Theory of Root-Raised Cosine Filter

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Abstract The raised cosine filter is used in wireless transmission (e.g. 3GPP) to pulse-shape the chip stream output before it is modulated to the RF. The spectrum is bandwidth limited in order to avoid interferences with neighbour symbols.

Keywords: digital filter;raised-cosine;root-raised;3GPP;UMTS

I. INTRODUCTION

The amplitude steps in a digital chip stream are the cause for high-frequency spectral components. Since the signal is transmitted on a bandwidth-limited channel, smearing of adjacent symbols may happen, known as inter symbol interference (ISI). In order to avoid such interference, the signal is low-pass filtered.

The raised-cosine filter satisfies the Nyquist criterion of suppressing the spectral distortion at integral multiples of the sampling rate. To improve noise cancellation, the filter is usually split into two parts, the root-raised-cosine filter, one at the sender side and the other at the receiver side.

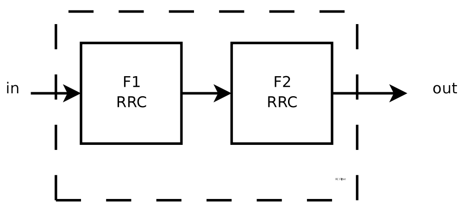

FIG. 1: Split Filter

The transfer function of each of the two filter parts is the root-raised cosine (RRC) function, which is the square root of the raised cosine filter function. The combination of the two root-raised cosine filters yields the raised cosine transfer function.

A. Raised Cosine Filter

The transfer function of the raised cosine filter is (acc. [1]):

$$H_{RC}(\omega) = \begin{cases} A & \text{for } |\omega| \le \omega_1 \\ \frac{A}{2}\left(1+\cos\left(\pi\frac{|\omega|-\omega_1}{r\omega_c}\right)\right) & \text{for } \omega_1 \le |\omega| \le \omega_2 \\ 0 & \text{for } |\omega| > \omega_2 \end{cases}$$

$$\text{with} \quad \omega_1 = \frac{1-r}{2}\omega_c \quad , \quad \omega_2 = \frac{1+r}{2}\omega_c \qquad \text{and} \quad A = \frac{2\pi}{\omega_c} = T_c \tag{1}$$

In this notation the total energy is (see Appendix 1):

$$\|H_{RC}\|^2 = \int_{-\infty}^{\infty} |H_{RC}(\omega)|^2 \, \mathrm{d}\omega = \frac{\pi^2}{\omega_c}(4 - r) \tag{2}$$

B. Root-Raised Cosine Filter

The transfer function of the root-raised cosine filter is:

$$H_{RRC}(\omega) = \begin{cases} B & \text{for } |\omega| \le \omega_1 \\ \frac{B}{\sqrt{2}}\sqrt{1+\cos\left(\pi\frac{|\omega|-\omega_1}{r\omega_c}\right)} & \text{for } \omega_1 \le |\omega| \le \omega_2 \\ 0 & \text{for } |\omega| > \omega_2 \end{cases}$$

$$\text{with} \quad B = \sqrt{A} = \sqrt{\frac{2\pi}{\omega_c}} = \sqrt{T_c} \tag{3}$$

In this notation the total energy is (see Appendix 2):

$$\|H_{RRC}\|^2 = \int_{-\infty}^{\infty} |H_{RRC}(\omega)|^2 \, \mathrm{d}\omega = 2\pi \tag{4}$$

The (normalized) spectrum of the RC and RRC filter is shown in Fig 2.

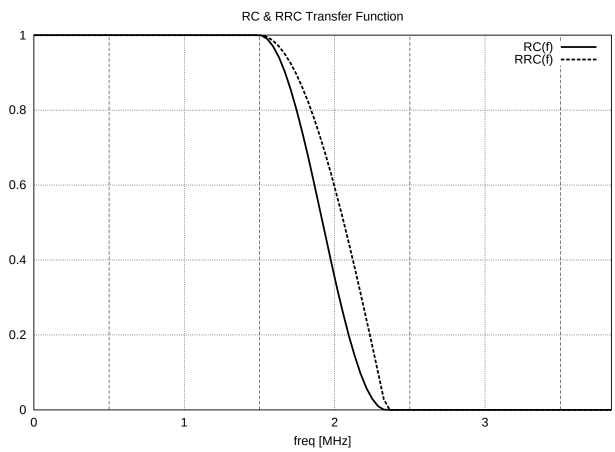

FIG. 2: RC & RRC Transfer Function

*http://www.michael-joost.de/tech.html;

To obtain the impulse response $f_{RRC}(t)$ from the transfer function it is assumed that the impulse response is a real, even function, hence, $f_{RRC}(t) = f_{RRC}(-t)$.

Then, $f_{RRC}(t)$ can be calculated from the Fourier integral:

$$
\begin{aligned}
f_{RRC}(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} H_{RRC}(\omega)e^{i\omega t}\,\mathrm{d}\omega \\
&= \frac{1}{\pi}\int_{0}^{\infty} H_{RRC}(\omega)\cos(\omega t)\,\mathrm{d}\omega \\
&= \frac{B}{\pi}\int_{0}^{\omega_1} \cos(\omega t)\,\mathrm{d}\omega \\
&+ \frac{1}{\pi}\int_{\omega_1}^{\omega_2} H_{RRC}(\omega)\cos(\omega t)\,\mathrm{d}\omega \\
&+ \frac{1}{\pi}\int_{\omega_2}^{\infty} 0\cos(\omega t)\,\mathrm{d}\omega \\
&= \frac{B}{t\pi}\sin(\omega t)\Big|_{0}^{\omega_1} \\
&+ \frac{1}{\pi}\int_{\omega_1}^{\omega_2} H_{RRC}(\omega)\cos(\omega t)\,\mathrm{d}\omega
\end{aligned} \tag{5}
$$

The signal is a superposition of three spectral components. The first component is a constant spectrum, resulting in a sinc component in the time domain. The second spectral component results from the root-raised cosine function, and the third component (at higher frequencies) is zero.

