Math40003 Linear Algebra and Groups Problem Sheet 1

1. Find all solutions of the following systems of linear equations.

(a)
$$\begin{aligned} x_1 - 2x_2 + x_3 - x_4 &= 8 \\ 3x_1 - 6x_2 + 2x_3 &= 18 \\ x_3 - 2x_4 &= 5 \\ 2x_1 - 2x_2 + 3x_4 &= 4 \end{aligned}$$

(b)
$$\begin{aligned} x_1 - 3x_2 + x_3 &= 2 \\ 3x_1 - 8x_2 + 2x_3 &= 5 \\ 2x_1 - 5x_2 + x_3 &= 1 \end{aligned}$$

(c)
$$\begin{aligned} x_1 - 2x_3 + x_4 &= 0 \\ 2x_1 - x_2 + x_3 - 3x_4 &= 0 \\ 4x_1 - 3x_2 - x_3 - 7x_4 &= 4 \end{aligned}$$

(d)
$$\begin{aligned} -x_2 + x_3 - 3x_4 &= 0 \\ x_1 + 3x_2 + x_3 - x_4 &= 0 \\ 2x_1 + 5x_2 + 3x_3 - 5x_4 &= 0 \end{aligned}$$

In each case write down all solutions with $x_2 = 5$.

(a) Performing row operations on the augmented matrix gives

$$\left(\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 8 \\ 3 & -6 & 2 & 0 & 18 \\ 0 & 0 & 1 & -2 & 5 \\ 2 & -2 & 0 & 3 & 4 \end{array}\right) \xrightarrow[R_4 \mapsto R_4 - 2R_1]{R_2 \mapsto R_2 - 3R_1} \left(\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 8 \\ 0 & 0 & -1 & 3 & -6 \\ 0 & 0 & 1 & -2 & 5 \\ 0 & 2 & -2 & 5 & -12 \end{array}\right) \xrightarrow[R_4 \mapsto R_4 + R_3]{R_2 \leftrightarrow R_4} \left(\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 8 \\ 0 & 2 & -2 & 5 & -12 \\ 0 & 0 & 1 & -2 & 5 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right)$$

From the bottom up we read off $x_4 = -1,\ x_3 = 5 + 2x_4 = 3,\ x_2 = \frac{1}{2}(-12 + 2x_3 - 5x_4) = -\frac{1}{2},\ x_1 = 8 + 2x_2 - x_3 + x_4 = 3$**. So there is a unique solution** $(x_1, \ldots, x_4) = (3, -\frac{1}{2}, 3, -1)$**. For (b) we have**

$$\left(\begin{array}{ccc|c} 1 & -3 & 1 & 2 \\ 3 & -8 & 2 & 5 \\ 2 & -5 & 1 & 1 \end{array}\right) \xrightarrow[R_3 \mapsto R_3 - 2R_1]{R_2 \mapsto R_2 - 3R_1} \left(\begin{array}{ccc|c} 1 & -3 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & -1 & -3 \end{array}\right) \xrightarrow{R_3 \mapsto R_3 - R_2} \left(\begin{array}{ccc|c} 1 & -3 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & -2 \end{array}\right)$$

giving the contradiction $0x_3 = -2$**, so there are no solutions. Next (c):**

$$\left(\begin{array}{cccc|c} 1 & 0 & -2 & 1 & 0 \\ 2 & -1 & 1 & -3 & 0 \\ 4 & -3 & -1 & -7 & 4 \end{array}\right) \xrightarrow[R_3 \mapsto R_3 - 4R_1]{R_2 \mapsto R_2 - 2R_1} \left(\begin{array}{cccc|c} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & 5 & -5 & 0 \\ 0 & -3 & 7 & -11 & 4 \end{array}\right) \xrightarrow{R_3 \mapsto R_3 - 3R_2} \left(\begin{array}{cccc|c} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & 5 & -5 & 0 \\ 0 & 0 & -8 & 4 & 4 \end{array}\right)$$

The last row gives $-2x_3 + x_4 = 1$ so if we set $x_3 = a$ then $x_4 = 1 + 2a$, then from the second row $x_2 = 5x_3 - 5x_4 = -5a - 5$ and from the first, $x_1 = 2x_3 - x_4 = -1$. So $(x_1, \dots, x_4) = (-1, -5a - 5, a, 1 + 2a)$ is the general solution, where $a \in \mathbb{R}$. Finally (d):

$$\left(\begin{array}{cccc|c} 0 & -1 & 1 & -3 & 0 \\ 1 & 3 & 1 & -1 & 0 \\ 2 & 5 & 3 & -5 & 0 \end{array}\right) \xrightarrow[R_3 \mapsto R_3 - 2R_1' - R_2']{R_1 \leftrightarrow R_2} \left(\begin{array}{cccc|c} 1 & 3 & 1 & -1 & 0 \\ 0 & -1 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The second row gives $x_2 = x_3 - 3x_4$ so if we set $x_3 = a$, $x_4 = b$ then $x_2 = a - 3b$ and $x_1 = -3x_2 - x_3 + x_4 = -4a + 10b$ so the general solution has $(x_1, \dots, x_4) = (10b - 4a, a - 3b, a, b)$ for $a, b \in \mathbb{R}$.

