CS621/CSL611
Quantum Computing For Computer Scientists

Quantum Circuits and Protocols

Dhiman Saha

Winter 2024

IIT Bhilai

Partial measurements

Problem Definition

We have a system consisting of two or more qubits and we only measure one of them.

Example (2-qubit system)

- What is the difference between the case where **both** qubits are measured and when **only one** is measured?
  - Probabilities of measurement-outcomes of first qubit is *same* irrespective of whether second qubit was measured or not.
  - However, the **superposition** of the system after the measurement may be *different*

Both Measured

Resulting superposition is one of $|00\rangle$, $|01\rangle$, $|10\rangle$, or $|11\rangle$

One Measured

May lead to a different superposition

Example - 2-Qubit System

First Qubit Measured

- Initial superposition: $|\psi\rangle = \frac{1}{2}|00\rangle - \frac{i}{2}|10\rangle + \frac{1}{\sqrt{2}}|11\rangle$

First Qubit Measured

- Initial superposition: $|\psi\rangle = \frac{1}{2}|00\rangle - \frac{i}{2}|10\rangle + \frac{1}{\sqrt{2}}|11\rangle$

Measuring 0

Measuring 1

Example - 2-Qubit System — First Qubit Measured

- Initial superposition: $|\psi\rangle = \frac{1}{2}|00\rangle - \frac{i}{2}|10\rangle + \frac{1}{\sqrt{2}}|11\rangle$

Measuring 0

- Consistent terms

$|\psi\rangle = \frac{1}{2}|00\rangle - \cancel{\frac{i}{2}|10\rangle} + \cancel{\frac{1}{\sqrt{2}}|11\rangle}$

Measuring 1

- Consistent terms

$|\psi\rangle = \cancel{\frac{1}{2}|00\rangle} - \frac{i}{2}|10\rangle + \frac{1}{\sqrt{2}}|11\rangle$

- Initial superposition: $|\psi\rangle = \frac{1}{2}|00\rangle - \frac{i}{2}|10\rangle + \frac{1}{\sqrt{2}}|11\rangle$

Measuring 0

- Consistent terms

  $|\psi\rangle = \frac{1}{2}|00\rangle - \cancel{\frac{i}{2}|10\rangle} + \cancel{\frac{1}{\sqrt{2}}|11\rangle}$

- Outcome 0 Prob.

  $\left|\left|\frac{1}{2}|00\rangle\right|\right|^2 = \frac{1}{4}$

Measuring 1

- Consistent terms

  $|\psi\rangle = \cancel{\frac{1}{2}|00\rangle} - \frac{i}{2}|10\rangle + \frac{1}{\sqrt{2}}|11\rangle$

- Outcome 1 Prob.

  $\left\|-\frac{i}{2}|10\rangle + \frac{1}{\sqrt{2}}|11\rangle\right\|^2 = \left|\frac{-i}{2}\right|^2 + \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{3}{4}$

Example - 2-Qubit System First Qubit Measured

- Initial superposition: $|\psi\rangle = \frac{1}{2}|00\rangle - \frac{i}{2}|10\rangle + \frac{1}{\sqrt{2}}|11\rangle$

Measuring 0

- Consistent terms

  $|\psi\rangle = \frac{1}{2}|00\rangle - \cancel{\frac{i}{2}|10\rangle} + \cancel{\frac{1}{\sqrt{2}}|11\rangle}$

- Outcome 0 Prob.

  $\left|\left| \frac{1}{2}|00\rangle \right|\right|^2 = \frac{1}{4}$

Measuring 1

- Consistent terms

  $|\psi\rangle = \cancel{\frac{1}{2}|00\rangle} - \frac{i}{2}|10\rangle + \frac{1}{\sqrt{2}}|11\rangle$

- Outcome 1 Prob.

  $\left\| -\frac{i}{2}|10\rangle + \frac{1}{\sqrt{2}}|11\rangle \right\|^2 = \left|\frac{-i}{2}\right|^2 + \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{3}{4}$

- The state of the two qubits after the measurement **conditioned** on the measurement outcome **being 0 or 1** is the vector itself **renormalized** respectively.

Example - 2-Qubit System First Qubit Measured

- Initial superposition: $|\psi\rangle = \frac{1}{2}|00\rangle - \frac{i}{2}|10\rangle + \frac{1}{\sqrt{2}}|11\rangle$

Measuring 0

- Consistent terms

$|\psi\rangle = \frac{1}{2}|00\rangle - \cancel{\frac{i}{2}|10\rangle} + \cancel{\frac{1}{\sqrt{2}}|11\rangle}$

- Outcome 0 Prob.

$\left\| \frac{1}{2}|00\rangle \right\|^2 = \frac{1}{4}$

- Renormalization

$\frac{\frac{1}{2}|00\rangle}{\left\| \frac{1}{2}|00\rangle \right\|} = |00\rangle$

Measuring 1

- Consistent terms

$|\psi\rangle = \cancel{\frac{1}{2}|00\rangle} - \frac{i}{2}|10\rangle + \frac{1}{\sqrt{2}}|11\rangle$

- Outcome 1 Prob.

$\left\| -\frac{i}{2}|10\rangle + \frac{1}{\sqrt{2}}|11\rangle \right\|^2 = \left| \frac{-i}{2} \right|^2 + \left| \frac{1}{\sqrt{2}} \right|^2 = \frac{3}{4}$

- Renormalization

$\frac{-\frac{i}{2}|10\rangle + \frac{1}{\sqrt{2}}|11\rangle}{\left\| -\frac{i}{2}|10\rangle + \frac{1}{\sqrt{2}}|11\rangle \right\|} = -\frac{i}{\sqrt{3}}|10\rangle + \sqrt{\frac{2}{3}}|11\rangle$

- The state of the two qubits after the measurement **conditioned** on the measurement outcome **being 0 or 1** is the vector itself **renormalized** respectively.

