Optimal trend following portfolios

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Abstract

This paper derives an optimal portfolio that is based on trend-following signal. Building on an earlier related article, it provides a unifying theoretical setting to introduce an autocorrelation model with the covariance matrix of trends and risk premia. We specify practically relevant models for the covariance matrix of trends. The optimal portfolio is decomposed into four basic components that yield four basic portfolios: Markowitz, risk parity, agnostic risk parity, and trend following on risk parity. The overperformance of the proposed optimal portfolio, applied to cross-asset trading universe, is confirmed by empirical backtests. We provide thus a unifying framework to describe and rationalize earlier developed portfolios.

Keywords: Portfolio Management; Trend following; Risk Parity; Sharpe Ratio

JEL: G11, G15, G4, C6.

1. Introduction

In systematic trading on exchange markets, detecting minuscule trends in asset price fluctuations is like looking for a needle in a haystack. While the chance of a correct forecast of the next price move is fairly close to 50%, even a minor excess above this value can lead to significant profits after multiple transactions supervised by an algoritmic trading system according to prescribed rules. A key feature of trend following investment strategies is that they can be implemented by applying simple rules [1]. For instance, trend following (TF) strategies adjust their market exposure by assuming the next move of an asset price to be in a trend with its past variations [2, 3, 4, 5, 6]. While its profitability is debatable (as it contradicts the market efficiency hypothesis) [7, 8, 9, 10, 11], trend following remains a widely used strategy among professional asset managers.

Most explanations on trend following success rely on behavioral theory of asset pricing, which includes boundedly rational investors' initial underreaction to new information that allows momentum traders to take profit of any under-reaction by trend chasing [12]. It also includes well known herding behavior, which gives rise to a collective decision-making process by investors' beliefs that are either optimistic or pessimistic [13]. More intuitively, Bhansali et al. explain the ability of trend following to deliver substantial returns because it is a cousin of the cross-sectional momentum anomaly [14]. Hence, a trend-following strategy typically takes long positions in securities with positive past returns and short positions in securities with negative past returns. As such, Moskowitz et al. document that a portfolio of time-series momentum strategies, or equivalently trend-following strategies, across all asset classes delivers substantial abnormal returns performs best during extreme markets [11]. In the same vein, Hurst et al. find that trend following has been consistently profitable throughout the past 137 years, which makes them conclude that based on their long-term out-of-sample evidence that price trends in markets is not a product of statistical randomness or even data mining [15]. Even though forecasting individual asset prices is rather hopeless, the statistical analysis of the ensemble of numerous cross-correlated asset prices can reveal more reliably profitable trends in the market. Typically, fund managers build diversified portfolios to decorrelate constituent TF strategies as much as possible in order to enhance their profit and reduce risk. Relying on a Gaussian model with both auto-correlation and crosscorrelation structures of asset returns, it was shown that conventional allocation schemes lead to sub-optimal portfolios [16]. In particular, inter-asset cross-correlations, if accounted for properly, can facilitate trend detection and thus significantly improve the risk-adjusted portfolio returns. However, the optimal allocation of trend following strategies developed in [16] remains too sophisticated for direct applications in finance industry. In particular, the optimal solution has to be obtained by solving a very large system of nonlinear equations that limits its implementation for large trading universes. Moreover, numerous parameters linked to the asset autocorrelation structures are unknown and very difficult to estimate.

In the present paper, we extend the optimal allocation scheme developed in [16] in two directions. On one hand, we relax the former assumption of zero mean returns and include the effect of small but always present net returns (risk premia). While their contribution is negligible at short time scales, mean returns affect the allocation weights and thus the overall profitability of the optimal portfolio at longer time horizons. On the other hand, we simplify as much as possible the covariance matrices accounting for auto- and cross-correlations of assets. Our goal here is to propose a minimal theoretical setting that can produce explicit, easily interpretable and practically implementable solutions of the allocation problem. In particular, we show that, under certain assumptions specified below, the optimal solution can be seen as a linear combination of four basic portfolios: risk parity, naive Markowitz solution (when expectations of returns are based on trends), trend on risk parity, and agnostic risk portfolio [17]. The choice of involving risk parity in the research design is motivated by a more efficient way of allocating assets according to their risk contribution to the portfolio because weights are proportional to inverse volatility, which seeks more or less equal risk-exposure between all the asset classes within a portfolio. Risk parity strategies are founded on the intuition of Black [18] that safer assets should offer higher risk-adjusted returns than riskier assets [19]. Moreover, a benefit over meanvariance optimization, is that investors are not required to formulate any assumption on the distribution of the returns [20].

The paper is organized as follows. In Sec. 2, we introduce the autocorrelation model with the covariance matrix of trends and risk premia. We derive the main formula of the paper that describes the optimal portfolio depending on the covariance matrix of returns, the covariance matrix of trends and the risk premia. In Sec. 3, we introduce the specifications for the covariance matrix of trends that make several basic portfolios reported in the literature optimal from the theoretical point of view. The optimal portfolio is obtained

as a linear combination of these basic portfolios. In Sec. 4, we present the empirical backtest for different portfolios. We compare them to the simulated performance of their optimal linear combination. Section 5 summarizes the main results, while technical derivations are reported in Appendices.

