Exercise 1

Using Mathematica

```
iseq = y == a[i, y, f, q];
lmeq = m == l[i, y];
derivs = {Dt[iseq], Dt[lmeq]};
notationRules = {Dt[q] → dQ, Dt[f] → dF, Dt[y] → dY, Dt[i] → di, Dt[m] → dm,
   a^(0,0,0,1)[i, y, f, q] → a_q, a^(0,0,1,0)[i, y, f, q] → a_f, a^(0,1,0,0)[i, y, f, q] → a_y,
   a^(1,0,0,0)[i, y, f, q] → a_i, l^(0,1)[i, y] → l_y, l^(1,0)[i, y] → l_i};
derivs /. notationRules // FullSimplify // MatrixForm // TraditionalForm
Solve[derivs, {Dt[q], Dt[m]}] /. notationRules // FullSimplify // TraditionalForm
```

$$\begin{pmatrix} \mathrm{dF}\, a_f + \mathrm{di}\, a_i + \mathrm{dQ}\, a_q + \mathrm{dY}\, a_y = \mathrm{dY} \\ \mathrm{dm} = \mathrm{di}\, l_i + \mathrm{dY}\, l_y \end{pmatrix}$$

$$\left\{\left\{\mathrm{dQ} \to -\frac{\mathrm{dF}\, a_f + \mathrm{di}\, a_i + \mathrm{dY}\,(a_y - 1)}{a_q},\ \mathrm{dm} \to \mathrm{di}\, l_i + \mathrm{dY}\, l_y\right\}\right\}$$

Using Matrix Methods

$$\begin{pmatrix} A_q & 0 \\ -L_y & 1 \end{pmatrix} \begin{pmatrix} \mathrm{dQ} \\ \mathrm{dm} \end{pmatrix} = \begin{pmatrix} A_F\,\mathrm{dF} + A_i\,\mathrm{di} + (1 - A_y)\,\mathrm{dY} \\ L_i\,\mathrm{di} + L_y\,\mathrm{dY} \end{pmatrix}$$

```
cInverse = Inverse[( A_q   0 )]
                   ( -L_y  1 )
```

$$\left\{\left\{\frac{1}{A_q}, 0\right\}, \left\{\frac{L_y}{A_q}, 1\right\}\right\}$$

```
cInverse . ( A_F dF + A_i di + (1 - A_y) dY ) // MatrixForm // TraditionalForm
           (        L_i di + L_y dY         )
```

$$\begin{pmatrix} \frac{\mathrm{dF}\, A_F + \mathrm{di}\, A_i + \mathrm{dY}\,(1 - A_y)}{A_q} \\ \frac{L_y\,(\mathrm{dF}\, A_F + \mathrm{di}\, A_i + \mathrm{dY}\,(1 - A_y))}{A_q} + \mathrm{di}\, L_i + \mathrm{dY}\, L_y \end{pmatrix}$$

The sign of $\frac{\partial Q}{\partial F}$ is obtained by solving the above for $\frac{\partial Q}{\partial F}$ (that is holdiing all the other partials constant). We obtain $\frac{\partial Q}{\partial F} = \frac{-A_F}{A_Q}$. We assume that $A_Q > 0$ and $A_F > 0$. We obtain $\frac{\partial Q}{\partial F} < 0$.

Exercise 2

```
Maximize[{c1^(1/2) + c2^(1/2), c1*1 + 3*c2 ≤ 10 000}, {c1, c2}]
```

$\left\{\frac{200}{\sqrt{3}}, \left\{c1 \to 7500,\ c2 \to \frac{2500}{3}\right\}\right\}$

Using the substitituttion method

We solve (c_1 + 3$*c_2$ = 10000) for c_1 to obtain (c_1 = 10000 - 3 c_2)

Substituting into the utility function, we obtain:

$U(c_2) = (10000 - 3\ c_2)^{\frac{1}{2}} + c_2{}^{\frac{1}{2}}$

$U' = \frac{-3}{2}(10000 - 3\ c_2)^{\frac{-1}{2}} + \frac{1}{2}\ c_2{}^{\frac{-1}{2}} = 0$

