

Notes on Richards growth in Stock Synthesis

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Table of contents

1	Introduction	1
2	Methods	2
2.1	ss3_code	2
2.2	schnute	2
2.3	von_bertalanffy	2
2.4	gompertz	3
3	Results	3
3.1	Stock Synthesis model output	3
3.2	Special cases	6
3.3	Richards growth curves with different k and b combinations	7
4	Discussion	8
4.1	Range of permissible b values	8
4.2	Proof of <code>ss3_code()</code> being equivalent to <code>schnute()</code>	9
4.3	FSA package	10
5	References	10

1 Introduction

The equations used in the Stock Synthesis implementation of the Richards curve, as found in the ADMB source code, are different from the equations in the Stock Synthesis User Manual (Methot et al. 2024). This vignette (1) shows that the equations are equivalent and (2) demonstrates the various shapes of the Richards curve shapes using different values for the k and b parameters.

2 Methods

Load the `r4ss` package, so we can import Stock Synthesis model output.

```
library(r4ss)
```

2.1 `ss3_code`

The following R function replicates the Richards growth implementation from the Stock Synthesis ADMB source code.

```
ss3_code <- function(t, L1, L2, k, b, A1, A2)
{
  LminR <- L1^b
  LmaxR <- L2^b
  LinfR <- LminR + (LmaxR - LminR) / (1 - exp(-k*(A2-A1)))
  temp <- LinfR + (LminR - LinfR) * exp(-k*(t-A1))
  temp^(1/b)
}
```

2.2 `schnute`

The Schnute parametrization of the Richards curve is from Schnute (1981, Eq. 15, case 1).

```
schnute <- function(t, L1, L2, k, b, A1, A2)
{
  (L1^b + (L2^b-L1^b) * (1-exp(-k*(t-A1))) / (1-exp(-k*(A2-A1))))^(1/b)
}
```

2.3 `von_bertalanffy`

The reparametrized von Bertalanffy function is from Schnute and Fournier (1980, Eq. 7).

```
von_bertalanffy <- function(t, L1, L2, k, A1, A2)
{
  L1 + (L2-L1) * (1-exp(-k*(t-A1))) / (1-exp(-k*(A2-A1)))
}
```

2.4 gompertz

Finally, a standard Gompertz function is found in Schnute (1981, Eq. 2).

```
gompertz <- function(t, Linf, tau)
{
  Linf * exp(-exp(-k*(t-tau)))
}
```

3 Results

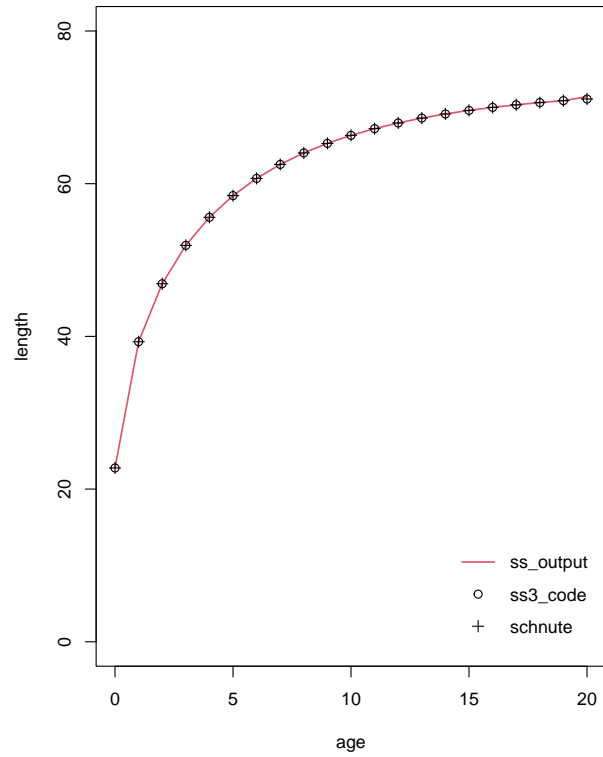
3.1 Stock Synthesis model output

Read estimated growth parameters from a fitted Stock Synthesis model.

```
model <- SS_output("simple_small_richards")
A1 <- subset(model$Growth_Parameters, Sex==1, A1, drop=TRUE)
A2 <- subset(model$Growth_Parameters, Sex==1, A2, drop=TRUE)
L1 <- subset(model$Growth_Parameters, Sex==1, L_a_A1, drop=TRUE)
L2 <- subset(model$Growth_Parameters, Sex==1, L_a_A2, drop=TRUE)
k <- subset(model$Growth_Parameters, Sex==1, K, drop=TRUE)
b <- model$parameters["Richards_Fem_GP_1", "Value"]
Linf <- subset(model$Growth_Parameters, Sex==1, Linf, drop=TRUE)
```

