Homework 3

Exercise 1

Non-mathematica

$P_A = f(c_A, P_B)$
$P_B = g(c_B, P_A)$

$dP_A = f_{C_A} dC_A + f_{P_B} dP_B$
$dP_B = g_{C_B} dC_B + g_{P_A} dP_A$
where $h_x = \frac{\partial h}{\partial x}$

We can rewrite these as:
$d P_A - f_{P_B} \, d P_B = f_{C_A} \, d C_A$
$d P_B - g_{P_A} \, d P_A = g_{C_B} \, d C_B$

Or in matrix form:
$$\begin{pmatrix} 1 & -f_{P_B} \\ -g_{P_A} & 1 \end{pmatrix} \begin{pmatrix} d P_A \\ d P_B \end{pmatrix} = \begin{pmatrix} f_{C_A} \, d C_A \\ g_{C_B} \, d C_B \end{pmatrix}$$

If we invert the matrix of coefficients we obtain

$\frac{1}{\Delta} \begin{pmatrix} 1 & f_{P_B} \\ g_{P_A} & 1 \end{pmatrix}$, where $\Delta = 1 - (f_{P_B}) \, (g_{P_A})$

Multiplying by the inverse, we obtain
$$\begin{pmatrix} d P_A \\ d P_B \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} 1 & f_{P_B} \\ g_{P_A} & 1 \end{pmatrix} \begin{pmatrix} f_{C_A} \, d C_A \\ g_{C_B} \, d C_B \end{pmatrix}$$

$$\begin{pmatrix} d P_A \\ d P_B \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} f_{C_A} \, d C_A + f_{P_B} \, g_{C_B} \, d C_B \\ g_{P_A} \, f_{C_A} \, d C_A + g_{C_B} \, d C_B \end{pmatrix}$$

The comparative statics are as follows (assuming, $\frac{d C_B}{d C_A} = 0$, $\frac{d C_A}{d C_B} = 0$):

$\frac{d P_A}{d C_A} = f_{C_A} + f_{P_B} \left(g_{C_B} \frac{d C_B}{d C_A} + g_{P_A} \frac{d P_A}{d C_A}\right)$
$\frac{d P_A}{d C_A} = f_{C_A} + f_{P_B} \, g_{P_A} \frac{d P_A}{d C_A}$

$\frac{dP_A}{dC_A}$ - $f_{P_B}\, g_{P_A} \frac{dP_A}{dC_A} = f_{C_A}$

$\left(\frac{dP_A}{dC_A}\right)(1 -\ f_{P_B}\, g_{P_A}) =\ f_{C_A}$

$\frac{dP_A}{dC_A} = \frac{f_{C_A}}{(1 - f_{P_B}\, g_{P_A})}$

Similarly

$\frac{dP_B}{dC_B} = \frac{f_{C_B}}{(1 - g_{P_B}\, f_{P_A})}$

$\frac{dP_A}{dC_B} = f_{C_A} \frac{dC_A}{dC_B} + f_{P_B}(g_{C_B} + g_{P_A} \frac{dP_A}{dC_B})$

$\frac{dP_A}{dC_B} = f_{P_B}(g_{C_B} + g_{P_A} \frac{dP_A}{dC_B})$

$\frac{dP_A}{dC_B} - g_{P_A} \frac{dP_A}{dC_B} = f_{P_B} g_{C_B}$

$\left(\frac{dP_A}{dC_B}\right)(1 - g_{P_A}) = f_{P_B} g_{C_B}$

$\frac{dP_A}{dC_B} = \frac{f_{P_B}\, g_{C_B}}{(1 - g_{P_A})}$

Similarly

$\frac{dP_B}{dC_A} = \frac{g_{P_A}\, f_{C_A}}{(1 - f_{P_B})}$

Assuming

$f_{C_A} \geq 0$ (Firm A increases [weakly] prices due to higher production costs)

$f_{P_B} \geq 0$ (Firm A increases [weakly] prices due to higher prices from firm B)

$g_{C_B} \geq 0$ (Firm B increases [weakly] increases prices due to higher production costs)

$g_{P_A} \geq 0$ (Firm B increases [weakly] prices due to higher prices from firm A)

