Lecture 7

State Space Models and the Kalman Filter

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Outline

1. Motivation: Forward-looking betas
    - Bayes rule
    - Sequential Learning
2. A simple state space model
3. State space models and the Kalman Filter
    - Filtering Probabilities
    - Smoothing Probabilities
    - Maximum Likelihood Estimation

Motivation

- We have discussed, but not attempted to estimate, forward-looking conditional betas
  - Consider the conditional market beta of firm i at time t:

$$\beta_{it} = \frac{cov_t\left(R_{m,t+1}, R_{i,t+1}\right)}{var_t\left(R_{m,t+1}\right)}.$$

  - This is the *ex ante* forward-looking beta at time t for returns realized at time $t+1$

- Contrast this with a *realized beta* (akin to a realized variance), which is the *ex post* beta for returns from time t to $t+1$
  - Here, assume t counts months and the realized beta is based on daily data and N_{t+1} is the number of days in month $t+1$:

$$\beta_{i,t+1}^{\text{realized}} = \frac{\sum_{j=1}^{N_{t+1}}\left(R_{m,t+1,j} - \frac{1}{N_{t+1}}\Sigma_{k=1}^{N_{t+1}} R_{m,t+1,k}\right)\left(R_{i,t+1,j} - \frac{1}{N_{t+1}}\Sigma_{k=1}^{N_{t+1}} R_{i,t+1,k}\right)}{\sum_{j=1}^{N_{t+1}}\left(R_{m,t+1,j} - \frac{1}{N_{t+1}}\Sigma_{k=1}^{N_{t+1}} R_{m,t+1,k}\right)^2}$$

Motivation (cont'd)

- Since $\beta_{i,t+1}^{\text{realized}}$ is a regression coefficient (of daily returns on firm i on the market over month $t+1$), it is an unbiased estimator of β_{it} (the true forward-looking beta), assuming the conditional beta is constant for each day of the month.
  - Thus, we can write:
  $$\beta_{i,t+1}^{\text{realized}} = \beta_{it} + \eta_{i,t+1},$$
  where $\eta_{i,t+1}$ is a noise term due to estimation error with standard deviation equal to the standard error of $\beta_{i,t+1}^{\text{realized}}$. The latter is obtained from the within-month daily return regression.
  - If we assume the number of daily returns are sufficient for asymptotic theory to be a decent approximation, the error term is also normally distributed.

- Thus, we can view the realized beta as a noisy, normally distributed signal with mean equal to the true conditional beta.

- Even noisy signals are informative and therefore knowing the realized beta should inform our belief about what the true forward-looking beta was.

Estimation method: Project on lagged instruments

- Below are two ways to proceed in order to estimate β_{it}.

  1. Regress the realized beta at time $t+1$ on a set of variables known at time t over a long sample, $t = 1, ..., T$. Since the noise term $\eta_{i,t+1}$ is a function of data realized after time t, it is uncorrelated with instruments known at time t. Thus, predicted value from this forecasting regression is a way to get an estimate of the current conditional beta.

  2. Use Bayes rule to update your belief about the current conditional beta upon observing the current realized beta.

- In this lecture, we will consider approach 2 in which we are sequentially learning about a latent (hidden) time series variable

Bayes Rule and Updates in Beliefs

- Assume you have a prior about firm beta. For instance:

$$\beta_{it} \sim N\left(1, 0.5^2\right) \quad \text{for all } i$$

- That is, your mean belief about a firm's true β_{it} is that it equals the (value-weighted) market average. You think it is unlikely that betas are less than 0 and greater than 2.
  - This is a reasonable belief if you do not have any other information about the firm.

- Assume you then observe that the realized beta in month t (using daily data realized between time $t-1$ and t was 1.8, with a standard error of 0.4. We will assume the realized beta is normally distributed.
  - How should you optimally update your belief after having received this new information?
  - Use Bayes Rule!

Bayes Rule and Updates in Beliefs

- Bayes Rule:

$$P(A|B)\,P(B) = P(A,B) \Leftrightarrow P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}$$

- So, in our example:

$$P\left(\beta_{it}|\beta_{it}^{\text{realized}}\right) = \frac{P\left(\beta_{it}^{\text{realized}}|\beta_{it}\right)P(\beta_{it})}{P\left(\beta_{it}^{\text{realized}}\right)}$$

The lingo:

- $P(\beta_{it})$ is the **prior distribution** of β_{it}
- $P\left(\beta_{it}|\beta_{it}^{\text{realized}}\right)$ is the **posterior distribution** after having observed the new information
- $P\left(\beta_{it}^{\text{realized}}|\beta_{it}\right)$ is the probability distribution for the observed data conditional on the true value of β_{it}: the **likelihood function**
- $P\left(\beta_{it}^{\text{realized}}\right)$ is the marginal distribution of the data observation, which we do not need to know (more on this in a bit)

Bayes Rule and Updates in Beliefs

- From the previous slide:

$$P\left(\beta_{it}|\beta_{it}^{\text{realized}}\right) \propto P\left(\beta_{it}^{\text{realized}}|\beta_{it}\right) P\left(\beta_{it}\right).$$

Let's do some math!

- A preliminary calculation. Start with two known distributions:

$$\begin{aligned} x &\sim N\left(\mu_X, \sigma_X^2\right) \\ y|x &\sim N\left(x, \sigma_{Y|X}^2\right) \end{aligned}$$

- Here x corresponds to β_{it} and y corresponds to $\beta_{it}^{\text{realized}} = 1.8$
  - Further: μ_X is 1, $\sigma_X^2 = 0.5^2$, $\sigma_{Y|X}^2 = 0.4^2$ (the standard error squared of the realized beta)

Bayes Rule and Updates in Beliefs

- We want to get to the distribution of $x|y$, (really, $\beta_{it}|\beta_{it}^{\text{realized}}$), so let's first multiply these two pdf's:

$$\frac{1}{\sqrt{2\pi\sigma_X^2}}\exp\left\{\frac{(x-\mu_X)^2}{2\sigma_X^2}\right\}\frac{1}{\sqrt{2\pi\sigma_{Y|X}^2}}\exp\left\{\frac{(y-x)^2}{2\sigma_{Y|X}^2}\right\}$$

$$= \frac{1}{2\pi\sqrt{\sigma_X^2\sigma_{Y|X}^2}}\exp\left\{\frac{(x-\mu_X)^2}{2\sigma_X^2}+\frac{(y-x)^2}{2\sigma_{Y|X}^2}\right\}$$

$$= \frac{1}{2\pi\sqrt{\sigma_X^2\sigma_{Y|X}^2}}\exp\left\{\begin{array}{c}\frac{(x-\mu_X)^2}{2\sigma_X^2}\frac{\left(\sigma_X^{-2}+\sigma_{Y|X}^{-2}\right)^{-1}/\sigma_X^2}{\left(\sigma_X^{-2}+\sigma_{Y|X}^{-2}\right)^{-1}/\sigma_X^2}\ldots\\ +\frac{(y-x)^2}{2\sigma_{Y|X}^2}\frac{\left(\sigma_X^{-2}+\sigma_{Y|X}^{-2}\right)^{-1}/\sigma_{Y|X}^2}{\left(\sigma_X^{-2}+\sigma_{Y|X}^{-2}\right)^{-1}/\sigma_{Y|X}^2}\end{array}\right\}$$

Oh yeah... Algebra!

Continuing...

