Problem Sheet 0

Scalar and vector products

1. If $\mathbf{A} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ and $\mathbf{B} = 4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$. find

$$\text{(i) } \mathbf{A}\cdot\mathbf{B}; \quad \text{(ii) } |\mathbf{A}|\,; \quad \text{(iii) } \mathbf{A}\times\mathbf{B}; \quad \text{(iv) } (2\mathbf{A}+\mathbf{B})\cdot(\mathbf{A}-2\mathbf{B}).$$

2. Show that $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$ is the volume of a parallelepiped with sides $\mathbf{A},\mathbf{B},\mathbf{C}$, where $\mathbf{A},\mathbf{B},\mathbf{C}$ form a right-handed system. What is the answer for the volume if the system is not right-handed?

Partial differentiation

3. Let $x = u^3 + uv + v^3,\ y = u^2 - v^2$.

(i) Calculate the mixed second derivatives of x and y with respect to u and v and show that they commute;

(ii) Calculate the first order partial derivatives of u and v with respect to x and y at $(u, v) = (1, 0)$.

4. Suppose $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $r = |\mathbf{r}|$ and let $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$. Calculate for $r \neq 0$:

$$\text{(i) } \nabla^2(r); \quad \text{(ii) } \nabla^2(1/r).$$

Integration

5. Evaluate the following integrals

$$\text{(i)}\int_0^{2\pi}\sin^2\theta\ d\theta; \quad \text{(ii)}\int_0^{2\pi}\cos^2\theta\ d\theta; \quad \text{(iii)}\int_0^{\pi}\cos^2\theta\sin\theta\ d\theta; \quad \text{(iv)}\int_0^{2\pi}\cos^4\theta\ d\theta; \quad \text{(v)}\int_0^{2\pi}\sin^4\theta\ d\theta.$$

6. Calculate

$$\int_R (x^2 - 2y^2)\ dx\ dy$$

where R is the circular disc $x^2 + y^2 \leq a^2$, by converting to polar coordinates.

Differential equations

7. Solve the following ordinary differential equations

$$\text{(i) } y' - \frac{3y}{x+1} = (x+1)^4; \quad \text{(ii) } y'' + y = \cos 2x; \quad \text{(iii) } y'' + y = \cos x.$$

8. Use the substitution $x = e^t$ to solve

$$x^2y'' + 2xy' - 2y = 1/x.$$

9. Find the values of λ for which there exists a non-trivial solution to the boundary-value problem

$$y'' + (1-\lambda)y = 0, \quad y(0) = y(\pi) = 0.$$

Sheet 0 Answers

1. (i) -10; (ii) $\sqrt{14}$; (iii) $8\mathbf{i} - 12\mathbf{j} - 14\mathbf{k}$; (iv) -14.
2. $|\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})|$.
3. (i) $\partial^2x/\partial u\partial v = \partial^2x/\partial v\partial u = 1; \partial^2y/\partial u\partial v = \partial^2y/\partial v\partial u = 0$;
(ii) $\partial u/\partial x = 0, \partial v/\partial x = 1, \partial u/\partial y = 1/2, \partial v/\partial y = -3/2$.
4. (i) $2/r$; (ii) zero.
5. (i) π; (ii) π; (iii) $2/3$; (iv) $3\pi/4$; (v) $3\pi/4$.
6. $-\pi a^4/4$.
7. (i) $y = (x+1)^3(x^2/2 + x + C)$; (ii) $y = A\cos x + B\sin x - (1/3)\cos 2x$;
(iii) $y = A\cos x + B\sin x + (1/2)x\sin x$.
8. $y = A/x^2 + Bx - 1/(2x)$.
9. $\lambda = 1 - n^2$ where $n = 1, 2, \ldots$.