CS621/CSL611
Quantum Computing For Computer Scientists

Quantum Architecture

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Winter 2024

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Classical Gates From Quantum Perspective

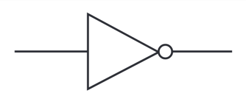
How should we represent our logical gates?

- We represent n input bits as a $(2^n \times 1)$ matrix and m output bits as a $(2^m \times 1)$ matrix
- When one multiplies a $(2^m \times 2^n)$ matrix with a $(2^n \times 1)$ matrix, the result is a $(2^m \times 1)$ matrix.

$$(2^m \times 2^n) \star (2^n \times 1) = (2^m \times 1)$$

- Bits $\rightarrow$ **column vectors**
- Logic gates $\rightarrow$ **matrices**

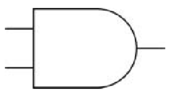
- To study classical gates from the point of view of matrices.



$$\mathrm{NOT}\,|0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$$

$$\mathrm{NOT} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\mathrm{NOT}\,|1\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$$



$$\text{AND} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- $\text{AND}\ |11\rangle = |1\rangle$

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- $\text{AND}\ |01\rangle = |0\rangle$

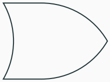
$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- Similarly, $\text{AND}\ |10\rangle = |0\rangle$ and $\text{AND}\ |00\rangle = |0\rangle$[1]

[1] We are **allowed only** to multiply these classical gates with vectors that represent classical states, i.e., column matrices with a single 1 entry and all other entries 0.

OR — Exercise

- Find the matrix for the OR Gate



- The matrix:

$$\mathrm{OR} = \begin{pmatrix} \underline{\hspace{2em}} & \underline{\hspace{2em}} & \underline{\hspace{2em}} & \underline{\hspace{2em}} \\ \\ \underline{\hspace{2em}} & \underline{\hspace{2em}} & \underline{\hspace{2em}} & \underline{\hspace{2em}} \end{pmatrix}$$

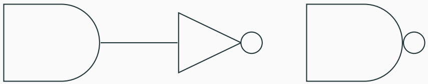
NAND

$$\text{NAND} = \begin{array}{c c} & \begin{array}{cccc} \mathbf{00} & \mathbf{01} & \mathbf{10} & \mathbf{11} \end{array} \\ \begin{array}{c} \mathbf{0} \\ \mathbf{1} \end{array} & \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Note

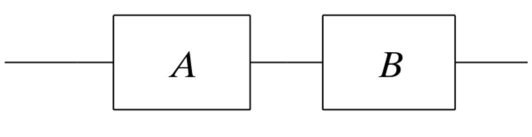
Column names correspond to the inputs and the row names correspond to the outputs. 1 in the jth column and ith row means that on entry j the matrix/gate will output i.

- Another way to determine the NAND gate



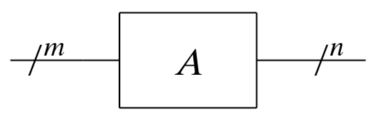
$$NOT \star AND = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \star \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} = \text{NAND}.$$

- Time goes from left to right.

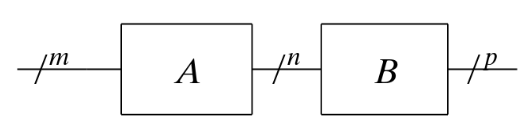


- A computation of A followed by B
- What does this mean in terms of the corresponding matrix multiplications?

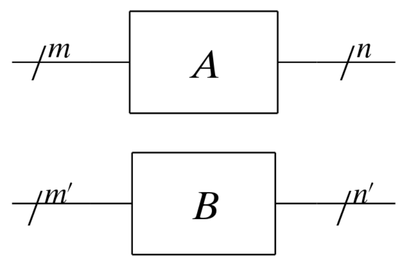
- What is size of the matrix for A?



- What are the sizes for A and B?



- Above sequential operation corresponds to $B \star A$
- What is the size of the matrix corresponding to $B \star A$?



