Topics on Data and Signal Analysis

Task I

Problem 1. Consider the following three vectors in $\mathbb{R}^3$:

$$\varphi_1 = [1 \ -1 \ 2]^T \quad , \quad \varphi_2 = [2 \ 3 \ -2]^T \quad , \quad \varphi_3 = [3 \ 1 \ 1]^T \ .$$

a) Does the set $\Phi = \{\varphi_1, \varphi_2, \varphi_3\}$ form a basis for $\mathbb{R}^3$?

b) If $\Phi = \{\varphi_1, \varphi_2, \varphi_3\}$ forms a basis, find its biorthogonal basis $\Psi = \{\psi_1, \psi_2, \psi_3\}$.

c) For an arbitrary $x = [x_1 \ x_2 \ x_3]^T$ in $\mathbb{R}^3$ describe the procedure and the expression for finding coefficients c_1, c_2 and c_3 such that

$$x = c_1\varphi_1 + c_2\varphi_2 + c_3\varphi_3 \ .$$

d) Find the largest number $A > 0$ and the smallest number $B < \infty$ such that

$$A\|x\|^2 \leq \sum_{i=1}^{3} |\langle \varphi_i, x \rangle|^2 \leq B\|x\|^2$$

for all $x \in \mathbb{R}^3$.

Problem 2. Consider the systems in Figures 1-4.

$x[n] \rightarrow 2\downarrow \rightarrow 3\uparrow \rightarrow y[n]$

Figure 1

$x[n] \rightarrow 3\uparrow \rightarrow 2\downarrow \rightarrow y[n]$

Figure 2

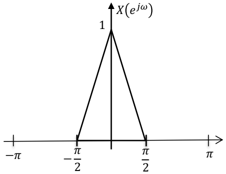
$x[n] \rightarrow 2\downarrow \rightarrow 4\uparrow \rightarrow y[n]$

Figure 3

$x[n] \rightarrow 4\uparrow \rightarrow 2\downarrow \rightarrow y[n]$

Figure 4

a) For the signal $x[n]$ which is given in the Fourier domain by the figure below, sketch $Y(e^{j\omega})$ for each of the four systems in the above.



b) For any arbitrary signal $x[n]$ express samples of $y[n]$ in terms of samples of $x[n]$ for each of the four systems in the above.

Problem 3. Consider a filter $h[n]$.

a) Find the Fourier transform of its autocorrelation sequence

$$a[n] = \sum_k h[k]h^*[k-n] \ .$$

b) Show that if $\langle h[k-n], h[k] \rangle = \delta[n]$, then

$$|H(e^{j\omega})| = 1, \ \forall \omega.$$

c) Show that if $|H(e^{j\omega})| = 1$, then

$$\langle h[k-n], h[k] \rangle = \delta[n] \ .$$

d) Show that if $|H(e^{j\omega})| = 1, \forall \omega$, then $\{h[k-n], n \in \mathbb{Z}\}$ is an orthonormal basis for $\ell^2(\mathbb{Z})$.

Problem 4. Consider two waveforms $\varphi_0[n]$ and $\varphi_1[n]$ and two waveform $\psi_0[n]$ and $\psi_1[n]$ in $\ell^2(\mathbb{Z})$. Let $h_0[n]$ and $h_1[n]$ be two filters such that $h_0[n] = \varphi_0^*[-n]$ and $h_1[n] = \varphi_1^*[-n]$, and $g_0[n]$ and $g_1[n]$ two filters such that $g_0[n] = \psi_0[n]$ and $g_1[n] = \psi_1[n]$.

a) Show that if $\langle \psi_0[n], \varphi_0[n-2k] \rangle = \delta[k]$ then $H_0(z)G_0(z) + H_0(-z)G_0(-z) = 2$ and that if $\langle \psi_1[n], \varphi_1[n-2k] \rangle = \delta[k]$ then $H_1(z)G_1(z) + H_1(-z)G_1(-z) = 2$.

b) Show that if $\langle \psi_0[n], \varphi_1[n-2k] \rangle = 0$ for all $k \in \mathbb{Z}$ then $H_1(z)G_0(z) + H_1(-z)G_0(-z) = 0$ and that if $\langle \psi_1[n], \varphi_0[n-2k] \rangle = 0$ for all $k \in \mathbb{Z}$ then $H_0(z)G_1(z) + H_0(-z)G_1(-z) = 0$.

