**Generalized Multi-Asset Model v3.0 (GMAM 3.0)**

**Methodological Document**

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| *An example of a fund’s returns as generated by GMAM 3.0. The fund was launched in 2019. GMAM 3.0 backcasted the returns to 2004 using hierarchical Bayesian techniques. These included joint estimation of de-smoothed returns and systematic factor exposures, among other model paramets. The backcast helps investors pair the fund with a portfolio of publicly available securities.* |

# Introduction and Motivation

Alternative investments such as private equity, hedge funds, and structured products are being traded at all-time highs. For instance, Bain & Company[[1]](#footnote-2) valued private equity (PE) buyout deals for 2022 at $654 billion globally. This was the second-highest valuation since 2008, and brought the total deal volume to $2.4 trillion[[2]](#footnote-3). Additionally, total private markets AUM reached $11.7 trillion as of June 2022. Structured products also grew to reach $1.5 trillion in new issuance in 2021, according to S&P and Bloomberg states that structured products still outsize the total ETF market at $5.3tn[[3]](#footnote-4). All of this makes alternatives a major asset class, with institutions such as colleges, foundations, pension funds, and more investing in it.

On the investor-side, SEC rules defining what an accredited investor is have recently been broadened to include “individual investors that have the knowledge and expertise to participate in [financial] markets.” One no longer needs to pass previously stringent income requirements to be able to avail of such investment opportunities.

iCapital has hastened the market expansion and liquidity of alternative investments and made it a viable option for mass affluent investors and private wealth funds[[4]](#footnote-5).

Optimal wealth allocation requires data on the risk, return, and covariance of asset classes. Un- fortunately, PE is largely exempt from public disclosure requirements. The issue is particularly pernicious with respect to performance metrics based on actual transactions. The limited data impedes the investment process for portfolios containing such assets.

iCapital solves this problem and other issues associated with returns on illiquid and alternative investments via our Generalized Multi-Asset Model version 3.0 (henceforth, GMAM 3). GMAM 3 is a returns-based model that uses hierarchical Bayesian modeling techniques to generate the underlying economic returns for any single fund that trades on iCapital’s marketplace. The model helps educate investors on fund suitability based on their investment needs and risk preferences.

This technical manual documents the important economic features of the model, and explains the economic and econometric motivation behind choices made in model construction. The intended audience for this document has a working familiarity with Bayesian hierarchical models, econometric time series, expected return beta models, and other similar topics.

## Why Bayesian?

Bayesian techniques offer a more robust and flexible approach to capture the complex dynamics of private equity returns due to their ability to:

* incorporate prior knowledge,
* handle limited data,
* account for illiquidity, and
* update estimates as new information becomes available.

Bayesian methodologies allow for the integration of prior beliefs or information about specific funds into the estimation of returns. The prior information effectually supplements limited data. These techniques directly incorporate uncertainty into the return estimates by treating all pa- rameters as stochastic. The framework aligns well with the unique characteristics of alternative investments, allowing for more accurate and informed returns estimation.

It is well-known in the academic finance literature that for private equity (henceforth, PE) there is a dearth of performance metrics based on actual transactions (Kaplan and Schoar [2005]). The available time series data often relies on non-market valuations or multiyear internal rates of return (IRR), often segmented by the vintage years of the funds. For investors seeking to optimally allocate their wealth across public securities and private equity, the lack of data is a significant barrier. Even simple Markowitz-style (Markowitz [1952]) mean-variance optimization requires a sufficiently long history of returns, which is unavailable in the case of PE.

As mentioned above, one commonly used metric in the PE world to gauge a fund’s performance is the IRR. It provides a convenient way by which to compare and benchmark various funds. Furthermore, it is an easy way to account for the timing and magnitude of cash flows since PE investments are typically illiquid and longer investment horizons. And finally, it helps investors assess the risk of a fund.

While these are good reasons to use the IRR, there are numerous measurement concerns related to it. One implicit assumption when calculating IRRs is that the cash proceeds will be reinvested to achieve a particular return. This assumption does not coincide with the reality of cash flow distributions by PE. For example, if a PE fund reports a 50% IRR and has returned cash early in its life, the assumption is that the cash will be reinvested at 50%. However, it is unlikely that the fund will find such an investment opportunity every time cash is distributed.

Furthermore, IRR can distort management incentives, upwardly bias performance measures, and misrepresent volatility estimates[[5]](#footnote-6). As stated in Sorensen and Jagannathan (2013)[[6]](#footnote-7), “The IRR may not exist, and it may not be unique.” Even though the Modified IRR (MIRR) largely tackles some well-known pitfalls of IRR, its practical implementation in a private partnership is not obvious.

GMAM 3, on the other hand, uses time-weighted returns.

