Analysis 2 - Concise Notes

MATH50001

Total Content

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Colour Code - Definitions are **green** in these notes, **Consequences** are **red** and **Causes** are **blue**

Content from MATH40002 assumed to be known.

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March 15, 2022

Contents

1 Holomorphic Functions

1.1 Complex Numbers

Definition 1.1. i

$$i = \sqrt{-1}, \quad i^2 = -1$$

Root of $x^2 + 1 = 0$

Basic properties

$$z = x + iy, \quad Re(z) = x,\ Im(z) = y$$

The complex conjugate:

$$\bar{z} = x - iy$$

Polar Coordinates
$z = x + iy$

$$r = |z| = \sqrt{x^2 + y^2}$$

$$x = r\cos\theta,\ y = r\sin\theta$$

$$z = r(\cos\theta + i\sin\theta$$

De-Moivre's Formula

$$z^n = r^n(\cos(n\theta) + i\sin(n'\theta)),\ n \in \mathbb{Z}^+$$

Eulers's Formula

$$e^{i\theta} = (\cos\theta + i\sin\theta)$$

1.2 Sets in Complex Plane

Definition 1.2. Discs in $\mathbb{C}$

Open Disc : $D_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$
Boundary of Disc : $C_r(z_0) = \{z \in \mathbb{C} : |z - z_0| = r\}$
Unit Disc : $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$

Definition 1.3. **Interior Point**

$\Omega \in \mathbb{C}, z_0$ an **interior point** of Ω if $\exists r > 0$ s.t $D_r(z_0) \subset \Omega$

Definition 1.4.

Set Ω **open** if $\forall \omega \in \Omega$, ω an interior point

Definition 1.5.

Set Ω **closed** if $\Omega^C = \mathbb{C} \backslash \Omega$ open
Closed $\iff$ contains all its limit points.

Definition 1.6. Closure

Closure of $\Omega = \bar{\Omega} = \{\Omega \cup$ limit points of $\Omega\}$

Definition 1.7. Boundary

Boundary of $\Omega = \underbrace{\bar{\Omega}}_{\text{Closure}} \backslash \underbrace{\partial\Omega}_{\text{interior}}$

Definition 1.8. Bounded

Ω bounded if $\exists M > 0$ s.t $|\omega| < M\ \forall \omega \in \Omega$

Definition 1.9. Diameter

$$diam(\Omega) = \sup_{z,w\in\Omega} |z-w|$$

Definition 1.10. Compact

Ω compact if closed and bounded

Theorem 1.1.

$$\begin{aligned}\Omega \text{ compact} &\iff \text{every sequence}\{z_n\} \subset \Omega \text{ has a subsequence convergent in } \Omega \\ &\iff \text{every open covering of } \Omega \text{ has a finite subcover}\end{aligned}$$

Theorem 1.2.

if $\Omega_1 \supset \Omega_2 \supset \dots \Omega_n \supset \dots$ a sequence of non-empty compact sets
s.t $\lim_{n\to\infty} diam(\Omega_n) \to 0$

$$\implies \exists! w \in \mathbb{C},\ w \in \Omega_n\ \forall n$$

Definition 1.11. Connected

Open set Ω **connected** $\iff$ any 2 points in Ω joined by curve γ entirely contained in Ω

1.3 Complex Functions

Definition 1.12.

$\Omega_1, \Omega_2 \subset \mathbb{C}$

$$f : \Omega_1 \to \Omega_2$$

a **mapping** $\Omega_1 \to \Omega_2$ if

$$\forall z = x + iy \in \Omega_1$$

$$\exists! w = u + iv \in \Omega_2,\ s.t\ w = f(z)$$

We have $w = f(z) = u(x,y) + iv(x,y)$
$u, v : \mathbb{R}^2 \to \mathbb{R}$

Definition 1.13.

