TODO

Main Body:
- ☐ Update/Verify tables
- ☐ Add GREXIT Analysis
- ☐ Calculate Roll and Amihud Measures for GREXIT

Appendices:
- ☐ Appendix: Reanalysis w/ outliers included
- ☐ Appendix: Sensitivity to CARA utility assumptions
- ☐ Appendix: Alternative estimators for DiD
- ☐ Appendix: Alternative estimator for volatility

Optimal Order Qty under CRRA utility:

The final wealth of the liquidity trader is given by:

$$w(Q) = Q \cdot p_b + (L-Q)\cdot(1+z) = Q \cdot p_b + L - Q + (L-Q)z$$
$$= Q(p_b - 1) + L + (L-Q)z$$

Let the trader's utility fn over wealth be:

$$u(w) = \begin{cases} \dfrac{w^{1-\gamma} - 1}{1-\gamma}, & \gamma \neq 1 \\ \ln w, & \gamma = 1 \end{cases}$$

Case 1: $\gamma \neq 1$

Let $\mu_w = \mathbb{E}[w(Q)] = Q \cdot (p_b - 1) + L$ be the expected wealth. A second-order Taylor-series approximation of the utility fn around μ_w is given by

$$u'(w) = w^{-\gamma}$$
$$u''(w) \; -\gamma w^{-\gamma-1}$$

$$u(w)\Big|_{w=\mu_w} \approx u(\mu_w) + (w-\mu_w)u(\mu_w) + \frac{1}{2}(w-\mu_w) u''(\mu_w)$$

Aside: Measure of relative risk aversion

$$RRA = -\frac{w u''}{u'} = -\left[-\frac{\gamma w^{-\gamma-1} \cdot w}{w^{-\gamma}} \right]$$

$$u(w)\Big|_{w=\mu_w} \approx u(\mu_w) + (w-\mu_w)u'(\mu_w) + \frac{1}{2}(w-\mu_w)^2 u''(\mu_w)$$

Aside: RRA measure

$$RRA = -\left[\frac{-\gamma w^{-\gamma-1} \cdot w}{w^{-\gamma}}\right]$$

$RRA = \gamma$ constant.

$$\mathbb{E}[u(w)] \approx u(\mu_w) + \frac{1}{2}(w-\mu_w)^2 u''(\mu_w)$$

where I have used the fact that $\mathbb{E}[w-\mu_w]=0$

Note that $\mathbb{E}(w-\mu_w)^2 = (L-Q)^2\sigma^2$

$$\therefore\ \mathbb{E}\left[u(w)\Big|_{w=\mu_w}\right] \approx \frac{\mu_w(Q)^{1-\gamma}-1}{1-\gamma} - \frac{\gamma}{2}\mu_w(Q)^{-\gamma-1}(L-Q)^2\sigma^2$$

where I write $\mu_w = \mu_w(Q)$ to highlight that the expected wealth is a fn. of the order qty.

To find the optimal Q, take the derivative of EU and set it to 0:

FOC:

$$0 = \frac{d}{dQ}\mathbb{E}[u(w(Q))] = \frac{d}{dQ}\left[\frac{\mu_w(Q)^{1-\gamma}-1}{1-\gamma} - \frac{\gamma}{2}\mu_w(Q)^{-\gamma-1}(L-Q)^2\sigma^2\right]$$

Note: $w(Q) = Q(p_b-1) + L + (L-Q)\varepsilon \rightarrow \mathbb{E}[w(Q)] = \mu_w = Q(p_b-1) + L$

$$\frac{d\mu_w(Q)}{dQ} = \frac{d}{dQ}\left[Q(p_b-1)+L\right] = p_b - 1$$

$$\frac{d}{dQ}\left[\mathbb{E}[u(w(Q))]\right] = \frac{\cancel{(1-\gamma)}\cdot\mu_w(Q)^{-\gamma}(p_b-1)}{\cancel{1-\gamma}} - \frac{\gamma}{2}(-\gamma-1)\mu_w(Q)^{-\gamma-2}(p_b-1)\cdot(L-Q)^2\sigma^2$$

$$-\gamma\cdot\mu_w(Q)^{-\gamma-1}(L-Q)(-1)\sigma^2 = 0$$

which simplifies to: $\mu_w(Q)^{-\gamma}(p_b-1) + \dfrac{\gamma(L-Q)\sigma^2}{\mu_w(Q)^{\gamma+1}} = 0$

Solving this for Q one obtains: ~~Aside RRA measure~~

$$Q = L + \frac{(P_b - 1)\mu_W(Q)^{2\gamma+1}}{\gamma\sigma^2}$$

Note that this expression is similar to the optimal order quantity under CARA (previously derived):

$Q_{CARA} = \frac{1}{\alpha\sigma^2}(P_b^{CARA} - 1) + L$ w/ an additional

$\mu_W(Q)^{2\gamma+1}$ term in the numerator of the fraction showing order qty. is sensitive to the (expected) level of wealth of the trader.