Though these have their own caveats[[7]](#footnote-8), they also have their strengths. For instance, this method of calculation accounts for the contributions and distributions, as well as their timing. Also, it allows for a more standardized comparison of performance across different investment vehicles and asset classes. This applies especially to investors who wish to combine PE with publicly available investments. And finally, it is a widely understood and commonly used method in the broader investment industry. Its use in measuring PE performance can facilitate understanding and communication about fund performance, especially for stakeholders more familiar with traditional investment vehicles.

One of the strengths of the model’s Bayesian estimation process provides measures of uncertainty around the estimated returns, and this allows the user to decide for themselves the degree to which they can rely on these estimated, time-weighed returns.

# The Model

At its core, the model predicts fund returns from factor returns. The essence of the GMAM 3 returns-generating process can be modeled as:

where is a length vector of factor returns including an intercept dummy, is a matrix of exposures, is the predicted return, is the risk-free rate of return, and is the error. The subscripts here are for an individual fund () for a given period (). Henceforth, assume that all measures are for an individual fund, and so the -subscript will be dropped.

We use a superset of thirteen factors: Alt Commodities, Alt Hedge Fund Crowding, Alt Oil, Alt Trend, Emerging Markets, Equity Market, Equity Momentum, Equity Quality, Equity SmallCap, Equity Value, Fixed Credit, Fixed Duration, and US Dollar. Detailed information about factor construction can be found in the Appendix.

While this may seem like a conventional way to estimate returns, there are two important things to be noted about the equation above:

1. is NOT the final return provided to the user since these returns are *de-smoothed*. Alternative investment returns are both, serially correlated and lagged. The returns generated are the true economic returns. To mimic the underlying characteristics of the reported returns to a greater degree[[8]](#footnote-9), we perform a re-smoothing of these returns. Section 1.3 below discusses the economic rationale and mathematical process in greater detail. The Appendix has more details on this as well.
2. We start with a superset of thirteen economic factors as listed above. Rather than include them all in the regression, we use the stochastic search variable selection (SSVS) technique[[9]](#footnote-10) to select a subset of regressors that are economically meaningful to a particular fund. SSVS is a type of Bayesian linear regression which is useful when the number of predictors is larger than the number of observations – a problem frequently encountered in alternative investment data. More on this can be found in Section 1.4, and the Appendix.

It suffices to say here, that GMAM 3 uses Markov Chain Monte Carlo (MCMC) to compute estimates of as well as the *distributional* parameters of . The final product displays only point estimates – which are calculated as averages across draws in the MCMC process. For MCMC, each draw is a *distribution* and not a point estimate (more details on this in the Appendix). So, the expected, predicted, de-smoothed returns estimated from simulations are given by:

Since this is the result of a hierarchical Bayesian regression, the distribution of the error terms is hard to characterize precisely. However, we can approximately say that[[10]](#footnote-11):

Here, is derived from model parameters sampled at each iteration and is a t-distribution degree of freedom parameter likewise sampled. In other words, the error terms approximate a t-distribution to a great degree.

## Re-Smoothing

It is well-known in the finance practitioner literature that returns from alternative investment such as PE, hedge funds, or real estate funds, are highly serially correlated[[11]](#footnote-12). In other words, past values are correlated with present values. The serial correlation mentioned here occurs because of the lack of liquidity in the fund itself or some of the assets held within it. For instance, these illiquid assets could be ones that don't trade frequently, making it challenging to always determine their market prices. When funds contain such illiquid assets, their reported returns may seem steadier than their actual economic returns (returns that consider all available market information about those securities). This can lead to a downward bias in estimating return variance and result in positive serial return correlation.

It is also well-known that returns from real estate funds, are lagged[[12]](#footnote-13). For instance, for many funds, returns are compiled and reported quarterly. But valuations of many properties included in the funds are effectively updated only annually. Each quarter some properties have their valuations updated, and others do not. Those properties that don't get a new valuation within a quarter carry over their last known value into the current quarter.

Finally, Financial Accounting Standard 157, released by the FASB in 2006 during the run-up to the financial crisis of that time, and now called Accounting Standards Code Topic 820, required companies to mark their assets to market. This was a radical change from historic cost accounting and required LPs to periodically mark the assets to market, even though markets in many portfolio companies are illiquid. This backwards appraisal may also result in a managerial bias towards smoothing asset values.

