

Modeling the Cash Flow Dynamics of Private Equity Funds – Theory and Empirical Evidence*

Axel Buchner, Christoph Kaserer and Niklas Wagner

This Version: February 2009

*Axel Buchner is at the Technical University of Munich and at the Center of Private Equity Research (CEPRES); Christoph Kaserer is at the Technical University of Munich and at the Center for Entrepreneurial and Financial Studies (CEFS); Niklas Wagner is at Passau University. Earlier versions of this paper have benefited from comments by seminar participants at the 15th CEFS-ODEON Research Colloquium, Munich, the 6. Kölner Finanzmarktkolloquium "Asset Management", Cologne, the 14th Annual Meeting of the German Finance Association (DGF), Dresden, the XVI International Tor Vergata Conference on Banking and Finance, Rome, the Faculty Seminar in Economics and Management of the University Fribourg, the 11. Finanzwerkstatt at the University of Passau, and the Finance Seminar of the Santa Clara University, Leavey School of Business. We are also grateful to the European Venture Capital and Private Equity Association (EVCA) and Thomson Venture Economics for making the data used in this study available. All errors are our responsibility. Corresponding author: Axel Buchner; phone: +49-89-232 495616; email address: axel.buchner@wi.tum.de.

Modeling the Cash Flow Dynamics of Private Equity Funds – Theory and Empirical Evidence

Abstract

We present a novel continuous-time approach to modeling the typical cash flow dynamics of private equity funds. Our model consists of two independent components. First is a mean-reverting square-root process is applied to model the rate at which capital is drawn over time. Second are capital distributions which are assumed to follow an arithmetic Brownian motion with a time-dependent drift component that incorporates the typical time-pattern of the repayments of private equity funds. Our empirical analysis shows that the model can easily be calibrated to real world fund data by the method of *conditional least squares* (CLS) and nicely fits historical data. We use a data set of mature European private equity funds as provided by Thomson Venture Economics (TVE). Our model explains up to 99.6 percent of the variation in average cumulated net fund cash flows and provides a good approximation of the empirical distribution of private equity fund cash flows over a typical fund's lifetime. Overall, the empirical results indicate that our model is of economic relevance in an effort to accurately model the cash flows dynamics of private equity funds.

Keywords:

Venture capital, private equity, stochastic modeling

JEL classification code: G24

1 Introduction

Illiquid alternative assets, such as private equity, have become an increasingly significant portion of institutional asset portfolios as investors seek diversification benefits relative to traditional stock and bond investments. Various recent studies deal with the problem of estimating the risk and return characteristics of private equity investments.¹ So far, only little research has focused on the cash flow dynamics of private equity investments.

The present article proposes a novel stochastic model on the typical cash flow dynamics of private equity funds. The uncertain timing of capital drawdowns and proceeds poses a challenge to the management of future investment cash flows. Our stochastic model of the cash flow dynamics consists of two components. First is the stochastic model of drawdowns from the committed capital. Second is the stochastic model of the distribution of dividends and proceeds. Our model differs from the work of Takahashi and Alexander (2002) and Malherbe (2004, 2005) in that it solely relies on observable cash flow data.² Based on cash flow data, our analysis reveals that our model is flexible and can well match the various typical drawdown and distribution patterns that are observed empirically. Our model is easy to understand as well as to implement and it can be used in various directions. For example, an institutional investor may employ it for the purpose of liquidity planning. He may also measure the sensitivity of some ex-post performance measure – such as the IRR – to changes in typical drawdown or distribution patterns. Moreover, the model may be used as a tool for many risk management applications.

In our empirical analysis, we use a dataset of 203 mature European private equity funds of which 95 were fully liquidated during the January 1980 to June 2003 sample period. Our analysis shows that the model can easily be calibrated to real world fund data. The results of two consistency tests underline the economic relevance of our model, which explains up to 99.6 percent of the variation in average cumulated net cash flows of the liquidated funds. Overall, our model provides a good approximation of the empirical distribution of typical cash flows over the most relevant periods of fund lifetime.

The remainder of this article is organized as follows. In Section 2, we present our model. Section 3 illustrates the model dynamics and analyzes the impact of the various parameters on the timing and magnitude of fund cash flows. Section 4 presents our calibration results and shows a risk management application of the model. Finally, Section 5 gives a conclusion and identifies areas for future research.

¹Recent studies include, among others, Cochrane (2005), Diller and Kaserer (2009), Kaplan and Schoar (2005), Ljungquist and Richardson (2003a,b), Moskowitz and Vissing-Jorgenson (2002), Peng (2001a,b) and Phalippou and Gottschalg (2009).

²Takahashi and Alexander (2002) propose modeling the cash flow dynamics of a private equity fund. However, their model is deterministic and thus fails to reproduce the erratic nature of real world private equity cash flows. Malherbe (2004, 2005) develops a continuous-time stochastic version of the model of Takahashi and Alexander (2002). While his model considers randomness, it relies on the specification of the dynamics of an unobservable fund value and therefore has to account for an inaccurate fund valuation by incorporating an error term.

2 Modeling the Cash Flow Dynamics of Private Equity Funds

2.1 Institutional Framework

Private equity investments are typically intermediated through private equity funds. Thereby, a private equity fund denotes a pooled investment vehicle whose purpose is to negotiate purchases of common and preferred stocks, subordinated debt, convertible securities, warrants, futures and other securities of companies that are usually unlisted. The vast majority of private equity funds are organized as limited partnerships in which the private equity firm serves as the general partner (GP). The bulk of the capital invested in private equity funds is typically provided by institutional investors, such as endowments, pension funds, insurance companies, and banks. These investors, called limited partners (LPs), commit to provide a certain amount of capital to the private equity fund – the committed capital denoted as C . The GP then has an agreed time period in which to invest this committed capital – usually on the order of five years. This time period is referred to as the commitment period of the fund and will be denoted by T_c in the following. In general, when a GP identifies an investment opportunity, it “calls” money from its LPs up to the amount committed, and it can do so at any time during the prespecified commitment period. That is, we assume that capital calls of the fund occur unscheduled over the commitment period T_c , where the exact timing does only depend on the investment decisions of the GPs. However, total capital calls over the commitment period T_c can never exceed the total committed capital C . As those drawdowns occur, the available cash is immediately invested in managed assets and the portfolio begins to accumulate. When an investment is liquidated, the GP distributes the proceeds to its LPs either in marketable securities or in cash. The GP also has an agreed time period in which to return capital to the LPs – usually on the order of ten to fourteen years. This time period is also called the total legal lifetime of the fund and will be referred to by T_l in the following, where obviously $T_l \geq T_c$ must hold. In total, the private equity fund to be modeled is essentially a typical closed-end fund with a finite lifetime.³

2.2 Modeling Capital Drawdowns

We begin by assuming that the fund to be modeled has a total initial committed capital given by C as defined above. Cumulated capital drawdowns from the LPs up to some time t during the commitment period T_c are denoted by D_t , undrawn committed capital up to time t by U_t . When the fund is set up, at time $t = 0$, $D_0 = 0$ and $U_0 = C$ are given by definition. Furthermore, at any time $t \in [0, T_c]$, the simple identity

$$D_t = C - U_t \quad (2.1)$$

must hold. In the following, we assume that the dynamics of the cumulated capital drawdowns, D_t , can be described by the ordinary differential equation

$$dD_t = \delta_t(C - D_t)1_{\{0 \leq t \leq T_c\}}dt, \quad (2.2)$$

³For a more thorough introduction on the subject of private equity funds, for example, refer to Gompers and Lerner (1999), Lerner (2001) or to the recent survey article of Phalippou (2007).

where $\delta_t \geq 0$ denotes the rate of contribution or simply the fund's drawdown rate at time t and $1_{\{0 \leq t \leq T_c\}}$ is an indicator function. This modeling approach is similar to the assumption that capital is drawn over time at some non-negative rate δ_t from the remaining undrawn committed capital $U_t = C - D_t$.

