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The valuation of uncertain income streams and the pricing of options

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A simple formula is developed for the valuation of uncertain income streams consistent with rational risk averse investor behavior and equilibrium in financial markets. Applying this formula to the pricing of an option as a function of its associated stock, the Black-Scholes formula is derived even though investors can only trade at discrete points in time.

■ Since the theoretical breakthrough leading to a simple single-period valuation formula for uncertain income consistent with rational risk averse investor behavior and equilibrium in financial markets (Sharpe, 1964), there have been various attempts to extend this model to a multiperiod context. Fama (1970) proved that even though a risk averter maximized the expected utility from the stream of consumption over his lifetime, his choices in each period would be indistinguishable from that of a properly specified rational risk averse investor with a single-period horizon. Moreover, if his investment (and consumption) opportunities followed a (possibly nonstationary) random walk, then the induced single-period utility would be a nonstochastic function. These observations implied that the simple single-period model could apply to successive periods in a multiperiod setting. Subsequently, Merton (1973b) and later Long (1974) generalized the single-period formula to intertemporally stochastically dependent opportunity sets. While their resulting formulae were more complex, they still possessed the merit of empirical promise.

However, of greater interest than the anchoring of a single-period valuation formula in a multiperiod setting, is development of a simple multiperiod formula, that is, a method of determining the present value of a series of cash flows received over many future dates. The early certainty-equivalent and risk-adjusted discount rate approaches (Robichek and Myers, 1965) were flawed by failing to specify determinants of the adjustment parameters. In effect, they remained little more than definitions of these parameters. More recently, implicitly making use of Fama's result that the simple single-period model could be applied successively, Bogue and Roll (1974) have indeed derived a method for discounting an uncertain income stream consistent with rational risk averse investor behavior. In the spirit of dynamic pro-

1. Introduction

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gramming, they apply the single-period mean-variance model to the last period determining the date $T - 1$ present value of income received at date T . Knowing the determinants of this uncertain present value and the additional uncertain income received at date $T - 1$, the single-period model is again applied determining the date $T - 2$ present value, etc. By this means, the date zero present value of the income stream is determined. Unfortunately, the resulting formula, even for convenient special cases, is far from simple. The comfortable elegance of the formula for discounting a certain income stream does not seem to carry over to uncertainty.

This is all the more discouraging considering the obvious importance of valuing uncertain income streams for research in financial markets. The primary purpose of this paper is to develop the needed valuation formula that satisfies the tests of uncertain income received over time, consistency with rational risk averse investor behavior and equilibrium in financial markets, as well as simplicity and statement in terms of empirically observable inputs. Section 2 presents a general approach for valuation of uncertain income streams, but the formula contains inputs which are not easily observable. Section 3 remedies this defect for the special case of constant proportional risk aversion (CPRA). With a simple valuation formula in hand, in Section 4 it is applied to the multiperiod problem of the valuation of an option in terms of its associated stock. To my surprise, the resulting option pricing formula is *identical* to the Black-Scholes (1973) formula even though only costless *discrete-time* trading opportunities are available so that investors cannot create a perfect hedge, investors are risk averse, and must simultaneously choose among a large number of securities.

2. Uncertain income streams: general case

■ Assume there exist states of nature so that given the revelation of the state at any date, the cash flow (i.e., dividend) received by the owner of any security is known with certainty. Let $t = 0, 1, 2, \dots$ denote dates, $s(t)$ (random variable) denote the state at date t , $X[s(t)]$ the dividend received from a security in state $s(t)$, and $P[s(t)]$ the price of the security in state $s(t)$.

The requirement that securities with identical returns should have the same value is probably the most basic condition for equilibrium in financial markets. In particular, assume

(1) *Single-price law of markets*: If two securities, or more generally two portfolios of securities,¹ yield the same dividends for every future state, then their current prices are the same.

Since there then can be no more linearly independent securities than states, there must exist a set of random variables $\{Z[s(t)]\}$, the same for all securities, such that for any security

$$P_0 = \sum_{t=1}^{\infty} \sum_{s(t)} Z[s(t)] X[s(t)].$$

¹ Extension of the single-price law of markets to portfolios of securities (i.e. convex combinations of dividends) is inconsistent with transactions costs which vary with the scale of investment in any security. However, it insures that any pricing formula for arbitrary securities will possess the "portfolio property" that it applies to arbitrary portfolios as well.

Of course, the random variables will not be unique unless the number of linearly independent securities equals the number of states. A normalized riskless security for any date t has a certain dividend $X[s(t)] = 1$ at date t and a certain dividend of zero at any other date. Therefore, denoting R_{Ft}^{-1} as the current price of a date t normalized riskless security, then $R_{Ft}^{-1} = \sum_{s(t)} Z[s(t)]$. If, additionally, we assume

(2) *Nonsatiation: Ceteris paribus*, the larger its dividends for any state, the greater the current price of a security.

then $Z[s(t)] > 0$ for all states.²

These two assumptions motivate the following theorem:

Theorem 1: Given assumptions 1 and 2, there exists a positive random variable $Y[s(t)]$, the same for all securities, such that for any security³

$$P_0 = \sum_{t=1}^{\infty} \frac{E(X_t) + \text{Cov}(X_t, Y_t)/E(Y_t)}{R_{Ft}}. \quad (1)$$

Proof: Let $\pi[s(t)]$ be a probability,⁴ assessed at date $t = 0$, that state $s(t)$ will occur. Define random variable $Z'[s(t)] \equiv Z[s(t)]/\pi[s(t)]$; then $P_0 = \sum_t E(X_t Z'_t)$. Second, define random variable $Y[s(t)] \equiv E(Y_t) R_{Ft} Z'[s(t)]$; then $P_0 = \sum_t E(X_t Y_t) / [R_{Ft} E(Y_t)]$. The result follows from the definition of covariance. Q.E.D.

The valuation formula implies that the current price of a security (or portfolio of securities) equals the sum of the discounted certainty equivalents of its dividends discounted at the riskless rate. The risk adjustment factor applied to the mean dividend at each date is its ‘‘coefficient of covariation’’ with a random variable common to all securities. Not only will this random variable \tilde{Y}_t not be unique if the number of states exceeds the number of different securities, but even if the number of states and different securities are equal, Y_t will be unique only up to a positive multiplicative constant.⁵

However, until more is said about \tilde{Y}_t other than its sign, the theory has little empirical content. While we shall shortly provide a simple characterization of \tilde{Y}_t , first we develop the natural uncertainty analogue to the Williams-Gordon (1938, pp. 87–89) perpetual dividend growth model. Let one plus the rate of growth of \tilde{X}_t be denoted by \tilde{g}_t and one plus the rate of growth of \tilde{Y}_t be denoted by \tilde{y}_t so that $\tilde{X}_t = X_0(\tilde{g}_1 \tilde{g}_2 \dots \tilde{g}_t)$ and $\tilde{Y}_t = Y_0(\tilde{y}_1 \tilde{y}_2 \dots \tilde{y}_t)$. Letting $R_{Ft} = r_{F1} r_{F2} \dots r_{Ft}$, equation (1) becomes

$$P_0 = X_0 \left\{ \sum_{t=1}^{\infty} \frac{E[g_1 g_2 \dots g_t y_1 y_2 \dots y_t] / E[y_1 y_2 \dots y_t]}{r_{F1} r_{F2} \dots r_{Ft}} \right\}.$$

Analogous to the stationary riskless rate and stationary dividend

² If it were also assumed that a security existed insuring a non-negative rate of return to which all investors had access (i.e., cash), then $0 < Z[s(t)] < 1$ and indeed $\sum_{s(t)} Z[s(t)] \leq 1$ for all dates. Moreover, $R_1 \leq R_2 \leq R_3 \dots$

³ $E(\cdot)$ denotes expectation, $\text{Cov}(\cdot)$ covariance, $\text{Var}(\cdot)$ variance, and $\text{Std}(\cdot)$ standard deviation.

