4 Appendix

The complete generative process is given by:

$$p\left(y|rest\right) \sim MN\left(X_L R\phi + x_S, \frac{1}{\tau_y} I\right) \tag{14}$$

$$p\left(x|rest\right) \sim MN\left(F\beta + r, \frac{1}{\tau_x \tau_y} \Psi^{-1}\right) \tag{15}$$

$$p\left(\phi|rest\right) \sim MN\left(\phi_0, \frac{1}{\tau_y \tau_\phi} M_0^{-1}\right) \tag{9}$$

$$p\left(\beta|rest\right) \sim MN\left(\beta_0 + D^{-1}\beta_0^{\Delta}, \frac{1}{\tau_x \tau_y \tau_\beta} \left[DA_0 D\right]^{-1}\right) \tag{12}$$

$$d_k = (\gamma_k + (1-\gamma_k)\frac{1}{v^2})^{0.5} \tag{13}$$

$$p\left(\gamma_k\right) \sim Bern\left(\omega\right) \tag{11}$$

$$p\left(\omega\right) \sim Beta\left(\kappa_0, \delta_0\right) \tag{16}$$

$$p\left(\tau_y\right) \sim Gamma\left(\alpha_{y0}, \zeta_{y0}\right) \tag{17}$$

$$p\left(\tau_x\right) \sim Gamma\left(\alpha_{x0}, \zeta_{x0}\right) \tag{18}$$

$$p\left(\psi_t\right) \sim Gamma\left(\nu/2, \nu/2\right) \tag{19}$$

$$p\left(\nu\right) \sim Gamma\left(\alpha_{\nu 0}, \zeta_{\nu 0}\right) \tag{20}$$

$$p\left(\tau_\phi\right) \sim Gamma\left(\alpha_{\phi 0}, \zeta_{\phi 0}\right) \tag{21}$$

$$p\left(\tau_\beta\right) \sim Gamma\left(\alpha_{\beta 0}, \zeta_{\beta 0}\right) \tag{22}$$

Note that here y can be written in two forms using matrix notation. When x and y have the same frequency:

$$y = \Phi x = X_L R\phi + x_S \tag{4}$$

Where

$$X_L R\phi + x_S = \begin{bmatrix} x_1 & x_2 & \cdots & x_{P-1} & x_P & x_{P+1} \\ x_2 & x_3 & \cdots & x_P & x_{P+1} & x_{P+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ x_{T-P-1} & x_{T-P} & \cdots & x_{T-3} & x_{T-2} & x_{T-1} \\ x_{T-P} & x_{T-P+1} & \cdots & x_{T-2} & x_{T-1} & x_T \end{bmatrix} [R] \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_{P-1} \\ \phi_P \end{bmatrix} + \begin{bmatrix} x_{P+1} \\ x_{P+2} \\ \vdots \\ x_{T-1} \\ x_T \end{bmatrix} \tag{23}$$

And

$$R = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -1 & -1 & \cdots & -1 & -1 \end{bmatrix}$$

To generalize to other frequencies, recall that $t[s] \equiv P + s * \Delta t$ and define:

$$\begin{aligned}
\Phi_{sj} \equiv & \begin{cases} \phi_{P-(t[s]-\Delta t-j)} & 1 \leq P - (t[s] - \Delta t - j) \leq P \\ 1 - \left(\iota^{\phi}_{\Delta t-(t[s]-j)}\right)' \phi & t[s] - \Delta t < j \leq t[s] \\ 0 & \text{otherwise} \end{cases} \\
X_{Lsj} \equiv & x_{t[s]-(P+\Delta t-j)} \\
\iota^{\phi}_{pl} \equiv & \begin{cases} 1 & p - l \mod \Delta t = 0 \\ 0 & \text{otherwise} \end{cases}
\end{aligned} \tag{24}$$

The above matrix formulation (23) can be generalized by only including rows of X_L where $t \in \{t[1...S]\}$ and adjusting the restriction matrix. The general version is then:

$$y = \Phi x = X_L R \phi + x_S, \quad \text{s.t. } x_S = X_L \iota_{\Delta t} \tag{4}$$

where $\iota_{\Delta t}$ is a vector of P zeros followed by Δt ones, making x_S the sum of the last Δt columns of X_L.