Define $k \equiv \left(\sigma_X^{-2} + \sigma_{Y|X}^{-2}\right)^{-1}$:

$$
\begin{aligned}
& \frac{1}{2\pi\sqrt{\sigma_X^2 \sigma_{Y|X}^2}} \exp\left\{ \frac{(x-\mu_X)^2}{2\sigma_X^2} \frac{k/\sigma_X^2}{k/\sigma_X^2} + \frac{(y-x)^2}{2\sigma_{Y|X}^2} \frac{k/\sigma_{Y|X}^2}{k/\sigma_{Y|X}^2} \right\} \\
= \quad & \frac{1}{2\pi\sqrt{\sigma_X^2 \sigma_{Y|X}^2}} \exp\left\{ \frac{\left(x^2 - 2x\mu_X + \mu_X^2\right) k/\sigma_X^2 + \left(y^2 - 2yx + x^2\right) k/\sigma_{Y|X}^2}{2k} \right\} \\
= \quad & \frac{1}{2\pi\sqrt{\sigma_X^2 \sigma_{Y|X}^2}} \exp\left\{ \frac{x^2 k/\sigma_X^2 - 2x\mu_X k/\sigma_X^2 + \mu_X^2 k/\sigma_X^2 + y^2 k/\sigma_{Y|X}^2 - 2yxk/\sigma_{Y|X}^2 + x^2 k/\sigma_{Y|X}^2}{2k} \right\} \\
= \quad & \frac{1}{2\pi\sqrt{\sigma_X^2 \sigma_{Y|X}^2}} \exp\left\{ \frac{x^2 k \left(\sigma_X^{-2} + \sigma_{Y|X}^{-2}\right) - 2x \left(yk/\sigma_{Y|X}^2 + \mu_X k/\sigma_X^2\right) + \mu_X^2 k/\sigma_X^2 + y^2 k/\sigma_{Y|X}^2}{2k} \right\}
\end{aligned}
$$

Finger-lickin'!

Continuing... ...

Note that $k\left(\sigma_X^{-2} + \sigma_{Y|X}^{-2}\right) = 1$. So:

$$\frac{1}{2\pi\sqrt{\sigma_X^2 \sigma_{Y|X}^2}} \exp\left\{\frac{x^2 - 2x\left(yk/\sigma_{Y|X}^2 + \mu_X k/\sigma_X^2\right) + \mu_X^2 k/\sigma_X^2 + y^2 k/\sigma_{Y|X}^2}{2k}\right\}.$$

Next, complete the square:

$$\frac{1}{\sqrt{2\pi k}} \exp\left\{\frac{\left(x - \left(yk/\sigma_{Y|X}^2 + \mu_X k/\sigma_X^2\right)\right)^2}{2k}\right\}$$

$$\times \frac{\sqrt{2\pi k}}{2\pi\sqrt{\sigma_X^2 \sigma_{Y|X}^2}} \exp\left\{\frac{-\left(yk/\sigma_{Y|X}^2 + \mu_X k/\sigma_X^2\right)^2 + \mu_X^2 k/\sigma_X^2 + y^2 k/\sigma_{Y|X}^2}{2k}\right\}.$$

Note that the first line says $x|y$ is normally distributed with mean $\left(yk/\sigma_{Y|X}^2 + \mu_X k/\sigma_X^2\right)$ and variance k.

- The second line is a constant (not a function of x), conditional on y. Since we only were given the distribution up to a proportion (recall the Bayes Rule equation), we can ignore it for our purposes.

Learning with Normal Distributions

In sum, we are looking for the distribution of x conditional on a data point, y.

We found that $x|y$ is normally distributed using Bayes Rule.

- The mean of this distribution is:

$$\begin{aligned} yk/\sigma^2_{Y|X} + \mu_X k/\sigma^2_X &= y\frac{\sigma^{-2}_{Y|X}}{\sigma^{-2}_X + \sigma^{-2}_{Y|X}} + \mu_X \frac{\sigma^{-2}_X}{\sigma^{-2}_X + \sigma^{-2}_{Y|X}} \\ &= y \times (1 \text{ - weight on prior}) + \mu_X \times (\text{weight on prior}) \end{aligned}$$

  Note that the more precise the signal is (the higher $\sigma^{-2}_{Y|X}$ is) and the less precise the prior is (the lower σ^{-2}_X is), the more weight is given to the signal when updating the mean belief about x.

- The variance is $k = \left(\sigma^{-2}_X + \sigma^{-2}_{Y|X}\right)^{-1} < \sigma^2_X$.

Reverting to our beta-example

Given our prior on the distribution of β_{it} and the observed realized beta, we have that

$$\beta_{it}|\beta_{it}^{\text{realized}} \sim N\left(1.4878, 0.3124^2\right)$$

since

$$\begin{aligned}
y\frac{\sigma_{Y|X}^{-2}}{\sigma_X^{-2}+\sigma_{Y|X}^{-2}} + \mu_X\frac{\sigma_X^{-2}}{\sigma_X^{-2}+\sigma_{Y|X}^{-2}} &= 1.8\frac{0.4^{-2}}{0.5^{-2}+0.4^{-2}} + 1\frac{0.5^{-2}}{0.5^{-2}+0.4^{-2}} \\
&= 1.4878
\end{aligned}$$

and

$$\begin{aligned}
\left(\sigma_X^{-2}+\sigma_{Y|X}^{-2}\right)^{-1} &= \left(0.5^{-2}+0.4^{-2}\right)^{-1} \\
&= 0.3124^2
\end{aligned}$$

Sequential Learning

We update beliefs every time new information arrives

- This is called **sequential learning**

Let y^t denote available data up until and including time t

- Assume we are trying to infer the value of a latent variable, s_t. If the data only provides noisy signals, we are looking for the distribution of $s_t|y^t$.
- Note that $s_t|y^t = s_t|y_t, y^{t-1}$

Sequential learning and Bayes Rule:

$$p\left(s_t|y^t\right) = p\left(s_t|y_t, y^{t-1}\right) \propto p\left(y_t|s_t, y^{t-1}\right) p\left(s_t|y^{t-1}\right)$$

- $p\left(s_t|y^{t-1}\right)$ is the *prior distribution* of the state s_t, conditional on data up until time $t-1$
- $p\left(y_t|s_t, y^{t-1}\right)$ is the *likelihood* of the empirical observation y_t, given your prior
- $p\left(s_t|y^t\right)$ is the *posterior distribution* about the latent state, s_t

Simplest State Space Model

State space models provide an analytically tractable framework for sequential learning about latent variables

- Adds dynamics to the latent variable, a VAR(1) setup in general

The simplest state space model is as follows:

$$\begin{aligned} \text{observation equation} \quad &: \quad y_t = a + b x_t + \varepsilon_t \\ \text{state transition equation} \quad &: \quad x_t = c + d x_{t-1} + \eta_t \end{aligned}$$

where $\varepsilon_t \sim N\left(0, \sigma_\varepsilon^2\right)$, $\eta_t \sim N\left(0, \sigma_\eta^2\right)$, and $E\left[\eta_t \varepsilon_t\right] = 0$.

- The latent state, x_t, is hidden (unobserved).
  - E.g., the true beta or the state of the economy
- The data gives us y_t.
- We will consider the general specification later in this lecture note

Sequential Learning in a State Space Model

Let's say our current (prior) belief about the current value of x_t is:

$$p\left(x_t | y^t\right) = N\left(\hat{x}_t, \sigma^2_{\hat{x},t}\right).$$

- Then:

$$p\left(x_{t+1} | y_{t+1}, y^t\right) \propto p\left(y_{t+1} | x_{t+1}, y^t\right) p\left(x_{t+1} | y^t\right)$$

In this model, we have that:

$$\begin{aligned} p\left(x_{t+1} | y^t\right) &= N\left(c + d\hat{x}_t, \sigma^2_{\hat{\eta},t}\right), \\ p\left(y_{t+1} | x_{t+1}, y^t\right) &= N\left(a + b\hat{x}_t, \sigma^2_{\hat{\varepsilon},t}\right), \end{aligned}$$

where $E\left[x_t | y^t\right] = \hat{x}_t$, $\sigma^2_{\hat{\varepsilon},t} = var_t\left(y_{t+1} - a - b\hat{x}_t\right)$, and $\sigma^2_{\hat{\eta},t} = var_t\left(x_{t+1} - c - d\hat{x}_t\right)$.

