Homework 5

Exercise 1

$$\begin{pmatrix} -a_r & 0 \\ -l_{\Pi+r} & 1 \end{pmatrix} \begin{pmatrix} dr \\ dm \end{pmatrix} = \begin{pmatrix} a_f\, df - dy \\ l_{\Pi+r}\, d\Pi + l_y\, dy \end{pmatrix}$$

$$b^{-1} == \begin{pmatrix} -\frac{1}{a_r} & 0 \\ -\frac{l_{r+\Pi}}{a_r} & 1 \end{pmatrix}$$

$$\begin{pmatrix} dr \\ dm \end{pmatrix} == \begin{pmatrix} -\frac{1}{a_r} & 0 \\ -\frac{l_{r+\Pi}}{a_r} & 1 \end{pmatrix} \begin{pmatrix} a_f\, df - dy \\ l_{\Pi+r}\, d\Pi + l_y\, dy \end{pmatrix}$$

$$\begin{pmatrix} dr \\ dm \end{pmatrix} == \begin{pmatrix} \frac{dy - df\, a_f}{a_r} \\ dy\, l_y + \frac{(dy - df\, a_f + d\Pi\, a_r)\, l_{r+\Pi}}{a_r} \end{pmatrix}$$

We can find $\frac{dr}{df}$ by dividing the first row of the matrix by df.

Exercise 2

Maximize x y s.t. p_x x + p_y y = 100

$\mathcal{L}$ = x y + λ (p_x x + p_y y - 100)

$\frac{\partial \mathcal{L}}{\partial x}$ = y + λ p_x = 0

$\frac{\partial \mathcal{L}}{\partial y}$ = x + λ p_y = 0

$\frac{\partial \mathcal{L}}{\partial \lambda}$ = p_x x + p_y y - 100 = 0

```
l = x y + λ (px x + py y - 100)
lx = ∂x l == 0
ly = ∂y l == 0
ll = ∂λ l == 0
soln = Solve[{lx, ly, ll}, {x, y, λ}]
```

$x\,y + (-100 + px\,x + py\,y)\,\lambda$

$y + px\,\lambda == 0$

$x + py\,\lambda == 0$

$-100 + px\,x + py\,y == 0$

$\left\{\left\{x \to \frac{50}{px},\ y \to \frac{50}{py},\ \lambda \to -\frac{50}{px\,py}\right\}\right\}$

The elasticity of the functions are:

Exercise 3

Corbae 2.5.3

We write the relations in matrix form for convenience:

The ≤ relation) on the set {0, 1, 2, 3,4}:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

We can see that the matrix is upper traingular. For every element in {0, 1, 2, 3,4} either aRb or bRa. We also see that the diagonals are all 1, therefore it is reflexive.

The = relation on {0, 1, 2, 3, 4}:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

We can see that this relation is not complete because (for example) neither (0, 1) nor (1, 0) are in the set of =. This relationship is transitive because it is only defined for elements for identical elements.

The < relation on {0, 1, 2, 3, 4}:

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This relation is not complete because (0, 0) is not in the set of >.
The relation is transitive because for any aRb ∧ bRc ⇒ aRc (we can see this in the matrix representation, where all elements to the right of the first “1” in a row are also 1.

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

We can see the relation is not complete because elements such as (0, 0) are not in the set of ≠. We can see the set is not transitive because 3 ≠ 1, 1 ≠ 3 does not imply 3 ≠ 3 (that is, (1,3) and (3, 1) are in the set of the relation ≠, but (1, 1) is not).

For example 2.5.6 we have:

□	chicken	fish	beef	pork
pork	1	0	0	1
beef	0	0	1	1
fish	0	1	1	1
chicken	1	1	1	0

The following relations are in the set $\lesssim$ {(pork, chicken), (pork, pork), (beef, beef), (beef, pork), (fish, fish), (fish, beef), (fish, pork), (chicken, chicken), (chicken, fish), (chicken, beef)}. We can see that from the set {pork, beef, fish, chicken} either a $\lesssim$ b or b $\lesssim$ a.

2.5.5

~ is rational means ~ is complete and transitive.
This means $x \gtrsim x$ by completeness. Therefore, we can rule out $\neg[x \gtrsim x]$. This is only possible if $x \sim x$.
Suppose ~ is not transitive. This implies $\exists_{x,y,z \in \mathbb{A}} x \sim y, y \sim z$ but $\neg(x \sim z)$. This means $(x > z) \vee (z < z)$.
Thus we have $x \sim y > z$. But we have $y \sim z$. Thus we have a contratdiction.