We obtain the following signs

$\frac{dP_A}{dC_A} \geq 0$

$\frac{dP_B}{dC_B} \geq 0$

$\frac{dP_A}{dC_B} \geq 0$

$\frac{dP_B}{dC_A} \geq 0$

Mathematica

Exercise 2

Maximize $x^{0.5}\, y^{0.5}$ such that 2 x + y = k

$\mathcal{L}(x, y, \lambda) = x^{0.5}\, y^{0.5} + \lambda\,(2x + y - k)$

$\frac{\partial \mathcal{L}}{\partial x} = \frac{\sqrt{y}}{2\sqrt{x}} + 2\,\lambda = 0$

$\frac{\partial \mathcal{L}}{\partial y} = \left(\frac{\sqrt{x}}{2\sqrt{y}}\right) + \lambda = 0$

$\frac{\partial \mathcal{L}}{\partial \lambda} = 2\,x + y - k = 0$

$\lambda = \frac{-\sqrt{x}}{2\sqrt{y}}$

$2\,\lambda = \frac{-\sqrt{x}}{\sqrt{y}}$

$\frac{\partial \mathcal{L}}{\partial x} = \left(\frac{\sqrt{y}}{2\sqrt{x}}\right) - \frac{\sqrt{x}}{\sqrt{y}} = 0$

$\left(\frac{\sqrt{y}}{2\sqrt{x}}\right) = \frac{\sqrt{x}}{\sqrt{y}}$

y = 2 x

-2 x + y = 0

2 x + y - k = 0

2x + y = k

$$\begin{pmatrix} -2 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ k \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{-1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 \\ k \end{pmatrix} = \begin{pmatrix} \frac{-1}{4}\,0 + \frac{1}{4}\,k \\ \frac{1}{2}\,0 + \frac{1}{2}\,k \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{k}{4} \\ \frac{k}{2} \end{pmatrix}$$

Exercise 3

$C(n+1, k+1) = \frac{(n+1)!}{(k+1)!\,(n-k)!}$

$C(n, k) = \frac{n!}{k!\,(n-k)!} = \frac{n!}{k!\,(n-k)\,(n-k-1)!} = \frac{n!}{k!\,(n-k-1)!}\ \frac{1}{n-k}$

$C(n, k+1) = \frac{n!}{(k+1)!\,(n-k-1)!} = \frac{n!}{(k+1)\,k!\,(n-k-1)!} = \frac{n!}{k!\,(n-k-1)!}\ \frac{1}{k+1}$

$C(n, k) + C(n, k+1) = \frac{n!}{k!\,(n-k-1)!}\ \frac{1}{k+1} + \frac{n!}{k!\,(n-k-1)!}\ \frac{1}{n-k} = \frac{n!}{k!\,(n-k-1)!}\left(\frac{1}{k+1} + \frac{1}{n-k}\right)$

$= \frac{n!}{k!\,(n-k-1)!}\left(\frac{n-k+k+1}{(k+1)\,(n-k)}\right) = \frac{n!}{k!\,(n-k-1)!}\left(\frac{n+1}{(k+1)\,(n-k)}\right) = \frac{(n+1)!}{(k+1)!\,(n-k)!} = C(n+1, k+1)$

Exercise 4

An algebra of sets (σ algebra) of a set X is a collection Σ of subsets of X that is closed under complementation, union of countably many sets and intersection of countably many sets.

Ω = {1, 2, 3, 4, 5, 6}

E1 = {2, 4, 6}

E2 = {1, 3, 5}
Σ = {∅, E1, E2, Ω} is an algebra of sets over Ω because:
$\varnothing^C = \Omega$ (member of Σ)
$\Omega^C = \varnothing$ (member of Σ)
$E1^C = E2$ (member of Σ)
$E2^C = E1$ (member of Σ)
The intersection of ∅ with any of the other elements in the collection is ∅
The intersection of Ω with any of the other elements in the collection is that element
The union of ∅ with any other of the other elements in the collection is that element
The union of Ω with any other element in the collection is Ω
$E1 \cup E2 = \Omega$ (member of Σ)
$E1 \cap E2 = \varnothing$ (member of Σ)

Thus the set is closed under the required operations.

Ω = {1, 2, 3, 4, 5, 6}
E1 = {2, 4}
E2 = {1, 3}
T = {∅, E1, E2, Ω} is not an algebra of sets over Ω because $E1^C = \{1, 3, 5, 6\}$ is not a member of T.