Sequential Learning in a State Space Model

The posterior is then:

$$p\left(x_{t+1} | y_{t+1}, y^{t}\right) = p\left(x_{t+1} | y^{t+1}\right)$$

$$= N\left(\frac{y_{t+1}-a}{b}\frac{\sigma_{\hat{\varepsilon},t}^{-2}}{\sigma_{\hat{\eta},t}^{-2}+\sigma_{\hat{\varepsilon},t}^{-2}} + \left(c + d\hat{x}_t\right)\frac{\sigma_{\hat{\eta},t}^{-2}}{\sigma_{\hat{\eta},t}^{-2}+\sigma_{\hat{\varepsilon},t}^{-2}}, \quad \left(\sigma_{\hat{\eta},t}^{-2}+\sigma_{\hat{\varepsilon},t}^{-2}\right)^{-1}\right)$$

Thus, the prior expectation of x_{t+1} is $c + d\hat{x}_t$, whereas the updated expectation is

$$\begin{aligned} \hat{x}_{t+1} &= \frac{y_{t+1}-a}{b}\frac{\sigma_{\hat{\varepsilon},t}^{-2}}{\sigma_{\hat{\eta},t}^{-2}+\sigma_{\hat{\varepsilon},t}^{-2}} + \left(c + d\hat{x}_t\right)\frac{\sigma_{\hat{\eta},t}^{-2}}{\sigma_{\hat{\eta},t}^{-2}+\sigma_{\hat{\varepsilon},t}^{-2}} \\ &= \text{signal} \times (\text{1 - weight on prior}) \\ &\quad + \text{prior mean} \times (\text{weight on prior}) \end{aligned}$$

Note that you will never learn the true value of x_t at any time given the presence of ε_{t+1} and η_{t+1}

Why State Space Models?

Quite general:

1. Vector representation allowing a set of observation equations and a VAR for the state transition equations.
2. Can accommodate missing information (set $\sigma^2_{\varepsilon,t} = \infty$ for observations of y_t that are missing).
3. Well-developed both in terms of theory and code: a workhorse model
4. VARs, ARMAs, $+++$: Many known models can be mapped into the State Space Model framework
5. Maximum likelihood estimation is standard and already coded up for most applications

Great reference (in addition to Hamilton; more in depth on State Space Model)

- Durbin and Koopman: "Time Series Analysis by State Space Models"

Remainder of lecture note goes over the State Space Model and the Kalman Filter in more detail

- The Kalman Filter is a set of matrix equations that operationalizes the sequential learning

State Space Models and the Kalman Filter in Detail

Local Level Model

Consider the **local level model**, for $t = 1, \ldots, T$:

$$\begin{aligned} y_t &= \mu_t + \eta_t, \qquad \eta_t \sim \mathrm{N}(0, \sigma_\eta^2) \\ \mu_{t+1} &= \mu_t + \varepsilon_{t+1}, \qquad \varepsilon_{t+1} \sim \mathrm{N}(0, \sigma_\varepsilon^2) \end{aligned}$$

- We observe the data y_t for $t = 1, \ldots, T$
- But! The trend μ_t is not observable. It is latent.
- The Kalman filter provides a way to estimate the trend μ_t.
- The trend μ_t is also known as a **state variable**.
- In this lecture, I will use the notation: $y_{1:t} = (y_1, \ldots, y_t)$

Local Level Model

$$
\begin{aligned}
y_t &= \mu_t + \eta_t, \qquad \eta_t \sim \mathrm{N}(0, \sigma_\eta^2) \\
\mu_{t+1} &= \mu_t + \varepsilon_{t+1}, \qquad \varepsilon_{t+1} \sim \mathrm{N}(0, \sigma_\varepsilon^2)
\end{aligned}
$$

- The Kalman filter, Kalman (1960), is a recursive algorithm that performs:
  1. Filtering
  2. Prediction
  3. Smoothing
  4. Evaluation of the log-likelihood
- Let's introduce each of these ideas one-by-one.
- For now, assume we know the parameters: $\boldsymbol{\theta} = (\sigma_\eta^2, \sigma_\varepsilon^2)$

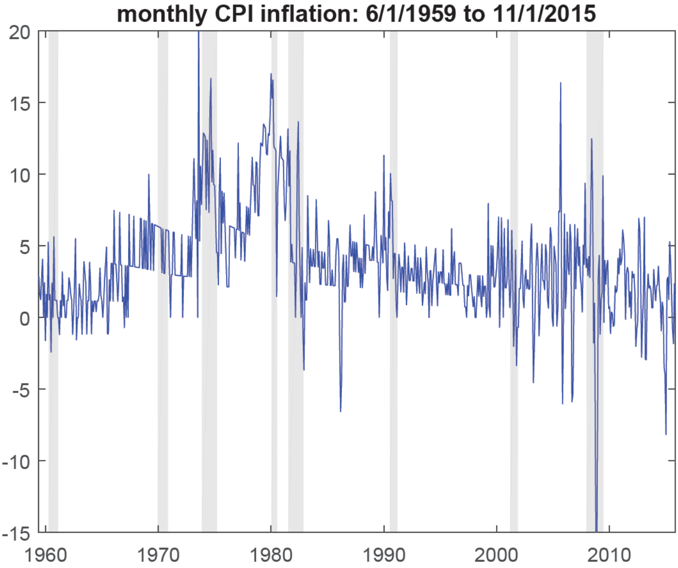
U.S. inflation

- Let $y_t = \pi_t$ denote monthly U.S. CPI inflation
- Stock and Watson (2007) use the local level model to forecast inflation.

$$
\begin{aligned}
\pi_t &= \bar{\pi}_t + \eta_t, \qquad \eta_t \sim \mathrm{N}(0, \sigma_\eta^2) \\
\bar{\pi}_{t+1} &= \bar{\pi}_t + \varepsilon_{t+1}, \qquad \varepsilon_{t+1} \sim \mathrm{N}(0, \sigma_\varepsilon^2)
\end{aligned}
$$

- The state variable $\mu_t = \bar{\pi}_t$ represents the trend in inflation and is what many economists would call '*expected inflation*.'

U.S. inflation



U.S. inflation, monthly continuously compounded.

Bayes Theorem

- Let's start at time $t = 0$ before we have seen the first observation y_1.
- Let $p(\mu_1; \boldsymbol{\theta})$ denote a prior distribution that describes our beliefs about μ_1 before we have observed y_1.
- Let's assume that our prior $p(\mu_1; \boldsymbol{\theta})$ is a normal distribution

$$p(\mu_1; \boldsymbol{\theta}) \quad = \quad \mathsf{N}(\mu_{1|0}, \Sigma_{1|0})$$

- The notation $\mu_{1|0}$ means that it is our guess (prediction) of the value of μ_1 at time $t = 1$ but with time $t = 0$ information.
- The variance $\Sigma_{1|0}$ measures our uncertainty about μ_1 with time $t = 0$ information.
- Recall that, for the normal distribution, the mean and covariance matrix are **sufficient statistics**.

