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Exact critical line of an angle-modulated triangular-lattice Ising model and Monte Carlo verification

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Abstract We consider a nearest-neighbor Ising model on the triangular lattice with direction-dependent couplings $J_{ij} = J + J_a \cos(n\phi_{ij})$, motivated by bond-directional modulation. When the bond angles take only the three lattice directions, the model reduces to an anisotropic triangular Ising model with three couplings J_k. Using the exact triangular-lattice critical manifold, we compute the critical temperature T_c as a function of the dimensionless modulation ratio J_a/J across both the ferromagnetic and stripe regimes, including the gauge-equivalent two-negative-bond sector. We identify a singular decoupling point (for the paper case $n = 2$) where two couplings vanish and prove that the finite-temperature transition disappears, $T_c = 0$. Cluster Monte Carlo simulations with bootstrap uncertainty provide a numerical check of the analytic critical line and quantify finite-size effects. *The exact critical manifold itself is classic* [2,4]; our contributions are its explicit application to the angle-modulated parametrization, the extension to the stripe sector via a gauge map, a transparent proof and asymptotic analysis of the $J_a = 2J$ singularity, and a reproducible code+plot pipeline enabling direct analytic–MC overlays.

Keywords Ising model · triangular lattice · anisotropy · exact critical manifold · Monte Carlo

1 Introduction

Nearest-neighbor Ising models on the triangular lattice are a standard laboratory for geometric effects and anisotropy. Recent work has emphasized bond-directional modulations inspired by spin-lattice couplings, leading to angle-dependent exchanges of the form $J_{ij} = J + J_a \cos(n\phi_{ij})$ [3]. For bonds

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restricted to the three triangular directions, this model becomes an anisotropic triangular Ising model, for which the exact critical manifold is known [2,4].

The analytic input is the anisotropic triangular Ising critical manifold from early exact work [2,4]. The purpose of this manuscript is twofold. First, we connect that classic result to the angle-modulated parametrization used in recent bond-directional models [3]. Second, we provide a clean, reproducible analytic–numerical comparison.

Contributions. This work makes the following contributions:

1. An explicit mapping from the angle-modulated rule $J_{ij} = J + J_a \cos(n\phi_{ij})$ to a one-parameter critical line $T_c(J_a)$ by solving the exact manifold (4) for the three directional couplings $J_k(J_a)$.
2. An extension of the same critical manifold to the *stripe* sector (exactly two negative couplings) via a gauge transformation $J_k \mapsto |J_k|$, clarifying when the closed-form condition remains applicable.
3. A transparent proof that the decoupling point ($J_a = 2J$ in the paper case $n = 2$) has no finite-temperature solution of (4), hence $T_c = 0$, together with an asymptotic explanation of the logarithmically slow collapse toward this point.
4. A reproducible workflow (scripts + Makefile targets) that generates (a) MC estimates with uncertainty and (b) high-resolution analytic curves (including arbitrary-precision zooms near the singularity), enabling direct overlays.

2 Model

We consider Ising spins $s_i = \pm 1$ on a periodic $L \times L$ rhombus of the triangular lattice with Hamiltonian

$$\mathcal{H} = -\sum_{\langle i,j \rangle} J_{ij}\, s_i s_j, \qquad J_{ij} = J + J_a \cos(n\phi_{ij}). \tag{1}$$

If the bond angles take only three values ϕ_k corresponding to the three lattice directions, the model reduces to three couplings

$$J_k = J + J_a \cos(n\phi_k), \qquad k \in \{1, 2, 3\}. \tag{2}$$

We set $k_B = 1$ and define $K_k = J_k/T$.

Dimensionless control parameter. Since an overall rescaling of energies rescales temperature, we report the phase boundary as T_c/J versus the dimensionless modulation $a \equiv J_a/J$. In the figures we plot T_c in units where $J = 1$, and label the horizontal axis as J_a/J for clarity.

Paper case ($n = 2$). For the conventional triangular directions with angles $\phi_1 = 0$, $\phi_2 = 2\pi/3$, and $\phi_3 = 4\pi/3$, and for $n = 2$, one obtains

$$J_1 = J + J_a, \qquad J_2 = J_3 = J - \frac{J_a}{2}. \tag{3}$$

For $J_a < 2J$ all couplings are ferromagnetic. For $J_a > 2J$, two couplings are negative and the ordered phase corresponds to stripes [3].