In most cases, capital drawdowns of private equity funds are heavily concentrated in the first few years or even quarters of a fund's life. After high initial investment activity, drawdowns of private equity funds are carried out at a declining rate, as fewer new investments are made, and follow-on investments are spread out over a number of years. This typical time-pattern of the capital drawdowns is well reflected in the structure of equation (2.2). Under the specification (2.2), cumulated capital drawdowns D_t are given by

$$D_t = C - C \exp \left(- \int_0^{t \leq T_c} \delta_u du \right) \quad (2.3)$$

and instantaneous capital drawdowns $d_t = dD_t/dt$, i.e. the (annualized) capital drawdowns that occur over an infinitesimally short time interval from t to $t + dt$, are equal to

$$d_t = \delta_t C \exp \left(- \int_0^{t \leq T_c} \delta_u du \right) 1_{\{0 \leq t \leq T_c\}}. \quad (2.4)$$

Equation (2.4) shows that the initially very high capital drawdowns d_t at the start of the fund converge to zero over the commitment period T_c . This condition leads to the realistic feature that capital drawdowns are highly concentrated in the early years of a fund's lifetime under this specification. Furthermore, equation (2.3) shows that the cumulated drawdowns D_t can never exceed the total amount of capital C that was initially committed to the fund under this model setup, i.e., $D_t \leq C$ for all $t \in [0, T_c]$.

Usually, the capital drawdowns of real world private equity funds show an erratic feature as investment opportunities do not arise constantly over the commitment period T_c . As it stands, the model for the capital drawdowns is purely deterministic. A stochastic component can easily be introduced into the model by defining a continuous-time stochastic process for the drawdown rate δ_t . We assume that the drawdown rate δ_t follows a mean-reverting square-root process given by the stochastic differential equation

$$d\delta_t = \kappa(\theta - \delta_t)dt + \sigma_\delta \sqrt{\delta_t} dB_{\delta,t}, \quad (2.5)$$

where $\theta > 0$ is the long-run mean of the drawdown rate, $\kappa > 0$ governs the rate of reversion to this mean and $\sigma_\delta > 0$ reflects the volatility of the drawdown rate; $B_{\delta,t}$ is a standard Brownian motion.⁴ The drawdown rate behavior implied by the structure of this process has the following two relevant properties: (i) Negative values of the drawdown rate are precluded under this specification.⁵ This is a necessary

⁴This process is known in the financial literature as a square-root diffusion and was initially proposed by Cox et al. (1985) as a model of the short rate, generally referred to as the CIR model.

⁵If $\kappa, \theta > 0$, then δ_t will never be negative; if $2\kappa\theta \geq \sigma_\delta^2$, then δ_t remains strictly positive for all t , almost surely. See Cox et al. (1985), p. 391.

condition, as we model capital distributions and capital drawdowns separately and must, therefore, restrict capital drawdowns to be strictly non-negative at any time t during the commitment period T_c . (ii) Furthermore, the mean-reverting structure of the process reflects the fact that we assume the drawdown rate to fluctuate randomly around some mean level θ over time.

Under the specification of the square-root diffusion (2.5), the conditional expected cumulated and instantaneous capital drawdowns can be inferred. Given that $E_s[\cdot]$ denotes the expectations operator conditional on the information set available at time s , expected cumulated drawdowns at some time $t \geq s$ are given by

$$E_s[D_t] = C - U_s \exp[A(s, t) - B(s, t)\delta_s] \quad (2.6)$$

and the expected instantaneous capital drawdowns by

$$E_s[d_t] = -U_s[A'(s, t) - B'(s, t)\delta_s] \exp[A(s, t) - B(s, t)\delta_s], \quad (2.7)$$

where $A(s, t)$ and $B(s, t)$ are deterministic functions depending on the model parameters and the time subscripts s and t . In addition, $A'(s, t) = \partial A(s, t)/\partial t$ and $B'(s, t) = \partial B(s, t)/\partial t$.⁶

2.3 Modeling Capital Distributions

As capital drawdowns occur, the available capital is immediately invested in managed assets and the portfolio of the fund begins to accumulate. As the underlying investments of the fund are gradually exited, cash or marketable securities are received and finally returns and proceeds are distributed to the LPs of the fund. We assume that cumulated capital distributions up to some time $t \in [0, T_l]$ during the legal lifetime T_l of the fund are denoted by P_t and $p_t = dP_t/dt$ denotes the instantaneous capital distributions, i.e., the (annualized) capital distributions that occur over infinitesimally short time interval from t to $t + dt$.

We model distributions and drawdowns separately and, therefore, must also restrict instantaneous capital distributions p_t to be strictly non-negative at any time $t \in [0, T_l]$. The second constraint that needs to be imposed on the distributions model is the addition of a stochastic component that allows a certain degree of irregularity in the cash outflows of private equity funds. An appropriate assumption that meets both requirements is that the logarithm of instantaneous capital distributions, $\ln p_t$, follows an arithmetic Brownian motion of the form

$$d \ln p_t = \mu_t dt + \sigma_P dB_{P,t}, \quad (2.8)$$

where μ_t denotes the time dependent drift and σ_P is the constant volatility of the stochastic process. $B_{P,t}$ is a second standard Brownian motion, which, for simplicity, is assumed to be uncorrelated with $B_{\delta,t}$, i.e., $d\langle B_{P,t} B_{\delta,t} \rangle = 0$.⁷

From (2.8) it follows that the instantaneous capital distribution p_t must exhibit a lognormal distribution. Therefore, the process (2.8) has the relevant property that it precludes instantaneous capital distributions p_t from becoming negative at any time $t \in [0, T_l]$ and is therefore an economically reasonable assumption. For

⁶Both functions are well known from the bond pricing literature. For explicit expressions see Cox et al. (1985), p.393.

⁷This simplifying assumption can easily be relaxed to incorporate a positive or negative correlation coefficient ρ between the two processes.

an initial value p_s at time $s \leq t$, the solution to the stochastic differential equation (2.8) is given by

$$p_t = p_s \exp \left[\int_s^t \mu_u du + \sigma_P (B_{P,t} - B_{P,s}) \right]. \quad (2.9)$$

Taking the expectation $E_s[\cdot]$ of (2.9) conditional on the available information at time $s \leq t$ yields

$$E_s[p_t] = p_s \exp \left[\int_s^t \mu_u du + \frac{1}{2} \sigma_P^2 (t - s) \right]. \quad (2.10)$$

The dynamics of (2.9) and (2.10) both depend the specification of the integral over the time-dependent drift μ_t . The question posed now is to find a reasonable and parsimonious yet realistic way to model this parameter.