⁴ These probabilities need not be held homogeneously by all investors, or indeed by any investor. They may be viewed as a purely mathematical construct. However, every state must be interpreted as possible so that $\pi[s(t)] > 0$.

⁵ Beja (1971) has developed similar results to equation (1).

growth rate assumptions adopted by Williams and Gordon under certainty, we assume that $r_{Ft} \equiv r_F$ is stationary, that $\tilde{g}_t \equiv \tilde{g}$ follows a stationary random walk with a stationary correlation with \tilde{y}_t , and that $\tilde{y}_t \equiv \tilde{y}$ follows a stationary random walk.⁶

Theorem 2: Given assumptions 1 and 2, if the riskless rate is stationary and the rate of growth of a security's dividend stream follows a stationary random walk with stationary correlation with the rate of growth of \tilde{Y}_t , which also follows a stationary random walk, then

$$P_0 = X_0 \left[\frac{\mu_g + \sigma_{gy}/\mu_y}{r_F - (\mu_g + \sigma_{gy}/\mu_y)} \right].$$

Proof: From the premise, \tilde{g}_t is serially uncorrelated, \tilde{y}_t is serially uncorrelated, and lagged values of \tilde{g}_t and \tilde{y}_t are uncorrelated. Therefore, $E[g_1 g_2 \dots g_t y_1 y_2 \dots y_t] = E(g_1 y_1) E(g_2 y_2) \dots E(g_t y_t)$; moreover $E[y_1 y_2 \dots y_t] = E(y_1) E(y_2) \dots E(y_t)$. Applying the stationary assumptions,

$$P_0 = X_0 \sum_{t=1}^{\infty} \left(\frac{\mu_{gy}/\mu_y}{r_F} \right)^t.$$

The result then follows from the formula for the sum of an infinite geometric series and the definition of covariance. Q.E.D.

Observe that, excepting the stationarity of the stochastic process of the dividend stream, the formula is quite general relying primarily on the single-price law of markets. If a security has no "nondiversifiable risk" so that $\sigma_{gy} = 0$, then the formula simplifies to $P_0 = X_0 \mu_g / (r_F - \mu_g)$. However, again since \tilde{y} is not well-specified, the formula appears empty of empirical content. But this is not quite the case. Let the expression in brackets be denoted ϕ so that $P_0 = X_0 \phi$. At date $t = 1$, $\tilde{P}_1 = \tilde{X}_1 \phi$. Moreover, since $\tilde{X}_1 = X_0 \tilde{g}_1$, then the rate of return of the security $(\tilde{X}_1 + \tilde{P}_1)/P_0 = \tilde{g}_1 [1 + \phi]/\phi$ follows a stationary random walk. In short, we have derived important properties of the stochastic process of the rate of return of a security from the stochastic process of its dividend stream.⁷

Corollary: Under the conditions of Theorem 2, the rate of return of a security follows a stationary random walk.⁸

Define $\tilde{X}_1/(\tilde{X}_1 + \tilde{P}_1 - P_0)$ as the date $t = 1$ "payout ratio" from market-determined earnings. Since this equals $[1 + \phi - (\phi/g)]^{-1}$, the payout ratio also follows a stationary random walk, varying *inversely* with the growth rate of dividends. That is, although the absolute level of dividends increases, it must be accompanied by a lower payout to

⁶ By definition, $\mu_g \equiv E(g)$, $\mu_y \equiv E(y)$, $\sigma_{gy} \equiv \text{Cov}(g, y)$, $\mu_{gy} \equiv E(gy)$.

⁷ From much of the literature on "efficient markets," it may have been expected that security rates of return should follow a random walk irrespective of the stochastic process of dividends. For example, see Granger and Morgenstern (1970, p. 26). Clearly, this is not true for default-free pure-discount bonds. Moreover, I have argued elsewhere (1974b) that in an important case, the rate of return of the market portfolio will be serially independent if and only if the rate of growth of aggregate consumption (i.e., the social dividend) is serially independent. This result is generalized in Section 3.

⁸ Samuelson (1973) has derived a more general proposition, which unlike the simple case here, does not require stationary discount rates over time. However, like the present case, it does require that discount rates be nonstochastic functions of time; that is, future interest rates are assumed to be known in advance. I have shown elsewhere (1974a) that this may lead to unrealistic implications.

assure continuation of the stationary dividend process. This harmonizes with Lintner's (1956) empirical observation that in order to maintain a target payout ratio (i.e., $[1 + \phi - (\phi/\mu_g)]^{-1}$), firms will under (over) shoot it in times of unusually high (low) earnings.

Corollary: Under the conditions of Theorem 2, the payout ratio varies inversely with the growth rate of dividends.

Returning now to general dividend processes, we shall identify the *positive random variable* \tilde{Y}_t by *enriching the economic environment*. Specifically, we shall assume

(3) *Perfect, competitive and Pareto-efficient financial markets:* There are no scale economies in the purchase of any security, every participant acts as if he cannot influence the price of any security, all investors can purchase the same security at the same date for the same price, and there exist a sufficient diversity of securities such that no investor desires the costless creation of a new security.⁹

(4) *Rational time-additive tastes:* Every investor acts as if he maximizes his expected utility over his lifetime dollar value of consumption which is concave and additive in consumption at each date.

(5) *Weak aggregation:*¹⁰ There exists an average investor such that

- (i) *homogeneity:* every homogeneous economic characteristic also characterizes the average investor
- (ii) *commensurability:* if an economic characteristic is denominated in units of wealth, then this characteristic for the average investor is an unweighted arithmetic average over the corresponding characteristic of every investor
- (iii) *consensus:* prices are determined as if every investor were average.

Under these assumptions, we can meaningfully define an average investor who

$$\max_{\{C[s(t)]\}} \sum_t \sum_{s(t)} \pi[s(t)] U_t\{C[s(t)]\}$$

subject to $W_0 = \sum_t \sum_{s(t)} Z[s(t)] C[s(t)]$,

where U_t denotes his date t utility over consumption $C[s(t)]$ and $W_0 \equiv \sum_t \sum_{s(t)} Z[s(t)] \bar{C}[s(t)]$ is his initial wealth defined in terms of endowed claims to consumption $\bar{C}[s(t)]$. With these assumptions, we can identify \tilde{Y}_t .

Theorem 3: Given assumptions 1–5, $\tilde{Y}_t = U'_t(\tilde{C}_t)$ for all dates and states.

Since this is a classic and easily obtainable result from the first-order

⁹ This last condition would be satisfied if the financial market were complete, or if, as in the familiar mean-variance model, a riskless security exists and all investors divide their wealth after consumption between it and the market portfolio. The condition is required for Pareto-efficient exchange arrangements.

¹⁰ If all investors were identical this assumption is trivially satisfied. I have developed elsewhere (1974a, 1974b) more general sets of conditions. Perhaps the most appealing is the case where investors are heterogeneous with respect to the scale and composition of endowed resources, lifetime, time-preference, beliefs, and whether proportional risk aversion is increasing, constant, or decreasing. Additive generalized logarithmic utility is the only homogeneity requirement.

conditions to the programming problem, no proof will be given. The economic content of this theorem is that (1) the randomness of \tilde{Y}_t is solely determined by per capita consumption \tilde{C}_t and (2) \tilde{Y}_t is a decreasing function of \tilde{C}_t (since U_t is concave). From equation (1), this implies that securities tend to be more valuable if they tend to have high dividends in dates and states with relatively low per capita consumption.

