HW9

Exercise 1

12.17 Since x is in the interior of S, there exists an open ball B(x, ϵ) that is a subset of S. There is a limit to the sequence $\{x_n\}_{n=1}^{\infty}$ (meaning there exists an N such that d(x_n, x) < ϵ for sufficiently large n). Therefore, $x_n \in$ B(x, ϵ).

12.20 In a finite metric space S, all single points are closed (consider any $\{x_n\}_{n=1}^{\infty}$ in S that converges to x. x ∈ S). The finite union of all single points in S is alsio closed. The set of integers proof is the same as with the distance/metric function == 1.

12.24 ?

12. 29
Let C be a closed subset of a compact set T. U is an open cover of C. Since C is closed, therefore T\C is open in T. if we add T\C to U, we see that U∪(T\C) is also an open cover of T. As T is compact, there is a finite subcover of U, say V={U1,U2,…,Ur}. This covers C by the fact that it covers T. If T\C is an element of V, then it can be removed from V and the rest of V still covers C.
Thus we have a finite subcover of U which covers C, and hence C is compact. (taken from Wikipedia)

Exercise 2

Assume that there is a R-maximal element x^+ in the set S.
$\forall_x \in S \neg$ (x R x^+)

The strict lower contour set of the relationship on S is:
P_s = {y ∈ S | s P y} (where P is the asymmetric subrelation of R)

Since P_s is a cover of S
S $\subseteq \cup_{a \in A} P_{s_a}$

But by the strict lower contour set y ∈ S iff s P y. Thus S $\subseteq \cup_{a \in A} P_{s_a}$. We have a contradiction, so it must be the case that there exists no maximal element.

It is not the case that there is no R-maximal element in the set S iff the lower contour sets of the points x ∈ S form a cover of S because in this case the maximal element can be in the contour set. Thus, there is no contradiction when the comntour set covers S.

Exercise 3

Theorem:
Suppose that on a set X, an acyclic binary relation P has open lower contour sets. Then there exists a P-maximal element in any non-empty compact subset $S \subset X$.

Proof:
If there is no P-maximal element in S then every element in S is dominated by some other element in S:
$\forall_{s \in S} \exists_{s' \in S}$ s' < s

The lower contour sets form a cover of S
$S \subseteq \cup_{x \in S}$ (-∞ ... x)

The lower contour sets are open. Thus the lower contour sets are an open cover of S.

S is compact. Thus, there is a finite subcover of S. For each set in the subcover, pick any point in S for which it is the lower contour set. The collection of such points is a finite set, A, of points whose lower contour sets cover S:
$S \subseteq \cup_{x \in A}(-\infty$ .. x).

A is also covered by this finite subcover of S. So each point in A is “dominated” by some point in A. Since A is a finite set, which is only possible if P cycles.

Definitions:
A P-Maximal element is an element x^M such that $\forall_{x \in X}$ (x^M P x) $\vee$ ¬(x P x^M)
A cover C of a set S is a collection of sets such that $S \subseteq \cup_{x \in A} C_x$. That is, some arbitrary union of the C_x contains S.
A finite subcover is a subset of C that covers S under finite union.
A set is compact iff there exists a finite subcover for any open subcover.
The strict lower contour set is the $P \subseteq X$ such that: $P = \exists_{s \in X}$ {y ∈ S | s P y}. That is every element on the lower contour set is strictly dominated by some element in the original set.
An open set $S \in \tau$ of the topology (X, τ) which is defined by:
x, $\emptyset \in \tau$
$V_1, V_2 \in \tau \Rightarrow V_1 \cap V_2 \in \tau$
$(\forall_{\alpha \in I} V_\alpha \in \tau) \Rightarrow \cup_{\alpha \in I} V_\alpha \in \tau$.

Exercise 4

Consider a point q ∈ K. Then by Hausdorf condition, there exists open neighborhoods N(q) and N(p) such that N(q) ∩ N(p) = ∅. $K \subseteq \cup_{q \in K}$ N(q). That is N(q) is an open subcover of N(q). Since K is compact set, there is a finite subcover of K.n Let U be the union of the finite subcover of A. For eah set in the finite subcover there is a collection of open neighborhoods of p. Let U be the intersection of all these neighborhoods. Then V is an open set containing K that does not intersect U.

Exercise 5

A metric space is a set for which, for any pair of elements in the set, a distance function is defined. The distance function must follow these properties:

d(x, y) ≥ 0

d(x, y) = d (y, x)

d(x, y) = 0 ⇔ x = y

d(x, z) ≤ d(x, y) + d(y, z)

The function f(x, x') given for this problem satisfies these properties. f(x, x') can only take on the values 0 and 1 and is equal to 0 if x = x' and 1 otherwise. It is the case that if x = x' then d(x, x') = d(x' x) = 0. Otherwise d(x, x') = 1. Finally if y, z = x then d(x, y) = 0, d(y, z) = 0 ≤ d(x, z) = 0. Otherwise if x ≠ y then d(x, z) = ≤ d(x, y) + d(y, z) ⟶ 1 ≤ 0 + 1 or if x ≠ z then d(x, z) = ≤ d(x, y) + d(y, z) ⟶ 1 ≤ 0 + 1 or x ≠ y ≠ z then d(x, z) ≤ d(x,y) + d(x,z) ⟶ 1 ≤ 2.