4.1 Complete Posterior Distribution

The complete posterior distribution is given by:

$$\begin{aligned}
p(\Theta|y, F, r) \propto & \; p(y|x, \gamma, \omega, \beta, \phi, \tau_x, \tau_y, \tau_\phi, \tau_\beta, \psi, \nu, F) \times p(x|\beta, \phi, \tau_x, \tau_y, \psi, F) \\
& \times p(\phi|\tau_y, \tau_\phi) \times p(\beta|\gamma, \tau_x, \tau_y, \tau_\beta) \times p(\gamma|\omega) \times p(\omega) \\
& \times p(\psi|\nu) \times p(\nu) \times p(\tau_x) \times p(\tau_y) \times p(\tau_\phi) \times p(\tau_\beta) \\
= & \; MN\left(y; \Phi x, \frac{1}{\tau_y} I\right) \times MN\left(x; F\beta + r, \frac{1}{\tau_x \tau_y} \Psi^{-1}\right) \\
& \times MN\left(\phi; \phi_0, \frac{1}{\tau_y \tau_\phi} M_0^{-1}\right) \times MN\left(\beta; \beta_0 + D^{-1}\beta_0^{\Delta}, \frac{1}{\tau_x \tau_y \tau_\beta} [DA_0 D]^{-1}\right) \\
& \times \prod_{k=1}^{K} Bern(\gamma_k; \omega) \times Beta(\omega; \kappa_0, \delta_0) \\
& \times \prod_{t=1}^{T} Gamma(\psi; \nu/2, \nu/2) \times Gamma(\nu; \alpha_{\nu 0}, \zeta_{\nu 0}) \\
& \times Gamma(\tau_x; \alpha_{x0}, \zeta_{x0}) \times Gamma(\tau_y; \alpha_{y0}, \zeta_{y0}) \\
& \times Gamma(\tau_\phi; \alpha_{\phi 0}; \zeta_{\phi 0}) \times Gamma(\tau_\beta; \alpha_{\beta 0}, \zeta_{\beta 0})
\end{aligned} \tag{25}$$

4.2 Posterior of ϕ

First, define:

$$\begin{aligned}
\tilde{y} &\equiv y - x_S \\
\tilde{X}_L &\equiv X_L R
\end{aligned}$$

$$\log p(\phi|rest) = \log MN\left(\phi; \mu_\phi, \Lambda_\phi^{-1}\right) + c_4^\phi \tag{26}$$

where

$$
\begin{aligned}
\Lambda_\phi &= \tau_y \left(\tilde{X}_L' \tilde{X}_L + \tau_\phi M_0\right) \\
\mu_\phi &= \tau_y \Lambda_\phi^{-1} \left(\tilde{X}_L' \tilde{y} + \tau_\phi M_0' \phi_0\right) \\
c_1^\phi &= \frac{S+P}{2} \log\left(\frac{\tau_y}{2\pi}\right) + \frac{P}{2} \log \tau_\phi + \frac{1}{2} \log det\left(M_0\right) \\
&\quad + \log \Bigg[MN\left(x; F\beta + r, \frac{T}{\tau_x \tau_y} \Psi^{-1}\right) \times MN\left(\beta; \beta_0 + D^{-1}\beta_0^\Delta, \frac{1}{\tau_x \tau_y \tau_\beta} \left[DA_0D\right]^{-1}\right) \\
&\quad \times \prod_{k=1}^{K} Bern\left(\gamma_k; \omega\right) \times Beta\left(\omega; \kappa_0, \delta_0\right) \times \prod_{t=1}^{T} Gamma\left(\psi; \nu/2, \nu/2\right) \\
&\quad \times Gamma\left(\nu; \alpha_{\nu 0}, \zeta_{\nu 0}\right) \times Gamma\left(\tau_x; \alpha_{x0}, \zeta_{x0}\right) \times Gamma\left(\tau_y; \alpha_{y0}, \zeta_{y0}\right) \\
&\quad \times Gamma\left(\tau_\beta; \alpha_{\beta 0}, \zeta_{\beta 0}\right) \times Gamma\left(\tau_\phi; \alpha_{\phi 0}, \zeta_{\phi 0}\right) \Bigg] + c^{ev} \\
c_2^\phi &= c_1^\phi - \frac{\tau_y}{2} \left[\tau_\phi \phi_0' M_0 \phi_0 + \tilde{y}' \tilde{y}\right] \\
c_3^\phi &= c_2^\phi + \frac{1}{2} \mu_\phi' \Lambda_\phi \mu_\phi \\
c_4^\phi &= c_3^\phi + \frac{P}{2} \log 2\pi - \frac{1}{2} \log det \Lambda_\phi \\
c_{ev} &= -\log p\left(y_t\right)
\end{aligned}
$$

The last constant makes the full posterior into a valid probability distribution.

4.3 Posterior of x

$$
\log p\left(x|rest\right) = \log MN\left(x; \mu_x, \Lambda_x^{-1}\right) + c_4^x \tag{27}
$$