Time-additive tastes prevent \tilde{Y}_t from depending as well on past per capita consumption, and in general, weak aggregation is required to prevent \tilde{Y}_t from depending on the distribution of consumption across investors. Since we are aiming at simple formulas, these two assumptions are useful. However, in one well-known case we can obtain even greater simplicity without requiring weak aggregation, but at the price of an alternative assumption: the joint distribution of \tilde{X}_t and \tilde{C}_t is bivariate normal for each date t and all investors have homogeneous beliefs. The Appendix shows that, subject to a mild regulatory condition, if x and y are bivariate normal and $g(y)$ is any at least once differentiable function of y , then $\text{Cov}(x, g(y)) = E[g'(y)]\text{Cov}(x, y)$. Applying this to equation (1), individual i would choose his lifetime consumption $C_0^i, \tilde{C}_1^i, \dots, \tilde{C}_t^i, \dots$ such that for every security

$$P_0 = \sum_{t=1}^{\infty} \frac{E(X_t) - \theta_t^i \text{Cov}(X_t, C_t^i)}{R_{Ft}},$$

where $\theta_t^i \equiv -E[U_t^{i'}(C_t^i)]/E[U_t^{i''}(C_t^i)]$. Since financial markets are Pareto-efficient, prices are set as if there were complete markets. For every date t , we can therefore "manufacture" a risky security which pays off only at that date, so that $P_0 R_{Ft} = E(X_t) - \theta_t^i \text{Cov}(X_t, C_t^i)$. Summing this over all investors i and dividing by the number of investors I in the economy, produces a similar result to that found in single-period models, $P_0 R_{Ft} = E(X_t) - \theta_t \text{Cov}(X_t, C_t)$, where $\tilde{C}_t \equiv \Sigma_i \tilde{C}_t^i / I$ and $\theta_t \equiv [\Sigma_i (\theta_t^i)^{-1} / I]^{-1}$. Applying this for each date,

$$P_0 = \sum_{t=1}^{\infty} \frac{E(X_t) - \theta_t \text{Cov}(X_t, C_t)}{R_{Ft}}.$$

Again, since $\theta_t > 0$, securities are penalized which, *ceteris paribus*, have high dividends in dates and states with relatively high per capita consumption.

3. Uncertain income streams: special case

■ While our description of the economic environment has identified \tilde{Y}_t , the resulting valuation formula is not easily empirically useful. With weak aggregation, we need to determine the average utility function and even with the apparently simplified result from homogeneous joint normality, we require measurement of the stochastic process of per capita consumption. Moreover, to obtain the simple relationships of the familiar single-period model, we need to characterize $U'_t(\tilde{C}_t)$ as a function of the market rates of return $\tilde{r}_{Mt} \equiv \tilde{W}_t / (\tilde{W}_{t-1} - \tilde{C}_{t-1})$, where \tilde{W}_t is per capita wealth at date t . \tilde{C}_t has the natural interpretation of a social dividend on the market portfolio. In other words, we need to inquire under what circumstances does there exist a nonrandom function g_t such that $g_t(\tilde{r}_{M1}, \tilde{r}_{M2}, \dots, \tilde{r}_{Mt}) = U'_t(\tilde{C}_t)$?

In particular, $g_t(\cdot)$ cannot be separately dependent on per capita wealth \tilde{W}_τ for any date $0 < \tau \leq t$. For then, $g_t(\cdot)$ would be a random

function. It is known from portfolio theory that optimal portfolio composition is independent of wealth if and only if utility is constant proportional risk averse (CPRA); that is, if and only if

$$U_t(\tilde{C}_t) = \rho_1 \rho_2 \dots \rho_t \frac{1}{1-b} \tilde{C}_t^{1-b},$$

where $b > 0$ and $\rho_t > 0$.¹¹ ρ_t is a measure of time-preference and since $b = -\tilde{C}_t U''_t(\tilde{C}_t)/U'_t(\tilde{C}_t)$, b is the measure of CPRA. Since only in this case can $U'_t(\tilde{C}_t)$ be expressed in terms of $\tilde{r}_{M1}, \tilde{r}_{M2}, \dots, \tilde{r}_{Mt}$ and otherwise independent of \tilde{W}_τ for $0 < \tau \leq t$, CPRA acquires empirical significance.

Theorem 4: Given assumptions 1–5, and if average utility is CPRA and if either $b = 1$ or the rate of growth of per capita consumption follows a (possibly nonstationary) random walk, then

$$\tilde{Y}_t = \tilde{R}_{Mt}^{-b}$$

for all dates and states, where $\tilde{R}_{Mt} \equiv \tilde{r}_{M1} \tilde{r}_{M2} \dots \tilde{r}_{Mt}$.

Proof: Hakansson (1971) and Rubinstein (1974b) have shown that for an investor with CPRA his average propensity to consume wealth at any date is independent of his wealth. Indeed, this independence holds only under CPRA. In particular, in this case, there exists a random variable \tilde{k}_t independent of wealth at any date such that $\tilde{C}_t = \tilde{k}_t \tilde{W}_t$. Moreover, if $b = 1$ (logarithmic utility) then while k_t depends on the date (through date dependent time-preference $\rho_1 \rho_2 \dots \rho_t$ and the time remaining until his death), it is nonstochastic. When $b \neq 1$, then k_t will also be nonstochastic if the rate of growth of per capita consumption follows a (possibly nonstationary) random walk.¹² The assumptions of the theorem therefore imply k_t is a nonstochastic function of time. Since $\tilde{W}_t = (\tilde{W}_{t-1} - \tilde{C}_{t-1})\tilde{r}_{Mt}$, then $\tilde{C}_t = (\tilde{W}_{t-1} - \tilde{C}_{t-1})k_t \tilde{r}_{Mt}$. Similarly, $\tilde{C}_t = (\tilde{W}_{t-2} - \tilde{C}_{t-2})(1 - k_{t-1})k_t \tilde{r}_{Mt-1} \tilde{r}_{Mt}$. Continuing to work backwards

$$\tilde{C}_t = [W_0(1 - k_0)(1 - k_1) \dots (1 - k_{t-1})k_t]\tilde{R}_{Mt}.$$

Substituting this into equation (1), noting $[\cdot]$ is nonstochastic and $U'_t(\tilde{C}_t) = \rho_1 \rho_2 \dots \rho_t \tilde{C}_t^{-b}$, yields the conclusion of the theorem. Q.E.D.

In brief, given the conditions of the theorem, for any security (or portfolio)

$$P_0 = \sum_{t=1}^{\infty} \frac{E(X_t) - \lambda_t \kappa(X_t, -R_{Mt}^{-b}) \text{Std } X_t}{R_{Ft}}, \quad (2)$$

¹¹ In the limiting case of $b = 1$, $U_t(\tilde{C}_t) = \rho_1 \rho_2 \dots \rho_t \ln \tilde{C}_t$.