Thus, the impulse response function is

$$
\begin{aligned}
\Longrightarrow\ & \boxed{\begin{aligned} f(t) = \frac{B}{\pi}\Bigg(&\frac{1}{t}\sin(\omega_1 t) \\ &+ \frac{1}{\sqrt{2}}\int_{\omega_1}^{\omega_2}\sqrt{1+\cos(a|\omega|-b)}\cos(\omega t)\,\mathrm{d}\omega\Bigg)\end{aligned}} \\
&\text{with } a = \frac{\pi}{r\omega_c} = \frac{T_c}{2r} \\
&\text{and } \ b = \frac{\pi\omega_1}{r\omega_c} = \frac{\frac{1-r}{2}\pi\omega_c}{r\omega_c} = \pi\frac{1-r}{2r}
\end{aligned} \tag{6}
$$

To calculate the integral in (6) we consider that

$$
\begin{aligned}
\cos(x) &= \cos(\frac{x}{2}+\frac{x}{2}) = \cos^2(\frac{x}{2}) - \sin^2(\frac{x}{2}) \\
&= 1-\sin^2(\frac{x}{2}) - \sin^2(\frac{x}{2}) = 1 - 2\sin^2(\frac{x}{2})
\end{aligned} \tag{7}
$$

and therefore

$$
\begin{aligned}
1+\cos(x) &= 1+1-2\sin^2(\frac{x}{2}) = 2(1-\sin^2(\frac{x}{2})) \\
&= 2\cos^2(\frac{x}{2}) \\
&\Longrightarrow \quad \boxed{\sqrt{1+\cos(x)} = \sqrt{2}\,|\cos(\frac{x}{2})|}
\end{aligned} \tag{8}
$$

The primitive of the integral from (6) can therefore be evaluated as

$$
\begin{aligned}
&\int \sqrt{1+\cos(a|\omega|-b)}\cos(\omega t)\,\mathrm{d}\omega \\
&\qquad = \sqrt{2}\int \underbrace{\cos(\frac{1}{2}(a|\omega|-b))}_{f(\omega)}\underbrace{\cos(\omega t)}_{g'(\omega)}\,\mathrm{d}\omega
\end{aligned} \tag{9}
$$

With the partial integration rule

$$
\int f(x)g'(x)\,\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\,\mathrm{d}x \tag{10}
$$

and

$$
f(\omega) = \cos(\frac{1}{2}(a|\omega|-b)) \qquad g(\omega) = \frac{1}{t}\sin(\omega t) \tag{11}
$$

$$
f'(\omega) = -\frac{a}{2}\sin(\frac{1}{2}(a|\omega|-b)) \quad g'(\omega) = \cos(\omega t) \tag{12}
$$

we get from (9)

$$
\begin{aligned}
&\int \underbrace{\cos(\frac{1}{2}(a|\omega|-b))}_{f(\omega)}\underbrace{\cos(\omega t)}_{g'(\omega)}\,\mathrm{d}\omega = \\
&\underbrace{\cos(\frac{1}{2}(a|\omega|-b))}_{f(\omega)}\underbrace{\frac{1}{t}\sin(\omega t)}_{g(\omega)} \\
&\qquad - \int \underbrace{-\frac{a}{2}\sin(\frac{1}{2}(a|\omega|-b))}_{f'(\omega)}\underbrace{\frac{1}{t}\sin(\omega t)}_{g(\omega)}\,\mathrm{d}\omega = \\
&\frac{1}{t}\Bigg(\cos(\frac{1}{2}(a|\omega|-b))\sin(\omega t) \\
&\qquad + \frac{a}{2}\int \underbrace{\sin(\frac{1}{2}(a|\omega|-b))}_{f_2(\omega)}\underbrace{\sin(\omega t)}_{g_2'(\omega)}\,\mathrm{d}\omega\Bigg)
\end{aligned} \tag{13}
$$

Applying the partial integration rule again with

$$
f_2(\omega) = \sin(\frac{1}{2}(a|\omega|-b)) \qquad g_2(\omega) = -\frac{1}{t}\cos(\omega t) \tag{14}
$$

$$
f_2'(\omega) = \frac{a}{2}\cos(\frac{1}{2}(a|\omega|-b)) \quad g_2'(\omega) = \sin(\omega t) \tag{15}
$$

we get

$$\begin{aligned}
&\int \cos(\frac{1}{2}(a|\omega|-b))\ \cos(\omega t)\,\mathrm{d}\omega \\
&= \frac{1}{t}\Bigg(\cos(\frac{1}{2}(a|\omega|-b))\sin(\omega t) \\
&\qquad + \frac{a}{2}(-\sin(\frac{1}{2}(a|\omega|-b))\ \frac{1}{t}\cos(\omega t) \\
&\qquad - \int \frac{a}{2}\cos(\frac{1}{2}(a|\omega|-b))(-\frac{1}{t}\cos(\omega t))\,\mathrm{d}\omega)\Bigg) \\
&= \frac{1}{t}\Bigg(\cos(\frac{1}{2}(a|\omega|-b))\sin(\omega t) \\
&\qquad - \frac{a}{2t}\sin(\frac{1}{2}(a|\omega|-b))\ \cos(\omega t)\Bigg) \\
&\qquad + \frac{a^2}{4t^2}\int \cos(\frac{1}{2}(a|\omega|-b))\cos(\omega t)\,\mathrm{d}\omega
\end{aligned} \tag{16}$$

Resolving for $\int \cos(\frac{1}{2}(a|\omega|-b))\ \cos(\omega t)\,\mathrm{d}\omega$ yields

$$\begin{aligned}
&(1-\frac{a^2}{4t^2})\int \cos(\frac{1}{2}(a|\omega|-b))\ \cos(\omega t)\,\mathrm{d}\omega \\
&= \frac{1}{t}\Bigg(\cos(\frac{1}{2}(a|\omega|-b))\sin(\omega t) \\
&\qquad - \frac{a}{2t}\sin(\frac{1}{2}(a|\omega|-b))\ \cos(\omega t)\Bigg)
\end{aligned} \tag{17}$$

$$\Longrightarrow \boxed{\begin{aligned}
&\int \cos(\frac{1}{2}(a|\omega|-b))\ \cos(\omega t)\,\mathrm{d}\omega \\
&= \frac{1}{t(1-\frac{a^2}{4t^2})}\Bigg(\cos(\frac{1}{2}(a|\omega|-b))\sin(\omega t) \\
&- \frac{a}{2t}\sin(\frac{1}{2}(a|\omega|-b))\ \cos(\omega t)\Bigg)
\end{aligned}} \tag{18}$$

From (9) we get

$$\begin{aligned}
&\int \sqrt{1+\cos(a|\omega|-b)}\cos(\omega t)\,\mathrm{d}\omega \\
&\quad = \frac{\sqrt{2}}{t(1-\frac{a^2}{4t^2})}\Bigg(\cos(\frac{1}{2}(a|\omega|-b))\sin(\omega t) \\
&\qquad - \frac{a}{2t}\sin(\frac{1}{2}(a|\omega|-b))\ \cos(\omega t)\Bigg)
\end{aligned} \tag{19}$$

and the integral value

$$\begin{aligned}
&\int_{\omega_1}^{\omega_2} \sqrt{1+\cos(a\omega-b)}\cos(\omega t)\,\mathrm{d}\omega \\
&\quad = \frac{\sqrt{2}}{t(1-\frac{a^2}{4t^2})}\Bigg(\Bigg(\cos(\frac{1}{2}(a\omega_2-b))\sin(\omega_2 t) \\
&\qquad - \frac{a}{2t}\sin(\frac{1}{2}(a\omega_2-b))\ \cos(\omega_2 t)\Bigg) \\
&\qquad - \Bigg(\cos(\frac{1}{2}(a\omega_1-b))\sin(\omega_1 t) \\
&\qquad - \frac{a}{2t}\sin(\frac{1}{2}(a\omega_1-b))\ \cos(\omega_1 t)\Bigg)\Bigg) \\
&\quad = \frac{\sqrt{2}}{t(1-\frac{a^2}{4t^2})}\Bigg(\cos(\frac{1}{2}(a\omega_2-b))\sin(\omega_2 t) \\
&\qquad - \frac{a}{2t}\sin(\frac{1}{2}(a\omega_2-b))\ \cos(\omega_2 t) \\
&\qquad - \cos(\frac{1}{2}(a\omega_1-b))\sin(\omega_1 t) \\
&\qquad + \frac{a}{2t}\sin(\frac{1}{2}(a\omega_1-b))\ \cos(\omega_1 t)\Bigg)
\end{aligned} \tag{20}$$