Example - 2-Qubit System — First Qubit Measured

- Initial superposition: $|\psi\rangle = \frac{1}{2}|00\rangle - \frac{i}{2}|10\rangle + \frac{1}{\sqrt{2}}|11\rangle$

Measuring 0

- Consistent terms

  $|\psi\rangle = \frac{1}{2}|00\rangle - \cancel{\frac{i}{2}|10\rangle} + \cancel{\frac{1}{\sqrt{2}}|11\rangle}$

- Outcome 0 Prob.

  $\left\| \frac{1}{2}|00\rangle \right\|^2 = \frac{1}{4}$

- Renormalization

  $\frac{\frac{1}{2}|00\rangle}{\left\| \frac{1}{2}|00\rangle \right\|} = |00\rangle$

Measuring 1

- Consistent terms

  $|\psi\rangle = \cancel{\frac{1}{2}|00\rangle} - \frac{i}{2}|10\rangle + \frac{1}{\sqrt{2}}|11\rangle$

- Outcome 1 Prob.

  $\left\| -\frac{i}{2}|10\rangle + \frac{1}{\sqrt{2}}|11\rangle \right\|^2 = \left| \frac{-i}{2} \right|^2 + \left| \frac{1}{\sqrt{2}} \right|^2 = \frac{3}{4}$

- Renormalization

  $\frac{-\frac{i}{2}|10\rangle + \frac{1}{\sqrt{2}}|11\rangle}{\left\| -\frac{i}{2}|10\rangle + \frac{1}{\sqrt{2}}|11\rangle \right\|} = -\frac{i}{\sqrt{3}}|10\rangle + \sqrt{\frac{2}{3}}|11\rangle$

- Thus, conditioned on the measurement outcome, the state of the two qubits becomes one of the following:

$$|00\rangle \qquad \text{or,} \qquad -\frac{i}{\sqrt{3}}|10\rangle + \sqrt{\frac{2}{3}}|11\rangle$$

Exercise — Second Qubit Measured

- Find the state of the two qubits if the second qubit is measured.

$$|\psi\rangle = \frac{1}{2}\,|0\overset{\text{Measure}}{0}\rangle - \frac{i}{2}\,|1\overset{\text{Measure}}{0}\rangle + \frac{1}{\sqrt{2}}\,|1\overset{\text{Measure}}{1}\rangle$$

Example 3-Qubit System

- Initial State:

$$|\psi\rangle = \frac{1}{2}\overset{\text{Measure}}{|00\underline{0}\rangle} + \frac{1}{2}\overset{\text{Measure}}{|10\underline{0}\rangle} + \frac{1}{2}\overset{\text{Measure}}{|10\underline{1}\rangle} - \frac{1}{2}\overset{\text{Measure}}{|11\underline{1}\rangle}$$

- Partial measurement: Qubit-3
- What are the probabilities of the two possible measurement outcomes and what are the resulting superpositions of the three qubits for each case?
- We write:

$$|\psi\rangle = \left(\frac{1}{2}|00\rangle + \frac{1}{2}|10\rangle\right)|0\rangle + \left(\frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle\right)|1\rangle$$

- Inconsistent Terms

$$|\psi\rangle = \left(\frac{1}{2}|00\rangle + \frac{1}{2}|10\rangle\right)|0\rangle + \cancel{\left(\frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle\right)}|1\rangle$$

- $\Pr[M(Qubit-3)=0]$

$$\left\|\frac{1}{2}|00\rangle + \frac{1}{2}|10\rangle\right\|^2 = \frac{1}{2}$$

- Resulting Superposition:

$$\underbrace{\frac{\left(\frac{1}{2}|00\rangle + \frac{1}{2}|10\rangle\right)|0\rangle}{\left\|\frac{1}{2}|00\rangle + \frac{1}{2}|10\rangle\right\|}}_{\text{Renormalization}} = \sqrt{2}\left(\frac{1}{2}|00\rangle + \frac{1}{2}|10\rangle\right)|0\rangle$$

$$= \frac{1}{\sqrt{2}}|000\rangle + \frac{1}{\sqrt{2}}|100\rangle$$

- Inconsistent Terms

$$|\psi\rangle = \xcancel{\left(\frac{1}{2}|00\rangle + \frac{1}{2}|10\rangle\right)}|0\rangle + \left(\frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle\right)|1\rangle$$

- $\Pr[M(Qubit-3)=1]$

$$\left\|\frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle\right\|^2 = \frac{1}{2}$$

- Resulting Superposition:

$$\underbrace{\frac{\left(\frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle\right)|1\rangle}{\left\|\frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle\right\|}}_{\text{Renormalization}} = \sqrt{2}\left(\frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle\right)|1\rangle$$

$$= \frac{1}{\sqrt{2}}|101\rangle - \frac{1}{\sqrt{2}}|111\rangle$$

References

1. Quantum Computing for Computer Scientists, by Noson S. Yanofsky, Mirco A. Mannucci
2. Quantum Computing Explained, David Mcmahon. John Wiley & Sons
3. Lecture Notes on Quantum Computation, John Watrous, University of Calgary
    - https://cs.uwaterloo.ca/~watrous/QC-notes/