Compare three growth curves: length-at-age reported in the Stock Synthesis output, predicted lengths using the `ss3_code()` function, and predicted lengths using the `schnute()` function.

```
t <- 0:20
plot(Len_Beg~Age_Beg, model$endgrowth, subset=Sex==1, ylim=c(0,80),
     xlab="age", ylab="length", type="l", lwd=1.5, col=2)
points(t, ss3_code(t, L1, L2, k, b, A1, A2))
points(t, schnute(t, L1, L2, k, b, A1, A2), pch=3)
legend("bottomright", c("ss_output", "ss3_code", "schnute"),
     pch=c(NA, 1, 3), lty=c(1, NA, NA), lwd=c(1.5, NA, NA),
     col=c(2, 1, 1), bty="n", inset=0.02, y.intersp=1.5)
```



The predicted length from the `ss3_code()` and `schnute()` functions are in perfect agreement with each other. They are the same predicted lengths as reported in the Stock Synthesis output, except for the oldest age which is treated as a plus group inside the model, containing age 20 and older.

```

comparison <- data.frame(
  age = t,
  ss_output = subset(model$endgrowth, Sex==1, Len_Beg, drop=TRUE),
  ss3_code = ss3_code(t, L1, L2, k, b, A1, A2),
  schnute = schnute(t, L1, L2, k, b, A1, A2)
)
print(comparison, row.names=FALSE)

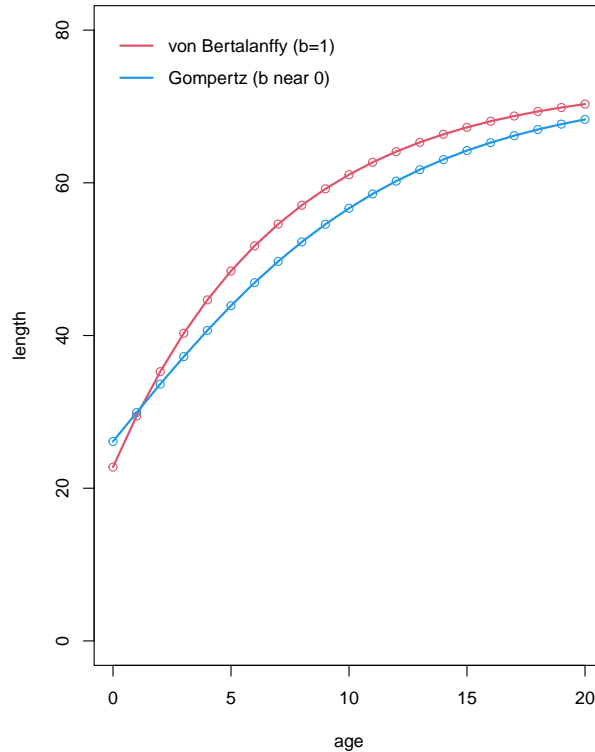
```

age	ss_output	ss3_code	schnute
0	22.7690	22.76900	22.76900
1	39.3016	39.30161	39.30161
2	46.8948	46.89477	46.89477
3	51.9159	51.91588	51.91588
4	55.5968	55.59675	55.59675
5	58.4357	58.43566	58.43566
6	60.6924	60.69238	60.69238
7	62.5224	62.52241	62.52241
8	64.0274	64.02738	64.02738
9	65.2779	65.27789	65.27789
10	66.3251	66.32514	66.32514
11	67.2075	67.20754	67.20754
12	67.9547	67.95467	67.95467
13	68.5897	68.58973	68.58973
14	69.1313	69.13126	69.13126
15	69.5942	69.59425	69.59425
16	69.9909	69.99095	69.99095
17	70.3315	70.33147	70.33147
18	70.6242	70.62422	70.62422
19	70.8762	70.87622	70.87622
20	71.3850	71.09338	71.09338

3.2 Special cases

Special cases of the Richards curve include the von Bertalanffy ($b=1$) and Gompertz ($b \rightarrow 1$) models. In the plot below, the lines are drawn using the actual von Bertalanffy and Gompertz functions, and the circles are drawn using the Richards curve with the appropriate values for the b parameter.

```
plot(NA, xlim=range(t), ylim=c(0,80), xlab="age", ylab="length")
lines(t, von_bertalanffy(t, L1, L2, k, A1, A2), ylim=c(0,80), lwd=2, col=2)
points(t, schnute(t, L1, L2, k, 1, A1, A2), col=2)
lines(t, gompertz(t, Linf, k), lwd=2, col=4)
Lg1 <- gompertz(A1, Linf, k)
Lg2 <- gompertz(A2, Linf, k)
points(t, schnute(t, Lg1, Lg2, k, 0.0001, A1, A2), col=4)
legend("topleft", c("von Bertalanffy (b=1)", "Gompertz (b near 0)"),
      lwd=2, col=c(2,4), bty="n", inset=0.02, y.intersp=1.5)
```



