Wave project, INF5620

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1 Mathematical problem

The project addresses the two-dimensional, standard, linear wave equation, with damping,

$$\frac{\partial^2 u}{\partial t^2} + b\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(q(x,y)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(q(x,y)\frac{\partial u}{\partial y}\right) + f(x,y,t)$$

The associated boundary condition is

$$\frac{\partial u}{\partial n} = 0$$

in a rectangular spatial domain $\Omega = [0, L_x] \times [0, L_y]$. The initial conditions are

$$u(x,y,0) = I(x,y), \quad u_t(x,y,0) = V(x,y).$$

2 Discretization

We discretize the time derivatives as

$$\frac{\partial^2 u}{\partial t^2} \approx [D_t D_t u]^n_{i,j} = \frac{u^{n+1}_{i,j} - 2u^n_{i,j} + u^{n-1}_{i,j}}{\Delta t^2},$$

$$\frac{\partial u}{\partial t} \approx [D_{2t} u]^n_{i,j} = \frac{u^{n+1}_{i,j} - u^{n-1}_{i,j}}{2\Delta t}.$$

The spatial derivatives are discretized as

$$\frac{\partial u}{\partial x}\left(q\frac{\partial u}{\partial x}\right) \approx [D_x q D_x u]^n_{i,j} = \frac{1}{\Delta x^2}\left(q_{i+\frac{1}{2},j}(u^n_{i+1,j} - u^n_{i,j}) - q_{i-\frac{1}{2},j}(u^n_{i,j} - u^n_{i-1,j})\right),$$

$$\frac{\partial u}{\partial y}\left(q\frac{\partial u}{\partial y}\right) \approx [D_y q D_y u]^n_{i,j} = \frac{1}{\Delta y^2}\left(q_{i,j+\frac{1}{2}}(u^n_{i,j+1} - u^n_{i,j}) - q_{i,j-\frac{1}{2}}(u^n_{i,j} - u^n_{i,j-1})\right).$$

Substituting for the corresponding terms in the differential equations we get thee following difference equation

$$
\begin{aligned}
&\frac{u_{i,j}^{n+1} - 2u_{i,j}^{n} + u_{i,j}^{n-1}}{\Delta t^2} + b\frac{u_{i,j}^{n+1} - u_{i,j}^{n-1}}{2\Delta t} \\
&\quad = \frac{1}{\Delta x^2}\left(q_{i+\frac{1}{2},j}(u_{i+1,j}^{n} - u_{i,j}^{n}) - q_{i-\frac{1}{2},j}(u_{i,j}^{n} - u_{i-1,j}^{n})\right) \\
&\quad + \frac{1}{\Delta y^2}\left(q_{i,j+\frac{1}{2}}(u_{i,j+1}^{n} - u_{i,j}^{n}) - q_{i,j-\frac{1}{2}}(u_{i,j}^{n} - u_{i,j-1}^{n})\right) + f_{i,j}^{n}.
\end{aligned}
$$

Multiplying through by Δt^2, collecting the $u_{i,j}^{n+1}$-terms on the left side, and dividing by the factor $1 + \frac{b}{2}\Delta t$ gives us

$$
\begin{aligned}
u_{i,j}^{n+1} = &\Bigg[2u_{i,j}^{n} - u_{i,j}^{n-1} + \frac{b}{2}u_{i,j}^{n-1}\Delta t + \frac{\Delta t^2}{\Delta x^2}\left(q_{i+\frac{1}{2},j}(u_{i+1,j}^{n} - u_{i,j}^{n}) - q_{i-\frac{1}{2},j}(u_{i,j}^{n} - u_{i-1,j}^{n})\right) \\
&+ \frac{\Delta t^2}{\Delta y^2}\left(q_{i,j+\frac{1}{2}}(u_{i,j+1}^{n} - u_{i,j}^{n}) - q_{i,j-\frac{1}{2}}(u_{i,j}^{n} - u_{i,j-1}^{n})\right) + f_{i,j}^{n}\Delta t^2\Bigg](1 + \frac{b}{2}\Delta t)^{-1}.
\end{aligned}
$$

This is the scheme that we will use to compute $u_{i,j}^{n+1}$ at interior points.

For the first time step, and at the boundaries, special care must be taken, since the scheme at these points asks for values of u outside the domain. The trick is to use the boundary conditions to express these values of u in terms of values inside the domain.

For the first time step we need an expression for the value of $u_{i,j}^{-1}$. Discretizing the boundary condition $u_t(x, y, 0) = V(x, y)$, we get

$$
[D_{2t}u]_{i,j}^{0} = \frac{u_{i,j}^{1} - u_{i,j}^{-1}}{2\Delta t} = V_{i,j},
$$

which gives

$$
u_{i,j}^{-1} = u_{i,j}^{1} - 2V_{i,j}\Delta t.
$$

Inserting this into the general scheme above, we obtain the modified scheme

$$
\begin{aligned}
u_{i,j}^{1} = &\Bigg[2u_{i,j}^{0} - (u_{i,j}^{1} - 2V_{i,j}\Delta t) + \frac{b}{2}(u_{i,j}^{1} - 2V_{i,j}\Delta t)\Delta t \\
&+ \frac{\Delta t^2}{\Delta x^2}\left(q_{i+\frac{1}{2},j}(u_{i+1,j}^{0} - u_{i,j}^{0}) - q_{i-\frac{1}{2},j}(u_{i,j}^{0} - u_{i-1,j}^{0})\right) \\
&+ \frac{\Delta t^2}{\Delta y^2}\left(q_{i,j+\frac{1}{2}}(u_{i,j+1}^{0} - u_{i,j}^{0}) - q_{i,j-\frac{1}{2}}(u_{i,j}^{0} - u_{i,j-1}^{0})\right) + f_{i,j}^{n}\Delta t^2\Bigg](1 + \frac{b}{2}\Delta t)^{-1}.
\end{aligned}
$$

for the first time step.

The von Neumann boundary conditions $\frac{\partial u}{\partial n} = 0$ give us

$$
u_{i-1,j}^{n} = u_{i+1,j}^{n}, \qquad i = 0, \quad i = N_x,
$$

and

$$u^n_{i,j-1} = u^n_{i,j+1}, \qquad j = 0, \quad j = N_y.$$

Instead of inserting these relations into our general scheme, we will just substitute the indices at the boundaries accordingly in our implementation, thus achieving the same effect.

3 Implementation