where

$$
\begin{aligned}
\Lambda_x &= \tau_y \left(\Phi'\Phi + \tau_x \Psi\right) \\
\mu_x &= \tau_y \Lambda_x^{-1} \left(\Phi' y + \tau_x \Psi \left(r + F\beta\right)\right) \\
c_1^x &= \frac{S+T}{2} \log\left(\frac{\tau_y}{2\pi}\right) + \frac{T}{2} \log \tau_x + \frac{1}{2} \log Det\left(\Psi\right) \\
&\quad + \log \Bigg[MN\left(\phi; \phi_0, \frac{1}{\tau_y \tau_\phi} M_0^{-1}\right) \times MN\left(\beta; \beta_0 + D^{-1}\beta_0^{\Delta}, \frac{1}{\tau_x \tau_y \tau_\beta}\left[DA_0D\right]^{-1}\right) \\
&\quad \times \prod_{k=1}^{K} Bern\left(\gamma_k; \omega\right) \times Beta\left(\omega; \kappa_0, \delta_0\right) \times \prod_{t=1}^{T} Gamma\left(\psi; \nu/2, \nu/2\right) \times Gamma\left(\nu; \alpha_{\nu 0}, \zeta_{\nu 0}\right) \\
&\quad \times Gamma\left(\tau_x; \alpha_{x0}, \zeta_{x0}\right) \times Gamma\left(\tau_y; \alpha_{y0}, \zeta_{y0}\right) \\
&\quad \times Gamma\left(\tau_x; \alpha_{x0}, \zeta_{x0}\right) \times Gamma\left(\tau_y; \alpha_{y0}, \zeta_{y0}\right) \\
&\quad \times Gamma\left(\tau_\beta; \alpha_{\beta 0}, \zeta_{\beta 0}\right) \times Gamma\left(\tau_\phi; \alpha_{\phi 0}, \zeta_{\phi 0}\right) \Bigg] + c^{ev} \\
c_2^x &= c_1^x - \frac{\tau_y \tau_x}{2} \left(r + F\beta\right)' \Psi \left(r + F\beta\right) - \frac{\tau_y}{2} y'y \\
c_3^x &= c_2^x + \frac{\mu_x' \Lambda_x \mu_x}{2} \\
c_4^x &= c_3^x + \frac{T}{2} \log 2\pi - \log det \Lambda_x
\end{aligned}
$$

4.4 Posterior of τ_y

- Let $\tilde{\beta} \equiv \beta - \beta_0$. Then the conditional posterior is:

$$
\log p\left(\tau_y | rest\right) = \log Gamma\left(\tau_y; \alpha_y, \zeta_y\right) + c_2^{\tau_y} \tag{28}
$$

where

$$
\begin{aligned}
\alpha_y &= \frac{S+T+P+K}{2} + \alpha_{y0} \\
\zeta_y &= \zeta_{y0} + \frac{1}{2}\left(\tilde{y} - \tilde{X}_L\phi\right)'\left(\tilde{y} - \tilde{X}_L\phi\right) + \frac{\tau_\phi}{2}\left(\phi - \phi_0\right)' M_0 \left(\phi - \phi_0\right) \\
&\quad + \frac{\tau_x}{2}\left((x-r) - F\beta\right)' \Psi \left((x-r) - F\beta\right) \\
&\quad + \frac{\tau_x \tau_\beta}{2}\left(\tilde{\beta} - D^{-1}\beta_0^{\Delta},\right)' DA_0D\left(\tilde{\beta} - D^{-1}\beta_0^{\Delta}\right) \\
c_1^{\tau_y} &= \alpha_{y0}\log\zeta_{y0} - \log\Gamma\left(\alpha_{y0}\right) - \frac{S+T+K+P}{2}\log 2\pi + \frac{T+K}{2}\log\tau_x \\
&\quad + \frac{1}{2}\log detM_0 + \frac{1}{2}\log det\Psi + \frac{1}{2}\log det\left(DA_0D\right) + \frac{P}{2}\log\tau_\phi + \frac{K}{2}\log\tau_\beta \\
&\quad + \log\Bigg(\prod_{k=1}^{K} Bern\left(\gamma_k;\omega\right) \times Beta\left(\omega;\kappa_0,\delta_0\right) \times \prod_{t=1}^{T} Gamma\left(\psi;\nu/2,\nu/2\right) \\
&\quad \times Gamma\left(\nu;\alpha_{\nu 0},\zeta_{\nu 0}\right) \times Gamma\left(\tau_x;\alpha_{x0},\zeta_{x0}\right) \\
&\quad \times Gamma\left(\tau_\beta;\alpha_{\beta 0},\zeta_{\beta 0}\right) \times Gamma\left(\tau_\phi;\alpha_{\phi 0},\zeta_{\phi 0}\right)\Bigg) + c^{ev} \\
c_2^{\tau_y} &= c_1^{\tau_y} + \log\Gamma\left(\alpha_y\right) - \alpha_y\log\zeta_y
\end{aligned}
$$

