Optimal Asset Sale Quantity

To solve for the optimal quantity Q using a Taylor Series approximation of the utility function, we proceed by expanding the utility function around the expected wealth. Here's the step-by-step process:

Step 1: Expand the Utility Function

The agent's final wealth is given by:

$$w(Q) = Q \cdot p_b + (L - Q) \cdot (1 + z)$$

This can be rewritten as:

$$w(Q) = (Q \cdot p_b + L - Q) + (L - Q) \cdot z = Q \cdot (p_b - 1) + L + (L - Q) \cdot z$$

We want to approximate the utility function $u(w(Q)) = \frac{w(Q)^{1-\gamma} - 1}{1-\gamma}$ using a Taylor series expansion around the expected wealth $\mathbb{E}[w(Q)]$.

Let's denote the expected wealth as $\mu_w(Q) = \mathbb{E}[w(Q)]$. The second-order Taylor series expansion of the utility function around $\mu_w(Q)$ is:

$$u(w(Q)) \approx u(\mu_w(Q)) + u'(\mu_w(Q)) \cdot (w(Q) - \mu_w(Q)) + \frac{1}{2} u''(\mu_w(Q)) \cdot (w(Q) - \mu_w(Q))^2$$

Step 2: Compute the Expected Utility

First, we calculate the expected wealth:

$$\mu_w(Q) = \mathbb{E}[w(Q)] = Q \cdot p_b + (L - Q)$$

Now, compute the first and second derivatives of the utility function with respect to wealth w:

$$u'(w) = \frac{d}{dw}\left(\frac{w^{1-\gamma} - 1}{1 - \gamma}\right) = w^{-\gamma}$$
$$u''(w) = \frac{d^2}{dw^2}\left(\frac{w^{1-\gamma} - 1}{1 - \gamma}\right) = -\gamma \cdot w^{-\gamma-1}$$

Substitute these into the Taylor expansion:

$$\mathbb{E}[u(w(Q))] \approx u(\mu_w(Q)) + \frac{1}{2} u''(\mu_w(Q)) \cdot \mathrm{Var}(w(Q))$$

Given that $u(\mu_w(Q)) = \frac{\mu_w(Q)^{1-\gamma} - 1}{1-\gamma}$ and $u''(\mu_w(Q)) = -\gamma \cdot \mu_w(Q)^{-\gamma-1}$, we have:

$$\mathbb{E}[u(w(Q))] \approx \frac{\mu_w(Q)^{1-\gamma} - 1}{1 - \gamma} - \frac{\gamma}{2} \cdot \mu_w(Q)^{-\gamma-1} \cdot \mathrm{Var}(w(Q))$$

Step 3: Calculate the Variance of Wealth

The variance of the final wealth $w(Q)$ is:

$$\mathrm{Var}(w(Q)) = \mathrm{Var}((L - Q) \cdot z) = (L - Q)^2 \cdot \mathrm{Var}(z) = (L - Q)^2 \cdot \sigma^2$$

Substitute this into the expected utility expression:

$$\mathbb{E}[u(w(Q))] \approx \frac{\mu_w(Q)^{1-\gamma} - 1}{1-\gamma} - \frac{\gamma}{2} \cdot \mu_w(Q)^{-\gamma-1} \cdot (L-Q)^2 \cdot \sigma^2$$

Step 4: Maximize the Expected Utility

To find the optimal Q, take the derivative of the expected utility with respect to Q and set it to zero:

$$\frac{d}{dQ}\mathbb{E}[u(w(Q))] = \frac{d}{dQ}\left(\frac{\mu_w(Q)^{1-\gamma} - 1}{1-\gamma} - \frac{\gamma}{2} \cdot \mu_w(Q)^{-\gamma-1} \cdot (L-Q)^2 \cdot \sigma^2\right) = 0$$

Differentiate $\mu_w(Q)$ with respect to Q:

$$\frac{d\mu_w(Q)}{dQ} = p_b - 1$$

Now differentiate the expected utility:

$$\frac{(1-\gamma) \cdot \mu_w(Q)^{-\gamma} \cdot (p_b - 1)}{1-\gamma} - \frac{\gamma}{2} \cdot (-\gamma - 1) \cdot \mu_w(Q)^{-\gamma-2} \cdot (p_b - 1) \cdot (L-Q)^2 \cdot \sigma^2 - \gamma \cdot \mu_w(Q)^{-\gamma-1} \cdot (L-Q) \cdot (-1) \cdot \sigma^2 = 0$$

This simplifies to:

$$\mu_w(Q)^{-\gamma} \cdot (p_b - 1) - \frac{\gamma \cdot (L-Q) \cdot \sigma^2}{\mu_w(Q)^{\gamma+1}} = 0$$

Simplifying further:

$$(p_b - 1) \cdot \mu_w(Q)^{2\gamma+1} = \gamma \cdot (L-Q) \cdot \sigma^2$$

Solve for Q:

$$Q = L - \frac{(p_b - 1) \cdot \mu_w(Q)^{2\gamma+1}}{\gamma \cdot \sigma^2}$$

Given the complexity of this equation, it can typically be solved numerically. However, this equation gives the optimal Q in terms of the parameters p_b, γ, σ^2, and L.