Minimum Edit Distance

How similar are two strings?

- Spell correction
  - The user typed “graffe”
    Which is closest?
    - graf
    - graft
    - grail
    - giraffe

- Also for Machine Translation, Information Extraction, Speech Recognition

Edit Distance

- The minimum edit distance between two strings
- Is the minimum number of editing operations
  - Insertion
  - Deletion
  - Substitution
- Needed to transform one into the other

Minimum Edit Distance

- Two strings and their **alignment**:

```
I N T E * N T I O N
| | | | | | | | | |
* E X E C U T I O N
```

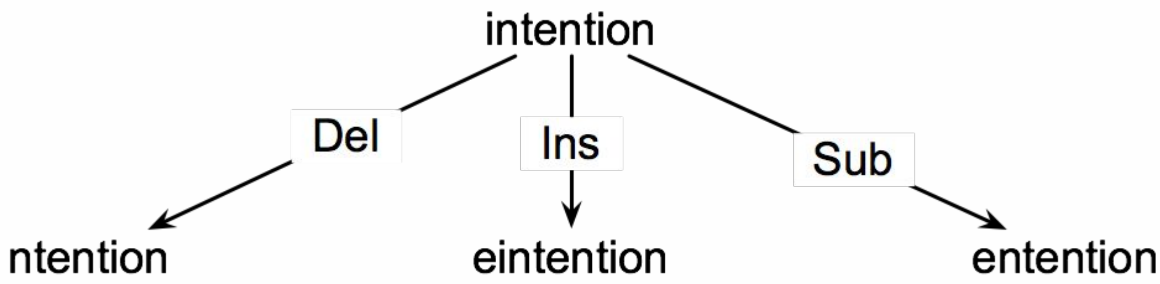
Minimum Edit Distance


```
I N T E * N T I O N
| | | | | | | | | |
* E X E C U T I O N
d s s   i s
```

- If each operation has cost of 1
  - Distance between these is 5
- If substitutions cost 2 (Levenshtein)
  - Distance between them is 8

How to find the Min Edit Distance?

- Searching for a path (sequence of edits) from the start string to the final string:
  - **Initial state**: the word we're transforming
  - **Operators**: insert, delete, substitute
  - **Goal state**: the word we're trying to get to
  - **Path cost**: what we want to minimize: the number of edits



Minimum Edit as Search

- But the space of all edit sequences is huge!
  - We can’t afford to navigate naïvely
  - Lots of distinct paths wind up at the same state.
    - We don’t have to keep track of all of them
    - Just the shortest path to each of those revisted states.

Defining Min Edit Distance

- For two strings
  - X of length n
  - Y of length m
- We define $D(i,j)$
  - the edit distance between $X[1..i]$ and $Y[1..j]$
    - i.e., the first i characters of X and the first j characters of Y
  - The edit distance between X and Y is thus $D(n,m)$

Dynamic Programming for Minimum Edit Distance

- **Dynamic programming**: A tabular computation of D(n,m)
- Solving problems by combining solutions to sub problems.
- Bottom-up
  - We compute D(i,j) for small i,j
  - And compute larger D(i,j) based on previously computed smaller values
  - i.e., compute D(i,j) for all i ($0 < i < n$) and j ($0 < j < m$)

Defining Min Edit Distance (Levenshtein)

- Initialization

```
D(i,0) = i
D(0,j) = j
```

- Recurrence Relation:

```
For each  i = 1…M
      For each  j = 1…N
                                    D(i-1,j) + 1
          D(i,j)=  min    D(i,j-1) + 1
                                     D(i-1,j-1)   +           2;
 if X(i) ≠ Y(j)

 0; if X(i) = Y(j)
```

- Termination:

```
D(N,M) is distance
```

The Edit Distance Table: Initialization

N	9									
O	8									
I	7									
T	6									
N	5									
E	4									
T	3									
N	2									
I	1									
#	0	1	2	3	4	5	6	7	8	9
	#	E	X	E	C	U	T	I	O	N

The Edit Distance Table

N	9									
O	8									
I	7									
T	6									
N	5									
E	4									
T	3									
N	2									
I	1	2								
#	**0**	1	2	3	4	5	6	7	8	9
	#	E	X	E	C	U	T	I	O	N

$$D(i,j) = \min \begin{cases} D(i-1,j) + 1 \\ D(i,j-1) + 1 \\ D(i-1,j-1) + \begin{cases} 2; & \text{if } S_1(i) \neq S_2(j) \\ 0; & \text{if } S_1(i) = S_2(j) \end{cases} \end{cases}$$

The Edit Distance Table

N	9	8	9	10	11	12	11	10	9	**8**
O	8	7	8	9	10	11	10	9	8	9
I	7	6	7	8	9	10	9	8	9	10
T	6	5	6	7	8	9	8	9	10	11
N	5	4	5	6	7	8	9	10	11	10
E	4	3	4	5	6	7	8	9	10	9
T	3	4	5	6	7	8	7	8	9	8
N	2	3	4	5	6	7	8	7	8	7
I	1	2	3	4	5	6	7	6	7	8
#	**0**	1	2	3	4	5	6	7	8	9
	#	E	X	E	C	U	T	I	O	N

Computing alignments

- Edit distance isn't sufficient
  - We often need to **align** each character of the two strings to each other
- We do this by keeping a "backtrace"
- Every time we enter a cell, remember where we came from
- When we reach the end,
  - Trace back the path from the upper right corner to read off the alignment

