1. You drive down a road whose speed limit is 60 miles per hour. An observer sees you at 12pm, and a second observer 35 miles away sees you at 12:30pm. Assuming they've watched their analysis lectures, how can they prove you were speeding?

   Solution. The mean value theorem guarantees that there is some time strictly between 12:00 and 12:30 when your velocity was

$$\frac{35 \text{ miles}}{\frac{1}{2} \text{ hour}} = 70 \text{ miles per hour.}$$

2. Prove using l'Hôpital's rule that $\lim_{x\to\infty} \left(1 + \frac{r}{x}\right)^x = e^r$. (Hint: take logs first.)

   Solution. We write the limiting term as

$$\left(1 + \frac{r}{x}\right)^x = e^{x\log\left(1+\frac{r}{x}\right)},$$

so by the continuity of $f(x) = e^x$ it will suffice to compute

$$\lim_{x\to\infty} \frac{\log\left(1 + \frac{r}{x}\right)}{1/x} = \lim_{y\downarrow 0} \frac{\log(1 + ry)}{y}.$$

The derivative of $\log(1 + ry)$ is $\frac{r}{1+ry}$, so we apply l'Hôpital's rule to get

$$\lim_{y\downarrow 0} \frac{\log(1 + ry)}{y} = \lim_{y\downarrow 0} \frac{r/(1 + ry)}{1} = r,$$

and so $\lim_{x\to\infty} x \log\left(1 + \frac{r}{x}\right) = r$ and it follows that $\lim_{x\to\infty} \left(1 + \frac{r}{x}\right)^x = e^r$.

3. Let H_n denote the harmonic sum $\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}$.

   (a) Show using the mean value theorem that $\frac{1}{n+1} < \log(n + 1) - \log(n) < \frac{1}{n}$ for all $n \in \mathbb{N}$.

   (b) Prove that $H_n - 1 < \log(n) < H_{n-1}$ for all $n \geq 2$, where $H_k = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{k}$, and deduce that $\log(n + 1) < H_n < \log(n) + 1$.

   (c) Prove that the sequence $\big(H_n - \log(n)\big)$ is decreasing, and that $\lim_{n\to\infty} (H_n - \log(n))$ exists. (This limit is called the *Euler–Mascheroni constant* $\gamma \approx 0.577\ldots$.)

   Solution. (a) Since $\log(x)$ has derivative $\frac{1}{x}$, there is some $z \in (n, n + 1)$ such that

$$\frac{\log(n + 1) - \log(n)}{(n + 1) - n} = \frac{1}{z} \in \left(\frac{1}{n + 1}, \frac{1}{n}\right),$$

or equivalently $\frac{1}{n+1} < \log(n + 1) - \log(n) < \frac{1}{n}$.

(b) We sum each side of the inequality $\frac{1}{k+1} < \log(k+1) - \log(k) < \frac{1}{k}$ from $k = 1$ to $n-1$ to get

$$\sum_{k=1}^{n-1} \frac{1}{k+1} < \log(n) - \log(1) < \sum_{k=1}^{n-1} \frac{1}{k},$$

noticing that lots of cancellation occurs in the middle. Since $\log(1) = 0$, this is equivalent to $H_n - 1 < \log(n) < H_{n-1}$. The left half of this gives us $H_n < \log(n) + 1$, and when we replace n with $n+1$ the right half gives us $\log(n+1) < H_n$, so we combine these to get $\log(n+1) < H_n < \log(n) + 1$.

(c) From part (b) we have

$$0 < \log(n+1) - \log(n) < H_n - \log(n) < 1,$$

so the sequence $a_n = H_n - \log(n)$ is bounded, and thus if it is monotone decreasing then it converges. We compute

$$a_{n+1} - a_n = \frac{1}{n+1} - \log(n+1) + \log(n),$$

so if we let $f(x) = \frac{1}{x+1} - \log(x+1) + \log(x)$, then we want to show that $f(n) < 0$ for all integers $n \geq 1$. We have

$$\begin{aligned} f'(x) &= -\frac{1}{(x+1)^2} - \frac{1}{x+1} + \frac{1}{x} \\ &= -\frac{1}{(x+1)^2} + \frac{1}{x(x+1)} = \frac{1}{x(x+1)^2} \end{aligned}$$

and so $f'(x) > 0$ for all $x > 0$, meaning that $f(x)$ is strictly increasing. But

