Task 1

Problem 1)

(a)

A subset s of a vector space V is called a basis if:

1) s is linearly independant
2) S is a spanning set

$\varphi_1 = [1 \ -1 \ 2]^T \quad \varphi_2 = [2 \ 3 \ -2]^T \quad \varphi_3 = [3 \ 1 \ 1]^T$

$$\varphi_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \quad \varphi_2 = \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix} \quad \varphi_3 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

Check if S is linearly independent set

Consider linear combination:

$$x_1 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = 0$$

Matrix Equation

$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Augmented Matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ -1 & 3 & 1 & 0 \\ 2 & -2 & 1 & 0 \end{array}\right] \xrightarrow[R_2 \to R_2 + R_1]{R_3 \to R_3 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 5 & 4 & 0 \\ 0 & -6 & -5 & 0 \end{array}\right]$$

$\downarrow R_3 \to R_3 + R_2$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 5 & 4 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] \xleftarrow{R_3 \to 5R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 5 & 4 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right]$$

$x_1 + 2x_2 + 3x_3 = 0, \quad 5x_2 + 4x_3 = 0, \quad -x_3 = 0 \Rightarrow x_3 = 0$

$\hookrightarrow x_1 = 0 \qquad \hookrightarrow x_2 = 0 \qquad \therefore \phi$ is linearly independent

So it must form a basis in R^3 $\Leftarrow$ as it consists of 3 linearly independent vectors in R^3

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(b)

To find biorthogonal basis $\Psi = \{\psi_1\ \psi_2\ \psi_3\}$ we use the formula:

$$\psi_i = (A^{-1})^T e_i$$

where A is the matrix with the vectors of ϕ as its columns

e_i is the ith standard basis vector

$(A^{-1})^T$ is the transpose of the inverse of A

Finding inverse of A:

$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix} = A$$

Augmented Matrix to make the left side identity matrix:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5/8 & 1/4 & -1/8 \\ 0 & 1 & 0 & 1/4 & 1/8 & -1/8 \\ 0 & 0 & 1 & -1/4 & -1/8 & 1/8 \end{array}\right]$$

Therefore, inverse of A is:

$$A^{-1} = \begin{bmatrix} 5/8 & 1/4 & -1/8 \\ 1/4 & 1/8 & -1/8 \\ -1/4 & -1/8 & 1/8 \end{bmatrix}$$

Now we can find biorthogonal basis:

$$\psi_1 = (A^{-1})^T [1\ 0\ 0]^T = [5/8\ \ 1/4\ \ -1/8]^T$$

$$\psi_2 = (A^{-1})^T [0\ 1\ 0]^T = [1/4\ \ 1/8\ \ -1/8]^T$$

$$\psi_3 = (A^{-1})^T [0\ 0\ 1]^T = [-1/8\ \ -1/8\ \ 1/8]^T$$

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(c)

To find coefficients c_1, c_2, c_3 for an arbitrary vector $x = [x_1\ x_2\ x_3]^T$ in R^3 we use the formula:

$$[x_1\ x_2\ x_3]^T = c_1[1\ -1\ 2]^T + c_2[2\ 3\ -2]^T + c_3[3\ 1\ 1]^T$$

This can be written as a system of linear equations:

$$c_1 + 2c_2 + 3c_3 = x_1$$
$$-c_1 + 3c_2 + c_3 = x_2$$
$$2c_1 - 2c_2 + c_3 = x_3$$

To solve this we use matrix inversion, which will give the values of c_1, c_2, c_3

(d)

To find the largest number $A > 0$ and smallest number $B < \infty$ such that $A\|x\|^2 \le \sum_{i=1}^{3} |\langle \varphi_i, x\rangle|^2 \le B\|x\|^2$, we can use the Cauchy-Schwarz inequality:

$$|\langle \varphi_i, x\rangle| \le \|\varphi_i\|\|x\|$$

square both sides and summing over all i:

$$\sum_{i=1}^{3} |\langle \varphi_i, x\rangle|^2 \le (\|\varphi_1\|^2 + \|\varphi_2\|^2 + \|\varphi_3\|^2)\|x\|^2$$

Find $\|\varphi_1\|^2 + \|\varphi_2\|^2 + \|\varphi_3\|^2$:

$$\|\varphi_1\|^2 + \|\varphi_2\|^2 + \|\varphi_3\|^2 = \|[1\ -1\ 2]^T\|^2 + \|[2\ 3\ -2]\|^2 + \|[3\ 1\ 1]\|^T$$
$$= 6 + 17 + 11 = 34$$

Find max and min eigenvalues of $A^{-1}A$ which is Gram matrix of ϕ:

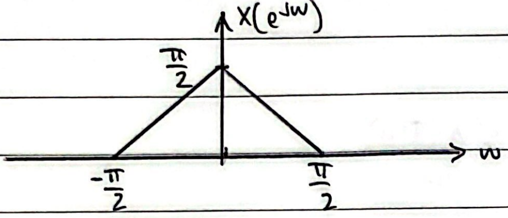
$$A^{-1}A = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{8} \\ \frac{1}{4} & \frac{5}{16} & -\frac{1}{16} \\ \frac{1}{8} & -\frac{1}{16} & \frac{1}{16} \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{7}{4} & -\frac{3}{4} & \frac{1}{4} \\ -\frac{3}{4} & \frac{13}{16} & -\frac{1}{16} \\ \frac{1}{4} & -\frac{1}{16} & \frac{9}{16} \end{bmatrix} \quad \det(\lambda I - A^{-1}A) = (\lambda-1)(\lambda-\tfrac{1}{2})(\lambda-\tfrac{11}{16})$$

largest eigenvalue = 1, smallest eigenvalue = $\frac{4}{16}$

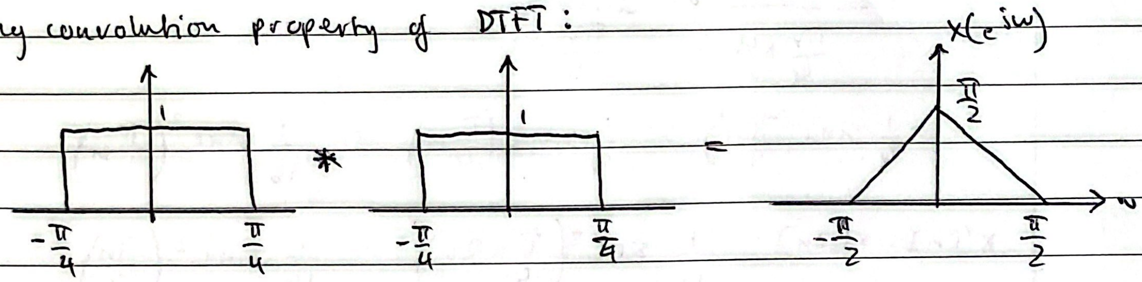
Therefore, $\frac{34}{16}\|x\|^2 \le \sum_{i=1}^{3} |\langle \varphi_i, x\rangle|^2 \le \|x\|^2$. $A = \frac{34}{16}$, $B = 1$

Problem 2)

(a) $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{jw})\, e^{jwn}\, dw$

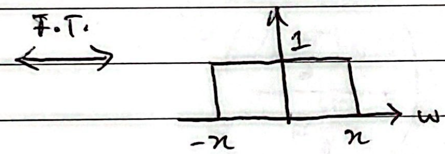


Using convolution property of DTFT:

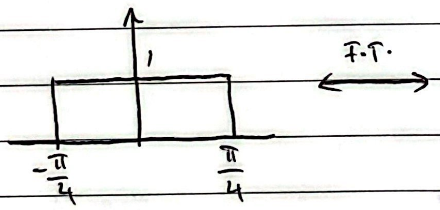


Convolving rectangular pulses of equal length results in the triangular pulse

$\frac{\sin xt}{\pi t}$

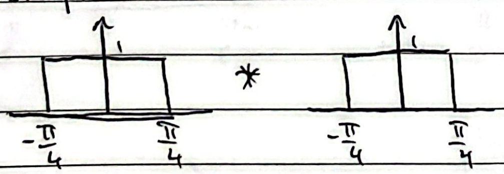


Therefore, the F.T. of the above rectangular pulse is given by:



$\frac{\sin \frac{\pi}{4} n}{\pi n}$

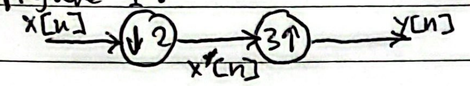
Since the frequency domain is convolved the time domain is multiplied:



$= \left(\frac{\sin \frac{\pi}{4} n}{\pi n}\right) \cdot \left(\frac{\sin \frac{\pi}{4} n}{\pi n}\right)$

$x[n] = \left(\frac{\sin \frac{\pi}{4} n}{\pi n}\right)^2$

figure 1:



$x'[n] = x[2n]$

$$\text{sinc}(n) = \frac{\sin(\pi n)}{\pi n}$$

$$x[n] = \frac{\sin\left(\frac{\pi}{4}n\right)}{\frac{\pi n}{4} \times 4} \cdot \frac{\sin\left(\frac{\pi}{4}n\right)}{\frac{\pi n}{4} \times 4}$$

$$= \frac{1}{4}\,\text{sinc}\left(\frac{\pi}{4}n\right) \cdot \frac{1}{4}\,\text{sinc}\left(\frac{\pi}{4}n\right) = \frac{1}{16}\,\text{sinc}^2\left(\frac{\pi}{4}n\right)$$