¹² Specifically, it follows from Rubinstein (1974b), that

$$\tilde{k}_t^{-1} = 1 + \rho_{t+1}^B \tilde{E}_t(r_{Ct+1}^{1-b}) + \rho_{t+1}^B \rho_{t+2}^B \tilde{E}_t(r_{Ct+1}^{1-b} r_{Ct+2}^{1-b}) + \dots,$$

where $B \equiv b^{-1}$, $\tilde{r}_{t,t} \equiv \tilde{C}_t/\tilde{C}_{t-1}$, and expectations are assessed with respect to information available at date t . When the rate of growth of per capita consumption follows a random walk, then k_t is nonstochastic, since

$$k_t^{-1} = 1 + \rho_{t+1}^B E_t(r_{Ct+1}^{1-b}) + \rho_{t+1}^B \rho_{t+2}^B E_t(r_{Ct+1}^{1-b}) E_t(r_{Ct+2}^{1-b}) + \dots$$

Moreover, it is easy to see when $b = 1$ that k_t is nonrandom even if the rate of growth of per capita consumption does not follow a random walk.

where $\lambda_t \equiv \text{Std}(R_{Mt}^{-b})/E(R_{Mt}^{-b})$ and $\kappa(\cdot)$ is a correlation coefficient.

Although no stochastic restrictions have been placed on the rates of return of securities, the premise that the rate of growth of per capita consumption follows a random walk is sufficiently strong to imply important stochastic properties for “basic” portfolios.

Corollary: Under the conditions of Theorem 4, if $b \neq 1$

- (1) *market portfolio:* the rate of return of the market portfolio follows a (possibly nonstationary) random walk;
- (2) *default-free bonds:* the term structure of interest rates is unbiased in the sense that at each date the next period expected rates of return of default-free pure discount bonds of all maturities are the same.

Moreover, if $b = 1$, then conclusion (1) holds if and only if the rate of growth of per capita consumption follows a (possibly nonstationary) random walk; and conclusion (2) holds if and only if the inverse one plus rate of growth of per capita consumption is serially uncorrelated.

Proof: To prove assertion (1), recall that $\tilde{C}_t = k_t \tilde{W}_t$, $\tilde{r}_{Mt} \equiv \tilde{W}_t/(\tilde{W}_{t-1} - \tilde{C}_{t-1})$ and one plus the rate of growth of per capita consumption $\tilde{r}_{Ct} \equiv \tilde{C}_t/\tilde{C}_{t-1}$ for all t . Therefore $\tilde{r}_{Mt} = \tilde{r}_{Ct}[k_{t-1}/k_t(1 - k_{t-1})]$ so that $\kappa(r_{Mt}, r_{Ct}) = 1$. When $b = 1$, we do not require the premise that \tilde{r}_{Ct} follow a random walk to keep k_t and k_{t-1} nonstochastic. To prove assertion (2), let ${}_{t-1}R_{Ft+1}^{-1}$ denote the price at date $t - 1$ of one dollar for sure at date $t + 1$. From the first-order conditions of the programming problem of Section 2, it follows that

$$r_{Ft}^{-1} = \rho_t E_{t-1}(r_{Ct}^{-b}), \quad r_{Ft+1}^{-1} = \rho_{t+1} \tilde{E}_t(r_{Ct+1}^{-b})$$

and

$${}_{t-1}R_{Ft+1}^{-1} = \rho_t \rho_{t+1} E_{t-1}(r_{Ct}^{-b} r_{Ct+1}^{-b})$$

if utility is CPRA, where \sim indicates random variables from the perspective of information available at date $t - 1$. Additionally, if ${}_{t-1}\tilde{r}_{Ft}$ denotes the one plus rate of return on the two-period bond from date $t - 1$ to date t , then in conformity with the single-price law of markets

$${}_{t-1}R_{Ft+1} = {}_{t-1}\tilde{r}_{Ft} \tilde{r}_{Ft+1}.$$

From this it follows that

$$E_{t-1}({}_{t-1}r_{Ft}) = [E_{t-1}(r_{Ct+1}^{-b})][\rho_t E_{t-1}(r_{Ct}^{-b} r_{Ct+1}^{-b})]^{-1}.$$

When $b \neq 1$ and by premise \tilde{r}_{Ct} and \tilde{r}_{Ct+1} follow a random walk, then this simplifies to $E_{t-1}({}_{t-1}r_{Ft}) = [\rho_t E_{t-1}(r_{Ct}^{-b})]^{-1}$ so that $r_{Ft} = E_{t-1}({}_{t-1}r_{Ft})$. When $b = 1$, this same conclusion is reached if and only if \tilde{r}_{Ct}^{-1} and \tilde{r}_{Ct+1}^{-1} are serially uncorrelated. This proof is easily extended to default-free bonds of all maturities. Q.E.D.

Whether or not (1) the market portfolio follows a random walk or (2) the term structure is unbiased depends critically on the stochastic process of per capita consumption over time. The underlying real *intertemporal* stochastic process of per capita consumption is mirrored in the equilibrium financial process governing the prices of “basic” portfolios. Moreover, when \tilde{r}_{Ct} follows a random walk or $b = 1$, then the real *contemporaneous* stochastic process of per capita

consumption is also mirrored in the market portfolio. From the above proof, $\tilde{r}_{Mt} = \tilde{r}_{Ct}[k_{t-1}/k_t(1 - k_{t-1})]$ so that \tilde{r}_{Ct} is normal (lognormal) if and only if \tilde{r}_{Mt} is normal (lognormal).

When the rate of growth of per capita consumption follows a stationary random walk and average time-preference is stationary, then equation (2) can be further simplified.

Corollary: Under the conditions of Theorem 4, if the rate of growth of per capita consumption follows a stationary random walk and average time-preference is stationary over an infinite life-time, then

$$P_0 = \sum_{t=1}^{\infty} \frac{E(X_t) - \lambda_t \kappa(X_t, -r_M^{-bt}) \text{Std } X_t}{r_F^t},$$

where

$$\lambda_t = \sqrt{(1 + \lambda^2)^t - 1} \text{ and } \lambda \equiv \text{Std}(r_M^{-b})/E(r_M^{-b}).$$

Proof: From the proof of the previous corollary, $\tilde{r}_{Mt} = \tilde{r}_{Ct}[k_{t-1}/k_t(1 - k_{t-1})]$. From the expression for k_t in note 12, under the premise of the above corollary since $\tilde{r}_{Ct} \equiv \tilde{r}_C$ and $\rho_t \equiv \rho$, then $k_t \equiv k$ is stationary. Therefore, $\tilde{r}_{Mt} \equiv \tilde{r}_M$ is likewise stationary. In general, if any random variable \tilde{x}_t follows a stationary random walk (or is merely stationary and serially uncorrelated), then

$$\text{Var}(x_1 x_2 \dots x_t) = (\sigma_x^2 + \mu_x^2)^t - (\mu_x^2)^t.$$

And, of course, $E(x_1 x_2 \dots x_t) = \mu_x^t$. From these properties, it follows that $\lambda_t = \sqrt{(1 + \lambda^2)^t - 1}$. Similarly, from the proof of the previous corollary, since $\tilde{r}_{Ft}^{-1} = \rho_t \tilde{E}_{t-1}(r_{Ct}^{-b})$, then $\tilde{r}_{Ft} = r_{Ft} \equiv r_F$ is a stationary constant. Q.E.D.

Under these stationarity conditions, the “market price of risk” λ_t increases with the time to the receipt of the cash flow so that risk averse investors tend to penalize more distant risks more than near risks. Compounding this effect is the tendency of $\text{Std } X_t$ to increase with t . For example, if the growth rate \tilde{g}_t of \tilde{X}_t follows a stationary random walk, then $\text{Std } X_t = X_0[(\sigma_g^2 + \mu_g^2)^t - (\mu_g^2)^t]^{\frac{1}{2}}$, which clearly increases with t . However, since r_F^t increases with t (assuming $r_F > 1$), the net effect on present value of the influence of time to receipt of income on risk is indefinite.

