CS621/CSL611
Quantum Computing For Computer Scientists

Quantum Circuits and Protocols

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Superdense Coding Protocol

Problem Definition

- Alice and Bob are in different parts of the world.
- **Need**: Alice wants to communicate two bits a and b to Bob
- **Constraint:** Alice can send just a single qubit.
- **Fact:** *Alice cannot encode two classical bits into a single qubit in any way that would give Bob more than just one bit of information about the pair* (a, b).
- **Verdict:** There is no way they can accomplish this task without *additional resources*

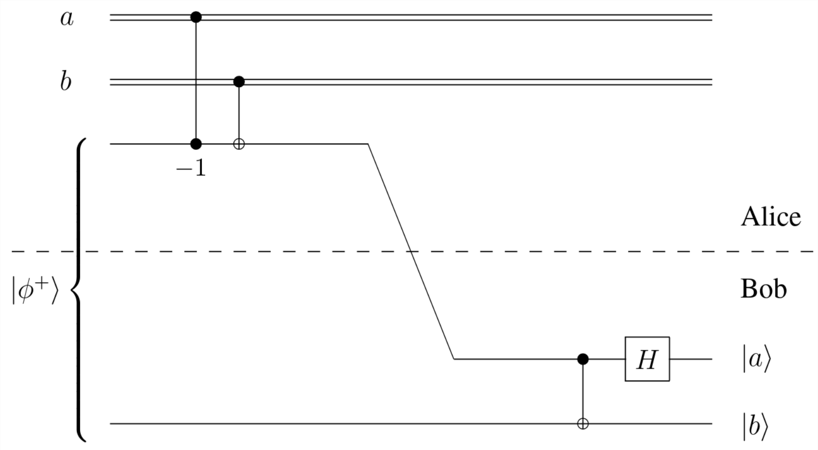
This protocol was first proposed by Bennett and Wiesner in 1970 and experimentally actualized in 1996 by Mattle, Weinfurter, Kwiat and Zeilinger using **entangled photon pairs**.

Pre-shared Entangled Resource

Share an $e-$bit

- **Additional Resource:** One pre-shared unit of entanglement
- Alice and Bob prepare two qubits A and B in the superposition:
$$\frac{1}{\sqrt{2}}\left|00\right\rangle + \frac{1}{\sqrt{2}}\left|11\right\rangle$$
Alice takes qubit A and Bob takes the qubit B.[1]
- **Implication:** *Given the additional resource of a shared e−bit of entanglement, Alice will be able to transmit both a and b to Bob by sending just one qubit.*

[1] This is independent of the knowledge of (a, b)



Note that the first two gates are acting on **one classical bit** and **one qubit**. Generally, this applies *only* when the *classical bit is a control bit* for some operation.

- Alice applies controlled$-\sigma_z$ gate[2] to qubit A with the **control bit** as a.

$$\text{controlled-}\sigma_z \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- Evolution of state of A, B

ab	After Step-1
00	$\frac{1}{\sqrt{2}}\lvert 00\rangle + \frac{1}{\sqrt{2}}\lvert 11\rangle$
01	$\frac{1}{\sqrt{2}}\lvert 00\rangle + \frac{1}{\sqrt{2}}\lvert 11\rangle$
10	$\frac{1}{\sqrt{2}}\lvert 00\rangle - \frac{1}{\sqrt{2}}\lvert 11\rangle$
11	$\frac{1}{\sqrt{2}}\lvert 00\rangle - \frac{1}{\sqrt{2}}\lvert 11\rangle$

[2] Recall the Pauli matrices

Step-2 — Superdense Coding Protocol

- Alice applies controlled$-\sigma_x$ gate[3] to qubit A with the **control bit** as b.

$$\text{controlled-}\sigma_x \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

- Evolution of state of A, B

ab	After Step-1	After Step-2
00	$\frac{1}{\sqrt{2}}\|00\rangle + \frac{1}{\sqrt{2}}\|11\rangle$	$\frac{1}{\sqrt{2}}\|00\rangle + \frac{1}{\sqrt{2}}\|11\rangle$
01	$\frac{1}{\sqrt{2}}\|00\rangle + \frac{1}{\sqrt{2}}\|11\rangle$	$\frac{1}{\sqrt{2}}\|10\rangle + \frac{1}{\sqrt{2}}\|01\rangle$
10	$\frac{1}{\sqrt{2}}\|00\rangle - \frac{1}{\sqrt{2}}\|11\rangle$	$\frac{1}{\sqrt{2}}\|00\rangle - \frac{1}{\sqrt{2}}\|11\rangle$
11	$\frac{1}{\sqrt{2}}\|00\rangle - \frac{1}{\sqrt{2}}\|11\rangle$	$\frac{1}{\sqrt{2}}\|10\rangle - \frac{1}{\sqrt{2}}\|01\rangle$