Defining an appropriate function for μ_t is not an easy task, as this parameter must incorporate the typical time pattern of the capital distributions of a private equity fund. In the early years of a fund, capital distributions tend to be of minimal size as investments have not had the time to be harvested. The middle years of a fund, on average, tend to display the highest distributions as more and more investments can be exited. Finally, later years are marked by a steady decline in capital distributions as fewer investments are left to be harvested. We model this behavior by first defining a fund multiple. If C denotes the committed capital of the fund, the fund multiple M_t at some time t is given by $M_t = P_t/C$, i.e., the cumulated capital distributions P_t are scaled by C . This variable will follow a continuous-time stochastic process as the multiple can also be expressed as $M_t = \int_0^t p_u du / C$. When the fund is set up, i.e. at time $t = 0$, $M_0 = 0$ holds by definition. As more and more investments of the fund are exited, the multiple increases over time. Thereby, we assume that its conditional expectation $\mathcal{M}_t^s = E_s^{\mathbb{P}}[M_t]$ at time t , given available information at time $s \leq t$, is

$$d\mathcal{M}_t^s = \alpha_t (m - \mathcal{M}_t^s) dt. \quad (2.11)$$

Under this specification, the expected multiple converges towards some long-run mean m over time, where this speed of convergence is governed by $\alpha_t = \alpha t$. Solving for \mathcal{M}_t^s by using the initial condition $\mathcal{M}_s^s = M_s$ yields

$$\mathcal{M}_t^s = m - (m - M_s) \exp \left[-\frac{1}{2} \alpha (t^2 - s^2) \right]. \quad (2.12)$$

With the condition, $p_t = (dM_t/dt)C$, the expected instantaneous capital distributions $E_s[p_t] = (d\mathcal{M}_t^s/dt)C$ turn out to be

$$E_s[p_t] = \alpha t (m C - P_s) \exp \left[-\frac{1}{2} \alpha (t^2 - s^2) \right]. \quad (2.13)$$

With equations (2.10) and (2.13) we are now equipped with two equations for the expected instantaneous capital distributions. Setting (2.10) equal to (2.13), we can finally solve for the integral $\int_s^t \mu_u du$. Substituting the result back into equation (2.9), the stochastic process for the instantaneous capital distributions at some time $t \geq s$ is given by

$$p_t = \alpha t (m C - P_s) \exp \left\{ -\frac{1}{2} [\alpha (t^2 - s^2) + \sigma_P^2 (t - s)] + \sigma_P \epsilon_t \sqrt{t - s} \right\}, \quad (2.14)$$

where $\epsilon_t \sqrt{t-s} = (B_{P,t} - B_{P,s})$ and $\epsilon_t \sim N(0,1)$. The stochastic process (2.14) can directly be used as a Monte-Carlo engine to generate sample paths of the capital distributions of a fund. In the next section, we illustrate the dynamics of both model components and analyze their sensitivity to changes in the main model parameters.

3 Model Analysis

Our model consists of two independent components that are governed by different model parameters. The only parameter that enters both model components is the committed capital C . This variable does not affect the timing of the capital drawdowns or distributions. Rather, it serves as a scaling factor to influence the magnitude of the overall expected cumulative drawdowns and distributions. As far as the capital drawdowns are concerned, the main model parameter governing the timing of the drawing process is the long-run mean drawdown rate θ . Increasing θ accelerates expected drawdowns over time. Thus, higher values of θ , on average, increase the capital drawn at the start of the fund and decreases the capital drawn in later phases of the fund's lifetime – a behavior which is in line with intuition. Compared to the impact of θ , the influence of the mean reversion coefficient κ and the volatility σ_δ on the expected drawing process are only small. In general, the effect of both parameters is about the same relative magnitude. However, the direction may differ in sign. Increases in σ_δ tend to slightly decelerate expected drawdowns, whereas increases in κ tend to slightly accelerate them. In contrast, σ_δ is the main model parameter governing the volatility of the capital drawdowns. The higher σ_δ , the more erratic the capital drawdowns will be over time.

The timing and magnitude of the capital distributions is determined by three main model parameters. The coefficient m is the long-run multiple of the fund, i.e., m times the committed capital C determines the total amount of capital that is expected to be returned to the investors over the fund's lifetime. The higher m the more capital per dollar committed is expected to be paid out. The coefficient σ_P governs the volatility of the capital distributions. Higher values of σ_P , hence, lead to more erratic capital distributions over time. Finally, α governs the speed at which capital is distributed over the fund's lifetime. To make this parameter easier to interpret, it can simply be related to the expected amortization period of a fund. Let t_A denotes the expected amortization period of the fund, i.e., the expected time needed until the cumulated capital distributions are equal to or exceed the committed capital C of the fund, then it follows from equation (2.12) that

$$E_0[M_{t_A}] \equiv 1 = m \left[1 - \exp \left(-\frac{1}{2} \cdot \alpha \cdot t_A^2 \right) \right] \quad (3.1)$$

must hold. Solving for α gives

$$\alpha = \frac{2 \ln \frac{m}{m-1}}{t_A^2}. \quad (3.2)$$

That is, α is inversely related to the expected amortization period t_A of the fund. Consequently, higher values of α lead to shorter expected amortization periods.

[Insert Table 1 about here]

To illustrate the model dynamics, Figure 1 compares the expected cash flows (drawdowns, distributions and net fund cash flows) for two different hypothetical funds.⁸ As the different sets of parameter values in Table 1 reveal, both hypothetical funds are assumed to have the same long-run multiple m and a committed capital C that is standardized to 1. That is, both funds are assumed to have cumulated capital drawdowns equal to 1 over their lifetime and expected cumulated capital distributions equal to 1.5. However, they differ in the timing of the capital drawdowns and distributions, as indicated by the different values of the other model parameters. For the first fund it is assumed that drawdowns occur rapid in the beginning, whereas capital distributions take place late. Conversely, for the second fund it is assumed that drawdowns occur more progressive and that distributions take place sooner. This is mainly achieved through a lower value of α and a higher value of θ for Fund 1. The effect can be inferred by comparing the different lines in Figure 1 (a) and (b). For this reason, both funds also have different expected amortization periods. From equation (3.2), the expected amortization periods of funds 1 and 2 are given by 8.6 years and 6.1 years, respectively.

[Insert Figure 1 and Figure 2 about here]

It is important to acknowledge that the basic patterns of the model graphs of the capital drawdowns, distributions and net cash flows in Figure 1 conform to reasonable expectations of private equity fund behavior. In particular, the cash flow streams that the model can generate will naturally exhibit a lag between the capital drawdowns and distributions, thus reproducing the typical development cycle of a fund and leading to the private equity characteristic J-shaped curve for the cumulated net cash flows that can be observed in Figure 1 (c). Furthermore, it is important to stress that our model is flexible enough to generate the potentially many different patterns of capital drawdowns and distributions. By altering the main model parameters, both timing and magnitude of the fund cash flows can be controlled in the model. Finally, our model captures well the erratic nature of real world private equity fund cash flows. Figure 2 illustrates this by giving simulated paths of the capital drawdowns and distributions for Fund 1. It shows that our model has the economically reasonable feature that volatility of the fund cash flows varies over time. Specifically, the volatility of the drawdowns (distributions) is high in times when average drawdowns (distributions) are high, and low otherwise. This is indicated in Figure 2 by the 99.5-percent confidence bounds given.

So far, our analysis was based on theoretical reasoning. To verify our intuition, the next section compares our model to historical fund data.

