

# Multivariable Calculus + Differential Equations

## Concise Notes

MATH50004

Term 1 Content

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Colour Code - **Definitions** are green in these notes, **Consequences** are red and **Causes** are blue

*Content from MATH40004 assumed to be known.*

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# 1 Vector Calculus

## 1.1 Prelim

Definition 1.1.1 - **Einstein Summation Convention**

$$a_i x_i = \sum_{i=1}^3 x_i$$

Definition 1.1.2 - **The Kronecker delta**

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Definition 1.1.3 - **The Permutation Symbol**

$$\epsilon_{ijk} = \begin{cases} 0, & \text{if any 2 elements } i, j, k \text{ equal} \\ 1, & \text{if } i, j, k \text{ a cyclic permutation of } 1, 2, 3 \\ -1, & \text{if } i, j, k \text{ an acyclic permutation } 1, 3, 2 \end{cases}$$

Formula - **Relation between Kroenecker Delta and Permutation Symbol**

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$$

$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$$

Definition 1.1.4 - **Vector Products**

**Here are some identities:**

- $\mathbf{a} \cdot \mathbf{b} = a_i b_i$
- $[\mathbf{a} \times \mathbf{b}]_i = \epsilon_{ijk} a_j b_k$
- $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \Rightarrow [a \times b]_i = \epsilon_{ijk} a_j b_k$
- $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \epsilon_{ijk} a_i b_j c_k$
- $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \Rightarrow [\mathbf{a} \times (\mathbf{b} \times \mathbf{c})]_i = (\mathbf{a} \cdot \mathbf{c})b_i - (\mathbf{a} \cdot \mathbf{b})c_i$

## 1.2 Gradient, Div, and Curl

### Definition 1.2 - Gradient, Directional Derivatives

$\phi = \text{constant}$ , defines a surface in  $3D$ , varying the constant yields a family of surfaces.

$$\hat{\mathbf{n}} \frac{\partial \phi}{\partial n} = \nabla = \left( \frac{\delta}{\delta x}, \frac{\delta}{\delta y}, \frac{\delta}{\delta z} \right) \Rightarrow \nabla \phi = \frac{\delta \phi}{\delta x} + \frac{\delta \phi}{\delta y} + \frac{\delta \phi}{\delta z}$$

Thus, directional derivative towards  $\mathbf{s} = \frac{\delta \phi}{\delta \mathbf{s}} = \nabla \phi \cdot \hat{\mathbf{s}}$

In cylindrical coordinates  $r, \theta, z$  parametrized by  $x = r \cos \theta$ ,  $y = r \sin \theta$  yields  $\nabla \phi = \hat{\mathbf{r}} \frac{\delta \phi}{\delta r} + \frac{\hat{\theta}}{r} \frac{\delta \phi}{\delta \theta} + \hat{\mathbf{k}} \frac{\delta \phi}{\delta z}$

### Definition 1.2.3 - Tangent Plane to $\phi(P)$

$$(\mathbf{r} - \mathbf{r}_P) \cdot (\nabla \phi)_P = 0$$

$$\left( \frac{\delta \phi}{\delta x} \right)_P (x - x_P) + \left( \frac{\delta \phi}{\delta y} \right)_P (y - y_P) + \left( \frac{\delta \phi}{\delta z} \right)_P (z - z_P) = 0$$

## 1.3 Divergence & Curl

### Definition 1.3.1 - Divergence and Curl

$\mathbf{A}$  a vector function of position

$$\text{Div } \mathbf{A} = \nabla \cdot \mathbf{A} = \frac{\delta A_1}{\delta x} + \frac{\delta A_2}{\delta y} + \frac{\delta A_3}{\delta z} \text{ where } \mathbf{A} = A_1 \hat{\mathbf{i}} + A_2 \hat{\mathbf{j}} + A_3 \hat{\mathbf{k}}$$

$$\text{Curl } \mathbf{A} = \nabla \times \mathbf{A} = \hat{\mathbf{i}} \left( \frac{\delta A_3}{\delta y} - \frac{\delta A_2}{\delta z} \right) - \hat{\mathbf{j}} \left( \frac{\delta A_3}{\delta x} - \frac{\delta A_1}{\delta z} \right) + \hat{\mathbf{k}} \left( \frac{\delta A_2}{\delta x} - \frac{\delta A_1}{\delta y} \right)$$

### Definition - Laplacian Operator

$$\nabla^2 \phi = \text{div}(\nabla \phi) = \frac{\delta^2 \phi}{\delta x^2} + \frac{\delta^2 \phi}{\delta y^2} + \frac{\delta^2 \phi}{\delta z^2}$$

## 1.4 Operations with Grad operator

### Resulting Equalities

- (i)  $\nabla(\phi_1 + \phi_2) = \nabla \phi_1 + \nabla \phi_2$
- (ii)  $\text{div}(\mathbf{A} + \mathbf{B}) = \text{div } \mathbf{A} + \text{div } \mathbf{B}$
- (iii)  $\text{curl}(\mathbf{A} + \mathbf{B}) = \text{curl } \mathbf{A} + \text{curl } \mathbf{B}$
- (iv)  $\nabla(\phi\psi) = \phi \nabla \psi + \psi \nabla \phi$
- (v)  $\text{div}(\phi \mathbf{A}) = \phi \text{div } \mathbf{A} + \nabla \phi \cdot \mathbf{A}$
- (vi)  $\text{curl}(\phi \mathbf{A}) = \phi \text{curl } \mathbf{A} + \nabla \phi \times \mathbf{A}$
- (vii)  $\text{div}(\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \text{curl } \mathbf{A} - \mathbf{A} \cdot \text{curl } \mathbf{B}$
- (viii)  $\text{curl}(\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla) \mathbf{A} - \mathbf{B} \text{div } \mathbf{A} - (\mathbf{A} \cdot \nabla) \mathbf{B} + \mathbf{A} \text{div } \mathbf{B}$
- (ix)  $\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{B} \cdot \nabla) \mathbf{A} + (\mathbf{A} \cdot \nabla) \mathbf{B} + \mathbf{B} \times \text{curl } \mathbf{A} + \mathbf{A} \times \text{curl } \mathbf{B}$
- (x)  $\text{curl}(\nabla \phi) = 0$
- (xi)  $\text{curl}(\text{curl } \mathbf{A}) = \nabla(\text{div } \mathbf{A}) - \nabla^2 \mathbf{A}$
- (xii)  $\text{div}(\text{curl } \mathbf{A}) = 0$

# 1 Integration

## Definition 1.4.6 - Scalar and Vector Fields

If at each point of region  $V$ , scalar function  $\phi$  defined -  $\phi$  a scalar field over  $V$

Similarly if vector function  $A$  defined  $\forall v \in V$ ,  $A$  a vector field.