With $a = \frac{\pi}{r\omega_c}$ and $b = \frac{\pi(1-r)}{2r}$ we can make these substitutions

$$\begin{aligned}
\frac{1}{2}(a\omega_1-b) &= \frac{1}{2}(\frac{\pi}{r\omega_c}\frac{1-r}{2}\omega_c - \pi\frac{1-r}{2r}) \\
&= \frac{\pi}{2}(\frac{(1-r)}{2r}-\frac{1-r}{2r}) \\
&= 0 \\
\Longrightarrow \sin(\frac{1}{2}(a\omega_1-b)) = 0 \quad &, \quad \cos(\frac{1}{2}(a\omega_1-b)) = 1
\end{aligned} \tag{21}$$

and

$$\begin{aligned}
\frac{1}{2}(a\omega_2-b) &= \frac{1}{2}(\frac{\pi}{r\omega_c}\frac{1+r}{2}\omega_c - \pi\frac{1-r}{2r}) \\
&= \frac{\pi}{2}(\frac{(1+r)}{2r}-\frac{1-r}{2r}) \\
&= \frac{\pi}{2} \\
\Longrightarrow \sin(\frac{1}{2}(a\omega_2-b)) = 1 \quad &, \quad \cos(\frac{1}{2}(a\omega_2-b)) = 0
\end{aligned} \tag{22}$$

and get for the integral value

$$\begin{aligned}
&\int_{\omega_1}^{\omega_2} \sqrt{1+\cos(a\omega-b)}\cos(\omega t)\,\mathrm{d}\omega \\
&= \frac{4t\sqrt{2}}{4t^2-(\frac{\pi}{r\omega_c})^2}\Bigg(\cos(\frac{\pi}{2})\sin(\frac{1+r}{2}\omega_c t)
\end{aligned}$$

$$-\frac{\pi}{2tr\omega_c}\sin(\frac{\pi}{2})\ \cos(\frac{1+r}{2}\omega_c t)-\sin(\frac{1-r}{2}\omega_c t)\Big)$$

$$=\frac{4t\sqrt{2}}{4t^2-(\frac{\pi}{r\omega_c})^2}\Big(-\frac{\pi}{2tr\omega_c}\cos(\frac{1+r}{2}\omega_c t)-\sin(\frac{1-r}{2}\omega_c t)\Big) \tag{23}$$

Now we have with (6)

$$f(t)=\frac{B}{\pi}\Big(\frac{1}{t}\sin(\frac{1-r}{2}\omega_c t)+\frac{4t}{4t^2-(\frac{\pi}{r\omega_c})^2}\\ \Big(-\frac{\pi}{2tr\omega_c}\cos(\frac{1+r}{2}\omega_c t)-\sin(\frac{1-r}{2}\omega_c t)\Big)\Big) \tag{24}$$

$$f(t)=\frac{B}{\pi}\Big(\frac{1}{t}\sin((1-r)\pi\frac{t}{T_c})+\frac{4t}{4t^2-(\frac{T_c}{2r})^2}\\ \Big(-\frac{T_c}{4tr}\cos((1+r)\pi\frac{t}{T_c})-\sin((1-r)\pi\frac{t}{T_c})\Big)\Big) \tag{25}$$

$$f(t)=\frac{B}{\pi}\Big(\frac{1}{t}\sin((1-r)\pi\frac{t}{T_c})+\\ \Big(\frac{4t}{(\frac{T_c}{2r})^2-4t^2}\frac{T_c}{4tr}\cos((1+r)\pi\frac{t}{T_c})\\ +\frac{4t}{(\frac{T_c}{2r})^2-4t^2}\sin((1-r)\pi\frac{t}{T_c})\Big)\Big) \tag{26}$$

$$f(t)=\frac{B}{\pi T_c}\Big(\frac{T_c}{t}\sin((1-r)\pi\frac{t}{T_c})\\ +\frac{4tT_c}{(\frac{T_c}{2r})^2-4t^2}\sin((1-r)\pi\frac{t}{T_c})\\ +\frac{4t}{(\frac{T_c}{2r})^2-4t^2}\frac{T_c^2}{4tr}\cos((1+r)\pi\frac{t}{T_c})\Big) \tag{27}$$

$$f(t)=\frac{B}{\pi T_c}\Big(\frac{1-(4r\frac{t}{T_c})^2}{\frac{t}{T_c}(1-(4r\frac{t}{T_c})^2)}\sin((1-r)\pi\frac{t}{T_c})\\ +\frac{4tT_c\,4(\frac{r}{T_c})^2\frac{t}{T_c}}{(\frac{T_c^2}{4r^2}-4t^2)4(\frac{r}{T_c})^2\frac{t}{T_c}}\sin((1-r)\pi\frac{t}{T_c})\\ +\frac{4tT_c^2}{\frac{T_c^2}{4r^2}-4t^2}\frac{\frac{r}{T_c^3}}{4tr\frac{r}{T_c^3}}\cos((1+r)\pi\frac{t}{T_c})\Big) \tag{28}$$

$$f(t)=\frac{B}{\pi T_c}\Big(\frac{1-(4r\frac{t}{T_c})^2}{\frac{t}{T_c}(1-(4r\frac{t}{T_c})^2)}\sin((1-r)\pi\frac{t}{T_c})\\ +\frac{(4r\frac{t}{T_c})^2}{(1-(4r\frac{t}{T_c})^2)\frac{t}{T_c}}\sin((1-r)\pi\frac{t}{T_c})\\ +\frac{4r\frac{t}{T_c}}{(1-(4r\frac{t}{T_c})^2)\frac{t}{T_c}}\cos((1+r)\pi\frac{t}{T_c})\Big) \tag{29}$$

$$f(t)=\frac{B}{\pi T_c}\Big(\frac{1-(4r\frac{t}{T_c})^2+(4r\frac{t}{T_c})^2}{\frac{t}{T_c}(1-(4r\frac{t}{T_c})^2)}\sin((1-r)\pi\frac{t}{T_c})\\ +\frac{4r\frac{t}{T_c}}{(1-(4r\frac{t}{T_c})^2)\frac{t}{T_c}}\cos((1+r)\pi\frac{t}{T_c})\Big) \tag{30}$$

$$f(t)\ =\ C\ \frac{\sin((1-r)\pi\frac{t}{T_c})+4r\frac{t}{T_c}\cos((1+r)\pi\frac{t}{T_c})}{\pi\frac{t}{T_c}(1-(4r\frac{t}{T_c})^2)}$$

$$\text{with}\qquad C=\frac{B}{T_c}=\frac{1}{\sqrt{T_c}} \tag{31}$$

The total energy is (see Appendix 3):

$$\|f_{RRC}\|^2\ =\ \int_{-\infty}^{\infty}|f_{RRC}(t)|^2\,\mathrm{d}t\ =\ 1 \tag{32}$$

