Generalized Richards model for fish growth

Overview of parametrizations and special cases

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Stock Synthesis Webinar
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Overview

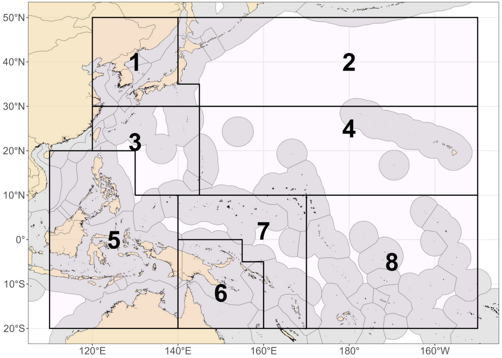
Introduction *beyond von Bert, Stock Synthesis, vignette*

Curves *von Bertalanffy, Gompertz, Schnute Case 3, (linear, exponential, logistic)*

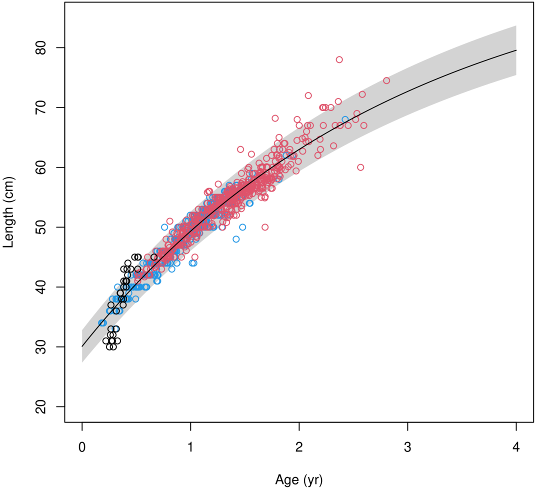
Equations *traditional parametrizations, Schnute & Fournier generalizations, special cases, mathematical properties*

Play *'FSA' package, 'fishgrowth' package, information from tag recaptures, the curious case of Schnute 3*

Skipjack tuna in the western and central Pacific



Skipjack tuna in the western and central Pacific



von Bertalanffy

Fitted to otoliths (black)
and tags (blue, red)

Used in the 2022 stock assessment

Can we explore curves with
a slightly different shape?

Looking for Richards

Where do I find the equation to use?

$$L = f(t, \theta)$$

Looking for Richards

- Richards (1959)

Looking for Richards

- Richards (1959)
- The 'FSA' package on CRAN (Derek Ogle)

R

```
> RichardsFuns(1)
function (t, Linf, k = NULL, a = NULL, b = NULL)
{
    if (length(Linf) == 4) {
        k <- Linf[[2]]
        a <- Linf[[3]]
        b <- Linf[[4]]
        Linf <- Linf[[1]]
    }
    Linf * (1 - a * exp(-k * t))^b
}
<bytecode: 0x558f2df30900>
<environment: 0x558f2d8fd048>
> RichardsFuns(2)
function (t, Linf, k = NULL, ti = NULL, b = NULL)
{
    if (length(Linf) == 4) {
        k <- Linf[[2]]
        ti <- Linf[[3]]
        b <- Linf[[4]]
        Linf <- Linf[[1]]
    }
    Linf * (1 - (1/b) * exp(-k * (t - ti)))^b
}
<bytecode: 0x558f2df2f390>
<environment: 0x558f2d9176e8>
> RichardsFuns(3)
function (t, Linf, k = NULL, ti = NULL, b = NULL)
{
    if (length(Linf) == 4) {
        k <- Linf[[2]]
        ti <- Linf[[3]]
        b <- Linf[[4]]
        Linf <- Linf[[1]]
    }
    Linf/((1 + b * exp(-k * (t - ti)))^(1/b))
}
<bytecode: 0x558f2df37238>
<environment: 0x558f2d920398>
> RichardsFuns(4)
function (t, Linf, k = NULL, ti = NULL, b = NULL)
{
    if (length(Linf) == 4) {
        k <- Linf[[2]]
        ti <- Linf[[3]]
        b <- Linf[[4]]
U:**-  *R*          Top   (1,0)      (iESS [R]: run ElDoc)
Mark set
```