$c_2 = \frac{2500}{3}$

Substituting into c_1 = 10000 - 3 c_2

$c_1 = 7500$

Using Langrangian method

$\mathcal{L} = c_1^{\frac{1}{2}} + c_2^{\frac{1}{2}} + \lambda\,(c_1 + 3\ c_2 - 10000)$

$\frac{\partial \mathcal{L}}{\partial \lambda} = c_1 + 3\ c_2 - 10000 = 0$

$\frac{\partial \mathcal{L}}{\partial c_1} = \frac{1}{2}\ c_1^{\frac{-1}{2}} + \lambda\ c_1 = 0$

$\frac{\partial \mathcal{L}}{\partial c_2} = \frac{1}{2}\ c_2^{\frac{-1}{2}} + 3\lambda\ c_2 = 0$

$c_1 = 7500$

$c_2 = \frac{2500}{3}$

Exercise 3

The identity relation on any set x is the set {(x, x) | x ∈ X}. It is

1. Reflexive. Because all $\forall_{x \in X}$, (x, x) ∈ i_x by definiction
2. Symmetric, becuase all members of the set of i_x are of the form (x, x). Which means that $\forall_{x,\,y \in X}$ (x, y) ∈ i_x ⇔ (y, x) ∈ i_y (because x = y)
3. Transitive. Because $\forall_{x,\,y \in X}$(x, y) ∈ i_x ⇒ x = y. ∴ (x, y) ∈ i_x ∧ (y, z) ∈ i_x ⇒ (x, z) ∈ i_x (since in this case y and z are also x)

Exercise 4

The symmetric component I of relation R are the subset of R such that {(x, x') ∈ I} ⇔ {(x', x) ∈ I}. We know the relation R is transitive, meaning $\forall_{x,y,\,z \in X}$({x, y) ∈ R} ∧ ({y, z) ∈ R} ⇔ {(x, z) ∈ R}. For the symeetric component, we have the subset of R which is {(x, x’) ∈ I} ⟺ {(x’, x) ∈ I}. This is a tautology.

For the asymmetric component P, we have the subset of R such that {(x, x’) ∈ P} ⇒ {(x’, x) ∉ P}. We know the relation R is transitive, meaning $\forall_{x,y,z\in X}$({x, y) ∈ R} ∧ ({y, z) ∈ R} ⟺ {(x, z) ∈ R}. For the asymettric component, we have the subset of R which is {(x, x’) ∈ P} ⇒ {(x’, x) ∉ P}. Replacing y with x’, and noting the transitive nature of the original we obtain that $\forall_{x,y\in X}$({x, x’) ∈ R} ∧ ({x’, z) ∈ R} ⟺ {(x, z) ∈ R}

By symmetry and transitivity we can replace y in the relationship (x P y) ∧ (y I z) to obtain (x P z). Similarly, by symmetry and transitivity, we can replace the second y in (x I y) ∧ (y P z) to obtain (x P z).

Exercise 5

I is a transitive binary relation. Assume that (x, z) ∉ I. Assume that (x, y) ∈ I ∧ (y, z) ∈ I. By transitivity, we have (x, z) ∈ I. But this is a contradiction. Thus we must have (x, y) ∉ I ∨ (y, z) ∉ I ∧ (x, z) ∈ I.

Exercise 6

R = {(1, 2), (1, 3), (1, 4), (1, 5), (1, 5), (1, 6), (1, 7), (1,8), (1, 9), (1, 10), (1, 11), (1, 12), (2, 4), (2, 6), (2, 8), (2, 10), (2, 12), (3, 6), (3, 9), (3, 12), (4, 8), (4, 12), (5, 10), (6, 12)}
We see that it is transitive because if x R y ∧ y R z ⇒ x R z (check each entry). We can also see this numerically, because if a is divisible by b and b is divisible by c then a is divisible by c.

Exercise 7

Exercise 8

We ae given that R is reflexive and transitive. The inverse relation R^{-1} is given by {(y, x) ∈ R^{-1}| (x, y) ∈ R}. The intersection of these two relations are the only elements that are symmetric. Therefore it is an equivalence relation.