2.5.8
We saw previously the relation is complete. However, the relation is not transitive. For example, we see “pork R chicken”, and “chicken R fish”, but no “pork R fish”. Thus it cannot be an equivalence relations.

2.7.6
The equivalence relation are the set of all vectors with the same length and direction.

2.7.10

$\sim = \left(\gtrsim \bigcap \gtrsim^{-1} \right)$?

$\succ = (\gtrsim \backslash \sim)$?

Exercise 4

<u>P 72</u>
1 a. Negate “Everyone who is majoring in math has a friend who needs help with his homework.”
Let A be the set of people majoring in math
Let B be the set of people who need help with homework
Let xFy be the relation x and y are friends
We can write the above symbolically as
$\forall_x \in A \ \exists_y \in B \wedge (x, y) \in F$
Negating this:
$\neg(\forall_x \in A \ \exists_y \in B$ such that $(x, y) \in F)$
$\Longleftrightarrow (\exists_x \in A \ \neg\ (\exists_y \in B$ such that $(x,y) \in F)$

$\Leftrightarrow$ ($\exists_x \in$ A such that $\forall_y \in$ B $\neg$(x, y) $\in$ F)
$\Leftrightarrow$ ($\exists_x \in$ such that A $\forall_y \in$ B $\neg$(x, y) $\notin$ F)

Which we can rewrite this as “there is someone who is majoring in math who doesn’t have any friends who needs help with math”.
1 b. Negate “Everyone has a roommate who dislikes everyone.”
Let xDy be the relation x dislikes y
Let xRy be the relation x and y are roommates
We can rewrite the above statement symbolically as:
$\forall_x \exists_y$ s.t. (x,y) $\in$ R $\wedge$ (x,y) $\in$ D
$\neg(\forall_x \exists_y$ s.t. (x,y) $\in$ R $\wedge$ (x,y) $\in$ D)
($\exists_x \neg(\exists_y$ s.t. (x,y) $\in$ R $\wedge$ (x,y) $\in$ D))
($\exists_x$ s.t. ($\forall_y$ $\neg$((x,y) $\in$ R $\wedge$ (x,y) $\in$ D)))
($\exists_x$ s.t. ($\forall_y$ ($\neg$(x,y) $\in$ R $\vee$ $\neg$(x,y) $\in$ D)))
($\exists_x$ s.t. ($\forall_y$ ((x,y) $\in$ R $\vee$ (x,y) $\in$ D)))
There is someone who has a roommate who doesn’t dislike everyone.

1 c.
Negate $A \cup B \subseteq C \setminus D$
$\neg(A \cup B \subseteq C \setminus D)$
$\Leftrightarrow \neg A \cap \neg B \subseteq C \setminus D)$
$\Leftrightarrow \neg A \cap B \nsubseteq (C \setminus D)$

1 d.
Negate $\exists_x \forall_y$ [y > x $\Rightarrow \exists_z \left(z^2 + 5 z = y\right)$
$\forall_x \exists_y$ [y > x $\Rightarrow \forall_z \left(z^2 + 5 z \neq y\right)$

2.
a. There is someone in the freshman class who doesn’t have a roommate.
Everyone in the freshman class has a roommate

b. Everyone likes someone, but no one likes everyone
There is someone who doesn’t like anyone or there is someone who likes everyone

c. $\forall_a \in A \; \exists_b \in B \; (a \in C \Leftrightarrow b \in C)$
$\exists_a \in A \; \forall_b \in B \; \neg(a \in C \Leftrightarrow b \in C)$
$\exists_a \in A \; \forall_b \in B \; (a \in C \wedge b \notin C) \vee (a \notin C \wedge b \in C)$

d. $\forall_y > 0 \; \exists_x (a\, x^2 + b\, x + c = y)$
$\exists_x > 0 \; \forall_x \; (a\, x^2 + b\, x + c \neq y)$

3.

a. False.
Consider the number 2
There are no a, b, c in the natural number ℕ = {1, 2, 3...} such that a^2 + b^2 + c^2 = 2 (however, the statement is false if 0 is in the natural numbers).