If  $\text{curl } A = 0$ ,  $A$  is an **irrotational vector field**. If  $\text{div } A = 0$ ,  $A$  a **solenoidal vector field**

## 1.5 Path Integrals

### Definition 1.5.1 - Definition of a Path Integral

$$\lim_{n \rightarrow \infty} \sum_{n=1}^N f_n \delta s_n = \int_{\gamma} f ds \Rightarrow \int_{\gamma} \mathbf{F} \cdot d\mathbf{r} = \int_{\gamma} \mathbf{F} \cdot \hat{\mathbf{t}} ds \text{ where } \hat{\mathbf{t}} \text{ is the normalized vector tangent to the path}$$

### Definition 1.5.3 - Conservative forces

If  $\mathbf{F} = \nabla \phi$  for a **differentiable scalar function**  $\phi$ ,  $\mathbf{F}$  is said to be a **conservative field**, which has the following properties:

$$\int_{\gamma} \mathbf{F} \cdot d\mathbf{r} = \phi(B) - \phi(A)$$

Result independent of path joining  $A$  and  $B$ , in particular for  $\gamma$  a closed curve ( $B \equiv A$ ) We have:

$$\oint_{\gamma} \mathbf{F} \cdot d\mathbf{r} = 0$$

Call this a **circulation of  $\mathbf{F}$**  around  $\gamma$

If a vector field  $\mathbf{F}$  s.t  $\oint_{\gamma} \mathbf{F} \cdot d\mathbf{r} = 0$ , for any closed curve  $\gamma$  say  **$\mathbf{F}$  a conservative field**, if  $\mathbf{F} = \nabla \phi \Rightarrow \mathbf{F}$  conservative.

If  $\mathbf{F}$  conservative  $\Rightarrow$  can always find differentiable scalar function  $\phi$  s.t  $\mathbf{F} = \nabla \phi$ , call  $\phi$  the **potential of field  $\mathbf{F}$**

### Definition 1.5.4 - Calculation of Path Integrals

When  $\mathbf{F} = \mathbf{F}(x, y, z)$  and the path  $\gamma$  can be parametrized by  $(x(t), y(t), z(t))$ , then:

$$\begin{aligned} \mathbf{r} &= x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}} \Rightarrow d\mathbf{r} = \frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}} + \frac{dz}{dt}\hat{\mathbf{k}} \\ \Rightarrow \int_{\gamma} \mathbf{F} \cdot d\mathbf{r} &= \int_{t_0}^{t_1} \left( \mathbf{F}_1 \frac{dx}{dt} + \mathbf{F}_2 \frac{dy}{dt} + \mathbf{F}_3 \frac{dz}{dt} \right) dt \end{aligned}$$

## 1.6 Surface Integrals

### Definition 1.6.1 - Surface Integral

Consider a surface  $S$ , where we find the surface integral of  $f = f(P)$  over  $S$ .

Dividing  $S$  into small elements of area  $\delta S_i$ , with  $f_i$  the values of  $f$  at typical points  $P_i$  of  $\delta S_i$

The **surface integral of  $f$  over  $S$**  is

$$\int_S f dS = \lim_{\substack{N \rightarrow \infty \\ \max(\delta S_n) \rightarrow 0}} \sum_{n=1}^N f_n \delta S_n$$

$f$  may be a vector or a scalar.

### 1.6.2 Types of Surfaces

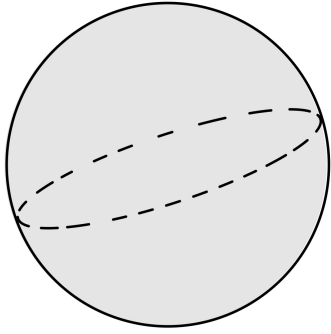


Figure 1: Closed Surface

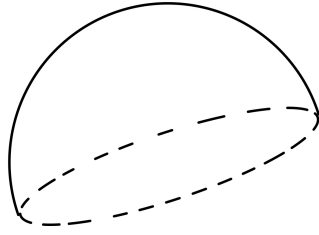


Figure 2: Open Surface

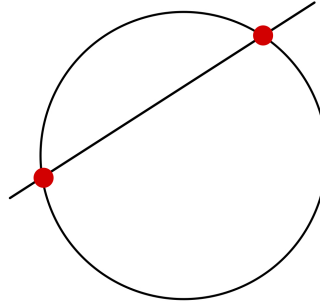


Figure 3: Convex Surface

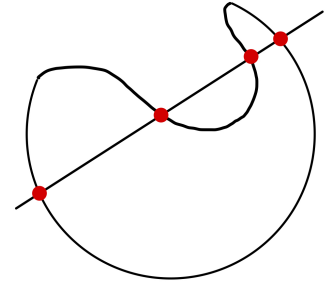
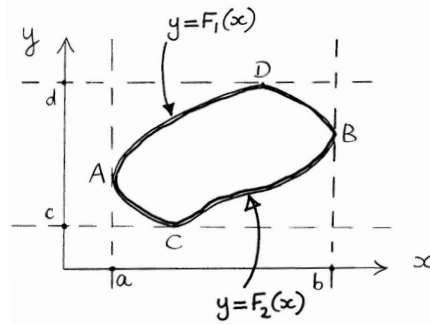


Figure 4: Non-Convex Surface

#### Definitions

1. **Closed Surface** - Divides 3D space into 2 non-connected regions; interior and exterior.
2. **Open Surface** - Does not divide 3D space into 2 non-connected regions - has a rim which can be represented by closed curve.  
Can think of closed surfaces as sum of 2 open surfaces.
3. **Convex Surface** - A surface which is crossed by a straight line at most twice

### 1.6.3 Evaluating surface integrals for plane surfaces in x-y plane



$dS$  infinitesimal area  $\Rightarrow$  think of as approx. plane.

**Vector areal element**  $d\mathbf{S}$  is the vector  $\hat{\mathbf{n}}dS$  for  $\hat{\mathbf{n}}$  the unit normal vector to  $dS$ .