The latent returns , henceforth just for notational simplicity, generated above are not smoothed. They are the true economic returns. Since GMAM 3 attempts to mimic true reported returns, we must re-smooth the latent returns to reflect the reported returns so that investors can appropriately compare their investments in PE with publicly available investments. In GMAM 3, this is done using a moving average (MA) process with smoothing parameters estimated using a Bayesian linear regression. Note that the MA is defined in an econometric sense, and not in a literal sense, although it may be interpreted as such. More specifically, the final returns presented to the user are re-smoothed using the following process:

where are the reported returns and is a matrix consisting of smoothing coefficients, and . In other words,

is a global precision parameter estimated for independent measurement using a Gamma-distributed prior: with given shape hyperparameter, , and given inverse scale hyperparameter, . is an identity matrix used for computational purposes.

is a vector of unrestricted coefficients, and is a vector of restricted and unrestricted coefficients. and represent terms that map the observation period of the reported, smoothed returns to the latent, de-smoothed returns . In particular,

The vector , we describe the process here using only for simplicity. The reported returns are s.t. , and the latent returns are s.t. . Here, may or may not equal , for example, when the reported returns are quarterly, and the factor returns are monthly. To resolve this, we have the following formula that maps to :

For example, if reported returns (are quarterly, and latent returns () are monthly, , and (since each quarter consists of three months). And so, for the first quarter, and . When , the , for must add up to the same value, . If not, months that fall earlier in the quarter will have a different long-run impact on NAV than months later in the quarter. In the current version of GMAM 3 in production, they add up to 3 when reported returns are quarterly, and they add up to 1 when reported returns are monthly.

The economic assumption behind this process is simple: that the observed fund returns () is a weighted average of the fund’s economic returns, , over the most recent () periods, including the current period. The econometric implications of this are that under the given assumption, the observed fund returns follow a MA process of order .

This restriction is similar to Getmansky, Lo, and Makarov (2004) – see previous Footnote 12 for full citation – where the observed return () for some period , is a weighted average of the “true” returns () over the most recent periods: , with to ensure that all information is eventually incorporated into observed returns, and for . In our case the observed returns are the reported returns, , the “true” returns are the latent returns generated using the factors, , and the terms (or as we notate them, terms) are generated using a multivariate normal distribution in the hierarchical Bayesian model, as follows:

Here, is as defined above, and is a global precision multiplier parameter, distributed as with given shape and inverse scale hyperparameters; is a matrix consisting of precision hyperparameters for ; and is a hyperparameter (given).

## Stochastic Search Variable Selection (SSVS)

As mentioned above, the regression performed to estimate is using the SSVS technique which defines prior regression coefficient variances consistent with inclusion or effective exclusion. In other words, it is a predictor selection technique. Such techniques commonly focus on which predictors to retain, though they also aim for improved predictive performance through developing an encompassing model, or model simplification without adversely affecting predictive accuracy.[[13]](#footnote-14) Formal model choice is simplified for normal linear regression, as marginal likelihoods may be obtained analytically, but for many predictors, comparison of the many possible models becomes infeasible.[[14]](#footnote-15)

An imperfect analogy is stepwise regression using AIC or BIC as a selection criterion. However, when the number of regressors is large (we have 13), the computational requirements for these procedures can be prohibitive. A common workaround is to use a heuristic method to restrict attention to a potential subset of regressors, i.e., include or exclude variables, based on considerations. With the intention of being more systematic and rigorous, we use the SSVS technique of George and McCulloch (1993) – see Footnote 10 for full citation, along with the spike and slab prior of Kuo and Mallick (1998).[[15]](#footnote-16)

SSVS uses a Bayesian approach with a normal mixture model for regression analysis. In this method, hidden variables are employed to pinpoint which subsets of predictors are worth considering. It works by identifying those predictors that have a higher chance of being relevant, based on their posterior probability. To do this, SSVS uses a technique called Gibbs sampling[[16]](#footnote-17). This approach helps to sample from the distribution of all possible subsets of predictors. The subsets that show up more often in these samples are considered promising because they have a higher probability of being relevant.

We start with a Bernoulli or binomial prior on , the selector variable:

Here, , which is the probability of factor selection with parameters and specified externally. , , is a vector and selector variable used to determine which of the thirteen factors are economically significant in a probabilistic sense. Initially, each of the factors have an equally likely chance of being included. Conditional on a factor being in the regression, we specify a prior distribution for the regression coefficient associated with that variable. In our case, this is multivariate normal:

Here, is the prior mean of conditional on exclusion (); is the shift in the prior mean of conditional on inclusion (); is a vector – a function of which adjusts for sparsity; is a matrix, with ; and is the variance of the spike distribution as a fraction of the slab variance. And finally, , , and are all gamma-distributed with shape and inverse-scale (hyper)parameters as detailed in the Appendix.

# Appendix

The complete data-generating process is as follows:

where:

Also:

* is a multivariate normally distributed variable; is a Bernoulli distributed variable; is a beta distributed variable; and is a gamma distributed variable with an inverse scale parametrization.
* All priors are conditionally conjugate except for .

