Imperial College London
Department of Mathematics

Solutions to Unseen Sheet 7 (week 10)

MATH40003 Linear Algebra and Groups Term 2, 2020/21

Unseen problem sheet for the tutorials in Week 10.

Question 1 Suppose G is a group. We say that $g, k \in G$ are *conjugate* if there exists $h \in G$ with $k = hgh^{-1}$.

(a) Prove that conjugacy is an equivalence relation on G.

The equivalence classes here are called the *conjugacy classes* in G. We will now determine the conjugacy classes in the symmetric group S_n.

Suppose $g, k \in S_n$ have the same disjoint cycle shape. We can define a bijection h of $\{1, \ldots, n\}$ which sends a cycle of g to a cycle of k simply by writing the disjoint cycle forms of g, k above each other (including fixed points) and sending the top row to the bottom row. For example in S_8, suppose:
$g = (1462)(357)(8)$ and
$k = (3571)(284)(6)$. Then let

$$h = \begin{pmatrix} 1 & 4 & 6 & 2 & 3 & 5 & 7 & 8 \\ 3 & 5 & 7 & 1 & 2 & 8 & 4 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 1 & 2 & 5 & 8 & 7 & 4 & 6 \end{pmatrix}.$$

(b) In the above example, check that $hgh^{-1} = k$. Turn this into a general argument: first show that $kh(x) = hg(x)$ (for all $x \in \{1, \ldots, n\}$).

Deduce that $g, k \in S_n$ are conjugate in S_n if and only if g, k have the same disjoint cycle shape.

Solution: (a) This is easy and has already been on a problem sheet.

(b) Here's the general argument. Let h be as constructed in the question. Let $x \in [n] = \{1, \ldots, n\}$ and $y = g(x)$. So in the disjoint cycle form of g we will see a cycle $(...xy....)$ (if $x = y$ this is just a 1-cycle). By definition of h, in the dcf of k we see the corresponding cycle $(...h(x)h(y)...)$. Thus $kh(x) = h(y) = hg(x)$, as required. So $kh = hg$, whence $k = hgh^{-1}$.

This shows us that if g, k have the same disjoint cycle shape, then they are conjugate. But it also shows the converse: if we start off with g, h in the above, we can write down the corresponding k and conclude that $hgh^{-1} = k$. The k which has been constructed is in dcf and has the same cycle shape as g.

Question 2 A subgroup H of a group G is a *normal subgroup* if for all $g \in G$, we have $gH = Hg$.

(a) Suppose $H \leq G$. Show that the following are equivalent:

(i) H is a normal subgroup of G;
(ii) for all $g \in G$ and $h \in H$ we have $ghg^{-1} \in H$;
(iii) H is a union of conjugacy classes in G.

(b) Find a normal subgroup of order 4 in S_4.

(c) Find the sizes of the conjugacy classes in S_5. Using this, together with Lagrange's theorem, prove that a normal subgroup of S_5 has order 1, 60 or 120. Is there an example in each case here?

Solution: (a) Easy checking using the definitions.

(b) The Klein four group V is a union of two conjugacy classes (the identity element and the class of the three double transpositions), so is a normal subgroup of S_4.

(c) The possible cycle shapes and number of elements of each shape are:

5^1: $4! = 24$
4^1: $5.3! = 30$
$3^1, 2^1$: 20
$3^1, 1^2$: 20
$2^2, 1^1$: 15
$2^1, 1^3$: 10
1^5: 1.

If H is a normal subgroup of S_5 its order is a sum of these numbers, including 1 (as H contains the identity element), which divides $|S_5| = 120$.

We also know from Problem sheet 8 that the set of 2-cycles generates the whole of S_5, so we can assume that H does not contain a 2-cycle.

Thus we need to consider when 1+ a subset of $24, 30, 20, 20, 15$ divides 120. The only possibilities are $1 + 24 + 15 = 40$ and $1 + 24 + 20 + 15 = 60$, so we need to exclude the first of these.

But if H contains $(12)(34)$ and $(15)(34)$ it contains a 3-cycle (152), as H is a subgroup. So it would have to contain all of the 3-cycles.

There are subgroups of orders $1, 60, 120$. The alternating group A_5 consisting of all even permutations is a subgroup of order 60. In fact, it is the only subgroup of order 60 (any such subgroup is of index 2 in S_5 and is therefore normal, by a question on problem sheet 7; we can then argue as above that it has to consist of the even permutations).