Exercise 5

If $\mathcal{A}$ is a σ algebra:
1. If $\mathcal{A}$ is an algebra of sets over Ω then $\Omega, \varnothing \in \mathcal{A}$
Proof:
$\Omega^C = \varnothing$ and vice versa. Ω must be in $\mathcal{A}$ by finite unions of elements of $\mathcal{A}$ for $\mathcal{A}$ to be cosed under finite unions (by definition of closure).
Another way to prove this is E1 and $E1^C$ must be in $\mathcal{A}$ (by definiion). $E1 \cap E1^C$ is ∅. Thus ∅ and $\varnothing^C = \Omega$ must also be in $\mathcal{A}$.
2. If $E1 \in \mathcal{A}$ and $E2 \in \mathcal{A}$ then $E1 \cap E2 \in \mathcal{A}$
Proof
If $E1 \cap E2$ were not in $\mathcal{A}$ then it would not be a σ algebra by definition.
Another proof is that $E1^C$ and $E2^C$ must be in $\mathcal{A}$. $E1 \cap E2 = (E1^C \cup E2^C)^C$. $(E1^C \cup E2^C)^C$ must also be in $\mathcal{A}$.
3. If $E1 \in \mathcal{A}$ and $E2 \in \mathcal{A}$ then $E1 \setminus E2 \in \mathcal{A}$
Proof
$E1 \setminus E2 = E1 \cap E2^C$.
Thus can be rewritten as $(E1^C \cup E2)^C$. $E1^C$ is in $\mathcal{A}$. Therefore $(E1^C \cup E2)$ and its complement $(E1^C \cup E2)^C$ must be as well.
4. If $E_n \in \mathcal{A}$, then $\cup_n^N E_n \in \mathcal{A}$
Proof by induction:
Base case: If $E_1 \in \mathcal{A} \land E_2 \in \mathcal{A} \longrightarrow E_1 \cup E_2 \in \mathcal{A}$ (by definition)

Inductive case: If $\bigcup_{n \in N} E_{n-1} \in \mathcal{A} \wedge E_n \in \mathcal{A} \longrightarrow \bigcup_{n \in N} E_{n-1} \cup E_n \in \mathcal{A}$ (by definition)
Therefore: $\bigcup_{n \in N} E_n$

If $E_n \in \mathcal{A}$, then $\bigcap_n^N E_n \in \mathcal{A}$
Proof by induction:
Base case: If $E_1 \in \mathcal{A} \wedge E_2 \in \mathcal{A} \longrightarrow E_1 \cap E_2 \in \mathcal{A}$ (previously proved)
Inductive case: If $\bigcup_{n \in N} E_{n-1} \in \mathcal{A} \wedge E_n \in \mathcal{A} \longrightarrow \bigcup_{n \in N} E_{n-1} \cap E_n \in \mathcal{A}$ (by previous proof)
Therefore: $\bigcap_n^N E_n$

Exercise 6

1. The probability of any event occuring is at least 0.
2. The probability that one of the events occuring (from the list of all possible events) occuring is 1.
3. If events don't overlap (example of overlapping event: roll an even number or 2), then the probability of the events occuring is the sum of their individual probabilities.

Properties
1. P(E1 $\cup$ E2) = P(E1) + P(E2)
Proof
$P(\bigcup_n^\infty E_N) = \sum_{n=1}^\infty P(E_n)$ (axiom 3)
Let $E_i = \emptyset \; \forall$ i $\in \mathbb{N}$, i > 2
$P(\bigcup_n^\infty E_N)$ = P(E1 $\cup$ E2 $\bigcup_3^\infty E_n$) = $P(E_1) + P(E_2) + P(E_3) + P(E_4) + \ldots$
P(∅) = P(∅) + P(∅) + ... iff P(∅) = 0 (otherwise the sum diverges)
$\therefore$ P(E1 $\cup$ E2) = $P(E_1) + P(E_2)$

2. $P(\bigcup_n^N E_n) = \sum_{i=1}^N P(E_n)$
by property 1 $P(E_{n-1} \cup E_n) = P(E_{n-1}) + P(E_n)$
by propertyb 4 of a σ algebra $\bigcup_n^N E_n \in \Sigma$
$\therefore P(\bigcup_n^N E_n) = \sum_{i=1}^N P(E_n)$

3. $P(E^C)$ = 1 - P(E)
$P(E^C \cup E) = P(\Omega) = 1 = P(E^C) + P(E)$
$\therefore P(E^C) = 1 - P(E)$