Thus we see there are no solutions of (a) or (b) with $x_2 = 5$, while in (c) they exist only for $a = -2$, i.e. the only solution is $(x_1, \dots, x_4) = (-1, 5, -2, -3)$. For (d) solutions have $a = 3b + 5$ so they are $(x_1, \dots, x_4) = (-2b - 20, 5, 3b + 5, b)$ for any $b \in \mathbb{R}$.

2. Which of these matrices A is invertible (and for which a)?

$$\begin{pmatrix} 6 & 7 \\ 8 & 9 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & -1 \\ 2 & -3 & 2 \\ -1 & 12 & -7 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 \\ a & 1 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

Calculate A^{-1} when it exists. **In order they are: invertible, non-invertible, and invertible iff $a \neq \frac{1}{2}$.**

You can check your inverses multiply the original matrix to give I. For the last example, when $a \neq \frac{1}{2}$, $A^{-1} = \frac{1}{1-2a} \begin{pmatrix} 1 & -2 & -1 \\ -2a & 2 & 1 \\ -a & 1 & 1 - a \end{pmatrix}$.

3. * How can you use Gaussian elimination to solve $\begin{pmatrix} 2 & 4 & 1 \\ 2 & 6 & 1 \\ 3 & 9 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$?

Find *all* solutions (x_1, x_2, x_3).

Notice that $Ax = x \iff Ax - x = 0 \iff (A - I)x = 0$ so apply Gaussian elimination to $A - I$ instead of A! Get

$$\begin{pmatrix} 1 & 4 & 1 \\ 2 & 5 & 1 \\ 3 & 9 & 2 \end{pmatrix} \xrightarrow[R_3 \mapsto R_3 - 3R_1]{R_2 \mapsto R_2 - 2R_1} \begin{pmatrix} 1 & 4 & 1 \\ 0 & -3 & -1 \\ 0 & -3 & -1 \end{pmatrix} \xrightarrow{R_3 \mapsto R_2 - R_3} \begin{pmatrix} 1 & 4 & 1 \\ 0 & -3 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

so bottom row gives $x_3 = a$ is arbitrary, second row gives $-3x_2 - x_3 = 0$, i.e. $x_2 = -a/3$, then top row gives $x_1 + 4x_2 + x_3 = 0$ so $x_1 = a/3$. Set $b = a/3$. Then solutions are $(b, -b, 3b)$ for any $b \in \mathbb{R}$.

4. Let A be an $n \times m$ matrix, and $b \in \mathbb{R}^n$. Suppose $Ax = b$ has at least one solution $x_0 \in \mathbb{R}^m$. Show all solutions are of the form $x = x_0 + h$, where h solves $Ah = 0$.

If $x = x_0 + h$ where $Ah = 0$ then $Ax = Ax_0 + Ah = b + 0 = b$ so it solves the equation.

It is important not to forget to prove the converse: If $Ax = b$ then $x = x_0 + h$ where $h := x - x_0$. But $Ah = Ax - Ax_0 = b - b = 0$, as required.

5. Let A and B be square $n \times n$ matrices with real entries. For each of the following statements, either give a **proof**, or find a **counterexample with** $n = 2$.

   (i) If $AB = 0$ then A and B cannot both be invertible. **True: assume A, B both invertible and $AB = 0$. Then $0 = A^{-1}(AB) = (A^{-1}A)B = B$, contradiction.**

   (ii) If A and B are invertible then $A+B$ is invertible. **False, eg $A = I, B = -I$.**

   (iii) If A and B are invertible then AB is invertible. **We show $B^{-1}A^{-1}$ is the inverse of AB by using associativity to show that their products either way round is I:**

   $$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I,$$
   $$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = I.$$

   (iv) If A and B are invertible and $(AB)^2 = A^2B^2$, then $AB = BA$. **True: $(AB)^2 = A^2B^2 \Rightarrow ABAB = AABB \Rightarrow A^{-1}(ABAB)B^{-1} = A^{-1}(AABB)B^{-1}$ $\Rightarrow BA = AB$.**

   (v) If $ABA = 0$ and B is invertible then $A^2 = 0$. **False, eg $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.**

