Homework 10

Exercise 1

Theorem 12.2: Suppose the sequence $\{x_n\}$ has a limit x. That means that if we go far enough out into the sequence (i.e. n > N_x), we can get arbitrariliy close to x. That is, $|x_{n>N_x}$ - x| < $\frac{\epsilon}{2}$. Similarly, for the sequence $\{y_n\}$, we can also get arbitrarily close to its limit y by going far enough out into the series (i.e $\forall_{n>N_y}$ $|y_n - y| < \frac{\epsilon}{2}$). We choose the larger of N_x and N_y and call this N. Then

$\forall_{n>N}$ $|x_n - x| < \frac{\epsilon}{2} \wedge |y_n - y| < \frac{\epsilon}{2}$

We add the n^{th} terms of the two sequence (where n > N) and subtract sum of their limits:

$|\{x_n\} + \{y_n\} - (x + y)| = |x_n - x + y_n - y|$. We know from previously that the first difference is < $\frac{\epsilon}{2}$ and the second difference is also < $\frac{\epsilon}{2}$.

Therefore:

$|x_n - x + y_n - y| < \left|\frac{\epsilon}{2} + \frac{\epsilon}{2}\right| = \epsilon$

Theorem 12.3: 12.2 By adding and subtracting x y_n and y y_n and rearraning terms we can obtain that

$|x (y - y_n| + (x - x_n)|\ |y_n\ y| + (x - x_n)y|$.

By the properties of the distance metric (i.e. ||) we know the above is less than

$\leq |x|\ |y - y_n| + |x\ x_n|\ |y_n\ y| + |x - x_n|\ |y|$. We pick an N large enough such that the terms with the factor x_n and y_n tend to be arbitrarily close to zero This can be done by picking ϵ such that

$|x - x_n| < \frac{\epsilon}{3(|y| + 1)}$

$|y - y_n| < \frac{\epsilon}{3(|x| + 1)}$

Substituting above, leads to $\lim_{n \to \infty} \{x_n\ y_n\} = xy$.

Theorem 12.4 Choose a positive ϵ such that it is less than xb. Then it must be the case that b < x ϵ. Construct an interval of width ϵ around x: (x - ϵ, x + ϵ), which will be to the right of b on the number line. For a large enough n, all x_n are inside this interval (that is b < x_n). This is a contradiction of x_n < b.

Exercise 2

Suppose that x_n converges to x. Pick some $\epsilon > 0$. Then $\exists_{N \in \mathbb{N}}$ n > N $\Rightarrow$ $|x_n - x| < \epsilon$. Then we can construct a ball B(x; ϵ) of radius ϵ centered on x such that all the terms for which $|x_n - x| < \epsilon$ is in this ball. Then the only terms not in this ball are the terms $x_1, x_2, ..., x_n$.

Exercise 3

Let g(x) = f(x) - x. Suppose g(a) $\geq$ 0 $\wedge$ g(b) $\leq$ 0. That is: g(a) - a $\geq$ 0 $\wedge$ g(b) - b $\leq$ 0. Since g is continuous, $\exists_c$ g(c) - c = 0. Therefore, there must be a c such that g(c) = c. so there must be a fixed point.

Intuitively, by f being continuous, we guarantee that f approaches x smoothly (that is, we can apply the intermediate value theorem). If f were discontinuous, g(x) might be 0 without “passing” the points in-between (that is taking on intermediate values around 0).

Exercise 4

Suppose that $\{s_n\}$ is a bounded, increasing monotone sequence. Let u be the supremum of $\{s_n\}$ and let ϵ be a positive number. Then $\exists_{N \in \mathbb{N}^+}$ u - ϵ < $s_N \leq$ u. Since $\{s_n\}$ is increasing, u - ϵ < $s_n \leq$ u for all n>N. Therefore, |u - s_n| < ϵ for all n>N and $s_n \to$ u. (http://math.uh.edu/~almus/4389_AN_ch2.pdf)

Similarly if $\{s_n\}$ is bounded decreasing monotone, there exists a lower limit.

