Abstract

Sammendrag

Preface

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In three dimensions, the matrix $\boldsymbol{L}$ is defined as

$$\boldsymbol{L} = \begin{pmatrix} \nabla^2 \boldsymbol{C}^{-1}[\xi_3, \xi_3, \xi_3, \xi_3] & 2\frac{\lambda_3 - \lambda_1}{\lambda_1 \lambda_3}\langle \xi_1, (\nabla \xi_3)\xi_3 \rangle & 2\frac{\lambda_3 - \lambda_2}{\lambda_2 \lambda_3}\langle \xi_2, (\nabla \xi_3)\xi_3 \rangle \\ 2\frac{\lambda_3 - \lambda_1}{\lambda_1 \lambda_3}\langle \xi_1, (\nabla \xi_3)\xi_3 \rangle & 2\frac{\lambda_3 - \lambda_1}{\lambda_1 \lambda_3} & 0 \\ 2\frac{\lambda_3 - \lambda_2}{\lambda_2 \lambda_3}\langle \xi_2, (\nabla \xi_3)\xi_3 \rangle & 0 & 2\frac{\lambda_3 - \lambda_2}{\lambda_2 \lambda_3} \end{pmatrix}, \tag{2.1}$$

where the dependence of $\boldsymbol{L}$ and its constituent parts on t, t_0 and $\mathbf{x}_0$ has been omitted for brevity. The distinction between so-called strong and weak hyperbolic LCSs is based upon whether or not $\boldsymbol{L}$ is positive definite (Haller 2010; Farazmand and Haller 2011). By Sylvester's theorem, a square matrix is positive definite if and only if all of the leading principal minors of $\boldsymbol{L}$ are positive. The term *leading principal minors* is defined as follows (Gilbert 1991):

Definition 1 (Principals of square matrices)**.** Let $\boldsymbol{A}$ be an $n \times n$ matrix. For $1 \leq k \leq n$, the k^{th} *principal submatrix* of $\boldsymbol{A}$ is the $k \times k$ submatrix formed from the first k rows and first k columns of $\boldsymbol{A}$. Its determinant is the k^{th} principal minor.

Thus, in three dimensions, the matrix $\boldsymbol{L}$ being positive definite amounts to the simultaneous fulfillment of the three requirements

$$\nabla^2 \boldsymbol{C}^{-1}[\xi_3, \xi_3, \xi_3, \xi_3] > 0, \tag{2.2a}$$

$$2\frac{\lambda_3 - \lambda_1}{\lambda_1 \lambda_3}\left\{\nabla^2 \boldsymbol{C}^{-1}[\xi_3, \xi_3, \xi_3, \xi_3] - 2\frac{\lambda_3 - \lambda_1}{\lambda_1 \lambda_3}\langle \xi_1, (\nabla \xi_3)\xi_3 \rangle^2\right\} > 0, \tag{2.2b}$$

$$\det(\boldsymbol{L}) > 0. \tag{2.2c}$$

By straightforward algebraic manipulations, the inequality (2.2c) is equivalent to

$$\begin{aligned} 4\frac{(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_1)}{\lambda_1 \lambda_2 \lambda_3^2}\Big\{\nabla^2 \boldsymbol{C}^{-1}[\xi_3, \xi_3, \xi_3, \xi_3] - \Big(2\frac{\lambda_3 - \lambda_1}{\lambda_1 \lambda_3}\langle \xi_1, (\nabla \xi_3)\xi_3 \rangle^2 \\ +2\frac{\lambda_3 - \lambda_2}{\lambda_2 \lambda_3}\langle \xi_2, (\nabla \xi_3)\xi_3 \rangle^2\Big)\Big\} > 0. \end{aligned} \tag{2.3}$$

Now, we make use of the following result from Haller (2010):

Lemma 1. *At each point of a weak LCS in three dimensions, the following identity holds:*

$$\begin{aligned} \nabla^2 \boldsymbol{C}^{-1}[\xi_3, \xi_3, \xi_3, \xi_3] = -\frac{1}{\lambda_3^2}\langle \xi_3, \nabla^2 \lambda_3 \xi_3 \rangle + 2\frac{\lambda_3 - \lambda_1}{\lambda_1 \lambda_3}\langle \xi_1, (\nabla \xi_3)\xi_3 \rangle^2 \\ + 2\frac{\lambda_3 - \lambda_2}{\lambda_2 \lambda_3}\langle \xi_2, (\nabla \xi_3)\xi_3 \rangle^2. \end{aligned} \tag{2.4}$$

Proof. See theorem 7 in Haller (2010). □

Using equation (2.4) in conjunction with equation (2.3) and the relationships $0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3$, the inequalities (2.2) can be expressed as follows: Foo

3 Method

4 Results

5 Discussion

6 Conclusions

References

Farazmand, M. and Haller, G. (2011). “Erratum and addendum to ‘A variational theory of hyperbolic Lagrangian coherent structures’ [Physica D 240 (2011) 547–598]”. In: *Physica D: Nonlinear Phenomena* 241.4, pp. 439–441. ISSN: 0167-2789.

Gilbert, G. T. (1991). “Positive Definite Matrices and Sylvester’s Criterion”. In: *The American Mathematical Monthly* 98.1, pp. 44–46.

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A Appendix A