4 Empirical Evidence

4.1 Data Description

We use a dataset of European private equity funds that has been provided by Thomson Venture Economics (TVE). It should be noted that TVE uses the term private equity to describe the universe of all venture investing, buyout investing and

⁸The corresponding expectations are obtained from equations (2.6) and (2.7) for the capital drawdowns and from equations (2.12) and (2.13) for the capital distributions. In addition, note that unconditional expectations are shown, i.e., we set $s = 0$.

mezzanine investing.⁹ We have been provided with various information related to the exact timing and size of cash flows, residual net asset values (NAV), fund size, vintage year, fund type, fund stage and liquidation status for a total of 777 funds over the period from January 1, 1980 through June 30, 2003. All cash flows and reported NAVs are net of management fees and carried interest.

Before presenting our empirical results, we have to deal with a problem caused by the limited number of liquidated funds included in our data set. The purpose of our study requires the knowledge of the full cash flow history of the analyzed funds. In principle, this is only possible for those funds that have already been fully liquidated at the end of our observation period. This reduces our data set to a total number of only 95 funds (47 venture capital funds and 48 buyout funds). So, only a small subset of the full data can be used for analysis. Furthermore, given that the average age of the liquidated funds in our sample is about 13 years, one can infer that restricting the analysis to liquidated funds could also limit our results as more recently founded funds would be systematically left out. In order to mitigate this problem, we follow a common approach in the literature to increase the data universe by adding largely liquidated funds to the working sample.¹⁰ In specific, we add non-liquidated funds to our sample if their residual value is not higher than 10 percent of the undiscounted sum of the absolute value of all previously accrued cash flows. In such cases, treating the current net asset value at the end of the observation period as a final cash flow will have a minor impact on our results. All funds that are not liquidated by 30 June 2003 and satisfy this condition are added to the liquidated funds to form an extended data sample of mature funds. This extended sample consists of a total of 203 funds and comprises 102 venture capital funds and 101 buyout funds. In the subsequent, we base our core empirical analysis on the sample of liquidated private equity funds. In addition, the broader sample of mature funds is used to test the robustness of our results.

4.2 Estimation Results

Our model parameters are estimated by using the concept of *Conditional Least Squares* (CLS).¹¹ To make funds of different investment size comparable, all capital drawdowns and distributions are first expressed as a percentage of the corresponding total committed capital.

[Insert Table 2 about here]

Table 2 shows the model parameters we estimated for the drawdown and distribution model. As for the capital drawdowns, the estimated annualized long-run mean drawdown rate θ of all $N = 95$ liquidated funds amounts to 0.57. This implies that in the long-run approximately 14.25 percent of the remaining committed capital is drawn on average in each quarter of a fund's lifetime. However, the exceptionally high value reported for the volatility σ_δ also indicates that the sample

⁹For a detailed overview on the TVE dataset and a discussion of its potential biases see Kaplan and Schoar (2005).

¹⁰See, for example, Ljungquist and Richardson (2003a,b), Kaplan and Schoar (2005), Diller and Kaserer (2009), and Phalippou and Gottschalg (2009).

¹¹The concept of conditional least squares, which is a general approach for estimating the parameters involved in a continuous-time stochastic process, was given a thorough treatment by Klimko and Nelson (1978). An application of the CLS method to the CIR process is given by Overbeck and Rydén (1997). See Buchner et al. (2008) for a detailed description of the estimation methodology applied.

private equity funds draw down their capital at a very fluctuating pace over time. As far as the capital distributions are concerned, the long-run multiple m of the sample of liquidated funds is estimated to equal 1.69. That is, on average, the funds in our sample distribute 1.69 times their committed capital over the total lifetime. The reported α coefficient further implies that the sample funds have an average amortization period of 7.8 years or around 94 month.¹² Further, it can be inferred from the reported value for σ_P that the capital distributions also show an exceptionally high volatility over time.

In addition, two interesting differences between the parameters values of our two data samples are apparent. First, it can be inferred from the higher long-run multiple m that the mature funds, on average, distribute more capital to their investors. Second, it also seem that these funds draw down capital at a somewhat slower pace, as indicated by the lower parameter θ . These two differences can be attributed to the fact that the extended sample of mature funds contains a larger fraction of funds that were founded more recently. These younger funds, on average, achieve higher multiples due to the favorable exit conditions private equity funds found especially in the late 1990s. In addition, the slower drawdown pace can potentially be explained by the higher competition for attractive deals during this period, which decreases the average speed at which capital is drawn by the fund managers.

4.3 Model Consistency Tests

Having thus calibrated the model to historical fund cash flow data, we now confront it with the empirical evidence. In particular, two simple consistency tests are considered in the following.

A. Consistency of the Implied Cash Flow Time-Patterns

The first simple way to gauge the specification of our model is to examine whether the model's implied cash flow patterns are consistent with those implicit in the time series of our defined data sample. That is, are the model expectations of the cash flows similar in magnitude and timing to the values derived from their data samples counterparts? The closer the values, the less misspecified is the model.

[Insert Figure 3 about here]

In this sense, the charts in Figure 3 compare the historical average capital draw-downs, capital distributions and net fund cash flows of the sample of liquidated funds to the corresponding expectations that can be constructed from our model by using the parameters reported in Table 2. Overall, the results from Figure 3 indicate an excellent fit of the model with the historical data of the sample liquidated funds. In particular, as measured by the coefficient of determination, R^2 , our model can explain a very high degree of 99.6 percent of the variation in average cumulated net fund cash flows and 94.7 percent of the variation in average yearly net fund cash flows. Although not shown in the figure, a similar fit can also be found for the broader sample of mature private equity funds. For this sample, our model can explain a very high degree of 97.73 percent of the variation in average yearly net

¹²The expected amortization period can directly be inferred by solving equation (3.2) for the amortization period t_A .

fund cash flows. That is, the model fit with this broader sample is even slightly better.

The economic relevance of this analysis stems from the fact that it shows that our model is capable to correctly forecast the cash flows of a broadly diversified portfolio of private equity funds. This is of particular interest for institutional investors with a large private equity exposure, for example, when analyzing their portfolios with the purpose of liquidity planning.

B. Consistency of the Implied Cash Flow Probability Distributions

Besides its economic relevance, the previous analysis can only provide an incomplete picture of the goodness of fit of our model. This follows as it is only based on a comparison of the first distributional moment of the fund's cash flows. Therefore, a second consistency test of the model is applied that is based on a comparison of the entire theoretical and empirical distributions of the cash flows.

[Insert Figure 4 and Figure 5 about here]

A simple graphical method to compare a sample distribution to a prespecified theoretical distribution is the *quantile-quantile* plot, for short *QQ*-plot. *QQ*-plots can be used to plot the quantiles of an empirical distribution against the quantiles of the prespecified theoretical reference distribution. *QQ*-plots for the yearly capital drawdowns and capital distributions of the sample of fully liquidated funds are illustrated in Figure 4 and 5, respectively.¹³ For illustrative purposes, Figure 4 covers the first four years and Figure 5 covers the first eight years of the lifetime of the sample funds. If the empirical data were generated from random samples of the model reference distributions, the plots should all look roughly linear. In addition, if the location and scale parameters of the distributions are also correctly specified, then the plotted values should fall on the 45-degree reference line with an intercept of zero. Both figures indicate a reasonable fit of the theoretical and empirical quantiles in some years, whereas the plotted values deviated more or less from the given reference lines in other years. For example, in year seven of Figure 5 the plotted values almost perfectly match the reference line. This result indicates that our model can only provide an approximation of the true distribution of the cash flows over the lifetime. This conjecture is also supported by a Kolmogorov–Smirnov test that reveals that the null hypothesis that the empirical capital drawdowns and capital distributions are generated by the model reference distribution can be rejected at a 10 percent, 5 percent and 1 percent level for all years. In order to assess the quality of the approximation, two statistical measures are employed in the following. The first is the coefficient of determination R^2 . The coefficient determination R^2 here gives the percentage of the variation in the empirical quantiles that can be explained by the model. In addition, an adapted coefficient of determination $R^2(45^\circ)$ is used. The idea behind this coefficient is that the plotted values are roughly on a straight line if the empirical values are generated from a random sample of the theoretical distribution. However, this must not necessarily be the 45-degree reference line if the scale and location parameter of the theoretical probability distributions are not correctly specified. The coefficient $R^2(45^\circ)$ accounts for this effect.