According to Parseval's Theorem we find confirmed that $\|H_{RRC}\|^2\ =\ 2\pi\,\|f_{RRC}\|^2$

II. IMPULSE RESPONSE FUNCTION

The impulse response of the root-raised cosine filter is according to (eq 31), and also as defined by the 3GPP standard [2], neglecting the constant factor C:

$$\boxed{h_0(tT_c)=\frac{\sin(\pi t(1-r))+4rt\cos(\pi t(1+r))}{\pi t\,(1-(4rt)^2)}}$$

with the chip-length T_c and $r=0.22$ (chosen by 3GPP) (33)

A. Singularities

This function has two removable singularities, one at $t=0$ and the other at $t=\pm\frac{1}{4r}$.

The function value at these singularities can be calculated using L'Hôpital's rule:

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)} \tag{34}$$

since $\lim\limits_{x\to a} f(x)=\lim\limits_{x\to a} g(x)=0$.

B. Zeroes Of Numerator and Denominator

The denominator function is obviously zero for arguments $t=0$ and $t=\pm\frac{1}{4r}$. The numerator function value at these arguments is also zero:

$$\begin{aligned}&\sin(\pi t(1-r))+4rt\cos(\pi t(1+r))\Big|_{t=0}=\\&\qquad\sin(\pi 0(1-r))+4r0\cos(\pi 0(1+r))=0\end{aligned}$$

and

$$\begin{aligned}&\sin(\pi t(1-r))+4rt\cos(\pi t(1+r))\Big|_{t=\pm\frac{1}{4r}}=\\&\qquad\sin(\frac{\pi(1-r)}{\pm 4r})\pm\cos(\frac{\pi(1+r)}{\pm 4r})=\end{aligned}$$

with $sin(-x)=-sin(x)$ and $cos(-x)=cos(x)$:

$$\begin{aligned}&\pm\sin(\frac{\pi(1-r)}{4r})\pm\cos(\frac{\pi(1+r)}{4r})=\\&\qquad\pm\sin(\frac{\pi}{4r}-\frac{\pi}{4})\pm\cos(\frac{\pi}{4r}+\frac{\pi}{4})=\end{aligned}$$

with $cos(x)=sin(\frac{\pi}{2}-x)$:

$$\begin{aligned}&\pm\sin(\frac{\pi}{4r}-\frac{\pi}{4})\pm\sin(\frac{\pi}{2}-\frac{\pi}{4r}-\frac{\pi}{4})=\\&\qquad\pm\sin(\frac{\pi}{4r}-\frac{\pi}{4})\pm\sin(\frac{\pi}{4}-\frac{\pi}{4r})=\\&\qquad\pm\sin(\frac{\pi}{4r}-\frac{\pi}{4})\mp\sin(\frac{\pi}{4r}-\frac{\pi}{4})\quad=\quad 0\end{aligned}$$

C. Derivatives

The derivative of the numerator function in eq. (33) is:

$$\begin{aligned}f'(t)=\pi(1-r)\cos(\pi t(1-r))+4r\cos(\pi t(1+r))\\-4rt\pi(1+r)\sin(\pi t(1+r))\end{aligned} \tag{35}$$

The derivative of the denominator function in eq. (33) is:

$$g'(t)=\pi(1-48(rt)^2) \tag{36}$$

D. Function Value

For $t\to 0$ the function value of the impulse response function is

$$\begin{aligned}&\lim_{t\to 0}\frac{f(t)}{g(t)}=\lim_{t\to 0}\frac{f'(t)}{g'(t)}=\\&\frac{\pi(1-r)\cos(\pi 0(1-r))+4r\cos(\pi 0(1+r))}{\pi(1-48(r0)^2)}\\&\qquad-\frac{4r0\pi(1+r)\sin(\pi 0(1+r))}{\pi(1-48(r0)^2)}=\\&\qquad\frac{\pi(1-r)1+4r1}{\pi}\end{aligned} \tag{37}$$

$$\Longrightarrow\qquad \boxed{h_0(0)=(1-r)+\frac{4r}{\pi}} \tag{38}$$

For $t\to\pm\frac{1}{4r}$ the function value is

$$\begin{aligned}&\lim_{t\to\pm\frac{1}{4r}}\frac{f(t)}{g(t)}=\lim_{t\to\pm\frac{1}{4r}}\frac{f'(t)}{g'(t)}=\\&\frac{\pi(1-r)\cos(\frac{\pi}{4r}(1-r))+4r\cos(\frac{\pi}{4r}(1+r))}{\pi(1-48(r\frac{1}{4r})^2)}\\&\qquad\mp\frac{4r\frac{1}{4r}\pi(1+r)\sin(\pm\frac{\pi}{4r}(1+r))}{\pi(1-48(r\frac{1}{4r})^2)}=\\&-\frac{1}{2\pi}\Bigg(\pi(1-r)\cos(\frac{\pi}{4r}(1-r))+4r\cos(\frac{\pi}{4r}(1+r))\\&\qquad-\pi(1+r)\sin(\frac{\pi}{4r}(1+r))\Bigg)=\\&\frac{1}{2\pi}\Bigg(\pi(1+r)\sin(\frac{\pi}{4r}(1+r))-\pi(1-r)\cos(\frac{\pi}{4r}(1-r))\\&\qquad-4r\cos(\frac{\pi}{4r}(1+r))\Bigg)\end{aligned} \tag{39}$$

with $\sin(x+y)=\sin(x)\cos(y)+\cos(x)\sin(y)$ and $\cos(x+y)=\cos(x)\cos(y)-\sin(x)\sin(y)$:

$$\begin{aligned}=\frac{1}{2\pi}\Bigg(&\pi(1+r)(\sin(\frac{\pi}{4r})\cos(\frac{\pi}{4})+\cos(\frac{\pi}{4r})\sin(\frac{\pi}{4}))\\&-\pi(1-r)(\cos(\frac{\pi}{4r})\cos(\frac{\pi}{4})+\sin(\frac{\pi}{4r})\sin(\frac{\pi}{4}))\\&-4r(\cos(\frac{\pi}{4r})\cos(\frac{\pi}{4})-\sin(\frac{\pi}{4r})\sin(\frac{\pi}{4}))\Bigg)\end{aligned} \tag{40}$$

with $\cos(\frac{\pi}{4})=\sin(\frac{\pi}{4})=\frac{1}{\sqrt{2}}$:

$$\begin{aligned}=\frac{1}{2\pi\sqrt{2}}\Bigg(&\pi(1+r)(\sin(\frac{\pi}{4r})+\cos(\frac{\pi}{4r}))\\&-\pi(1-r)(\cos(\frac{\pi}{4r})+\sin(\frac{\pi}{4r}))\\&-4r(\cos(\frac{\pi}{4r})-\sin(\frac{\pi}{4r}))\Bigg)\end{aligned} \tag{41}$$

$$= \frac{1}{2\pi\sqrt{2}}\Big(\sin(\frac{\pi}{4r})(\pi(1+r) - \pi(1-r) + 4r) + \cos(\frac{\pi}{4r})(\pi(1+r) - \pi(1-r) - 4r)\Big)$$

$$= \frac{1}{2\pi\sqrt{2}}\Big(\sin(\frac{\pi}{4r})(2\pi r + 4r) + \cos(\frac{\pi}{4r})(2\pi r - 4r)\Big) \quad (42)$$

$$= \frac{2r}{2\sqrt{2}}\Big(\sin(\frac{\pi}{4r})(1 + \frac{2}{\pi}) + \cos(\frac{\pi}{4r})(1 - \frac{2}{\pi})\Big)$$

$$\boxed{h_0(\pm\frac{1}{4r}T_c) = \frac{r}{\sqrt{2}}\Big((1 + \frac{2}{\pi})\sin(\frac{\pi}{4r}) + (1 - \frac{2}{\pi})\cos(\frac{\pi}{4r})\Big)} \quad (43)$$

E. Impulse Response

The impulse response graph[1] according to (eq 33) of the RRC filter is shown in Fig. 3.

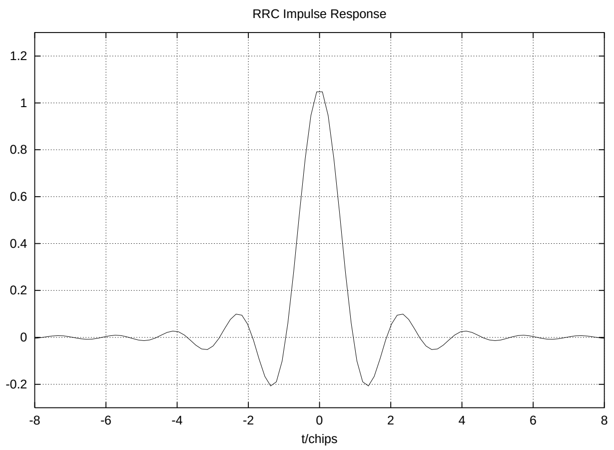

FIG. 3: Impulse Response

The impulse response of the root-raised cosine filter is not exactly zero at integral multiples of T_c, only the combined (raised cosine) filter can avoid inter-symbol interference.
For example, after one symbol length, hence for $t = 1$, the impulse response value is (from eq 33)

$$h_0(T_c) = \frac{\sin(\pi(1-r)) + 4r\cos(\pi(1+r))}{\pi\,(1-(4r)^2)} \underset{r\,\epsilon\,]0,1[}{\neq} 0 \quad (44)$$

[1] Interactive diagrams of the spectrum and the impulse response can be found at http://www.michael-joost.de/tech.html

III. SPECTRUM

The spectrum[1] of the RRC filter is shown in Fig. 4.

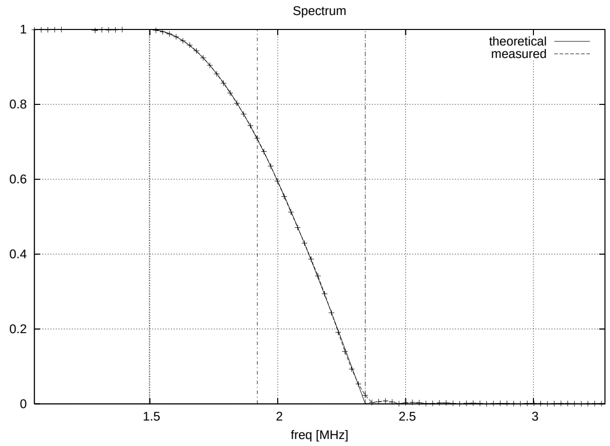

FIG. 4: Spectrum

IV. FINITE IMPULSE RESPONSE FILTER

A. Filter Structure

The RRC filter can be approximated by a Finite Impule Response (FIR) structure:

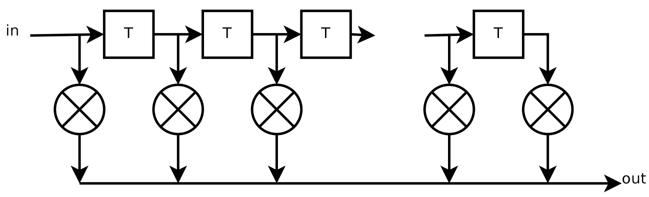

FIG. 5: FIR Filter Structure

The filter length is (for example) $n = 32$ taps, multiplied by the oversampling factor osf. The $n * osf + 1$ filter coefficients are determined according to (eq 33).
The FIR filter causes a signal delay of half it's length.

B. Signal Example

A FIR filter with oversampling factor 4 and 32*4+1 taps has a filtered response to a random digital input signal as shown in Fig. 6. The output signal has been shifted in this diagram by half the filter length in order to match the position of the input signal.
The relevant output samples are those located at the middle of each input pulse.

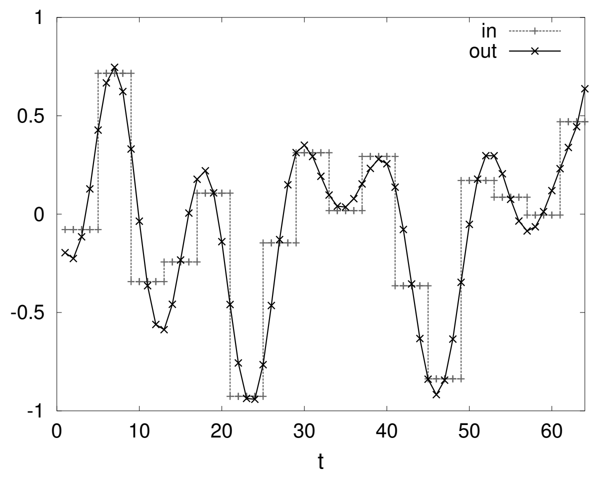

FIG. 6: RRC Time Domain Response

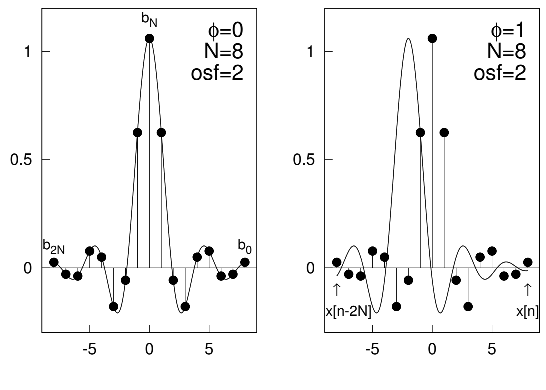

FIG. 7: Scenarios: ISI in FIR Filter

C. ISI Performance

While the ideal RRC filter has an infinite impulse response, a FIR filter necessarily has a finite length, as its name implies. To quantify the impact of the finite length on the inter-symbol interference, we consider an impulse filtered by an ideal sender-side RRC filter, hence, the signal described by (eq 31, 33), and feed this thru a finite-length receiver-side RRC FIR filter. The ratio of the maximum of output amplitudes at multiples of the symbol sampling times, related to the output amplitude of the symbol pulse ($t = 0$), indicates a measure for inter-symbol suppression.