For a plane lying in  $z = 0$ , we can say  $dS = dxdy$

For a rectangle,  $x = a, b$  and  $y = c, d$  circumscribing convex  $S$ . We let

$$y = \begin{cases} F_1(x) & \text{upper half ADB} \\ F_2(x) & \text{lower half ACB} \end{cases}$$

$$\text{Area of } S = \int_S dS = \int_{x=a}^{x=b} \int_{y=F_2(x)}^{y=F_1(x)} dy dx = \int_a^b [F_1(x) - F_2(x)] dx$$

For  $f(x, y)$  a function of position

$$\int_S f dS = \int_{x=a}^{x=b} \int_{y=F_2(x)}^{y=F_1(x)} f(x, y) dy dx$$

Equivalently;

$$x = \begin{cases} G_1(y) & \text{right half CBD} \\ G_2(y) & \text{left half CAD} \end{cases}$$

$$\text{Area of } S = \int_S dS = \int_c^d G_1(y) - G_2(y) dy$$

$$\int_S f dS = \int_{y=c}^{y=d} \int_{x=G_2(y)}^{x=G_1(y)} f(x, y) dx dy$$

### 1.6.5 Projection of an area onto a plane

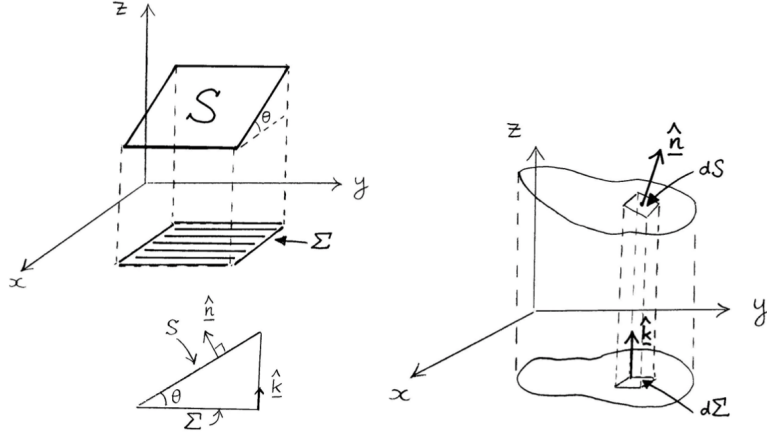


Figure 9: Left; Projection of plane area  $S$  onto  $x - y$  plane  
Figure 9: Right; Projection of curved surface  $S$  onto  $x - y$  plane

$$dS = \frac{d\Sigma}{|\hat{n} \cdot \hat{k}|}$$

### 1.6.6 The Projection Theorem

$P$  a point on surface  $S$ , which at no point is orthogonal to  $\mathbf{k}$

$$\int_S f(P) dS = \int_{\Sigma} f(P) \frac{dx dy}{|\hat{n} \cdot \hat{k}|}$$

For  $\Sigma$  a projection of  $S$  onto  $z = 0$ , with  $\hat{n}$  normal to  $S$

For  $S$  given by  $z = \phi(x, y)$

$$\int_S f(x, y, z) dS = \int_{\Sigma_z} f(x, y, \phi(x, y)) \frac{dx dy}{|\hat{n} \cdot \hat{k}|}$$

Projecting onto  $x = 0$  or  $y = 0$

$$\int_S f(P) dS = \int_{\Sigma_x} f(x, y, \phi(x, y)) \frac{dy dz}{|\hat{n} \cdot \hat{i}|} = \int_{\Sigma_y} f(x, y, \phi(x, y)) \frac{dx dz}{|\hat{n} \cdot \hat{j}|}$$

$\Sigma_x$ , projection onto  $x = 0$ ,  $\Sigma_y$ , projection onto  $y = 0$

## 1.7 Volume Integrals

### Definition 1.7.1 - Volume Integral

Considering a volume  $\tau$ , split into  $N$  subregions,  $\{\delta\tau_i\}$ , with  $\{P_i\}$  typical points of  $\{\delta\tau_i\}$ .

$$\int_{\tau} f d\tau = \lim_{\substack{N \rightarrow \infty \\ \max(\delta\tau_i) \rightarrow 0}} \sum_{i=1}^N f(P_i) \delta\tau_i$$

In Cartesian coordinates, the volume element  $d\tau = dx dy dz$

## 1.8 Results relating line, surface and volume integrals

### 1.8.1 Green's Theorem in the plane

$R$  a closed plane region bounded by a simple plane closed convex curve in  $x - y$  plane.

$L, M$  continuous functions of  $x, y$  with continuous derivatives throughout  $R$ . Then:

$$\oint_C (L dx + M dy) = \int_R \left( \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} \right) dx dy,$$

For  $C$  the boundary of  $R$  described in the counter-clockwise sense.

### 1.8.2 Vector forms of Green's Theorem

(i) *2D Stokes Theorem*

Let  $\mathbf{F} = L\mathbf{i} + M\mathbf{j}$  and  $d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j}$ . Then

$$\text{curl } \mathbf{F} = \left( \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} \right) \mathbf{k}$$

Over region  $R$  write  $dx dy = dS$ .

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_R \mathbf{k} \cdot \text{curl } \mathbf{F} dS \\ &= \int_R \text{curl } \mathbf{F} \cdot d\mathbf{S}, \quad d\mathbf{S} = \hat{\mathbf{k}} dS \end{aligned} \tag{1}$$

(ii) *Divergence Theorem in 2D*

Let  $\mathbf{F} = M\mathbf{i} - L\mathbf{j}$ . Then

$$\text{div } \mathbf{F} = \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}$$

So we can rewrite Green's Theorem as

$$\int_R \text{div } \mathbf{F} dx dy = \oint_C \mathbf{F} \cdot \hat{\mathbf{n}} ds$$

Green's Theorem holds for more complicated geometries too, if  $C$  not convex we can see it as the composition of 2 or more simple convex closed curves.