4.5 Posterior for τ_x

$$
\log p\left(\tau_x | rest\right) = \log Gamma\left(\tau_x;\alpha_x,\zeta_x\right) + c_2^{\tau_x} \tag{29}
$$

where

$$
\begin{aligned}
\alpha_x &= \frac{T+K}{2} + \alpha_{0x} \\
\zeta_x &= \frac{\tau_y}{2}\left((x-r) - F\beta\right)' \Psi \left((x-r) - F\beta\right) + \frac{\tau_y\tau_\beta}{2}\left(\tilde{\beta} - D^{-1}\beta_0^{\Delta}\right)' DA_0D\left(\tilde{\beta} - D^{-1}\beta_0^{\Delta}\right) + \zeta_{0x} \\
c_1 &= \frac{T+K}{2}\log\frac{\tau_y}{2\pi} + \frac{1}{2}\log detDA_0D + \frac{1}{2}\log det\Psi + \frac{K}{2}\log\tau_\beta \\
&\quad + \alpha_{x0}\log\zeta_{x0} - \log\Gamma\left(\alpha_{x0}\right) \\
&\quad + \log\Bigg(MN\left(\phi;\phi_0,\frac{1}{\tau_y\tau_\phi}M_0^{-1}\right) \times MN\left(y;\Phi x,\frac{1}{\tau_y}I\right) \\
&\quad \times \prod_{k=1}^{K} Bern\left(\gamma_k;\omega\right) \times Beta\left(\omega;\kappa_0,\delta_0\right) \\
&\quad \times \prod_{t=1}^{T} Gamma\left(\psi;\nu/2,\nu/2\right) \times Gamma\left(\nu;\alpha_{\nu 0},\zeta_{\nu 0}\right) \times Gamma\left(\tau_y;\alpha_{y0},\zeta_{y0}\right) \\
&\quad \times Gamma\left(\tau_\beta;\alpha_{\beta 0},\zeta_{\beta 0}\right) \times Gamma\left(\tau_\phi;\alpha_{\phi 0},\zeta_{\phi 0}\right)\Bigg) + c^{ev} \\
c_2 &= c_1 + \log\Gamma\left(\alpha_x\right) - \alpha_x\log\zeta_x
\end{aligned}
$$

4.6 Posterior for τ_ϕ

$$\log p\left(\tau_\phi|rest\right) = \log Gamma\left(\tau_\phi;\alpha_\phi,\zeta_\phi\right) + c_2^{\tau\phi} \tag{30}$$

where

$$\begin{aligned}
\alpha_\phi &= \alpha_{\phi_0} + \frac{P}{2} \\
\zeta_\phi &= \zeta_{\phi_0} + \frac{\tau_y}{2}\left(\phi-\phi_0\right)' M_0 \left(\phi-\phi_0\right) \\
c_1^{\tau\phi} &= \alpha_{\phi_0}\log\zeta_{\phi_0} - \log\Gamma\left(\alpha_{\phi_0}\right) + \frac{1}{2}\log Det\left(M_0\right) + \frac{P}{2}\log\left(\frac{\tau_y}{2\pi}\right) \\
&\quad + \log\Bigg[MN\left(y;\Phi x,\frac{1}{\tau_y}I\right)\times MN\left(x;F\beta+r,\frac{1}{\tau_x\tau_y}\Psi^{-1}\right)\times MN\left(\beta;\beta_0+D^{-1}\beta_0^{\Delta},\frac{1}{\tau_x\tau_y\tau_\beta}\left[DA_0D\right]^{-1}\right) \\
&\quad \times \prod_{k=1}^{K} Bern\left(\gamma_k;\omega\right)\times Beta\left(\omega;\kappa_0,\delta_0\right)\times\prod_{t=1}^{T} Gamma\left(\psi;\nu/2,\nu/2\right)\times Gamma\left(\tau_\beta;\tau_{\beta 0},\zeta_{\beta 0}\right) \\
&\quad \times Gamma\left(\nu;\alpha_{\nu 0},\zeta_{\nu 0}\right)\times Gamma\left(\tau_x;\alpha_{x0},\zeta_{x0}\right)\times Gamma\left(\tau_y;\alpha_{y0},\zeta_{y0}\right)\Bigg] + c^{ev} \\
c_2^{\tau\phi} &= c_1^{\tau\phi} - \alpha_\phi\log\zeta_\phi + \log\Gamma\left(\alpha_\phi\right)
\end{aligned}$$

4.7 Posterior for τ_β

$$\log p\left(\tau_\beta|rest\right) = \log Gamma\left(\tau_\beta;\alpha_\beta,\zeta_\beta\right) + c_2^{\tau\beta} \tag{31}$$

