Homework 7

Exercise 1

$Y = A(Y) + T$

$T = T(Q, Y, Y^*)$

$dY = \frac{T_Q\, dQ + T^*\, dY^*}{a_y + T_Y - 1}$

$dT = \frac{(a_Y - 1)\,(T_Q\, dQ + T^*\, dY^*)}{a_Y + T_Y - 1}$

In[75]:=
```
demand = y == a[y] + t[q, y, yStar]
tb  = t == t[q, y, yStar]
notationRules  =
  {Dt[y] -> dY, a[y] -> a_y, Dt[yStar] → dYStar, t^(0,0,1)[q, y, yStar] → T_Ystar,
   t^(0,1,0)[q, y, yStar] → T_y, Dt[q] → dQ, t^(1,0,0)[q, y, yStar] → T_Q, Dt[t] → dT};
Dt[demand] /. notationRules // TraditionalForm
Dt[tb] /. notationRules // TraditionalForm
Solve[Dt[demand] && Dt[tb], {Dt[y], Dt[t]}] /. notationRules // TraditionalForm
```

Out[75]= `y == a[y] + t[q, y, yStar]`

Out[76]= `t == t[q, y, yStar]`

Out[78]//TraditionalForm=

$$\mathrm{dY} = \mathrm{dY}\, a'(y) + \mathrm{dQ}\, T_Q + \mathrm{dY}\, T_y + \mathrm{dYStar}\, T_{\mathrm{Ystar}}$$

Out[79]//TraditionalForm=

$$\mathrm{dT} = \mathrm{dQ}\, T_Q + \mathrm{dY}\, T_y + \mathrm{dYStar}\, T_{\mathrm{Ystar}}$$

Out[80]//TraditionalForm=

$$\left\{\left\{\mathrm{dY} \to -\frac{\mathrm{dQ}\, T_Q + \mathrm{dYStar}\, T_{\mathrm{Ystar}}}{a'(y) + T_y - 1},\ \mathrm{dT} \to \frac{(a'(y) - 1)\,(\mathrm{dQ}\, T_Q + \mathrm{dYStar}\, T_{\mathrm{Ystar}})}{a'(y) + T_y - 1}\right\}\right\}$$

Exercise 2

$\mathcal{L} = \sqrt{xy} + \lambda\,(p_x\, x + p_y y - 100)$

$\frac{\partial \mathcal{L}}{\partial x} = \frac{y}{2\sqrt{x}} + \lambda\, p_x = 0$

$\frac{\partial \mathcal{L}}{\partial x} = \frac{x}{2\sqrt{y}} + \lambda\, p_y = 0$

$\frac{\partial \mathcal{L}}{\partial \lambda} = p_x\, x + p_y\, y - 100 = 0$

$\lambda = \frac{x}{-2\, p_Y \sqrt{y}}$

$x == \frac{50}{px},\ y == \frac{50}{py},\ \lambda == \frac{50}{px\, py}$

$\epsilon_{x,p_x} = \frac{-50}{x}$, $\epsilon_{y,p_y} = \frac{50}{y}$

$\epsilon_{x,p_y} = 0$, $\epsilon_{y,p_x} = 0$

Exercise 3

Assume that R is transitive.

Assume that R is asymmetric.

Assume that R is cyclic.

For some k > 1, x_1 R x_2, x_2 R x_3,... x_{k-1}R $x_{k-1} \wedge x_k$ R x_1 (a cycle of length k)

But from transitivity we have:

x_{k-2} R $x_{k-1} \wedge x_{k-1}$ R $x_k \Rightarrow x_{k-2}$ R $x_k\ \forall_k$

$\therefore x_1$R x_k

By asymmetry: x_1 R $x_k \Rightarrow \neg x_k$R x_1

but then contradiction!

$\therefore$ Either R is not cyclic or R is instransitive.

Exercise 4

By definition P is antisyemmtric. Thus, x_1 P $x_n \Rightarrow \neg\ x_n$ P $x_1 \vee x_1 = x_n$. Kreps says $x_1 \neq x_n$. $\therefore$ it must be the case that $\neg x_n$ P x_1.