[3] Recall $\sigma_x = \text{NOT}$

- Alice sends the qubit A to Bob
- This is the **only** qubit that is sent during the protocol
- Recall, this was the constraint given in the problem definition

- Bob applies a controlled-NOT operation to the pair (A, B), where A is the control and B is the target

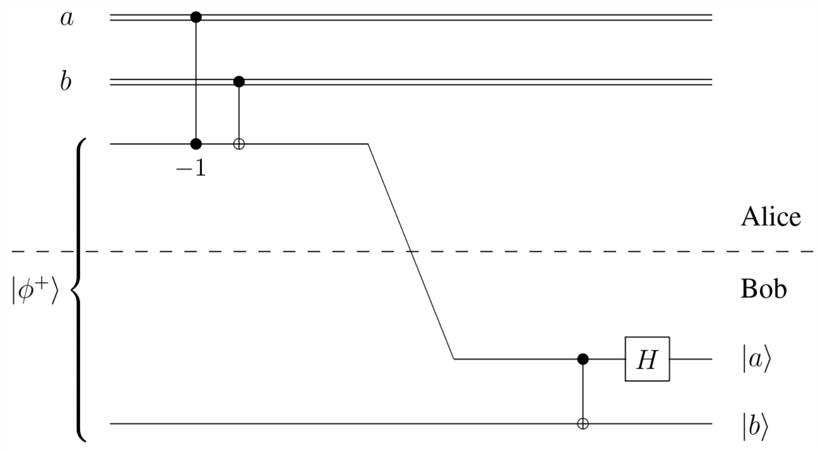
ab	After Step-2	After Step-4
00	$\frac{1}{\sqrt{2}}\|00\rangle + \frac{1}{\sqrt{2}}\|11\rangle$	$\frac{1}{\sqrt{2}}\|00\rangle + \frac{1}{\sqrt{2}}\|10\rangle = \left(\frac{1}{\sqrt{2}}\|0\rangle + \frac{1}{\sqrt{2}}\|1\rangle\right)\|0\rangle$
01	$\frac{1}{\sqrt{2}}\|10\rangle + \frac{1}{\sqrt{2}}\|01\rangle$	$\frac{1}{\sqrt{2}}\|11\rangle + \frac{1}{\sqrt{2}}\|01\rangle = \left(\frac{1}{\sqrt{2}}\|1\rangle + \frac{1}{\sqrt{2}}\|0\rangle\right)\|1\rangle$
10	$\frac{1}{\sqrt{2}}\|00\rangle - \frac{1}{\sqrt{2}}\|11\rangle$	$\frac{1}{\sqrt{2}}\|00\rangle - \frac{1}{\sqrt{2}}\|10\rangle = \left(\frac{1}{\sqrt{2}}\|0\rangle - \frac{1}{\sqrt{2}}\|1\rangle\right)\|0\rangle$
11	$\frac{1}{\sqrt{2}}\|10\rangle - \frac{1}{\sqrt{2}}\|01\rangle$	$\frac{1}{\sqrt{2}}\|11\rangle - \frac{1}{\sqrt{2}}\|01\rangle = \left(\frac{1}{\sqrt{2}}\|1\rangle - \frac{1}{\sqrt{2}}\|0\rangle\right)\|1\rangle$

- Bob applies a Hadamard transform to A

ab	After Step-4	After Step-5
00	$\left(\frac{1}{\sqrt{2}}\|0\rangle + \frac{1}{\sqrt{2}}\|1\rangle\right)\|0\rangle = H\|0\rangle\|0\rangle$	$\|00\rangle$
01	$\left(\frac{1}{\sqrt{2}}\|1\rangle + \frac{1}{\sqrt{2}}\|0\rangle\right)\|1\rangle = H\|0\rangle\|1\rangle$	$\|01\rangle$
10	$\left(\frac{1}{\sqrt{2}}\|0\rangle - \frac{1}{\sqrt{2}}\|1\rangle\right)\|0\rangle = H\|1\rangle\|0\rangle$	$\|10\rangle$
11	$\left(\frac{1}{\sqrt{2}}\|1\rangle - \frac{1}{\sqrt{2}}\|0\rangle\right)\|1\rangle = -H\|1\rangle\|1\rangle$	$-\|11\rangle$

- Bob measures both qubits A and B.
- The output will be (a, b) with certainty.

Note $H^2|0\rangle = |0\rangle$ and $H^2|1\rangle = |1\rangle$



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References

1. Quantum Computing for Computer Scientists, by Noson S. Yanofsky, Mirco A. Mannucci
2. Quantum Computing Explained, David Mcmahon. John Wiley & Sons
3. Lecture Notes on Quantum Computation, John Watrous, University of Calgary
    - https://cs.uwaterloo.ca/~watrous/QC-notes/