Exercise 5

If a series converges then:
$\lim_{n\to\infty} \sum_{k=1}^{n} a_k$ = L.
This can be rewritten as:
$\lim_{n\to\infty} a_n$ + $\lim_{n\to\infty} \sum_{k=1}^{n-1} a_k$
The second limit also tends to L thus the above can be rewritten as:
$\lim_{n\to\infty} a_n$ + L
If this limit is to equal L, then $\lim_{n\to\infty} a_n$ must equal 0.
(http://math.stackexchange.com/questions/107961/if-a-series-converges-then-the-sequence-of-terms-converges-to-0)

Exercise 6

Consider the series associated with the geometric distribution
a_k = P(K=k) $= p\,(1-p)^k$, 0 < p < 1
We find the limit of the sequence as k $\to \infty$
$\lim_{k\to\infty} a_k$ = $\lim_{k\to\infty} p\,(1-p)^k$
Since p is a constant, we can rewrite as:
$\lim_{k\to\infty} p\,(1-p)^k$ = p $\lim_{k\to\infty} (1-p)^k$ = 0 (because |1-p| < 1

Exercise 7

Consider the geometric series $\sum_{k=0}^{\infty} x^k$, |x| < 1

This series converges to:

$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$

Differentiating (term by term), we obtain:

$\frac{d}{dx}\sum_{k=0}^{\infty} x^k = \frac{d\frac{1}{1-x}}{dx}$

$\sum_{k=0}^{\infty} k\, x^{k-1} = \frac{1}{(1-x)^2}$

Multiplying by x:

$\sum_{k=0}^{\infty} k\, x^k = \frac{x}{(1-x)^2}$

Now consider the geometric series given by

Pr(X=k) = $p\,(1-p)^k$

The expected value is:

$\sum_{k=1}^{\infty} \mathrm{k}\,(1-p)^k \mathrm{p} = \frac{p(1-p)}{(1-(1-p))^2} = \frac{p(1-p)}{p^2} = \frac{1-p}{p}$

Exercise 8

$Q_t^d = d_0 - d_1 P_t$

$Q_t^s = s_0 - s_1 P_{t-1}$

In equilbrium, supply equals demand:

$d_0 - d_1 P_t = s_0 - s_1 P_{t-1}$

$P_t = \frac{s_0 - d_0 - s_1 P_{t-1}}{-d_1}$

or

$P_t = \frac{d_0 - s_0}{d_1} - \frac{s_1}{d_1} P_{t-1}$

$P_{\mathrm{ss}} = \frac{1}{1+\frac{s_1}{d_1}} \frac{d_0 - s_0}{d_1}$

Simplifying:

$P_{\mathrm{ss}} = \frac{d_0 - s_0}{d_1 + s_1}$

The economic interpretation of the condition that $P_{\mathrm{ss}} > 0$ iff $d_0 > s_0$ is that there must consumer surplus (at 1 unit of production) must be strictly greater than the fixed cost of production for the market to exist.

Exercise 9

Any fixed point on a contracting mapping must be unique, because if there are two unique fixed points, the distance between these points will be the same in the image as in the preimage. However, for a contraction mapping, two points in the image must be closer than they are in the preimage.

Construct the series (x_t) by function iteration: $(x_t) = f^{\circ t}(x)$. Because f is a contraction mapping: $d(f(x_t),\ f(x)) \leq \lambda d(x_t,\ x)$, $\lambda \in [0, 1)$

or

$d(x_{t+1}, x) \leq \lambda d(x_t,\ x) \leq \lambda^{t+1}\, d(x_0,\ x)$

which tends to 0 because λ^{t+1} tends to 0. Thus $d(x_t, x) \leq 0$. Since distance must be positive: $\lim_{t\to\infty} d(x_t, x) = 0$.

Exercise 10

A contraction mapping is a mapping from a set to itself such that the distance between two points is always greater than the distance between their images.

f(x) = x/2, x $\in \mathbb{R}$ is a contraction mapping because the distance between two points in the image is always 1/2 the distance between them in the preimage. By Banach's theorem, there is a unique fixed point.

f(X) = 1/x, 0 < x< 1 is not a contraction mapping because the distance between two points in the image is always greater than the distance between them in the preimage.

f(x) = 1/(1+x) 0 < x < 1. Note that if x > 0 then 1/(1+x). Thus the distance between two points in the image is always less than the distance between two points in the preimage:

$\|(f(x_0) - f(x_1)\| = \frac{1}{1+x_0} - \frac{1}{1+x_1} = \frac{1+x_1+1+x_0}{(1+x_0)(1+x_1)} = \frac{x_1 - x_0}{1+x_1+x_0+x_0\, x_1}$. Since $0 < x_0 < 1$ and $0 < x_1 < 1$, $(x_0\, x_1) \ll 1$ (i.e. very small). Ignoring this term, we have:

$\frac{x_1 - x_0}{1+x_1+x_0}$ which must be $< \|x_1 - x_0\|$

By Banach's theorem, there is a unique fixed point.