¹³In addition, note that the theoretical quantiles in Figure 4 and 5 are derived by the method of a Monte-Carlo simulation of discrete-time versions of the stochastic model components. Basis of this simulation are 500,000 iterations for each time interval, thus, making sure that the difference between the actual and approximated theoretical quantiles is only of negligible magnitude.

[Insert Table 3 about here]

Table 3 summarizes the results for both statistical measure of goodness of fit. We use a time period of seven years for the capital drawdowns and 15 years for the capital distributions.¹⁴ The results for the sample of liquidated funds are given in the first panel of the table. As for the capital drawdowns, values of R^2 and $R^2(45^\circ)$ are all above 75 percent. For example, in year three 95.11 percent (R^2) of the variation in the empirical quantiles can be explained by the model. Taking the 45-degree line as a reference, the explanatory power slightly decreases with $R^2(45^\circ)$ reaching 94.98 percent. However, both values indicate that the model provides a very good approximation of the empirical distribution in that year. After year five the values of the two coefficients seem to decrease, indicating that the quality of the approximation deteriorates. As far as the capital distributions are concerned, the results show that our model provides a very good approximation of the true empirical distribution in the years four to ten. In these years the R^2 and $R^2(45^\circ)$ values are almost all above 90 percent. Furthermore, the $R^2(45^\circ)$ are close to R^2 values, thus, indicating that the scale and location parameters of the theoretical distributions are also correctly specified. An important aspect revealed by the consistency test for the capital disbursements is that the values of the two coefficients R^2 and $R^2(45^\circ)$ decrease substantially in the first few years and in the later years of a fund's lifetime, with $R^2(45^\circ)$ even turning negative in the first year. This effect can be partially linked to the fact that the stochastic model assumes a lognormal distribution for the capital outflows over the fund's expected life. This assumption allows for arbitrarily low values of the capital out flows, but not values equal to zero. Many of the private equity funds in our sample do not show any capital disbursements at the beginning and/ or at the end of their lifespan. A behavior that cannot be adequately captured by the dynamics of our model. Despite this drawback, the high values of the two coefficients R^2 and $R^2(45^\circ)$ over most time periods show that the model provides a good approximation of the distributions of cash outflows. The second panel in Table 3 summarizes the same statistical measure for the broader sample of mature funds. As can be inferred, the results are robust to the sample that is used. The R^2 and $R^2(45^\circ)$ figures even seem to be higher in most years, which again indicates a slightly better fit of our model with this extended sample.

In conclusion, our model can provide a good approximation of the distribution of the capital drawdowns of a private equity fund – especially in the first years of the lifetime. A similar result holds for the approximation of the capital distributions in the middle phase of the fund's lifetime. In other years, the quality of the approximations more or less decreases. Considering however, that those are the most relevant time periods over which the largest fraction of the capital drawdowns and distributions occur, our model is of economical and practical relevance in an effort to accurately model the cash flows dynamics of private equity funds.

4.4 Model Application

Our model can be used in various directions. An investor can employ it to estimate expected future cash needed to meet capital commitments, as well as projected distributions that generate liquidity in the future. In this context, it is important to stress that the model is able to incorporate and respond to the actual capital

¹⁴Note that we restrict our analysis to these time periods as most of the funds in our sample do not show any capital drawdowns after year 7 and no capital distributions after year 15.

drawdown and distribution experience.¹⁵ It can be updated each period with actual data, allowing it to adjust to current events. For example, the capital drawdowns in the model for a given period depend on the outstanding undrawn amounts to the fund. This means that unusually large early capital drawdowns diminish future capital drawdowns. Likewise, estimates of future capital distributions in the model do also depend on prior periods' data, thus, allowing actual data to influence the future projections. In addition, it should be acknowledged here that the model can easily be adapted to changing investment environments. This flexibility is particularly helpful for an investor in analyzing the impact of varying market conditions on the future cash flows. For example, an investor can reflect the forecast of an unfavorable exit environment for private equity funds in the model by lowering the long-run multiple m or the parameter α that governs the speed at which capital distributions occur over time.

[Insert Table 4 about here]

In a more general context, the model can also be used for risk management purposes. The model constituents (i.e., the capital drawdowns and distributions) allow, as a by-product, ex-ante calculations of the expected risk and return profile of a private equity fund. These calculations can be carried out with the performance measures commonly employed in the private equity industry – such as the IRR or the DPI (Distribution to Paid-In) multiple. An illustrative example for this is provided in Table 4. The base case scenario in column 1 is constructed by using the estimated model parameters for the sample liquidated funds shown in Table 2. That is, we assume here, for simplicity, that the fund to be modeled has a drawdown and distribution schedule that conforms to the average historical behavior of our sample liquidated funds. All calculations are based on quarterly simulated fund cash flows. Various risk and return measures are provided in Table 4. The results show, for example, that the ex-ante expected IRR of this fund amounts to 8.94 percent per annum and that the probability of incurring a loss ($\text{Prob}(\text{IRR} < 0\%)$) is equal to 11.65 percent.

In addition, columns 2-5 in Table 4 provide a sensitivity analysis that illustrates how the long-run multiple m and the long-run drawdown rate θ affect the risk and return profile of this fund. Columns 2-3 give a best-case and worst-case scenario analysis for the long-run multiple m . The best-case (worst-case) scenario in column 2 (3) is constructed by using the base case parameter m plus (minus) two times the standard error of the estimated value given in Table 2. The results illustrate that a lower value of m than expected can substantially decrease expected returns and increase the risk of the fund. For example, the probability of incurring a loss amounts to 30.43 percent under this worst-case scenario. A similar analysis is carried out in columns 4-5 for the long-run drawdown rate θ . Generally, a higher value of θ (i.e. a faster drawdown schedule) leads to capital being tied in the fund longer than expected. This reduced the expected return of the fund, as can be inferred from the figures given in column 4 of the table. However, it is also important to note that this leaves the risk parameters (in particular the loss probability) of the fund almost unchanged.

¹⁵This ability arises from the fact that both model components are formulated conditional on the current available information set of the investor. In particular, the future capital drawdowns at some time $t > s$ depend on the undrawn amounts at time s (see equations (2.6) and (2.7)). Similarly, the future capital distributions depend on the cumulated capital drawdowns that have occurred up to time s (see equation (2.14)).

Note that a similar analysis can also be carried to study the impact of the other model parameters on the risk and return profile of a private equity fund. In conclusion, this short model application underlines that an investor can employ the model presented here to measure the sensitivity of the IRR (or any other ex-post performance measure) to changes in the drawdown or distribution schedule in a very clear and concise way.