3 Exact critical manifold and gaugeable stripe sector

For ferromagnetic couplings $J_k > 0$ the anisotropic triangular Ising critical manifold can be written as [2]

$$e^{-2(K_1+K_2)} + e^{-2(K_2+K_3)} + e^{-2(K_3+K_1)} = 1. \tag{4}$$

Two-negative-bond (stripe) case. If exactly two J_k are negative, the bond-sign pattern is locally unfrustrated: each elementary triangle contains one bond of each type and the product of bond signs is $\mathrm{sgn}(J_1 J_2 J_3) = +1$. Therefore a site-dependent gauge transformation $s_i \mapsto \eta_i s_i$ with $\eta_i = \pm 1$ can map the model to ferromagnetic couplings $|J_k|$ (equivalently, it relabels the ordered phase as stripes in the original variables). On a finite periodic $L \times L$ system, this gauge map can introduce a global twist (equivalently, change boundary conditions) when L is odd; in the thermodynamic limit the critical temperature is unchanged. Consequently, the stripe-sector transition temperature is obtained by applying Eq. (4) to $K_k = |J_k|/T$.

4 Singular decoupling point and proof that $T_c = 0$

In the paper case (3), the point $J_a = 2J$ yields

$$J_1 = 3J, \qquad J_2 = J_3 = 0. \tag{5}$$

Substituting into (4) gives

$$2e^{-2(3J+0)/T} + e^{-2(0+0)/T} = 2e^{-6J/T} + 1 = 1. \tag{6}$$

For any finite $T > 0$, $e^{-6J/T} > 0$ and therefore $2e^{-6J/T} + 1 > 1$, so the equality cannot hold. Hence there is *no* finite solution $T_c > 0$ at $J_a = 2J$, i.e.

$$T_c(J_a = 2J) = 0. \tag{7}$$

5 Asymptotics near $J_a = 2J$ (logarithmically slow collapse)

Let $\delta = |J_a - 2J|$ so that $|J_2| = |J_3| = \delta/2$ and $J_1 = J + J_a = 3J + \mathcal{O}(\delta)$ near $J_a = 2J$. Equation (4) reduces (using $J_2 = J_3$) to

$$2e^{-2(J_1+|J_2|)/T} + e^{-4|J_2|/T} = 1, \tag{8}$$

where the absolute values cover both the ferromagnetic side ($J_a < 2J$) and the gauge-mapped stripe side ($J_a > 2J$). For $\delta \to 0^+$, the second term approaches 1. Using $e^{-4|J_2|/T} = 1 - (4|J_2|/T) + o(|J_2|/T)$, the balance requires $e^{-2(J_1+|J_2|)/T_c} \sim 2|J_2|/T_c$, i.e. $e^{-6J/T_c} \sim \delta/T_c$ to leading order, implying a logarithmically slow approach

$$T_c(\delta) \sim \frac{6J}{\ln(J/\delta)} \qquad (\delta \to 0^+), \tag{9}$$

up to subleading $\ln T_c$ corrections. This explains why extremely small windows around $J_a = 2$ can still yield $T_c = \mathcal{O}(10^{-1})$.

6 Monte Carlo simulation

We simulate the model on an $L \times L$ periodic rhombus with $N = L^2$ spins. In regimes where a gauge map yields ferromagnetic effective couplings, we use Wolff single-cluster updates [5] to reduce critical slowing down; otherwise we fall back to Metropolis updates. We estimate the (pseudo-)critical temperature from the peak of the magnetic susceptibility, $\chi = (N/T)(\langle m^2\rangle - \langle|m|\rangle^2)$, with $m = \frac{1}{N}\sum_i s_i$. Uncertainties are estimated by block-averaging the time series and applying bootstrap resampling to the block estimates (see e.g. [1]). All MC results shown are at finite L; the resulting pseudo-critical point differs from the thermodynamic T_c by finite-size effects, but provides a practical numerical validation of the analytic curve.