c) Using the results of a) and b) show that if $\langle \psi_0[n], \varphi_0[n-2k] \rangle = \delta[k]$, $\langle \psi_1[n], \varphi_1[n-2k] \rangle = \delta[k]$, $\langle \psi_0[n], \varphi_1[n-2k] \rangle = 0$ for all $k \in \mathbb{Z}$, and $\langle \psi_1[n], \varphi_0[n-2k] \rangle = 0$ for all $k \in \mathbb{Z}$, then

$$H_0(z)G_0(z) + H_1(z)G_1(z) = 2 \text{ and } H_0(-z)G_0(z) + H_1(-z)G_1(z) = 0 \ .$$

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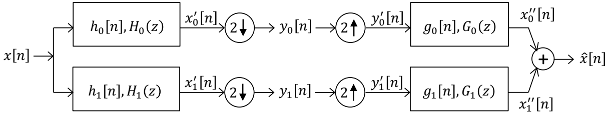
Task II

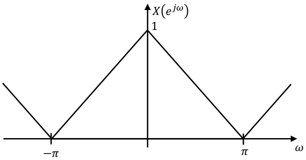
Problem 1. Consider a two-channel filter bank tree as shown in the figure



where

$$H_0(e^{j\omega}) = \begin{cases} \sqrt{2}, & |\omega| < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} \le |\omega| < \pi \end{cases} \quad , \quad H_1(e^{j\omega}) = \begin{cases} 0, & |\omega| < \frac{\pi}{2} \\ -e^{-j\omega}\sqrt{2}, & \frac{\pi}{2} \le |\omega| < \pi \end{cases} \quad ,$$

and $G_0(z) = H_0(z^{-1})$ and $G_1(z) = H_1(z^{-1})$. If the spectrum of $x[n]$ is as show in the figure below, sketch $|X_0'(e^{j\omega})|$, $|X_1'(e^{j\omega})|$, $|Y_0(e^{j\omega})|$, $|Y_1(e^{j\omega})|$, $|Y_0'(e^{j\omega})|$, $|Y_1'(e^{j\omega})|$, $|X_0''(e^{j\omega})|$, $|X_1''(e^{j\omega})|$.



Problem 2. Using the lattice factorisation of paraunitary matrices

$$\mathbf{G}_p(z) = \begin{bmatrix} G_{00}(z) & G_{10}(z) \\ G_{01}(z) & G_{11}(z) \end{bmatrix} = U_0 \left(\prod_{i=1}^{k-1} \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix} U_i \right)$$

where U_i, $i = 0, \ldots, k-1$ are unitary 2×2 matrices

$$U_i = \begin{bmatrix} \cos \alpha_i & -\sin \alpha_i \\ \sin \alpha_i & \cos \alpha_i \end{bmatrix},$$

write down expressions for $G_0(z)$ and $G_1(z)$ in terms of parameters α_i for $k = 2$. How should α_0 and α_1 be selected so that $G_0(e^{j\pi}) = 0$?

Problem 3. Show that in a filter bank with linear phase filters, the iterated filters are also linear phase. In particular, consider the case where $h_0[n]$ and $h_1[n]$ are of even length, symmetric and antisymmetric respectively. Consider a four-channel bank, with $H_a(z) = H_0(z)H_0(z^2)$, $H_b(z) = H_0(z)H_1(z^2)$, $H_c(z) = H_1(z)H_0(z^2)$, and $H_d(z) = H_1(z)H_1(z^2)$. What are the lengths and symmetries of these four filters?

Problem 4. In Section 5.1.2 of the textbook, it has been shown how the continuous wavelet transform can characterise the local regularity of a function. Take the Haar wavelet for simplicity.

a) Consider the function

$$f(t) = \begin{cases} t, & 0 \leq t \\ 0, & t < 0 \end{cases}$$

and show, using arguments similar to the ones used in the text, that

$$CWT_f(a, b) \simeq a^{3/2},$$

around $b = 0$ and for small a.

b) Show that if

$$f(t) = \begin{cases} t^n, & 0 \leq t \\ 0, & t < 0 \end{cases}$$

then

$$CWT_f(a, b) \simeq a^{(2n+1)/2},$$

around $b = 0$ and for small a.