CS621/CSL611
Quantum Computing For Computer Scientists

The Leap from Classical to Quantum

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Probabilistic Systems

- Instead of dealing with a bunch of marbles moving about, we shall work with a single marble.
- The state of the system will tell us the probabilities of the marble being on each vertex.
- For a three-vertex graph, a typical state might look like

$$X = \left[\frac{1}{5}, \frac{3}{10}, \frac{1}{2}\right]^T$$

- This will correspond to the fact that there is
  - a one-fifth chance that the marble is on vertex 0,
  - a three-tenths chance that the marble is on vertex 1,
  - a half chance that the marble is on vertex 2.
- Because the marble must be somewhere on the graph, the sum of the probabilities is 1

- Rather than exactly one arrow leaving each vertex, we will have several arrows shooting out of each vertex with real numbers between 0 and 1 as weights.
- These weights describe the probability of our marble moving from one vertex to another in one time click

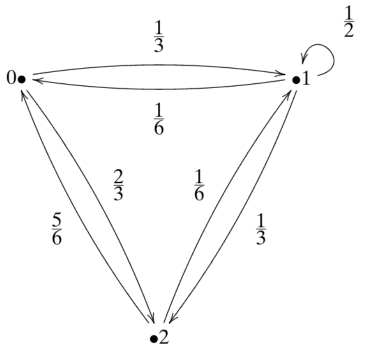
Restriction

- Weighted graphs satisfying two conditions
  - The sum of all the weights leaving a vertex is 1 and
  - The sum of all the weights entering a vertex is 1

- This will correspond to the fact that the marble must both go and come from someplace (there might be loops)

The Adjacency Matrix Revisited

$$M = \begin{bmatrix} 0 & \frac{1}{6} & \frac{5}{6} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{bmatrix}.$$



- The adjacency matrices for our graphs will have real entries between 0 and 1 where the sums of the rows and the sums of the columns are all 1.
- Such matrices are called **doubly stochastic**.

- Note that
  - The sum of the entries of Y is 1.
  - X expresses the probability of the position of a marble.
  - M expresses the probability of the way the marble moves around

$$MX = \begin{bmatrix} 0 & \frac{1}{6} & \frac{5}{6} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{21}{36} \\ \frac{9}{36} \\ \frac{6}{36} \end{bmatrix} = Y.$$

- $MX = Y = \left[\frac{21}{36}, \frac{9}{36}, \frac{6}{36}\right]^T$ expresses the probability of the marbles location after moving
- If X is the probability of the marble at time t, then MX is the probability of the marble at time $t+1$

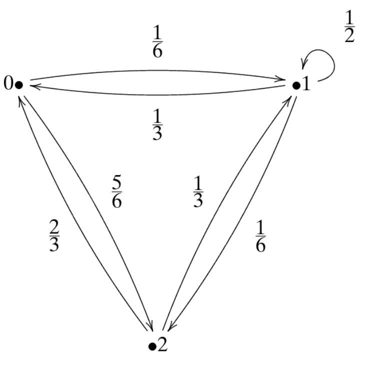
$$WM = \left[\frac{1}{3}, 0, \frac{2}{3}\right] \begin{bmatrix} 0 & \frac{1}{6} & \frac{5}{6} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{bmatrix} = \left[\frac{4}{9}, \frac{5}{18}, \frac{5}{18}\right] = Z.$$

- Notice that the sum of the entries of Z is 1

What can this possibly mean?

- M^T corresponds to our directed graph with the arrows reversed

$$M^T = \begin{bmatrix} 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ \frac{5}{6} & \frac{1}{6} & 0 \end{bmatrix}$$



- Reversing the arrows is like traveling back in time or having the marble roll backward.

$$M^T W^T = \begin{bmatrix} 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ \frac{5}{6} & \frac{1}{6} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ 0 \\ \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{4}{9} \\ \frac{5}{18} \\ \frac{5}{18} \end{bmatrix} = Z^T$$

$$M^T W^T = (WM)^T = Z^T$$

- If multiplying on the right of M takes states from time t to time $t+1$, then multiplying on the left of M takes states from time t to time $t-1$

Time Symmetry

Time symmetry

- Our description of system dynamics is entirely symmetric
- One of the fundamental concepts of quantum mechanics and quantum computation

- By replacing
  - Column vectors with row vectors, and
  - Forward evolution in time with backward evolution,
- The laws of dynamics still hold

What does this represent? $MM = M^2$

$$\begin{bmatrix} 0 & \frac{1}{6} & \frac{5}{6} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{6} & \frac{5}{6} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{bmatrix} = \begin{bmatrix} \frac{11}{18} & \frac{13}{36} & \frac{1}{36} \\ \frac{5}{18} & \frac{13}{36} & \frac{13}{36} \\ \frac{1}{9} & \frac{5}{18} & \frac{11}{18} \end{bmatrix}$$

- Is M^2 doubly stochastic?
- $M^2[i,j] =$ the probability of going from vertex j to vertex i in 2 time clicks
- In general, for an arbitrary positive integer k, we have $M^k[i,j] =$ the probability of going from vertex j to vertex i in k time clicks.