f defined on $\Omega_1 \subset \mathbb{C}$ f **continuous** at $z_0 \in \Omega$ if

$$\forall \epsilon > 0 \exists \delta > 0\ s.t\ |z - z_0| < \delta \implies |f(z) - f(z_0)| < \epsilon$$

f continuous if continuous $\forall z \in \Omega$

1.4 Complex Derivative

Definition 1.14. Holomorphic

$\Omega_1, \Omega_2 \subset \mathbb{C}$ open sets
$f : \Omega_1 \to \Omega_2$
Say f **differentiable/holomorphic** at z_0 if

$$\lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h} = f'(z_0) \textbf{ exists}$$

f holomorphic on open set Ω if holomorphic at every point of Ω
Lemma
f holomorphic at $z_0 \in \Omega \iff \exists\ a \in \mathbb{C}$ s.t

$$f(z_0+h) - f(z_0) - ah = h\Psi(h)$$

For Ψ a function defined for all small h with $\lim_{h\to 0} \Psi(h) = 0$, $a = f'(z_0)$
Corollary
f holomorphic $\implies$ f continuous

Proposition
f, g holomorphic in $\Omega \implies$

(i) $(f+g)' = f' + g'$

(ii) $(fg)' = f'g + fg'$

(iii) $g(z_0) \neq 0 \implies (\frac{f}{g})' = \frac{f'g - fg'}{g^2}$

(iv) $f : \Omega \to V,\ g : \to \mathbb{C}$ holomorphic
$\implies [g \circ f(z)]' = g'(f(z))f'(z)\ \forall z \in \Omega$

1.5 Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$
$$u'_x = v'_y \qquad u'_y = -v'_x$$

Definition 1.15.

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right) \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right)$$

Theorem 1.3.

$f(z) = u(x,y) + iv(x,y) \quad z = x + iy$
f holomorphic at $z_0 \implies$

$$\frac{\partial f}{\partial \bar{z}}(z_0) = 0 \qquad f'(z_0) = \frac{\partial f}{\partial z}(z_0) = 2\frac{\partial u}{\partial z}(z_0)$$

Theorem 1.4.

$f = u + iv$ complex-valued function on open set Ω
u, v continuously differentiable, satisfying Cauchy-Riemann equations $\implies f$ holomorphic on Ω with $f'(z) = \frac{\partial f}{\partial z}(z)$

1.6 Cauchy-Riemann equations in polar

For $f = u + iv$ we have

$$u'_r = \frac{1}{r}v'_\theta \qquad v'_r = -\frac{1}{r}u'_\theta$$

1.7 Power Series

Definition 1.16. Power Series

Of the form

$$\sum_{n=0}^{\infty} a_n z^n \quad a_n \in \mathbb{C}$$

Series converge at z if $S_N(z) = \sum_{n=0}^{N} a_n z^n$ has limit $S(z) = \lim_{n\to\infty} S_N(z)$

Theorem 1.5.

Given power series $\sum_{n=0}^{\infty} a_n z^n$, $\exists 0 \leq R \leq \infty$ s.t

(i) if $|z| < R \implies$ series converges absolutely

(ii) $|z| > R \implies$ series diverges

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n} \qquad \text{(Radius of Convergence)}$$

Theorem 1.6.

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

Defines holomorphic function on its disc of convergence. With

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$$

with same radius of convergence as f.
Power series infinitely differentiable in the disc of convergence, achieved through term-wise differentiation.

Definition 1.17. Entire

A function said to be **entire** if holomorphic $\forall z \in \mathbb{C}$

1.8 Elementary functions

1.8.1 Exponential function

$$e^z = e^x \cos y + ie^x \sin y \qquad z = x + iy \in \mathbb{C}$$

Properties

(i) $y = 0 \implies e^z = e^x$

(ii) e^z is entire

(iii) $g(z)$ holomorphic
$\implies \frac{\partial}{\partial z} e^{g(z)} = e^{g(z)} g'(z)$