The response of a FIR filter with $2N + 1$ taps is given by

$$y[n] = \sum_{i=0}^{2N} b_i \, x[n-i] \tag{45}$$

and in case of the RRC filter the coefficients b_i are chosen as

$$b_i = \frac{\sin(\pi t(1-r)) + 4rt\cos(\pi t(1+r))}{\pi t\,(1-(4rt)^2)} \tag{46}$$

with

$$t = \frac{N-i}{osf}\,, \qquad \text{osf} = \text{oversampling factor} \tag{47}$$

The input signal to this filter is

$$x_\phi[n-i] = \frac{\sin(\pi s(1-r)) + 4rs\cos(\pi s(1+r))}{\pi s\,(1-(4rs)^2)} \tag{48}$$

with

$$s = \frac{N-i}{osf} + \phi\,, \quad \phi = \text{signal offset} \tag{49}$$

In Fig. 7 the filter coefficients b_i are shown as dots, with the coefficient for the most recent sample $x[n]$, b_0, located on the right side.
We now consider two observations. The left diagram of Fig. 7 shows the situation where the input signal contour matches the filter coefficients contour ($\phi = 0$), thus, maximizing the filter's output. This is the reference output indicating the payload symbol.
The right diagram is the observation at some later time, when the peak of the input signal has already passed the filter coefficient's peak by one symbol length ($\phi = 1$), here: two samples. We expect the filter output to be minimal at this time, ideally it would be zero for an infinitely long filter.

The following diagrams quantify the ratio, hence, the ISI suppression performance, for various filter lengths, oversampling factors and symbol distances.

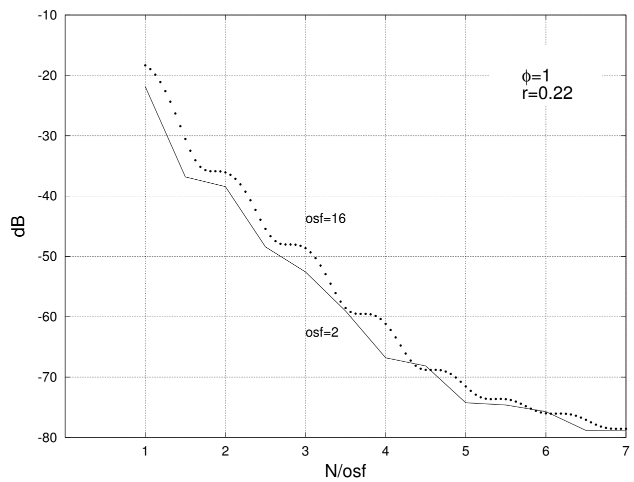

FIG. 8: ISI suppression after one symbol

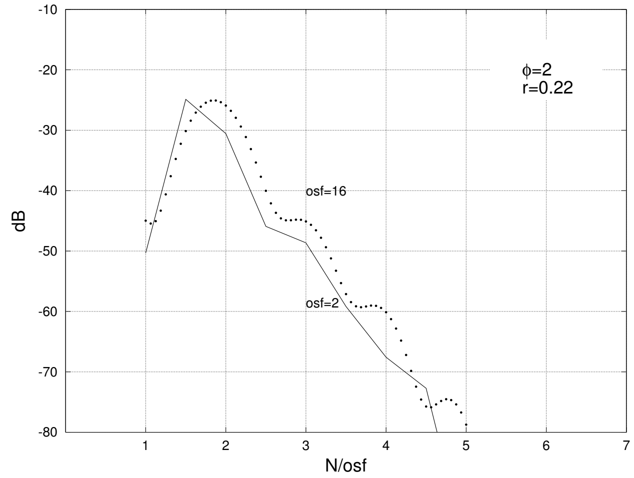

FIG. 9: ISI suppression after two symbols

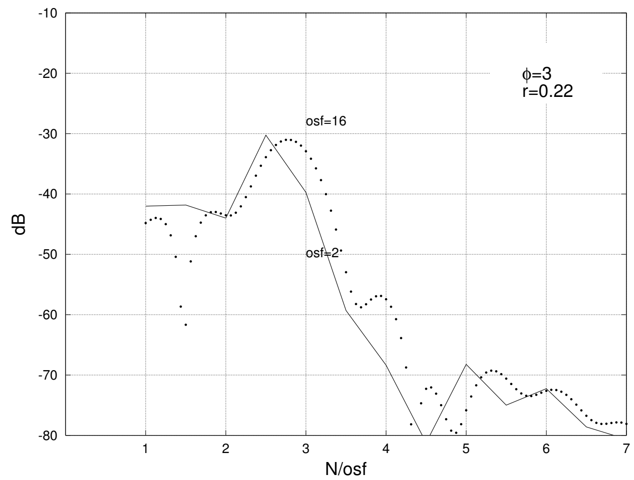

FIG. 10: ISI suppression after three symbols

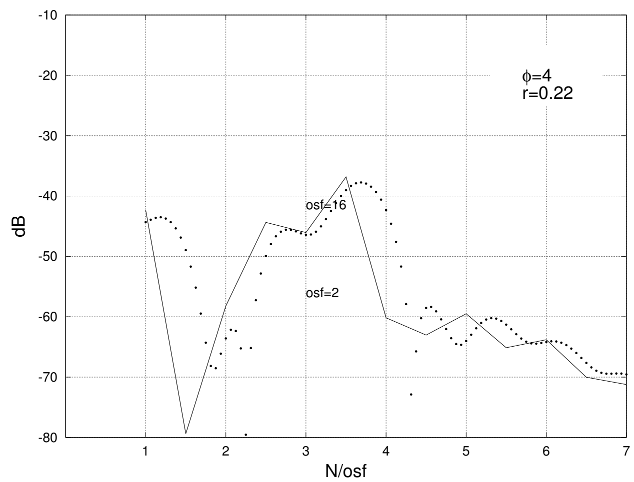

FIG. 11: ISI suppression after four symbols

A higher oversampling factor does not improve the ISI suppression, in fact, the results are slightly worse. The peaks for small filter lengths and higher symbol distances originate from the main pulse having (partly) left the finite-sized filter, resulting in some level of inaccuracy.