Joining  $A, A'$  form  $C_1, C_2$  enclosing  $R_1, R_2$  s.t  $R_1 + R_2 = R$

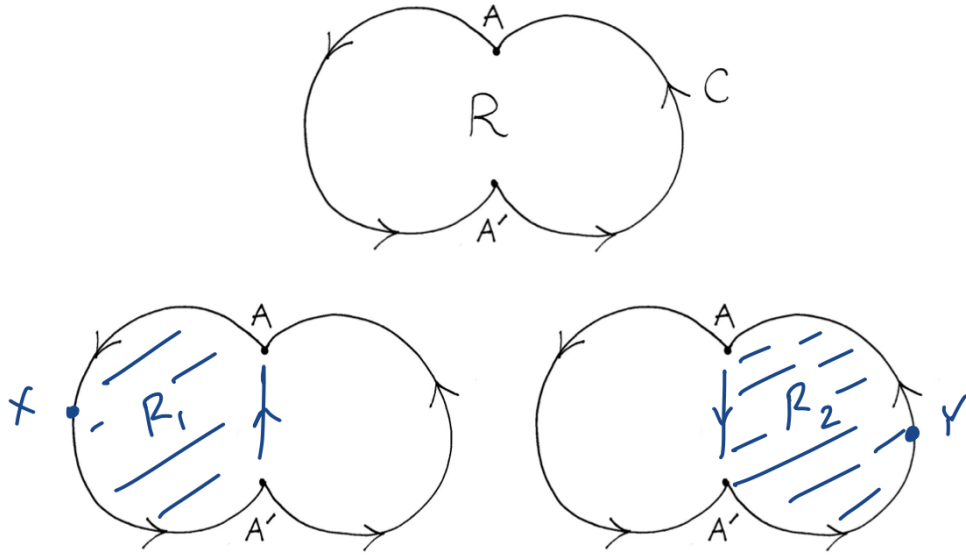


Figure 13: A non-convex boundary

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \oint_{C_1} \mathbf{F} \cdot d\mathbf{r} + \oint_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_R \text{curl } \mathbf{F} \cdot d\mathbf{S} \\ \oint_{C_1} &= \int_{AXA'} + \int_{A'A}^A \\ \oint_{C_2} &= \int_{A'YA} + \int_A^{A'} \end{aligned} \tag{2}$$



### 1.8.4 Green's Theorem in multiply-connected regions

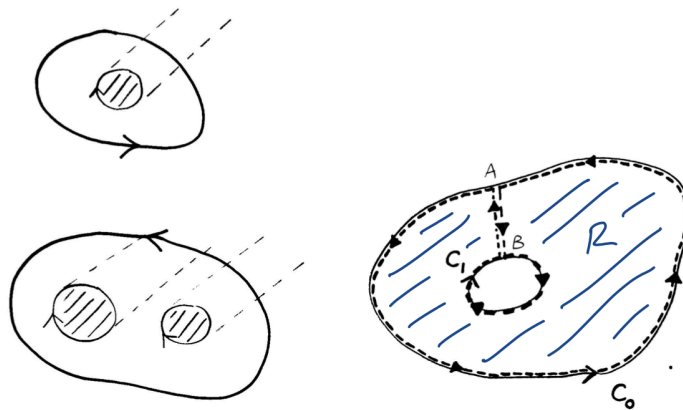


Figure 14: Left; Doubly- and triply- connected regions

Figure 14: Right; Green's Theorem in multiply-connected regions

$R$  **simply-connected** if any closed curve in  $R$  can be shrunk to a point without leaving  $R$ .

For 2D any region with a hole in it; **not simply connected**, we say it is **multiply-connected**

**Green's theorem still holds in multiply-connected regions.**  $C$  interpreted as the entire inner and outer boundary.

For doubly-connected region, describe outer  $C_0$  anti-clockwise,  $C_1$  clockwise, and join them via  $A$  on  $C_0$  and  $B$  on  $C_1$   
 $R$  now a simply connected region bounded by  $(C_0 + AB + C_1 + BA)$

$$\int_R \text{curl } \mathbf{F} \cdot d\mathbf{S} = \left( \oint_{C_0} + \int_A^B + \oint_{C_1} + \int_B^A \right) (\mathbf{F} \cdot d\mathbf{r})$$

$$\int_R \text{curl } \mathbf{F} \cdot d\mathbf{S} = \left( \oint_{C_0} + \oint_{C_1} \right) (\mathbf{F} \cdot d\mathbf{r}) = \left( \oint_C \mathbf{F} \cdot d\mathbf{r} \right)$$

Where  $C = C_0 + C_1$

### 1.8.5 Flux

If  $S$  is a surface then the flux of  $\mathbf{A}$  across  $S$  is defined as

$$\int_S \mathbf{A} \cdot \hat{\mathbf{n}} dS$$

If  $S$  a closed surface then by convention draw unit normal  $\hat{\mathbf{n}}$  **out** of  $S$ .

### 1.8.6 The divergence theorem

If  $\tau$  the volume enclosed by a closed surface  $S$  with unit outward normal  $\hat{\mathbf{n}}$  and  $\mathbf{A}$  is a vector field with continuous derivatives throughout  $\tau$ , then:

$$\int_S \mathbf{A} \cdot \hat{\mathbf{n}} dS = \int_\tau \text{div } \mathbf{A} d\tau$$

### 1.8.7 The Divergence theorem in more complicated geometries

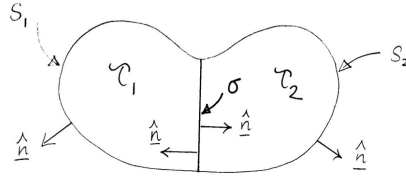


Figure 17: The divergence theorem for a non-convex surface

- (i) **Non-convex surfaces** *non-convex surface  $S$  can be divided by surface(s)  $\sigma$  into 2 (or more) parts  $S_1$  and  $S_2$  which together with  $\sigma$  form convex surfaces  $S_1 + \sigma, S_2 + \sigma$*

Applying divergence theorem to the convex parts, upon addition yields the same result as before.

- (ii) **A region with internal boundaries**

- (a) *Simply-connected regions* - e.g space between concentric spheres..

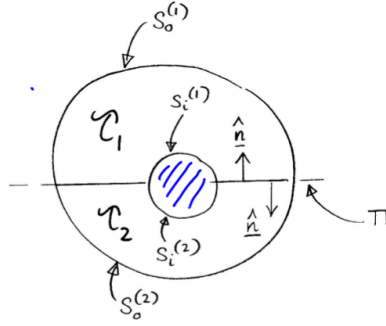


Figure 18: Simply-connected regions

Given interior surface  $S_i$  and outer surface  $S_o$ . A plane  $\Pi$  cutting both  $S_o, S_i$ , divides  $S_o, S_i$  into open  $S_o^{(1)}, S_o^{(2)}$  and  $S_i^{(1)}, S_i^{(2)}$  respectively.

