Optimal Allocation Analysis

To solve for the optimal allocation between a risky asset and a riskless asset, we generally follow the steps of mean-variance optimization, assuming the investor maximizes the expected utility of their wealth. Let's denote:

- W_0 as the initial wealth.
- θ as the fraction of wealth invested in the risky asset.
- The remaining fraction $(1-\theta)$ is invested in the riskless asset.

1. Wealth after Investment:

The final wealth W_1 after one period is given by:

$$W_1 = \theta W_0(1+z) + (1-\theta)W_0 p_b$$

Simplifying this, we get:

$$W_1 = W_0\left[\theta(1+z) + (1-\theta)p_b\right]$$

2. Expected Utility:

Assuming the investor has a utility function $U(W_1)$. For the purpose of this example, we'll assume a CRRA (constant relative risk aversion) utility function:

$$U(W_1) = \frac{W_1^{1-\gamma}}{1-\gamma}$$

where γ is the relative risk aversion coefficient.

3. Maximizing Expected Utility:

The expected utility $E[U(W_1)]$ is:

$$E[U(W_1)] = E\left[\frac{\left(W_0\left[\theta(1+z) + (1-\theta)p_b\right]\right)^{1-\gamma}}{1-\gamma}\right]$$

4. Approximations and Solving:

Given that $z \sim N(0, \sigma^2)$, we can approximate using a Taylor expansion for small σ or use the method of Lagrange multipliers for a general approach. The first step is to linearize the utility around the expected wealth:

$$E[U(W_1)] \approx U(E[W_1]) + \frac{1}{2}U''(E[W_1])\mathrm{Var}(W_1)$$

We can then substitute:

$$E[W_1] = W_0\left[\theta(1) + (1-\theta)p_b\right]$$

and

$$\mathrm{Var}(W_1) = W_0^2\theta^2\sigma^2$$

The first-order condition for the optimal θ comes from maximizing $E[U(W_1)]$ with respect to θ:

$$\frac{\partial\, E[U(W_1)]}{\partial\, \theta} = 0$$

For CRRA utility, the first-order condition simplifies to:

$$\frac{E[z]}{p_b - 1} = \gamma\theta\sigma^2$$

Since $E[z] = 0$ (as $z \sim N(0, \sigma^2)$), the expected return of the risky asset is zero, so the first-order condition is:

$$0 = \gamma\theta\sigma^2$$

This indicates that under the given assumptions, if there is no expected excess return ($E[z] = 0$), the optimal allocation to the risky asset is zero ($\theta = 0$).

5. Interpretation:

If the expected return on the risky asset is zero, the optimal strategy is to invest entirely in the riskless asset because there's no compensation for taking on risk.

However, if there was an expected positive return (i.e., if $E[z]$ were positive), we could solve for θ explicitly, yielding:

$$\theta = \frac{E[z]}{\gamma\sigma^2}$$

This result suggests that the optimal allocation θ to the risky asset increases with the expected excess return and decreases with both the risk (variance) and the investor's risk aversion γ.

Conclusion:

For the given problem, since z has a mean of 0, the optimal allocation θ to the risky asset is 0, meaning all wealth should be invested in the riskless asset p_b.