Exercise 9

Exercise 10

The transitive closure of R are R_t = {(g, q} ∈ P^2 | g is a grandparent of q or parent of q}. The relationship $R_t \circ R_t^{-1}$ = Ø

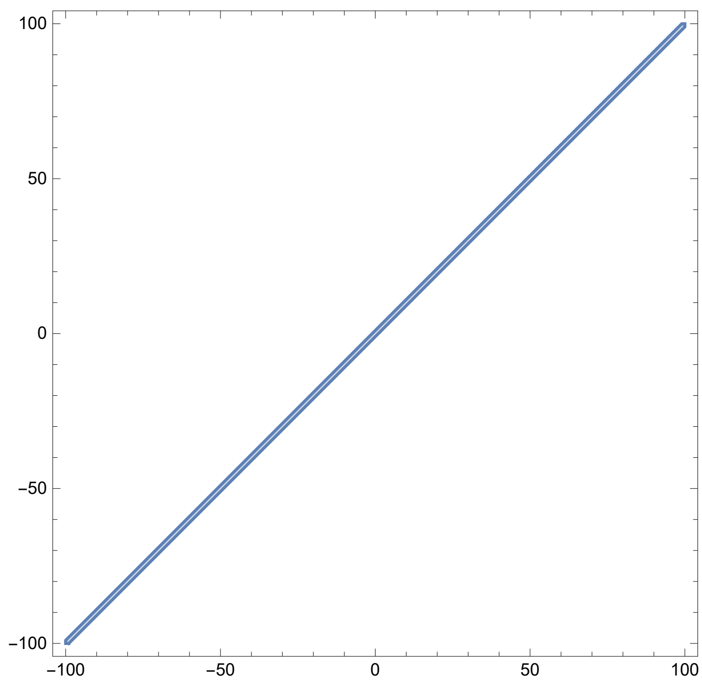
Exercise 11

1. X = {a, b, c}, R = {(a, a), (a, b), (c, b)}
In order to make the relation symmetric, we must add: (b, a) and (b, c).
R_s = {(a, a), (a, b), (c, b), (b, a), (b, c)}
The transitive closure of this set is R_t = {(a, a), (a, b), (c, b), (b, b), (c, c), (a, c)}
The reflexive closure for this set is R_r= {(a, a), (a, b), (c, b), (b, b), (c, c)}

2. The symmetric closure of this set is $\{(x, y) \in X^2 \mid x \leq y\}$
The transitive closure is the original set
The reflexive closure is $\{(x, y) \in X^2 \mid x \leq y\}$

3. This is the diagonal of width r.
The relationship is already reflexive and symmetric and transitive.

```
RegionPlot[Abs[j - k] < 1, {j, -100, 100}, {k, -100, 100}]
```