where

$$\begin{aligned}
\alpha_\beta \equiv& \alpha_{\beta 0} + \frac{K}{2} \\
\zeta_\beta \equiv& \zeta_{\beta 0} + \frac{\tau_x\tau_y}{2}\left(\tilde{\beta}-D^{-1}\beta_0^{\Delta}\right)' DA_0D\left(\tilde{\beta}-D^{-1}\beta_0^{\Delta}\right) \\
c_1^{\tau\beta} \equiv& \alpha_{\beta 0}\log\zeta_{\beta 0} - \log\Gamma\left(\alpha_{\beta 0}\right) + \frac{K}{2}\log\frac{\tau_x\tau_y}{2\pi} + \frac{1}{2}\log detDA_0D \\
&+ \log\Bigg(MN\left(\phi;\phi_0,\frac{1}{\tau_y\tau_\phi}M_0^{-1}\right)\times MN\left(y;\Phi x,\frac{1}{\tau_y}I\right)\times MN\left(x;F\beta+r,\frac{1}{\tau_x\tau_y}\Psi^{-1}\right) \\
&\times \prod_{k=1}^{K} Bern\left(\gamma_k;\omega\right)\times Beta\left(\omega;\kappa_0,\delta_0\right) \\
&\times \prod_{t=1}^{T} Gamma\left(\psi;\nu/2,\nu/2\right)\times Gamma\left(\nu;\alpha_{\nu 0},\zeta_{\nu 0}\right)\times Gamma\left(\tau_\phi;\tau_{\phi 0},\zeta_{\phi 0}\right) \\
&\times Gamma\left(\tau_y;\alpha_{y0},\zeta_{y0}\right)\times Gamma\left(\tau_x;\alpha_{x0},\zeta_{x0}\right)\Bigg) + c^{ev} \\
c_2^{\tau\beta} \equiv& c_1^{\tau\beta} - \alpha_\beta\log\zeta_\beta + \log\Gamma\left(\alpha_\beta\right)
\end{aligned}$$

4.8 Posterior for β

First, define $\tilde{\beta}_0^{\Delta} \equiv \beta_0 - D^{-1}\beta_0$. The the conditional posterior $p\left(\beta|rest\right)$ is:

$$\log p\left(\beta|rest\right) = \log MN\left(\beta;\mu_\beta,\Lambda_\beta\right) + c_4^\beta \tag{32}$$

where

$$\begin{aligned}
\Lambda_\beta \equiv& \tau_x\tau_y\left[F'\Psi F + \tau_\beta D A_0 D\right]\\
\mu_\beta \equiv& \tau_x\tau_y\Lambda_\beta^{-1}\left[F'\Psi\left(x-r\right) + \tau_\beta D A_0\left(D\beta_0 + \beta_0^\Delta\right)\right]\\
c_1^\beta \equiv& \frac{T+K}{2}\log\frac{\tau_x\tau_y}{2\pi} + \frac{1}{2}\log detDA_0D + \frac{1}{2}\log det\Psi + \frac{K}{2}\log\tau_\beta\\
&+\log\left(MN\left(\phi;\phi_0,\frac{1}{\tau_y\tau_\phi}M_0^{-1}\right)\times MN\left(y;\Phi x,\frac{1}{\tau_y}I\right)\right.\\
&\times\prod_{k=1}^{K}Bern\left(\gamma_k;\omega\right)\times Beta\left(\omega;\kappa_0,\delta_0\right)\\
&\times\prod_{t=1}^{T}Gamma\left(\psi;\nu/2,\nu/2\right)\times Gamma\left(\nu;\alpha_{\nu0},\zeta_{\nu0}\right)\\
&\times Gamma\left(\tau_y;\alpha_{y0},\zeta_{y0}\right)\times Gamma\left(\tau_x;\alpha_{x0},\zeta_{x0}\right)\\
&\left.\times Gamma\left(\tau_\beta;\alpha_{\beta0},\zeta_{\beta0}\right)\times Gamma\left(\tau_\phi;\alpha_{\phi0},\zeta_{\phi0}\right)\right) + c^{ev}\\
c_2^\beta =& c_1^\beta - \frac{\tau_x\tau_y}{2}\left(\left(x-r\right)'\Psi\left(x-r\right) + \tau_\beta\left(D\beta_0+\beta_0^\Delta\right)'A_0\left(D\beta_0+\beta_0^\Delta\right)\right)\\
c_3^\beta =& c_2^\beta + \frac{\mu_\beta'\Lambda_\beta\mu_\beta}{2}\\
c_4^\beta =& c_3^\beta - \frac{1}{2}\log det\Lambda_\beta + \frac{K}{2}\log 2\pi
\end{aligned}$$

Note that while $\beta_{0k} + \beta_{0k}^\Delta$ is the prior on the mean of variable k conditional on selection, β_{0k} is not the prior on the mean if variable k is not selected. While the difference is rarely material, it can be corrected. To derive the correction, let $\dot{\beta}_{0k}$ be the desired mean conditional on no selection and $\dot{\beta}_{0k} + \dot{\beta}_{0k}^\Delta$ is the prior mean conditional on selection. Then solve for the value conditional on $\gamma = 0$ and $\gamma = 1$:

$$\begin{aligned}
\dot{\beta}_{0k} &= \beta_{0k} + v\beta_0^\Delta\\
\dot{\beta}_{0k} + \dot{\beta}_{0k}^\Delta &= \beta_{0k} + \beta_{0k}^\Delta
\end{aligned}$$

Therefore:

$$\begin{aligned}
\beta_{0k} &= \dot{\beta}_{0k} - \frac{v}{1-v}\dot{\beta}_{0k}^\Delta\\
\beta_{0k}^\Delta &= \frac{1}{1-v}\dot{\beta}_{0k}^\Delta
\end{aligned}$$

4.9 Posterior for γ

What follows is the conditional posterior for γ_k in the scenario where each γ_k is conditionally independent of the other values of γ (denoted as γ_{-k}). In other words, $p(\gamma_k|\gamma_{-k}, rest) = p(\gamma_k|rest)$. Note that this does not imply unconditionally that $\gamma_k \perp \gamma_{-k}$ as other variables (e.g. β) influence both γ_k and γ_{-k}.