figs — Dolphin | richards_webinar.tex | *R* | richards_webinar.p... | us | 14:18

- Richards
  - Within FSA, Linf is the mean asymptotic length, ti is the age at the inflection point, k controls the slope at the inflection point (maximum relative growth rate), b is dimensionless but related to the vertical position (i.e., size) of the inflection point, a is dimensionless but related to the horizontal position (i.e., age) of the inflection point, and L0 is the mean length at age-0.
  - The parameterizations (1-6) correspond to functions/equations 1, 4, 5, 6, 7, and 8, respectively, in Tjorve and Tjorve (2010). Note that their A, S, k, d, and B are Linf, a, k, b, and L0, respectively, here (in FSA). Their (Tjorve and Tjorve 2010) K does not appear here.
- logistic
  - Within FSA, L0 is the mean length at age 0, Linf is the mean asymptotic length, ti is the age at the inflection point, and gninf is the instantaneous growth rate at negative infinity.

Author(s)

Derek H. Ogle, DerekOgle51@gmail.com, thanks to Gabor Grothendieck for a hint about using `get()`.

References

Ogle, D.H. 2016. Introductory Fisheries Analyses with R. Chapman & Hall/CRC, Boca Raton, FL.

Campana, S.E. and C.M. Jones. 1992. Analysis of otolith microstructure data. Pages 73-100 In D.K. Stevenson and S.E. Campana, editors. Otolith microstructure examination and analysis. Canadian Special Publication of Fisheries and Aquatic Sciences 117. [Was (is?) from https://waves-vagues.dfo-mpo.gc.ca/library-bibliotheque/141734.pdf.]

Fabens, A. 1965. Properties and fitting of the von Bertalanffy growth curve. Growth 29:265-289.

Francis, R.I.C.C. 1988. Are growth parameters estimated from tagging and age-length data comparable? Canadian Journal of Fisheries and Aquatic Sciences, 45:936-942.

Filter files...

NEWS.md

R

growthModels.R

2 files changed +354 -353 lines changed

Search within code

NEWS.md +1

```
@@ -1,4 +1,5 @@
1   1   # FSA 0.9.5.9000
    2 + * `GompertzFuns()`: Accepted pull request related to [#112](https://github.com/fishR-Core-Team/FSA/issues/112) that fixed several typos and dead links in the documentation ... thanks Arni. Corrected the erroneous reference to t* (should have been t0) in the documentation for the Gompertz function (fixes [#113](https://github.com/fishR-Core-Team/FSA/issues/113) ... thanks again to Arni).
2   3
3   4   # FSA 0.9.5
4   5   * Fixed FSA-package \alias problem using the "automatic approach" (i.e., adding a "_PACKAGE" line to FSA.R) suggested in an e-mail from Kurt Hornik on 19-Aug-2023.
```

R/growthModels.R +353 -353

```
@@ -26,7 +26,7 @@
26  26   #'   \itemize{
27  27   #'     \item The \sQuote{Ricker2} and \sQuote{QuinnDeriso1} are synonymous, as are \sQuote{Ricker3} and \sQuote{QuinnDeriso2}.
28  28   #'     \item The parameterizations and parameters for the Gompertz function are varied and confusing in the literature. I have attempted to use a uniform set of parameters in these functions, but that makes a direct comparison to the literature difficult. Common sources for Gompertz models are listed in the references below. I make some comments here to aid comparisons to the literature.
29     - #'     \item Within FSA, L0 is the mean length at age 0, Linf is the mean asymptotic length, ti is the age at the inflection point, gi is the instantaneous growth rate at the inflection point, t* is a dimensionless parameter related to time/age, and a is a dimensionless parameter related to growth.
    29 + #'     \item Within FSA, L0 is the mean length at age 0, Linf is the mean asymptotic length, ti is the age at the inflection point, gi is the instantaneous growth rate at the inflection point, t0 is a dimensionless parameter related to time/age, and a is a dimensionless parameter related to growth.
```

Looking for Richards

- Richards (1959)
- The 'FSA' package on CRAN (Derek Ogle)
- Tjørve and Tjørve (2010)

Looking for Richards

- Richards (1959)
- The 'FSA' package on CRAN (Derek Ogle)
- Tjørve and Tjørve (2010)
- Ricker (1979), Schnute (1981), Schnute and Fournier (1980)
- Stock Synthesis documentation

Schnute/Richards growth function

The Richards (1959) growth model as parameterized by Schnute (1981) provides a flexible growth parameterization that allows for not only asymptotic growth but also linear, quadratic or exponential growth. The Schnute/Richards growth is invoked by entering option 2 in the growth type field. The Schnute/Richards growth function uses the standard growth parameters (e.g., Lmin, Linf, and k) and a fourth parameter that is read after reading the von Bertalanffy growth coefficient parameter (k). When this fourth parameter has a value of 1.0, it is equivalent to the standard von Bertalanffy growth curve. When this function was first introduced, it was required that A0 parameter be set to 0.0.