   (vi) If $ABA = I$ then A is invertible and $B = (A^{-1})^2$. **True: we showed in lectures that if $AE = I$ then also $EA = I$ so E is the inverse of A. Since $ABA = I$, this means BA is the inverse of A, so A is invertible. Then $BA = A^{-1} \implies B = A^{-2}$.**

   (vii) If A has a left inverse B and a right inverse C then $B = C$. **Just compute that $BA = I \implies BAC = IC = C \implies BI = C \implies B = C$.**

6. Let $n \geq 2$ and let $A_n = (a_{ij})$ be the $n \times n$ matrix such that

   $$a_{i-1,i} = 1 \text{ for } i = 2, \ldots n,$$
   $$a_{i+1,i} = 1 \text{ for } i = 1, \ldots n-1,$$

   and $a_{ij} = 0$ for all other i, j. Write down A_2, A_3 and A_4. Prove that A_n is invertible for all even values of n, and is not invertible for all odd values of n. Find A_2^{-1} and A_4^{-1}.

   For n even you can reduce A_n to the identity by subtracting row n from row $n-2$, then row $n-2$ from row $n-4$ and so on, and then swapping rows; hence A_n is invertible. For n odd you can reduce to a matrix with a bottom row of zeros by subtracting row 1 from row 3, then row 3 from row 5 and so on; hence A_n is not invertible.

7.* For which $a, b \in \mathbb{R}$ does the system of equations

$$\begin{aligned} x_1 + x_2 + x_3 &= -1 \\ 2x_1 + x_2 + ax_3 &= 1 \\ 3x_1 + x_2 + x_3 &= b \end{aligned}$$

have (i) no solutions, (ii) exactly one solution, (iii) infinitely many solutions?

Apply the row operations

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 2 & 1 & a & 1 \\ 3 & 1 & 1 & b \end{array}\right) \xrightarrow[R_3 \mapsto R_3 - 3R_1 - 2R_2']{R_2 \mapsto R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & -1 & a-2 & 3 \\ 0 & 0 & 2-2a & b-3 \end{array}\right)$$

The final row gives $(2-2a)x_3 = b-3$, so there are no solutions if $a = 1, b \neq 3$. If $a = 1, b = 3$ then final row gives no information, but we have $x_2 = (a-2)x_3 - 3$ and $x_1 = -1 - x_2 - x_3$ from the second and first rows. Substituting $x_3 = c$ for any $c \in \mathbb{R}$ we find infinitely many solutions.

When $a \neq 1$ then $x_3 = \frac{b-3}{2-2a}$ is uniquely determined, as is x_2 from the second row and then x_1 from the first, so there is a unique solution.

What about the system

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 0 \\ x_1 - x_2 + ax_3 + x_4 &= 1 \\ 2x_1 + ax_2 + x_3 + 2x_4 &= b\ ? \end{aligned}$$

You have to be more careful with this one:

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & -1 & a & 1 & 1 \\ 2 & a & 1 & 2 & b \end{array}\right) \xrightarrow[R_3 \mapsto R_3 - 2R_1]{R_2 \mapsto R_2 - R_1} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & -2 & a-1 & 0 & 1 \\ 0 & a-2 & -1 & 0 & b \end{array}\right)$$

(If $a = 2$ this is already in echelon form and the last row – and then back substitution – shows there are infinitely many solutions. But actually don't need to make this a special case, as it is also included in the general case below.)

$R_3 \mapsto 2R_3 + (a-2)R_2$ **gives**

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & -2 & a-1 & 0 & 1 \\ 0 & 0 & a^2 - 3a & 0 & 2b + a - 2 \end{array}\right)$$

If $a \notin \{0, 3\}$ then from the last row we see we get infinitely many solutions.

If $a = 0$ the last row shows we get 0 solutions if $b \neq 1$ and infinitely many otherwise.

If $a = 3$ the last row shows we get 0 solutions if $b = -\frac{1}{2}$ and infinitely many otherwise.

8. Which of the following are possible, find examples if possible:

   (a) Two simultaneous equations in two unknowns which defines a line in $\mathbb{R}^2$.
   (b) Two simultaneous equations in two unknowns which defines the empty set in $\mathbb{R}^2$.
   (c) One equation in no unknowns which defines the empty set.
   (d) Two simultaneous equations in three unknowns which defines a point in $\mathbb{R}^3$.

(a)

$$\begin{aligned} x + y &= 1 \\ 2x + 2y &= 2 \end{aligned}$$

(b)

$$\begin{aligned} x + y &= 1 \\ x + y &= 2 \end{aligned}$$

(c) $0 = 1$

(d) **Not possible.**