A.1 Note on the Priors

Priors, while influencing the results with fewer datapoints and noisier data, do not ultimately matter much as data increases. While priors are essential in Bayesian analysis, their influence generally decreases as the amount of data increases, especially if the data is strongly informative. In theory, Bayesian models exhibit a property known as 'consistency.' This means that as the sample size grows to infinity, the posterior distribution of the parameter of interest will converge to the true parameter value, assuming the model is correctly specified. In such scenarios, the choice of prior becomes less critical as the sample size increases.

That said, the extensive use of the gamma distribution with an inverse scale parametrization above is convenient for several reasons:

* **Conjugacy**: conjugate priors are advantageous for computational and analytical simplicity.
* **Positive Values**: the gamma distribution is defined for positive values only.
* **Informative and Non-Informative Settings**: the gamma distribution can be used to encode both informative and non-informative priors. By adjusting its parameters, you can represent strong prior beliefs or specify a prior that has minimal influence on the posterior, allowing the data to drive the inference.
* **Regularization and Stability**: In Bayesian hierarchical models, the use of gamma priors can help in regularizing the estimates, particularly in complex models or in the presence of limited data. This can prevent overfitting and improve the model's stability and predictive performance.

The variables modeled as beta and Bernoulli distributions are used in the SSVS regression outlined earlier. In particular, , has only two values so modeling it as Bernoulli makes sense in this context.

A.2 Posterior Distributions

A.2.1 Posterior for

First, define

where and is a vector of zeros followed by ones, making the sum of the last columns of ; and

s.t.

1. Source: https://www.bain.com/insights/topics/global-private-equity-report/ [↑](#footnote-ref-2)
2. Source: https://www.mckinsey.com/industries/private-equity-and-principal-investors/our-insights/mckinseys-private-markets-annual-review/ [↑](#footnote-ref-3)
3. https://www.bloomberg.com/professional/blog/sure-time-to-grasp-the-potential-of-structured-products/ [↑](#footnote-ref-4)
4. Source: https://icapital.com/insights/practice-management/untapped-potential-alternative-investments-and-the-wealth-management-channel/ [↑](#footnote-ref-5)
5. See Ludovic Phalippou, “The Hazards of Using IRR to Measure Performance: The Case of Private Equity” posted to SSRN in March 2008 at <https://papers.ssrn.com/sol3/papers.cfm?abstract_id=1111796>. [↑](#footnote-ref-6)
6. Morten Sorensen and Ravi Jagannathan, “The Public Market Equivalent and Private Equity Performance,” *Financial Analysts Journal*, 71:4 (2015): 43-50. [↑](#footnote-ref-7)
7. They may not fully capture the complexities of private equity investments, such as the strategic timing of cash flows and the long-term, illiquid nature of these assets. [↑](#footnote-ref-8)
8. This re-smoothing, while replicating the essential attributes of the reported returns more closely, does come at the cost of not reflecting the true economic changes in value over time. [↑](#footnote-ref-9)
9. Edward I. George and Robert E. McCulloch, “Variable Selection Via Gibbs Sampling,” *Journal of the American Statistical Association*, 88: 423 (1993): 881-889. [↑](#footnote-ref-10)
10. Strictly speaking, this approximates a t-distribution since the final distribution is the result of several layers of Bayesian analysis. More information about this can be found in the Appendix. [↑](#footnote-ref-11)
11. This phenomenon is well-studied in the finance practitioner literature. See, for example, Mila Getmansky, Andrew W. Lo, and Igor Makarov, “An econometric model of serial correlation and illiquidity in hedge fund returns,” *Journal of Financial Economics*, 74: 3 (2004): 529–609. [↑](#footnote-ref-12)
12. David Geltner, “Estimating Market Values from Appraised Values without Assuming an Efficient Market ,” *The Journal of Real Estate Research*, 8: 3 (1993): 325-345. [↑](#footnote-ref-13)
13. Juho Piironen and Aki Vehtari, “Comparison of Bayesian predictive methods for model selection,” *Statistics and Computing* 27: (2017): 711–735. [↑](#footnote-ref-14)
14. Peter D. Congdon. *Bayesian hierarchical models: with applications using R*. CRC Press, 2020. [↑](#footnote-ref-15)
15. Lynn Kuo and Bani Mallick, “Variable Selection for Regression Models,” *Sankhyā: The Indian Journal of Statistics, Series B (1960-2002)*, 60:1 [Bayesian Analysis (Apr., 1998)]: 65-81. [↑](#footnote-ref-16)
16. Gibbs sampling is a statistical technique used for generating sequences of samples from the probability distribution of multiple variables. It's a kind of Markov Chain Monte Carlo (MCMC) method. It is particularly useful in scenarios where directly sampling from the joint distribution is difficult, but sampling from the conditional distribution of each variable is feasible. [↑](#footnote-ref-17)