The Schnute/Richards growth model is parameterized as:

$$L_t = L_{MIN}^b + (L_\infty^b - L_{MIN}^b)\frac{1-e^{-k(t-A_1)}}{1-e^{-k(A_2-A_1)}}^{1/b} \tag{6}$$

with parameters L_{MIN}, L_{MAX}, k, and b.

The Richards model has $b < 0$, the von Bertalanffy model has $b = 1$. The general case of $b > 0$ was called the "generalized von Bertalanffy" by Schnute (1981). The Gompertz model has $b = 0$, where the equation is undefined as written above and must be replaced with:

$$L_t = y_1 e\left[ln(y_2/y-1)\frac{1-e^{-k(t-A_1)}}{1-e-k(A_2-A_1)}\right] \tag{7}$$

Richards growth function

The Richards (1959) growth model as parameterized by Schnute (1981) provides a flexible growth parameterization that allows for a variety of growth curve shapes. The Richards growth is invoked by entering option 2 in the growth type field. The Richards growth function uses the standard growth parameters (L_1, L_2, k) and a fourth shape parameter b that is specified after the growth coefficient k.

The Richards growth model is parameterized as:

$$L_t = \left[L_1^b + (L_2^b - L_1^b)\frac{1-e^{-k(t-A_1)}}{1-e^{-k(A_2-A_1)}}\right]^{1/b} \tag{6}$$

with parameters L_1, L_2, k, and b.

The b shape parameter can be positive or negative but not precisely 0. When estimating b as a floating-point number, there is effectively no risk of the parameter becoming precisely zero during estimation, as long as the initial value is non-zero.

As special cases of the Richards growth model, $b=1$ is von Bertalanffy growth and b near 0 is Gompertz growth. To use a Gompertz growth curve, the b parameter can be fixed at a small value such as 0.0001.

When A_1 is greater than the youngest age in the model, some combinations of Richards growth parameters can lead to undefined (NaN) predicted length for the younger ages. The choice of A_1 and A_2 will affect the possible growth curve shapes.

New SS3 Richards vignette

https://nmfs-ost.github.io/ss3-website/qmds/richards_growth_curve.html

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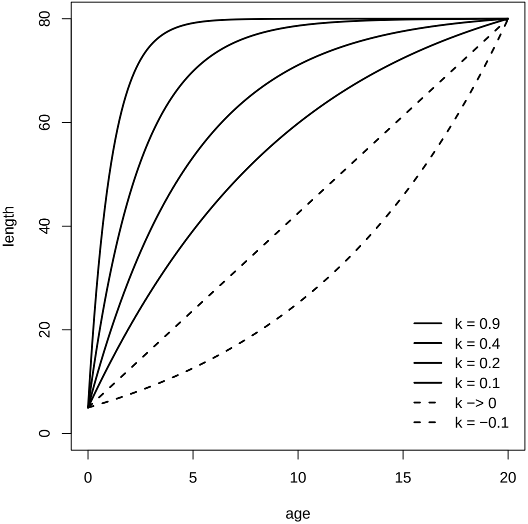
Von Bertalanffy



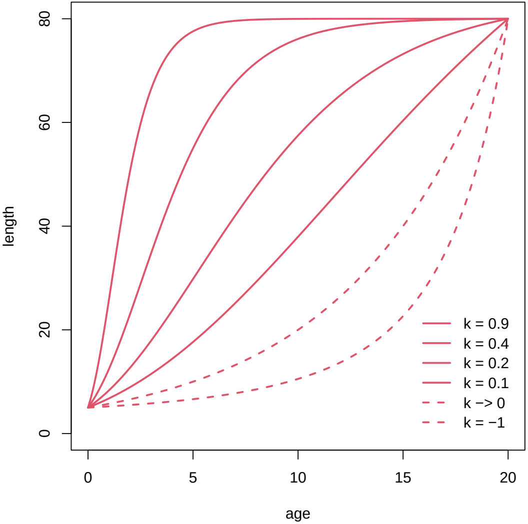
$$L = L_\infty \left(1 - e^{-k(t-t_0)}\right)$$