Edit Distance

N	9									
O	8									
I	7									
T	6									
N	5									
E	4									
T	3									
N	2									
I	1									
#	0	1	2	3	4	5	6	7	8	9
	#	E	X	E	C	U	T	I	O	N

$$D(i,j) = \min \begin{cases} D(i-1,j) + 1 \\ D(i,j-1) + 1 \\ D(i-1,j-1) + \begin{cases} 2; & \text{if } S_1(i) \neq S_2(j) \\ 0; & \text{if } S_1(i) = S_2(j) \end{cases} \end{cases}$$

MinEdit with Backtrace


```
I N T E * N T I O N
| | | | | | | | | |
* E X E C U T I O N
d s s   i s
```

n	9	↓ 8	↙←↓ 9	↙←↓ 10	↙←↓ 11	↙←↓ 12	↓ 11	↓ 10	↓ 9	↙ **8**	
o	8	↓ 7	↙←↓ 8	↙←↓ 9	↙←↓ 10	↙←↓ 11	↓ 10	↓ 9	↙ **8**	← 9	
i	7	↓ 6	↙←↓ 7	↙←↓ 8	↙←↓ 9	↙←↓ 10	↓ 9	↙ **8**	← 9	← 10	
t	6	↓ 5	↙←↓ 6	↙←↓ 7	↙←↓ 8	↙←↓ 9	↙ **8**	← 9	← 10	←↓ 11	
n	5	↓ 4	↙←↓ 5	↙←↓ 6	↙←↓ 7	↙←↓ **8**	↙←↓ 9	↙←↓ 10	↙←↓ 11	↙↓ 10	
e	4	↙ 3	← 4	↙← **5**	← **6**	← 7	←↓ 8	↙←↓ 9	↙←↓ 10	↓ 9	
t	3	↙←↓ 4	↙←↓ **5**	↙←↓ 6	↙←↓ 7	↙←↓ 8	↙ 7	←↓ 8	↙←↓ 9	↓ 8	
n	2	↙←↓ **3**	↙←↓ 4	↙←↓ 5	↙←↓ 6	↙←↓ 7	↙←↓ 8	↓ 7	↙←↓ 8	↙ 7	
i	**1**	↙←↓ 2	↙←↓ 3	↙←↓ 4	↙←↓ 5	↙←↓ 6	↙←↓ 7	↙ 6	← 7	← 8	
#	**0**	1	2	3	4	5	6	7	8	9	
	#	**e**	**x**	**e**	**c**	**u**	**t**	**i**	**o**	**n**	

Adding Backtrace to Minimum Edit Distance

- **Base conditions:**

```
D(i,0) = i
D(0,j) = j          D(N,M) is distance
```

- **Recurrence Relation:**

```
For each  i = 1…M
     For each  j = 1…N
```

insertion D(i-1,j) + 1

D(i,j)= min deletion D(i,j-1) + 1

D(i-1,j-1) +

2; if X(i) ≠ Y(j)

0; if X(i) = Y(j)

substitution

insertion

deletion

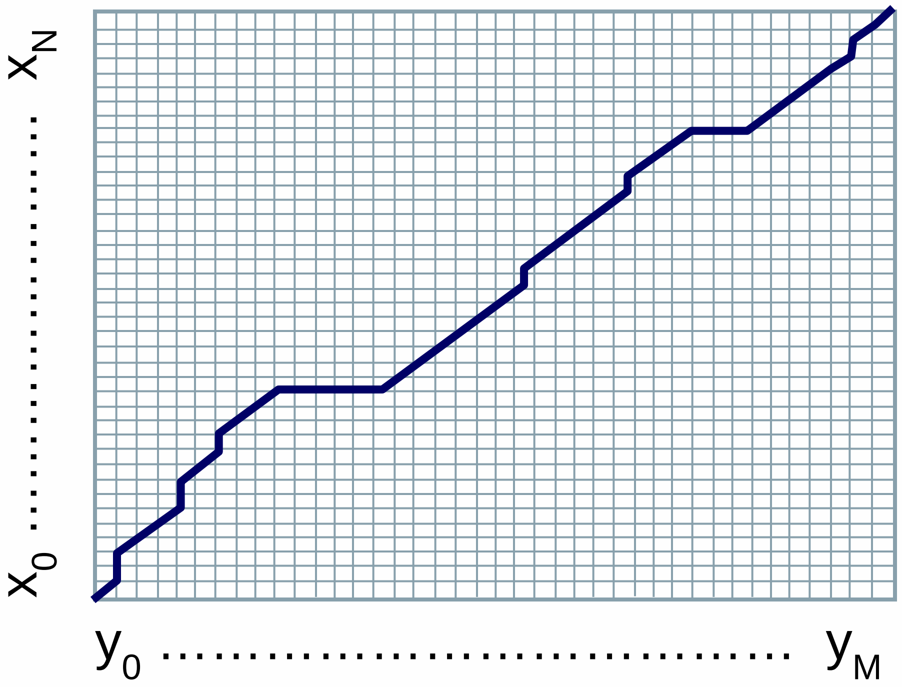
LEFT

ptr(i,j)= DOWN

substitution

DIAG

The Distance Matrix



Every non-decreasing path

from (0,0) to (M, N)

corresponds to
an alignment
of the two sequences

An optimal alignment is composed of optimal sub alignments

Result of Backtrace

- Two strings and their **alignment**:

```
I N T E * N T I O N
| | | | | | | | | |
* E X E C U T I O N
```

Performance

- Time:

  $O(nm)$

- Space:

  $O(nm)$

- Backtrace

  $O(n+m)$

THANK YOU