Bayes Theorem

- We have the prior: $p(\mu_1; \boldsymbol{\theta}) \; = \; \mathsf{N}(\mu_{1|0}, \Sigma_{1|0})$.
- We observe y_1 which is a noisy measure of μ_1.
- Our model says y_1 and μ_1 are related:

$$y_1 = \mu_1 + \eta_1, \qquad \eta_1 \sim \mathsf{N}(0, \sigma_\eta^2)$$

- The (conditional) likelihood is:

$$p(y_1 | \mu_1; \boldsymbol{\theta}) = \mathsf{N}(\mu_1, \sigma_\eta^2)$$

Bayes Theorem

- We have the prior: $p(\mu_1; \boldsymbol{\theta}) \;=\; \mathsf{N}(\mu_{1|0}, \Sigma_{1|0})$
- The (conditional) likelihood: $p(y_1|\mu_1; \boldsymbol{\theta}) \;=\; \mathsf{N}(\mu_1, \sigma_\eta^2)$
- After we observe the data y_1, how do we revise our beliefs about μ_1?
- We apply Bayes rule:

$$p(\mu_1|y_1; \boldsymbol{\theta}) \quad = \quad \frac{p(y_1|\mu_1; \boldsymbol{\theta})p(\mu_1; \boldsymbol{\theta})}{p(y_1; \boldsymbol{\theta})}$$

- The posterior $p(\mu_1|y_1; \boldsymbol{\theta})$ describes our beliefs about μ_1 after we observe y_1.

Filtering: local level model

- Applying Bayes Rule, we find

$$p(\mu_1|y_1;\boldsymbol{\theta}) \;=\; \frac{p(y_1|\mu_1;\boldsymbol{\theta})p(\mu_1;\boldsymbol{\theta})}{p(y_1;\boldsymbol{\theta})} \;=\; \mathrm{N}(\mu_{1|1},\Sigma_{1|1})$$

  where the mean and variance are

$$\mu_{1|1} \;=\; \mu_{1|0} + \Sigma_{1|0}F_1^{-1}(y_1 - \mu_{1|0}) \qquad \Sigma_{1|1} \;=\; \Sigma_{1|0} - \Sigma_{1|0}F_1^{-1}\Sigma_{1|0}$$

  and $F_1 = \Sigma_{1|0} + \sigma_\eta^2$.

- The notation $\mu_{1|1}$ indicates our estimate for μ_1 given $t = 1$ information.
- All distributions are normally distributed. Only update the mean and covariance matrix (the sufficient statistics).

One-step Ahead Prediction

- Suppose we want to forecast the value of μ_2 given data up to y_1.
- We want the one-step ahead predictive distribution: $p(\mu_2 | y_1; \boldsymbol{\theta})$
- The predictive distribution $p(\mu_2 | y_1; \boldsymbol{\theta})$ describes our uncertainty about μ_2 given we observe y_1.
- Our model says μ_2 and μ_1 are related:

$$\mu_2 \quad = \quad \mu_1 + \varepsilon_2, \qquad \varepsilon_2 \sim \mathsf{N}(0, \sigma_\varepsilon^2)$$

- This defines the (Markov) transition density

$$p(\mu_2 | \mu_1; \boldsymbol{\theta}) \quad = \quad \mathsf{N}(\mu_1, \sigma_\varepsilon^2)$$

One-step Ahead Prediction: local level model

- To calculate $p(\mu_2|y_1;\boldsymbol{\theta})$, we integrate out μ_1 by

$$p(\mu_2|y_1;\boldsymbol{\theta}) \quad = \quad \int p(\mu_2|\mu_1;\boldsymbol{\theta})p(\mu_1|y_1;\boldsymbol{\theta})d\mu_1$$

- Since all distributions are Gaussian, integral can be solved analytically

$$p(\mu_2|y_1;\boldsymbol{\theta}) = \mathsf{N}(\mu_{2|1}, \Sigma_{2|1})$$

where the mean and variance are

$$\mu_{2|1} \ = \ \mu_{1|1} \qquad \Sigma_{2|1} \ = \ \Sigma_{1|1} + \sigma_\varepsilon^2$$

- We just update the sufficient statistics.
- At time $t = 2$, this is our new prior: $p(\mu_2|y_1;\boldsymbol{\theta}) = \mathsf{N}(\mu_{2|1}, \Sigma_{2|1})$.

Filtering and one-step ahead prediction

- For $t = 2, \ldots, T$, we recursively repeat the two steps:

  Filtering

  $$p(\mu_t | y_1, \ldots, y_t; \boldsymbol{\theta}) = \frac{p(y_t | \mu_t; \boldsymbol{\theta}) p(\mu_t | y_{1:t-1}; \boldsymbol{\theta})}{p(y_{1:t}; \boldsymbol{\theta})}$$

  One-step ahead prediction

  $$p(\mu_{t+1} | y_{1:t}; \boldsymbol{\theta}) = \int p(\mu_{t+1} | \mu_t; \boldsymbol{\theta}) p(\mu_t | y_{1:t}; \boldsymbol{\theta}) d\mu_t$$

- The predictive distribution $p(\mu_{t+1} | y_{1:t}; \boldsymbol{\theta})$ is the prior at the next iteration, i.e. at time $t+1$

The Kalman filter for the Local Level Model

- The Kalman filter recursively calculates these two steps
- Start with the initial conditions $\mu_{1|0}$ and $\Sigma_{1|0}$
- For $t = 1, \ldots, T$

$$
\begin{aligned}
v_t &= y_t - \mu_{t|t-1}, \\
F_t &= \Sigma_{t|t-1} + \sigma_\eta^2, \\
K_t &= \Sigma_{t|t-1} / F_t, \\
\mu_{t|t} &= \mu_{t|t-1} + K_t v_t, \qquad \text{Filter step} \\
\Sigma_{t|t} &= \Sigma_{t|t-1} - K_t \Sigma_{t|t-1}
\end{aligned}
$$

$$
\begin{aligned}
\mu_{t+1|t} &= \mu_{t|t}, \qquad \text{Prediction step} \\
\Sigma_{t+1|t} &= \Sigma_{t|t} + \sigma_\varepsilon^2.
\end{aligned}
$$

Filtering and one-step ahead prediction

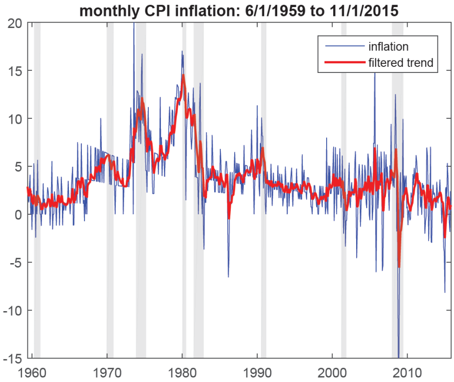
- These distributions describe our uncertainty about μ_t conditional on different information sets.
- All distributions are Gaussian!
- Only calculate means and covariance matrices (sufficient statistics)
- The filtering distribution

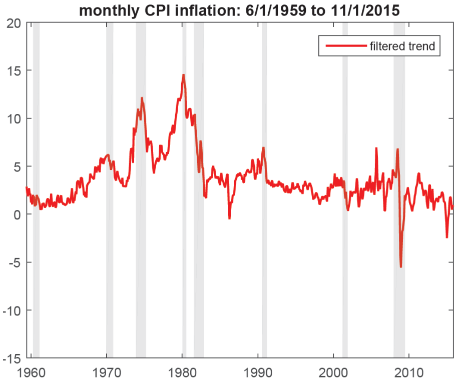
$$p(\mu_t | y_1, \ldots, y_t; \boldsymbol{\theta}) \quad = \quad \mathsf{N}(\mu_{t|t}, \Sigma_{t|t}) \qquad t = 1, \ldots, T$$

- One-step ahead predictive distribution

$$p(\mu_{t+1} | y_1, \ldots, y_t; \boldsymbol{\theta}) \quad = \quad \mathsf{N}(\mu_{t+1|t}, \Sigma_{t+1|t}) \qquad t = 0, \ldots, T$$

U.S. inflation





U.S. inflation, monthly continuously compounded. Left: inflation & filtered trend. Right: filtered trend.

