Math40003 Linear Algebra and Groups Term 2 Unseen 3

Definition 1. Let $T : V \to V$ be a linear transformation. a subspace $W \subseteq V$ is T*-invariant* if $T(W) \subseteq W$.

1. Let F be a field, let V be an F-vector space, let $T : V \to V$ be a linear transformation and let $\lambda \in F$. Prove that $W := \{ v \in V | T(v) = \lambda(v) \}$ is an invariant T-subspace.

   To see it is a subspace is standard and was shown in the previous unseen. Let $w \in W$. Then $T(w) = \lambda w \in W$.

2. Let V be an n-dimensional vector space and let $T : V \to V$ be a linear transformation. Let $0 < k < n$.

   (a) Prove that there is a k-dimensional T-invariant subspace if and only if there is some basis $\mathcal{E}$ of V and matrices $A \in M_{k\times k}(F), B \in M_{(n-k)\times(n-k)}(F), C \in M_{k\times(n-k)(F)}$ such that $[T]_{\mathcal{E}} = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$.

   $\implies$ **Let $B = (v_1, \ldots, v_n)$ such that $Span(v_1, \ldots, v_k) = W$. By definition, $[T(v_i)]_B$ is of the form $(\alpha_1, \ldots, \alpha_k, 0, \ldots, 0)^t$ for every $1 \le i \le k$.**

   $\impliedby$ **Assume $B = (v_1, \ldots, v_n)$ is such that $[T]_B = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$. Then $W := \operatorname{span}(v_1, \ldots, v_k)$ is T-invariant, since by definition of the representation matrix, $T(v_i) \in W$ for all $1 \le i \le k$. Therefore $T(W) = \operatorname{span}(T(v_1), \ldots, T(v_k)) \subseteq W$.**

   (b) Prove that there is are T-invariant subspaces W_1, W_2 such that $V = W_1 + W_2$, $W_1 \cap W_2 = \{0\}$, and $\dim(W_1) = k$ if and only if there is some basis $\mathcal{E}$ of V and matrices $A \in M_{k\times k}(F), B \in M_{(n-k)\times(n-k)}(F)$ such that $[T]_{\mathcal{E}} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$.

   $\implies$ **Let $\mathcal{E}_1 = (v_1, \ldots, v_k)$ be a basis for W_1, let $\mathcal{E}_2 = (v_{k+1}, \ldots, v_n)$ be a basis for W_1. $V = \operatorname{span}(v_1, \ldots, v_n)$ by the assumption that $V = W_1 + W_2$, so it is a basis for V.**
   Proceed as in Item 2a.

   $\impliedby$ **Assume $\mathcal{E} = (v_1, \ldots, v_n)$ is such that $[T]_B = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$. Define $W_1 = \operatorname{span}(v_1, \ldots, v_k), W_2 = \operatorname{span}(v_{k+1}, \ldots, v_n)$. Then $V = W_1 + W_2$ by spanning of $\mathcal{E}$ and $W_1 \cap W_2 = \{0\}$ by l.i. of $\mathcal{E}$. The proof that W_1 and W_2 follows by observing $[T]_{\mathcal{E}}$ as in Item 2a.**

Definition 2.

(i) A matrix $A \in M_n(F)$ is *upper triangular* if $a_{i,j} = 0$ for all $i < j$, i.e.

$$A = \begin{pmatrix} * & * & \dots & * \\ 0 & * & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & * \end{pmatrix}$$

(ii) A field F is *matrix-triangulable* if for all $n \in \mathbb{N}$, for all $A \in M_n(F)$ there is some invertible matrix $P \in M_n(F)$ and upper triangular $B \in M_n(F)$ such that $A = P^{-1}BP$.

(iii) A field F is *algebraically closed* if for every non-constant polynomial $p(x) \in F[x]$, there is some $a \in F$ such that $p(a) = 0$.

3. Prove that $\mathbb{R}$ is not matrix-triangularable.

   Notice that for an upper triangular matrix B, all entries of the diagonal $[B]_{i,i}$ are eigenvalues. We saw on the BlackBoard quiz that if $A = PBP^{-1}$ then A and B have the same characteristic polynomial, hence the same eigenvalues. (Also, not hard to prove on the spot.) So if $A = PDP^{-1}$ for some upper triangular $B \in M_n(\mathbb{R})$, then A has a real eigenvalue. But this does not hold for $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Theorem 1 (The Fundamental Theorem of Algebra). *$\mathbb{C}$ is algebraically closed.*

4. Prove that $\mathbb{C}$ is matrix-triangulable.

   We prove this by induction on n. For $n = 1$ the claim holds trivially. for $n > 1$, assume the property holds for every $k < n$. Let $A \in M_n(\mathbb{C})$ and let $p_A(x)$ be its characteristic polynomial. By Theorem 1, there is some $z \in \mathbb{C}$ such that $p_A(z) = 0$, i.e., z is an eigenvalue of A, so there are matrices $A_1 \in M_k(\mathbb{C})$, $A_2 \in M_{n-k}(\mathbb{C}), C \in M_{k\times(n-k)}(\mathbb{C})$ and a basis $\mathcal{E}'$ such that $A' := [T_A]_{\mathcal{E}'} = \begin{pmatrix} A_1 & C \\ 0 & A_2 \end{pmatrix}$. If $P' =_{\mathcal{E}} [I]_{\mathcal{E}'}$, then $A = P'A'P'^{-1}$.

   Now, by the induction hypothesis, there are invertible P_1, P_2 and upper triangular B_1, B_2 of appropriate sizes such that $A_i = P_iB_iP_i^{-1}$ for $i = 1, 2$. Observe that if $P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}$ and $B = \begin{pmatrix} B_1 & P_1^{-1}CP_2^{-1} \\ 0 & B_2 \end{pmatrix}$, then $P^-1 = \begin{pmatrix} P_1^{-1} & 0 \\ 0 & P_2^{-1} \end{pmatrix}$ and $A' = PBP^-1$. So $A = P'PB(P'P)^{-1}$ and B is upper triangular.

5. Prove that a field F is matrix-triangulable if and only if F is algebraically closed. ***hint:*** *use Question 9 from Problem Sheet 1 (term 2).*

   One direction is exactly the same as in Question 4. For the other, we prove the contrapositive: Assuming F is not algebraically closed, we'll

show that it is not matrix-triangularable. By the assumption, there is some non-constant polynomial $p(x) \in F[x]$ of degree $n > 0$ with no zero in F. Due to Question 9, Problem Sheet 1, there is some matrix $A \in M_n(F)$ such that p is the characteristic polynomial of A. Now, if A would have some B upper triangular and P such that $A = PBP^{-1}$, then A would have an eigenvalue. But then it would be a zero of p.