* VC performed well in the 90s (obv)
* Buyout underperformed on a PME basis in tehe 90s
* PME and IRR both both exhibit signficant hereogeneity

2 Benchmarking private equity: The direct alpha method

Gredil-Griffiths-Stucke 2023

2.1 Motivation

PE literature performance metrics are fragmented and heuristic.

Note- see Nagel and Korteweg for a generalized PME

Also, this paper provides a nice survey of the PME approaches.

2.2 Theory

The theory starts from the SDF:

$$NPV = E\left[\sum_{t\in 1:T} M_0^t CF\right]$$

They consider the CAPM SDF (referenced from Kortegeg and Nagel (2016):

$$CAPM - M_0^t = \exp\left[\delta t - \gamma_{\tau=1}^t r_{m\tau}\right]$$

Most of the theory discussion references other papers- notably, Korteweg and Nagel (2021,2016) and Kaplan and Schoar. While I am not replicating it here, they show that the Kaplan and Schoar PME is equivalent to the Ruebinstein (1977) log CAPM SDF when $\delta = 0$ and $\gamma = -1$. The $KS - PME$ then becomes:

$$KS_{PME} = \frac{\sum_{t\in 1:T} D_t \exp\left[-\sum_{0:t} r_\tau\right]}{\sum_{t\in 1:T} C_t \gamma_{\tau=1}^t \exp\left[-\sum_{0:t} r_\tau\right]}$$

where D_t is capital distribution and C_t is a capital committment both at time t. Let NAV_T be the residual value of the fund. This gives

$$KS_{PME} = \frac{PV\left(NAV_T\right) + \sum PV\left(D_t\right)}{\sum PV\left(C_t\right)}$$

where the present value operator is defined at time 0 as $\frac{m_0}{m_t}$ and m_t is the market price. This is equivalent to:

$$
\begin{aligned}
KS_{PME} =& \frac{\left[PV\left(NAV_T\right)+\sum PV\left(D_t\right)\right]\times\frac{m_T}{m_t}}{\sum PV\left(C_t\right)\times\frac{m_T}{m_t}} \\
=& \frac{\left[NAV_T+\sum FV\left(D_t\right)\right]}{\sum FV\left(C_t\right)\times\frac{m_T}{m_t}}
\end{aligned}
$$

Further note that the present value variant can be written as

$$
\begin{aligned}
KS_{PME} =& \frac{\left[PV\left(NAV_T\right)+\sum PV\left(D_t\right)\right]-\sum PV\left(C_t\right)+\sum PV\left(C_t\right)}{\sum PV\left(C_t\right)} \\
KS_{PME} =& \frac{NPV}{\sum PV\left(C_t\right)}+1
\end{aligned}
$$

The direct alpha metric is proposed as

$$
DA = IRR\left(FV\left(C\right), FV\left(D\right), NAV\right)
$$

If a fund has only made a single contribution, at time 0, this is equivalent at time t to

$$
\begin{aligned}
KS_{PME} =& \frac{NAV_t}{C_0\frac{m_t}{m_0}}+1 \\
=& \left(1+DA\right)^t
\end{aligned}
$$

Thus the KS-PME is not generally equivalent to the proposed DA metric except in the narrow circumstance above. Note that KS_{PME} can provide a duration by solving for t:

$$
t = \frac{\log KS_{PME}}{1+DA}
$$

Note there is a beta adjustment used from equation 3 that they say improves the estiamte.

2.3 Empirical Design

2.3.1 Data

Burgiss

2.4 Results

Buyout funds generally produce positive alpha. Venture funds do not, unless using the unadjusted variant without any industry adjustments. The resutls are senitive to the benchmark choice.

3 Stephan/Nagel 2022 (see 2016 paper for supplemental info)

The authors introduce a new metric for performance (GPME centric alpha), which they say has some advantages over GPME. They also provide a brief overview of GPME.

