



Pacific  
Community  
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du Pacifique

# Generalized Richards model for fish growth

*Overview of parametrizations and special cases*

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Oceanic Fisheries Program, Pacific Community (SPC)

Stock Synthesis Webinar

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# Overview

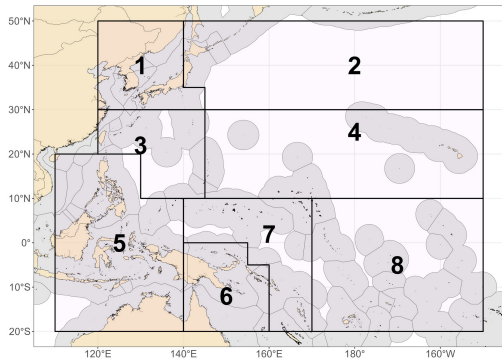
**Introduction**    *beyond von Bert, Stock Synthesis, vignette*

**Curves**    *von Bertalanffy, Gompertz, Schnute Case 3, (linear, exponential, logistic)*

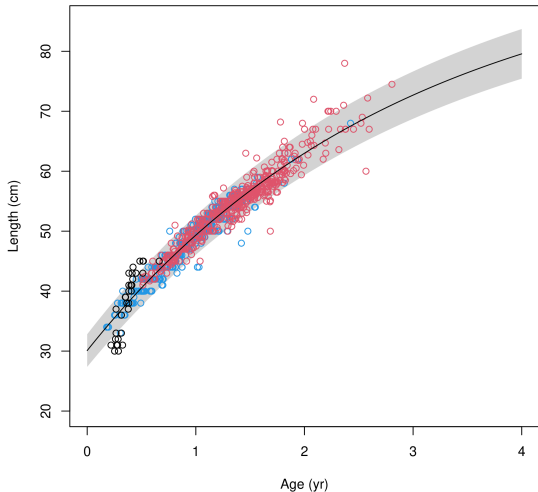
**Equations**    *traditional parametrizations, Schnute & Fournier generalizations,  
special cases, mathematical properties*

**Play**    *'FSA' package, 'fishgrowth' package, information from tag recaptures,  
the curious case of Schnute 3*

# Skipjack tuna in the western and central Pacific



# Skipjack tuna in the western and central Pacific



von Bertalanffy

Fitted to otoliths (black)  
and tags (blue, red)

Used in the 2022 stock assessment

Can we explore curves with  
a slightly different shape?

# Looking for Richards

Where do I find the equation to use?

$$L = f(t, \theta)$$

## Looking for Richards

- ▶ Richards (1959)

## Looking for Richards

- ▶ Richards (1959)
- ▶ The 'FSA' package on CRAN (Derek Ogle)

```
RichardsFuns(1)
function (t, Linf, k = NULL, a = NULL, b = NULL)
{
  if (length(Linf) == 4) {
    k <- Linf[[2]]
    a <- Linf[[3]]
    b <- Linf[[4]]
    Linf <- Linf[[1]]
  }
  Linf * (1 - a * exp(-k * t))^b
}
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<environment: 0x558f2d8fd048>
> RichardsFuns(2)
function (t, Linf, k = NULL, ti = NULL, b = NULL)
{
  if (length(Linf) == 4) {
    k <- Linf[[2]]
    ti <- Linf[[3]]
    b <- Linf[[4]]
    Linf <- Linf[[1]]
  }
  Linf * (1 - (1/b) * exp(-k * (t - ti)))^b
}
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<environment: 0x558f2d9176e8>
> RichardsFuns(3)
function (t, Linf, k = NULL, ti = NULL, b = NULL)
{
  if (length(Linf) == 4) {
    k <- Linf[[2]]
    ti <- Linf[[3]]
    b <- Linf[[4]]
    Linf <- Linf[[1]]
  }
  Linf/((1 + b * exp(-k * (t - ti)))^(1/b))
}
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<environment: 0x558f2d920398>
> RichardsFuns(4)
function (t, Linf, k = NULL, ti = NULL, b = NULL)
{
  if (length(Linf) == 4) {
    k <- Linf[[2]]
    ti <- Linf[[3]]
    b <- Linf[[4]]
  }
}
```

U:\*\*\* \*R\* Top (1,0) (iESS [R]: run ElDoc)