Apply divergence theorem to  $\tau_1, \tau_2$  bounded by closed  $S_o^{(1)} + S_i^{(1)} + \Pi$  and  $S_o^{(2)} + S_i^{(2)} + \Pi$ . Upon addition contribution from  $\Pi$  cancels.

$$\int_{S_o + S_i} \mathbf{A} \cdot \hat{\mathbf{n}} dS = \int_S \mathbf{A} \cdot \hat{\mathbf{n}} dS = \int_{\tau_1} \text{div} \mathbf{A} d\tau + \int_{\tau_2} \text{div} \mathbf{A} d\tau = \int_{\tau} \text{div} \mathbf{A} d\tau$$

- (b) *Multiply-connected regions*  
e.g. region between 2 cylinders.

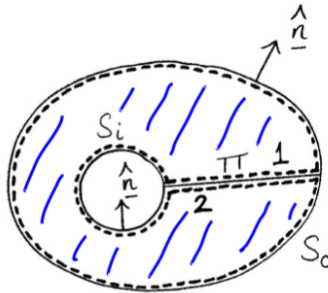


Figure 18: Multiply-connected regions

Given interior surface  $S_i$  and outer surface  $S_o$ , linked by plane  $\Pi$ .

Consider the closed surface, enclosing simply connected region  $\tau$

$$S_i + \text{side 1 of } \Pi + S_o + \text{side 2 of } \Pi$$

Applying divergence theorem to  $\tau$ . Once again gives

$$\int_{S_o + S_i} \mathbf{A} \cdot \hat{\mathbf{n}} dS = \int_{\tau} \text{div} \mathbf{A} d\tau$$

### 1.8.8 Green's identity in 3D

For  $\phi$  and  $\psi$  2 scalar fields with continuous derivatives. We consider  $\mathbf{A} = \phi \nabla \psi$ , for which we have

$$\begin{aligned} \text{div} \mathbf{A} &= \phi \nabla^2 \psi + (\nabla \phi) \cdot (\nabla \psi) \\ \hat{\mathbf{n}} \cdot \mathbf{A} &= \phi (\nabla \psi) \cdot \hat{\mathbf{n}} = \phi \frac{\partial \psi}{\partial n} \end{aligned}$$

**Green's first identity**

$$\int_S \left\{ \phi \frac{\partial \psi}{\partial n} \right\} dS = \int_\tau \phi \nabla^2 \psi + (\nabla \phi) \cdot (\nabla \psi) d\tau$$

**Green's Second identity**

$$\int_S \left\{ \phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right\} dS = \int_\tau \phi \nabla^2 \psi - \psi \nabla^2 \phi d\tau$$

### 1.8.9 Green's identities in 2D

Divergence theorem in 2D:  $\oint_F \text{div} \mathbf{F} dx dy = \oint_C \mathbf{F} \cdot \hat{\mathbf{n}} ds$

Giving the following Green's identities:

$$\oint_C \phi \frac{\partial \psi}{\partial n} ds = \int_R [\phi \nabla^2 \psi + (\nabla \psi) \cdot (\nabla \phi)] dx dy$$

and

$$\oint_C \left[ \phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right] ds = \int_R [\phi \nabla^2 \psi - \psi \nabla^2 \phi] dx dy$$

$$\int_R \phi \nabla^2 \psi dx dy = \oint_C \phi \frac{\partial \psi}{\partial n} ds - \int_R (\nabla \psi) \cdot (\nabla \phi) dx dy - \text{Looks like Integration by parts}$$

### 1.8.10 Gauss' Flux Theorem

Let  $S$  a closed surface with outward unit normal  $\hat{\mathbf{n}}$  and let  $O$  the origin of the coordinate system.

$\mathbf{A} = \frac{\mathbf{r}}{r^3}$  Then:

$$\int_S \frac{\hat{\mathbf{n}} \cdot \mathbf{r}}{r^3} = \begin{cases} 0, & \text{if } O \text{ is exterior to } S \\ 4\pi, & \text{if } O \text{ interior to } S \end{cases}$$

### 1.8.11 Stokes Theorem

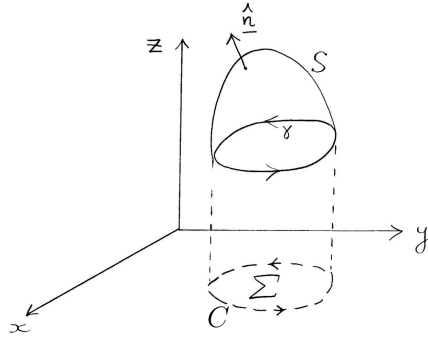


Figure 20: Diagram for proof of Stokes' Theorem

Suppose  $S$  is **open** surface with simple closed curve  $\gamma$  forming its boundary.

$\mathbf{A}$  a vector field with continuous partial derivatives, Then:

$$\oint_\gamma \mathbf{A} \cdot d\mathbf{r} = \int_S \text{curl} \mathbf{A} \cdot \hat{\mathbf{n}} dS$$

This holds for **any** open surface with  $\gamma$  as a boundary.

**Theorem**

For  $\mathbf{A}$  continuously differentiable and simply connected region:

$$\underbrace{\oint_\gamma \mathbf{A} \cdot d\mathbf{r}}_{\mathbf{A} \text{ conservative}} = 0 \iff \text{curl} \mathbf{A} = 0, \text{ throughout region for which } \gamma \text{ is drawn}$$

## 1.9 Curvilinear Coordinates

### 1.9.1 Intro + Definition

Consider generally cartesian coordinates:  $(x_1, x_2, x_3)$  with each expressible as single-valued differentiable functions of the new coordinates  $(u_1, u_2, u_3)$

$$x_i = x_i(u_1, u_2, u_3)$$

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij} = \frac{\partial x_i}{\partial u_1} \frac{\partial u_1}{\partial x_j} + \frac{\partial x_i}{\partial u_2} \frac{\partial u_2}{\partial x_j} + \frac{\partial x_i}{\partial u_3} \frac{\partial u_3}{\partial x_j}$$