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \left(\frac{1}{x+1} - \log\left(1 + \frac{1}{x}\right) \right) = 0.$$

Since $f(x)$ is strictly increasing, this implies that $f(x) < 0$ for all $x > 0$. (Proof: assuming otherwise, let $\epsilon = f(y)$ be a positive value of f. Then there is no $N > 0$ such that if $x \geq N$ then $|f(x) - 0| < \epsilon$, because as soon as $x > \max(y, N)$ we have $f(x) > f(y) = \epsilon$. This contradicts $f(x) \to 0$.) So in particular, for all integers $n \geq 1$ we have $a_{n+1} - a_n = f(n) < 0$, hence $a_{n+1} < a_n$.

4. (*) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable, and suppose there is a constant $C < 1$ such that $|f'(x)| \leq C$ for all $x \in \mathbb{R}$. We will prove that f has exactly one fixed point, meaning there is a unique $y \in \mathbb{R}$ such that $f(y) = y$. Pick some $x_0 \in \mathbb{R}$ and let

$$x_{n+1} = f(x_n) \text{ for all } n \geq 0.$$

(a) Prove that $|x_{n+2} - x_{n+1}| \leq C|x_{n+1} - x_n|$ for all n.

(b) Prove that the sequence (x_n) converges, and that if its limit is y then $f(y) = y$.

(c) Prove that f cannot have two different fixed points.

Solution. (a) If $x_{n+1} = x_n$ then $x_{n+2} = f(x_{n+1}) = f(x_n) = x_{n+1}$, and so both sides of the desired inequality are zero. Otherwise, the mean value theorem tells us that there is some t between x_n and x_{n+1} such that

$$\frac{f(x_{n+1}) - f(x_n)}{x_{n+1} - x_n} = f'(t) \ \Rightarrow\ \left|\frac{x_{n+2} - x_{n+1}}{x_{n+1} - x_n}\right| = |f'(t)| \le C.$$

Thus $|x_{n+2} - x_{n+1}| \le C|x_{n+1} - x_n|$ as desired.

(b) Write $d = |x_1 - x_0|$. Then $|x_{n+1} - x_n| \le C^n d$ by induction and part (a). The triangle inequality says that for any integers $m \ge n$,

$$\begin{aligned}|x_m - x_n| &\le |x_m - x_{m-1}| + \cdots + |x_{n+2} - x_{n+1}| + |x_{n+1} - x_n| \\ &\le C^{m-1}d + \cdots + C^{n+1}d + C^n d \\ &< \sum_{i=n}^{\infty} C^i d = \frac{C^n d}{1-C}.\end{aligned}$$

For any $\epsilon > 0$ we can find $N \ge 0$ such that $\frac{C^N d}{1-C} < \epsilon$, since the left side approaches 0 as $N \to \infty$. Then given $m, n \ge N$ we have shown that

$$|x_m - x_n| < \frac{C^N d}{1-C} < \epsilon,$$

which proves that the sequence $\big(x_n\big)$ is Cauchy and hence convergent, say with limit y. Since $x_n \to y$ and f is continuous, we have $f(x_n) \to f(y)$. But $f(x_n) = x_{n+1} \to y$, so it must be the case that $f(y) = y$.

(c) Suppose that y and z are distinct fixed points of f. By the mean value theorem, there is some t between y and z such that

$$\frac{f(y) - f(z)}{y - z} = f'(t) \ \Rightarrow\ \frac{y-z}{y-z} = f'(t) \ \Rightarrow\ f'(t) = 1.$$

But this contradicts the assumption that $|f'(x)| \le C < 1$ for all $x \in \mathbb{R}$.

5. (a) Compute the Taylor series $P(x)$ of $f(x) = \log(1+x)$ centered at $x = 0$, and prove that it converges absolutely on $(-1, 1)$.