b. False
$E_{!x}$ $(x - 4)^2 = 9$
$(x - 4)^2 - 9 = 0$
x == 1 || x == 7

c. $\exists_{!x}$ $(x - 4)^2 = 25$
False, x = 1 or x = 9

d. True. From previous example, if x = 1 and y = 9 then (x - 4)^2 = 25 and (y - 4)^2 = 25

4. Prove $\neg \forall_x P(x) \Leftrightarrow \exists_x \neg P(x)$
$\neg \exists_x P(x) \Leftrightarrow \forall_x \neg P(x)$
negate both side
$\neg(\neg \exists_x P(x)) \Leftrightarrow \neg (\forall_x \neg P(x))$
distribute
$E_x \neg P(x) \Leftrightarrow (\neg A_x P(x))$

5. $\neg E_x \in A\ P(x) \Leftrightarrow \forall_x \neg P(x)$
by quantifier negation law

<u>P. 170</u>
4.
A = {1, 2, 3}
B = {1, 4}
C = {3, 4}
D = {5}
B ∪ C = {1, 3, 4}
B ∩ C = {4}
A ∪ C = {1, 2, 3, 4}
A ∩ C = {3}
B ∪ D = {1, 4, 5}
B ∩ D = {}
A × B = {(1, 1), (1, 4), {2, 1), (2, 4), (3, 1), (3, 4)}
A x C = {(1, 3), (1, 4), (2,3), (2, 4), (3, 3), (3, 4)}
C × D = {(3, 5), (4, 5)}

A × (B ∩ C) = (A × B) ∩ (A × C)

{1, 2, 3} × {4} = {(1, 1), (1, 4), {2, 1), (2, 4), (3, 1), (3, 4)} ∩ {(1, 3), (1, 4), (2,3), (2, 4), (3, 3), (3, 4)}
{(1, 4), (2, 4), (3, 4)} = {(1,4), (2, 4), (3, 4)}

A × (B ∪ C) = (A × B) ∪ (A × C)
{1, 2, 3} × {1, 3, 4} = {(1, 1), (1, 4), {2, 1), (2, 4), (3, 1), (3, 4)} ∪ {(1, 3), (1, 4), (2,3), (2, 4), (3, 3), (3, 4)}
{(1, 1), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 3), (3, 4)} = {(1, 1), (1, 3), (1, 4), {2, 1), (2,3), (2, 4), (3, 1), (3, 3), (3, 4)}

(A × B) ∩ (C × D) = (A ∩ C) × (B ∩ D)
{(1, 1), (1, 4), {2, 1), (2, 4), (3, 1), (3, 4)} ∩ {(3, 5), (4, 5)} = {3} ∩ {}
{} = {}

(A × B) ∪ (C × D) ⊆ (A ∪ C) × (B ∪ D)
{(1, 1), (1, 4), {2, 1), (2, 4), (3, 1), (3, 4)} ∪ {(3, 5), (4, 5)} ⊆ {1, 2, 3, 4} × {1, 4, 5}
{(1, 1), (1, 4), {2, 1), (2, 4), (3, 1), (3, 4), (3, 5), (4, 5)} ⊆ {(1, 1), (1, 4), (1, 5), (2, 1), (2, 4), (2, 4), (3, 1), (3, 4), (3, 5), (4, 1), (4, 4), (4, 5)}

A × ∅ = ∅ × A = ∅
{1, 2, 3} × ∅ = ∅
∅ × {1, 2, 3} = ∅ (see page 166).

7. If A has m elements and B has n A × B will have m × n elements.

<u>P. 178</u>
1 a. {(p, q} ∈ P × P | the person p is a parent of the person q}, where P is the set of all living people}
The domain is the set of all living people.
The range is the set of all living people with a living parent.

b. {(x, y) $\mathbb{R}^2$ | $y > x^2$}
The doman is ℝ. The range is all non-negative real numbers.

2 a. {(p,q) ∈ P × P |the person p is a brother of the person q} P is the set of all living people.
The domain is the set of all living people. The range is the set of all living people with a living brother.

b. {(x, y) ∈ $\mathbb{R}^2$ | $y^2 = 1 - 2/(x^2 + 1)$}.
The domain is the set of all real numbers. the range is -1 < y < 1.