- $A \to 2^n \times 2^m$
- $B \to 2^{n'} \times 2^{m'}$

- Here we have A acting on some bits and B on others
- This is captured by tensor product: $A \otimes B$
- $A \otimes B \to 2^n 2^{n'} \times 2^m 2^{m'}$
- So $A \otimes B$ is of size:

$$2^{n+n'} \times 2^{m+m'}$$

- AKA Kronecker product
- In general, for any two matrices

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,m} \end{pmatrix}_{n\times m} \qquad B = \begin{pmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,l} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,l} \\ \vdots & \vdots & \ddots & \vdots \\ b_{k,1} & b_{k,2} & \cdots & b_{k,l} \end{pmatrix}_{k\times l}$$

- $A \otimes B$ is defined by the $nk \times ml$ matrix[2]

$$A \otimes B = \begin{pmatrix} a_{1,1}B & a_{1,2}B & \cdots & a_{1,m}B \\ a_{2,1}B & a_{2,2}B & \cdots & a_{2,m}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1}B & a_{n,2}B & \cdots & a_{n,m}B \end{pmatrix}_{nk\times ml}$$

[2] The definition works for vectors by thinking of them as one column matrices.

- Associative: For any choice of matrices A, B and C:

$$(A \otimes B) \otimes C = A \otimes (B \otimes C)$$

- For any choice of matrices A, B, C and D[3]

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD) \tag{1}$$

- Distributive: $A \otimes (B + C) = A \otimes B + A \otimes C$
- Scalar multiplication[4]: $(\alpha A) \otimes B = A \otimes (\alpha B) = \alpha(A \otimes B)$
- Not-commutative: $A \otimes B \neq B \otimes A$

[3] Juxtaposition implies multiplication

[4] Scalars "float freely" through the tensor product

Eq (1) is of special significance while dealing with quantum circuits

- Let X and Y be qubits having associated superpositions

$$v = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \qquad w = \begin{pmatrix} \gamma \\ \delta \end{pmatrix}$$

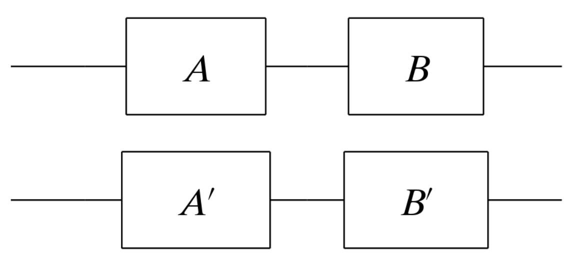
- The superposition of the pair (X, Y) is $v \otimes w = \begin{pmatrix} \alpha\gamma \\ \alpha\delta \\ \beta\gamma \\ \beta\delta \end{pmatrix}$

- Not every superposition can be written as a tensor product

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \leftarrow$$

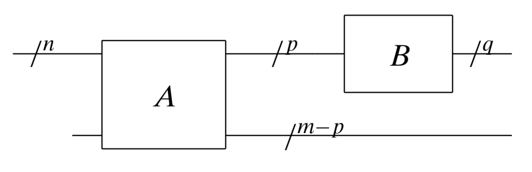
Recall: For quantum case, this type of correlation is called an <u>entanglement</u>

- $(B \otimes B')(A \otimes A') = (BA) \otimes (B'A')$



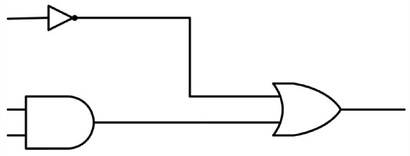
- To what does this correspond in terms of performing different operations on different (qu)bits?

- How to treat the $m - p$ bits that are not operated on by B?
- Here B is a $2^q \times 2^p$ matrix



- Use the $(2^{m-p} \times 2^{m-p})$ Identity matrix I_{m-p}
- The entire circuit can be represented by the following matrix:

$$(B \otimes I_{m-p})A$$



$$\text{NOT} \otimes \text{AND} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{OR} \star (\text{NOT} \otimes \text{AND}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$