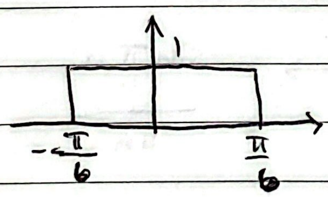
$$x'[n] = x[2n] = \frac{1}{16}\,\text{sinc}^2\left(\frac{\pi}{4} \times 2n\right) = \frac{1}{16}\,\text{sinc}^2\left(\frac{\pi n}{2}\right)$$

$$y[n] = x'[n/3] = \frac{1}{16}\,\text{sinc}^2\left(\frac{\frac{\pi n}{3}}{2}\right) = \frac{1}{16}\,\text{sinc}^2\left(\frac{\pi n}{6}\right)$$

$$= \frac{1}{16}\left(\frac{\sin\left(\frac{\pi}{6}n\right)}{\frac{\pi}{6}n}\right)^2$$

$$= \frac{36}{16}\left(\frac{\sin\left(\frac{\pi}{6}n\right)}{\pi n}\right)^2 = \frac{9}{4}\left(\frac{\sin\left(\frac{\pi}{6}n\right)}{\pi n}\right)^2$$

$$\frac{\sin\frac{\pi}{6}n}{\pi n} \overset{F.T.}{\longleftrightarrow}$$



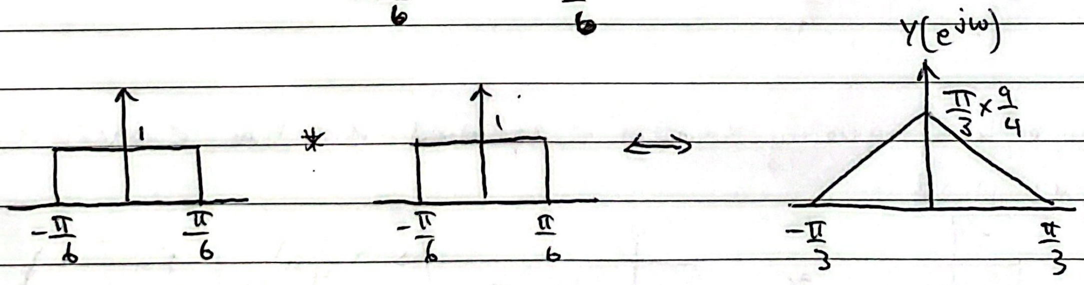


figure 2:

$x[n] \rightarrow \boxed{\uparrow 3} \xrightarrow{x'[n]} \boxed{\downarrow 2} \rightarrow y[n]$

$x'[n] = x[n/3]$

$x[n] = \frac{1}{16}\,\text{sinc}^2\left(\frac{\pi}{4}n\right)$

$x'[n] = \frac{1}{16}\,\text{sinc}^2\left(\frac{\pi}{4}\frac{n}{3}\right) = \frac{1}{16}\,\text{sinc}^2\left(\frac{\pi n}{12}\right)$

$y[n] = x'[2n] = \frac{1}{16}\,\text{sinc}^2\left(\frac{\pi\, 2n}{12}\right) = \frac{1}{16}\,\text{sinc}^2\left(\frac{\pi n}{6}\right)$

$= \frac{1}{16}\left(\frac{\sin\left(\frac{\pi n}{6}\right)}{\frac{\pi n}{6}}\right)^2 = \frac{36}{16}\left(\frac{\sin\left(\frac{\pi n}{6}\right)}{\pi n}\right)^2$

$= \frac{9}{4}\left(\frac{\sin\left(\frac{\pi n}{6}\right)}{\pi n}\right)^2$

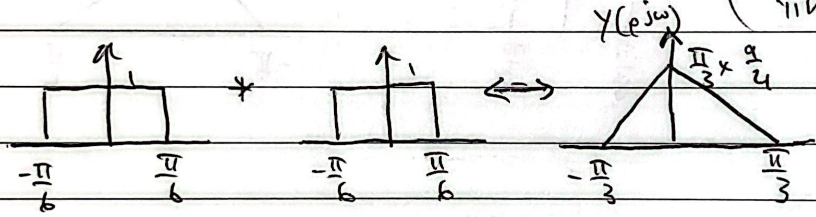


Figure 3:

$x[n] \rightarrow \boxed{\downarrow 2} \xrightarrow{x'[n]} \boxed{\uparrow 4} \rightarrow y[n]$

$x[n] = \frac{1}{16}\,\text{sinc}^2\left(\frac{\pi}{4}n\right)$

$x'[n] = x[2n] = \frac{1}{16}\,\text{sinc}^2\left(\frac{\pi n}{2}\right)$

$y[n] = x'[n/4] = \frac{1}{16}\,\text{sinc}^2\left(\frac{\pi}{2}\frac{n}{4}\right) = \frac{1}{16}\,\text{sinc}^2\left(\frac{\pi n}{8}\right)$

$= \frac{1}{16}\left(\frac{\sin\left(\frac{\pi n}{8}\right)}{\frac{\pi n}{8}}\right)^2 = \frac{64}{16}\left(\frac{\sin\left(\frac{\pi n}{8}\right)}{\pi n}\right)^2 = 4\left(\frac{\sin\left(\frac{\pi n}{8}\right)}{\pi n}\right)^2$