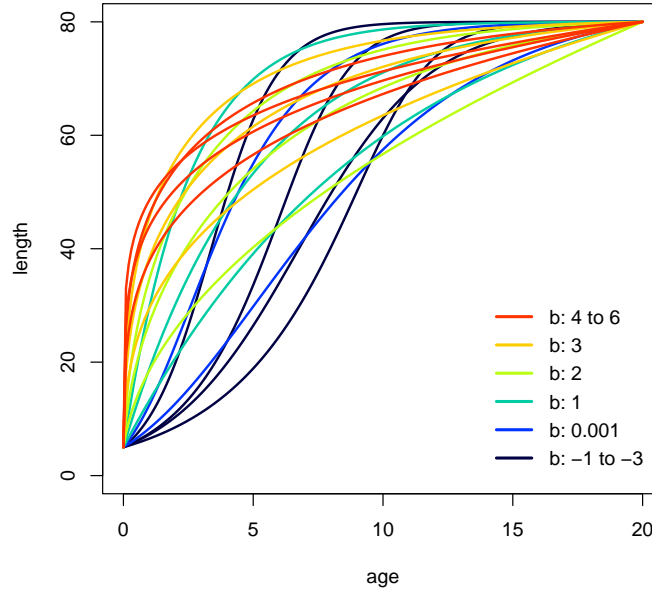
3.3 Richards growth curves with different k and b combinations

```
x <- seq(0, 20, by=0.1)
L1 <- 5
L2 <- 80
A1 <- 0
A2 <- 20

plot(NA, xlim=range(x), ylim=c(0, 80), xlab="age", ylab="length")
z <- rich.colors.short(6)

lines(x, schnute(x, L1, L2, k=0.8, b=-3, A1, A2), lwd=2, col=z[1])
lines(x, schnute(x, L1, L2, k=0.8, b=-2, A1, A2), lwd=2, col=z[1])
lines(x, schnute(x, L1, L2, k=0.8, b=-1, A1, A2), lwd=2, col=z[1])
lines(x, schnute(x, L1, L2, k=0.4, b=-1, A1, A2), lwd=2, col=z[1])
lines(x, schnute(x, L1, L2, k=0.4, b=0.0001, A1, A2), lwd=2, col=z[2])
lines(x, schnute(x, L1, L2, k=0.2, b=0.0001, A1, A2), lwd=2, col=z[2])
lines(x, schnute(x, L1, L2, k=0.4, b=1, A1, A2), lwd=2, col=z[3])
lines(x, schnute(x, L1, L2, k=0.2, b=1, A1, A2), lwd=2, col=z[3])
lines(x, schnute(x, L1, L2, k=0.1, b=1, A1, A2), lwd=2, col=z[3])
lines(x, schnute(x, L1, L2, k=0.2, b=2, A1, A2), lwd=2, col=z[4])
lines(x, schnute(x, L1, L2, k=0.1, b=2, A1, A2), lwd=2, col=z[4])
lines(x, schnute(x, L1, L2, k=0.0001, b=2, A1, A2), lwd=2, col=z[4])
lines(x, schnute(x, L1, L2, k=0.2, b=3, A1, A2), lwd=2, col=z[5])
lines(x, schnute(x, L1, L2, k=0.1, b=3, A1, A2), lwd=2, col=z[5])
lines(x, schnute(x, L1, L2, k=0.0001, b=3, A1, A2), lwd=2, col=z[5])
lines(x, schnute(x, L1, L2, k=0.1, b=4, A1, A2), lwd=2, col=z[6])
lines(x, schnute(x, L1, L2, k=0.0001, b=4, A1, A2), lwd=2, col=z[6])
lines(x, schnute(x, L1, L2, k=0.0001, b=5, A1, A2), lwd=2, col=z[6])
lines(x, schnute(x, L1, L2, k=0.0001, b=6, A1, A2), lwd=2, col=z[6])

legend("bottomright", c("b: 4 to 6", "b: 3", "b: 2", "b: 1", "b: 0.001",
  "b: -1 to -3"), lwd=2.5, col=rev(z), bty="n", inset=0.02, y.intersp=1.2)
```



4 Discussion

4.1 Range of permissible b values

The Schnute parametrization of the Richards curve used in Stock Synthesis can produce a wide variety of shapes, based on the value of the b parameter, as demonstrated in Section 3.3. When A_1 is greater than the youngest age in the model, some combinations of Richards growth parameters can lead to undefined (NaN) predicted length for the younger ages. The choice of A_1 and A_2 will affect the possible growth curves shapes.