Linear, Gaussian State Space Models

- Linear, Gaussian state space models are widely applicable
- For example, the following models can be placed in **state space form**
  1. local level model
  2. AR(1) observed in noise
  3. ARMA(p, q) models
  4. VAR(p) models
  5. Linear regression with serially correlated errors
  6. Time-varying parameter models
  7. Structural time series models
  8. Many more!
- First, we need to generalize the model.

Linear, Gaussian state space models

Definition

A **linear, Gaussian state space model** has observation equation

$$y_t = Z_t \alpha_t + d_t + \eta_t \qquad \eta_t \sim \mathrm{N}(0, H_t)$$

state transition equation

$$\alpha_{t+1} = T_t \alpha_t + c_t + R_t \varepsilon_{t+1} \qquad \varepsilon_{t+1} \sim \mathrm{N}(0, Q_t)$$

and initial conditions

$$\alpha_1 \sim \mathrm{N}(a_{1|0}, P_{1|0}).$$

-
- The **state variable** or **state vector** is α_t.
- The **system matrices** $Z_t, d_t, H_t, T_t, c_t, R_t, Q_t$ are often time invariant: Z, d, H, T, c, R, Q.

Local level model

Consider the local level model

$$
\begin{aligned}
y_t &= \mu_t + \eta_t, \qquad \eta_t \sim \mathrm{N}(0, \sigma_\eta^2) \\
\mu_{t+1} &= \mu_t + \varepsilon_{t+1}, \qquad \varepsilon_{t+1} \sim \mathrm{N}(0, \sigma_\varepsilon^2)
\end{aligned}
$$

The model can be written in state space form as:

$$
\alpha_t = \mu_t, \qquad Z = 1 \qquad d = 0 \qquad H = \sigma_\eta^2
$$

$$
T = 1 \qquad c = 0 \qquad R = 1 \qquad Q = \sigma_\varepsilon^2
$$

AR(1) observed in noise

$$
\begin{aligned}
y_t &= \mu_t + \eta_t & \eta_t \sim \mathrm{N}(0, \sigma_\eta^2) \\
\mu_{t+1} &= \phi_0 + \phi_1 \mu_t + \varepsilon_{t+1} & \varepsilon_{t+1} \sim \mathrm{N}(0, \sigma_\varepsilon^2)
\end{aligned}
$$

The model can be written in state space form as:

$$
\alpha_t = \mu_t \qquad Z = 1 \qquad d = 0 \qquad H = \sigma_\eta^2
$$

$$
T = \phi_1 \qquad c = \phi_0 \qquad R = 1 \qquad Q = \sigma_\varepsilon^2
$$

(Note: the local level model sets $\phi_0 = 0$ and $\phi_1 = 1$.)

ARMA(3,2) (version 1)

$$\begin{aligned} y_t &= \mu + \phi_1(y_{t-1} - \mu) + \phi_2(y_{t-2} - \mu) + \phi_3(y_{t-3} - \mu) + \varepsilon_t \\ &\quad + \vartheta_1 \varepsilon_{t-1} + \vartheta_2 \varepsilon_{t-2} \qquad \varepsilon_t \sim \mathrm{N}(0, \sigma_\varepsilon^2) \end{aligned}$$

Let $\phi_0 = (1 - \phi_1 - \phi_2 - \phi_3)\mu$. A state space form is:

$$Z = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad d = 0 \qquad H = 0 \qquad Q = \sigma_\varepsilon^2$$

$$\alpha_t = \begin{pmatrix} y_t \\ y_{t-1} \\ y_{t-2} \\ \varepsilon_t \\ \varepsilon_{t-1} \end{pmatrix} \; T = \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 & \vartheta_1 & \vartheta_2 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \; c = \begin{pmatrix} \phi_0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \; R = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

ARMA(3,2) (version 2)

$$\begin{aligned} y_t &= \mu + \phi_1(y_{t-1} - \mu) + \phi_2(y_{t-2} - \mu) + \phi_3(y_{t-3} - \mu) + \varepsilon_t \\ &\quad + \vartheta_1 \varepsilon_{t-1} + \vartheta_2 \varepsilon_{t-2} \qquad \varepsilon_t \sim \mathrm{N}(0, \sigma_\varepsilon^2) \end{aligned}$$

Let $\phi_0 = (1 - \phi_1 - \phi_2 - \phi_3)\mu$. An alternative state space form is:

$$Z = \begin{pmatrix} 1 & 0 & \ldots & 0 \end{pmatrix} \qquad d = 0 \qquad H = 0 \qquad Q = \sigma_\varepsilon^2$$

$$\alpha_t = \begin{pmatrix} y_t \\ \phi_2 y_{t-1} + \phi_3 y_{t-2} + \vartheta_1 \varepsilon_t + \vartheta_2 \varepsilon_{t-1} \\ \phi_3 y_{t-1} + \vartheta_2 \varepsilon_t \end{pmatrix}$$

$$T = \begin{pmatrix} \phi_1 & 1 & 0 \\ \phi_2 & 0 & 1 \\ \phi_3 & 0 & 0 \end{pmatrix} \qquad c = \begin{pmatrix} \phi_0 \\ 0 \\ 0 \end{pmatrix} \qquad R = \begin{pmatrix} 1 \\ \vartheta_1 \\ \vartheta_2 \end{pmatrix}$$

Note: the dimension of α_t is smaller than the last slide.

Remarks

- As the example of an ARMA$(3, 2)$ shows us, the state space form of a model is not unique.
- There are multiple ways to place the same model in state space form. The definition of the 'state vector' is not necessarily the same in each case.
- The ARMA$(3, 2)$ example also shows us that the state variable α_t is not always a latent variable.
- For examples on how to write ARMA(p, q) models in state space form; see Hamilton (1994) or Durbin and Koopman (2012).

$$y_t = \Phi_0 + \Phi_1 y_{t-1} + \ldots + \Phi_p y_{t-p} + \varepsilon_t \qquad \varepsilon_t \sim \mathrm{N}(0, \Sigma_\varepsilon)$$

The model can be written in state space form as:

$$Z = \begin{pmatrix} I & 0 & \ldots & 0 \end{pmatrix} \qquad d = 0 \qquad H = 0 \qquad Q = \Sigma_\varepsilon$$

$$\alpha_t = \begin{pmatrix} y_t \\ y_{t-1} \\ \vdots \\ y_{t-p} \end{pmatrix}, \; T = \begin{pmatrix} \Phi_1 & \Phi_2 & \ldots & \Phi_p \\ I & 0 & \ldots & 0 \\ 0 & I & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & I & 0 \end{pmatrix}, \; c = \begin{pmatrix} \Phi_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \; R = \begin{pmatrix} I \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Linear regression with serially correlated errors

$$
\begin{aligned}
y_t &= X_t\beta + \epsilon_t \\
\epsilon_{t+1} &= \phi\epsilon_t + \varepsilon_{t+1} \qquad \varepsilon_t \sim \mathrm{N}(0, \sigma_\varepsilon^2)
\end{aligned}
$$

The model can be written in state space form as:

$$
\alpha_t = \epsilon_t \qquad Z = 1 \qquad d_t = X_t\beta \qquad H = 0
$$

$$
T = \phi \qquad c = 0 \qquad R = 1 \qquad Q = \sigma_\varepsilon^2
$$

Time-varying parameter models

$$\begin{aligned} y_t &= X_t \beta_t + \eta_t & \eta_t \sim \mathrm{N}(0, \Omega) \\ \beta_{t+1} &= \Phi_0 + \Phi_1 \beta_t + \varepsilon_{t+1} & \varepsilon_t \sim \mathrm{N}(0, \Sigma_\varepsilon) \end{aligned}$$

The model can be written in state space form as:

$$\begin{array}{cccc} \alpha_t = \beta_t & Z_t = X_t & d = 0 & H = \Omega \\ T = \Phi_1 & c = \Phi_0 & R = I & Q = \Sigma_\varepsilon \end{array}$$

Note: a special case of this is a CAPM with time-varying β.