2. Mathematical model and its optimal solution

We first extend the mathematical model of assets returns and linear trend following strategies introduced in [16, 21]. We present then an approximate optimal solution that maximizes the squared Sharpe ratio of the portfolio.

2.1. Mathematical model

First, we extend the mathematical model introduced in [16] by adding drift terms to describe the risk premia that should be positive according to the theory. We postulate that the return¹ r_t^j of the j-th asset at time t has three contributions: a constant drift μ^j , an instantaneous fluctuation (noise) ε_t^j , and a stochastic trend, which is modeled as a linear combination of random fluctuations $\xi_{t'}^j$,

$$r_t^j = \mu^j + \varepsilon_t^j + \sum_{t'=1}^{t-1} \mathbf{A}_{t,t'}^j \xi_{t'}^j,$$
 (1)

where the matrix \mathbf{A}^j describes the stochastic trend of the j-th asset; in particular, when the elements of \mathbf{A}^j decay exponentially (see Appendix A), this is a discrete version of a stochastic multi-asset price model in which the trends follow unobservable correlated Ornstein-Uhlenbeck processes. In turn, $\varepsilon_1^j, \ldots, \varepsilon_t^j$ and ξ_1^j, \ldots, ξ_t^j are two sets of independent Gaussian variables with mean zero and the following covariance structure:

$$\langle \varepsilon_t^j \varepsilon_{t'}^k \rangle = \delta_{t,t'} \mathbf{C}_{\varepsilon}^{j,k}, \qquad \langle \xi_t^j \xi_{t'}^k \rangle = \delta_{t,t'} \mathbf{C}_{\varepsilon}^{j,k}, \qquad \langle \varepsilon_t^j \xi_{t'}^k \rangle = 0,$$
 (2)

¹Throughout this paper, we call by "returns" additive logarithmic returns resized by realized volatility which is a common practice on futures markets [8, 22]. Although asset returns are known to exhibit various non-Gaussian features (so-called "stylized facts" [23, 24, 25, 26, 27, 28]), resizing by realized volatility allows one to reduce, to some extent, the impact of changes in volatility and its correlations [29, 30, 31], and to get closer to the Gaussian hypothesis of returns [32].

where $\delta_{t,t'} = 1$ for t = t' and 0 otherwise, and $\langle \ldots \rangle$ denotes the expectation. Here \mathbf{C}_{ε} and \mathbf{C}_{ξ} are the covariance matrices that describe inter-asset correlations of noises ε_t^j and of stochastic trend components ξ_t^j , respectively. The covariance matrix of Gaussian asset returns is then

$$\mathbf{C}_{t,t'}^{j,k} \equiv \langle r_t^j r_{t'}^k \rangle = \delta_{t,t'} \mathbf{C}_{\varepsilon}^{j,k} + \mathbf{C}_{\xi}^{j,k} (\mathbf{A}^j \mathbf{A}^{k,\dagger})_{t,t'}, \tag{3}$$

where \dagger denotes the matrix transposition. For each asset, the stochastic trend induces auto-correlations due to a linear combination of exogenous random variables ξ_t^j which are independent from short-time noises ε_t^j . Moreover, these *auto-correlations* (described by the matrix \mathbf{A}^j) are considered to be independent from *inter-asset cross-correlations* (described by matrices \mathbf{C}_{ε} and \mathbf{C}_{ξ}). In particular, the covariance matrices \mathbf{C}_{ε} and \mathbf{C}_{ξ} do not depend on time.

A TF portfolio is composed of n assets with positive or negative weights Π_t^j , which are in general re-evaluated at each time t (e.g., on daily basis). Here Π_t^j is the position² of the TF strategy on the j-th asset at time t, which is evaluated as a weighted linear combination of the signals from all assets:

$$\Pi_t^j = \sum_{k=1}^n \omega_t^{j,k} \ s^k(r_1^k, \dots, r_{t-1}^k), \tag{4}$$

where $s^k(r_1^k, \ldots, r_{t-1}^k)$ is a signal based on past returns of the k-th asset, with weights $\omega_t^{j,k}$ to be determined at each time t. The incremental profit-and-loss (P&L) of a TF portfolio (i.e., the total return of the portfolio at time t) is

$$\delta \mathcal{P}_t = \sum_{j=1}^n r_t^j \ \Pi_t^j = \sum_{j,k=1}^n \omega_t^{j,k} \ r_t^j \ s^k(r_1^k, \dots, r_{t-1}^k), \tag{5}$$

where $\omega_t^{j,k}$ can thus be interpreted as the weight of the k-th signal onto the position of j-th asset. The particular case of diagonal weights (when $\omega_t^{j,k} = 0$ for $j \neq k$) corresponds to a portfolio of n TF strategies with weights $\omega_t^{j,j}$. Therefore, the standard portfolio allocation problem is included in our framework, in which the diagonal weight $\omega_t^{j,j}$ represents the amount of capital

²The term "position" refers to the exposure or investment in a given asset. It is generally used in futures trading where position can be either positive (long) or negative (short) [33].

allocated to the j-th asset. In general, non-diagonal terms allow one to benefit from inter-asset correlations to enhance the profitability of the TF portfolio.

