CS621/CSL611
Quantum Computing For Computer Scientists

Quantum Architecture

Dhiman Saha

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IIT Bhilai

Multi-Bit/Multi-Qubit Setting

Juxtaposition of Kets $\implies$ Tensor Product

- Recall: To combine quantum systems, one should use the tensor product: $|\psi\rangle\,|\phi\rangle \overset{def}{=} |\psi\rangle \otimes |\phi\rangle$
- For spaces indexed by $\{00, 01, 10, 11\}$ we define[1]

$$|00\rangle = |0\rangle\,|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$|01\rangle = |0\rangle\,|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

[1] The pattern continues in this way for any number of bits. For example, $|1010\rangle$ is a 16 dimensional vector with a 1 in the position indexed by 1010 in binary.

A Classical Byte

- Consider a classical byte (8 bits)

$$01101011$$

- How would you represent it as series of vectors.

$$\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}$$

- In order to combine quantum systems, one should use the tensor product.

$$|0\rangle \otimes |1\rangle \otimes |1\rangle \otimes |0\rangle \otimes |1\rangle \otimes |0\rangle \otimes |1\rangle \otimes |1\rangle$$

- As a qubit, this is an element of

$$\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$$

Exercise

Express the three bits 101 or

$$|1\rangle \otimes |0\rangle \otimes |1\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$$

as a vector in $(\mathbb{C}^2)^{\otimes 3} = \mathbb{C}^8$.

Do the same for 011 and 111.

Byte Vs Qubyte

- This is a complex vector space of dimension $2^8 = 256$.
- $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 = (\mathbb{C}^2)^{\otimes 8} \simeq \mathbb{C}^{256}$

$$
\begin{array}{l}
\mathbf{00000000} \\
\mathbf{00000001} \\
\vdots \\
\mathbf{01101010} \\
\mathbf{01101011} \\
\mathbf{01101100} \\
\vdots \\
\mathbf{11111110} \\
\mathbf{11111111}
\end{array}
\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}.
\qquad
\begin{array}{l}
\mathbf{00000000} \\
\mathbf{00000001} \\
\vdots \\
\mathbf{01101010} \\
\mathbf{01101011} \\
\mathbf{01101100} \\
\vdots \\
\mathbf{11111110} \\
\mathbf{11111111}
\end{array}
\begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{106} \\ c_{107} \\ c_{108} \\ \vdots \\ c_{254} \\ c_{255} \end{bmatrix},
$$

Where $\sum_{i=0}^{255} |c_i|^2 = 1$

- Note the exponential growth

- Two qubits in ket notation

$$|0\rangle \otimes |1\rangle$$

$$|0 \otimes 1\rangle$$

- Meaning: which means that the first qubit is in state $|0\rangle$ and the second qubit is in state $|1\rangle$.
- Another view:

$$\begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Exercise

- What vector corresponds to the state $3\left|01\right\rangle + 2\left|11\right\rangle$?

Denoting A Two-qubit System

- $\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ can be written as

$$\frac{1}{\sqrt{3}} |00\rangle - \frac{1}{\sqrt{3}} |10\rangle + \frac{1}{\sqrt{3}} |11\rangle = \frac{|00\rangle - |10\rangle + |11\rangle}{\sqrt{3}}$$

- General state of a two-qubit system:

$$|\psi\rangle = c_{0,0} |00\rangle + c_{0,1} |01\rangle + c_{1,0} |10\rangle + c_{1,1} |11\rangle$$

Note

Tensor product of two states is **not** commutative:

$$|0\rangle \otimes |1\rangle = |01\rangle \neq |10\rangle = |1\rangle \otimes |0\rangle$$

- Two qubits are **entangled** if the system is in the following state:

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

Interpretation

- If we measure the first qubit and it is found in state $|1\rangle$ then we automatically know that the state of the second qubit is $|1\rangle$.
- Similarly, if we measure the first qubit and find it in state $|0\rangle$ then we know the second qubit is also in state $|0\rangle$.

- Perform a measurement:

$$\begin{pmatrix}\alpha\\\beta\end{pmatrix} \xrightarrow{\text{Measurement}} \begin{pmatrix}1\\0\end{pmatrix} \text{or, } \begin{pmatrix}0\\1\end{pmatrix}$$

- Perform a unitary operation: For any unitary matrix U , the operation described by U transforms any superposition v into the superposition Uv.

$$v \xrightarrow{U} Uv, \quad \text{where } U : U^{\dagger}U = I$$

- Some unitary matrices: Hadamard (H), Identity (I), Not

$$H = \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{pmatrix}, I = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, Not = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$$

- Suppose your friend has a qubit that he knows is in one of the two superpositions but he isnt sure which.

$$v_0 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} v_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

- How can you help him determine which one it is?