March 26, 2023

A discussion on the morning-star two-step regression technique

The Morningstar technique (with plug-in estimators)

$$
\begin{aligned}
&\min_x E\left[e_t^2\right] \\
&s.t. \\
0 \leq& x \\
1 \geq& 1'x \\
e_t \equiv& r_{pt} - a_t'x \\
0 =& \lambda_k x_k \\
0 =& \gamma\left(1 - x'1\right)
\end{aligned}
$$

The Langrangian:

$$
\begin{aligned}
L =& E\left[\left(r_p - a_t'x\right)'\left(r_p - a_t'x\right)\right] - x'\lambda + \gamma\left(1 - x'1\right) \\
=& \sigma_p^2 + \mu_p^2 + x'E\left[a_t a_t'\right]x - E\left[r_{pt}\right]x'E\left[a_t\right] + E\left[r_{pt}\right]E\left[a_t\right]'x - x'\lambda + +\gamma\left(1 - x'1\right) \\
FOC&: \\
0 =& 2E\left[a_t a_t'\right]x - 2E\left[r_{pt}\right]E\left[a_t\right] - \lambda + \gamma 1
\end{aligned}
$$

For simplicity, assume that $\lambda_k = 0$. Then the FOC simplifies to:

$$
\begin{aligned}
\gamma 1 =& 2E\left[a_t a_t'\right]x - 2E\left[r_{pt}\right]E\left[a_t\right] \\
x =& \frac{\gamma}{2}E\left[a_t a_t'\right]^{-1}1 + E\left[a_t a_t'\right]^{-1}E\left[a_t\right]E\left[r_{pt}\right] \\
x =& \delta + \beta_a \\
s.t.& \\
\delta \equiv& \gamma\frac{E\left[a_t a_t'\right]^{-1}1}{2} \\
\beta_a \equiv& E\left[a_t a_t'\right]^{-1}E\left[a_t\right]E\left[r_p\right]
\end{aligned}
$$

This implies:

$$
\begin{aligned}
r_b =& a_t'\left(\delta + \beta_a\right) \\
E\left[r_b\right] =& \mu_a'\left(\delta + \beta_a\right) \\
\sigma_b^2 =& x'E\left[aa'\right]x - x'\mu_a\mu_a'x \\
=& x'\Sigma_a x \\
=& \left(\delta + \beta_a\right)'\Sigma_a\left(\delta + \beta_a\right)
\end{aligned}
$$

Note that under normality:

$$
\begin{aligned}
E\left[r_{pt}|r_{bt}\right] =& \mu_p + \frac{cov\left(r_{pt}, r_{bt}\right)}{\sigma_b^2}\left(r_{bt} - \mu_b\right) \\
cov\left(r_{pt}, r_{bt}\right) =& cov\left(r_p, a_t'\left(\delta + \beta_a\right)\right) \\
=& cov\left(a_t'\beta_a, \gamma a_t'\beta_a + \gamma a_t \delta\right) \\
=& \beta_a'\Sigma_a\left(\beta + \delta\right) \\
\implies \left[r_{pt}|r_{bt}\right] =& \mu_p + \frac{\beta_a'\Sigma_a\left(\delta + \beta_a\right)}{\left(\delta + \beta_a\right)'\Sigma_a\left(\delta + \beta_a\right)}\left(a_t - \mu_a\right)'\left(\delta + \beta_a\right) \\
=& \mu_a'\beta_a + \frac{\beta_a'\Sigma_a\left(\delta + \beta_a\right)}{\left(\delta + \beta_a\right)'\Sigma_a\left(\delta + \beta_a\right)}\left(a_t - \mu_a\right)'\left(\delta + \beta_a\right)
\end{aligned}
$$

This implies that conditioning on r_{bt} is NOT the same as conditioning on a_t and will lead to an inconsistent estimator of exposure. The exception occurs when both constraints are zero.