Carter (2001) definition says that if x_n is “chained to” x_1 by the asymmetric component, then it must be the case that x_n is (directly) related to x_1 (for the whole relation R, not the antisymmetric relation P). This implies $x_1 \sim x_n$ (because R = P + ~).

Exercise 5

It is not complete. Consider $(x_1, x_2) = (10, 11)$ and $(y_1, y_2) = (11, 10)$. This is not in $\gtrsim\ \subseteq \mathbb{R}^2$.

$\gtrsim$ is transitive. Consider the case (x_1, x_2) R (y_1, y_2) R (z_1, z_2). We can rewrite this as: $(x_1 \geq y_1 \wedge y_1 \geq z_1) \wedge (x_2 \geq y_2 \wedge y_2 \geq z_2)$. Since all x’s, y’s, and z’s are all real numbers, $x_1 \geq z_1$ and $x_2 \geq z_2$. Therefore (x_1, x_2) R (z_1, z_2).

The asymmetric subrelation $>$ is irreflexive. Consider the case $(x_1, x_2) > (y_1, y_2)$. This means $x_1 > y_1$ and $x_2 > y_2$. Since all x’s and y’s are real numbers, then $(x_1 > y_1) \wedge (x_2 > y_2) \Rightarrow\ \neg\,[x_1 = x_2] \wedge \neg\,[y_1 = y_2]$. Therefore $(x_1, x_2) > (y_1, y_2) \Rightarrow (y_1, y_2) \neq (x_1, x_2)$.

The asymmetric subrelation $>$ is asymmetric. Consider the case $(x_1, x_2) > (y_1, y_2)$. This means $x_1 > y_1$ and $x_2 > y_2$. Since all x’s and y’s are real numbers, then $(x_1 > y_1) \wedge (x_2 > y_2) \Rightarrow\ \neg[(y_2 > x_2) \wedge (y_1 > x_1)]$. Therefore $(x_1, x_2) > (y_1, y_2) \Rightarrow \neg\,(y_1, y_2) > (x_1, x_2)$.

The asymmetric subrelation is also acyclical (by transitivity).

Exercise 6

If a relation is reflexive, then it has diagonals equal to 1. When the matrix **R** is raised to the first power, we have the case i = j ⇒ $r_{i,j} = 1$. For the squared matrix the $r_{i,j}^2$ element is given by $r_i \cdot r_j^T$ where r_i is the i^{th} row of the original matrix and r_j^T is the (transpose of the) j^{th} column (and · is the dot product). Thus, we can see that if xRy and yRz in the original relationship, than xR^2z in the new relationship (and so on for larger relationships). By the time the matrix is raised to the N^{th} power, we will have the case that x_1 R $x_2 \wedge x_2$ R $x_3 \wedge \ldots x_{N-1}$ R $x_N \Rightarrow x_1\ R^N\ x_N$.

Exercise 7

A topological space (X, T) is a set X along with a collection of subsets T of X (called the open subsets of X) which satisfies the following condtions

1. $\emptyset \in T \wedge X \in T$.
3. $V_1, V_2 \in T \Rightarrow V_1 \cap V_2 \in T$
4. $\forall_\alpha \in I$, $V_\alpha \in T \Rightarrow \bigcup_{a \in I} V_\alpha \in T$

An algebra is a set X along with a collection of subsets of Σ of X such that;

1. $X \in \Sigma$.
2. $A \in \Sigma \Rightarrow A^C \in \Sigma$
3. If A_n is a sequence of elements of Σ, then $\bigcup_{n=1}^N A_n \in \Sigma$

The topological space is closed under arbitrary union, while an algebra is closed under countable union.

Exercise 8

1. T is (by definition of topology) a collection of open sets of X. By definition, ∅ and X are in T. Therefore, they must be open. However, X^c is ∅ and $\emptyset^C$ = X. Therefore these must also be open.

2. A closed set A^C is the open open set A. The complement of this set $(A^C)^C$, which is A. This is open.

3. Consider the two closed sets A^c and B^C. There union is $A^C \cup B^C$. By Demorgan's Law this is $(A^C)^C \cap (B^C)^C$ which is A ∩ B. By previous A, B are open. Therefore A ∩ B must be in T (by definition of topology). Thus the union of closed sets is in T.