5 Conclusion

In this paper, we present a new stochastic model for the dynamics of a private equity funds. Our work differentiates from previous research in the area of venture and private equity fund modeling in the sense that our model of a fund's capital drawdowns and distributions is based on observable economic variables only. That is, we do not specify a process for the dynamics of the unobservable value of a fund's assets over time, as done in the deterministic and stochastic models of Takahashi and Alexander (2002) and Malherbe (2004, 2005). The dynamics of our model are solely based on observable cash flows data, which seems to be a more promising stream for future research in the area of private equity fund modeling. A theoretical model analysis shows that our model is flexible enough to reproduce the many different drawdown and distribution patterns that can be observed for real world private equity funds. Furthermore, the economic relevance of our model is also underlined by the empirical analysis we performed. Overall, we can show that that our model fits the historical fund data nicely. At the same time, the main model parameters are easy to understand and the model is easy to implement. The model can be used in various directions. An institutional investor can, for example, employ it to estimate expected future cash needed to meet capital commitments, as well as projected distributions that generate liquidity in the future. More generally, the model could be used as a risk management tool in the private equity context. Furthermore, it should be acknowledged that although we focus on private equity funds in this article, our model can in principle also be extended to other alternative asset fund types. For instance, by altering the input parameters, the model can be utilized to represents funds that invest in other assets such as real estate and infrastructure. Our model should, therefore, constitute a promising approach for further research. In fact, the empirical performance of the model would have to be scrutinized in more detail by using a broader data set as well as more sophisticated calibration and benchmarking tools. Such research would constitute a basis for further developing the model proposed here.

References

- Buchner, A., C. Kaserer, and N. Wagner (2008). Stochastic modeling of private equity - an equilibrium based approach to fund valuation. Working paper, CEFS.
- Cochrane, J. H. (2005). The risk and return of venture capital. *Journal of Financial Economics* 75, 3–52.
- Cox, J. C., J. E. Ingersoll, and S. A. Ross (1985). A theory of the term structure of interest rates. *Econometrica* 53(2), 385–408.
- Diller, C. and C. Kaserer (2009). What drives cash flow based european private equity returns? - fund inflows, skilled gps and/or risk? *European Financial Management* 15(3), 643–675.
- Gompers, P. and J. Lerner (1999). *The Venture Capital Cycle*. Cambridge: MIT Press.
- Kaplan, S. N. and A. Schoar (2005). Private equity performance: Returns, persistence and capital flows. *Journal of Finance* 60(4), 1791–1823.
- Klimko, L. A. and P. I. Nelson (1978). On conditional least squares estimation for stochastic processes. *The Annals of Statistics* 6(3), 629–642.
- Lerner, J. (2001). Venture capital and private equity: A course overview. Technical report, Harvard Business School.
- Ljungquist, A. and M. Richardson (2003a). The cash flow, return and risk characteristics of private equity. Working Paper 9454, NBER.
- Ljungquist, A. and M. Richardson (2003b). The investment behaviour of private equity fund managers. Working paper, NYU Stern.
- Malherbe, E. (2004). Modeling private equity funds and private equity collateralised fund obligations. *International Journal of Theoretical and Applied Finance* 7(3), 193–230.
- Malherbe, E. (2005). A model for the dynamics of private equity funds. *Journal of Alternative Investments Winter*, 82–89.
- Moskowitz, T. J. and A. Vissing-Jorgenson (2002). The returns to entrepreneurial investment: A private equity premium puzzle? *American Economic Review* 92, 745–778.
- Overbeck, L. and T. Rydén (1997). Estimation in the cox-ingersoll-ross model. *Econometric Theory* 13(3), 430–461.
- Peng, L. (2001a). Building a venture capital index. Yale icf working paper 00-51.
- Peng, L. (2001b). A new approach of valuing illiquid asset portfolios. Yale icf working paper no. 00-24.
- Phalippou, L. (2007). Investing in private equity funds: A survey. Technical report, University of Amsterdam.

Phalippou, L. and O. Gottschalg (2009). The performance of private equity funds.
Review of Financial Studies 22(4), 1747–1776.

Takahashi, D. and S. Alexander (2002). Illiquid alternative asset fund modeling.
Journal of Portfolio Management Winter, 90–100.

Tables and Figures

Table 1: Model Parameters for the Capital Drawdowns and Distributions of Two Different Funds

Table 1 shows the sample model parameters for the capital drawdowns and distributions of two different funds. The committed capital C of both funds is standardized to 1. In addition, the starting values of the drawdown rates δ_0 are set equal their corresponding long-run means θ .

Model	Drawdowns			Distributions		
	κ	θ	σ_δ	m	α	σ_P
Fund 1	2.00	1.00	0.50	1.50	0.03	0.20
Fund 2	0.50	0.50	0.20	1.50	0.06	0.20

Table 2: Estimated Model Parameters for the Capital Drawdowns and Distributions

Table 2 shows the estimated model parameters for the sample of the $N = 95$ liquidated funds and the extended sample of the $N = 203$ mature funds. Standard errors of the estimates are given in parentheses. Standard errors of the estimated θ , κ and α coefficients are derived by a bootstrap simulation. In addition, note that we set $\delta_0 = 0$.

Model	Drawdowns			Distributions		
	κ	θ	σ_δ	m	α	σ_P
Liquidated Funds	4.2472 (3.1697)	0.5686 (0.1299)	4.0258 -	1.6871 (0.1694)	0.0295 (0.0007)	1.6589 -
Mature Funds	7.3259 (5.8762)	0.4691 (0.1043)	4.7015 -	1.8462 (0.1355)	0.0284 (0.0007)	1.6718 -

Table 3: Consistency of the Model Implied Probability Distributions of the Yearly Capital Drawdowns and Distributions

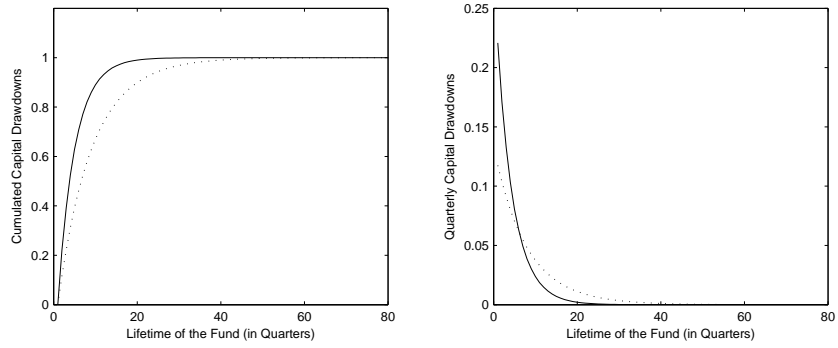
Table 3 shows the statistical measures of goodness of fit, R^2 and $R^2(45^\circ)$, for the yearly capital drawdowns and capital distributions of the core sample of $N = 95$ liquidated funds and the extended sample of $N = 203$ mature funds.