Exercise 5

The theorem and proof are not correct. Consider an empty relation (i.e. no two elements of the non-empty set are in the relation). Then R is trasnsitive and symmetric, but not reflexive. in order to fix the proof, we must add we must add a quantifier condition (that is for every element a in the set, there exists at least one b such that aRb).

Exercise 6

if relation R is irreflexive than $\forall_a \neg(aRa)$.
If relation R is antisymmetric, then $\forall_{a,b}$ (aRb ∧ bRa) ⇒ a = b
Therefore $\forall_{a,b}$ ((aRb ∧ bRa) ⇒ a = b) ∧ ¬(aRa) ⟺ (¬(aRb ∧ bRa) ∨ a = b) ∧ ¬ (aRa) ⟺ (¬aRb ∨ ¬bRa ∨ a = b) ∧ ¬ aRa⟺aRb ⇒ ¬bRa

Exercise 7

R is symmetric: iff a_{ij} = 1 then a_{ji} = 1 in the bolean matrix representation (where a_{ij} is the i^{th} row and j^{th} column of the boolean matrix representation of the relationship).
Asymettric relation: iff a_{ij} = 1 then a_{ji} = 0.
If R is complete, then the matrix is triangular.

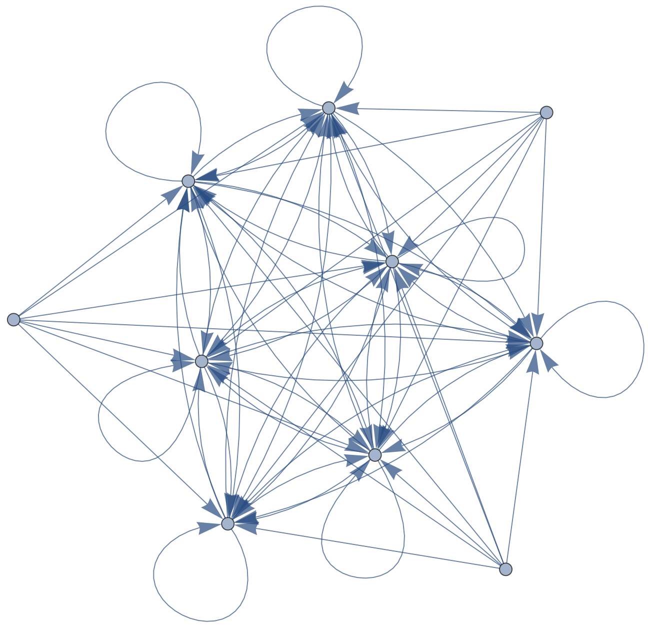
Computational Exercise 1

```
ClearAll[result]
subsym[x_] := Module[{result = Table[Table[0, Length[x]], Length[x]]},
  Do[If[x[[i]][[j]] == 1 && x[[j]][[i]] == 1,
    result[[i]][[j]] = 1,
    0], {i, 1, Length[x]}, {j, 1, Length[x]}];
  result
 ]
x = Table[{1, 0, 1, 1, 0, 1, 1, 1, 0, 1}, 10];
x // Labeled[TableForm[#], "Original Relation"] &
AdjacencyGraph[x, PlotLabel → "Graph of original adjacency matrix"]
s = subsym[x];
s // Labeled[TableForm[#], "Symmetric Subelation"] &
AdjacencyGraph[s, PlotLabel → "Symmetric Subrelation"]
```

1	0	1	1	0	1	1	1	0	1
1	0	1	1	0	1	1	1	0	1
1	0	1	1	0	1	1	1	0	1
1	0	1	1	0	1	1	1	0	1
1	0	1	1	0	1	1	1	0	1
1	0	1	1	0	1	1	1	0	1
1	0	1	1	0	1	1	1	0	1
1	0	1	1	0	1	1	1	0	1
1	0	1	1	0	1	1	1	0	1
1	0	1	1	0	1	1	1	0	1