The one value of b that is not allowed when using the Richards growth curve in Stock Synthesis is $b=0$. This also holds for the R functions `ss3_code()` and `schnute()` used in this vignette. When estimating b as a floating-point number, there is effectively no risk of the parameter having the value of precisely zero, as long as the initial value is non-zero. To use a Gompertz growth curve, the b parameter can be fixed at a small value such as 0.0001, as demonstrated in Section 3.2.

4.2 Proof of `ss3_code()` being equivalent to `schnute()`

The above demonstrations comparing the Stock Synthesis output, `ss3_code()` and `schnute()` indicate that they are mathematically equivalent, meaning that the same parameter input produces the same growth curve. The equivalence can also be proven algebraically.

In Stock Synthesis, the Richards growth curve is implemented as

```
ss3_code <- function(t, L1, L2, k, b, A1, A2)
{
  LminR <- L1^b
  LmaxR <- L2^b
  LinfR <- LminR + (LmaxR - LminR) / (1 - exp(-k*(A2-A1)))
  temp <- LinfR + (LminR - LinfR) * exp(-k*(t-A1))
  temp^(1/b)
}
```

$$L_t = \left[L_{\infty R} + (L_1^b - L_{\infty R}) e^{-k(t-A_1)} \right]^{1/b}$$

where:

$$L_{\infty R} = L_1^b + \frac{L_2^b - L_1^b}{1 - e^{-k(A_2-A_1)}}$$

We proceed by replacing $L_{\infty R}$ with its definition and then gradually simplify the equation:

$$\begin{aligned} L_t &= \left[L_1^b + \frac{L_2^b - L_1^b}{1 - e^{-k(A_2-A_1)}} + \left(L_1^b - \left[L_1^b + \frac{L_2^b - L_1^b}{1 - e^{-k(A_2-A_1)}} \right] \right) e^{-k(t-A_1)} \right]^{1/b} \\ L_t &= \left[L_1^b + \frac{L_2^b - L_1^b}{1 - e^{-k(A_2-A_1)}} + \left(L_1^b - L_1^b - \frac{L_2^b - L_1^b}{1 - e^{-k(A_2-A_1)}} \right) e^{-k(t-A_1)} \right]^{1/b} \\ L_t &= \left[L_1^b + \frac{L_2^b - L_1^b}{1 - e^{-k(A_2-A_1)}} - \frac{L_2^b - L_1^b}{1 - e^{-k(A_2-A_1)}} e^{-k(t-A_1)} \right]^{1/b} \\ L_t &= \left[L_1^b + (L_2^b - L_1^b) \frac{1}{1 - e^{-k(A_2-A_1)}} - (L_2^b - L_1^b) \frac{e^{-k(t-A_1)}}{1 - e^{-k(A_2-A_1)}} \right]^{1/b} \\ L_t &= \left[L_1^b + (L_2^b - L_1^b) \frac{1 - e^{-k(t-A_1)}}{1 - e^{-k(A_2-A_1)}} \right]^{1/b} \end{aligned}$$

```
schnute <- function(t, L1, L2, k, b, A1, A2)
{
  (L1^b + (L2^b-L1^b) * (1-exp(-k*(t-A1))) / (1-exp(-k*(A2-A1))))^(1/b)
}
```

4.3 FSA package

The FSA package on CRAN (Ogle et al. 2023) is a useful reference to navigate through the plethora of alternative parametrizations of different growth curves. Among the FSA function calls that are relevant to this vignette are:

```
library(FSA)
Schnute(t, case=1, ...)
Schnute(t, case=2, ...)
```

Schnute case 1 (Schnute 1981, Eq. 15) is the same as `schnute()` used in this vignette.

Schnute case 2 (Schnute 1981, Eq. 16) is a Gompertz growth model parametrized with L_1 , L_2 , and k , similar to `ss3_code()` and `schnute()` when b is fixed at nearly 0.

5 References

- Methot, R.D., Jr., C.R. Wetzel, I.G. Taylor, K.L. Doering, E.F. Perl, and K.F. Johnson. 2024. Stock Synthesis User Manual Version 3.30.23. <https://github.com/nmfs-ost/ss3-doc/releases>
- Ogle D.H, J.C. Doll, A.P. Wheeler, and A. Dinno. 2023. FSA: Simple Fisheries Stock Assessment Methods. R package version 0.9.5. <https://cran.r-project.org/package=FSA>
- Schnute, J. 1981. A versatile growth model with statistical stable parameters. Can. J. Fish. Aquat. Sci. 38:1128–1140. <https://doi.org/10.1139/f81-153>
- Schnute, J. and D. Fournier. 1980. A new approach to length-frequency analysis: Growth structure. Can. J. Fish. Aquat. Sci. 37:1337–1351. <https://doi.org/10.1139/f80-172>