(iv) $z_1, z_2 \in \mathbb{C} \quad e^{z_1 + z_2} = e^{z_1} e^{z_2}$

(v) $|e^z| = |e^x||e^{iy}| = e^x \sqrt{\cos^2 x + \sin^2(x)} = e^x$

(vi) $(e^{iy})^n = e^{iny}$

(vii) $arg(z) = \arctan(y/x)$
$arg(e^z) = y + 2\pi k, \quad k \in \mathbb{Z}$

1.8.2 Trigonometric functions

Definition 1.18.

$$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right) \qquad \sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$$

Properties

(i) $\sin z, \cos z$ are entire

(ii) $\frac{\partial}{\partial z} \sin z = \cos z \quad \frac{\partial}{\partial z} \cos z = -\sin z$

(iii) $\sin^2 z + \cos^2 z = 1$

(iv) $\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$
$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$

1.8.3 Logarithmic Functions

Definition 1.19.

$$log(z) = ln|z| + i\arg(z) = log(r) + i(\theta + 2\pi k) \quad z \neq 0,\ k \in \mathbb{Z}$$

$\log(z)$ a multi-valued function

Definition 1.20.

$\text{Log}(z) = \ln|z| + i\,\text{Arg}(z)$ for $\text{Arg}(z)$ principal value $\in [-\pi, \pi]$
Properties

(i) $\log(z_1 z_2) = \log(z_1) + \log(z_2)$

(ii) $\text{Log}(z)$ holomorphic in $\mathbb{C}\backslash\{(\infty, 0]\}$

1.8.4 Powers

Definition 1.21.

$\alpha \in \mathbb{C}$ define $z^\alpha := e^{\alpha\log(z)}$ as a multi-valued function

Definition 1.22.

Principal value of $z^\alpha,\ \alpha \in \mathbb{C}$ as $z^\alpha = e^{\alpha\,\text{Log}(z)}$ ***Properties***

(i) $z^{a_1} z^{a_2} = z^{a_1 + a_2}$

2 Cauchy's Integral Formula

2.1 Parametrised Curve

Definition 2.1.

Parametrised curve a function $z(t) : [a, b] \to \mathbb{C}$

Smooth if $z'(t)$ exists and is continuous on $[a, b]$ with $z'(t) \neq 0 \forall t \in [a, b]$ Taking one-sided limits for $z'(a), z'(b)$.

Piecewise-smooth if z continuous on $[a, b]$ and if $\exists$ finitely many points $a = a_0 < a_1 < \cdots < a_n = b$ s.t $z(t)$ smooth on $[a_k, a_{k+1}]$

$$z : [a, b] \to \mathbb{C} \quad \tilde{z} : [c, d] \to \mathbb{C}$$

equivalent if $\exists$ continuously differentiable bijection $s \to t(s)$ from $[c, d]$ to $[a, b]$ s.t $t'(s) > 0$ and $\tilde{z}(s) = z(t(s)) =$

Definition 2.2. Path Integrals

Path integral given smooth $\gamma \subset \mathbb{C}$ parametrised by $z : [a, b] \to \mathbb{C}$.
f continuous function on γ

$$\int_\gamma f(z)dz = \int_a^b f(z(t))z'(t)dt$$

independent of choice of parametrization.
If γ piece-wise smooth

$$\int_\gamma f(z)dz = \sum_{k=0}^{n-1} \int_{a_k}^{a_{k+1}} f(z(t))z'(t)dt$$

Definition 2.3.

Define curve γ^- obtained by reversing orientation of γ
Can take $z^- : [a, b] \to \mathbb{C}$ s.t $z^-(t) = z(b + a - t)$

Definition 2.4. Closed Curve

Smooth/piece-wise smooth curve closed if $z(a) = z(b)$ for any parametrisation.

