Math40003 Linear Algebra and Groups
Term 2 Unseen 4A, Linear Algebra (Week 7)

In this exercise, we generalize the definition of the inner product you saw in the lectures. To avoid confusion we refer to this new definition as a "general inner product"

Definition 1. Let V be an $\mathbb{R}$-vector space. A *general inner product* on V is a binary function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ such that the following properties hold for all $v, u, w \in V$, $\alpha \in \mathbb{R}$:

(i) $\langle v, u \rangle = \langle u, v \rangle$.

(ii) $\langle \alpha v + u, w \rangle = \alpha \langle v, w \rangle + \langle u, w \rangle$.

(iii) $\langle v, v \rangle \geq 0$.

(iv) if $\langle v, v \rangle = 0$, then $v = 0$.

Similarly, we define a general norm on V: $\|v\| := \sqrt{\langle v, v \rangle}$.

1. For each of the following, determine whether $\langle \cdot, \cdot \rangle$ is a general inner product:
   (a) Let $V = M_{n \times m}(\mathbb{R})$ and let $\langle A, B \rangle := \mathrm{trace}(AB^\top)$.
   (b) Let V be the space of all continuous functions on the interval $[a, b]$, with point-wise addition and scalar multiplication. Let $\langle g, f \rangle := \int_a^b f(x)g(x)dx$.
   (c) Let V be the set of random variables on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$. Let $\langle X, Y \rangle := \mathbb{E}[X \cdot Y]$.

2. Prove that every n-dimensional inner product space is isomorphic to $\mathbb{R}^n$ with the inner product from class. Namely: Let $\langle (a_1, \ldots, a_n), (b_1, \ldots, b_n) \rangle_s := \sum_{i=1}^n a_i b_i$ (we call this the *standard inner product*, a.k.a the dot product). Let V be an $\mathbb{R}$-vector space with general inner product $\langle \cdot, \cdot \rangle$. Prove that there is an invertible linear transformation $T : \mathbb{R}^n \to B$ such that $\forall v, u \in \mathbb{R}^n : \langle v, u \rangle_s = \langle T(v), T(u) \rangle$.

3. Prove the following for general inner products: Let V be an $\mathbb{R}$-vector space and let $\langle \cdot, \cdot \rangle$ be a general inner product on V. Prove:
   (a) Cauchy-Schwarz inequality:
      i. $\forall v, u \in V : \langle v, u \rangle \leq \|v\| \cdot \|u\|$
      ii. $\forall v, u \in V : \langle v, u \rangle = \|v\| \cdot \|u\| \iff \{ v, u \}$ are linearly dependent.
   (b) The triangle inequality: $\forall v, u \in V : \|v + w\| \leq \|v\| + \|w\|$.
   (c) The Pythagorean theorem:
   $\forall v, u \in V : \langle v, u \rangle = 0 \iff \|v\|^2 + \|u\|^2 = \|u + v\|^2$.
   (d) The Parallelogram law: $\forall v, u \in V : \|v + u\|^2 + \|v - u\|^2 = 2\|v\|^2 + 2\|u\|^2$.