(b) Prove using Taylor's theorem that $f(x) = P(x)$ on some open neighborhood of 0, by showing that the sequence of nth order Taylor polynomials $P_n(x)$ converges uniformly to $f(x)$. Show that the same is true at $x = 1$, and so $\log(2) = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$.

Solution. (a) We have $f'(0) = \frac{1}{1+x}$, and we claim by induction that

$$f^{(n)}(x) = (-1)^{n-1}(n-1)!(1+x)^{-n}$$

for all $n \ge 1$: if it's true for $n = k$ then we have

$$f^{(k+1)}(x) = (-1)^{k-1}(k-1)! \cdot (-k)(1+x)^{-k-1} = (-1)^k k!(1+x)^{-(k+1)}$$

as desired. Then $f^{(n)}(0) = (-1)^{n-1}(n-1)!$ for $n \geq 1$, and $f(0) = \log(1) = 0$, so $f(x)$ has Taylor series

$$P(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!x^n}{n!} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n},$$

which has the form $\frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$. Absolute convergence follows from the comparison test, since $|\frac{(-1)^{n-1}x^n}{n}| \leq |x^n|$ and $\sum_{n=1}^{\infty} x^n$ is a geometric series which converges absolutely on $(-1, 1)$.

(b) By Taylor's theorem, if $x > -1$ is nonzero then we have

$$f(x) = P_n(x) + \frac{f^{(n+1)}(t)}{(n+1)!}x^{n+1}$$

for some t between 0 and x. The same computation as in part (a) says that $f^{(n+1)}(t) = (-1)^n n!(1+t)^{-(n+1)}$, so

$$|f(x) - P_n(x)| = \left| \frac{(-1)^n n!(1+t)^{-(n+1)}}{(n+1)!}x^{n+1} \right| = \frac{1}{n+1} \left| \frac{x}{1+t} \right|^{n+1}.$$

Now if $0 < x \leq 1$ then we have $0 < t < x$, so $1+t > 1 \geq x$ and hence $\left|\frac{x}{1+t}\right| < 1$. If instead $-\frac{1}{2} \leq x < 0$ then we have $1+t > 1+x \geq \frac{1}{2} > |x|$, so again $\left|\frac{x}{1+t}\right| < 1$. Thus for any nonzero $x \in [-\frac{1}{2}, 1]$ we have

$$|f(x) - P_n(x)| \leq \frac{1}{n+1},$$

and so P_n converges uniformly to $f(x) = \log(1+x)$ on this interval.

Remark: In fact $f(x) = P(x)$ on all of $(-1, 1)$, but we need better control over t to prove this on the interval $(-1, \frac{1}{2})$.

6. Suppose that $f : \mathbb{R} \to \mathbb{R}$ has at least six continuous derivatives, and that $f^{(i)}(0) = 0$ for $i = 1, 2, 3, 4, 5$ but $f^{(6)}(0) = 1$. Prove that $f(x)$ has a local minimum at $x = 0$.

Solution. We apply Taylor's theorem to see that if $x \in (-\delta, \delta)$ is nonzero, then there is some t between 0 and x such that

$$f(x) = \sum_{i=0}^{5} \frac{f^{(i)}(0)x^i}{i!} + \frac{f^{(6)}(t)x^6}{6!} = f(0) + \frac{f^{(6)}(t)x^6}{6!}.$$

Since $f^{(6)}(x)$ is continuous, there is some $\delta > 0$ such that

$$|y - 0| < \delta \;\Rightarrow\; |f^{(6)}(y) - f^{(6)}(0)| < 1,$$

hence $f^{(6)}(y) > 0$ for all $y \in (-\delta, \delta)$. If we take $x \in (-\delta, \delta)$ above then $t \in (-\delta, \delta)$ as well, so $f^{(6)}(t) > 0$, and then since $\frac{x^6}{6!} \geq 0$ we conclude that $f(x) \geq f(0)$ for all $x \in (-\delta, \delta)$.