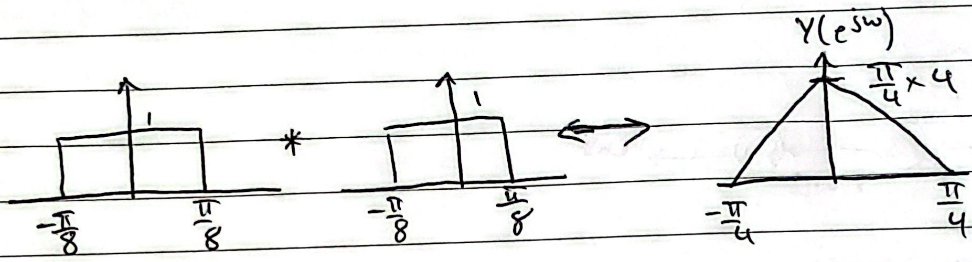


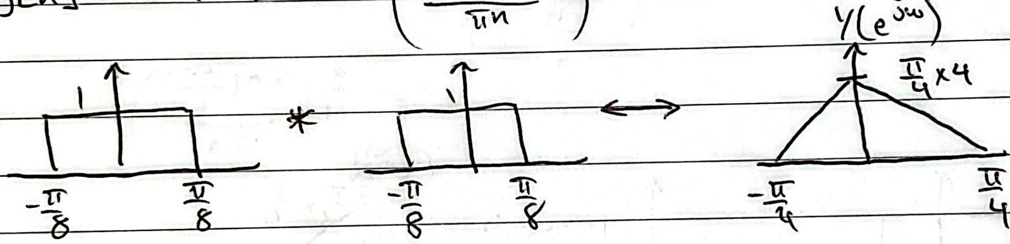
figure 4:

$$x[n] \longrightarrow \boxed{4\uparrow} \xrightarrow{x'[n]} \boxed{2\downarrow} \longrightarrow y[n]$$

$$x[n] = \frac{1}{16}\,\text{sinc}^2\left(\frac{\pi}{4}n\right)$$

$$x'[n] = x[n/4] = \frac{1}{16}\,\text{sinc}^2\left(\frac{\pi n}{16}\right)$$

$$y[n] = x'[2n] = 4\left(\frac{\sin\left(\frac{\pi n}{8}\right)}{\pi n}\right)^2$$



(b)

figure 1:

$x[n] \rightarrow 2\downarrow \xrightarrow{x[n]} 3\uparrow \rightarrow y[n]$

$x[n] = \{5, 4, 4, 3, 2, 1, 1, 1, 1, 2, 2, 2, 2\}$

Downsample by 2:

$x[n] = \{4, 3, 1, 1, 2, 2\}$

Upsample by 3:

$y[n] = \{4, 0, 0, 3, 0, 0, 1, 0, 0, 1, 0, 0, 2, 0, 0, 2\}$

n	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9

figure 2:

$x[n] \rightarrow 3\uparrow \xrightarrow{x[n]} 2\downarrow \rightarrow y[n]$

$x[n] = \{3, 4, 1, 2, 2\}$

Upsample by 3:

$x[n] = \{3, 0, 0, 4, 0, 0, 1, 0, 0, 2, 0, 0, 2\}$

Downsample by 2:

$y[n] = \{3, 0, 0, 1, 0, 0, 2\}$

n	-3	-2	-1	0	1	2	3

figure 3:

$x[n] \rightarrow 2\downarrow \xrightarrow{x[n]} 4\uparrow \rightarrow y[n]$

$x[n] = \{1, 1, 2, 3, 1, 2, 2\}$

Downsample by 2:

$x[n] = \{1, 3, 2\}$

Upsample by 4:

$y[n] = \{1, 0, 0, 0, 3, 0, 0, 0, 2\}$

n	-4	-3	-2	-1	0	1	2	3	4

figure 4:

$$x[n] \longrightarrow 4\uparrow \xrightarrow{x[n]} 2\downarrow \longrightarrow y[n]$$

$x[n] = \{3, 1, 1, 2, 2\}$

Upsample by 4:

$x[n] = \{3,0,0,0,1,0,0,0,1,0,0,0,2,0,0,0,2\}$

Downsample by 2:

$y[n] = \{3, 0, 1, 0, 1, 0, 2, 0, 2\}$

n	-4	-3	-2	-1	0	1	2	3	4

Problem 3)

(a)

$$a[n] = \sum_{k=-\infty}^{\infty} h[k]h^*[k-n]$$

$$= D.T.F.T\{a[n]\} = \sum_{n=-\infty}^{\infty} a[n]e^{-jwn}$$

$$= \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} h[k]h^*[k-n]e^{-jwn}$$