Original Relation

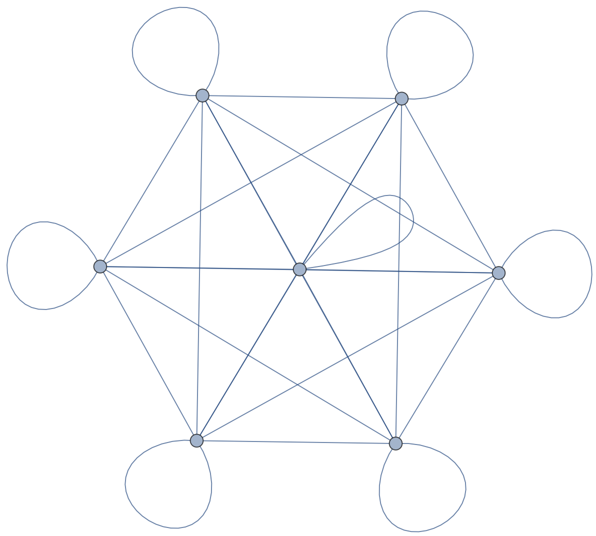
Graph of original adjacency matrix



1	0	1	1	0	1	1	1	0	1
0	0	0	0	0	0	0	0	0	0
1	0	1	1	0	1	1	1	0	1
1	0	1	1	0	1	1	1	0	1
0	0	0	0	0	0	0	0	0	0
1	0	1	1	0	1	1	1	0	1
1	0	1	1	0	1	1	1	0	1
1	0	1	1	0	1	1	1	0	1
0	0	0	0	0	0	0	0	0	0
1	0	1	1	0	1	1	1	0	1

Symmetric Subelation

Symmetric Subrelation



Compuational Exercise 2

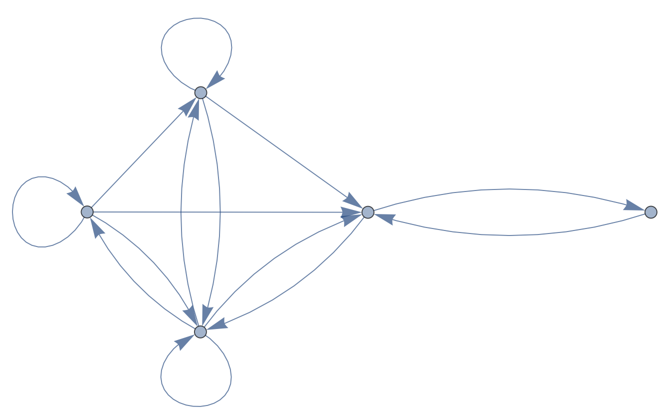
```
adjMatrix = Table[RandomChoice[{0, 1}, 5], 5]
adjMatrix // Labeled["Adjacency Matrix", TableForm[#]] &
adjGraph = AdjacencyGraph[adjMatrix, PlotLabel → "Adjacency Graph"]
```

```
{{1, 0, 1, 1, 0}, {1, 1, 1, 1, 0}, {0, 0, 0, 1, 1}, {1, 1, 1, 1, 0}, {0, 0, 1, 0, 0}}
```

Adjacency Matrix

1	0	1	1	0
1	1	1	1	0
0	0	0	1	1
1	1	1	1	0
0	0	1	0	0

Adjacency Graph



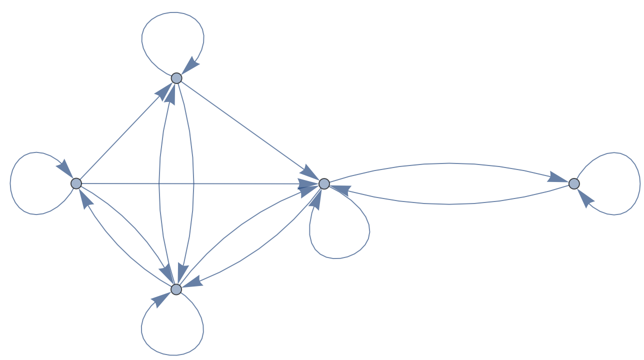
```
StringForm["Is the relation reflexive? ``",
 AllTrue[Diagonal[adjMatrix], # == 1 &]]
symmetric[x_] := Module[{},
   Do[
    If
     [x[[i]][[j]] == 1 && x[[j]][[i]] == 0 ,
     Return[False]
    ];
    Return[True],
    {i, 1, Length[x]},
    {j, 1, Length[x]}
   ]
  ]
StringForm["Is the relation symmetric? ``", symmetric[adjMatrix]]
```

```
Is the relation reflexive? False

Is the relation symmetric? True
```