Definition 2.5. Simple Curve

Smooth/piece-wise smooth curve simple if not **self-intersecting**

$$z(t) \neq z(s) \text{ unless } s = t \in [a, b]$$

2.2 Integration along Curves

Definition 2.6. Length of smooth curve

$$\text{Length}(\gamma) = \int_a^b |z'(t)|dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}dt$$

Theorem 2.1. Properties of Integration

(i) $\int_\gamma af(z) + bg(z)dz = a\int_\gamma f(z)dz + b\int_\gamma g(z)dz$

(ii) γ^- reverse orientation of γ

$$\implies \int_\gamma f(z)dz = -\int_{\gamma^-} f(z)dz$$

(iii) **M-L inequality**

$$\left|\int_\gamma f(z)dz\right| \leq \sup_{z\in\gamma}|f(z)| \cdot \text{length}(\gamma) = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}dt$$

2.3 Primitive Functions

Definition 2.7. Primitive

A **Primitive** for f on $\Omega \subset \mathbb{C}$ a function F holomorphic on Ω s.t $F'(z) = f(z)\ \forall z \in \Omega$

Theorem 2.2.

Continuous function f with primitive F in open set Ω and curve γ in Ω from $w_1 \to w_2$

$$\int_\gamma f(z)dz = F(w_2) - F(w_1)$$

Corollary
γ closed curve in open set Ω f continuous and has primitive in $\Omega \implies$

$$\oint_\gamma f(z)dz = 0$$

Corollary
$Omega$ with $f' = 0 \implies f$ constant

2.4 Properties of Holomorphic functions

Theorem 2.3.

Let $\Omega \subset \mathbb{C}$ open set
$T \subset \Omega$ a triangle whose interior contained in Ω

$$\implies \oint_T f(z)dz = 0$$

for f holomorphic in Ω
Corollary
f holomorphic on open set Ω containing rectangle R in its interior

$$\implies \oint_R f(z)dz = 0$$

2.5 Local existence of primitives and Cauch-Goursat theorem in a disc

Theorem 2.4.

Holomorphic functions in open disc have a primitive in that disc
***Corollary* - (Cauchy-Goursat Theorem for a disc)**
f holomorphic in disc $\implies \oint_\gamma f(z)dz = 0$
for any closed curve γ in that disc
Corollary
Suppose f holomorphic in open set containing circle C and its interior

$$\implies \oint_C f(z)dz = 0$$

2.6 Homotopies and simply connected domains

Definition 2.8. Homotopic

γ_0, γ_1 **homotopic** in Ω if $\forall s \in [0,1], \exists$ curve $\gamma \subset \Omega$ with $\gamma_s(t)$ s.t

$$\gamma_s(a) = \alpha \qquad \gamma_s(b) = \beta$$

$\forall t \in [a,b] : \gamma_s(t)|_{s=0} = \gamma_0(t) \quad \gamma_s(t)|_{s=1} = \gamma_1(t)$
With $\gamma_s(t)$ jointly continuous in $s \in [0,1]$ and $t \in [a,b]$

Theorem 2.5.

γ_0, γ_1 homotopic, f holomorphic

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz$$

Definition 2.9.

Open set $\Omega \subset \mathbb{C}$ **simply connected** if any 2 pair of curves in Ω with shared end-points homotopic.

Theorem 2.6.

Any holomorphic function in simply connected domain has a primitive.

***Corollary* - (Cauchy-Goursat Theorem)**

f holomorphic in simply connected open set Ω

$$\implies \oint_\gamma f(z)dz = 0$$

for any closed piecewise-smooth curve $\gamma \subset \Omega$

Theorem 2.7. (Deformation Theorem)

γ_1 and γ_2, 2 simple closed piecewise-smooth curves with γ_2 lying wholly inside γ_1
f holomorphic in domain containing region between γ_1, γ_2

$$\implies \oint_{\gamma_1} f(z)dz = \oint_{\gamma_2} f(z)dz$$

2.7 Cauchy's Integral Formulae

Theorem 2.8. (Cauchy's Integral Formula)

f holomorphic inside and on simple closed piecewise-smooth curve γ
$\forall z_0$ interior $to \gamma$

$$f(z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - z_0} dz$$

Theorem 2.9. *(Generalised cauchy's integral formula)*

f holomorphic in open set Ω.
γ simple,closed piecewise-smooth Ω
$\forall z$ interior to γ

$$\implies \frac{d^n f(z)}{dz^n} = \frac{n!}{2\pi i} \oint_\gamma \frac{f(t)}{(t-z)^{n+1}} dt$$

Corollary

f holomorphic $\implies$ all its derivatives are too.