With the following matrix equation

$$\begin{pmatrix} \partial x_1 / \partial u_1 & \partial x_1 / \partial u_2 & \partial x_1 / \partial u_3 \\ \partial x_2 / \partial u_1 & \partial x_2 / \partial u_2 & \partial x_2 / \partial u_3 \\ \partial x_3 / \partial u_1 & \partial x_3 / \partial u_2 & \partial x_3 / \partial u_3 \end{pmatrix} \begin{pmatrix} \partial u_1 / \partial x_1 & \partial u_1 / \partial x_2 & \partial u_1 / \partial x_3 \\ \partial u_2 / \partial x_1 & \partial u_2 / \partial x_2 & \partial u_2 / \partial x_3 \\ \partial u_3 / \partial x_1 & \partial u_3 / \partial x_2 & \partial u_3 / \partial x_3 \end{pmatrix} = I$$

Or more succinctly

$$J(x_u) \cdot J(u_x) = I$$

We say  $J(x_u)$  the **Jacobian matrix** for the  $(x_1, x_2, x_3)$  system.

$$\det(J(x_u)) \neq 0 \implies J(u_x) \text{ exists}$$

$$\det(J(x_u)) = \frac{1}{\det(J(u_x))}$$

We say  $(u_1, u_2, u_3)$  define a curvilinear coordinate system.

With each  $u_i = \text{constant}$ , defining a family of surfaces, with a member of each family passing through each  $P(x, y, z)$   
Let  $(\hat{\mathbf{a}}_1, \hat{\mathbf{a}}_2, \hat{\mathbf{a}}_3)$  unit vectors at  $P$  in the direction normal to  $u_i = u_i(P)$ , s.t  $u_i$  increasing in the direction  $\hat{\mathbf{a}}_i$

$$\hat{\mathbf{a}}_i = \frac{\nabla \mathbf{u}_i}{|\nabla \mathbf{u}_i|}$$

if we have that  $(\hat{\mathbf{a}}_1, \hat{\mathbf{a}}_2, \hat{\mathbf{a}}_3)$  mutually orthogonal  $\implies$  **orthogonal curvilinear coordinate system.**

$$\frac{\partial \mathbf{r}}{\partial u_i} = \hat{\mathbf{e}}_i h_i$$

For which we define  $h_i = |\partial \mathbf{r} / \partial u_i|$ . We call these the **length scales**

### 1.9.2 Path element

$\mathbf{r} = \mathbf{r}(u_1, u_2, u_3)$  **path element**  $d\mathbf{r}$  given by

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u_1} du_1 + \frac{\partial \mathbf{r}}{\partial u_2} du_2 + \frac{\partial \mathbf{r}}{\partial u_3} du_3$$

$$= h_1 du_1 \hat{\mathbf{e}}_1 + h_2 du_2 \hat{\mathbf{e}}_2 + h_3 du_3 \hat{\mathbf{e}}_3$$

For an orthongal system

$$(ds)^2 = (d\mathbf{r}) \cdot (d\mathbf{r}) = h_1 (du_1)^2 + h_2 (du_2)^2 + h_3 (du_3)^2$$

$$\hat{\mathbf{e}}_i = \hat{\mathbf{a}}_i = \frac{\nabla \mathbf{u}_i}{|\nabla \mathbf{u}_i|}$$

### 1.9.3 Volume Element

$$d\tau = (h_1 du_1)(h_2 du_2)(h_3 du_3)$$

$$= h_1 h_2 h_3 du_1 du_2 du_3$$

### 1.9.4 Surface element

For  $u_1$  constant.

$$dS = h_2 h_3 du_2 du_3$$

similarly for  $u_2, u_3$

### 1.9.5 Properties of various orthogonal coordinates

(i) **Cartesian coordinates**  $(x, y, z)$

$$d\tau = dx dy dz \quad d\mathbf{r} = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}}$$

$$(ds)^2 = (d\mathbf{r}) \cdot (d\mathbf{r}) = (dx)^2 + (dy)^2 + (dz)^2$$

We have  $h_1 = h_2 = h_3$

(ii) **Cylindrical polar coordinates**  $(r, \phi, z)$

Related to cartesian by

$$x = r \cos \theta \quad y = r \sin \phi \quad z = z$$

$$\begin{aligned} \frac{\partial \mathbf{r}}{\partial r} &= \left(\frac{\partial x}{\partial r}\right)\hat{\mathbf{i}} + \left(\frac{\partial y}{\partial r}\right)\hat{\mathbf{j}} + \left(\frac{\partial z}{\partial r}\right)\hat{\mathbf{k}} = (\cos \phi)\hat{\mathbf{i}} + (\sin \phi)\hat{\mathbf{j}} & \left(\frac{\partial \mathbf{r}}{\partial r}\right) \cdot \left(\frac{\partial \mathbf{r}}{\partial \phi}\right) &= 0 & h_1 &= \left|\frac{\partial \mathbf{r}}{\partial r}\right| = 1 \\ \frac{\partial \mathbf{r}}{\partial \phi} &= \left(\frac{\partial x}{\partial \phi}\right)\hat{\mathbf{i}} + \left(\frac{\partial y}{\partial \phi}\right)\hat{\mathbf{j}} + \left(\frac{\partial z}{\partial \phi}\right)\hat{\mathbf{k}} = -(r \sin \phi)\hat{\mathbf{i}} + (r \cos \phi)\hat{\mathbf{j}} & \left(\frac{\partial \mathbf{r}}{\partial r}\right) \cdot \left(\frac{\partial \mathbf{r}}{\partial z}\right) &= 0 & h_2 &= \left|\frac{\partial \mathbf{r}}{\partial \phi}\right| = r \\ \frac{\partial \mathbf{r}}{\partial z} &= \hat{\mathbf{k}} & \left(\frac{\partial \mathbf{r}}{\partial \phi}\right) \cdot \left(\frac{\partial \mathbf{r}}{\partial z}\right) &= 0 & h_3 &= \left|\frac{\partial \mathbf{r}}{\partial z}\right| = 1 \end{aligned}$$