Year	Liquidated Funds ($N = 95$)				Mature Funds ($N = 203$)			
	Drawdowns		Distributions		Drawdowns		Distributions	
	R^2	$R^2(45^\circ)$	R^2	$R^2(45^\circ)$	R^2	$R^2(45^\circ)$	R^2	$R^2(45^\circ)$
1	96.76%	91.17%	81.62%	<0	96.98%	92.91%	79.93%	<0
2	96.30%	82.22%	83.91%	71.26%	97.07%	91.05%	81.07%	66.49%
3	95.11%	94.98%	86.43%	60.36%	94.79%	94.49%	93.86%	79.27%
4	94.95%	93.52%	97.76%	91.76%	95.28%	93.12%	98.11%	93.12%
5	89.94%	84.88%	97.70%	93.39%	93.47%	88.74%	96.03%	93.25%
6	86.07%	75.75%	88.80%	83.42%	94.98%	84.34%	92.45%	85.81%
7	80.46%	77.36%	98.76%	95.45%	82.64%	72.71%	99.39%	97.30%
8	-	-	98.57%	93.66%	-	-	97.61%	95.47%
9	-	-	95.50%	91.12%	-	-	97.92%	84.75%
10	-	-	93.74%	91.25%	-	-	97.73%	92.48%
11	-	-	96.78%	73.92%	-	-	95.62%	72.18%
12	-	-	94.36%	53.72%	-	-	92.38%	60.80%
13	-	-	92.03%	60.80%	-	-	94.56%	72.40%
14	-	-	83.51%	56.05%	-	-	81.61%	67.80%
15	-	-	64.91%	42.16%	-	-	79.43%	65.45%

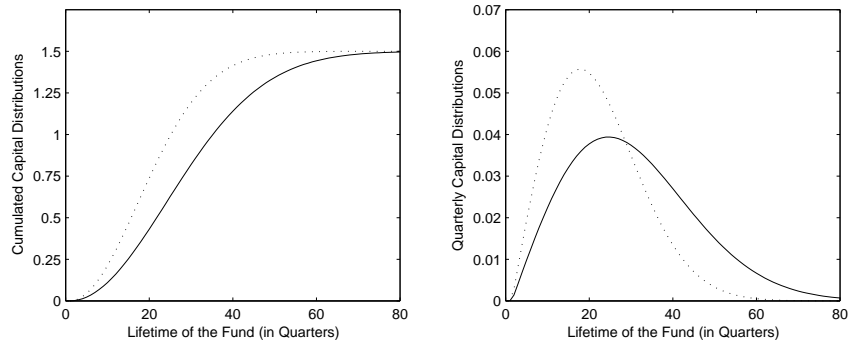
Table 4: Sensitivity Analysis for the Risk Profile of a Private Equity Fund

Table 4 illustrates the risk profile of the private equity fund and provides a sensitivity analysis. The base case in column 1 is constructed by using the estimated model parameters for the sample liquidated funds shown in Table 2. Columns 2-5 show how the results change by altering the long-run multiple m and the long-run drawdown rate θ . High Dist. (Low Dist.) corresponds to the case when m is equal to the base case parameter plus (minus) two times the standard error of the estimator. Similarly, Fast Draw. (Slow Draw.) corresponds to the case when θ is equal to the base case parameter plus (minus) two times the standard error of the estimator. All calculations are based on quarterly simulated fund cash flows.

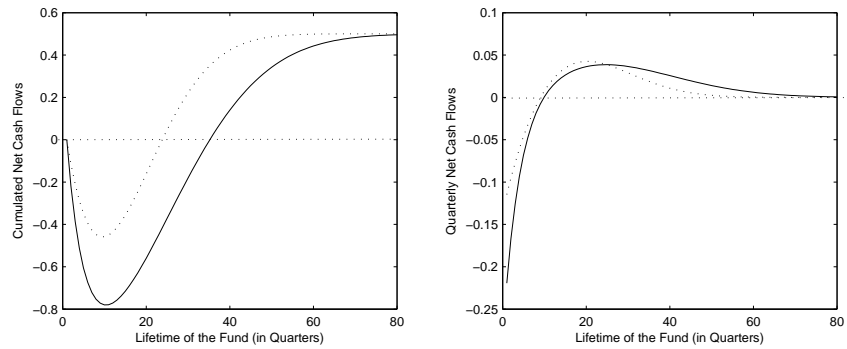
	Internal Rate of Return (in % p.a.)				
	Base Case	High Dist.	Low Dist.	Fast Draw.	Slow Draw.
Mean	8.94%	13.04%	4.72%	8.67%	9.42%
Median	6.66%	10.12%	2.81%	6.53%	6.82%
Std.	13.84%	20.06%	12.09%	13.01%	16.34%
Lower 99th Quantile	-4.52%	-2.01%	-7.16%	-4.47%	-4.66%
Lower 95th Quantile	-1.88%	0.68%	-4.89%	-1.87%	-2.05%
Probability of a Loss (Prob(IRR<0%))	11.65%	3.55%	30.43%	11.65%	11.78%
Average IRR given a Loss	-2.00%	-1.54%	-2.81%	-1.97%	-2.09%



(a) Expected Cumulated Capital Drawdowns (Left) and Quarterly Capital Drawdowns (Right)



(b) Expected Cumulated Capital Distributions (Left) and Quarterly Capital Distributions (Right)



(c) Expected Cumulated Net Fund Cash Flows (Left) and Quarterly Net Fund Cash Flows (Right)

Figure 1: Model Expectations for Two Different Funds (Solid Lines represent Fund 1; Dotted Lines represent Fund 2)

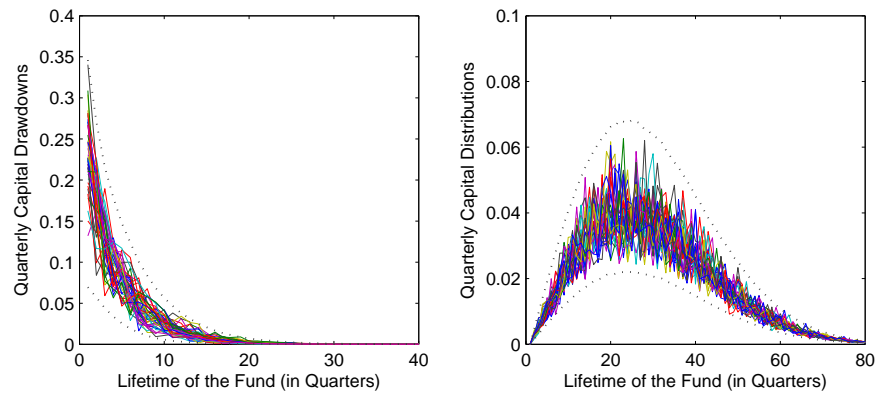
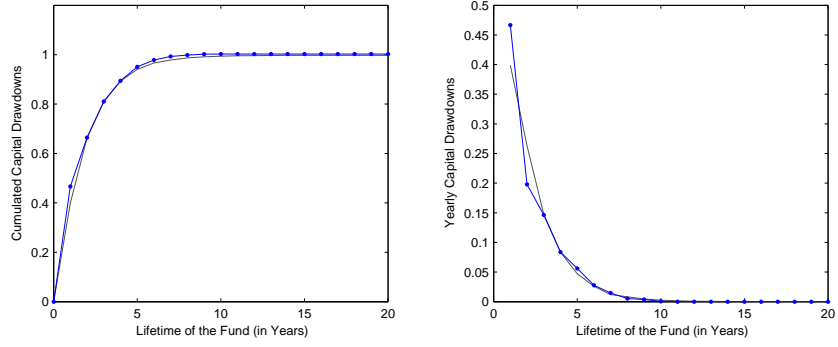
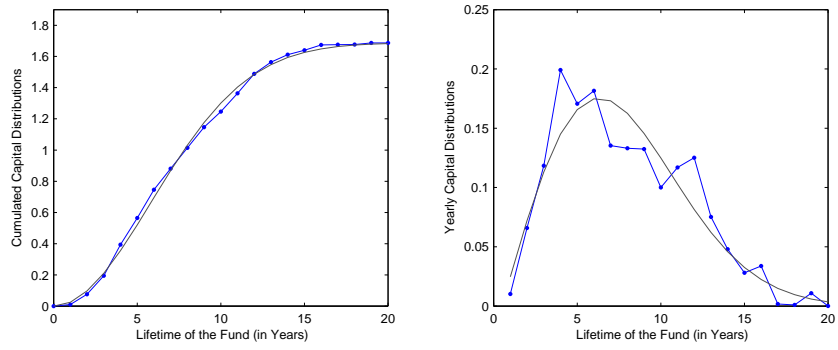