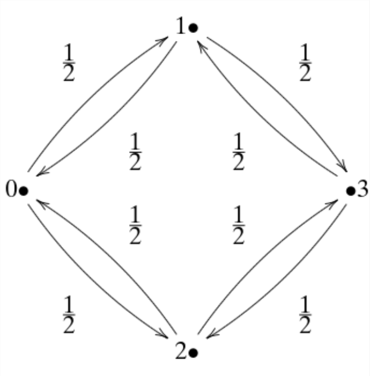
$$M = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

- Given the above two doubly stochastic matrices, calculate $M \star N$ and show that this is again a doubly stochastic matrix.

HomeWork

Prove that the product of a doubly stochastic matrix with another doubly stochastic matrix is also a doubly stochastic matrix.

- If M is an $n - by - n$ doubly stochastic matrix and X is an $n - by - 1$ column vector whose entries sum to 1, then $M^k X = Y$ is expressing the probability of the position of a marble after k time clicks.
- We are not constrained to multiply M by itself. We may also multiply M by another doubly stochastic matrix.
- If M and N each describe some probability transition for going from one time click to the next, $M \star N$ will then describe a probability transition of going from time t to $t + 1$ to $t + 2$



$$A = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

- Initial state $= [1, 0, 0, 0]^T$
- After one time click, $\left[0, \frac{1}{2}, \frac{1}{2}, 0\right]^T$
- After another time click, $\left[\frac{1}{2}, 0, 0\frac{1}{2}\right]^T$

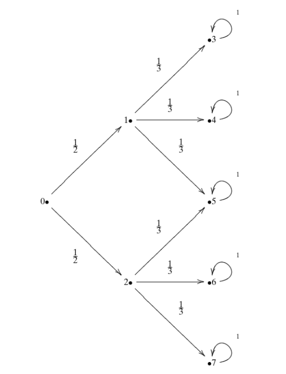
[1] Quantum version of this example will be coming up soon.

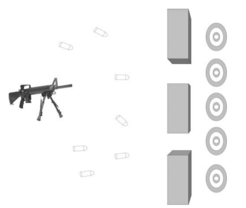
Probabilistic Double-Slit Experiment

The Set-up

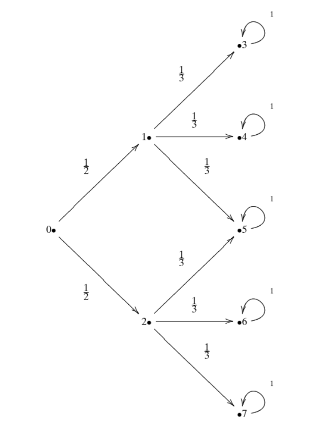
- There are two slits in the wall. The shooter is a good enough shot to always get the bullets through one of the two slits. There is a 50% chance that the bullet will travel through the top slit. Similarly, there is a 50% chance the bullet will travel through the bottom slit.
- Once a bullet is through a slit, there are three targets to the right of each slit that the bullet can hit with equal probability. The middle target can get hit in one of two ways: from the top slit going down or from the bottom slit going up.
- It is assumed that it takes the bullet one time click to travel from the gun to the wall and one time click to travel from the wall to the targets.

Double-slit experiment with bullets





- Notice that the vertex marked 5 can receive bullets from either of the two slits.
- Also notice that once a bullet is in position 3, 4, 5, 6, or 7, it will, with probability 1, stay there



$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

- The matrix B is not a doubly stochastic matrix. The sum of the weights entering vertex 0 is not 1. The sum of weights leaving vertices 3, 4, 5, 6, and 7 are more than 1.

$$B \star B = B^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{3} & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{3} & 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 1 & 0 & 0 \\ \frac{1}{6} & 0 & \frac{1}{3} & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{6} & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

- Initial sate $X = [1,0,0,0,0,0,0,0]^T$,
- After two time clicks

$$B^2 X = \left[0, 0, 0, \frac{1}{6}, \frac{1}{6}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6}\right]^T$$

Summary

- The vectors that represent states of a probabilistic physical system express a type of **indeterminacy** about the exact physical state of the system.
- The matrices that represent the dynamics express a type of **indeterminacy** about the way the physical system will change over time.
- Their entries enable us to compute the likelihood of transitioning from one state to the next.
- The way in which the **indeterminacy** progresses is simulated by matrix multiplication, just as in the deterministic scenario.