7. (a) Prove that $f(x) = e^x$ is convex on all of $\mathbb{R}$.
   (b) Let $a, b > 0$. Prove the *arithmetic mean–geometric mean inequality*
   $$\frac{a+b}{2} \geq \sqrt{ab}$$
   by using the convexity of e^x. (Hint: think about $\alpha = \log(a)$ and $\beta = \log(b)$.)
   (c) Prove for any $a, b > 0$ and $s \in [0, 1]$ that $sa + (1-s)b \geq a^s b^{1-s}$.
   (d) Prove *Young's inequality*: for any $x, y \geq 0$ and $p, q > 0$ with $\frac{1}{p} + \frac{1}{q} = 1$,
   $$\frac{x^p}{p} + \frac{y^q}{q} \geq xy.$$

Solution. (a) It suffices to check that that $f''(x) \geq 0$ for all x, and this is certainly true since $f''(x) = e^x$.

(b) Assuming $\alpha < \beta$ without loss of generality, the convexity of e^x implies for $\alpha < \frac{\alpha+\beta}{2} < \beta$ that
$$\frac{e^\alpha + e^\beta}{2} \geq e^{(\alpha+\beta)/2} = \sqrt{e^\alpha \cdot e^\beta}$$
which is equivalent to $\frac{a+b}{2} \geq \sqrt{ab}$.

(c) Since e^x is convex, we know that
$$se^\alpha + (1-s)e^\beta \geq e^{s\alpha+(1-s)\beta},$$
and the left side is $sa + (1-s)b$ while the right side is $(e^\alpha)^s(e^\beta)^{1-s} = a^s b^{1-s}$.

(d) We may assume that $x, y > 0$, since otherwise the inequality reduces to $\frac{x^p}{p} + \frac{y^q}{q} = 0$, which is true. We now use part (c), setting $s = \frac{1}{p}$ (so $1 - s = \frac{1}{q}$) and $(a, b) = (x^p, y^q)$, to get
$$\frac{x^p}{p} + \frac{y^q}{q} \geq (x^p)^{1/p}(y^q)^{1/q} = xy.$$

8. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0. \end{cases}$
   (a) Prove that for all integers $n \geq 0$, there is a polynomial $p_n(x)$ such that
   $$f^{(n)}(x) = \frac{p_n(x)}{x^{3n}} e^{-1/x^2} \text{ for all } x \neq 0.$$
   (b) Prove that $f^{(n)}(0) = 0$ for all n, and hence that $f(x)$ does not equal its Taylor series (centered at $a = 0$) at any nonzero x.
   (c) Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = \begin{cases} 0, & x \leq 0 \\ e^{-1/x^2}, & x > 0. \end{cases}$ Prove that $g^{(n)}(x)$ exists for all $n \geq 0$ and all $x \in \mathbb{R}$, and that $g^{(n)}(0) = 0$ for all n.

(d) Define $h : \mathbb{R} \to \mathbb{R}$ by $h(x) = g(x)g(1-x)$. Prove that $h^{(n)}(x)$ exists for all $n \geq 0$ and all $x \in \mathbb{R}$, and that $h(x) \neq 0$ if and only if $0 < x < 1$.

The function h is called a *bump function*: it is infinitely differentiable, and it is zero outside a compact set (namely $[0,1]$) but also takes positive values.

Solution. (a) When $n = 0$ we take $p_0(x) = 1$. If this holds for $n = k$, we compute

$$\begin{aligned} f^{(k+1)}(x) &= \frac{d}{dx}\left(\frac{p_k(x)}{x^{3k}}e^{-1/x^2}\right) \\ &= \frac{p_k'(x)x^{3k} - 3kx^{3k-1}p_k(x)}{x^{6k}}e^{-1/x^2} + \frac{p_k(x)}{x^{3k}}\left(\frac{2}{x^3}e^{-1/x^2}\right) \\ &= \left(\frac{p_k'(x)x^3 - 3kx^2p_k(x) + 2p_k(x)}{x^{3k+3}}\right)e^{-1/x^2}, \end{aligned}$$

so we can take $p_{k+1} = x^3p_k' - (3kx^2 - 2)p_k$ and the proof follows by induction.