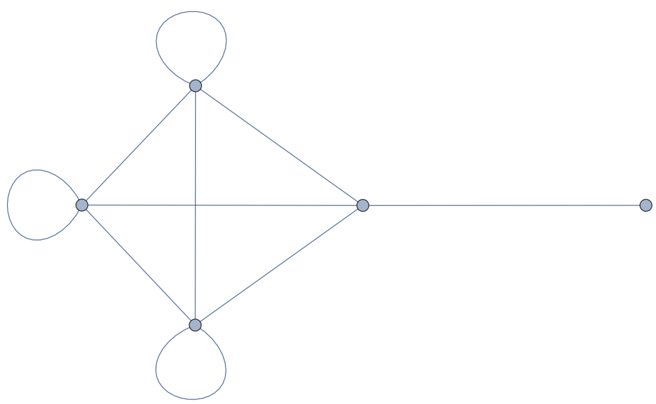
```
reflexivize[x_] := Module[{result = x},
  Do[
   result[[i]][[i]] = 1,
   {i, 1, Length[x]}];
  Return[result]]
reflexiveAdjMatrix = reflexivize[adjMatrix];
StringForm["Is the new relation (with reflexive closure) reflexive? ``",
 AllTrue[Diagonal[reflexiveAdjMatrix], # == 1 &]]
AdjacencyGraph[reflexiveAdjMatrix]
```

```
Is the new relation (with reflexive closure) reflexive? True
```



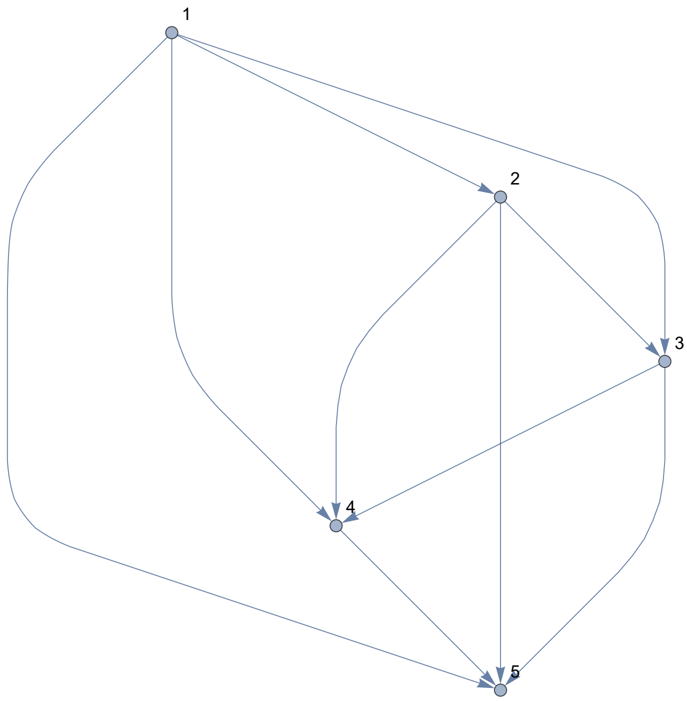
```
symmetrize[x_] := Module[{result = x}, Do[
   If[result[[i]][[j]] == 1, result[[j]][[i]] = 1, ],
   {i, 1, Length[result]}, {j, 1, Length[result]}];
  Return[result]]
symmetricAdjMatrix = symmetrize[adjMatrix] ;
StringForm["Is the new relation (with symmtric closure)? ``",
 symmetric[symmetricAdjMatrix]]
AdjacencyGraph[symmetricAdjMatrix]
```

```
Is the new relation (with symmtric closure)? True
```



Computational Exercise 3

```
g = RelationGraph[Less, Range[1, 5]]
a = AdjacencyMatrix[g]
a //
 Labeled["<", TableForm[#, TableHeadings → {{1, 2, 3, 4, 5}, {1, 2, 3, 4, 5}}]] &
```

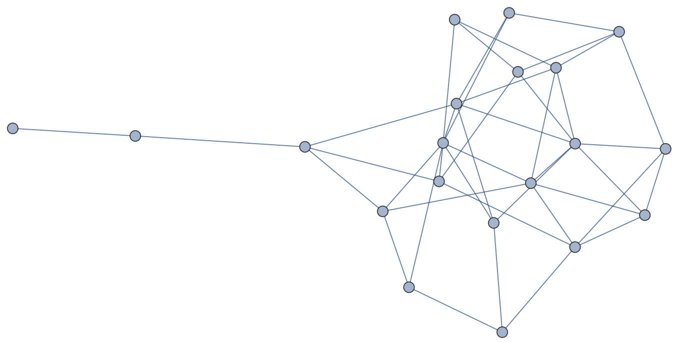


SparseArray[ Specified elements: 10 Dimensions: {5, 5}]