Although Theorem 4 identifies \tilde{Y}_t as \tilde{R}_{Mt}^{-b} , unless b , the level of CPRA, were known in advance, empirical application of the valuation formula (2) would prove difficult. Even the implied single-period formula

$$P_{t-1} = \frac{E(\tilde{X}_t) - \hat{\lambda}_t \text{Cov}(\tilde{X}_t, -r_{Mt}^{-b}) \text{Std } \tilde{X}_t}{r_{Ft}} \quad (3)$$

where $\tilde{\tilde{X}}_t \equiv \tilde{X}_t + \tilde{P}_t$ and $\hat{\lambda}_t \equiv \text{Std}(r_{Mt}^{-b})/E(r_{Mt}^{-b})$ is difficult to apply. Standard linear regression techniques cannot be used to estimate b , since the formula is nonlinear in b . Nonetheless, there are three interesting methods to overcome this difficulty. As Blume and Friend (1975) and Cohen *et al.* (1975) have recently attempted, it may be possible to measure b from empirical surveys of consumer wealth allocation. Second, and possibly more promising, is to infer b by trial and error from the behavior of the rate of return of the market portfolio. Since equation (3) holds for the market portfolio, dividing by P_{t-1} and rearranging, $r_{Ft} = E(r_{Mt}^{1-b})/E(r_{Mt}^{-b})$. Partially differentiat-

ing the right-hand side by b , the Schwartz inequality can be used to show that $\partial[E(r_{Mt}^{1-b})/E(r_{Mt}^{-b})]/\partial b > 0$. Therefore, a trial-and-error procedure will quickly converge to a good estimate of b which approximately satisfies the above equality. Third, is to infer b from the rate of return of the market portfolio when it is assumed to conform to a given stochastic process.

Corollary: Under the conditions of Theorem 4, if the single-period one plus rate of return of the market portfolio is lognormal, then

$$b = \frac{E(\ln r_{Mt}) - \ln r_{Ft}}{\text{Var}(\ln r_{Mt})} + 1/2.$$

Proof: Since equation (3) holds for the market portfolio, dividing by P_{t-1} and rearranging, $r_{Ft} = E(r_{Mt}^{1-b})/E(r_{Mt}^{-b})$. From the Appendix, if \tilde{r}_{Mt} is lognormal, then \tilde{r}_{Mt}^{1-b} and \tilde{r}_{Mt}^{-b} are lognormal. Therefore, $E(r_{Mt}^{1-b}) = \exp[(1-b)E(\ln r_{Mt}) + \frac{1}{2}(1-b)^2\text{Var}(\ln r_{Mt})]$ and $E(r_{Mt}^{-b}) = \exp[-bE(\ln r_{Mt}) + \frac{1}{2}b^2\text{Var}(\ln r_{Mt})]$. The result follows by dividing these two expressions, setting the quotient equal to r_{Ft} and taking logarithms. Q.E.D.

Equation (2) with b estimated from the corollary, requires only that \tilde{r}_{Mt} be lognormal and (unless $b = 1$) follow a (possibly nonstationary) random walk; no distributional restrictions have been placed on the rates of return of other securities. This is fortunate since the rates of return of many securities such as options and default-free bonds near maturity, as well as the rates of return of capital budgeting projects, are neither lognormal nor follow a random walk. However, for securities with one plus rates of return \tilde{r}_t jointly lognormal with \tilde{r}_{Mt} , a similar argument to the corollary can be used to show

$$\ln[E(r_t)] = \ln r_{Ft} + b \text{Cov}(\ln r_t, \ln r_{Mt}).$$

Although this formula has been derived in discrete time, it is consistent with Merton's (1973b) continuous-time model as interpreted over discrete intervals by Jensen (1972, p. 386). CPRA has substituted for continuous trading opportunities to achieve the same end.¹³

4. Options

■ The similarity between the valuation results derived in discrete and continuous time suggests that other security pricing relationships may also be immune from this distinction. In particular, it is known that the Black-Scholes (1973) option pricing equation for dividend-protected options,

$$Q = SN(z^* + \sigma\sqrt{t}) - Kr_F^{-t}N(z^*), \quad (4)$$

where

$$z^* \equiv \frac{\ln(S/K) + (\ln r_F - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}},$$

holds either with continuous trading opportunities or risk neutrality.¹⁴

¹³ See Kraus and Litzenberger (1975) for similar remarks.

¹⁴ Merton (1973a) mentions that Samuelson and Merton (1969) have uncovered yet a third set of assumptions for equation (4): (a) the average investor has CRPA, (b) only three securities exist—a default-free bond, the option, and its associated stock, and (c) the net supply of both the options and the bonds is zero.

Clearing up the notation:

Q = current price of option (i.e., call)

S = current price of associated stock

K = striking price

t = time to expiration

r_F = one plus the interest rate

σ^2 = the variance of the logarithm of one plus the rate of return of the associated stock

$N(\cdot)$ = the standard normal cumulative density function.

Although proofs are available elsewhere, to ease understanding of the subsequent discrete-time analysis under risk aversion, it will be useful to sketch a proof here. Following Sprenkle (1961), we first derive a simplified expression for the expected future value of the option at expiration and then discount this value back to the present. Let \tilde{Q}_t and \tilde{S}_t be the respective values of the option and its associated stock at expiration. Since¹⁵ $\tilde{Q}_t = 0$ if $\tilde{S}_t < K$ and $\tilde{Q}_t = \tilde{S}_t - K$ if $\tilde{S}_t \geq K$, then $E(Q_t) = E[S_t - K | \tilde{S}_t \geq K]$. Defining $\tilde{R} \equiv \tilde{S}_t/S$, $E(Q_t) = SE[\tilde{R} - (K/S) | \tilde{R} \geq (K/S)]$. Defining $\tilde{r} \equiv \ln \tilde{R}$, $E(Q_t) = SE[e^{\tilde{r}} - (K/S) | \tilde{r} \geq \ln(K/S)]$. Since \tilde{S}_t is lognormal, \tilde{R} is lognormal and \tilde{r} is normal, therefore

$$E(Q_t) = S \int_{\ln(K/S)}^{\infty} \left(e^{\tilde{r}} - \frac{K}{S} \right) f(\tilde{r}) d\tilde{r},$$

where $f(\cdot)$ is the normal density function. Breaking this apart into the difference between two integrals, using equations (1) and (2) of the Appendix to evaluate them, and noting the transformation of the mean of a lognormal variable, we obtain

$$E(Q_t) = S\mu_R N(z^* + \sigma_r) - KN(z^*),$$

where

$$z^* \equiv \frac{\ln(S/K) + \mu_r}{\sigma_r}.$$

Since $\mu_R = \exp[\mu_r + \frac{1}{2} \sigma_r^2]$, then $\mu_r = \ln \mu_R - \frac{1}{2} \sigma_r^2$. Finally discounting $E(Q_t)$ back to the present by the expected compound rate of return μ_Q of the option through expiration, we obtain

$$Q = [S\mu_R N(z^* + \sigma_r) - KN(z^*)]/\mu_Q$$

where

$$z^* \equiv \frac{\ln(S/K) + \ln \mu_R - \frac{1}{2} \sigma_r^2}{\sigma_r}.$$

The usefulness of this result is hampered primarily by measurement of μ_Q and secondarily by measurement of μ_R . However, assuming risk neutrality, the expected rates of return of the stock and option are equal and moreover equal to the riskless rate so that $R_F = \mu_Q = \mu_R$. If, additionally, the riskless rate is constant over time so that $r_F = (\mu_R)^{1/t} = (\mu_Q)^{1/t}$, and the rate of return of the associated stock follows a stationary random walk so that $\sigma_r^2 = t\sigma^2$, then the Black-Scholes equation follows exactly.