Important things to remember

- There is more than one way to place a model in **state space form**
- Consequently, the definition of the '**state variable**' may change depending on how you do it.
- No matter how you place a model in **state space form** some things will not change:
  1. forecasts of future data y_{t+h} for $h > 0$
  2. the log-likelihood of the model

The Kalman filter calculates....

- The filtering distribution

$$p(\alpha_t | y_{1:t}; \boldsymbol{\theta}) = \frac{p(y_t | \alpha_t; \boldsymbol{\theta}) p(\alpha_t | y_{1:t-1} \boldsymbol{\theta})}{p(y_{1:t}; \boldsymbol{\theta})}$$

- One-step ahead predictive distribution

$$p(\alpha_{t+1} | y_{1:t}; \boldsymbol{\theta}) = \int p(\alpha_{t+1} | \alpha_t; \boldsymbol{\theta}) p(\alpha_t | y_{1:t}; \boldsymbol{\theta}) d\alpha_t$$

- All distributions are Gaussian! We only need their means and covariance matrices (their sufficient statistics).

- Our notation:

$$\begin{aligned} p(\alpha_t | y_{1:t}; \boldsymbol{\theta}) &= \mathrm{N}(a_{t|t}, P_{t|t}) \\ p(\alpha_{t+1} | y_{1:t}; \boldsymbol{\theta}) &= \mathrm{N}(a_{t+1|t}, P_{t+1|t}) \end{aligned}$$

The Kalman Filter

- The Kalman filter recursively calculates these two steps.
- Start with the initial conditions $a_{1|0}$ and $P_{1|0}$.
- For $t = 1, \ldots, T$

$$
\begin{aligned}
v_t &= y_t - Z_t a_{t|t-1} - d_t, \\
F_t &= Z_t P_{t|t-1} Z_t' + H_t, \\
K_t &= P_{t|t-1} Z_t' F_t^{-1}, \\
a_{t|t} &= a_{t|t-1} + K_t v_t, \qquad \text{Filter step} \\
P_{t|t} &= P_{t|t-1} - K_t Z_t P_{t|t-1},
\end{aligned}
$$

$$
\begin{aligned}
a_{t+1|t} &= T_t a_{t|t} + c_t, \qquad \text{Prediction step} \\
P_{t+1|t} &= T_t P_{t|t} T_t' + R_t Q_t R_t'
\end{aligned}
$$

Prediction and then filtering

- Some researchers reverse the order of the steps.
- Start with the initial (filtering) conditions $a_{0|0}$ and $P_{0|0}$.
- For $t = 1, \ldots, T$

$$\begin{aligned} a_{t|t-1} &= T_{t-1} a_{t-1|t-1} + c_{t-1}, \qquad \text{Prediction step} \\ P_{t|t-1} &= T_{t-1} P_{t-1|t-1} T'_{t-1} + R_{t-1} Q_{t-1} R'_{t-1} \end{aligned}$$

$$\begin{aligned} v_t &= y_t - Z_t a_{t|t-1} - d_t, \\ F_t &= Z_t P_{t|t-1} Z'_t + H_t, \\ K_t &= P_{t|t-1} Z'_t F_t^{-1}, \\ a_{t|t} &= a_{t|t-1} + K_t v_t, \qquad \text{Filter step} \\ P_{t|t} &= P_{t|t-1} - K_t Z_t P_{t|t-1}, \end{aligned}$$

The Kalman Predictor

- **Kalman predictor**: The filtered values $a_{t|t}$ and $P_{t|t}$ are never calculated.
- Start with the initial conditions $a_{1|0}$ and $P_{1|0}$.
- For $t = 1, \ldots, T$

$$
\begin{aligned}
v_t &= y_t - Z_t a_{t|t-1} - d_t, \\
F_t &= Z_t P_{t|t-1} Z_t' + H_t, \\
M_t &= T_t P_{t|t-1} Z_t' F_t^{-1}, \\
L_t &= T_t - M_t Z_t, \\
a_{t+1|t} &= T_t a_{t|t-1} + c_t + M_t v_t, \\
P_{t+1|t} &= T_t P_{t|t-1} L_t' + R_t Q_t R_t'
\end{aligned}
$$

- Computationally faster because we $a_{t|t}$ and $P_{t|t}$ are not calculated.

Initializing the Kalman filter

- We need values for $a_{1|0}$ and $P_{1|0}$ to start the Kalman filtering recursions.
- There are many suggestions in the literature for how to choose these values. **You should think of them as part of your model!**
- In practice, we encounter two common situations:
  1. α_t is stationary
  2. α_t is non-stationary
- Case 1 is pretty easy. Case 2 is not.
- In the literature, $a_{1|0}$ and $P_{1|0}$ are often called **initial conditions**.

Stationarity of the state equation

- Suppose the state equation α_t is a stationary process.
- This means there exists a stationary (marginal) distribution $\mathrm{N}(\mu_\alpha, V_\alpha)$.
- Stationarity means that all the eigenvalues of the matrix T are inside the unit circle.
- Explain: you can check this condition by taking an eigendecomposition of T.

Initializing the Kalman filter (stationary case)

- Suppose the state equation α_t is a stationary process.
- Let $\mathrm{N}(\mu_\alpha, V_\alpha)$ denote the stationary distribution of α_t.
- Taking unconditional expectations, we find

$$\begin{aligned} E[\alpha_{t+1}] &= TE[\alpha_t] + c + RE[\varepsilon_t] \\ \mu_\alpha &= T\mu_\alpha + c \\ \\ \Rightarrow \mu_\alpha &= (I - T)^{-1} c \end{aligned}$$

- If we use the stationary distribution, we set the mean to be

$$a_{1|0} = \mu_\alpha$$

Initializing the Kalman filter (stationary case)

- Let $\mathrm{N}(\mu_\alpha, V_\alpha)$ denote the stationary distribution of α_t.
- Taking unconditional variances, we find

$$\begin{aligned}
V[\alpha_{t+1}] &= TV[\alpha_t]T' + RV[\varepsilon_t]R' \\
V_\alpha &= TV_\alpha T' + RQR' \\
\mathrm{vec}(V_\alpha) &= \mathrm{vec}(TV_\alpha T') + \mathrm{vec}(RQR') \\
\mathrm{vec}(V_\alpha) &= (T \otimes T)\mathrm{vec}(V_\alpha) + \mathrm{vec}(RQR') \\
\\
\Rightarrow \mathrm{vec}(V_\alpha) &= [I - (T \otimes T)]^{-1}\mathrm{vec}(RQR')
\end{aligned}$$

- If we use the stationary distribution, we set the covariance matrix as

$$P_{1|0} = V_\alpha$$

Initializing the Kalman filter (non-stationary)

- In some models of interest, elements of the state vector α_t are non-stationary.
- These models have unit roots; e.g. the local level model.
- The stationary distribution of α_t does not exist.
- The simple way to initialize the Kalman filter is set the variance $P_{1|0}$ to a really large number; e.g. 10^4;
- Koopman (1997) gives an exact initialization. Effectively, it calculates a conditional log-likelihood function that drops the initial parts of the likelihood. Not easy to implement though. See also Chapter 5 of Durbin and Koopman (2012).