$$L = L_1 + (L_2 - L_1) \frac{1 - e^{-k(t-t_1)}}{1 - e^{-k(t_2-t_1)}}$$

$k \rightarrow 0$ is linear

$k < 0$ is exponential

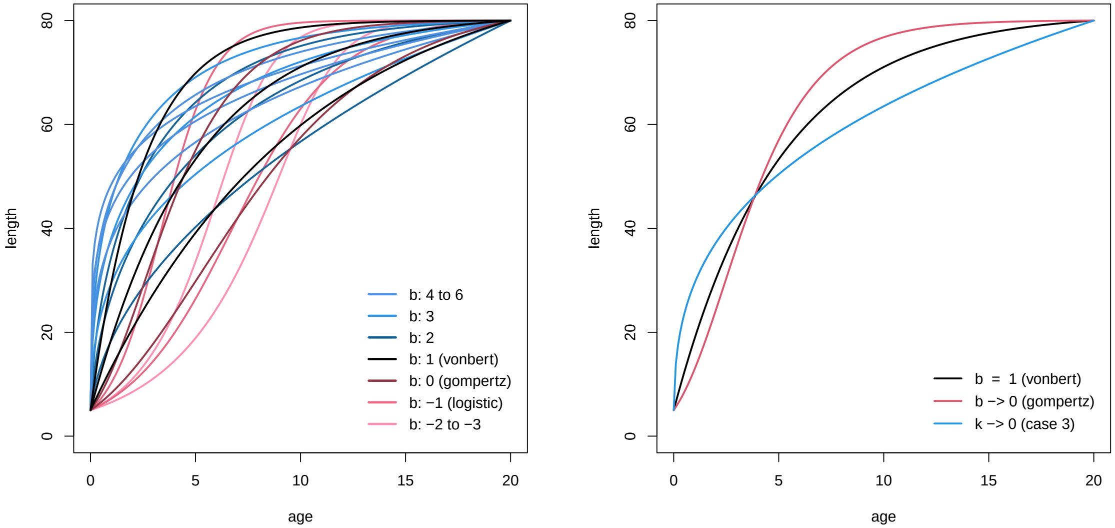
Gompertz



$$L = L_\infty \exp\left(-e^{-k(t-\tau)}\right)$$

$$L = L_1 \exp\left[\log(L_2/L_1)\,\frac{1-e^{-k(t-t_1)}}{1-e^{-k(t_2-t_1)}}\right]$$

Richards



length
age
b: 4 to 6
b: 3
b: 2
b: 1 (vonbert)
b: 0 (gompertz)
b: −1 (logistic)
b: −2 to −3
b = 1 (vonbert)
b −> 0 (gompertz)
k −> 0 (case 3)

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Von Bertalanffy = Pütter Growth Curve 1

August Pütter developed and used the 'von Bertalanffy' curve, fourteen years before von Bertalanffy

Pütter (1920, Eq. 3) predicts length L at a given age t as

$$L \;=\; L_\infty \left(1 \,-\, a e^{-kt}\right)$$

Von Bertalanffy (1934, Eq. 2 and 1938, Eq. 6) reparametrized as

$$L \;=\; L_\infty \,-\, (L_\infty - L_0)\, e^{-kt}$$

Von Bertalanffy reparametrizations

Brody (1945, pp. 527) presented the simplest form of the von Bertalanffy curve:

$$L = L_\infty - a e^{-kt}$$

Beverton and Holt (1957, pp. 33–34) and Ricker (1958, Eq. 9.5) reparametrized as

$$L = L_\infty \left(1 - e^{-k(t-t_0)}\right)$$

Von Bertalanffy derivative

From the traditional von Bertalanffy function,

$$L = L_\infty \left(1 - e^{-k(t-t_0)}\right)$$

we can calculate the derivative:

$$\begin{aligned} L'(t) &= kL_\infty \, e^{-k(t-t_0)} \\ &= k(L_\infty - L_t) \end{aligned}$$

This linear relationship is the basis of all parametrizations of the von Bertalanffy curve, as highlighted by von Bertalanffy (1934, Eq. 1) and later authors:

$$L'(t) = \alpha - \beta L_t$$

Von Bertalanffy initial slope

From the derivative,

$$L'(t) \quad = \quad kL_\infty \, e^{-k(t-t_0)}$$

we can calculate the initial slope:

$$L'(0) \quad = \quad kL_\infty \, e^{kt_0}$$

The initial slope is kL_∞ at age t_0.