7 Results

Figure 1 overlays the analytic $T_c(J_a/J)$ curve with MC estimates obtained from the susceptibility peak at fixed L. Finite-size effects shift the MC pseudo-critical point relative to the thermodynamic T_c; nevertheless the overall trend agrees with the exact line. In the phase-diagram annotations we use the standard shorthand: FE (ferromagnetic), PM (paramagnetic), and ST (stripe-ordered).

Figure 2 shows a high-precision zoom around $J_a = 2J$ illustrating the cusp and the slow decay toward $T_c = 0$.

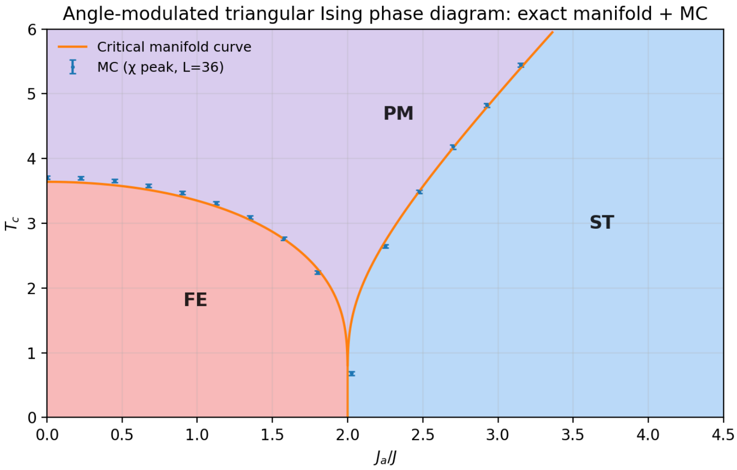

Fig. 1 Phase diagram from the exact triangular-lattice critical manifold (T_c vs. J_a/J), with MC estimates for $L = 36$; shaded regions indicate FE/PM/ST phases.

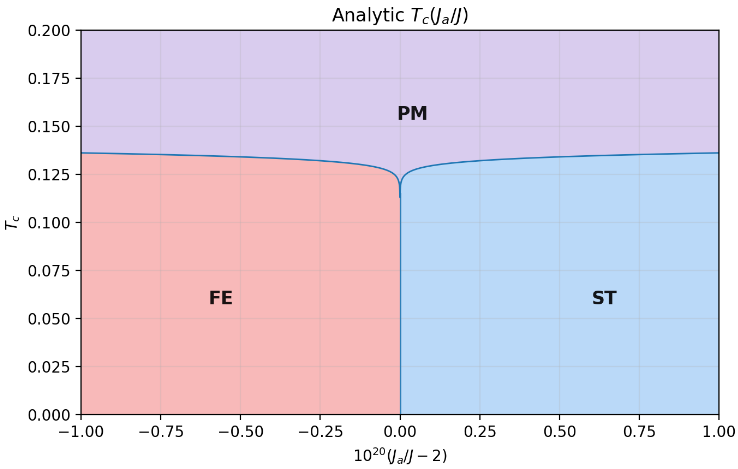

Fig. 2 High-precision analytic zoom around $J_a = 2J$.

8 Conclusion

We provided an analytic critical line for an angle-modulated triangular-lattice Ising model by reducing to an anisotropic triangular Ising model and applying the exact critical manifold. The stripe sector (two negative couplings) is covered via a gauge map to $|J_k|$. At the decoupling point $J_a = 2J$ (for $n = 2$ and the standard triangular directions), we proved that the finite-temperature transition disappears, $T_c = 0$, and the approach is logarithmically slow. Monte

Carlo simulations with cluster updates corroborate the analytic curve and provide a practical numerical check.

Data and code availability

All plots and numerical results in this manuscript are generated by scripts in the accompanying repository directory. The plotting scripts output PNG figures and also export CSV/JSON data for reproducibility. All code is available at https://github.com/arnocandel/angle-modulated-triangular-lattice-ising.

Use of AI tools

AI tools were used for editorial purposes. All responsibility for the content lies with the authors.

Author contributions

Arno Candel and Michael Rey contributed equally to conceptualization, software, analysis, visualization, and writing. Both authors approved the final manuscript.

Competing interests

The authors declare no competing interests.

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