Mark set



figs — Dolphin



richards\_webinartex



\*R\*



richards\_webinartex



- Richards

- Within FSA, Linf is the mean asymptotic length, ti is the age at the inflection point, k controls the slope at the inflection point (maximum relative growth rate), b is dimensionless but related to the vertical position (i.e., size) of the inflection point, a is dimensionless but related to the horizontal position (i.e., age) of the inflection point, and L0 is the mean length at age-0.
- The parameterizations (1-6) correspond to functions/equations 1, 4, 5, 6, 7, and 8, respectively, in Tjorve and Tjorve (2010). Note that their A, S, k, d, and B are Linf, a, k, b, and L0, respectively, here (in FSA). Their (Tjorve and Tjorve 2010) K does not appear here.

- logistic

- Within FSA, L0 is the mean length at age 0, Linf is the mean asymptotic length, ti is the age at the inflection point, and gninf is the instantaneous growth rate at negative infinity.

### Author(s)

Derek H. Ogle, [DerekOgle51@gmail.com](mailto:DerekOgle51@gmail.com), thanks to Gabor Grothendieck for a hint about using `get()`.

### References

Ogle, D.H. 2016. [Introductory Fisheries Analyses with R](#). Chapman & Hall/CRC, Boca Raton, FL.

Campana, S.E. and C.M. Jones. 1992. Analysis of otolith microstructure data. Pages 73-100 In D.K. Stevenson and S.E. Campana, editors. Otolith microstructure examination and analysis. Canadian Special Publication of Fisheries and Aquatic Sciences 117. [Was (is?) from <https://waves-vagues.dfo-mpo.gc.ca/library-bibliotheque/141734.pdf>]

Fabens, A. 1965. Properties and fitting of the von Bertalanffy growth curve. *Growth* 29:265-289.

Francis, R.I.C.C. 1988. Are growth parameters estimated from tagging and age-length data comparable? *Canadian Journal of Fisheries and Aquatic Sciences*, 45:936-942.

Filter files...

- NEWS.md
- R
  - growthModels.R

2 files changed +354 -353 lines changed Search within code

```
NEWS.md
... @@ -1,4 +1,5 @@
1 1 # FSA 0.9.5.9000
2 + * 'GompertzFuns()': Accepted pull request related to [#112](https://github.com/fishR-Core-Team/FSA/issues/112) that fixed several typos and dead
links in the documentation ... thanks Arni. Corrected the erroneous reference to t* (should have been t0) in the documentation for the Gompertz
function (fixes [#113](https://github.com/fishR-Core-Team/FSA/issues/113) ... thanks again to Arni).
2 3
3 4 # FSA 0.9.5
4 5 * Fixed FSA-package \alias problem using the "automatic approach" (i.e., adding a "_PACKAGE" line to FSA.R) suggested in an e-mail from Kurt
Hornik on 19-Aug-2023.
```

```
R/growthModels.R
@@ -26,7 +26,7 @@
26 26 #' \itemize{
27 27 #' \item The \sQuote{Ricker2} and \sQuote{QuinnDeriso1} are synonymous, as are \sQuote{Ricker3} and \sQuote{QuinnDeriso2}.
28 28 #' \item The parameterizations and parameters for the Gompertz function are varied and confusing in the literature. I have attempted to
use a uniform set of parameters in these functions, but that makes a direct comparison to the literature difficult. Common sources for
Gompertz models are listed in the references below. I make some comments here to aid comparisons to the literature.
29 - #' \item Within FSA, L0 is the mean length at age 0, Linf is the mean asymptotic length, ti is the age at the inflection point, gi is the
instantaneous growth rate at the inflection point, t* is a dimensionless parameter related to time/age, and a is a dimensionless parameter
related to growth.
29 + #' \item Within FSA, L0 is the mean length at age 0, Linf is the mean asymptotic length, ti is the age at the inflection point, gi is the
instantaneous growth rate at the inflection point, t0 is a dimensionless parameter related to time/age, and a is a dimensionless parameter
related to growth.
```

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- ▶ Richards (1959)
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- ▶ Richards (1959)
- ▶ The 'FSA' package on CRAN (Derek Ogle)
- ▶ Tjørve and Tjørve (2010)
- ▶ Ricker (1979), Schnute (1981), Schnute and Fournier (1980)
- ▶ Stock Synthesis documentation

## Schnute/Richards growth function

The Richards (1959) growth model as parameterized by Schnute (1981) provides a flexible growth parameterization that allows for not only asymptotic growth but also linear, quadratic or exponential growth. The Schnute/Richards growth is invoked by entering option 2 in the growth type field. The Schnute/Richards growth function uses the standard growth parameters (e.g.,  $L_{min}$ ,  $L_{inf}$ , and  $k$ ) and a fourth parameter that is read after reading the von Bertalanffy growth coefficient parameter ( $k$ ). When this fourth parameter has a value of 1.0, it is equivalent to the standard von Bertalanffy growth curve. When this function was first introduced, it was required that  $A_0$  parameter be set to 0.0.