Yielding length and volume elements:

$$(ds)^2 = (dr)^2 + r^2(d\phi)^2 + (dz)^2 \quad d\tau = r dr d\phi dz$$

(iii) **Spherical polar coordinates**  $(r, \theta, \phi)$

Related to cartesian by:

$$x = r \sin \theta \cos \phi \quad y = r \sin \theta \sin \phi \quad z = r \cos \theta$$

$$\begin{aligned} \frac{\partial \mathbf{r}}{\partial r} &= (\sin \theta \cos \phi)\hat{\mathbf{i}} + (\sin \theta \sin \phi)\hat{\mathbf{j}} + (\cos \theta)\hat{\mathbf{k}} & \left(\frac{\partial \mathbf{r}}{\partial r}\right) \cdot \left(\frac{\partial \mathbf{r}}{\partial \theta}\right) &= 0 & h_1 &= \left|\frac{\partial \mathbf{r}}{\partial r}\right| = 1 \\ \frac{\partial \mathbf{r}}{\partial \theta} &= (r \cos \theta \cos \phi)\hat{\mathbf{i}} + (r \cos \theta \sin \phi)\hat{\mathbf{j}} + (-r \sin \theta)\hat{\mathbf{k}} & \left(\frac{\partial \mathbf{r}}{\partial r}\right) \cdot \left(\frac{\partial \mathbf{r}}{\partial \phi}\right) &= 0 & h_2 &= \left|\frac{\partial \mathbf{r}}{\partial \theta}\right| = r \\ \frac{\partial \mathbf{r}}{\partial \phi} &= (-r \sin \theta \sin \phi)\hat{\mathbf{i}} + (r \sin \theta \cos \phi)\hat{\mathbf{j}} + (0)\hat{\mathbf{k}} & \left(\frac{\partial \mathbf{r}}{\partial \theta}\right) \cdot \left(\frac{\partial \mathbf{r}}{\partial \phi}\right) &= 0 & h_3 &= \left|\frac{\partial \mathbf{r}}{\partial \phi}\right| = r \sin \theta \end{aligned}$$

Volume element:

$$d\tau = r^2 \sin \theta dr d\theta d\phi$$

### 1.9.6 Gradient in orthogonal curvilinear coordinates

Let  $\nabla \Phi = \lambda_1 \hat{\mathbf{e}}_1 + \lambda_2 \hat{\mathbf{e}}_2 + \lambda_3 \hat{\mathbf{e}}_3$ .

In a general coordinate system for  $\lambda_i$ s to be found.

$$\begin{aligned} d\mathbf{r} &= h_1 du_1 \hat{\mathbf{e}}_1 + h_2 du_2 \hat{\mathbf{e}}_2 + h_3 du_3 \hat{\mathbf{e}}_3 \\ d\Phi &= \left(\frac{\partial \Phi}{\partial u_1}\right) du_1 + \left(\frac{\partial \Phi}{\partial u_2}\right) du_2 + \left(\frac{\partial \Phi}{\partial u_3}\right) du_3 \\ &= \left(\frac{\partial \Phi}{\partial x}\right) dx + \left(\frac{\partial \Phi}{\partial y}\right) dy + \left(\frac{\partial \Phi}{\partial z}\right) dz \\ &= \boxed{(\nabla \Phi) \cdot d\mathbf{r} = \lambda_1 h_1 du_1 + \lambda_2 h_2 du_2 + \lambda_3 h_3 du_3} \\ h_i \lambda_i &= \frac{\partial \Phi}{\partial u_i} \\ \Rightarrow \nabla \Phi &= \frac{\hat{\mathbf{e}}_1}{h_1} \frac{\partial \Phi}{\partial u_1} + \frac{\hat{\mathbf{e}}_2}{h_2} \frac{\partial \Phi}{\partial u_2} + \frac{\hat{\mathbf{e}}_3}{h_3} \frac{\partial \Phi}{\partial u_3} \end{aligned}$$

(i) **Cylindrical polars**  $(r, \phi, z)$

We have:  $h_1 = 1$   
 $h_2 = r \Rightarrow \nabla = \hat{\mathbf{r}} \frac{\partial}{\partial r} + \frac{\hat{\phi}}{r} \frac{\partial}{\partial \phi} + \hat{\mathbf{z}} \frac{\partial}{\partial z}$   
 $h_3 = 1$

(ii) **Spherical polars**  $(r, \theta, \phi)$

$$\begin{aligned} h_1 &= 1 \\ \text{We have: } h_2 &= r \quad \Rightarrow \quad \nabla = \hat{r} \frac{\partial}{\partial r} + \frac{\hat{\theta}}{r} \frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r \sin \theta} \frac{\partial}{\partial \phi} \\ h_3 &= r \sin \theta \end{aligned}$$

### 1.9.7 Expressions for unit vectors

$$\hat{\mathbf{e}}_i = h_i \nabla u_i$$

Alternatively, unit vectors orthogonal  $\Rightarrow$  if we know 2 already then

$$\hat{\mathbf{e}}_1 = (\hat{\mathbf{e}}_2 \times \hat{\mathbf{e}}_3) = h_2 h_3 (\nabla u_2 \times \nabla u_3)$$

### 1.9.8 Divergence in orthogonal curvilinear coordinates

Suppose we have vector field

$$\begin{aligned} \mathbf{A} &= A_1 \hat{\mathbf{e}}_1 + A_2 \hat{\mathbf{e}}_2 + A_3 \hat{\mathbf{e}}_3 \\ \Rightarrow \nabla \cdot \mathbf{A} &= \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial u_1} (A_1 h_2 h_3) + \frac{\partial}{\partial u_2} (A_2 h_3 h_1) + \frac{\partial}{\partial u_3} (A_3 h_1 h_2) \right\} \end{aligned}$$

So we have divergence in other coordinate systems as follows:

(i) **Cylindrical polars**  $(r, \phi, z)$

$$\begin{aligned} h_1 &= 1 \\ \text{We have: } h_2 &= r \quad \Rightarrow \quad \nabla \cdot \mathbf{A} = \frac{\partial A_1}{\partial r} + \frac{A_1}{r} + \frac{1}{r} \frac{\partial A_2}{\partial \phi} + \frac{\partial A_3}{\partial z} \\ h_3 &= 1 \end{aligned}$$

(ii) **Spherical polars**  $(r, \theta, \phi)$

$$\begin{aligned} h_1 &= 1 \\ \text{We have: } h_2 &= r \quad \Rightarrow \quad \nabla \cdot \mathbf{A} = \frac{1}{r^2 \sin \theta} \left\{ \frac{\partial}{\partial r} (r^2 \sin \theta A_1) + \frac{\partial}{\partial \theta} (r \sin \theta A_2) + \frac{\partial}{\partial \phi} (r A_3) \right\} \\ h_3 &= r \sin \theta \end{aligned}$$