Following [21], we consider a TF strategy whose signal is determined by a *linear* combination of earlier returns:

$$s^{k}(r_{1}^{k}, \dots, r_{t-1}^{k}) = \sum_{t'=1}^{t-1} \mathbf{S}_{t,t'}^{k} r_{t'}^{k}, \tag{6}$$

with given matrices \mathbf{S}^k . In summary, the mathematical model is fixed by choosing the vector μ of drifts μ^j and the matrices \mathbf{C}_{ε} , \mathbf{C}_{ε} , \mathbf{A}^j , and \mathbf{S}^k .

2.2. Optimal solution

Relying on the Gaussian character of the model, the mean, $\langle \delta \mathcal{P}_t \rangle$, and the variance, $\operatorname{var}\{\delta \mathcal{P}_t\}$, of the incremental profit-and-loss $\delta \mathcal{P}_t$ can be computed [16]. In Appendix A, we provide general formulas for these two quantities for our extended model from Sec. 2.1. Using these formulas, one can therefore search for the weights $\omega_t^{j,k}$ that optimize a chosen criterion (e.g., to minimize the variance under a fixed expected return for the Markowitz theory). In this paper, we aim at finding the optimal weights $\omega_t^{j,k}$ that maximize the squared Sharpe ratio (or squared risk-adjusted return of the portfolio),

$$S^2 \equiv \frac{\langle \delta \mathcal{P}_t \rangle^2}{\text{var}\{\delta \mathcal{P}_t\}} \tag{7}$$

(note that S^2 is used instead of S just for convenient notations, the optimization results are identical in both cases). It was shown in [16] that this optimization problem is equivalent to solving a set of n^2 quadratic equations onto n^2 unknown weights $\omega_t^{j,k}$ (see Appendix A for details). Since the mean and the variance of the increment P&L depend on time due to the dynamic character of TF strategies, the optimal weights need to be re-evaluated at each time step of the TF strategy. Unfortunately, this formal solution is impractical due to its computational costs for realistic trading universes with many hundred of assets. Moreover, the solution depends on numerous model parameters (matrices \mathbf{C}_{ε} , \mathbf{C}_{ξ} , \mathbf{A}^j and \mathbf{S}^k) whose accurate calibration from empirical data is not feasible.

These limitations motivated us to search for simplifications under which practically relevant explicit solutions are possible. The fundamental challenge in forecasting next price moves follows from the fact that the short-time

noises ε_t^j provide the dominant contributions to the returns. In other words, the covariance matrix of returns, \mathbf{C} , is essentially given by the matrix \mathbf{C}_{ε} , whereas the matrices \mathbf{C}_{ξ} and $(\mathbf{M})_{ij} = \mu^i \mu^j$ are negligible in comparison to \mathbf{C}_{ε} . In this situation (and under some other, more technical simplifications described in Appendix A), we derive in Appendix A the explicit approximate expression for the matrix ω of the optimal weights:

$$\omega_t \approx \mathbf{C}^{-1} (h_{\xi}(t) \mathbf{C}_{\xi} + h_{\mu}(t) \mathbf{M}) \mathbf{C}^{-1},$$
 (8)

where $h_{\xi}(t)$ and $h_{\mu}(t)$ are two explicitly known time-dependent functions that determine relative contributions of auto-correlation induced stochastic trends and net returns, respectively. At t goes to infinity, functions $h_{\xi}(t)$ and $h_{\mu}(t)$ tend to constants h_{ξ} and h_{μ} that describe relative contributions of stochastic trends and net returns in the steady state regime:

$$\omega \approx \mathbf{C}^{-1} (h_{\xi} \mathbf{C}_{\xi} + h_{\mu} \mathbf{M}) \mathbf{C}^{-1}. \tag{9}$$

In this regime, the matrix ω of optimal weights does not depend on time anymore.

This approximate optimal solution is the main theoretical result of the paper. At first thought, a linear superposition of two contributions, stochastic trends (asset autocorrelations) and drifts, is rather surprising given that, without our simplifications, one would have to solve a large system of nonlinear equations. The linearity of the solution is a very appealing property. In fact, it simplifies the determination of the optimal portfolio, even though the matrices \mathbf{M} and \mathbf{C}_{ξ} , which are very difficult to estimate, remain partly unknown. Indeed, the optimal portfolio is a linear combination of two basic portfolios, each of which can be determined easier. The first portfolio is based on the risk premia, \mathbf{M} , as if stochastic trends (autocorrelation) did not exist. The second portfolio depends only on the covariance matrix of trends, \mathbf{C}_{ξ} , as if the risk premia did not exist.

In the following, we focus on the steady state solution (9). In the next section, we discuss specifications for the matrices \mathbf{M} and \mathbf{C}_{ξ} , under which the solution leads to basic portfolios referred in the literature as optimal. We will then show that it is their linear combination that is optimal.