The Schnute/Richards growth model is parameterized as:

$$L_t = L_{MIN}^b + (L_{\infty}^b - L_{MIN}^b) \frac{1 - e^{-k(t-A_1)}}{1 - e^{-k(A_2-A_1)}}^{1/b} \quad (6)$$

with parameters  $L_{MIN}$ ,  $L_{MAX}$ ,  $k$ , and  $b$ .

The Richards model has  $b < 0$ , the von Bertalanffy model has  $b = 1$ . The general case of  $b > 0$  was called the “generalized von Bertalanffy” by Schnute (1981). The Gompertz model has  $b = 0$ , where the equation is undefined as written above and must be replaced with:

100

$$L_t = y_1 e^{\left[ \ln(y_2/y - 1) \frac{1 - e^{-k(t-A_1)}}{1 - e^{-k(A_2-A_1)}} \right]} \quad (7)$$

## Richards growth function

The Richards (1959) growth model as parameterized by Schnute (1981) provides a flexible growth parameterization that allows for a variety of growth curve shapes. The Richards growth is invoked by entering option 2 in the growth type field. The Richards growth function uses the standard growth parameters ( $L_1$ ,  $L_2$ ,  $k$ ) and a fourth shape parameter  $b$  that is specified after the growth coefficient  $k$ .

The Richards growth model is parameterized as:

$$L_t = \left[ L_1^b + (L_2^b - L_1^b) \frac{1 - e^{-k(t-A_1)}}{1 - e^{-k(A_2-A_1)}} \right]^{1/b} \quad (6)$$

with parameters  $L_1$ ,  $L_2$ ,  $k$ , and  $b$ .

The  $b$  shape parameter can be positive or negative but not precisely 0. When estimating  $b$  as a floating-point number, there is effectively no risk of the parameter becoming precisely zero during estimation, as long as the initial value is non-zero.

As special cases of the Richards growth model,  $b = 1$  is von Bertalanffy growth and  $b$  near 0 is Gompertz growth. To use a Gompertz growth curve, the  $b$  parameter can be fixed at a small value such as 0.0001.

100

When  $A_1$  is greater than the youngest age in the model, some combinations of Richards growth parameters can lead to undefined (NaN) predicted length for the younger ages. The choice of  $A_1$  and  $A_2$  will affect the possible growth curve shapes.

## New SS3 Richards vignette

[https://nmfs-ost.github.io/ss3-website/qmds/richards\\_growth\\_curve.html](https://nmfs-ost.github.io/ss3-website/qmds/richards_growth_curve.html)

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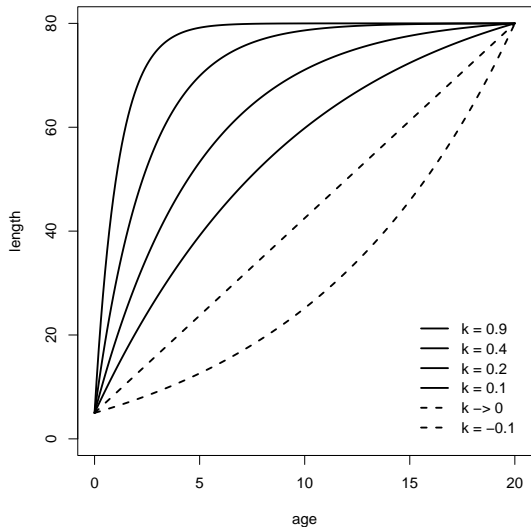
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# Von Bertalanffy