Von Bertalanffy reparametrizations

Schnute and Fournier (1980, Eqs 7, 8, and 10) introduced a reparametrization that reduces the parameter correlation, estimating L_1, L_2, and k:

$$L \;=\; L_1 \;+\; (L_2 - L_1)\,\frac{1 - e^{-k(t-t_1)}}{1 - e^{-k(t_2-t_1)}}$$

Transforming back to traditional von Bertalanffy parameters:

$$L_\infty \;=\; L_1 \;+\; \frac{L_2 \;-\; L_1}{1 \;-\; e^{-k(t_2-t_1)}}$$

$$t_0 \;=\; t_1 \;+\; \frac{1}{k}\,\log\!\left(\frac{L_2 \;-\; L_1}{L_2 \;-\; L_1\,e^{-k(t_2-t_1)}}\right)$$

Von Bertalanffy reparametrizations

Francis (1988, Eq. 4) introduced a reparametrization where the estimated parameters are L_1, L_{mid}, and L_2:

$$L_t \;=\; L_1 + (L_2 - L_1)\,\frac{1 - r^{2(t-t_1)/(t_2-t_1)}}{1 - r^2}$$

where:

$$r \;=\; \frac{L_2 - L_{\mathrm{mid}}}{L_{\mathrm{mid}} - L_1}$$

Von Bertalanffy reparametrizations

The Stock Synthesis user manual (Methot et al. 2024, Eqs 4 and 5) parametrizes the von Bertalanffy growth curve as

$$L_t \;=\; L_\infty \,+\, (L_1 - L_\infty)\, e^{-k(t-t_1)}$$

The user can either specify an age t_2 to use a growth curve with L_2 or use a dummy value 999 to use L_∞ instead. If t_2 is specified with a lower value than 999, then L_∞ is calculated as:

$$L_\infty \;=\; L_1 \,+\, \frac{L_2 \,-\, L_1}{1 \,-\, e^{-k(t_2-t_1)}}$$

Richards original

The original Richards (1959, p. 292) growth model was presented as:

$$L^{1-m} = L_\infty^{1-m} \times (1 - be^{-kt}) \qquad m < 1$$

$$L^{1-m} = L_\infty^{1-m} \times (1 + be^{-kt}) \qquad m > 1$$

Fletcher (1975, Eq. 3.8) and Ricker (1979, Eq. 49) combined both cases into one equation:

$$L^{1-n} \;=\; L_\infty^{1-n} \;+\; k\left[\exp\left(-tmn^{n/(n-1)}/L_\infty\right)\right]$$

Richards as parametrized by Schnute (1981)

Case 1: $k \neq 0, \quad b \neq 0 \quad$ (Richards)

$$L \;=\; \left[L_1^b \;+\; (L_2^b - L_1^b)\,\frac{1 - e^{-k(t-t_1)}}{1 - e^{-k(t_2-t_1)}} \right]^{1/b}$$

Four estimated parameters: L_1, L_2, k, b

If b is estimated close to 1, you can fix $b = 1$ and you have von Bertalanffy

If b is estimated close to 0, you can use Case 2 for Gompertz (or fix $b = 0.0001$ in SS3)

If k is estimated close to 0 you can use Case 3 growth curve

Richards as parametrized by Schnute (1981)

Three-parameter special cases of the Richards curve

Case 2: $k \neq 0, \quad b = 0 \quad \text{(Gompertz)}$

$$L \;=\; L_1 \exp\left[\log(L_2/L_1) \, \frac{1 - e^{-k(t-t_1)}}{1 - e^{-k(t_2-t_1)}} \right]$$

Case 3: $k = 0, \quad b \neq 0$

$$L \;=\; \left[L_1^b \;+\; (L_2^b - L_1^b) \, \frac{t - t_1}{t_2 - t_1} \right]^{1/b}$$

Richards as parametrized by Schnute (1981)

Backcalculate L_∞ and t_0:

$$L_\infty \;=\; \left[\frac{L_2^b\, e^{kt_2} \;-\; L_1^b\, e^{kt_1}}{e^{kt_2} \;-\; e^{kt_1}}\right]^{1/b}$$

$$t_0 \;=\; t_1 \;+\; t_2 \;-\; \frac{1}{k}\,\log\left[\frac{L_2^b\, e^{kt_2} \;-\; L_1^b\, e^{kt_1}}{L_2^b \;-\; L_1^b}\right]$$