$k-n=l \Rightarrow n=k-l$

$$A(e^{jw}) = \sum_{l=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} h[k]\, h^*[l]e^{-jw(k-l)}$$

$$= \sum_{l=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty} h[k]e^{-jwk}\right)e^{jwl}h^*[l]$$

$$= \sum_{l=-\infty}^{\infty} H(e^{jw})h^*[l]e^{jwl}$$

$$= H(e^{jw})\left[\sum_{l=-\infty}^{\infty} h[l]e^{-jwl}\right]^* = H(e^{jw})H^*(e^{jw})$$

$$\Rightarrow A(e^{jw}) = |H(e^{jw})|^2 = D.T.F.T\{a[n]\}$$

(b) If $a[n] = \sum_{k=-\infty}^{\infty} h[k]h^*[k-n] = \delta[n]$

$\Updownarrow$ D.T.F.T from part (a)

$$A(e^{jw}) = |H(e^{jw})|^2 = 1$$

because $\delta(n) \xleftrightarrow{D.T.F.T} 1$

$\therefore |H(e^{jw})| = 1$ for $|w| < \pi$

(c) $|H(e^{jw})| = 1 \Rightarrow |H(e^{jw})|^2 = 1$

$\Updownarrow$ I. D.T.F.T.

$$a[n] = \sum_{k=-\infty}^{\infty} h[k]h^*[k-n] = \delta[n] \quad \text{from part (b)}$$

$\Rightarrow h[k]$ is orthogonal to its shifted version $h^*[k-n]$ and using the D.T.F.T pair

$$\frac{\sin(\pi n)}{\pi n} = \sin(n) \xleftrightarrow{D.T.F.T} 1, \quad |w| < \pi$$

and time shifting property,

$$\delta(n-n_0) \xleftrightarrow{\text{D.T.F.T.}} e^{-jwn_0}, \quad |w|<\pi$$

$$\Rightarrow h[k] = \delta(k-k_0) \longleftrightarrow H(e^{jw}) = e^{-jwk_0}$$

$$\|h[k]\|^2 = \|\delta(k-k_0)\|^2 = \sum_{k=-\infty}^{\infty} [\delta[k-k_0)]^2$$

because $\delta(k-k_0) = \begin{cases} 1, & k=k_0 \\ 0, & k\neq k_0 \end{cases}$

(d) For a orthonormal basis:

1) Orthogonality: The elements of the set $\{h[k-n], n\in\mathbb{Z}\}$ are orthogonal to each other
2) Completeness: The set $\{h[k-n], n\in\mathbb{Z}\}$ spans the entire $\ell^2(\mathbb{Z})$

Orthogonality:
From part (c), we know $H(e^{jw}) = 1, \forall w$, then $\langle h[k-n], h[k]\rangle = \delta[n]$.
This shows elements of the set $\{h[k-n], n\in\mathbb{Z}\}$ are orthogonal to each other.

Completeness:

$$x[k] = \sum_{n=-\infty}^{\infty} c_n h[k-n]$$

where c_n are complex coefficients to be determined
To find the coefficients c_n, we need to take the inner product of $x[k]$ with each element of the set:

$$\langle x[k], h[k-n]\rangle = \sum_k x[k] h^*[k-n]$$

Remember, $\langle h[k-n], h[k]\rangle = \delta[n]$, we have:

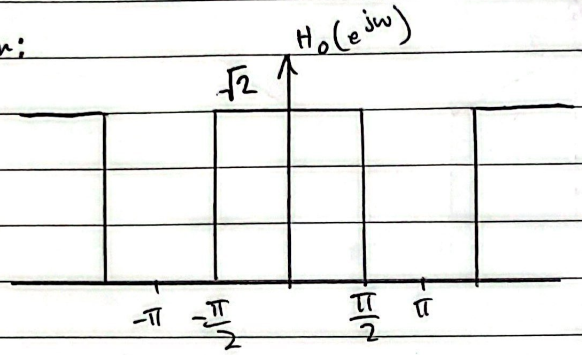
$$\langle x[k], h[k-n]\rangle = \sum_k x[k] h^*[k-n] = c_n$$

$\therefore$ the coefficient c_n is the inner product of $x[k]$ with $h[k-n]$.
Since the set $\{h[k-n], n\in\mathbb{Z}\}$ spans all possible shifts of $h[k]$, we have shown that it spans $\ell^2(\mathbb{Z})$

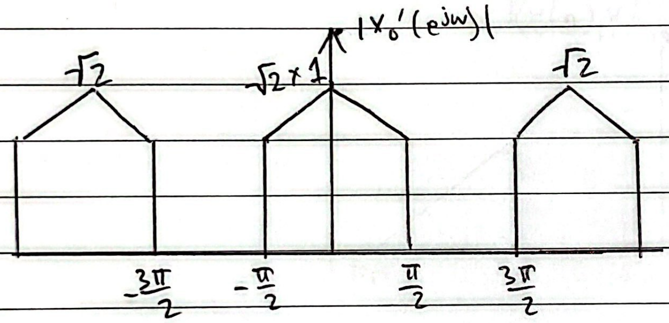
Task 2

Problem 1)

Given:

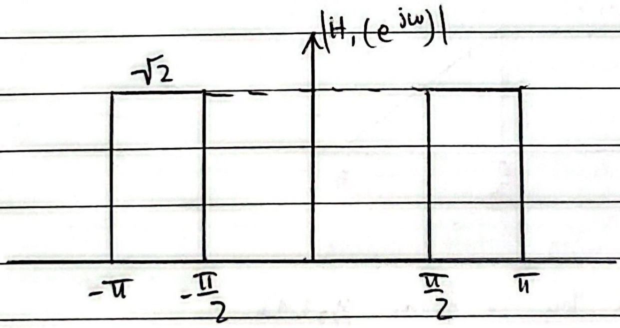


$X_0'(e^{jw}) = X(e^{jw})\,H_0(e^{jw})$



$H_1(e^{jw}) = -e^{-jw}\sqrt{2}$, magnitude of $H_1(e^{jw}) = \sqrt{2}$

$|X_1'(e^{jw})| = |X(e^{jw})| \cdot |H_1(e^{jw})|$

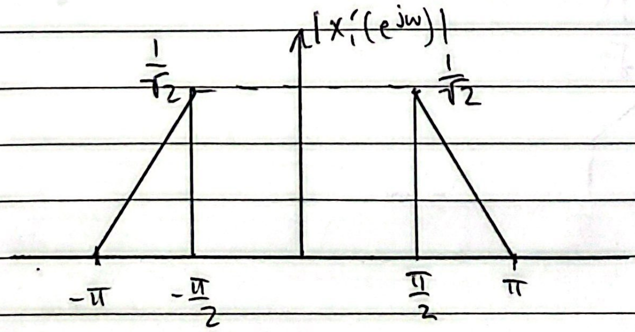


$X(e^{jw}) = -\frac{w}{\pi} + 1$

$(0 \le w < \pi)$

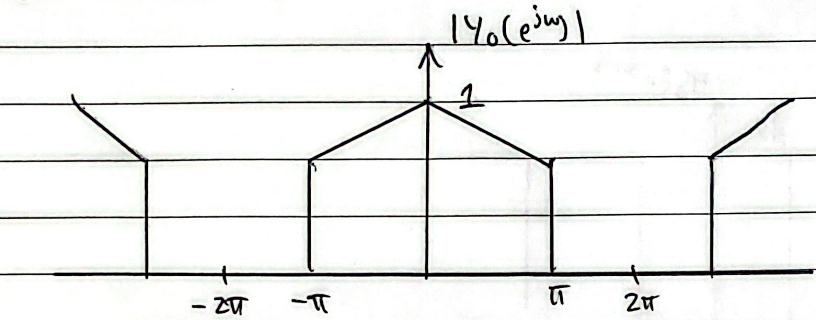
at $w = \frac{\pi}{2}$

$X(e^{jw}) = 0.5$



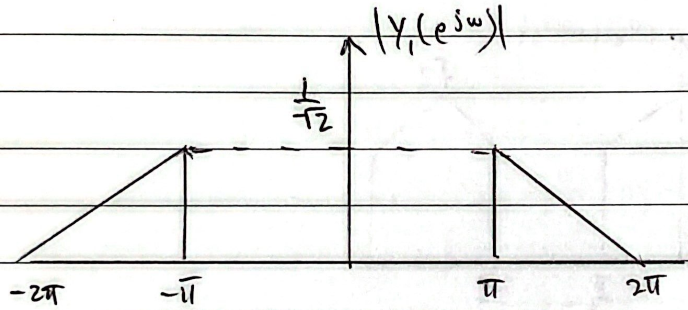
$Y_0[n] = X_0'[kn]$, $k = 2$

$Y_0(e^{jw}) = X_0'(e^{jw})$

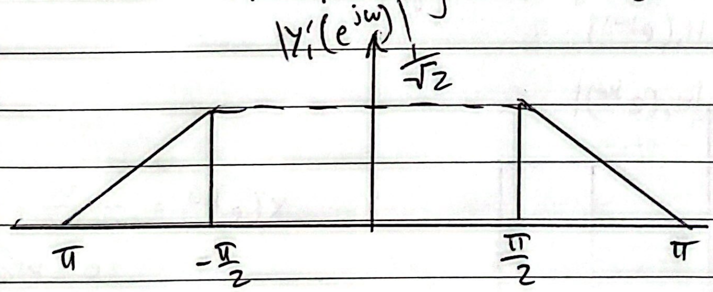


$y_1[n] = x_1'[2n]$

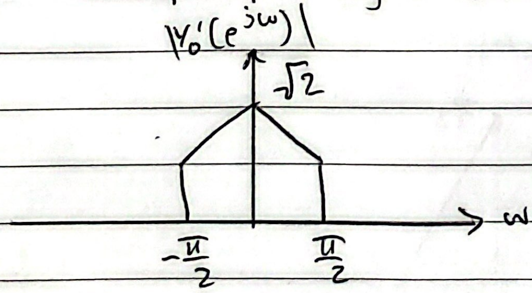
$Y_1(e^{jw}) = X_1'(e^{j\times 2})$



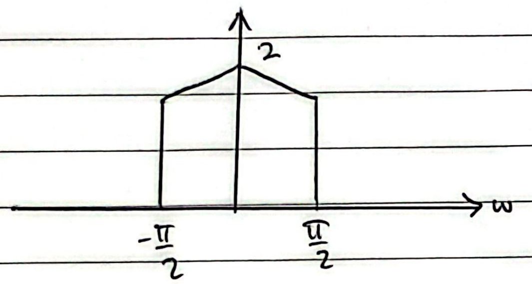
$y_1'[n] \Rightarrow$ Upsampled by 2 from $y_1[n]$