$$L = L_{\infty} \left( 1 - e^{-k(t-t_0)} \right)$$

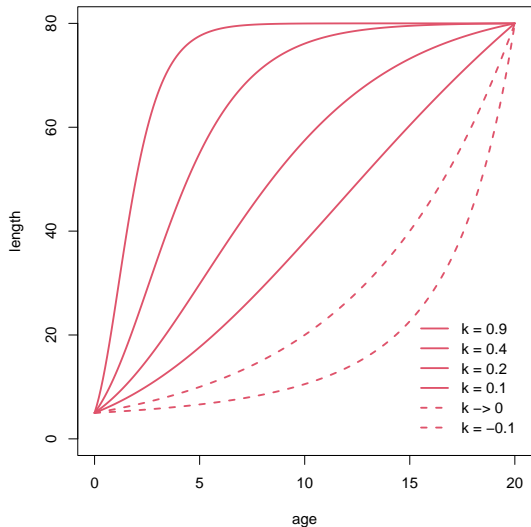
$$L = L_1 + (L_2 - L_1) \frac{1 - e^{-k(t-t_1)}}{1 - e^{-k(t_2-t_1)}}$$

$k \rightarrow 0$  is linear

$k < 0$  is exponential



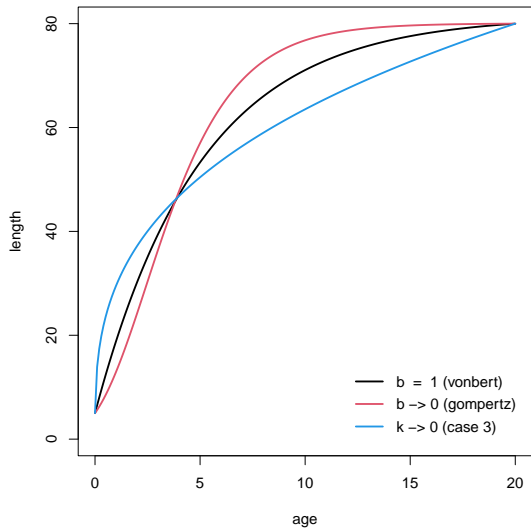
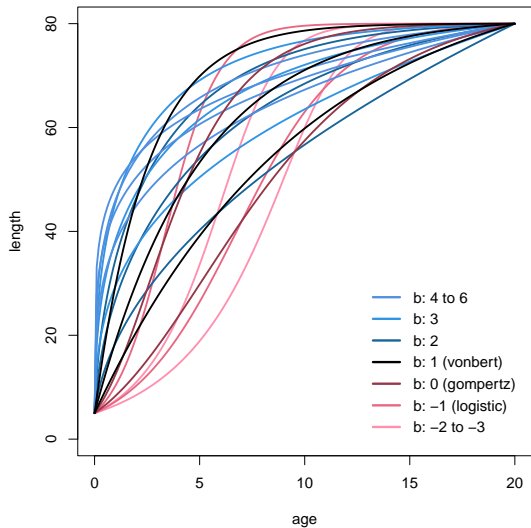
# Gompertz



$$L = L_{\infty} \exp(-e^{-k(t-\tau)})$$

$$L = L_1 \exp \left[ \log(L_2/L_1) \frac{1 - e^{-k(t-t_1)}}{1 - e^{-k(t_2-t_1)}} \right]$$

# Richards



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## Von Bertalanffy = Pütter Growth Curve 1

August Pütter developed and used the 'von Bertalanffy' curve, fourteen years before von Bertalanffy

Pütter (1920, Eq. 3) predicts length  $L$  at a given age  $t$  as

$$L = L_{\infty} \left(1 - ae^{-kt}\right)$$

Von Bertalanffy (1934, Eq. 2 and 1938, Eq. 6) reparametrized as

$$L = L_{\infty} - (L_{\infty} - L_0) e^{-kt}$$

## Von Bertalanffy reparametrizations

Brody (1945, pp. 527) presented the simplest form of the von Bertalanffy curve:

$$L = L_{\infty} - ae^{-kt}$$

Beverton and Holt (1957, pp. 33–34) and Ricker (1958, Eq. 9.5) reparametrized as

$$L = L_{\infty} \left(1 - e^{-k(t-t_0)}\right)$$

## Von Bertalanffy derivative

From the traditional von Bertalanffy function,

$$L = L_{\infty} \left( 1 - e^{-k(t-t_0)} \right)$$

we can calculate the derivative:

$$\begin{aligned} L'(t) &= kL_{\infty} e^{-k(t-t_0)} \\ &= k(L_{\infty} - L_t) \end{aligned}$$

This linear relationship is the basis of all parametrizations of the von Bertalanffy curve, as highlighted by von Bertalanffy (1934, Eq. 1) and later authors:

$$L'(t) = \alpha - \beta L_t$$

## Von Bertalanffy initial slope

From the derivative,

$$L'(t) = kL_{\infty} e^{-k(t-t_0)}$$

we can calculate the initial slope:

$$L'(0) = kL_{\infty} e^{kt_0}$$

The initial slope is  $kL_{\infty}$  at age  $t_0$ .