<

	1	2	3	4	5
1	0	1	1	1	1
2	0	0	1	1	1
3	0	0	0	1	1
4	0	0	0	0	1
5	0	0	0	0	0

```
g0 = RandomGraph[BernoulliGraphDistribution[20, 0.2]]
```



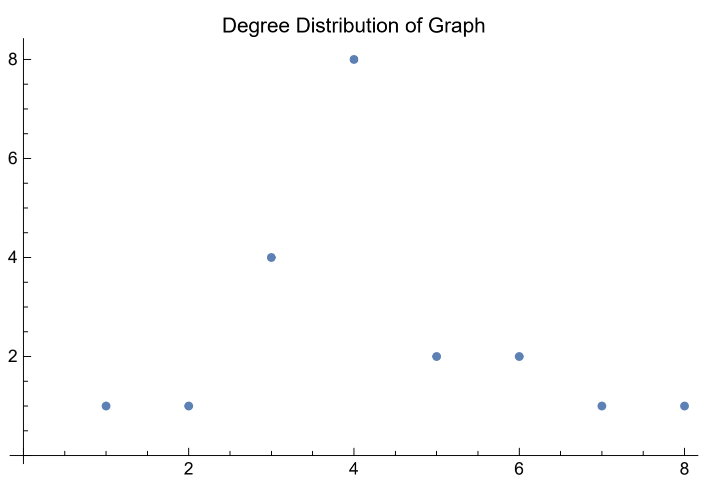
```
d = VertexDegree[g0]
```

{3, 4, 5, 4, 8, 6, 7, 6, 4, 4, 4, 3, 5, 4, 4, 3, 1, 2, 4, 3}

```
Tally[d]
```

{{3, 4}, {4, 8}, {5, 2}, {8, 1}, {6, 2}, {7, 1}, {1, 1}, {2, 1}}

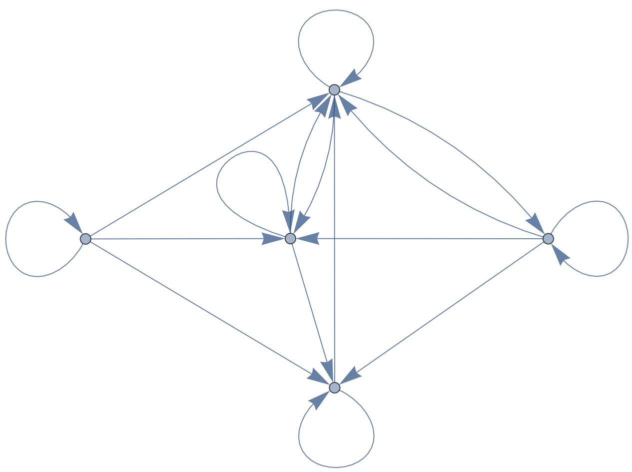
```
ListPlot[Tally[d], PlotLabel → "Degree Distribution of Graph"]
```



Computational Exercise 4

```
adjList = {1 ⟼ 1, 1 ⟼ 3, 2 ⟼ 1, 2 ⟼ 2, 2 ⟼ 3, 3 ⟼ 2, 3 ⟼ 3,
   3 ⟼ 5, 4 ⟼ 1, 4 ⟼ 2, 4 ⟼ 3, 4 ⟼ 4, 5 ⟼ 1, 5 ⟼ 2, 5 ⟼ 3, 5 ⟼ 5}
g1 = Graph[adjList]
AdjacencyMatrix[g1] // MatrixForm
```

```
{1 ⟼ 1, 1 ⟼ 3, 2 ⟼ 1, 2 ⟼ 2, 2 ⟼ 3, 3 ⟼ 2, 3 ⟼ 3,
 3 ⟼ 5, 4 ⟼ 1, 4 ⟼ 2, 4 ⟼ 3, 4 ⟼ 4, 5 ⟼ 1, 5 ⟼ 2, 5 ⟼ 3, 5 ⟼ 5}
```



$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$