Practically this implies A_0 is diagonal, such that $a_0 \equiv diag(A_0)$. Also recall $d_k^2 \equiv \gamma_k + \frac{1-\gamma_k}{v^2}$. As the only discrete distribution, the derivation for $p(\gamma)$ proceeds somewhat differently than others. The distribution for p_k with a conditionally independent prior for β is given by $\frac{\tilde{p}(\gamma_k=1)}{\tilde{p}_k(\gamma_k=0)+\tilde{p}_k(\gamma_k=1)}$.

$$\log p(\gamma_k) = \gamma_k \log p_{\gamma k} + (1-\gamma_k)\log(1-p_{\gamma k}) + c_2^{\gamma k} \tag{33}$$

where

$$\begin{aligned}
p_{\gamma k} =& \frac{\tilde{p}_{\gamma k}|_1}{\tilde{p}_{\gamma k}|_0 + \tilde{p}_{\gamma k}|_1} \\
\tilde{p}_{\gamma k}|_1 \equiv& \exp\left(-\frac{\tau_x\tau_y\tau_\beta a_{0k}}{2}\left(\tilde{\beta}_k^2 - 2\beta_{0k}^\Delta\tilde{\beta}_k\right)\right)\omega \\
\tilde{p}_{\gamma k}|_0 \equiv& \exp\left(-\frac{\tau_x\tau_y\tau_\beta a_{0k}}{2}\left(\frac{\tilde{\beta}_k^2}{v^2} - \frac{2\beta_{0k}^\Delta\tilde{\beta}_k}{v}\right)\right)\frac{1-\omega}{v} \\
c_1^{\gamma k} \equiv& \frac{1}{2}\log\frac{a_{0k}\tau_x\tau_y\tau_\beta}{2\pi} \\
&+ \log\Bigg(MN\left(\phi;\phi_0,\frac{1}{\tau_y\tau_\phi}M_0^{-1}\right)\times MN\left(y;\Phi x,\frac{1}{\tau_y}I\right) \\
&\times MN\left(x;F\beta+r,\frac{1}{\tau_x\tau_y}\Psi^{-1}\right) \\
&\times \prod_{j=1,j\neq k}^{K}\left(N\left(\beta_j;\frac{\beta_0^\Delta}{d_j}+\beta_0,\frac{1}{\tau_x\tau_y\tau_\beta a_{0j}d_j^2}\right)\times Bern(\gamma_j;\omega)\right)\times Beta(\omega;\kappa_0,\delta_0) \\
&\times \prod_{t=1}^{T} Gamma(\psi;\nu/2,\nu/2)\times Gamma(\nu;\alpha_{\nu 0},\zeta_{\nu 0}) \\
&\times Gamma(\tau_y;\alpha_{y0},\zeta_{y0})\times Gamma(\tau_x;\alpha_{x0},\zeta_{x0}) \\
&\times Gamma(\tau_\beta;\alpha_{\beta 0},\zeta_{\beta 0})\times Gamma(\tau_\phi;\alpha_{\phi 0},\zeta_{\phi 0})\Bigg) + c^{ev} \\
c_2^{\gamma k} \equiv& c_1^{\gamma k} - \frac{\tau_x\tau_y\tau_\beta a_{0k}\left(\beta_{0k}^\Delta\right)^2}{2} \\
c_3^{\gamma k} \equiv& c_2^{\gamma k} + \log\left(\tilde{p}_{\gamma k}|_1 + \tilde{p}_{\gamma k}|_0\right)
\end{aligned}$$

Note that the normalization is accounted for in $c_3^{\gamma k}$. The normalization is fully revealed as the true probabilities must add to one. Similarly, the approximate distribution for any γ_k is given by $\frac{\tilde{q}_k(\gamma_k=1)}{\tilde{q}_k(\gamma_k=0)+\tilde{q}_k(\gamma_k=1)}$.