3 Applications of Cauchy's integral formula

Corollary - (Liouville's theorem)
if an entire function bounded $\implies f$ constant

Theorem 3.1. (Fundamental theorem of algebra)

Every polynomial of degree > 0 with complex coefficients has at least one zero.
Corollary Every polynomial $P(z) = a_n z^n + \cdots + a_0$ of degree $n \geq 1$ has precisely n roots in C

Theorem 3.2. (Morera's theorem)

Suppose f continuous in open disc D s.t $\forall$ triangle $T \subset D$

$$\int_T f(z)dz = 0 \implies f \text{ holomorphic}$$

3.1 Taylor + Maclaurin Series

Theorem 3.3. (Taylor's expansion theorem

f holomorphic in Ω, $z_0 \in \Omega$

$$\implies f(z) = f(z_0) = f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \ldots$$

Valid in all circles $\{z : |z - z_0| < r\} \subset \Omega$

Definition 3.1. (Taylor Series)

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots = \sum_{i=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

Definition 3.2. (Maclaurin Series)

Taylor series for $z_0 = 0$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} z^n$$

3.2 Sequences of holomorphic functions

Theorem 3.4.

if $\{f_n\}_{n=1}^{\infty}$ a sequence of holomorphic functions converging uniformly to f in every compact subset of $\Omega \implies f$ holomorphic in Ω
Corollary

$$F(z) = \sum_{n=1}^{\infty} f_n(z)$$

f_n holomorphic in $\Omega \subset \mathbb{C}$
Given series converges uniformmly in compact subsets of $\Omega \implies F(z)$ holomorphic

Theorem 3.5.

Sequence $\{f_n\}_{n=1}^{\infty} \xrightarrow[unif]{} f$ in every compact subset of $\Omega \implies$ sequence $\{f'_n\}_{n=1}^{\infty} \xrightarrow[unif]{} f'$ in every compact subset of Ω

3.3 Holomorphic functions defined in terms of integrals

Theorem 3.6.

Let $F(z, s)$ defined for $(z, s) \in \Omega \times [0, 1]$
$\Omega \subset \mathbb{C}$ open set. Given F satisfies

(i) $F(z, s)$ holomorphic in $\Omega \forall s$

(ii) F continuous on $\Omega \times [0, 1]$

$\implies f(z) := \int_0^1 F(z, s)ds$ holomorphic

3.4 Schwarz reflection principle

Definition 3.3.

$\Omega \subset \mathbb{C}$ open and **symmetric** w.r.t real line

$$z \in \Omega \iff \bar{z} \in \Omega$$

Definition 3.4.

$$\Omega^+ = \{z \in \Omega : Im(z) > 0\} \quad \Omega^- = \{z \in \Omega : Im(z) < 0\} \quad I = \{z \in \Omega : Im(z) = 0\}$$

Theorem 3.7. (Symmetry Principle)

f^+, f^- holomorphic in Ω^+, Ω^- respectively.
Extend continuously to I s.t $f^+(x) = f^-(x) \quad \forall x \in I$

$$f(z) := \begin{cases} f^+(z), & z \in \Omega^+ \\ f^+(z) = f^-(z), & z \in I \\ f^-(z), & z \in \Omega^- \end{cases} \qquad \text{holomorphic}$$

Theorem 3.8. (Schwarz relfection principle)

f holomorphic in Ω^+ extend continuously to I s.t f real-valued on I
$\implies \exists F$ holomorphic in Ω s.t $F|_{\Omega^+} = f$

4 Meromorphic Functions

4.1 Complex Logarithm

Theorem 4.1.

Ω simply connected, $1 \in \Omega, 0 \notin \Omega$
$\implies$ in Ω there is a branch of logarithm

$$F(z) = \log_\Omega(z)$$

Satisfying

(i) F holomorphic in Ω

(ii) $e^{F(z)} = z \ \forall z \in \Omega$

(iii) $F(r) = \log(r),\ r \in \mathbb{R}$ close to 1

Theorem 4.2.