## Von Bertalanffy reparametrizations

Schnute and Fournier (1980, Eqs 7, 8, and 10) introduced a reparametrization that reduces the parameter correlation, estimating  $L_1$ ,  $L_2$ , and  $k$ :

$$L = L_1 + (L_2 - L_1) \frac{1 - e^{-k(t-t_1)}}{1 - e^{-k(t_2-t_1)}}$$

Transforming back to traditional von Bertalanffy parameters:

$$L_{\infty} = L_1 + \frac{L_2 - L_1}{1 - e^{-k(t_2-t_1)}}$$
$$t_0 = t_1 + \frac{1}{k} \log \left( \frac{L_2 - L_1}{L_2 - L_1 e^{-k(t_2-t_1)}} \right)$$



## Von Bertalanffy reparametrizations

Francis (1988, Eq. 4) introduced a reparametrization where the estimated parameters are  $L_1$ ,  $L_{\text{mid}}$ , and  $L_2$ :

$$L_t = L_1 + (L_2 - L_1) \frac{1 - r^{2(t-t_1)/(t_2-t_1)}}{1 - r^2}$$

where:

$$r = \frac{L_2 - L_{\text{mid}}}{L_{\text{mid}} - L_1}$$

## Von Bertalanffy reparametrizations

The Stock Synthesis user manual (Methot et al. 2024, Eqs 4 and 5) parametrizes the von Bertalanffy growth curve as

$$L_t = L_\infty + (L_1 - L_\infty) e^{-k(t-t_1)}$$

The user can either specify an age  $t_2$  to use a growth curve with  $L_2$  or use a dummy value 999 to use  $L_\infty$  instead. If  $t_2$  is specified with a lower value than 999, then  $L_\infty$  is calculated as:

$$L_\infty = L_1 + \frac{L_2 - L_1}{1 - e^{-k(t_2-t_1)}}$$

## Richards original

The original Richards (1959, p. 292) growth model was presented as:

$$L^{1-m} = L_{\infty}^{1-m} \times (1 - be^{-kt}) \quad m < 1$$

$$L^{1-m} = L_{\infty}^{1-m} \times (1 + be^{-kt}) \quad m > 1$$

Fletcher (1975, Eq. 3.8) and Ricker (1979, Eq. 49) combined both cases into one equation:

$$L^{1-n} = L_{\infty}^{1-n} + k \left[ \exp \left( -tmn^{n/(n-1)} / L_{\infty} \right) \right]$$

## Richards as parametrized by Schnute (1981)

**Case 1:**  $k \neq 0$ ,  $b \neq 0$  (Richards)

$$L = \left[ L_1^b + (L_2^b - L_1^b) \frac{1 - e^{-k(t-t_1)}}{1 - e^{-k(t_2-t_1)}} \right]^{1/b}$$

Four estimated parameters:  $L_1$ ,  $L_2$ ,  $k$ ,  $b$

If  $b$  is estimated close to 1, you can fix  $b = 1$  and you have von Bertalanffy

If  $b$  is estimated close to 0, you can use Case 2 for Gompertz (or fix  $b = 0.0001$  in SS3)

If  $k$  is estimated close to 0 you can use Case 3 growth curve

## Richards as parametrized by Schnute (1981)

Three-parameter special cases of the Richards curve

**Case 2:**  $k \neq 0, \quad b = 0$  (Gompertz)

$$L = L_1 \exp \left[ \log(L_2/L_1) \frac{1 - e^{-k(t-t_1)}}{1 - e^{-k(t_2-t_1)}} \right]$$