Impact of the initial conditions

- If the model is stationary, the **initial conditions** $a_{1|0}$ and $P_{1|0}$ typically do not have a large influence on the results.
- Filtered estimates $a_{t|t}$ and $P_{t|t}$ will converge to the same thing (they are equal) even if we start the Kalman filter from different initial conditions!!
- This is due to the stationarity of the model.
- The early estimates $a_{t|t}$ and $P_{t|t}$ will be different during the first few iterations: $t < 20$ or so.
- For non-stationary models, the initial conditions can have an impact if the overall (time-series) sample size T is small.

Smoothing distributions

- During the forwards pass of the Kalman filter we calculate the filtering and one-step ahead predictive distributions.

$$
\begin{aligned}
p(\alpha_t | y_{1:t}; \boldsymbol{\theta}) &= \mathrm{N}(a_{t|t}, P_{t|t}) \\
p(\alpha_{t+1} | y_{1:t}; \boldsymbol{\theta}) &= \mathrm{N}(a_{t+1|t}, P_{t+1|t})
\end{aligned}
$$

  for $t = 1, \ldots, T$

- These distributions describe our uncertainty about the state α_t conditional on different information sets.

- For many time series models, there is still uncertainty about the state vector α_t even after we observe **all** the data.

$$
p(\alpha_t | y_1, \ldots, y_T; \boldsymbol{\theta})
$$

- This is called the **smoothed distribution**.

Smoothing distributions

- At time $t = T$, the filtered and smoothed distributions are equal!

$$p(\alpha_T | y_{1:T}; \boldsymbol{\theta}) \quad = \quad \mathrm{N}(a_{T|T}, P_{T|T})$$

- At $t = T$, we know the mean $a_{T|T}$ and covariance matrix $P_{T|T}$.
- We can write the sufficient statistics $a_{t|T}, P_{t|T}$ as a recursive function of $a_{t+1|T}, P_{t+1|T}$.
- The **Kalman smoother** recursively calculates the smoothing distributions backwards

$$p(\alpha_t | y_{1:T}; \boldsymbol{\theta}) \quad = \quad \mathrm{N}(a_{t|T}, P_{t|T})$$

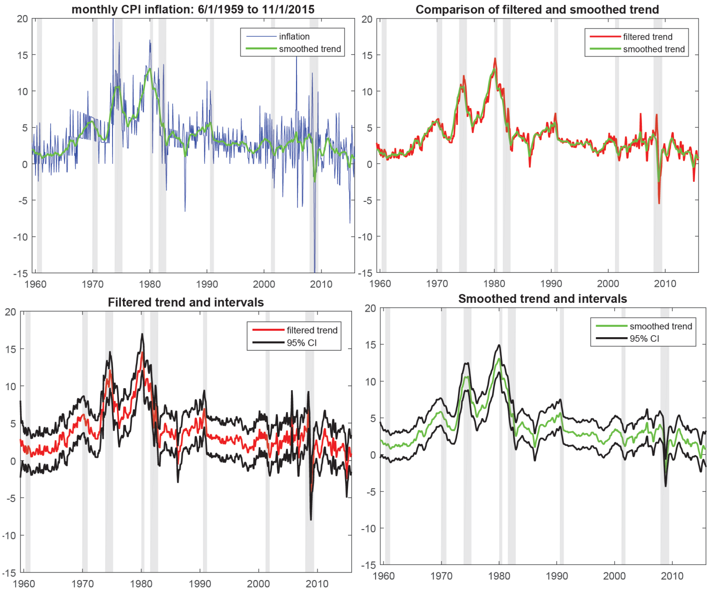
for $t = T - 1, \ldots, 1$

Kalman smoother (Rauch-Tung-Striebel)

- Run the Kalman filter forward in time for $t = 1, \ldots, T$.
- Store the quantities $\left\{a_{t+1|t}, P_{t+1|t}, v_t, F_t, L_t\right\}_{t=1}^{T}$.
- Set $r_{T+1} = 0$ and $N_{T+1} = 0$
- For $t = T, \ldots, 1$

$$
\begin{aligned}
r_t &= Z_t' F_t^{-1} v_t + L_t r_{t+1} \\
N_t &= Z_t' F_t^{-1} Z_t + L_t' N_{t+1} L_t \\
a_{t|T} &= a_{t+1|t} + P_{t+1|t} r_t \\
P_{t|T} &= P_{t+1|t} - P_{t+1|t} N_t P_{t+1|t}
\end{aligned}
$$

Example: U. S. inflation



U.S. inflation, monthly continuously compounded.

Comments on the Kalman smoother

- There are different versions of the **Kalman smoother**.
- All of them compute the means $a_{t|T}$ and covariance matrices $P_{t|T}$.
- They differ depending on what you store in computer memory on the forward pass of the Kalman filter.
- In the statistics/econometrics/engineering literature, the terms '**filtering**' and '**smoothing**' are used in different ways.
  - in state space models, filtering means to use data only up to time t.
  - in state space models, smoothing means to use ALL the data T.
  - however, different parts of science use the terms '**filtering**' and '**smoothing**' to mean other things.

Forecasting

- We often want to forecast:
  1. the state variable h steps ahead: α_{t+h}
  2. future data h steps ahead: y_{t+h}
- We also want to characterize our uncertainty of these variables.
- This is easy to do in a state space model.

Forecasting the state variable

- To forecast the state variable α_t at time $t+h$, we need to calculate the predictive distribution.

$$p(\alpha_{t+h}|y_{1:t};\boldsymbol{\theta})$$

- Under the assumption that the errors η_t and ε_t are Gaussian, the predictive distribution is Gaussian.

$$p(\alpha_{t+h}|y_{1:t};\boldsymbol{\theta}) \;=\; \mathrm{N}(a_{t+h|t}, P_{t+h|t})$$

- We can calculate the mean $a_{t+h|t}$ and covariance matrix $P_{t+h|t}$.
- These can be calculated recursively: $t+1$, then $t+2$, then...$t+h$

Forecasting the state variable

- Let's assume that the model is time-invariant. The **system matrices** are constant

$$Z,\ d,\ H,\ T,\ c,\ R,\ Q$$

- At the end of the Kalman filter, we already have

$$p(\alpha_{t+1}|y_{1:t};\boldsymbol{\theta}) \ = \ \mathrm{N}(a_{t+1|t}, P_{t+1|t})$$

- The next predictive distribution is:

$$p(\alpha_{t+2}|y_{1:t};\boldsymbol{\theta}) \ = \ \int p(\alpha_{t+2}|\alpha_{t+1};\boldsymbol{\theta})p(\alpha_{t+1}|y_{1:t};\boldsymbol{\theta})d\alpha_{t+1}$$

- To calculate $a_{t+2|t}, P_{t+2|t}$, we apply the recursion

$$\begin{aligned} a_{t+2|t} &= Ta_{t+1|t} + c, \\ P_{t+2|t} &= TP_{t+1|t}T' + RQR' \end{aligned}$$

Forecasting the state variable

- To calculate, these quantities at longer horizons, we simply iterate

$$
\begin{aligned}
a_{t+h|t} &= Ta_{t+h-1|t} + c, \\
P_{t+h|t} &= TP_{t+h-1|t}T' + RQR'
\end{aligned}
$$

- If the system matrices are time-varying, you need to know their future values:

$$
H_{t+j}, T_{t+j}, c_{t+j}, R_{t+j}, Q_{t+j} \qquad j = 0, \ldots, h-1
$$

Forecasting future data

- Let's assume that the model is time-invariant.
- To forecast future data, we write the model h-steps ahead

$$y_{t+h} = Z\alpha_{t+h} + d + \eta_{t+h}$$

- Take the conditional expectation of both sides:

$$\begin{aligned} E_t[y_{t+h}] &= ZE_t[\alpha_{t+h}] + d + E_t[\eta_{t+h}] \\ &= Z\ a_{t+h|t} + d \end{aligned}$$

- The predicted value is:

$$y_{t+h|t} = Z\ a_{t+h|t} + d$$

- We just showed how to calculate $a_{t+h|t}$!