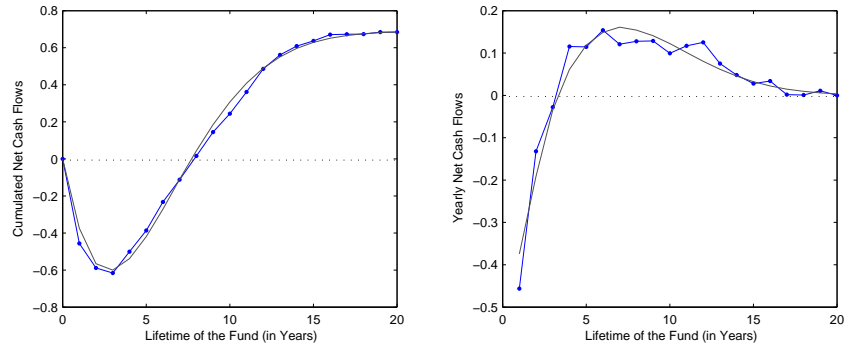
Figure 2: Simulated Paths of the Capital Drawdowns (Left) and Capital Distributions (Right) for Fund 1 (Dotted Lines represent 99.5% Confidence Bounds)



(a) Cumulated Capital Drawdowns (Left) and Yearly Capital Drawdowns (Right)



(b) Cumulated Capital Distributions (Left) and Yearly Capital Distributions (Right)



(c) Cumulated Net Fund Cash Flows (Left) and Yearly Net Fund Cash Flows (Right)

Figure 3: Model Expectations Compared to Historical Data of the 95 Liquidated Private Equity Funds (Solid Lines represent Model Expectations; Dotted Lines represent Historical Data)

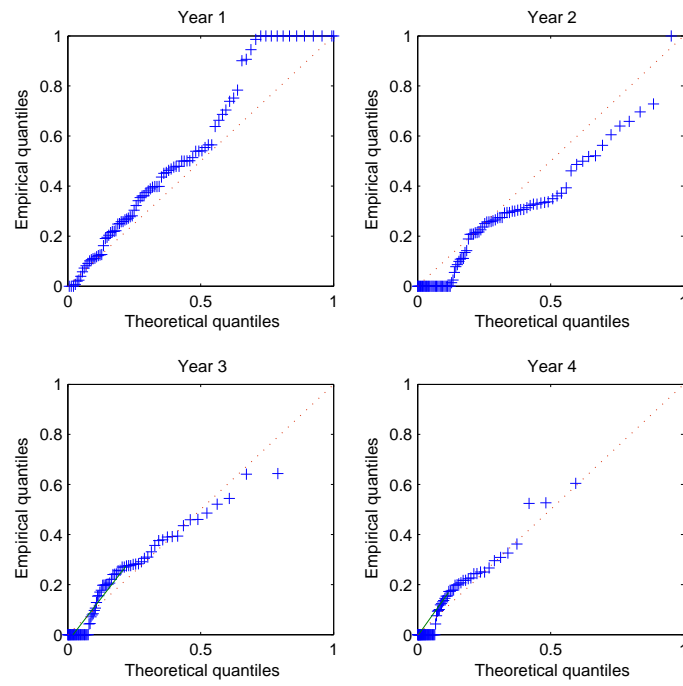


Figure 4: *Quantile-Quantile-Plots* of the Yearly Capital Drawdowns for the Sample of the 95 Liquidated Private Equity Funds; Years 1 to 4

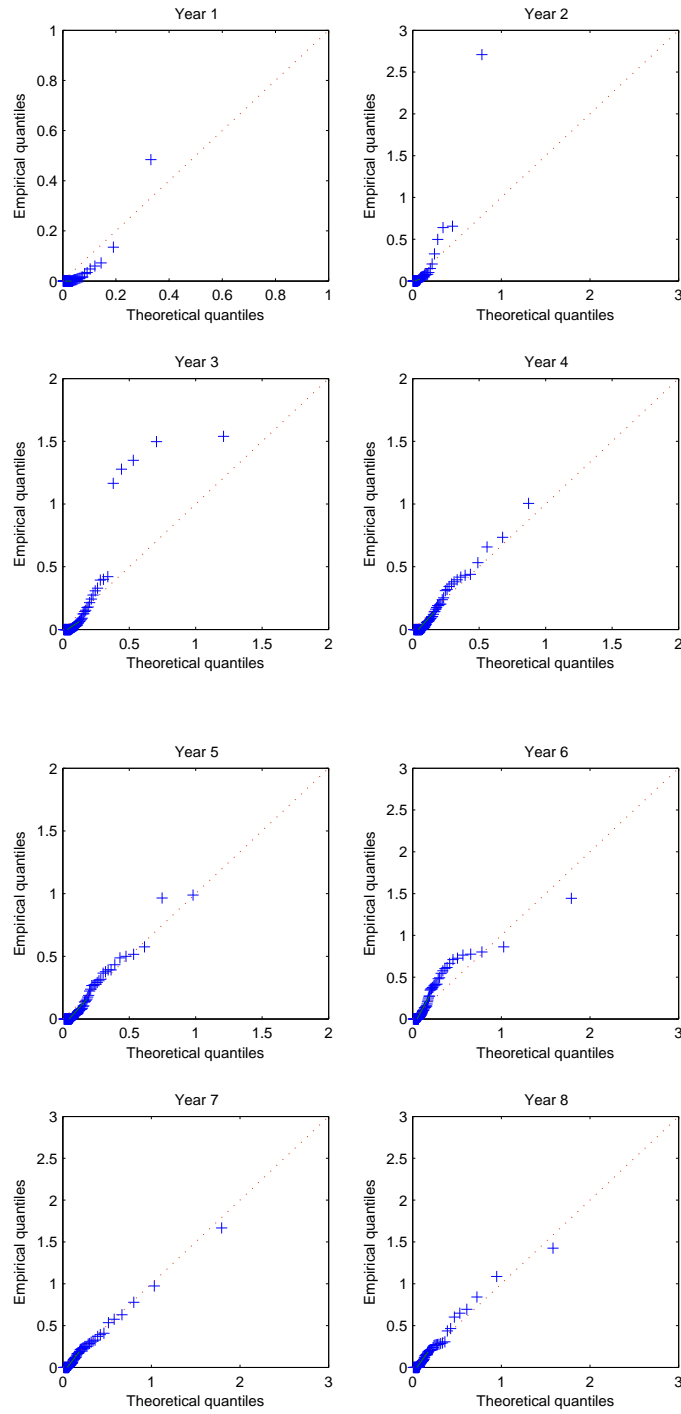


Figure 5: *Quantile-Quantile-Plots of the Yearly Capital Distributions for the Sample of 95 Liquidated Private Equity Funds; Years 1 to 8*