Exercise 6

Case 1: Assume z > x and y > z

$|\frac{1}{x} - \frac{1}{z}| \leq |\frac{1}{x} - \frac{1}{y}| + |\frac{1}{y} - \frac{1}{z}|$

We rearrange the above using assumptions so it is strictly positive

$\frac{1}{x} - \frac{1}{z} \leq \frac{1}{x} - \frac{1}{y} + \frac{1}{z} - \frac{1}{y}$

$\frac{1}{x} - \frac{1}{x} - \frac{1}{z} - \frac{1}{z} \leq \frac{-2}{y}$

$\frac{-2}{z} \leq \frac{-2}{y}$

-2y ≤ -2z

y ≥ z

x = 1, y = 3, z = 2

$|\frac{1}{1} - \frac{1}{2}| \leq |\frac{1}{1} - \frac{1}{3}| + |\frac{1}{3} - \frac{1}{2}|$

$\frac{1}{2} \leq \frac{2}{3} + \frac{1}{6}$

Case 2: Assume z > x and z > y

$|\frac{1}{x} - \frac{1}{z}| \leq |\frac{1}{x} - \frac{1}{y}| + |\frac{1}{3} - \frac{1}{2}|$

We drop the absolute value signs because the differencses above are all strictly positive given the assumptions

$\frac{1}{x} - \frac{1}{z} \leq \frac{1}{x} - \frac{1}{y} + \frac{1}{y} - \frac{1}{z}$

which simplifies to

$\frac{1}{x} - \frac{1}{z} \leq \frac{1}{x} - \frac{1}{z}$ which is true

Case 3: Assume z > x and x > y

$|\frac{1}{x} - \frac{1}{z}| \leq |\frac{1}{x} - \frac{1}{y}| + |\frac{1}{3} - \frac{1}{2}|$

We rearrange the above using assumptions so it is strictly positive

$\frac{1}{x} - \frac{1}{z} \leq \frac{1}{y} - \frac{1}{x} + \frac{1}{z} - \frac{1}{y}$

which simplifies to

$\frac{1}{x} - \frac{1}{z} \leq \frac{1}{z} - \frac{1}{x}$

$\frac{2}{x} \leq \frac{2}{z}$

$2z \leq 2x$

$z \leq x$ which is true.

The other cases are trivially true:

$|x| \geq 0$ by definition of $|x|$

$\left| \frac{1}{x} - \frac{1}{x} \right| = 0$

$\left| \frac{1}{x} - \frac{1}{y} \right| = \left| \frac{1}{y} - \frac{1}{x} \right|$ by symmetry property of $|x - y|$ (that is $|x - y| = |y - x|$

Exercise 7

Let S be a compact set and let $x \in S$. The set S is bounded, if $\forall_{s \in S}$ d(x, s) < M where M is a positive integer.

Let $B(x, r) = \{y \in S | d(x, y) < r\}$ be an open ball with center x and radius r.

For any $y \in S$, $y \in \bigcup_{i=1}^{\infty} B(x, i)$ since $d(x, y) < \infty$. Therefore $\{B(x, i)\}$ is an open cover of S. Since S is compact, there is a finite subcover such that:

$S \subset \bigcup_{j=1}^{m} B(x, i_j)$.

This means $S \subset B(x, \max\{i_1, \ldots i_j\})$. Therefore S is bounded.

Exercise 8

Suppose (x_{t_k}) is subsequence of (x_t) that converges to $\overline{x}$. There exists a positive real number such that:

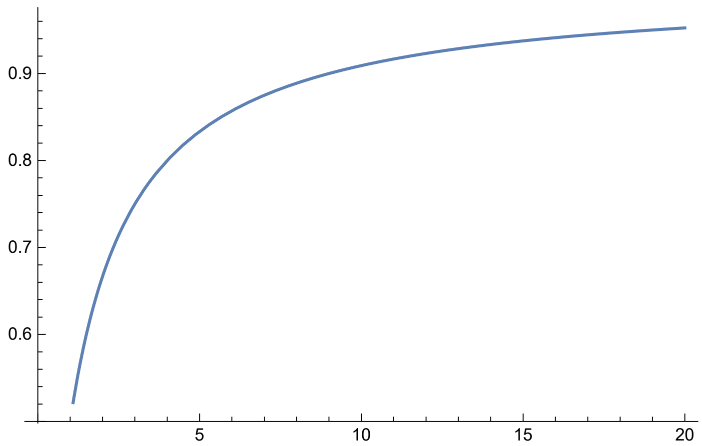
$\forall_{i,j \in \mathbb{N}}\ i,\ j > M \Longrightarrow d(x_i,\ x_j) < \frac{\epsilon}{2}$. By definition of convergence $\forall k \in N\ k > N \Longrightarrow d(xnk,x) < \frac{e}{2}$. By the triangle inequality, we have

$\forall_{m \in \mathbb{N}}\ m > K \Longrightarrow d(x_m,x) \leq d(x_m,x) + d(x_{nK},x) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

Comnputation I

In[1]:=
```
Plot[t/(t+1), {t, 0, 20}]
```

Out[1]=



A series is monotone if $\forall_{m>n}\ s_m > s_n$ (monotone increasing) or $\forall_{m>n}\ s_m < s_n$ (monotone decreasing). A sequence is bounded if $\exists_{x \in R}\ s_n < x$. A sequence is convergent if there exists a $N \in \mathbb{N}$ such that for any $n > N$, $|\ x_n - x\ | < \epsilon$.
My seuqnce is monotone bounded and convergent.

In[2]:=
```
Limit[t / t + 1, t → ∞]
```

Out[2]= 2

Computational Exercise 2

In[39]:=
```
pdv[savings_, rate_] := (
  dv = {};
  Do[AppendTo[dv, (savings[[i]] / (1 + rate)^(i-1))], {i, 1, Length[savings]}];
  Return [Total[dv]])
```

In[45]:=
```
pdv[Table[200, 60], .01]
Print[StringForm[
  "The Present value of 60 payments of 200 at a 1% monthly interest rate is ``",
  Round[%, 2]]]
```

Out[45]= 9080.92

```
The Present value of 60 payments of 200 at a 1% monthly interest rate is 9080
```