3. Derivation of basic particular and generalized optimal portfolios

3.1. Risk-parity portfolio (RP)

We assume here that there is no autocorrelation in the returns (i.e., $\mathbf{C}_{\xi} = 0$) and that only drifts contribute (i.e., $\mathbf{M} \neq 0$). The approximate optimal

solution is then

$$\omega \approx h_{\mu} \left(\mathbf{C}^{-1} \mu \right) (\mu^{\dagger} \mathbf{C}^{-1}) = h_{\mu} \left(\mathbf{C}^{-1} \mu \right) (\mathbf{C}^{-1} \mu)^{\dagger}, \tag{10}$$

where we used a direct product representation $\mathbf{M} = \mu \mu^{\dagger}$ with the vector μ of mean returns μ^{j} . Denoting by $\pi = \mathbf{C}^{-1}\mu$ the vector of standard Markowitz weights, one gets $\omega^{j,k} = h_{\mu}\pi^{j}\pi^{k}$. As a consequence, the optimal portfolio weights read

$$\Pi_t^j = \sum_{k=1}^n \omega^{j,k} \, s^k = h_\mu \, \pi^j \left(\sum_{k=1}^n \pi^k s^k \right). \tag{11}$$

As the weights are determined up to an arbitrary multiplicative factor, the sum in parentheses can be included into this constant, yielding

$$\Pi_t \propto \pi = \mathbf{C}^{-1} \mu. \tag{12}$$

To interpret this result as risk-parity portfolio, one can assume that the Sharpe ratio is the same for all the instruments, i.e., the drift μ^j is proportional to the volatility $\Sigma_{jj} = \sqrt{C_{jj}}$ of the asset j (except for exchanges rates instruments whose drift could be better assumed to be zero). In that way the risk is fairly rewarded and the drift describes a risk premium. This is the usual assumption made in the literature on the "Maximum Diversification" equity portfolio [34]. However, when it is applied to different asset classes (stocks, bonds, commodities), the optimal portfolio is similar to the better known risk parity portfolio that allows financial leverage and targets the same risk on every asset classes, with the constraint to hold only long positions. This portfolio investment category, named "risk parity", regroups massive investment. For this reason, we use the term "risk parity portfolio" instead of "maximum diversification portfolio". The weights of this portfolio can thus be written as

$$\Pi_t \propto \mathbf{C}^{-1} \Sigma.$$
 (13)

The interpretation of risk parity is rather simple: if the inter-asset correlations could be neglected, i.e., \mathbf{C}^{-1} would be diagonal, and thus $\Pi_t \propto 1/\Sigma_{jj}$, i.e., it would be close to the equally weighted (in volalility) portfolio. It should be highly correlated to the "market mode" of the correlation matrix, which is most of the time the second eigenmode, when bonds and stocks are negatively correlated.

Even though the assumption that average asset returns increase proportionally with volatility is very approximative, it is a way to get a proxy of

the multi asset global market portfolio (the resulting optimal portfolio will correspond to the market portfolio; the equilibrium requires that all assets have a beta which is proportional to its volatility divided by the volatility of the market portfolio). We could have used a more complex assumption as for example in Ref. [35] but our assumption has the advantage to give an explicit solution and one can expect that this optimal portfolio is a decent proxy for the global market portfolio (see an attempt to measure the inventory of a large universe of assets worldwide to proxy for a theoretical market portfolio in [36]; the portfolio was called "global market portfolio" as it is composed of all risky assets in the world in proportion to their market capitalization). Note that the maximum diversification portfolios were found to have an excess return quite similar to the capitalization weighted market portfolio [37]. Therefore the risk parity portfolio (13) captures the global average risk premium and plays a very special role in asset management.

The drawback of this portfolio is its very high sensitivity to the estimation of the correlation matrix: if cleaning of this matrix is not good enough, some long-short positions, capturing fictitious correlations, can appear.

3.2. Naive Markowitz porfolio (NM)

When expectations are based exclusively on trends, the optimal Markowitz solution can be retrieved. In fact, we consider here that $\mathbf{M} = 0$, but conditional drifts can be represented through stochatistic trends. If we assume that \mathbf{C}_{ξ} is proportional to \mathbf{C} , then the naive Markowitz portfolio reads

$$\Pi \propto \mathbf{C}^{-1} \mathbf{C}_{\xi} \mathbf{C}^{-1} \mathbf{s} \propto \mathbf{C}^{-1} \mathbf{s},$$
 (14)

where **s** is the vector of signals. The portfolio is easy to interpret as the result of independent trend-following strategies applied to the eigenvectors u_k of **C** with allocations in the realized risk defined to be proportional to the inverse of the square root of the eigenvalues λ_k of **C**. In fact, using the spectral decomposition of the matrix **C**, one gets

$$\Pi \propto \sum_{k} \lambda_k^{-1}(u_k^{\dagger} \mathbf{s}) u_k, \tag{15}$$

i.e., the portfolio is a linear combination of eigenvectors u_k , invested with the weights $\lambda_k^{-1}(\mathbf{s}^{\dagger}u_k)$. Since the return of a portfolio with weights given by an eigenvector u_k has the variance $\operatorname{var}\{\sum_j u_k^j r^j\} = (u_k \mathbf{C} u_k) = \lambda_k$, the factor $(\mathbf{s}^{\dagger}u_k)$ scales as $\lambda_k^{1/2}$ so that the above weights are proportional to $\lambda_k^{-1/2}$.

According to the naive Markowitz portfolio, the allocation would be optimal if there were more trends on eigenvectors with small eigenvalues that is not realistic from the financial point of view.