¹⁵ Merton (1973a) has shown that rational nonsatiated investors will not exercise a call option prior to expiration if it is properly dividend protected and if the striking price is fixed. He also proves this will *not* generally be true for puts.

Recently, Black and Scholes have stimulated academic interest in options by showing that the valuation formula derived under risk neutrality holds under seemingly much more general conditions. They argue that if the associated stock price follows a continuous stochastic process in time¹⁶ and if investors can costlessly continuously revise their portfolios, then investors can create a perfectly hedged (i.e., riskless) portfolio by selling a call against a long position in the associated stock. Cox and Ross (1975) have argued that since, in equilibrium, this portfolio must earn the riskless rate, then given the current price of the stock, the interest rate, and agreement about the relevant statistics characterizing the rate of return of the stock, the current price of the call will be independent of the characteristics of other securities and the preferences of investors. Therefore, the same current price of the option will be derived irrespective of investor preferences. To simplify matters, we are then free to assume risk neutrality without loss of generality. But, by the previous analysis, the Black-Scholes equation again holds.

On the surface, the existence of continuous trading opportunities to insure the riskless hedge seems critical to the Black-Scholes analysis. How else (except in the trivial risk neutrality case) could the pricing formula be independent of investor preference datum and opportunities for diversification? Although Merton (1975, p. 2) claims that the continuous trading formula appears in the limit as the trading interval shrinks to zero, it remains unclear what kind of an approximation is involved even for small finite trading intervals.

The following theorem proves that the Black-Scholes pricing formula is robust within lognormality conditions, even with only discrete trading opportunities and risk aversion.

Theorem 5: Given assumptions 1 and 2, if S_t and Y_t are jointly lognormal, investors agree on σ , and the associated stock pays no dividends through expiration of the option, then

$$Q = SN(z^* + \sigma) - KR_F^{-1}N(z^*), \quad (5)$$

where

$$z^* \equiv \frac{\ln(S/K) + \ln R_F}{\sigma} - \frac{1}{2} \sigma$$

and R_F is the riskless discount rate through the expiration of the option and σ is the standard deviation of the logarithm of the compound one plus rate of return of the associated stock through the expiration of the option (i.e., $\text{Std}(\ln R)$).

Proof: Recalling the proof to Theorem 1, we can define a random variable \tilde{Z}'_t such that $\tilde{Z}'_t \equiv \tilde{Y}_t R_{Ft}^{-1} / E(Y_t)$. Since the stock is assumed to pay no dividends, $S = E(S_t Z'_t)$. Moreover $Q = E[(S_t - K)Z'_t | \tilde{S}_t \geq K]$ and $R_F^{-1} = E(Z'_t)$. By defining $\tilde{R} \equiv \tilde{S}_t / S$, these relationships become

$$E(RZ'_t) = 1, R_{Ft}^{-1} = E(Z'_t)$$

and

$$Q = SE[(R - (K/S))Z'_t | \tilde{R} \geq (K/S)].$$

¹⁶ See Cox and Ross (1975) and Merton (1975) for analyses of option pricing when its associated stock price follows a jump process.

Defining $\tilde{r} \equiv \ln \tilde{R}$ and $\tilde{y} \equiv \ln \tilde{Z}'_t$, $Q = SE[(e^r - (K/S))e^y | \tilde{r} \geq \ln(K/S)]$. Since \tilde{S}_t and \tilde{Z}'_t are jointly lognormal, then \tilde{r} and \tilde{y} are jointly normal, therefore

$$Q = S \int_{-\infty}^{\infty} \int_{\ln(K/S)}^{\infty} \left(e^r - \frac{K}{S} \right) e^y f(r, y) dr dy,$$

where $f(\cdot)$ is the bivariate normal density function. Breaking this apart into the difference between two integrals, using equations (3) and (4) of the Appendix to evaluate them, and noting the transformations of the mean of a lognormal variable and the mean of the product of two lognormal variables, we obtain

$$Q = SE(RZ'_t)N(z^* + \sigma_r) - KE(Z'_t)N(z^*),$$

where

$$z^* \equiv \frac{\ln(S/K) + \mu_r + \kappa\sigma_r\sigma_y}{\sigma_r}.$$

Since $E(RZ'_t) = 1$ and $E(Z'_t) = R_F^{-1}$, this simplifies to

$$Q = SN(z^* + \sigma_r) - KR_F^{-1}N(z^*).$$

Since $E(RZ'_t) = \exp[\mu_r + \frac{1}{2}\sigma_r^2 + \mu_y + \frac{1}{2}\sigma_y^2 + \kappa\sigma_r\sigma_y] = 1$ and $E(Z'_t) = \exp[\mu_y + \frac{1}{2}\sigma_y^2] = R_F^{-1}$, then $\mu_r + \frac{1}{2}\sigma_r^2 - \ln R_F + \kappa\sigma_r\sigma_y = 0$. Substituting this into the expression for z^* proves the theorem. Q.E.D.

Appending the Black-Scholes stationarity conditions converts equation (5) into equation (4). However, the derivation presented here shows that under the strengthened lognormality assumption, these stationarity conditions are in no way necessary to retain the important features of equation (4).¹⁷ Under what conditions will \tilde{Y}_t be jointly lognormal with \tilde{S}_t ? Theorem 4 supplies an interesting sufficient condition. $\tilde{Y}_t = \tilde{R}_{Mt}^{-b}$ will be jointly lognormal with \tilde{S}_t if \tilde{R}_{Mt} and \tilde{S}_t are jointly lognormal.¹⁸

To extend equation (5) to dividend paying stock, assume through the life of the option, the dividend yield of the associated stock is a known nonstochastic function of time. Let $\delta_t \equiv \tilde{X}_t/\tilde{S}_t$ denote the dividend yield of the stock at any date t through the life of the option. Then the one plus the compound rate of return of the stock to the expiration date, $\tilde{R} = (\tilde{S}_t/S)(1 + \delta_1)(1 + \delta_2) \dots (1 + \delta_t)$. Defining $\Delta \equiv (1 + \delta_1)(1 + \delta_2) \dots (1 + \delta_t)$, then $\tilde{R} = \tilde{S}_t/(S\Delta^{-1})$. In the proof of Theorem 5, replacing S with $S\Delta^{-1}$,

¹⁷ Merton (1973a) has relaxed the stationarity conditions on the interest rate and allows the standard deviation of the logarithm of one plus the rate of return of the associated stock (σ) to be a known nonstochastic function of time; here σ can be stochastic as well, extending the model to more realistic stochastic processes. See Rosenberg (1972).

¹⁸ The Samuelson-Merton (1969) assumptions (see note 14) are clearly a special case. Their assumptions (b) and (c) imply that $Y_t = \tilde{R}^{-b}$, since the associated stock is the market portfolio. The risk neutrality justification is a degenerate special case where \tilde{Y}_t is nonstochastic. It should also be noted that the premise of Theorem 5 is stronger than necessary. We actually require a random variable \tilde{Y}_t shared *only* by the option, its associated stock, and the default-free bond, which is jointly lognormal with \tilde{S}_t . In effect, as in the Black-Scholes analysis, equation (5) will apply even if S and R_{Ft} are not in equilibrium with the other securities in the market. Finally, agreement on $\text{Var}(\ln \tilde{S}_t)$ is not required if heterogeneous investor beliefs can be meaningfully aggregated; see Rubinstein (1976).