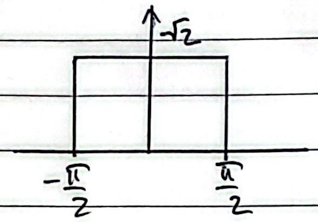
$y_0'[n] \Rightarrow$ Upsampled by 2 from $y_0[n]$



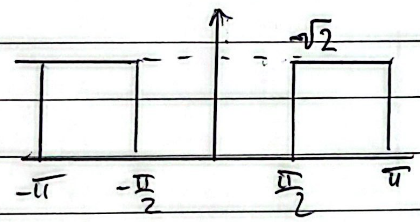
$\|X_0''(e^{jw})\| = \|Y_0'(e^{jw})\| \|G_0(e^{jw})\|$



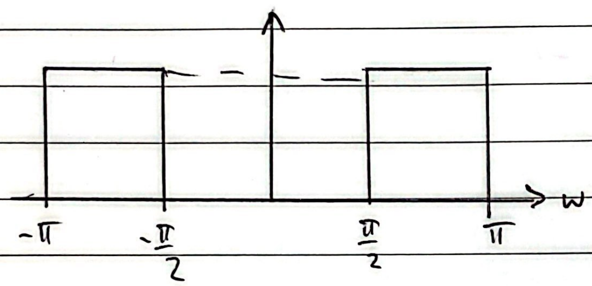
$G_0(e^{jw}) = H_0(e^{-jw})$



$G_1(e^{jw}) = H_1(e^{-jw})$



$\|X_1''(e^{jw})\| = \|Y_1'(e^{jw})\| \|G_1(e^{jw})\|$



Problem 2

$$G_p(z) = \begin{bmatrix} G_{00}(z) & G_{10}(z) \\ G_{01}(z) & G_{11}(z) \end{bmatrix} = U_0 \left(\prod_{i=1}^{k-1} \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix} U_i \right)$$

where U_i are unitary 2×2 matrices defined as:

$$U_i = \begin{bmatrix} \cos\alpha_i & -\sin\alpha_i \\ \sin\alpha_i & \cos\alpha_i \end{bmatrix}$$

For $k=2$:

$$G_p(z) = U_0 \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix} U_1$$

$$G_p(z) = \begin{bmatrix} \cos\alpha_0 & -\sin\alpha_0 \\ \sin\alpha_0 & \cos\alpha_0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix} \begin{bmatrix} \cos\alpha_1 & -\sin\alpha_1 \\ \sin\alpha_1 & \cos\alpha_1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix} \begin{bmatrix} \cos\alpha_1 & -\sin\alpha_1 \\ z^{-1}\sin\alpha_1 & z^{-1}\cos\alpha_1 \end{bmatrix} = \begin{bmatrix} \cos\alpha_1 & -\sin\alpha_1 \\ z^{-1}\sin\alpha_1 & z^{-1}\cos\alpha_1 \end{bmatrix}$$

Multiply with U_0:

$$\begin{bmatrix} \cos\alpha_0 & -\sin\alpha_0 \\ \sin\alpha_0 & \cos\alpha_0 \end{bmatrix} \begin{bmatrix} \cos\alpha_1 & -\sin\alpha_1 \\ z^{-1}\sin\alpha_1 & z^{-1}\cos\alpha_1 \end{bmatrix}$$

$$G_p(z) = \begin{bmatrix} \cos\alpha_0\cos\alpha_1 - \sin\alpha_0 z^{-1}\sin\alpha_1 & -\cos\alpha_0\sin\alpha_1 - \sin\alpha_0 z^{-1}\cos\alpha_1 \\ \sin\alpha_0\cos\alpha_1 + \cos\alpha_0 z^{-1}\sin\alpha_1 & -\sin\alpha_0\sin\alpha_1 + \cos\alpha_0 z^{-1}\cos\alpha_1 \end{bmatrix}$$

$$G_0(z) = \cos\alpha_0\cos\alpha_1 - \sin\alpha_0 z^{-1}\sin\alpha_1$$
$$G_1(z) = -\cos\alpha_0\sin\alpha_1 - \sin\alpha_0 z^{-1}\cos\alpha_1$$