4. Again consider A^C and B^C. $(A^C) \cap (B^C)$ may be rewritten as A ∪ B. By previous A, B are open. Therefore A ∪ B must be in T (by definition of topology)

Computational Exercise 1

In[210]:=
```
ClearAll[x, r, r2, r3]
r = {{0, 1, 0}, {0, 0, 1}, {1, 0, 0}};
r2 = MatrixPower[r, 2];
r3 = MatrixPower[r, 3];
Print["The original relationship R in matrix form"]
r // MatrixForm
Print["R^2 is produced by multplying the R with itself"]
r2 // MatrixForm
Print["R^3 is produced by multplying R with R^2"]
r3 // MatrixForm
Print[
 "The transitive closure of R is produced by adding R^2 (all the relations that can
   be gotten to in 2 steps) with R"]
rt = r + r2;
rt // MatrixForm
```

The original relationship R in matrix form

Out[215]//MatrixForm=

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

R^2 is produced by multplying the R with itself

Out[217]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

R^3 is produced by multplying R with R^2

Out[219]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The transitive closure of R is produced by adding
R^2 (all the relations that can be gotten to in 2 steps) with R

Out[222]//MatrixForm=

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Computational Exercise 2

In[223]:=
```
preferred[{x1_, x2_}, {y1_, y2_}] := If[x1 ≥ y1 ∧ x2 ≥ y2, True, False]
```

In[224]:=
```
preferred[{100, 100}, {0, 0}]
```
Out[224]=
```
True
```
In[225]:=
```
preferred[{49, 51}, {51, 49}]
```
Out[225]=
```
False
```
In[279]:=
```
incomes = Table[{RandomInteger[{0, 10}], RandomInteger[{0, 10}]}, 1000];

(* The maximal elements are the ones where x1 = x2 *)
symmetric[{x1_, x2_}] := x1 == x2
```
In[281]:=
```
symmetric[{1, 1}]
```
Out[281]=
```
True
```
In[282]:=
```
symmetric[{1, 2}]
```
Out[282]=
```
False
```
```
maximal = Select[incomes, symmetric] (* These are all the maximal components *)
Last[SortBy[maximal, First]] (* This is the best element *)
```
Out[286]=
```
{{7, 7}, {0, 0}, {9, 9}, {5, 5}, {1, 1}, {10, 10}, {7, 7}, {0, 0}, {5, 5}, {0, 0}, {6, 6},
 {9, 9}, {0, 0}, {3, 3}, {7, 7}, {1, 1}, {9, 9}, {4, 4}, {7, 7}, {7, 7}, {8, 8},
 {2, 2}, {3, 3}, {10, 10}, {3, 3}, {5, 5}, {10, 10}, {1, 1}, {3, 3}, {2, 2}, {4, 4},
 {5, 5}, {6, 6}, {7, 7}, {9, 9}, {6, 6}, {3, 3}, {2, 2}, {3, 3}, {7, 7}, {1, 1},
 {0, 0}, {5, 5}, {8, 8}, {7, 7}, {7, 7}, {2, 2}, {5, 5}, {5, 5}, {0, 0}, {1, 1},
 {0, 0}, {4, 4}, {2, 2}, {10, 10}, {1, 1}, {9, 9}, {4, 4}, {5, 5}, {1, 1}, {0, 0},
 {4, 4}, {3, 3}, {9, 9}, {2, 2}, {3, 3}, {3, 3}, {1, 1}, {6, 6}, {10, 10}, {10, 10},
 {3, 3}, {6, 6}, {1, 1}, {8, 8}, {2, 2}, {3, 3}, {7, 7}, {4, 4}, {8, 8}, {10, 10},
 {6, 6}, {0, 0}, {1, 1}, {4, 4}, {8, 8}, {4, 4}, {9, 9}, {2, 2}, {5, 5}, {9, 9}}
```
Out[287]=
```
{10, 10}
```