Corollary: Under the conditions of Theorem 5, except that the associated stock, through the expiration of the option, has a dividend yield which is a known nonstochastic function of time, and the option cannot be exercised prior to its expiration date, then

$$Q = S\Delta^{-1}N(z^* + \sigma) - KR_F^{-1}N(z^*)$$

where

$$z^* \equiv \frac{\ln(S\Delta^{-1}/K) + \ln R_F}{\sigma} - \frac{1}{2}\sigma$$

and R_F is the riskless discount rate through the expiration of the option and σ is the standard deviation of the logarithm of the compound one plus rate of return of the associated stock through the expiration of the option (i.e., $\text{Std}(\ln R)$).

When will the dividend yield be nonstochastic? Theorem 2 provides a sufficient condition in its premise. An "American" option, one that can be exercised prior to its expiration date, will be worth more than Q as given by the above formula.¹⁹

5. Conclusion

■ This paper has developed a simple and practical formula for the valuation of uncertain income streams, consistent with rational risk averse investor behavior and equilibrium in financial markets. Since the formula places virtually no stochastic restrictions on the income stream, it can be used to value securities or other assets such as capital budgeting projects with serially correlated cash flows or rates of return, securities with or without limited liability, and derivative securities such as options in terms of their underlying securities. The formula was used to derive simple sufficient restrictions on real economic variables for the rate of return of the market portfolio to follow a random walk and the term structure to be unbiased. When applied to the pricing of options, it was shown, given appropriate stochastic restrictions, to imply the Black-Scholes option pricing formula, even though a perfect hedge cannot be constructed.

Appendix

The bivariate normal density function

■ Random variables x and y are said to be bivariate *normal* if their joint density function is

$$f(x, y) = \frac{1}{2\pi\sqrt{\sigma_x^2\sigma_y^2(1-\kappa^2)}} \times \exp \left\{ \frac{-1}{2(1-\kappa^2)} \left[\frac{(x-\mu_x)^2}{\sigma_x^2} - 2\kappa \frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} + \frac{(y-\mu_y)^2}{\sigma_y^2} \right] \right\},$$

¹⁹ See Samuelson and Merton (1969). The corollary generalizes Merton (1973a, p. 171) by allowing the dividend yield to depend on time. In a succeeding working paper, Geske (1976) has further generalized the option pricing model for underlying stock with a stochastic but lognormal dividend yield. Since a perfect hedge cannot be constructed in this case, even if investors can make continuous revisions, his generalization requires the preference-specific approach developed in this paper.

where $\mu_x \equiv E(x)$, $\mu_y \equiv E(y)$, $\sigma_x^2 \equiv \text{Var } x$, $\sigma_y^2 \equiv \text{Var } y$, $\kappa \equiv \kappa(x, y)$. Defining the *marginal* density function of x , $f(x) \equiv \int_{-\infty}^{\infty} f(x, y) dy$ (and similarly for y), it is not difficult to prove that $f(x)$ (and $f(y)$) is a normal density function; that is,

$$f(x) = \frac{1}{\sigma_x \sqrt{2\pi}} \exp \left[\frac{-1}{2\sigma_x^2} (x - \mu_x)^2 \right].$$

Defining the *conditional* density function of y given x , $f(y|x) \equiv f(x, y)/f(x)$, it follows that

$$f(y|x) = \frac{1}{\sqrt{2\pi(1 - \kappa^2)\sigma_y^2}} \times \exp \left\{ \frac{-1}{2(1 - \kappa^2)\sigma_y^2} \left[(y - \mu_y) - \kappa \frac{\sigma_y}{\sigma_x} (x - \mu_x) \right]^2 \right\}.$$

Observe that this conditional density is itself a normal density with mean and variance, respectively,

$$E(y|x) = \mu_y + \kappa(\sigma_y/\sigma_x)(x - \mu_x)$$

and

$$\text{Var}(y|x) = (1 - \kappa^2)\sigma_y^2.$$

One very useful but little known²⁰ property of the bivariate normal distribution is that if x and y are bivariate normal and $g(y)$ is any at least once differentiable function of y , then, subject to mild regularity conditions,

$$\text{Cov}(x, g(y)) = E[g'(y)]\text{Cov}(x, y).$$

To see this, since

$$\text{Cov}(x, g(y)) = \iint x g(y) f(x, y) dx dy - \mu_x E[g(y)],$$

then using the conditional density

$$\text{Cov}(x, g(y)) = \int g(y) E(x|y) f(y) dy - \mu_x E[g(y)]$$

or, more particularly, for bivariate normal variables

$$\begin{aligned} \text{Cov}(x, g(y)) &= \kappa \left(\frac{\sigma_x}{\mu_y} \right) \int g(y) (y - \mu_y) f(y) dy \\ &= \text{Cov}(x, y) \int g(y) \left[\frac{y - \mu_y}{\sigma_y^2} \right] f(y) dy. \end{aligned}$$

Defining $h'(y) \equiv \left[\frac{y - \mu_y}{\sigma_y^2} \right] f(y)$ and integrating by parts, we obtain

$$\text{Cov}(x, g(y)) = \text{Cov}(x, y) \left[g(\infty)h(\infty) - g(-\infty)h(-\infty) - \int g'(y)h(y) dy \right].$$

Since $f(y) = \frac{1}{\sigma_y \sqrt{2\pi}} e^{-(y - \mu_y)^2 / 2\sigma_y^2}$, then $h(y) = -f(y)$. In addition, since then $f(-\infty) = f(\infty) = 0$ and if $g(y)$ is bounded above and below or, more generally, if

$$\lim_{y \rightarrow -\infty} g(y)f(y) = \lim_{y \rightarrow \infty} g(y)f(y) = 0,$$

²⁰ To my knowledge, this property of bivariate normal variables was first noted in Rubinstein (1971). Stein (1973) has independently made its discovery.

then

$$\text{Cov}(x, g(y)) = \text{Cov}(x, y) \left[\int g'(y) f(y) dy \right] = E[g'(y)] \text{Cov}(x, y).$$

In the process of deriving an analytic solution for the discrete-time valuation of options, we need to evaluate several definite integrals over the marginal and conditional normal density functions:

$$\int_a^\infty f(x) dx = N\left(\frac{-a + \mu_x}{\sigma_x}\right), \quad (\text{A1})$$

where $N(z^*)$ is the standard cumulative normal distribution from $-\infty$ to z^* ; that is, $N(z^*) \equiv \int_{-\infty}^{z^*} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$. This is proved by first converting $f(x)$ into the standard normal density function $f(z) \equiv \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$ and then observing that since z is symmetric around zero, the limits of integration can be interchanged. That is,

$$\int_a^\infty f(x) dx = \int_{\frac{a - \mu_x}{\sigma_x}}^\infty f(z) dz = \int_{-\infty}^{\frac{-a + \mu_x}{\sigma_x}} f(z) dz = N\left(\frac{-a + \mu_x}{\sigma_x}\right).$$

Second, we show that

$$\int_a^\infty e^x f(x) dx = (e^{\mu_x + \frac{1}{2} \sigma_x^2}) N\left(\frac{-a + \mu_x}{\sigma_x} + \sigma_x\right). \quad (\text{A2})$$

To see this,

$$\int_a^\infty e^x f(x) dx = \int_a^\infty \frac{1}{\sigma_x \sqrt{2\pi}} \exp\left[\frac{-1}{2\sigma_x^2} (x - \mu_x)^2 + x\right] dx.$$

Since the exponent of e is equal to $\mu_x + \frac{1}{2} \sigma_x^2 - \frac{1}{2\sigma_x^2} (x - \mu_x - \sigma_x^2)^2$,

$$\int_a^\infty e^x f(x) dx = (e^{\mu_x + \frac{1}{2} \sigma_x^2}) \int_a^\infty \frac{1}{\sigma_x \sqrt{2\pi}} \exp\left\{\frac{-1}{2\sigma_x^2} [x - (\mu_x + \sigma_x^2)]^2\right\} dx.$$

Again, the result follows by converting this density into the standard normal density and interchanging the limits.