4.9.1 Posterior for γ (General Case)

To get the off-diagonal terms for A_0, we use the below generalization:

$$p\left(\gamma_k\right)=\gamma_k \log p_{\gamma k}+\left(1-\gamma_k\right) \log \left(1-p_{\gamma k}\right)+c_3^{\gamma k} \tag{34}$$

where

$$\begin{aligned}
p_{\gamma k} \equiv & \frac{\tilde{p}_{\gamma k}|_1}{\tilde{p}_{\gamma k}|_0+\tilde{p}_{\gamma k}|_1} \\
\tilde{p}_{\gamma k}|_1 \equiv & \exp \left(-\frac{\tau_x \tau_y \tau_\beta}{2}\left[\left(\tilde{\beta}^{\prime} D A_0 D \tilde{\beta}-2 \tilde{\beta}^{\prime} D A_0 \beta_0^{\Delta}\right)\right]_{d_k=1}\right) \omega \\
\tilde{p}_{\gamma k}|_0 \equiv & \exp \left(-\frac{\tau_x \tau_y \tau_\beta}{2}\left[\left(\tilde{\beta}^{\prime} D A_0 D \tilde{\beta}-2 \tilde{\beta}^{\prime} D A_0 \beta_0^{\Delta}\right)\right]_{d_k=v^{-1}}\right) \frac{1-\omega}{v} \\
c_1^{\gamma k} \equiv & \frac{K}{2} \log \frac{\tau_x \tau_y \tau_\beta}{2 \pi}+\frac{1}{2} \log \det A_0+\sum_{j=1, j \neq k}^K \log d_j \\
& +\log \left(M N\left(\phi ; \phi_0, \frac{1}{\tau_y \tau_\phi} M_0^{-1}\right) \times M N\left(y ; \Phi x, \frac{1}{\tau_y} I\right)\right. \\
& \times M N\left(x ; F \beta+r, \frac{1}{\tau_x \tau_y} \Psi^{-1}\right) \\
& \times \prod_{j=1, j \neq k}^K \operatorname{Bern}\left(\gamma_j ; \omega\right) \times \operatorname{Beta}\left(\omega ; \kappa_0, \delta_0\right) \\
& \times \prod_{t=1}^T \operatorname{Gamma}(\psi ; \nu / 2, \nu / 2) \times \operatorname{Gamma}\left(\nu ; \alpha_{\nu 0}, \zeta_{\nu 0}\right) \\
& \times \operatorname{Gamma}\left(\tau_y ; \alpha_{y 0}, \zeta_{y 0}\right) \times \operatorname{Gamma}\left(\tau_x ; \alpha_{x 0}, \zeta_{x 0}\right) \\
& \left.\times \operatorname{Gamma}\left(\tau_\beta ; \alpha_{\beta 0}, \zeta_{\beta 0}\right) \times \operatorname{Gamma}\left(\tau_\phi ; \alpha_{\phi 0}, \zeta_{\phi 0}\right)\right)+c^{ev} \\
c_2^{\gamma k} \equiv & c_1^{\gamma k}-\frac{\tau_x \tau_y \tau_\beta}{2}\left(\beta_0^{\Delta}\right)^{\prime} A_0 \beta_0^{\Delta} \\
c_3^{\gamma k} \equiv & c_2^{\gamma k}+\log \left(\tilde{p}_{\gamma k}|_1+\tilde{p}_{\gamma k}|_0\right)
\end{aligned}$$

4.10 Posterior for ω

$$\log p(\omega)=\log \operatorname{Beta}(\kappa, \delta)+c_3^\omega \tag{35}$$

where

$$
\begin{aligned}
\kappa \equiv& \kappa_0 + \sum_{k=1}^{K} \gamma_k \\
\delta \equiv& \delta_0 + K - \sum_{k=1}^{K} \gamma_k \\
c_1^{\omega} \equiv& -\log B\left(\kappa_0, \delta_0\right) + \log \left(MN\left(\phi; \phi_0, \frac{1}{\tau_y \tau_\phi} M_0^{-1}\right) \times MN\left(y; \Phi x, \frac{1}{\tau_y} I\right) \right. \\
& \times MN\left(x; F\beta + r, \frac{1}{\tau_x \tau_y} \Psi^{-1}\right) \times MN\left(\beta; \beta_0 + D^{-1}\beta_0^{\Delta}, \frac{1}{\tau_x \tau_y \tau_\beta}\left[DA_0 D\right]^{-1}\right) \\
& \times \prod_{t=1}^{T} Gamma\left(\psi; \nu/2, \nu/2\right) \times Gamma\left(\nu; \alpha_{\nu 0}, \zeta_{\nu 0}\right) \\
& \times Gamma\left(\tau_y; \alpha_{y0}, \zeta_{y0}\right) \times Gamma\left(\tau_x; \alpha_{x0}, \zeta_{x0}\right) \\
& \left. \times Gamma\left(\tau_\beta; \tau_{\beta 0}, \zeta_{\beta 0}\right) \times Gamma\left(\tau_\phi; \tau_{\phi 0}, \zeta_{\phi 0}\right) \right) + c^{ev} \\
c_2^{\omega} \equiv& c_1^{\omega} + \log B\left(\kappa, \delta\right)
\end{aligned}
$$