3.3. Agnostic risk parity portfolio (ARP)

To target the same unconditional risk on any eigenvector of the correlation matrix, Benichou et al. proposed the agnostic risk parity portfolio [17]. This specific asset allocation allows to balance the risk between all the principal components of the correlation matrix. However, the optimality of the Sharpe ratio of this portfolio was not discussed. Here we suggest a simple sufficient condition that makes the agnostic risk parity portfolio optimal. We assume that the correlation matrix of trends, C_{ξ} , has only one eigenvalue different from zero and that the associated eigenvector is unknown. This is coherent with the assumption of Benichou et al. who considered the identity matrix to be the best estimation of the correlation matrix of signals and not the correlation matrix of returns. In other words, the specification of \mathbf{C}_{ξ} means that trends are concentrated on only one risk factor as if herding behavior could be efficient and amplified in only one dimension in the each time moment (in Appendix B, we introduce and discuss a very simple interaction model between agents that generates such a pattern). The trend on every instrument shares therefore the same common but unknown factor, which is likely to change from period to period and be for example the risk parity factor, the fly-to-quality factor or more specific factor linked to a local event as Brexit or specific initial trend as oil crash or bubble.

Neglecting the matrix \mathbf{M} of biases in returns, our optimal solution (9) implies

$$\Pi \approx h_{\xi} \mathbf{C}^{-1} \mathbf{C}_{\xi} \mathbf{C}^{-1} \mathbf{s}, \tag{16}$$

where h_{ξ} is a normalization constant. The covariance matrices \mathbf{C} and \mathbf{C}_{ξ} can be expressed in terms of the associated correlation matrix $\tilde{\mathbf{C}}$ and the normalized covariance matrix $\tilde{\mathbf{C}}_{\xi}$ as

$$\mathbf{C} = \Sigma \tilde{\mathbf{C}} \Sigma, \qquad \mathbf{C}_{\varepsilon} = \Sigma \tilde{\mathbf{C}}_{\varepsilon} \Sigma,$$
 (17)

where Σ is again the diagonal matrix of volatilities: $(\Sigma)_{ij} = \delta_{ij} \sqrt{(\mathbf{C})_{ii}}$. We denote by \mathbf{U} the matrix whose columns are composed of eigenvectors of the matrix $\tilde{\mathbf{C}}_{\xi}$,

$$\tilde{\mathbf{C}}_{\xi} = \mathbf{U}\mathcal{V}\mathbf{U}^{\dagger},\tag{18}$$

where \mathcal{V} is the diagonal matrix formed by the eigenvalues of $\tilde{\mathbf{C}}_{\xi}$: $\nu_1, 0, 0, \ldots, 0$. Given the structure of the matrix \mathcal{V} , it is convenient to split the matrix of eigenvectors as

$$\mathbf{U} = \mathbf{U}_{||} + \mathbf{U}_{\perp},\tag{19}$$

where $\mathbf{U}_{||}$ contains only the eigenvector u_1 (corresponding to ν_1), while \mathbf{U}_{\perp} contains the remaining eigenvectors. The optimal weights read then as

$$\Pi \approx h_{\xi} \mathbf{C}^{-1} \Sigma (\mathbf{U}_{||} + \mathbf{U}_{\perp}) \mathcal{V} (\mathbf{U}_{||} + \mathbf{U}_{\perp})^{\dagger} \Sigma \mathbf{C}^{-1} \mathbf{s}$$

$$= h_{\xi} \nu_{1} (v_{1}^{\dagger} \tilde{\mathbf{s}}) \Sigma^{-1} v_{1}, \qquad (20)$$

where $v_1 = \tilde{\mathbf{C}}^{-1}u_1$, $\tilde{\mathbf{s}} = \Sigma^{-1}\mathbf{s}$, and we used the particular structure of the matrix \mathcal{V} . Under the final assumption that u_1 is an L_2 -normalized eigenvector of the correlation matrix $\tilde{\mathbf{C}}$, one gets $v_1 = \lambda_1^{-1}u_1$, where λ_1 is the corresponding eigenvalue of $\tilde{\mathbf{C}}$. The optimal weights are then

$$\Pi \approx h_{\xi} \,\nu_1 \,\lambda_1^{-2} (u_1^{\dagger} \,\tilde{\mathbf{s}}) \Sigma^{-1} u_1. \tag{21}$$

We fix the normalization constant h_{ξ} by requiring that the variance of the portfolio,

$$V = (\Pi^{\dagger} \mathbf{C} \Pi) \approx h_{\xi}^{2} \nu_{1}^{2} \lambda_{1}^{-4} (u_{1}^{\dagger} \tilde{\mathbf{s}})^{2} \underbrace{(u_{1}^{\dagger} \Sigma^{-1} \mathbf{C} \Sigma^{-1} u_{1})}_{=\lambda_{1}}$$
(22)

(here we neglected smaller contribution from \mathbf{C}_{ξ}), is equal to 1 on average over all directions of u_1 :

$$1 = \langle V \rangle \approx h_{\xi}^2 \,\nu_1^2 \,\lambda_1^{-3} \,\langle (u_1^{\dagger} \,\tilde{\mathbf{s}})^2 \rangle, \tag{23}$$

We get then from Eq. (21):

$$\Pi \approx \frac{(u_1^{\dagger} \tilde{\mathbf{s}})}{\langle (u_1^{\dagger} \tilde{\mathbf{s}})^2 \rangle^{1/2}} \Sigma^{-1} \lambda_1^{-1/2} u_1 = \frac{(u_1^{\dagger} \tilde{\mathbf{s}})}{\langle (u_1^{\dagger} \tilde{\mathbf{s}})^2 \rangle^{1/2}} \Sigma^{-1} \tilde{\mathbf{C}}^{-1/2} u_1. \tag{24}$$