Exercise 11

Consider the present value of the cash stream if the buyout is accepted:

$15000 - 200\left(\frac{1-(1.01)^{-60}}{.01}\right) = 15000 - 8991.01 = 6009.00$

Consider the present value of the alternative:

$200\left(\frac{1-(1.01)^{-60}}{.01}\right) = 8991.01$

Not accepting the buyout offer is better.

```
15000 - TimeValue[Annuity[200, 60], .01, 0]
```

```
6008.99
```

Computational Exercise 1

The effective annual yield is the rate that equates the return based on a nominal rate that is compounded at a frequency other than annually with the return with annual compounding.

For example, if a nominal rate is 10% a year compounded monthly, the effective annual rate is 10.47%.

```
pv1eay[i0_, t0_] := Module[{i = i0, t = t0}, (1 + i / 100) ^t]
```

```
pv1eay[10, 1]
```

$\frac{11}{10}$

```
pvstream[lst_, i_, pv1fn_] := Module[{list = lst, rate = i, f = pv1fn},
  Do[Print[N[f[rate, time] * list[[time]]]], {time, Length[list]}]]
```

```
pvstream[Table[1000, 10], 5, pv1eay]
```

```
1050.
1102.5
1157.63
1215.51
1276.28
1340.1
1407.1
1477.46
1551.33
1628.89
```

```
pv1cont[i0_, t0_] := Module[{i = i0, t = t0}, Exp[i / 100 * t]]
```

```
pv1cont[5, 10]
```

$\sqrt{e}$

```
pvstream[Table[1000, 10], 5, pv1cont]
```

```
1051.27
1105.17
1161.83
1221.4
1284.03
1349.86
1419.07
1491.82
1568.31
1648.72
```

```
payments = FoldList[#1 + 50 &, Table[1000, 10]]
```

```
{1000, 1050, 1100, 1150, 1200, 1250, 1300, 1350, 1400, 1450}
```

```
pvstream[payments, 5, pv1eay]
```

1050.

1157.63

1273.39

1397.83

1531.54

1675.12

1829.23

1994.56

2171.86

2361.9

```
pvstream[payments, 5, pv1cont]
```

1051.27

1160.43

1278.02

1404.61

1540.83

1687.32

1844.79

2013.96

2195.64

2390.65

Computational Exercise 2

```
Solve[5000 (1 + r)^2 == 5408 && r ≥ 0]
```

$$\left\{\left\{r \to \frac{1}{25}\right\}\right\}$$

```
10000. * (1 + 1 / 25)^3
```