For an ISI suppression of 40dB an oversampling factor of 2 and a filter length of $N/osf = 4 \Rightarrow N = 8$ is sufficient, resulting in a filter with $2 * N + 1 = 17$ taps.

V. CONCLUSIONS

A digital FIR filter can be used to reduce the band width of the step-affected chip stream. The filter has to use oversampling ($osf \geq 2$), as no frequencies above half the filter's sampling frequency can be handled (Nyquist), while the upper frequency point ω_2 is at more than half the chip rate.

VI. REFERENCES

[1] Wikipedia:
Raised cosine
`http://en.wikipedia.org/wiki/Raised\_cosine`

[2] 3GPP
standard 25.104, sect 6.8.1
`http://www.3gpp.org`

[3] M.Joost
Interactive diagrams of RC and RRC filter:
spectrum, impulse response, eye diagram
`http://www.michael-joost.de/tech.html`

Appendix

1. Energy in Frequency-Domain of Raised Cosine Filter

$$\|H_{RC}\|^2 = \int_{-\infty}^{\infty} |H_{RC}(\omega)|^2 \,\mathrm{d}\omega = 2\int_{0}^{\infty} |H_{RC}(\omega)|^2 \,\mathrm{d}\omega$$
$$= 2A^2\omega_1 + \frac{A^2}{2}\int_{\omega_1}^{\omega_2} (1+\cos(\pi\frac{\omega-\omega_1}{r\omega_c}))^2 \,\mathrm{d}\omega \tag{50}$$

The integral can be written as

$$\int (1+\cos(\pi\frac{\omega-\omega_1}{r\omega_c}))^2 \,\mathrm{d}\omega$$

$$= \int (1+2\cos(a\omega+b)+\cos^2(a\omega+b)) \,\mathrm{d}\omega$$
$$\text{with} \quad a = \frac{\pi}{r\omega_c} \quad \text{and} \quad b = \frac{r-1}{2r}\pi$$

$$= \int \mathrm{d}\omega + 2\int \cos(a\omega+b)\,\mathrm{d}\omega + \int \cos^2(a\omega+b)\,\mathrm{d}\omega \tag{51}$$

Now we need to find the primitives for those three integrals. The first integral yields ω, obviously.
For the second integral, using the substitution rule

$$\int f(g(x))\,\mathrm{d}x = \int f(z)\frac{1}{g'(x)}\,\mathrm{d}z \quad \text{with } z = g(x) \tag{52}$$

we get the primitive:

$$\text{with } f(z) = \cos(z),\ z = g(\omega) = a\omega+b,\ g'(\omega) = a$$
$$\int \cos(a\omega+b)\,\mathrm{d}\omega = \int cos(z)\frac{1}{a}\,\mathrm{d}z$$
$$= \frac{1}{a}\sin(z) = \frac{1}{a}\sin(a\omega+b) \tag{53}$$

For the third integral, considering the partial integration rule

$$\int f(x)g'(x)\,\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\,\mathrm{d}x \tag{54}$$

and

$$f(\omega) = \cos(a\omega+b) \qquad g(\omega) = \frac{1}{a}\sin(a\omega+b) \tag{55}$$
$$f'(\omega) = -a\sin(a\omega+b) \qquad g'(\omega) = \cos(a\omega+b) \tag{56}$$

we get

$$\int \cos^2(a\omega+b)\,\mathrm{d}\omega = \frac{1}{a}\sin(a\omega+b)\cos(a\omega+b)$$
$$+ \int sin^2(a\omega+b)\,\mathrm{d}\omega \tag{57}$$

Since $\quad \sin(x)\cos(x) = \frac{1}{2}\sin(2x)$
and $\quad \sin^2(x) = 1 - cos^2(x)$

$$\int \cos^2(a\omega+b)\,\mathrm{d}\omega = \frac{1}{2a}\sin(2(a\omega+b))$$
$$+ \int (1 - cos^2(a\omega+b))\,\mathrm{d}\omega \tag{58}$$

$$\int \cos^2(a\omega+b)\,\mathrm{d}\omega = \frac{1}{2a}\sin(2(a\omega+b))$$
$$+ \omega - \int \cos^2(a\omega+b)\,\mathrm{d}\omega \tag{59}$$

$$\Rightarrow 2\int \cos^2(a\omega+b)\,\mathrm{d}\omega = \frac{1}{2a}\sin(2(a\omega+b)) + \omega$$
$$\Rightarrow \int \cos^2(a\omega+b)\,\mathrm{d}\omega = \frac{1}{4a}\sin(2(a\omega+b)) + \frac{\omega}{2} \tag{60}$$

Inserting the integral parts from (53) and (60) into (51) yields the primitive

$$F(\omega) = \omega + 2\frac{1}{a}\sin(a\omega+b)$$
$$+ \frac{1}{4a}\sin(2(a\omega+b)) + \frac{\omega}{2}$$
$$= \frac{3\omega}{2} + \frac{2}{a}\sin(a\omega+b)$$
$$+ \frac{1}{4a}\sin(2(a\omega+b)) \tag{61}$$

The definite integral in (50) has the value

$$F(\omega_2) - F(\omega_1) = \frac{3\omega_2}{2} + \frac{2}{a}\sin(a\omega_2+b)$$
$$+ \frac{1}{4a}\sin(2(a\omega_2+b))$$
$$- \frac{3\omega_1}{2} - \frac{2}{a}\sin(a\omega_1+b)$$
$$- \frac{1}{4a}\sin(2(a\omega_1+b))$$
$$= \frac{3(\omega_2-\omega_1)}{2}$$
$$+ \frac{2}{a}\Big(\sin(a\omega_2+b) - \sin(a\omega_1+b)\Big)$$
$$+ \frac{1}{4a}\Big(\sin(2(a\omega_2+b)) - \sin(2(a\omega_1+b))\Big) \tag{62}$$

With $\sin(x) - \sin(y) = 2\sin(\frac{x-y}{2})\cos(\frac{x+y}{2})$:

$$\begin{aligned} &= \frac{3r\omega_c}{2} \\ &+ \frac{2}{a} 2\sin(\frac{a}{2}(\omega_2 - \omega_1))\cos(\frac{a}{2}(\omega_1 + \omega_2) + b) \\ &+ \frac{1}{4a} 2\sin(a(\omega_2 - \omega_1))\cos(a(\omega_1 + \omega_2) + 2b) \end{aligned} \tag{63}$$