The average of the portfolio over all possible (uniformly chosen) directions u_1 yields

$$\langle \Pi \rangle \propto \Sigma^{-1} \tilde{\mathbf{C}}^{-1/2} \Sigma^{-1} \mathbf{s},$$
 (25)

where we omitted the proportionality constant $\langle (u_1^{\dagger} \tilde{\mathbf{s}})^2 \rangle^{-1/2}$. This is the agnostic risk parity portfolio [17], which differs from the naive Markowitz portfolio by the power -1/2 of the correlation matrix $\tilde{\mathbf{C}}$.

We note that the goal here was to provide a simple sufficient condition under which the agnostic risk-parity portfolio would be optimal. To get Eq. (25) with the matrix $\tilde{\mathbf{C}}^{-1/2}$, we employed numerous assumptions that make the sufficient condition too restrictive. We emphasize that the sufficient condition is not the necessary one, and we expect the ARP portfolio to be optimal under (much) weaker restrictions.

3.4. Trend-on-risk-parity portfolio (ToRP)

In this case, we assume that the special direction u_1 of the normalized covariancen matrix of trends, C_{ξ} , which was supposed to be unknown in the agnostick risk parity portfolio, is known and corresponds to the risk parity portfolio, $u_1 \propto 1$, where 1 is a vector composed of 1 for all stocks except for exchange rates instruments. Earlier studies have not shown any conclusive evidence for the direction of causality between interest rates and stock prices for US markets [38, 39]. Moreover, Chan et al. questioned the existence of a common trend between stock and bond prices [40]. At the same time, other works bring empirical evidences of a common part in stochastic trends in international stock markets [41] and explain why the market mode can be an eigenvector of the matrix C_{ξ} [43, 42, 44]. These works justify our consideration of the risk parity factor as the first eigenvector of the matrix \mathbf{C}_{ε} . As we just saw, this direction plays a special role by capturing the risk premium and thus helping to capture and to amplify the herding behavior. This means that trends on bonds and stocks are positively correlated, and the best way to capture these trends is to measure the trend on the risk parity portfolio.

Using Eq. (20) with $v_1 = \tilde{\mathbf{C}}^{-1}u_1$, we get

$$\Pi \propto ((\Sigma \mathbf{1})^{\dagger} \mathbf{C}^{-1} \mathbf{s}) \mathbf{C}^{-1} \Sigma \mathbf{1},$$
 (26)

where the proportionality constant can be chosen by fixing the variance of the portfolio, as in the agnostic risk-parity case. Here, we kept explicitly the scalar factor $((\Sigma 1)^{\dagger} \mathbf{C}^{-1} \mathbf{s})$, which depends on the signal \mathbf{s} . This factor corresponds to asset trends projected onto the risk parity portfolio and thus represents the trend of risk parity portfolio.

The risk parity portfolio is a very particular portfolio as it captures very well both the risk premia (or the carry) and a large part of the trends. Bhansali *et al.* confirm the link between the carry and the trends as they show that the trend has a better forecasting power when the carry is high

[14]. We can also extent this section to the trend on other factors that can capture the residual part of the trends. As an example, the Value and Momentum factors that could be good candidates are profitable in the equity world but also in the cross asset world [6]. Another possible extension would be to implement optimal equity market neutral trend following strategies on factors listed in [45, 46]. Hodges et al. show that trend on factor is the most efficient way to make factor timing as the trend is the best indicator to forecast the returns of the Value, Quality, Momentum and the Low volality factors, among other indicators including valuation, business cycle indicators [47].

3.5. The optimal generalized portfolio

In Sec. 2.2, we have shown how the weights of the optimal portfolio can be expressed via Eq. (9) through the matrices \mathbf{C} , \mathbf{C}_{ξ} , and \mathbf{M} . While the covariance matrix \mathbf{C} (or \mathbf{C}_{ε}) can be estimated from empirical data, both matrices \mathbf{C}_{ξ} and \mathbf{M} are very difficult to estimate. In this situation, it may be convenient to model the covariance matrix of trends, \mathbf{C}_{ξ} as a linear combination of the covariance matrices of three basic portfolios discussed in this section: (i) the naive Markowitz case ($\mathbf{C}_{\xi} \propto \mathbf{C}$); (ii) the agnostic risk parity case ($\tilde{\mathbf{C}}_{\xi}$ has only one nonzero eigenvalue), and (iii) the trend on the risk parity case (\mathbf{C}_{ξ} has one eigenvector corresponding to the risk parity). Including also the matrix of net returns, \mathbf{M} , the linearity of Eq. (9) implies that the optimal portfolio can be studied as a linear combination of the four basic portfolios. The empirical optimal weights could therefore give a clue to estimate the covariance matrix of trends \mathbf{C}_{ξ} .

4. Empirical backtest

4.1. Description of data and parameters

We select the most liquid futures that include 24 futures on stock index, 14 futures on bonds index and 9 futures on FOREX. The period starts from 8th May 1985 and ends at 31st December 2018. The Sharpe ratio and backtest statistics are computed based on the period from 1st January 1993 to 27st August 2020 (see Table 1). We do not take into account transactions cost and market impact. In practice, other constraints should be included to ensure the liquidity of the portfolio and to minimize the market impact. As we do not include these constraints in the optimization, the implemented portfolio can be different from the theoretical formula.