The optimal bid price for dealer 1

I now derive the optimal bid price for dealer 1 (Note: the problem is symmetric so dealer 2 will have the same solution).

Dealer 1 receives random inventory pos'n $I_\$^1 \sim U(-R,R)$. If s/he has the best bid price and serves order qty Q, his/her wealth at the end of the game is:

$$W = (I_\$^1 + Q)(1+z) - P_b^1 Q$$

If s/he does not receive the order flow, s/he will have wealth

$$W^{NT} = I_\$^1(1+z)$$

At some price $P_{b,r}^1$ the dealer will be indifferent between trading and not trading. This happens when $U(W^{NT}) = U(W)$.

Assume dealer 1's utility is:

$$u_1(w) = \begin{cases} \dfrac{w^{1-\gamma}-1}{1-\gamma}, & \gamma \neq 1 \\ \ln w, & \gamma = 1 \end{cases}$$

Case 1: $\gamma \neq 1$

Let $\mu_w(p^1_{b,r})$ ~~be the~~ $= \mathbb{E}\left[(I^1_1+Q)(1+z) - p^1_{b,r}Q\right]$
$= I^1_1 + Q - p_{b,r}Q = I^1 + Q(1-p^1_{b,r})$

(Note: I^1_1 has been realized and is a constant)

The 2^{nd}-order Taylor-series approx of $u(w)$ about μ_w is:

$$u(w)\Big|_{w=\mu_w} \approx u(\mu_w) + (w-\mu_w)u'(\mu_w) + \frac{1}{2}(w-\mu_w)^2 u''(\mu_w)$$

Taking expectations:

$$\mathbb{E}[u(w)] \approx \mathbb{E}[u(\mu_w)] + \frac{1}{2}u''(\mu_w)(w-\mu_w)^2$$

where: I have used $\mathbb{E}[w-\mu_w]=0$. Note that $\mathbb{E}[(w-\mu_w)^2] = (I^1+Q)^2\sigma^2$. $\therefore$ $\mathbb{E}[u(w)] \approx \dfrac{\left[I^1+Q(1-p^1_{b,r})\right]^{1-\gamma}-1 \ldots}{1-\gamma}$

$$\ldots + \frac{1}{2}(I^1+Q)^2\sigma^2 \cdot -\gamma\left(I^1+Q[1-p^1_{b,r}]\right)^{-\gamma-1}$$

The (expected utility) from no trade is:

$$\mathbb{E}[u(w^{NT})] = \mathbb{E}\left[\frac{I^1(1+z)^{1-\gamma}-1}{1-\gamma}\right]$$

Taking the Taylor-series approx. around $\mu_w(p^1_{b,r})$:

$$u(w^{NT})\Big|_{w=\mu_w} \approx u(\mu_w) + (w^{NT}-\mu_w)u'(\mu_w) + \frac{1}{2}u''(\mu_w)(w^{NT}-\mu_w)^2$$

Taking expectations: $\mathbb{E}[u(w^{NT})] \approx \dfrac{I^1+Q(1-p^1_{b,r})^{1-\gamma}-1}{1-\gamma} + \dfrac{1}{2}I^{1\,2}\sigma^2 \ldots$

$$\ldots -\gamma \mu_W^{1-\gamma-1}$$

$$\mathbb{E}[U(W^{NT})] \approx \frac{\mu_W(p^1_{b,r})^{1-\gamma} - 1}{1-\gamma} - \frac{\gamma}{2} I^{1^2} \sigma^2 \mu_W (p^1_{b,r})^{-\gamma-1}$$

Dealer 1 ~~The trader~~ is indifferent when:

$$\mathbb{E}[U(W^{NT})] = \mathbb{E}[U(W)]$$

$$\cancel{\frac{[\mu_W(p^1_{b,r})]^{1-\gamma} - 1}{1-\gamma}} - \frac{\gamma}{\cancel{2}} I^{1^2} \cancel{\sigma^2} [\mu_W(p^1_{b,r})]^{-\gamma-1}$$

$$= \cancel{\frac{[\mu_W(p^1_{b,r})]^{1-\gamma} - 1}{1-\gamma}} - \frac{\gamma}{\cancel{2}} (I^1+Q)^2 \cancel{\sigma^2} \mu_W^{-\gamma-1}$$

$$I^{1^2} [\cancel{\mu_W(p^1_{b,r})}]^{\cancel{-\gamma-1}} = (I^{1^2} + 2QI^1 + Q^2)[\cancel{\mu_W(p^1_{b,r})}]^{\cancel{-\gamma-1}}$$

$$(I^1)^2 = (I^1)^2 + 2QI^1 + Q^2$$

$$(I^1)^2 - (I^1)^2 - 2QI = 0$$

~~This ha~~

Note that the reservation price is

$$p^1_{b,r} - \frac{\gamma\sigma^2}{2} \mu_W (p^1_{r,b})$$

w/o solving explicity

I argue that under perfect competition, the dealer's markup (not shown) will tend to zero and prices will tend to the reservation price.