11248.6

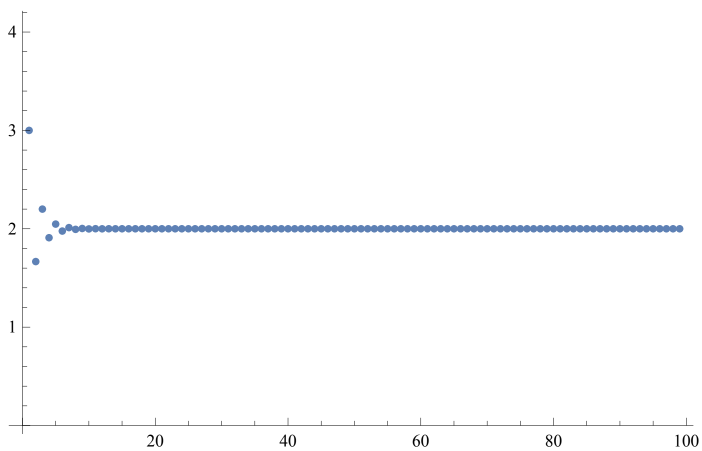
Computational Exercise 3

```
picard[fn_, p0_, eps_, t_] :=
  Module[{limit = t, f = fn, p = p0, e = eps, times = 0},
   result = {};
   While[True, (
     times = times + 1;
     If[times ≥ limit, Return[result]];
     p1 = p;
     p = f[p];
     If[Abs[p1 - p] < e, Return[result], AppendTo[result, p]];
    )]] ;
```

```
fp1 = picard[x Ì (x + 2.0) / x, 1, 1 E - 6, 100]
```

```
{3., 1.66667, 2.2, 1.90909, 2.04762, 1.97674, 2.01176, 1.99415, 2.00293, 1.99854,
 2.00073, 1.99963, 2.00018, 1.99991, 2.00005, 1.99998, 2.00001, 1.99999, 2.,
 2., 2., 2., 2., 2., 2., 2., 2., 2., 2., 2., 2., 2., 2., 2., 2., 2., 2., 2., 2.,
 2., 2., 2., 2., 2., 2., 2., 2., 2., 2., 2., 2., 2., 2., 2., 2., 2., 2., 2., 2.,
 2., 2., 2., 2., 2., 2., 2., 2., 2., 2., 2., 2., 2., 2., 2., 2., 2., 2., 2., 2.,
 2., 2., 2., 2., 2., 2., 2., 2., 2., 2., 2., 2., 2., 2., 2., 2., 2., 2., 2., 2.}
```

```
ListPlot[fp1]
```




```
fp2 = picard[x Ì (x + 1.0) / x, 1, 1 E - 6, 100]
```

```
{2., 1.5, 1.66667, 1.6, 1.625, 1.61538, 1.61905, 1.61765, 1.61818, 1.61798, 1.61806,
 1.61803, 1.61804, 1.61803, 1.61803, 1.61803, 1.61803, 1.61803, 1.61803,
 1.61803, 1.61803, 1.61803, 1.61803, 1.61803, 1.61803, 1.61803, 1.61803,
 1.61803, 1.61803, 1.61803, 1.61803, 1.61803, 1.61803, 1.61803, 1.61803,
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 1.61803, 1.61803, 1.61803, 1.61803, 1.61803, 1.61803, 1.61803, 1.61803}
```
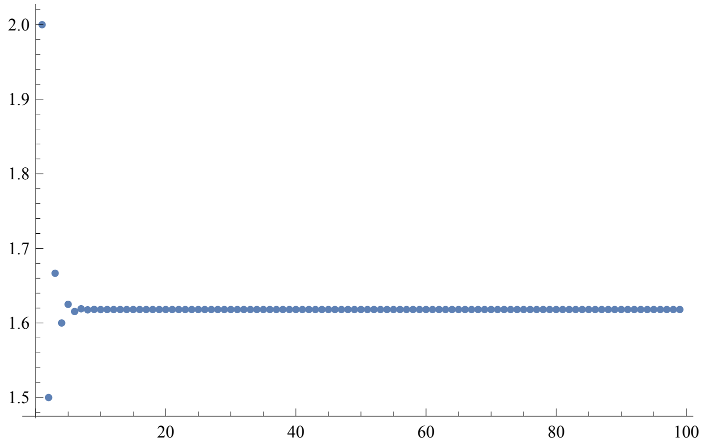

```
ListPlot[fp2]
```



Computational Exercise 4

When markets clear we have $Q_d = Q_s$. That is:

$500 - 2\,P_t = 50 + P_{t-1}$

$P_t = \frac{-P_{t-1}}{2} + 225$

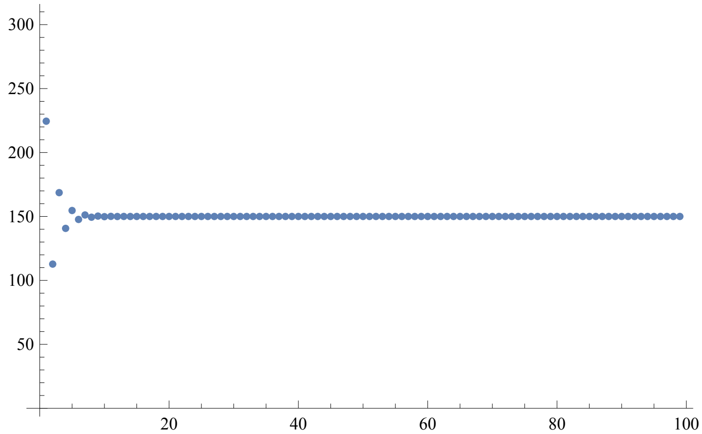
```
FullSimplify[Solve[500 - 2 pt == 50 + pt1, pt]]
```

$$\left\{\left\{\text{pt} \to 225 - \frac{\text{pt1}}{2}\right\}\right\}$$

```
fp3 = picard[p Ì -0.5 p + 225, 1, 1 E - 3, 100]
```

```
{224.5, 112.75, 168.625, 140.688, 154.656, 147.672, 151.164, 149.418, 150.291,
 149.854, 150.073, 149.964, 150.018, 149.991, 150.005, 149.998, 150.001, 149.999,
 150., 150., 150., 150., 150., 150., 150., 150., 150., 150., 150., 150., 150., 150.,
 150., 150., 150., 150., 150., 150., 150., 150., 150., 150., 150., 150., 150., 150.,
 150., 150., 150., 150., 150., 150., 150., 150., 150., 150., 150., 150., 150., 150.,
 150., 150., 150., 150., 150., 150., 150., 150., 150., 150., 150., 150., 150.,
 150., 150., 150., 150., 150., 150., 150., 150., 150., 150., 150., 150., 150.,
 150., 150., 150., 150., 150., 150., 150., 150., 150., 150., 150., 150., 150.}
```

```
ListPlot[fp3]
```



There is a steady state. The price convereges to ~$150.