4. E1 $\subseteq$ E2, P(E1) $\leq$ P(E2)
E1 = E2 $\cup$ (E1 $\cup$ E2^C)
P(E1) = P(E2) + P(E1 $\cup$ E2^C) $\geq$ P(E2)

5. P(A $\cup$ B) = P(A) + P(B) - P(A $\cap$ B)
Proof
P(A $\cup$ B) =

P((A \ B) ∪ (B \ A) ∪ (A ∩ B)) =
P((A ∩ B^C) ∪ (B ∪ A^C) ∪ (A ∩ B)) =
P(A ∩ B^C) + (B ∪ A^C) + P (A ∩ B)
P(A) + P(B) - P (A∩B)

Computational Exercise 1

```
In[84]:= ClearAll[lst, x, l, square, newList]
lst = Range[0, 15]
newList = {};
Do[AppendTo[newList, x^2], {x, lst}];
StringForm["The squared list using loops: ``", newList]
square [l_] := (
  y = {};
  Do[AppendTo[y, x^2], {x, l}];
  Return[y]
 )
StringForm["The squared list using a function: ``", square[lst]]
StringForm["The squared list using map: ``", Map[#^2 &, lst]]
isEven[x_] := If[EvenQ[x], 1, 0]
StringForm["isEven[9] = ``", isEven[9]]
StringForm["isEven[10] = ``", isEven[10]]
StringForm["Picking out the even itmes from the list ``",
 Pick[lst, Map[isEven, lst], 1]]
```

```
Out[85]= {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15}

Out[88]= The squared list using loops:
  {0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225}

Out[90]= The squared list using a function:
  {0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225}

Out[91]= The squared list using map:
  {0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225}

Out[93]= isEven[9] = 0

Out[94]= isEven[10] = 1

Out[95]= Picking out the even itmes from the list {0, 2, 4, 6, 8, 10, 12, 14}
```

Computational Exercise 2

```
ClearAll[powerSet, x, y, subsets]
powerSet[s_List] := (
  subsets = {{}};
  Do[(
    newSubsets = {};
    Do[(
      newSubset = Union[oldSubset, {element}];
      AppendTo[newSubsets, newSubset];
     ), {oldSubset, subsets}
    ];
    subsets = Join[subsets, newSubsets];
   ), {element, s}];
  Return[subsets]
 )
powerSet[{e1, e2, e3}]
```

```
{{}, {e1}, {e2}, {e1, e2}, {e3}, {e1, e3}, {e2, e3}, {e1, e2, e3}}
```

Computational Exercise 3

```
In[277]:= ClearAll[fiveFlips]
fiveFlips := RandomChoice[{45, 100 - 45} → {0, 1}, 5];
results = {};
Do[AppendTo[results, fiveFlips], 1000];
headCounts = Sort[Map[Total, results]];
counts = 1/1000. * BinCounts[headCounts, {0, 6, 1}]
StringForm["The empirical PMF is as follows \nP(0) = `1`,
   P(1) = `2`, P(2) = `3`, P(3) = `4`, P(4) = `5`, P(5) = `6`",
 counts[[1]], counts[[2]], counts[[3]], counts[[4]], counts[[5]], counts[[6]]]
```

```
Out[282]= {0.015, 0.121, 0.283, 0.333, 0.199, 0.049}
```

```
Out[283]= The empirical PMF is as follows
P(0) = 0.015`, P(1) = 0.121`, P(2) =
  0.28300000000000003`, P(3) = 0.333`, P(4) = 0.199`, P(5) = 0.049`
```

Computational Exercise 4

In[284]:=

```
fiveDraws = RandomVariate[NormalDistribution[], 5]; (* Take five draws *)
twentyFiveDraws = RandomVariate[NormalDistribution[], 25];
(* Take twenty five draws *)
oneTwentyFiveDraws = RandomVariate[NormalDistribution[], 125];
(* Take one hunder and twenty five draws *)
d5 = EmpiricalDistribution[fiveDraws]; (* Convert to a random distribution *)
d25 = EmpiricalDistribution[twentyFiveDraws];
(* Convert to a random distribution *)
d125 = EmpiricalDistribution[oneTwentyFiveDraws];
(* Convert to a random distribution *)
(* Plot *)
Plot[{CDF[d5, x], CDF[d25, x], CDF[d125, x], CDF[NormalDistribution[], x]},
 {x, -6, 6}, PlotLegends → {"5 Draws", "25 Drsws", "125 Draws", "Theoretical"},
 PlotStyle → Thick]
```

Out[290]=