surjective or bijective. Justify your answer.

   (d) Show that $\mathbb{C}[x]$ is an infinite dimensional vector space over $\mathbb{C}$.

   (e) Does Question 5 on Problem Sheet 6 admit the same answer for infinite dimensional vector spaces?

2. Let $\mathbb{C}[x]_{\leq 4}$ be the $\mathbb{C}$-vector space of polynomials in x with complex coefficients and degree less than or equal to 4. Let $D : \mathbb{C}[x]_{\leq 4} \to \mathbb{C}[x]_{\leq 4}$ be the linear transformation defined by the usual derivative of polynomials.

   (a) Write the matrix of D in the canonical basis
   $$\mathcal{E} = \{e_0 = 1, e_1 = x, e_2 = x^2, e_3 = x^3, e_4 = x^4\}.$$

   (b) Find a basis of $\mathbb{C}[x]_{\leq 4}$ consisting of polynomials of degree 4 and write the matrix of D in this basis.

   (c) Write the basis change matrix from $\mathcal{E}$ to the basis found in part **??**.

   (d) Write the matrix of D with respect to the basis $\mathcal{B}$.

   (e) Is D left or right invertible? In the affirmative case write the matrix of a left (or right) inverse to D (with respect to a basis of your choice).

   (f) Repeat **??** re-defining $D : \mathbb{C}[x]_{\leq 4} \to \mathbb{C}[x]_{\leq 3}$.

3. Let $V = M_{3\times 3}(\mathbb{C})$ be the space of 3×3 complex matrices with the usual structure of vector space over $\mathbb{C}$. The canonical basis of V is
   $$\mathcal{E} = \{E_{ij}\}_{i,j\in\{1,2,3\}}$$
   where E_{ij} is the complex matrix that has 1 in position (i, j) and 0 everywhere else. Consider the map $\operatorname{tr} : V \to \mathbb{C}$ defined by $\operatorname{tr}(A) = a_{11} + a_{22} + a_{33}$ for $A = (a_{ij}) \in V$.

   (a) Show that tr is a linear transformation.

   (b) Write the matrix of tr in the canonical bases of V and $\mathbb{C}$.

   (c) For two matrices $A, B \in M_{3\times 3}(\mathbb{C})$ set
   $$[A, B] = AB - BA.$$
   Compute the dimension of the subspace of V spanned by the matrices $\{[E_{ij}, E_{k\ell}] \mid i, j, k, \ell \in \{1, 2, 3\}\}$. (Hint: Consider $[E_{ii} - E_{jj}, E_{ij}]$ for $i \neq j$, $[E_{12}, E_{21}]$, and $[E_{23}, E_{32}]$)

(d) Compute the dimension of the subspace of V spanned by the matrices $\{AB - BA \mid A, B \in V\}$.