The signal of a TF strategy is chosen to be an EMA [48, 49]:

$$\mathbf{S}_{t,t'} = \begin{cases} (1-\eta)^{t-t'-1}, & t > t', \\ 0, & t \le t', \end{cases}$$
 (27)

where η is the rate of the TF strategy that we fix to be $\eta = 1/100$ on daily basis [21]. Setting the elements of these matrices to 0 for $t \leq t'$ implements the causality: the signal at time t relies only upon the earlier returns with t' < t. Moreover, the same rate η is used for all assets.

As mentioned in the footnote 1, it is convenient to consider the daily returns resized by the realized volatility. This resizing makes the diagonal elements of the covariance matrix \mathbf{C} to be very close to 1 so that \mathbf{C} can be understood as the correlation matrix. Although theoretical formulas in Sec. 3 were derived in the stationary regime (with a constant \mathbf{C}), it is more practical to update the matrix \mathbf{C} with time to render the portfolio more reactive and sensitive to the latest changes in the market. For this reason, we estimate the matrix \mathbf{C} as follows. First, we estimate the covariance matrix $\hat{\mathbf{C}}$ of weekly returns \hat{r}_t^j to offset different trading hours used worldwide. For this purpose, we use an EMA with $\eta' = 1/750$:

$$\hat{\mathbf{C}}_{t}^{ij} = (1 - \eta')\hat{\mathbf{C}}_{t-1}^{ij} + \eta' \hat{r}_{t}^{i} \hat{r}_{t}^{j}.$$
(28)

The covariance matrix is then rescaled by its diagonal elements:

$$\hat{\hat{\mathbf{C}}}_t^{ij} = \frac{\hat{\mathbf{C}}_t^{ij}}{\sqrt{\hat{\mathbf{C}}_t^{ii} \, \hat{\mathbf{C}}_t^{jj}}} \,. \tag{29}$$

The latter is cleaned with the aid of the rotational invariant estimator [50, 51] to finally get \mathbb{C} .

The variances v_t^j are estimated from the daily returns. In fact, as the volatilities characterizes a single asset, the issue of different trading hours is less relevant, and it is preferable to estimate with more returns. Here, we use an EMA with $\eta=1/100$

$$v_t^j = (1 - \eta)v_{t-1}^j + \eta [r_t^j]^2.$$
(30)

For the risk parity portfolio, we set $\mu_i = \sqrt{v_t^i}$ for stock index and bonds, and $\mu_i = 0$ for exchange rates because exchange rates present a long-short, completely neutral investment.

		FOREA
	BUND 10Yr	AUD/USD Fut.
S&F Mideap 400 Idx e-mini	CAD Bond 10Yr	GBP/USD Fut.
Idx e-mini	Bobl	CAD/USD Fut.
Cac 40 S	Schatz	EUR/USD Fut.
	Long-term Euro-btp	US DOLLAR Idx
	Euro-buxl Futures	JPY/USD Fut.
STOXX Europe 600 Index Futures	LONG Gilt 10Yr	MXN/USD Fut.
	10yr Fr Gov Bond	NZD/USD Fut.
_	US T-NOTE 5Yr	CHF/USD Fut.
	JGB 10Yr	
_	US T-NOTE 10Yr	
_	US T-Note 2Yr	
_	Ultra T-Bonds Combined	
_	US T-BOND 30Yr	
Mini MSCI Emerging Markets Index Future		
Idx-S&P CNX Nifty		
Nasdaq E-mini		
MSCI Singapore Index Futures		
S&P 500 e-mini		
MSCI Taiwan Index Futures		
Dj Euro Stoxx		
S&P Canada 60-ME		
SPI 200 Idx		
Mini Dow Futures		

Table 1: List of the 47 instruments.

4.2. Interpretation of the empirical results

In the practical implementation of the above portfolios, we adjust the proportionality coefficient in Eqs. (12, 14, 25, 26) with time to target the same conditional volatility. Figure 1 shows the simulated performance for the following portfolios: ARP (agnostic risk parity), NM (naive Markowitz), EW (equally weighted), RP (risk parity), and ToRP (Trend on Risk parity).

One can see that RP has the highest Sharpe ratio (1.32) as the risk premia are significant and thus easier to capture as compared to trends. However, RP does not have appealing diversifying property within the aggregated portfolio of all investors. Among the trend following portfolios that are decorrelated from RP (see Table 2), the ARP (0.76) performs much better than NM (0.51), suggesting that the assumption for the NM portfolio being optimal is not realistic. We see also that the ToRP (1.19) is the best among the trend following ones, meaning that the common factor for trends is most likely the risk parity portfolio.