**Case 3:**  $k = 0, \quad b \neq 0$

$$L = \left[ L_1^b + (L_2^b - L_1^b) \frac{t - t_1}{t_2 - t_1} \right]^{1/b}$$

## Richards as parametrized by Schnute (1981)

Backcalculate  $L_\infty$  and  $t_0$ :

$$L_\infty = \left[ \frac{L_2^b e^{kt_2} - L_1^b e^{kt_1}}{e^{kt_2} - e^{kt_1}} \right]^{1/b}$$

$$t_0 = t_1 + t_2 - \frac{1}{k} \log \left[ \frac{L_2^b e^{kt_2} - L_1^b e^{kt_1}}{L_2^b - L_1^b} \right]$$

## General derivatives

von Bertalanffy

$$L'(t) = \alpha - \beta L_t$$

Gompertz

$$L'(t) = \alpha L_t - \beta L_t (\log L_t)$$

Richards

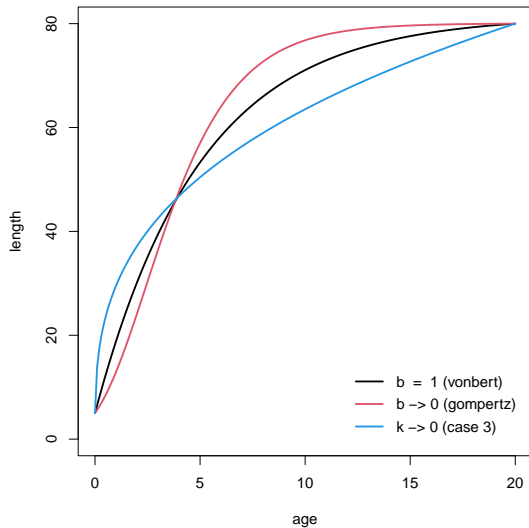
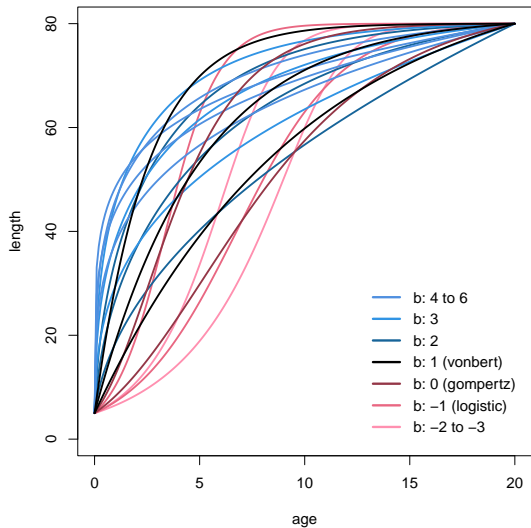
$$L'(t) = \alpha L_t + \beta L_t^n$$

# Derivative of Schnute (1981) Case 1

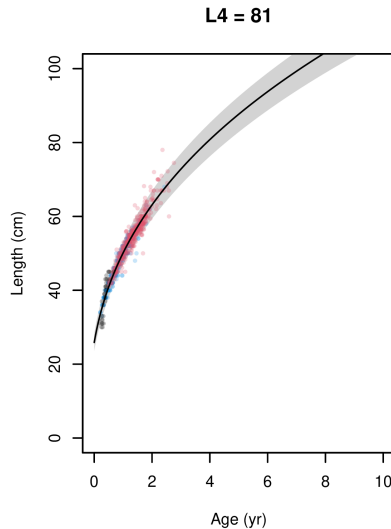
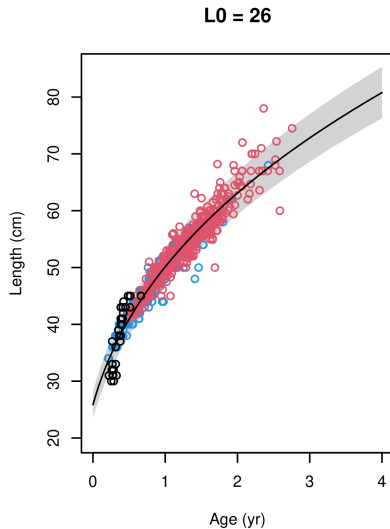
$$L'(t) = k(L_2^b - L_1^b) \frac{e^{-k(t-t_1)}}{b(1 - e^{-k(t_2-t_1)})} \left[ \frac{(L_2^b - L_1^b)(1 - e^{-k(t-t_1)})}{1 - e^{-k(t_2-t_1)}} + L_1^b \right]^{1/b-1}$$



# Richards curves



# Skipjack tuna: Schnute Case 3 model fit



# Summary

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