Set $z = e^{j\pi} \Rightarrow e^{j\pi} = -1$

$G_0(-1) = \cos\alpha_0\cos\alpha_1 + \sin\alpha_0\sin\alpha_1$

To make $G_0(-1) = 0 \Rightarrow \cos(\alpha_0 + \alpha_1) = 0$

$\Rightarrow \alpha_0 + \alpha_1 = \frac{\pi}{2}$

The expressions are:

$$G_0(z) = \cos\alpha_0 \cos\alpha_1 - \sin\alpha_0 z^{-1} \sin\alpha_1$$

$$G_1(z) = -\cos\alpha_0 \sin\alpha_1 - \sin\alpha_0 z^{-1} \cos\alpha_1$$

To satisfy $G_0(e^{j\pi}) = 0$, select α_0 and α_1 such that:

$$\alpha_0 + \alpha_1 = \frac{\pi}{2}$$

Problem 3)

$h_0[n] = h_0[N-1-m]$ for $0 \leq n \leq N-1$ (symmetric)
$h_1[n] = -h_1[N-1-n]$ for $0 \leq n \leq N-1$ (antisymmetric)

Since $h_0[n]$ and $h_1[n]$ have even lengths lets say L_0 and L_1 respectively, the lengths of the iterated filters will be as follows:

1) $H_a(z) = H_0(z) \times H_0(z^2)$

$$\text{Length}(h_a[n]) = h_0[n] + h_0[2n] - 1$$
$$= L_0 + \frac{L_0}{2} - 1 = \frac{3}{2} L_0 - 1$$

2) $H_b(z) = H_0(z) \times H_1(z^2)$

$$\text{Length}(h_b[n]) = h_0[n] + H_1[2n] - 1$$
$$= L_0 + \frac{L_1}{2} - 1$$

3) $H_c(z) = H_1(z) \times H_0(z^2)$

$$\text{Length}(h_c[n]) = h_1[n] + h_0[2n] - 1$$
$$= L_1 + \frac{L_0}{2} - 1$$

4) $H_d(z) = H_1(z) \times H_1(z^2)$

$$\text{Length}(h_d[n]) = h_1[n] + h_1[2n] - 1$$
$$= L_1 + \frac{L_1}{2} - 1 = \left(\frac{3}{2}\right) L_1 - 1$$

$H_a(z)$: Symmetric, $\text{length}(h_a[n]) = \frac{3}{2} L_0 - 1$
$H_b(z)$: Antisymmetric, $\text{length}(h_b[n]) = L_0 + \frac{L_1}{2} - 1$
$H_c(z)$: Antisymmetric, $\text{length}(h_c[n]) = L_1 + \frac{L_0}{2} - 1$
$H_d(z)$: Symmetric, $\text{length}(h_d[n]) = \frac{3}{2} L_1 - 1$

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Problem 4)

(a) $f(t) = \begin{cases} t & 0 \le t \\ 0 & t < 0 \end{cases}$

$$CWT_f(a,b) \simeq a^{3/2}$$

$$CWT_f(a,b) = \int_{-\infty}^{\infty} \psi_{a,b}\, f(t)\, dt$$

$$\psi(t) = \begin{cases} 1 & -\frac{1}{2} \le t \le 0 \\ -1 & 0 \le t \le \frac{1}{2} \end{cases}$$

$$CWT_f(a,b) = \begin{cases} -\frac{1}{\sqrt{a}} \int_0^{\frac{a}{2}+b} f(t)\, dt, & b < 0 \\ -\frac{1}{\sqrt{a}} \int_0^{b} f(t)\, dt - \frac{1}{\sqrt{a}} \int_b^{\frac{a}{2}+b} f(t)\, dt & b \ge 0 \end{cases}$$

for $f(t) = \begin{cases} 0 & t < 0 \\ t^n & t \ge 0 \end{cases}$

$$CWT_f(a,b) = \begin{cases} -\frac{a}{\sqrt{a}} \frac{1}{n+1} \left(\frac{a}{2}+b\right)^{n+1} & b < 0 \\ -\frac{1}{\sqrt{a}} \frac{1}{n+1} \left(\left(\frac{a}{2}+b\right)^{n+1} - 2b^{n+1}\right) & b \ge 0 \end{cases}$$

for $b \ll \frac{a}{2}$, this gives

$$CWT_f(a,b) \approx \frac{a^{\left(\frac{2n+1}{2}\right)}}{(n+1)2^{n+1}} \sim a^{\left(\frac{2n+1}{2}\right)} \sim a^{\frac{2n+1}{2}}$$

sub $n = 1$:

$$CWT_f(a,b) = \frac{a^{3/2}}{(2)(2)^2} \sim a^{\frac{3}{2}} \sim a^{\frac{3}{2}}$$ Hence proved

(b) $f(t) = \begin{cases} t^n & 0 \le t, \quad n = 0, 1, 2, \dots \\ 0 & t < 0 \quad t < 0 \end{cases}$

$$CWT_f(a,b) \simeq a^{(2n+1)/2}$$

The derivation in part (a) itself shows that $CWT \sim a^{\frac{2n+1}{2}}$