Forecasting future data

- To forecast future data, we write the model h-steps ahead

$$y_{t+h} = Z\alpha_{t+h} + d + \eta_{t+h}$$

- Take the conditional variance of both sides:

$$\begin{aligned} V_t[y_{t+h}] &= ZV_t[\alpha_{t+h}]Z' + d + V_t[\eta_{t+h}] \\ &= ZP_{t+h|t}Z' + H \end{aligned}$$

- We just showed how to calculate $P_{t+h|t}$!
- We can construct 95% forecasting intervals using the mean and covariance.

Forecasting future data

- To forecast future data y_{t+h}, do the following:
  1. calculate the predictive distribution of α_{t+h} using the earlier recursion

$$\begin{aligned} a_{t+h|t} &= Ta_{t+h-1|t} + c, \\ P_{t+h|t} &= TP_{t+h-1|t}T' + RQR' \end{aligned}$$

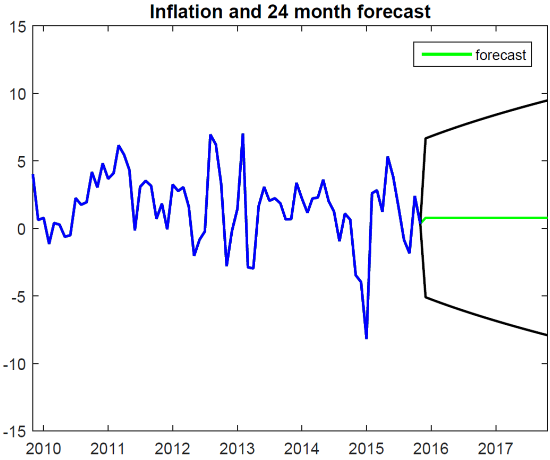
  2. calculate the predictive mean and variance of y_{t+h} via

$$\begin{aligned} \hat{y}_{t+h|t} &= Za_{t+h|t} + c \\ F_{t+h|t} &= ZP_{t+h|t}Z' + H \end{aligned}$$

- If the system matrices are time-varying, you need to know their future values:

$$Z_{t+j}\ d_{t+j}\ H_{t+j}, T_{t+j}, c_{t+j}, R_{t+j}, Q_{t+j} \qquad j = 0, \ldots, h-1$$

Example: U.S. inflation and the local level model



U.S. inflation, monthly continuously compounded.

Example: U.S. inflation: ARMA(1,1)

Suppose we consider an ARMA(1,1) for inflation

$$\pi_t = \mu + \phi_1(\pi_{t-1} - \mu) + \varepsilon_t + \vartheta_1 \varepsilon_{t-1} \qquad \varepsilon_t \sim \mathrm{N}(0, \sigma_\varepsilon^2)$$

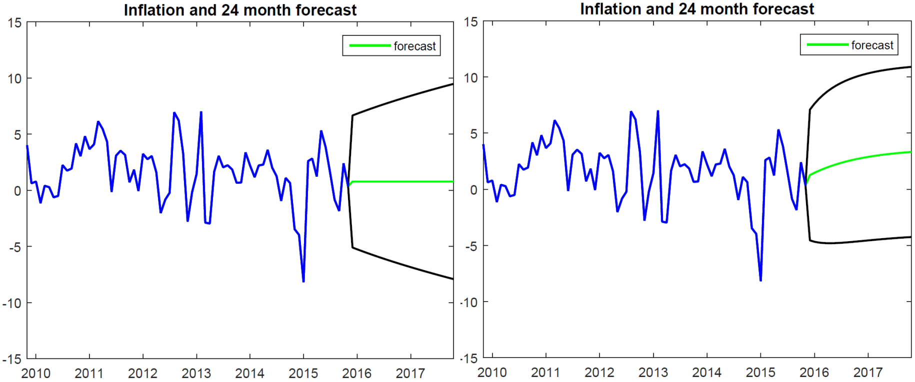
Let $\phi_0 = (1 - \phi_1)\mu$. I will use the state space form:

$$Z = \begin{pmatrix} 0 & 1 \end{pmatrix} \qquad d = 0 \qquad H = 0 \qquad Q = \sigma_\varepsilon^2$$

$$\alpha_t = \begin{pmatrix} \pi_t \\ \phi_1 \pi_t + \vartheta_1 \varepsilon_{t-1} \end{pmatrix} \quad T = \begin{pmatrix} 0 & 1 \\ 0 & \phi_1 \end{pmatrix} \quad c = \begin{pmatrix} 0 \\ \phi_0 \end{pmatrix} \quad R = \begin{pmatrix} 1 \\ \phi_1 + \vartheta_1 \end{pmatrix}$$

Note: the filtered and smoothed estimates of α_t are the same.

Example: U.S. inflation



U.S. inflation, monthly continuously compounded.

Parameter estimation

- Thus far, we have assumed that we know the parameters $\boldsymbol{\theta}$.
- The log-likelihood of the model is the joint distribution of the data:

$$\ln p(y_1, y_2, \ldots, y_T | \boldsymbol{\theta}) \;=\; \sum_{t=2}^{T} \ln p(y_t | y_{1:t-1}; \boldsymbol{\theta}) + \ln p(y_1; \boldsymbol{\theta})$$

- The Kalman filter calculates the time t contribution to the log-likelihood

$$\ln p(y_t | y_{1:t-1}; \boldsymbol{\theta})$$

  at each iteration during the forward pass.
- We can maximize the log-likelihood (numerically) to estimate the parameters $\boldsymbol{\theta}$.

What is the likelihood?

- The likelihood at time t is just our predicted value of y_t given information up to time $t-1$

$$p(y_t | y_1, \ldots, y_{t-1}; \boldsymbol{\theta})$$

- Under the assumption that the errors are Gaussian, this distribution is Gaussian. Calculate the mean and covariance matrix.
- In the last section, we just showed how to calculate the forecast of y_{t+h} given information up to time t!!!
- We need to calculate the forecast of y_t given information up to time $t-1$.

How to calculate the log-likelihood

- Start with the initial conditions $a_{1|0}$ and $P_{1|0}$
- Initialize the log-likelihood: $\ell_0 = 0$
- For $t = 1, \ldots, T$

$$
\begin{aligned}
v_t &= y_t - Z_t a_{t|t-1} - d_t, \\
F_t &= Z_t P_{t|t-1} Z_t' + H_t, \\
M_t &= T_t P_{t|t-1} Z_t' F_t^{-1}, \\
L_t &= T_t - M_t Z_t, \\
a_{t+1|t} &= T_t a_{t|t-1} + c_t + M_t v_t, \\
P_{t+1|t} &= T_t P_{t|t-1} L_t' + R_t Q_t R_t'
\end{aligned}
$$

$$
\ell_t = \ell_{t-1} - \frac{N}{2}\log(2\pi) - \frac{1}{2}\log|F_t| - \frac{1}{2} v_t' F_t^{-1} v_t
$$

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