3.1 Motivation

- Allocations to PE continue to increase, currently ~9% of DB plans.
- GPME (discussed in the 2016 paper_ provides a way of assessing benchmark returns
- However, they beleive GPME is too noisy for individual fund inference(!)
  - Specifically, because the residuals of a regression remove systematic risk, the alpha metric is more fficient that standard PME

3.2 The model

- The authors define an expoententially affine SDF as

$$M_{t+h} = \exp\left(\delta h - \gamma r^m_{t+h}\right)$$

where h is the number of periors, r^m is the market return and γ and δ are parameters. An advantage of thsi approach is it allows for the SDF to compound forward:

$$\begin{aligned} M_{t+h} &= \exp\left(\delta h - \gamma r^m_{t+h}\right) \\ &= \exp\left(\sum_{i\in 1:h}\left(\delta - \gamma r^m_{t+i}\right)\right) \end{aligned}$$

- They define the GPME as

$$GPME = \sum_{j=1}^{J} M^{h(j)}_{t+h(j)} C_{i,t+h(j)}$$

where C is the net cashflow. In the 2016 paper, a and b are chosen to perfectly price the stock market and risk-free asset payoffs.

  - Note in the 2016 paper, they estimate the model by setting up two benchmark funds.
    * Note they assume a 10-year life- they make major assumptions regarding the payoff
    * The parameters are set so that

$$\begin{gathered} u = \sum_{j=1}^{J} M_{t+h(j)} Y_{t+h(j)} \\ s.t. \\ Y_{t+h(j)} = \begin{bmatrix} C_{t+h(j)} \\ C^f_{t+h(j)} \\ C^m_{t+h(j)} \end{bmatrix} \end{gathered}$$

    * Here, the first element is the cashflow of an individual fund, while the second is the risk free rate fund and the third the market return benchmark fund.

    - The parameters are estimated via GMM. All pricing errors are expected to be 0. The above is weighted such that the individual fund moments have a weight of 0, and thus identification is only with reference to teh benchmarkfund.

- Suppose the market is log-normally distributed. Furthermore, define

$$\begin{aligned}\mu \equiv& \log E\left[R^m_{t+1}\right] - r_f \\ \sigma^2 \equiv& var\left[r^m_{t+1}\right]\end{aligned}$$

Then because

$$\log E\left[R^m_{t+1}\right] h = E\left[r^m_{t+h}\right] + \frac{1}{2}h\sigma^2$$

Adopting the h notation to represent $t+h$ as in the paper, the log market return is distributed as

$$r^m_{t+h} \sim N\left(\mu h + hr_f - \frac{1}{2}h\sigma^2, h\sigma^2\right)$$

    - Similarly, define x as the log payoff. Then:

$$\begin{bmatrix} r^m_h \\ x_j \end{bmatrix} \sim N\left(\begin{bmatrix} \mu h + hr_f - \frac{1}{2}h\sigma^2 \\ E\left[x_j\right] \end{bmatrix}, \begin{bmatrix} h\sigma^2 & \beta\sigma^2 h \\ \beta\sigma^2 h & var\left(x_j\right) \end{bmatrix}\right)$$

    Note that the use of j instead of h maintains generality in that it makes no assumption that the risk of x scales

    - The next step is to solve for the parameters. To do this, plug in the risk free rate and the market index into the pricing equation. Starting with the risk-free rate:

$$\begin{aligned}1 =& \log E\left[M_h\right] R^f_h \\ =& \delta h - \gamma E\left[r^m_h\right] + \frac{\gamma^2}{2} var\left(r^m_h\right) + r^f h \\ & \delta - \gamma\left(\mu + r^f - \frac{1}{2}\sigma^2\right) + \frac{1}{2}\gamma^2\sigma^2\end{aligned}$$

    - Similarly:

$$\begin{aligned}0 =& \log E\left[M_h R^m_h\right] \\ =& \log E\left(\exp\left(\delta h - (\gamma - 1) r^m_h\right)\right) \\ =& \delta h - (\gamma - 1) E\left[r^m_h\right] + \frac{(\gamma - 1)^2}{2} var\left(r^m_h\right)\end{aligned}$$

    - Subtracting:

$$0 = -\gamma\sigma^2 + \mu$$

* Hence

$$\delta = \gamma\left(\mu + r^f - \frac{1}{2}\sigma^2\right) - \frac{1}{2}\gamma^2\sigma^2 - r^f$$
$$\gamma = \frac{\mu}{\sigma^2}$$