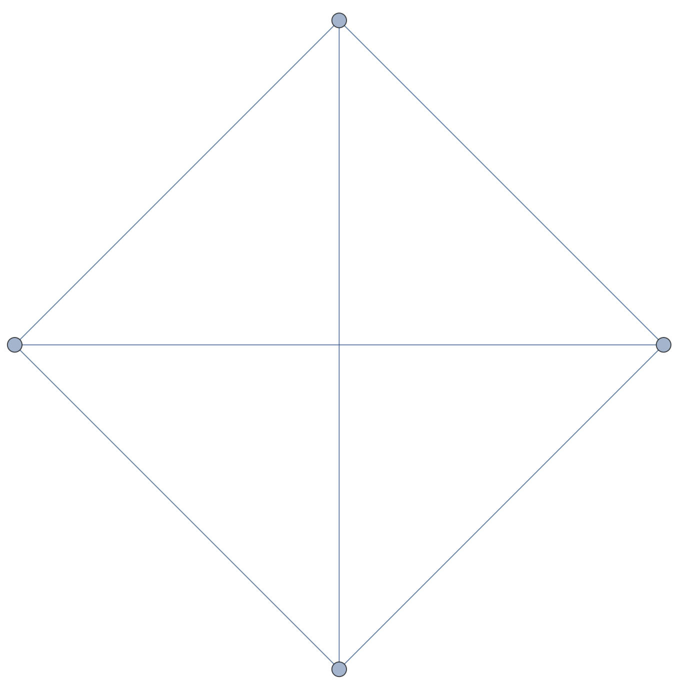
Execrise 12

Computational Exercise 1

```
prefMatrix = {{1, 0, 0, 1}, {0, 0, 1, 1}, {0, 1, 1, 1}, {1, 1, 1, 0}}
```

```
{{1, 0, 0, 1}, {0, 0, 1, 1}, {0, 1, 1, 1}, {1, 1, 1, 0}}
```

```
TransitiveClosureGraph[AdjacencyGraph[prefMatrix]]
AdjacencyMatrix[%]
% // MatrixForm
```



```
SparseArray[ Specified elements: 12
             Dimensions: {4, 4} ]
```

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

```
MatrixForm[prefMatrix]
```

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Computational Exercise 2

```
p = {{1.0, 1.0, 1.0}, {0.5, 2.0, 1.0}, {1.0, 1.0, 1.0},
   {2.0, 0.5, 1.0}, {2.0, 0.5, 1.0}, {1.0, 1.0, 1.0}, {1.0, 1.0, 1.0}};
q = {{30, 9, 9}, {34, 10, 0}, {10, 4, 0}, {4, 8, 5},
   {2, 15, 0}, {0, 15, 9}, {4, 7, 12}};
costx = p . Transpose[q] ;
% // MatrixForm
```

$$\begin{pmatrix} 48. & 44. & 14. & 17. & 17. & 24. & 23. \\ 42. & 37. & 13. & 23. & 31. & 39. & 28. \\ 48. & 44. & 14. & 17. & 17. & 24. & 23. \\ 73.5 & 73. & 22. & 17. & 11.5 & 16.5 & 23.5 \\ 73.5 & 73. & 22. & 17. & 11.5 & 16.5 & 23.5 \\ 48. & 44. & 14. & 17. & 17. & 24. & 23. \\ 48. & 44. & 14. & 17. & 17. & 24. & 23. \end{pmatrix}$$

```
rdMatrix = Table[0, {i, 1, 7}, {j, 1, 7}]
Do[rdMatrix[[j, k]] = costx[[j, j]] ≥ costx[[j, k]],
 {j, Range[1, 7]}, {k, Range [1, 7]}]
```

```
{{0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0}}
```

```
rdMatrix = Boole[rdMatrix]
```

```
{{1, 1, 1, 1, 1, 1, 1}, {0, 1, 1, 1, 1, 0, 1}, {0, 0, 1, 0, 0, 0, 0},
 {0, 0, 0, 1, 1, 1, 0}, {0, 0, 0, 0, 1, 0, 0}, {0, 0, 1, 1, 1, 1, 1}, {0, 0, 1, 1, 1, 0, 1}}
```

```
% // MatrixForm
```

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$

```
prefFunc[p_, q_] := Module[{costx, rdMatrix}, costx = p . Transpose[q];
  rdMatrix = Table[0, {i, 1, Length[p]}, {j, 1, Length[q]}];
  Do[rdMatrix[[j, k]] = Boole[costx[[j, j]] ≥ costx[[j, k]]],
   {j, Range[1, Length[rdMatrix]]}, {k, Range [1, Length[rdMatrix]]}];
  rdMatrix]
```

```
prefFunc[p, q] // MatrixForm
```

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$

Computational Exercise 3

See previous exercise.

```
(* To create the composition relation R ∘ R == R^2,
we square the matrix R and then only pick those elements ≠ 0*)
rdMatrix2 = MatrixPower[rdMatrix, 2]
Do[rdMatrix2[[j, k]] = Boole[rdMatrix2[[j, k]] ≠ 0],
 {j, 1, Length[rdMatrix]}, {k, 1, Length[rdMatrix]}]
```

```
{{1, 2, 5, 5, 6, 3, 4}, {0, 1, 3, 3, 4, 1, 2}, {0, 0, 1, 0, 0, 0, 0},
 {0, 0, 1, 2, 3, 2, 1}, {0, 0, 0, 0, 1, 0, 0}, {0, 0, 3, 3, 4, 2, 2}, {0, 0, 2, 2, 3, 1, 1}}
```

```
rdMatrix2 // MatrixForm
```

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

We see that rdMatrix 2 is a subset of rdMatrix