The argument to the first of the sin() terms is

$$\begin{aligned} \frac{a}{2}(\omega_2 - \omega_1) &= \frac{\pi}{2r\omega_c}(\frac{1+r}{2}\omega_c - \frac{1-r}{2}\omega_c) \\ &= \frac{\pi}{2r\omega_c}\frac{\omega_c}{2}(1 + r - 1 + r) = \frac{\pi}{2} \end{aligned} \tag{64}$$

so, $\sin(\frac{\pi}{2}) = 1$, and accordingly for the second sin() term: $\sin(\pi) = 0$.
The definite integral is therefore

$$\begin{aligned} &= \frac{3r\omega_c}{2} \\ &+ \frac{4}{a}\cos(\frac{\pi}{2r\omega_c}(\frac{1+r}{2}\omega_c + \frac{1-r}{2}\omega_c) + \frac{r-1}{2r}\pi) \\ &= \frac{3r\omega_c}{2} + \frac{4}{a}\cos(\frac{\pi}{2r\omega_c}\frac{2\omega_c}{2} + \frac{r-1}{2r}\pi) \\ &= \frac{3r\omega_c}{2} + \frac{4}{a}\cos(\frac{\pi}{2r}(1 + r - 1)) \\ &= \frac{3r\omega_c}{2} + \frac{4}{a}\cos(\frac{\pi}{2}) \\ &= \quad \frac{3r\omega_c}{2} \end{aligned} \tag{65}$$

Now we can calculate the enery from (50):

$$\begin{aligned} \|H_{RC}\|^2 &= 2A^2\omega_1 + \frac{A^2}{2}\frac{3r\omega_c}{2} \\ &= \frac{4\pi^2}{\omega_c^2}(2\frac{1-r}{2}\omega_c + \frac{3r\omega_c}{4}) \\ &= \frac{\pi^2}{\omega_c}(4(1-r) + 3r) \end{aligned} \tag{66}$$

$$\Rightarrow \qquad \boxed{\|H_{RC}\|^2 \quad = \quad \frac{\pi^2}{\omega_c}(4 - r)} \tag{67}$$

2. Energy in Frequency-Domain of Root-Raised Cosine Filter

Calculating the energy for the RRC filter is surprisingly simple.

$$\begin{aligned} \|H_{RRC}\|^2 &= \int_{-\infty}^{\infty} |H_{RRC}(\omega)|^2 \,\mathrm{d}\omega \\ &= 2\int_0^{\infty} |H_{RRC}(\omega)|^2 \,\mathrm{d}\omega \\ &= 2B^2\omega_1 + B^2\int_{\omega_1}^{\omega_2} (1 + \cos(\frac{\pi}{r\omega_c}(\omega - \omega_1))\,\mathrm{d}\omega \\ &= \frac{2\pi}{\omega_c}\Bigg(2\omega_1 + (\omega_2 - \omega_1) \\ &\quad + \int_{\omega_1}^{\omega_2} \cos(\frac{\pi}{r\omega_c}(\omega - \omega_1)\,\mathrm{d}\omega\Bigg) \end{aligned} \tag{68}$$

For the integral we know the primitive from (eq 53):

$$\begin{aligned} F(\omega) &= \frac{1}{a}\sin(a\omega + b) \\ &= \frac{r\omega_c}{\pi}\sin(\frac{\pi}{r\omega_c}\omega + \frac{\pi}{2r}(r-1)) \end{aligned} \tag{69}$$

Thus, the definite integral is:

$$\begin{aligned} F(\omega_2) - F(\omega_1) &= \frac{r\omega_c}{\pi}\sin(\frac{\pi}{r\omega_c}\omega_2 + \frac{\pi}{2r}(r-1)) \\ &\quad - \frac{r\omega_c}{\pi}\sin(\frac{\pi}{r\omega_c}\omega_1 + \frac{\pi}{2r}(r-1)) \\ &= \frac{r\omega_c}{\pi}\sin(\frac{\pi}{r\omega_c}\omega_c\frac{1+r}{2} + \frac{\pi}{2r}(r-1)) \\ &- \frac{r\omega_c}{\pi}\sin(\frac{\pi}{r\omega_c}\omega_c\frac{1-r}{2} + \frac{\pi}{2r}(r-1)) \\ &= \frac{r\omega_c}{\pi}\sin(\frac{\pi}{2r}(1 + r + r - 1)) \\ &- \frac{r\omega_c}{\pi}\sin(\frac{\pi}{2r}(1 - r + r - 1)) \\ &= \frac{r\omega_c}{\pi}\sin(\frac{\pi}{2r}2r) - \frac{r\omega_c}{\pi}\sin(\frac{\pi}{2r}2) \\ &= \frac{r\omega_c}{\pi}\sin(\pi) - \frac{r\omega_c}{\pi}\sin(0) \quad = \quad 0 \end{aligned} \tag{70}$$

That leaves

$$\|H_{RRC}\|^2 = \frac{2\pi}{\omega_c}(\omega_1 + \omega_2) = \frac{2\pi}{\omega_c}\omega_c \tag{71}$$

$$\Rightarrow \qquad \boxed{\|H_{RRC}\|^2 \quad = \quad 2\pi} \tag{72}$$

3. Energy in Time-Domain of Root-Raised Cosine Filter

$$
\begin{aligned}
\|f_{RRC}\|^2 &= \int_{-\infty}^{\infty} |f_{RRC}(t)|^2 \, \mathrm{d}t \\
&= \frac{2T_c}{\pi^2} \int_0^{\infty} \frac{(\sin((1-r)\pi\frac{t}{T_c}) + 4r\frac{t}{T_c}\cos((1+r)\pi\frac{t}{T_c})^2}{t^2(1-(4r\frac{t}{T_c})^2)^2} \, \mathrm{d}t \\
&= \frac{2T_c}{\pi^2} \int_0^{\infty} \frac{(\sin(\omega_1 t) + kt\cos(\omega_2 t))^2}{t^2\,(1-(kt)^2)^2} \, \mathrm{d}t
\end{aligned}
$$

with $\omega_1 = \dfrac{(1-r)\pi}{T_c}$ $\quad \omega_2 = \dfrac{(1+r)\pi}{T_c}$

$k = \dfrac{4r}{T_c}$

$$
\begin{aligned}
\|f_{RRC}\|^2 &= \frac{2T_c}{\pi^2}\Bigg(\int_0^{\infty} \frac{\sin^2(\omega_1 t)}{t^2\,(1-(kt)^2)^2} \, \mathrm{d}t \\
&+ 2k \int_0^{\infty} \frac{t\sin(\omega_1 t)\cos(\omega_2 t)}{t^2\,(1-(kt)^2)^2} \, \mathrm{d}t \\
&+ k^2 \int_0^{\infty} \frac{t^2\cos^2(\omega_2 t)}{t^2\,(1-(kt)^2)^2} \, \mathrm{d}t\Bigg) \\
&= \frac{2T_c}{\pi^2}\Bigg(\int_0^{\infty} \frac{\sin^2(\omega_1 t)}{t^2\,(1-(kt)^2)^2} \, \mathrm{d}t \\
&+ k \int_0^{\infty} \frac{t(\sin(\omega_c t) - \sin(r\omega_c t))}{t^2\,(1-(kt)^2)^2} \, \mathrm{d}t \\
&+ k^2 \int_0^{\infty} \frac{t^2\cos^2(\omega_2 t)}{t^2\,(1-(kt)^2)^2} \, \mathrm{d}t\Bigg)
\end{aligned} \tag{73}
$$

...

$$= 1$$