* The CAPM is given as:

$$\begin{aligned}
0 =& \log E\left[M_h R_h\right] \\
=& \delta h - \gamma E\left[r_h^m\right] + E\left[r_h\right] + \frac{1}{2}var\left(r_h - \gamma r_h^m\right) \\
=& \delta h - E\left[r_h^m \gamma\right] + E\left[r_h\right] + \frac{1}{2}var\left(r_h\right) + \frac{1}{2}var\left(\gamma r_h^m\right) - \gamma cov\left(r_h, r_h^m\right) \\
=& \delta h - \gamma E\left[r_h^m\right] + E\left[r_h\right] + \frac{1}{2}var\left(r_h\right) + \frac{\gamma^2}{2}\sigma^2 - \gamma cov\left(r_h, r_h^m\right) \\
=& \delta h - \gamma E\left[r_h^m\right] + E\left[r_h\right] + \frac{1}{2}var\left(r_h\right) + \frac{\gamma^2}{2}\sigma^2 - \gamma\beta\sigma^2 h \\
=& \gamma\left(\mu + r^f - \frac{1}{2}\sigma^2\right)h - r^f h - \gamma E\left[r_h^m\right] + E\left[r_h\right] + \frac{1}{2}var\left(r_h\right) - \beta\left(\log E\left[R_h^m\right] - r_h^f\right) \\
\log E\left[R_h\right] - r_h^f =& -\gamma\left(\mu + r^f - \frac{1}{2}\sigma^2\right)h + \gamma E\left[r_h^m\right] + \beta\left(\log E\left[R_h^m\right] - r_h^f\right) \\
\log E\left[R_h\right] - r_h^f =& -\gamma \log E\left[R_h^m\right] + \frac{1}{2}\sigma^2 h\gamma + \gamma E\left[r_h^m\right] + \beta\left(\log E\left[R_h^m\right] - r_h^f\right) \\
\log E\left[R_h\right] - r_h^f =& \beta\left(\log E\left[R_h^m\right] - r_h^f\right)
\end{aligned}$$

– To price any asset, plug into the SDF (this is derived in the appendix:

$$E\left[M_h X_j\right] = E\left[\exp\left(-r_h^f - \beta\left(r_h^m - r_h^f\right) + \frac{h}{2}\beta\left(\beta - 1\right)\sigma^2\right)X_j\right]$$

– The GPME is given as:

$$\begin{aligned}
GPME =& E\sum_{j=1}^{J}\frac{X_{h(j)}}{R_{h(j)}^b} \\
& s.t. \\
R_{h(j)}^b \equiv& \exp\left(r_h^f + \beta\left(r_h^m - r_h^f\right) - \frac{h}{2}\beta\left(\beta - 1\right)\sigma^2\right)
\end{aligned}$$

THeir performance metric however is defined in terms of abnormal realiszed returns (Q: is the key dif here that ex-post returns are used to define the metric as opposed to computing the expectation?)

$$\alpha = \sum_{j=1}^{J}\frac{X_{h(j)}}{R_{h(j)}^b}$$

– Note that R^b is the benchmark portfolio, which invests β in the market and $1 - \beta$ in the risk-free asset. Campbell and Viceira (2002) chapter 2 provides another interpretation. It is no longer an SDF due to the asset specific parameter β

* NOTE- despite this statement, they critically assume that β is identical within a (sub) asset class

- They show this metric is not as noisy
- Estimation:
  1. Create artifial PE funds which invest in T-bills and the market to fit the SDF
  2. Use this SDF to estimate the GPME exactly as in the 2016 paper- this is asset class level abnormal performance
  3. Estimate the β at the asset class level following the techniques in the paper
- Suppose the market is log-normally distributed. Then:

$$
\begin{aligned}
\frac{dR^m}{R^m} &= \left(\mu + r^f\right) dh + \sigma dZ \\
\implies dr^m &= \left(\mu + r^f - \frac{1}{2}\sigma^2\right) dh + \sigma dZ \\
r_h^m &= \left(\mu + r^f - \frac{1}{2}\sigma^2\right) h + \int_0^h \sigma^2 dZ
\end{aligned}
$$

- Suppose the market is log-normally distributed, such that

$$
r_{t+h}^m \sim logN\left(\mu h + r_f, \sigma^2 h\right)
$$

- then

$$
\begin{aligned}
\log E\left[R_{t+h}^m\right] =& E\left[r_{t+h}\right] + \frac{1}{2}\sigma_h^2 \\
E\left[r_{t+h}\right] =& \mu + r_f - \frac{1}{2}\sigma_h^2
\end{aligned}
$$