Holomorphic f has 0 of order m at z_0
$\iff$ can be written in form

$$f(z) = (z - z_0)^m g(z)$$

g holomorphic at z_0, $g(z_0) \neq 0$

Corollary
Os of non-constant holomorphic function are isolated.
Every zero has neighbourhood, inside of which it is the only 0

4.2 Laurent Series

Definition 4.1.

Laurent Series for f at z_0, where series converge

$$f(z) = \sum_{-\infty}^{\infty} a_n (z-z_0)^n = \dots + a_{-2}(z-z_0)^{-2} + a_{-1}(z-z_0)^{-1} + a_0 + a_1(z-z_0)^1 + a_2(z-z_0)^2 + \dots$$

Theorem 4.3. (Laurent Expansion theorem)

f holomorphic in anunulus $D = \{z : r < |z-z_0| < R\}$
$\implies f(z)$ expressed in form $f(z) = \sum_{-\infty}^{\infty} a_n(z-z_0)^n$

$$a_n = \frac{1}{2\pi i} \oint_\gamma \frac{f(\eta)}{(\eta - z_0)^{n+1}} d\eta$$

γ simple,closed piecewise smooth curve in D with z_0 in its interior.

4.3 Poles of holomorphic functions

Definition 4.2.

z_0 a **singularity** of complex function f
if f not holomorphic at z_0, but every neighbourhood of z_0 has at least 1 holomorphic point.

Definition 4.3.

Singulartiy z_0 is **isolated** if $\exists$ neighbourhood of z_0, where it is the only singularity.

Definition 4.4.

f holomorphic with isolated singularity z_0
Considering Laurent expansion valid in some annulus

$$f(z) = \sum_{-\infty}^{\infty} a_n (z-z_0)^n$$

$\implies$

- $a_n = 0 \ \forall n < 0 \implies z_0$ a **removable singularity**
- $a_n = 0 \forall n < -m, m \in \mathbb{Z}^+, a_{-m} \neq 0 \implies z_0$ pole of order m
- $a_n \neq 0$ for infinitely many negative $n \implies z_0$ a **essential singularity**

Theorem 4.4.

f has pole of order m at $z_0 \iff$ written in form

$$f(z) = \frac{g(z)}{(z-z_0)^m}$$

g holomorphic at z_0, $g(z_0) \neq 0$

4.4 Residue Theory

Definition 4.5.

Let $f(z) = \sum_{-\infty}^{\infty} a_n(z-z_0)^n$ for $0 < |z-z_0| < R$ the Laurent series for f at z_0
Residue of f at z_0 is

$$\implies Res[f, z_0] - a_{-1}$$

Theorem 4.5.

$\gamma \subset \{z : 0 < |z-z_0| < R\}$ simple closed piecewise-smooth curve containing z_0

$$\implies Res[f, z_0] = \frac{1}{2\pi i} \oint_\gamma f(z) dz$$

Theorem 4.6.

f holomorphic function inside and on simple closed piecewise-smooth curve γ except at the singularities $z_1, \dots, z_n$ in its interior

$$\implies \oint_\gamma f(z) dz = 2\pi i \sum_{j=1}^{n} Res[f, z_j]$$

4.5 The argument principle

Theorem 4.7. (Princple of argument)

f holomorphic in open Ω, except for finitely many poles.
γ simple closed piecewise-smooth curve in Ω not passing through poles or zeroes of f

$$\implies \oint_\gamma \frac{f'(z)}{f(z)} dz = 2\pi i(N - P)$$

$$N = \sum order(zeroes) \quad P = \sum order(poles)$$

Theorem 4.8. (Rouche's Theorem)

f, g holomorphic in open Ω
$\gamma \subset \Omega$ simple closed piecewise-smooth curve with interior containing only points of Ω

if $|g(z)| < |f(z)|, z \in \gamma$

$$\implies \sum_{0s \text{ of } f+g \text{ in } \gamma} order(zeros) = \sum_{0s \text{ of } f \text{ in } \gamma} order(zeros)$$

Definition 4.6.