(b) When $n = 0$ it is true by definition. If we have proved it for all $n \leq k$, then for $n = k+1$ we have

$$f^{(k+1)}(0) = \lim_{x\to 0} \frac{f^{(k)}(x) - f^{(k)}(0)}{x - 0} = \lim_{x\to 0} \frac{f^{(k)}(x)}{x},$$

assuming the limit exists, and so using part (a) we wish to prove that

$$\lim_{x\to 0} \frac{p_k(x)}{x^{3k+1}e^{1/x^2}} = 0.$$

Since $p_k(x)$ is continuous, it will suffice to prove that $\lim_{x\to 0} |x^{3k+1}e^{1/x^2}| = \infty$. By the substitution $y = \frac{1}{x}$ we have

$$\lim_{x\to 0} |x^{3k+1}e^{1/x^2}| = \lim_{|y|\to\infty} \left|\frac{e^{y^2}}{y^{3k+1}}\right| = \lim_{y\to\infty} \frac{e^{y^2}}{y^{3k+1}}.$$

But since $y^2 \geq 0$, every term in the power series $e^{y^2} = \sum_{i=0}^{\infty} \frac{(y^2)^i}{i!}$ is nonnegative, and so if we single out the $i = 2k+1$ term then

$$e^{y^2} \geq \frac{(y^2)^{2k+1}}{(2k+1)!} \Rightarrow \frac{e^{y^2}}{y^{3k+1}} \geq \frac{y^{4k+2}/(2k+1)!}{y^{3k+1}} = \frac{y^{k+1}}{(2k+1)!}.$$

The right side certainly goes to ∞ as $y \to \infty$, hence $\lim_{x\to 0} |x^{3k+1}e^{1/x^2}| = \infty$ and this proves that $f^{(k+1)}(0) = 0$. The proof follows for all n by induction.

The Taylor series of f at $a = 0$ is $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = 0$, but clearly for all $x \neq 0$ this is not equal to $f(x) = e^{-1/x^2} > 0$.

(c) For all n, we have

$$g^{(n)}(x) = \begin{cases} 0, & x < 0 \\ f^{(n)}(x), & x > 0. \end{cases},$$

so we only need to check that $g^{(n)}(0)$ exists and is zero for all n. Again, we induct: it is true when $n = 0$, and if it is true for for $n = k$ then

$$\frac{g^{(k)}(x) - g^{(k)}(0)}{x - 0} = \frac{g^{(k)}(x)}{x} = \begin{cases} 0, & x < 0 \\ f^{(k)}(x), & x > 0. \end{cases}$$

Thus $\lim_{x \uparrow 0} \dfrac{g^{(k)}(x) - g^{(k)}(0)}{x - 0} = 0$ by inspection, and

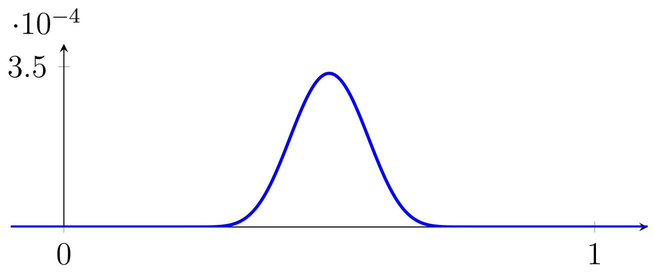
$$\lim_{x \downarrow 0} \frac{g^{(k)}(x) - g^{(k)}(0)}{x - 0} = \lim_{x \downarrow 0} \frac{f^{(k)}(x)}{x} = f^{(k+1)}(0) = 0$$

by part (b), so $g^{(k+1)}(0) = \lim_{x \to 0} \dfrac{g^{(k)}(x) - g^{(k)}(0)}{x - 0}$ exists and is zero as well.