4.11 Posterior for ψ_t

Conditional posterior:

$$\log p\left(\psi_t\right) = \log Gamma\left(\psi_t, \alpha_{\psi t}, \zeta_{\psi t}\right) + c_2^{\psi t} \tag{36}$$

where

$$
\begin{aligned}
\alpha_{\psi t} \equiv& \frac{\nu+1}{2} \\
\zeta_{\psi t} \equiv& \frac{\nu}{2} + \frac{\tau_x \tau_y}{2} \left((x_t - r_t) - f_t' \beta \right)^2 \\
c_1^{\psi t} \equiv& \frac{\nu}{2} \log \frac{\nu}{2} - \log \Gamma \left(\frac{\nu}{2} \right) + \frac{1}{2} \log \left(\frac{\tau_x \tau_y}{2\pi} \right) \\
&+ \log \left(\prod_{j=1, j \neq t}^{T} \left[N \left(x_t; f_t' \beta + r, \frac{1}{\tau_x \tau_y \psi_t} \right) \times Gamma \left(\psi_j; \frac{\nu}{2}, \frac{\nu}{2} \right) \right] \right. \\
&\times MN \left(\phi; \phi_0, \frac{1}{\tau_y \tau_\phi} M_0^{-1} \right) \times MN \left(y; \Phi x, \frac{1}{\tau_y} I \right) \\
&\times MN \left(\beta; \beta_0 + D^{-1} \beta_0^{\Delta}, \frac{1}{\tau_x \tau_y \tau_\beta} \left[D A_0 D \right]^{-1} \right) \\
&\times \prod_{k=1}^{K} Bern \left(\gamma_k; \omega \right) \times Beta \left(\omega; \kappa_0, \delta_0 \right) \times Gamma \left(\nu; \alpha_{\nu 0}, \zeta_{\nu 0} \right) \\
&\times Gamma \left(\tau_y; \alpha_{y0}, \zeta_{y0} \right) \times Gamma \left(\tau_x; \alpha_{x0}, \zeta_{x0} \right) \\
&\left. \times Gamma \left(\tau_\beta; \alpha_{\beta 0}, \zeta_{\beta 0} \right) \times Gamma \left(\tau_\phi; \alpha_{\phi 0}, \zeta_{\phi 0} \right) \right) + c^{ev} \\
c_2^{\psi t} \equiv& c_1^{\psi t} - \alpha_{\psi t} \log \zeta_{\psi t} + \log \Gamma \left(\alpha_{\psi t} \right)
\end{aligned}
$$

4.12 Posterior for ν

Conditional:

$$
p(\nu) = \left(\frac{\nu}{2} \right)^{\frac{T\nu}{2} + \alpha_{\nu 0} - 1} \Gamma^{-T} \left(\frac{\nu}{2} \right) \exp \left(\frac{\nu \eta_1}{2} \right) \eta_2 \exp c_3^{\nu} \tag{37}
$$

where

$$
\begin{aligned}
\eta_1 \equiv& \sum_{t\in 1:T} (\log\psi_t - \psi_t) - 2\zeta_{\nu 0} \\
\eta_2 \equiv& \left[\int_{\nu^-}^{\infty} \left(\frac{\nu}{2}\right)^{\frac{T\nu}{2}+\alpha_{\nu 0}-1} \Gamma^{-T}\left(\frac{\nu}{2}\right) \exp\left(\frac{\nu\eta_1}{2}\right) d\nu\right]^{-1} \\
c_1^{\nu} \equiv& \alpha_{\nu 0}\log\zeta_{\nu 0} - \log\Gamma\left(\alpha_{\nu 0}\right) - \log\int_{\nu-}^{\infty}\left[1 - Gamma\left(z;\alpha_{\nu 0},\zeta_{\nu 0}\right)\right] dz \\
&+ \log\left(MN\left(y;\Phi x, \frac{1}{\tau_y} I\right) \times MN\left(\phi;\phi_0, \frac{1}{\tau_y\tau_\phi} M_0^{-1}\right)\right. \\
&\times MN\left(\beta;\beta_0 + D^{-1}\beta_0^{\Delta}, \frac{1}{\tau_x\tau_y\tau_\beta}\left[DA_0D\right]^{-1}\right) \times MN\left(x; F\beta + r, \frac{1}{\tau_x\tau_y}\Psi^{-1}\right) \\
&\times \prod_{k=1}^{K} Bern\left(\gamma_k;\omega\right) \times Beta\left(\omega;\kappa_0,\delta_0\right) \\
&\times Gamma\left(\tau_x;\alpha_{y0},\zeta_{y0}\right) \times Gamma\left(\tau_y;\alpha_{y0},\zeta_{y0}\right) \\
&\left.\left.\times Gamma\left(\tau_\beta;\alpha_{\beta 0},\zeta_{\beta 0}\right) \times Gamma\left(\tau_\phi;\alpha_{\phi 0},\zeta_{\phi 0}\right)\right)\right] + c^{ev} \\
c_2^{\nu} \equiv& c_1^{\nu} + (\alpha_{\nu 0} - 1)\log 2 - \sum_{t\in 1:T}\log\psi_t \\
c_3^{\nu} \equiv& c_2^{\nu} - \log\eta_2
\end{aligned}
$$