General derivatives

Pacific Community
Communauté du Pacifique

von Bertalanffy

$$L'(t) = \alpha - \beta L_t$$

Gompertz

$$L'(t) = \alpha L_t - \beta L_t(\log L_t)$$

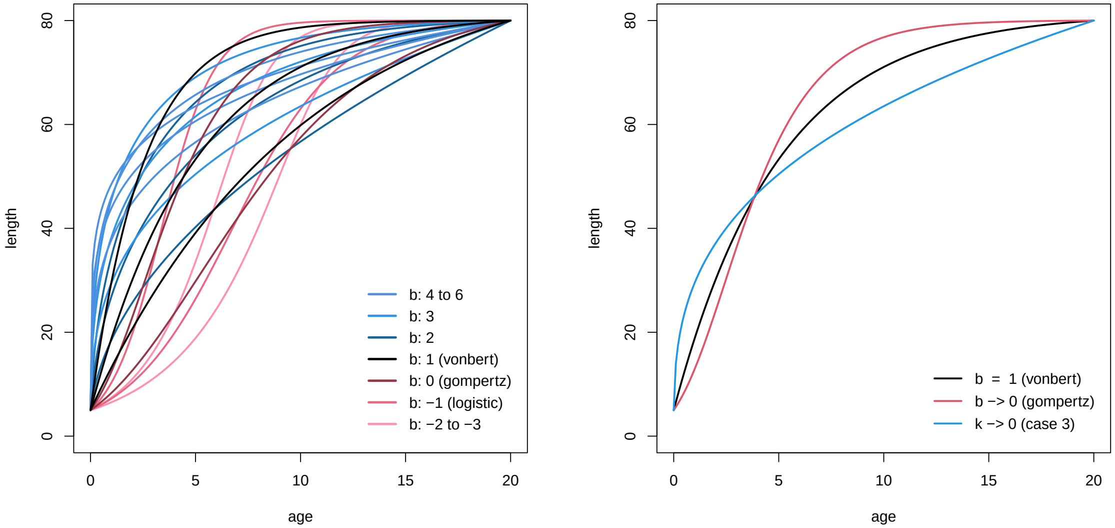
Richards

$$L'(t) = \alpha L_t + \beta L_t^n$$

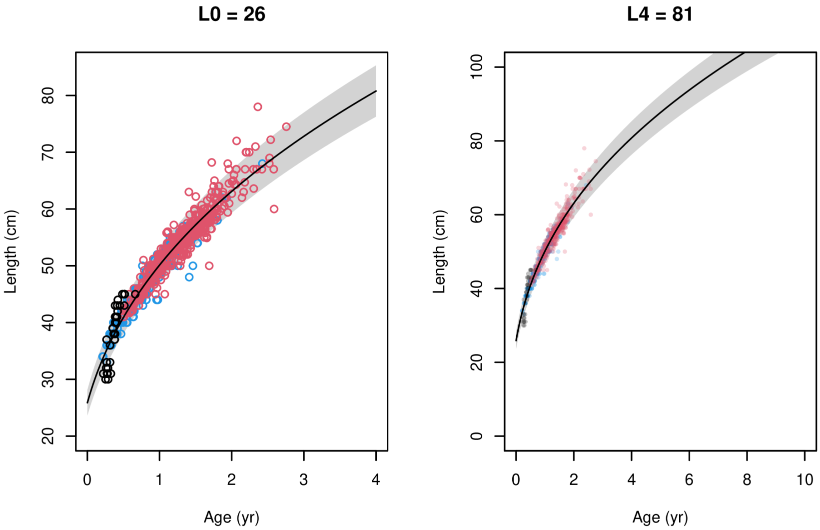
Derivative of Schnute (1981) Case 1

$$L'(t) \;=\; k(L_2^b - L_1^b)\,\frac{e^{-k(t-t_1)}}{b\left(1 - e^{-k(t_2-t_1)}\right)}\left[\frac{(L_2^b - L_1^b)\left(1 - e^{-k(t-t_1)}\right)}{1 - e^{-k(t_2-t_1)}} + L_1^b\right]^{1/b-1}$$

Richards curves



Skipjack tuna: Schnute Case 3 model fit



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