(d) Since $g(x)$ and $g(1-x)$ are infinitely differentiable, repeated application of the product rule says that $h(x) = g(x)g(1-x)$ has n derivatives for all n as well. Moreover, we have $g(x) = 0$ for $x \le 0$ and $g(1-x) = 0$ for $x \ge 1$, so $h(x) = 0$ for all $x \notin (0,1)$; and if $0 < x < 1$ then

$$h(x) = g(x)g(1-x) = e^{-1/x^2} \cdot e^{-1/(1-x)^2} > 0.$$

Here is a graph of $h(x)$:



Note that $h(x)$ is very small on the interval $(0,1)$ – the maximum value is $h(\frac{1}{2}) = e^{-8} \approx 0.000335\ldots$ – and it decays to zero so quickly that it's hard to see from the graph that $h(x) > 0$ for most x on this interval, but $h(x)$ is indeed positive there.

9. Define functions $f_n : \mathbb{R} \to \mathbb{R}$ by $f_n(x) = \sqrt{x^2 + \dfrac{1}{n^2}}$ for all $n \ge 1$.

(a) Prove that f_n is continuously differentiable, and that $|x| \le f_n(x) \le |x| + \frac{1}{n}$.

(b) Prove that (f_n) converges uniformly to a continuous function f.

(c) Prove that (f_n') doesn't converge uniformly on $[-1,1]$, so the theorem from lecture about limits of differentiable functions doesn't apply to tell us that f should be differentiable on $[-1,1]$. (Is f differentiable there?)

Solution. (a) Since $\sqrt{x} = x^{1/2}$ is differentiable at $x^2 + \frac{1}{n^2} > 0$, we can apply the chain rule:

$$f_n'(x) = \frac{d}{dx}\left(\left(x^2 + \frac{1}{n^2}\right)^{1/2}\right) = \frac{1}{2}\left(x^2 + \frac{1}{n^2}\right)^{-1/2}(2x) = \frac{x}{\sqrt{x^2 + \frac{1}{n^2}}}.$$

We also use the fact that $\sqrt{x}$ is monotone increasing to check that

$$|x| = \sqrt{|x|^2} < \sqrt{|x|^2 + \frac{1}{n^2}} \leq \sqrt{|x|^2 + \frac{2|x|}{n} + \frac{1}{n^2}} = |x| + \frac{1}{n}.$$

The middle term is $f_n(x)$ since $|x|^2 = x^2$, so we have $|x| < f_n(x) \leq |x| + \frac{1}{n}$.

(b) The continuous function is $f(x) = |x|$, since for any $\epsilon > 0$, part (a) says that

$$n > \frac{1}{\epsilon} \implies |f_n(x) - f(x)| \leq \frac{1}{n} < \epsilon$$

for all $x \in \mathbb{R}$.

(c) If f_n' converges uniformly on $[-1,1]$, then its pointwise limit should be continuous. But a quick computation shows that

$$\lim_{n\to\infty} f_n'(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

is discontinuous at $x = 0$, which is exactly where $f(x) = |x|$ is not differentiable.

10. In an upcoming lecture, we'll need to know that $\lim_{x\to\infty} xs^{x-1} = 0$ for all $s \in (0,1)$.

(a) Prove that for all $c > 0$, there exists $N > 0$ such that $\log(x) < cx$ for all $x \geq N$.

(b) Prove for $s \in (0,1)$ that $\lim_{x\to\infty} xs^x = 0$, and that this implies the above claim.

Solution. (a) It's enough to prove that $\lim_{x\to\infty} \frac{\log(x)}{x} = 0$, since then there's an $N > 0$ such that $0 < \frac{\log(x)}{x} < c$ for all $x \geq N$. This limit exists by l'Hôpital's rule, which says that it is equal to $\lim_{x\to\infty} \frac{1/x}{1} = 0$.

(b) For any $c > 0$, part (a) says that $0 < xs^x < e^{cx}s^x = (e^c s)^x$ for all large enough x. Since $0 < s < 1$, we can choose a positive $c < \log(1/s)$ so that $0 < e^c s < 1$, and then

$$\lim_{x\to\infty} (e^c s)^x = 0.$$

Thus the squeeze theorem says that $\lim_{x\to\infty} xs^x = 0$ as well. We conclude that

$$\lim_{x\to\infty} xs^{x-1} = \frac{1}{s}\left(\lim_{x\to\infty} xs^x\right) = 0.$$