Mapping **open** if maps open sets $\mapsto$ open sets

Theorem 4.9. (Open mapping theorem)

if f holomorphic and non-constant in open $\Omega \subset \mathbb{C}$

$$\implies f \text{ open}$$

Remark
f open $\implies$ $|f|$ open

Theorem 4.10. (Max modulus principle)

f non-constant holomorphic in open $\Omega \subset \mathbb{C}$
$\implies$ f cannot attain maximum in Ω
Corollary
Ω open with closure $\bar{\Omega}$ compact
f holomorphic on Ω and continuous on $\bar{\Omega}$

$$\sup_{z\in\Omega} |f(z)| \leq \sup_{z\in \bar{Omega}\setminus\Omega} |f(z)|$$

4.6 Evaluation of definite integrals

5 Harmonic Functions

5.1 Harmonic functions

Definition 5.1.

$\varphi = \varphi(x, y) : \mathbb{R}^2 \to \mathbb{R},\ x, y \in \mathbb{R}$
φ **harmonic** in open $\Omega \subset \mathbb{R}^2$ if

$$\begin{aligned}\underbrace{\Delta\varphi(x,y)}_{\text{laplace operator}} &:= \frac{\partial^2\varphi}{\partial x^2}(x,y) + \frac{\partial^2\varphi}{\partial y^2}(x,y) \\ &:= \varphi''_{xx}(x,y) + \varphi''_{yy}(x,y) \\ &:= 0\end{aligned}$$

Theorem 5.1.

$f(z) = u(x, y) + iv(x, y)$ holomorphic in open $\Omega \subset \mathbb{C}$
$\implies$ u, v harmonic

Theorem 5.2. (Harmonic conjugate)

u harmonic in open disc $D \subset \mathbb{C}$
$\implies \exists$ harmonic v s.t $f = u + iv$ holomorphic in D
v the **harmonic conjugate** to u
Remark
In simply connected domain $\Omega \subset \mathbb{R}^2$ every harmonic function u has harmonic conjugate v s.t

$$v(x,y) = \int_\gamma \left(-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \right)$$

Integral independent of path, by Green's theorem as u harmonic and Ω simply connected.

5.2 Properties of real + imaginary parts of holomorphic function

Theorem 5.3.

Assume $f = u + iv$ holomorphic on open connected $\Omega \subset \mathbb{C}$

$$u(x,y) = C \tag{1}$$
$$v(x,y) = K \tag{2}$$
$$C, K \in \mathbb{R} \tag{3}$$

If (1) and (2) have same solution (x_0, y_0) and $f'(x_0 + iy_0) \neq 0$
$\implies$ curve defined by (1) orthogonal to curve defined by (2)

5.3 Preservation of angles

Definition 5.2.

Consider smooth curve $\gamma \subset \mathbb{C}$
$z(t) = x(t) + iy(t) \quad t \in [a,b]$
$\forall t_0 \in [a,b]$ we have direction vector

$$\begin{aligned} L_{t_0} &= \{z(t_0) + tz'(t_0) : t \in \mathbb{R}\} \\ &= \{x(t_0) = tx'(t_0 + i(y(t_0) + ty'(t_0)) : t \in \mathbb{R}\} \end{aligned}$$

For γ_1, γ_2 curves parameterised by functions $z_1(t), z_2(t)$, $t \in [0,1]$ s.t $z_1(0) = z_2(0)$

Define angle between γ_1, γ_2 as angle between tangents

$$\arg z_2'(0) - \arg z_1'(0)$$

Theorem 5.4. (Angle preservation theorem)

f holomorphic in open $\Omega \subset \mathbb{C}$
Given γ_1, γ_2 inside Ω, parameterised by $z_1(t), z_2(t)$

Take $z_0 = z_1(0) = z_2(0)$ with $z_1'(0), z_2'(0), f'(z_0) \neq 0$

$$\underbrace{\arg z_2'(t) - \arg z_1'(t)}_{\text{angle between } z_1(0), z_2(0)} |_{t=0} = \underbrace{\arg f(z_2'(t)) - \arg f(z_1'(t))}_{\text{angle between } f(z_1(0)), f(z_2(0))}|_{t=0} (\mod 2\pi)$$

Definition 5.3.