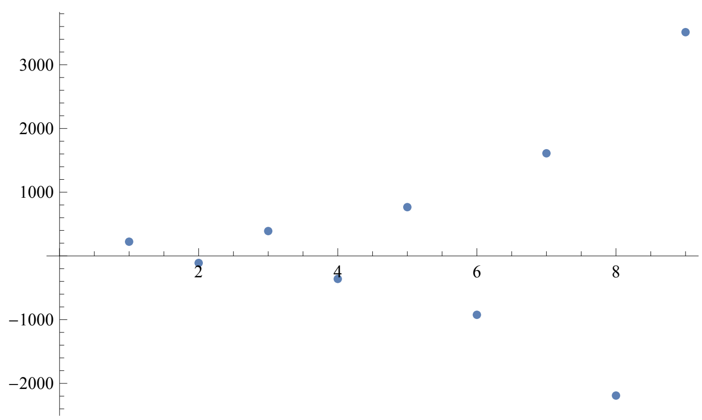
```
FullSimplify[Solve[500 - 2 pt == 50 + 3 pt1, pt]]
```

$$\left\{\left\{pt \to 225 - \frac{3\,pt1}{2}\right\}\right\}$$

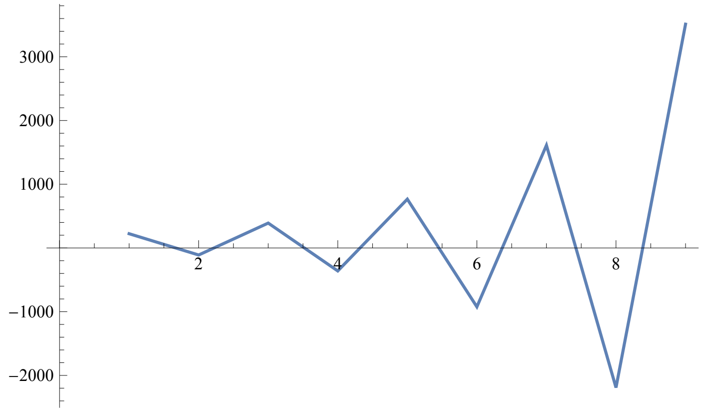
```
fp4 = picard[p Ì 225 - (3. * p / 2), 1, 1 E - 3, 10]
```

```
{223.5, -110.25, 390.375, -360.563, 765.844, -923.766, 1610.65, -2190.97, 3511.46}
```

```
ListPlot[fp4]
```



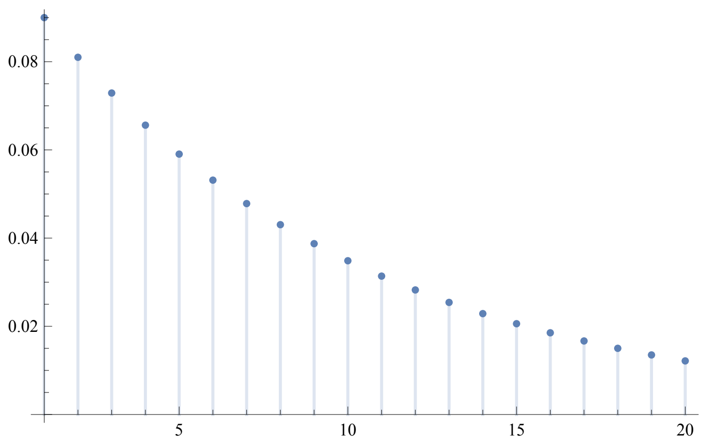
```
ListLinePlot[fp4]
```




```
GeometricDistribution[0.10]
```

```
GeometricDistribution[0.1]
```

```
DiscretePlot[PDF[GeometricDistribution[0.1], x], {x, 20}]
```




```
Histogram[RandomVariate[GeometricDistribution[0.1], 1000]]
```