As an interim result, we will need to prove that

$$\int_{-\infty}^\infty e^y f(y|x) dy = \exp\left[\mu_y + \kappa \frac{\sigma_y}{\sigma_x} (x - \mu_x) + \frac{1}{2} (1 - \kappa^2) \sigma_y^2\right].$$

This is equivalent to finding the expected value of e^y where y is normally distributed with mean $\mu_y + \kappa(\sigma_y/\sigma_x)(x - \mu_x)$ and variance $(1 - \kappa^2)\sigma_y^2$. This is analogous to integral (2) where $a = -\infty$. In this case, $N(\cdot) = N(\infty) = 1$, so that the exponent of e may be replaced by $\mu_y + \kappa(\sigma_y/\sigma_x)(x - \mu_x) + \frac{1}{2} (1 - \kappa^2) \sigma_y^2$.

Now we are prepared to prove

$$\int_{-\infty}^\infty \int_a^\infty e^y f(x, y) dx dy = (e^{\mu_y + \frac{1}{2} \sigma_y^2}) N\left(\frac{-a + \mu_x}{\sigma_x} + \kappa \sigma_y\right). \quad (\text{A3})$$

From our interim result

$$\begin{aligned} \int_{-\infty}^\infty \int_a^\infty e^y f(x, y) dx dy &= \int_a^\infty f(x) \left[\int_{-\infty}^\infty e^y f(y|x) dy \right] dx \\ &= \int_a^\infty e^{\mu_y + \kappa \frac{\sigma_y}{\sigma_x} (x - \mu_x) + \frac{1}{2} (1 - \kappa^2) \sigma_y^2} f(x) dx. \end{aligned}$$

Combining terms in the exponents of e , this, in turn, equals

$$\int_a^\infty \frac{1}{\sigma_x \sqrt{2\pi}} \exp \left[\mu_y + \kappa \frac{\sigma_y}{\sigma_x} (x - \mu_x) + \frac{1}{2} (1 - \kappa^2) \sigma_y^2 - \frac{1}{2\sigma_x^2} (x - \mu_x)^2 \right] dx.$$

Since the exponent of e is equal to $\mu_y + \frac{1}{2} \sigma_y^2 - \frac{1}{2\sigma_x^2} [(x - \mu_x) - \kappa \sigma_x \sigma_y]^2$,

$$\int_{-\infty}^\infty \int_a^\infty e^{yf(x,y)} dx dy = (e^{\mu_y + \frac{1}{2}\sigma_y^2}) \times \int_a^\infty \frac{1}{\sigma_x \sqrt{2\pi}} \exp \left\{ \frac{-1}{2\sigma_x^2} [x - (\mu_x + \kappa \sigma_x \sigma_y)]^2 \right\} dx.$$

Again, the result follows by converting this density into the standard normal density and interchanging the limits.

Finally,

$$\int_{-\infty}^\infty \int_a^\infty e^x e^{yf(x,y)} dx dy = (e^{\mu_x + \mu_y + \frac{1}{2}(\sigma_x^2 + 2\kappa\sigma_x\sigma_y + \sigma_y^2)}) \times N\left(\frac{-a + \mu_x}{\sigma_x} + \kappa\sigma_y + \sigma_x\right). \quad (A4)$$

Again from our interim result,

$$\begin{aligned} \int_{-\infty}^\infty \int_a^\infty e^x e^{yf(x,y)} dx dy &= \int_a^\infty e^x f(x) \left[\int_{-\infty}^\infty e^{yf(y|x)} dy \right] dx \\ &= \int_a^\infty e^{\mu_y + \kappa \frac{\sigma_y}{\sigma_x} (x - \mu_x) + \frac{1}{2}(1 - \kappa^2)\sigma_y^2} e^x f(x) dx. \end{aligned}$$

Combining terms in the exponents of e , this, in turn, equals

$$\int_a^\infty \frac{1}{\sigma_x \sqrt{2\pi}} \exp \left[\mu_y + \kappa \frac{\sigma_y}{\sigma_x} (x - \mu_x) + \frac{1}{2} (1 - \kappa^2) \sigma_y^2 + x - \frac{1}{2\sigma_x^2} (x - \mu_x)^2 \right] dx.$$

Since the exponent of e is equal to

$$\begin{aligned} \mu_x + \mu_y + \frac{1}{2} (\sigma_x^2 + 2\kappa\sigma_x\sigma_y + \sigma_y^2) \\ - \frac{1}{2\sigma_x^2} [(x - \mu_x) - (\kappa\sigma_x\sigma_y + \sigma_x^2)]^2, \end{aligned}$$

substituting this into the integral, converting this density into the standard normal density, and interchanging the limits yields (A4).

If x and y are bivariate normal, then random variables $X \equiv e^x$ and $Y \equiv e^y$ are said to be bivariate *lognormal*. Since sums of jointly normal variables are normal, products of jointly lognormal variables are lognormal. Moreover, if a and b are constants, since $y = a + bx$ is normally distributed if x is normally distributed, then $Y = aX^b$ is lognormally distributed if X is lognormally distributed. Since $x = \ln X$ and $y = \ln Y$ are normally distributed and since $d(\ln X) = dX/X$ and $d(\ln Y) = dY/Y$, then from the bivariate normal density function, the bivariate lognormal density function of X and Y is $F(X, Y) = \frac{1}{XY} f(x, y)$, where $f(x, y)$ is again the bivariate normal density function. Consequently, the *marginal* lognormal density function of X , usually called simply the lognormal density, is $F(X) = \frac{1}{X} f(x)$.

Using the lognormal density and integrating

$$\mu_X \equiv \int_0^\infty XF(X)dX \text{ and } \sigma_X^2 \equiv \int_0^\infty (X - \mu_X)^2 F(X)dX,$$

it can be shown that the mean and variance of the lognormal variables, μ_X and σ_X^2 , are related to the mean and variance of their corresponding normal variables, μ_x and σ_x^2 by²¹

$$\mu_X = e^{\mu_x + \frac{1}{2}\sigma_x^2} \text{ and } \sigma_X^2 = (e^{\mu_x + \frac{1}{2}\sigma_x^2})^2(e^{\sigma_x^2} - 1).$$

With these relationships, since the product XY is itself lognormal,

$$E(XY) = \exp[\mu_x + \mu_y + \frac{1}{2}\sigma_x^2 + \sigma_{xy} + \frac{1}{2}\sigma_y^2].$$

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²¹ The general formula for the n th order central moment of a lognormal variable is

$$E[(X - \mu_X)^n] = (\mu_X)^n \left[\sum_{r=0}^n \frac{n!}{r!(n-r)!} (-1)^{n-r} e^{\frac{1}{2}r(r-1)\sigma_x^2} \right]$$

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