Ω open $\subset \mathbb{C}$
$f : \Omega \to \mathbb{C}$ **conformal** if holomorphic in Ω and if $f'(z) \neq 0 \forall z \in \Omega$
Conformal mappings preserve angles.

Definition 5.4.

Holomorphic function a **local injection** on open $\Omega \subset \mathbb{C}$
if

$$\forall z_0 \in \Omega, \exists D = \{z : |z - z_0| < r\} \subset \Omega \text{ s.t } f : D \to f(D) \text{ an injection}$$

Theorem 5.5.

$f : \Omega \to \mathbb{C}$ local injection and holomorphic

$$\implies f'(z) \neq 0 \quad \forall z \in \Omega$$

Inverse of f defined on its range holomorphic
$\implies$ inverse of conformal mapping also holomorphic

5.4 Möbius Transformations

Definition 5.5.

Mobius Transformation/ Bilinear transformation a map

$$f(z) = \frac{az+b}{cz+d} \quad a, b, c, d \in \mathbb{C}, ad - bc \neq 0$$

Remark
Mobius Transformations holomorphic except for simple pole $z = -\frac{d}{c}$ with derivative

$$f'(z) = \frac{ad - bc}{(cz + b)^2}$$

$\implies$ mapping conformal for $\mathbb{C} \backslash \{-\frac{d}{c}\}$

Theorem 5.6.

(i) Inverse of mobius transformation a mobius transformation

(ii) Composition of mobius transformations a mobius transformations

Corresponding to matrix multiplication and inverses

Definition 5.6. (Special/Simple mobius tranformations)

(M1) $f(z) = az$ Scaling and rotation by a

(M2) $f(z) = z + b$ Translation by b

(M3) $f(z) = \frac{1}{z}$ Inverse and reflection w.r.t real axis

Theorem 5.7.

Every mobius transformations a composition of $M1, M2, M3$

Corollary
Mobius transformations:

$$\begin{aligned} &\text{circles} \mapsto \text{circles} \\ &\text{interior points} \mapsto \text{interior points} \end{aligned}$$

Straight lines, considered to be circles of infinite radius

5.5 Cross-ratios Mobius Transformations

Theorem 5.8.

$w = f(z)$ a Mobius Transformation
s.t distinct $(z_1, z_2, z_3) \mapsto (w_1, w_2, w_3)$

$$\implies \left(\frac{z - z_1}{z - z_3}\right)\left(\frac{z_2 - z_3}{z_2 - z_1}\right) = \left(\frac{w - w_1}{w - z_3}\right)\left(\frac{w_2 - w_3}{w_2 - w_1}\right) \quad \forall z$$

5.6 Conformal mapping of half-plane to unit disc

Theorem 5.9.

$$\begin{aligned} &\mathbb{D} = \{z : |z| < 1\} \quad &&\mathbb{H} = \{z = x + iy : Im(z) = y > 0\} \\ &w = f(z) = \tfrac{i-z}{i+z} &&g(w) = \tfrac{1-w}{1+w} \end{aligned}$$

5.7 Riemann mapping theorem

Definition 5.7.

$\Omega \subset \mathbb{C}$ **proper** if non-empty and $\Omega \neq \mathbb{C}$

Theorem 5.10.

Ω proper and simply connected
if $z_0 \in \Omega \implies \exists!$ conformal $f : \Omega \to \mathbb{D}$ s.t $f(z_0) = 0$ and $f'(z_0) > 0$
Corollary
Any 2 simply connected open subsets in $\mathbb{C}$ conformally equivalent.