### 1.9.9 Curl in orthogonal curvilinear coordinates

$$\text{curl} \mathbf{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{\mathbf{e}}_1 & h_2 \hat{\mathbf{e}}_2 & h_3 \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

(i) **Cylindrical polars**

$$\text{curl} \mathbf{A} = \frac{1}{r} \begin{vmatrix} \hat{\mathbf{r}} & r \hat{\phi} & \hat{\mathbf{k}} \\ \partial/\partial r & \partial/\partial \phi & \partial/\partial z \\ A_1 & A_2 & A_3 \end{vmatrix}$$

(ii) **Spherical polars**

$$\text{curl} \mathbf{A} = \frac{1}{r^2 \sin \theta} \begin{vmatrix} \hat{\mathbf{r}} & r \hat{\phi} & r \sin \theta \hat{\phi} \\ \partial/\partial r & \partial/\partial \phi & \partial/\partial z \\ A_1 & r A_2 & r \sin \theta A_3 \end{vmatrix}$$

### 1.9.10 The Laplacian in orthogonal curvilinear coordinates

From the above grad and div;

$$\begin{aligned}\nabla^2\Phi &= \nabla \cdot (\nabla\Phi) \\ &= \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial u_1} \left( \frac{h_2 h_3}{h_1} \frac{\partial\Phi}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left( \frac{h_1 h_3}{h_2} \frac{\partial\Phi}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left( \frac{h_1 h_2}{h_3} \frac{\partial\Phi}{\partial u_3} \right) \right\}\end{aligned}$$

(i) **Cylindrical polars**  $(r, \phi, z)$

$$\begin{aligned}\nabla^2\Phi &= \frac{1}{r} \left\{ \frac{\partial}{\partial r} \left( r \frac{\partial\Phi}{\partial r} \right) + \frac{\partial}{\partial \phi} \left( \frac{1}{r} \frac{\partial\Phi}{\partial \phi} \right) + \frac{\partial}{\partial z} \left( r \frac{\partial\Phi}{\partial z} \right) \right\} \\ &= \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r} \frac{\partial\Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2\Phi}{\partial \phi^2} + \frac{\partial^2\Phi}{\partial z^2}\end{aligned}$$

(ii) **Spherical polars**  $(r, \theta, \phi)$

$$\begin{aligned}\nabla^2\Phi &= \frac{1}{r^2 \sin\theta} \left\{ \frac{\partial}{\partial r} \left( r^2 \sin\theta \frac{\partial\Phi}{\partial r} \right) + \frac{\partial}{\partial \theta} \left( \sin\theta \frac{\partial\Phi}{\partial \theta} \right) + \frac{\partial}{\partial \phi} \left( \frac{1}{\sin\theta} \frac{\partial\Phi}{\partial \phi} \right) \right\} \\ &= \frac{\partial^2\Phi}{\partial r^2} + \frac{2}{r} \frac{\partial\Phi}{\partial r} + \frac{\cot\theta}{r^2} \frac{\partial\Phi}{\partial \theta} + \frac{1}{r^2} \frac{\partial^2\Phi}{\partial \theta^2} + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2\Phi}{\partial \phi^2}\end{aligned}$$

### 1.10 Changes of variables in surface integration

Suppose we have surface  $S$ , parametrized by quantities  $u_1, u_2$ . We can write:

$$x = x(u_1, u_2), \quad y = y(u_1, u_2), \quad z = z(u_1, u_2)$$

Consider surface to be comprised of arbitrarily small parallelograms, its sides given by keeping either  $u_1$  or  $u_2$

$$\begin{aligned}dS &= \text{Area of parallelogram with sides } \frac{\partial \mathbf{r}}{\partial u_1} du_1 \text{ and } \frac{\partial \mathbf{r}}{\partial u_2} du_2 \\ &= |\mathbf{J}| du_1 du_2\end{aligned}$$

**Vector Jacobian  $\mathbf{J}$**  given by  $\mathbf{J} = \frac{d\mathbf{r}}{du_1} \times \frac{d\mathbf{r}}{du_2}$ .

Useful in substitution of surface integrals:

$$\int_S f(x, y, z) dS = \int_S F(u_1, u_2) |\mathbf{J}| du_1 du_2$$

$$F(u_1, u_2) = f(x(u_1, u_2), y(u_1, u_2), z(u_1, u_2))$$

For  $S$  a region  $R$  in the  $x - y$  plane we can write:

$$\begin{aligned}\int_R f(x, y) dx dy &= \int_R F(u_1, u_2) |\det(J(x_u))| du_1 du_2 \\ |\mathbf{J}| &= \left| \frac{d\mathbf{r}}{du_1} \times \frac{d\mathbf{r}}{du_2} \right| = \det(J(x_u)) = \begin{vmatrix} \partial x / \partial u_1 & \partial x / \partial u_2 \\ \partial y / \partial u_1 & \partial y / \partial u_2 \end{vmatrix}\end{aligned}$$

For a surface described by  $z = f(x, y)$ . We have  $x = u_1, y = u_2$  and  $\mathbf{r} = (x, y, f(x, y))$

We have:

$$\begin{aligned}\frac{\partial \mathbf{r}}{\partial u_1} &= \frac{\partial \mathbf{r}}{\partial x} = \hat{\mathbf{i}} + \frac{\partial f}{\partial x} \hat{\mathbf{k}} \\ \frac{\partial \mathbf{r}}{\partial u_2} &= \frac{\partial \mathbf{r}}{\partial y} = \hat{\mathbf{j}} + \frac{\partial f}{\partial y} \hat{\mathbf{k}} \\ \left| \frac{\partial \mathbf{r}}{\partial u_1} \times \frac{\partial \mathbf{r}}{\partial u_2} \right| &= \left\| \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 0 & \partial f / \partial x \\ 0 & 1 & \partial f / \partial y \end{vmatrix} \right\| \\ &= \sqrt{1 + |\nabla f|^2}\end{aligned}$$

So we have area of surface given by

$$\int_{\Sigma} \sqrt{1 + |\nabla f|^2} dx dy$$

for  $\Sigma$  the projection of  $S$  onto the  $x - y$  plane.