By excluding the RP that should be avoided to offer diversification to investors, we also determined the optimal combination of ARP (27%) and ToRP (73%) that improves the Sharpe ratio to 1.25. In practice, ToRP has a shorter holding period (the Sharpe ratio is expected to be smaller when including the market impact). If one allocates too much on the ToRP, it will reduce the capacity to manage big assets of the portfolio, and increase the correlation between the portfolio and RP. Morever, the optimal weights are not so robust and are very sensitive to the estimation of the Sharpe ratio of each strategy that can change from period to period. It is therefore more robust to add a moderate contribution of ToRP to ARP to improve the Sharpe ratio of ARP. The weights of ARP at 75% and of ToRP at 25% appear to be a good compromise. The Sharpe ratio of the mixture remains above 1 (Fig. 2).

The EW portfolio works pretty well by two reasons: first, the universe is well equilibrated between the number of stocks indices and the number of bond indices; second, the EW portfolio, whose realized risk is in theory proportional to the square root of the eigenvalue, could be interpreted as a combination between ARP (the same realized risk for any eigenmode, in theory) and ToRP (concentration of realized risk in the first eigenmodes). This interpretation is confirmed by Fig. 3 that shows empirical realized risk depending on the square root of the eigenvalues. The disagreement with the backtest could be explained by the challenge to measure properly realized

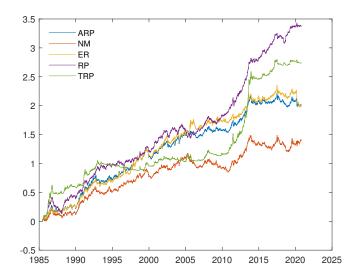


Figure 1: Equity curves of five portfolios: agnostic risk parity (ARP), naive Markowitz (NM), equally weighted (EW), risk parity (RP), and trend following on risk parity (ToRP). Their Sharpe ratios are respectively 0.75, 0.52, 0.65, 1.24, and 1.13. The Sharpe ratio of the optimal combination with the risk parity portfolio is 1.37, with the optimal weights 19.5% (ARP), 51% (RP), and 30% (ToRP). The Sharpe ratio of the optimal combination without the risk parity portfolio is 1.18, with the optimal weights 28% (ARP) and 72% (ToRP).

risk on small eigenvalues and by the deviation between the model and the market (in particular, correlations are not constant in time).

5. Conclusion

We derive a theoretical setting to yield implementable solutions of the allocation problem of trend following portfolios. The main formula of the paper describes the optimal portfolio as depending on the covariance matrix of returns, the covariance matrix of trends and the risk premia.

We implement the formula to gauge the performance of five well established portfolios (Agnostic Risk Parity, Markowitz, Equally Weighted, Risk Parity and Trend on Risk Parity), using daily data from futures markets of 24 stock indexes, 14 bonds indexes and 9 FX, from 1985 to 2020.

Our main empirical finding is the optimal combination of the three best

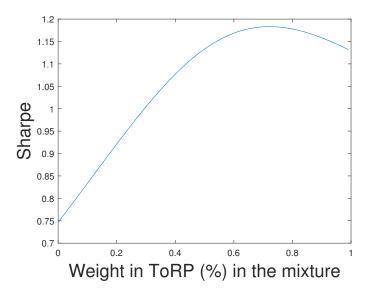


Figure 2: The Sharpe ratio of the combination between ARP and ToRP portfolios as a function of the relative weight of ToRP. The Sharpe ratio is above 1 for any mixture, in which the weight of ToRP is above 75%.

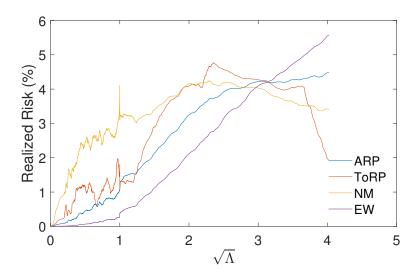


Figure 3: Empirical realized risk as a function of the square root of the eigenvalue of the correlation matrix $\tilde{\mathbf{C}}$ of four portfolios: agnostic risk parity (ARP), naive Markowitz (NM), equally weighted (EW), trend following on risk parity (ToRP). Disagreements with the theoretically expected realized risk are observed (for example, the risk is expected to be constant for ARP, inversely proportional to $\sqrt{\Lambda}$ for NM, and proportional to $\sqrt{\Lambda}$ for EW). Note that the NM is more allocated on small eigenvalues, while ToRP is invested on intermediate eigenvalues (mainly the second one).

	ARP	RP	ToRP
ARP		0.23	0.34
RP	0.23		0.59
ToRP	0.34	0.59	

Table 2: Correlations between the returns of three best portfolios: agnostic risk parity (ARP), risk parity (RP), and trend following on risk parity (ToRP). The portfolio that is the most decorrelated to RP is ARP, whereas ToRP is more correlated because it detected more often buy signals than sell signals in the past.

portfolios produces a Sharpe ratio of 1.37, with their respective optimal weights of 19.5% (ARP), 51% (RP), and 30% (ToRP) which combines both traditional and alternative approach. Consistent with related recent literature, we confirm that RP portfolio, which is a proxy of the traditional and well diversified portfolio is a important driver of performance. Furthermore, we show that the combination between ARP and ToRP is the best solution in term of Sharpe ratio for the trend following approach and the alternative benchmark as they tend to minimize the correlation among assets.

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Appendix A. Derivation of the main results

In [16], we considered the model without drifts, $\mu^{j} = 0$, for which the mean and the variance of the incremental profit-and-loss $\delta \mathcal{P}_{t}^{(0)}$ were derived

$$\langle \delta \mathcal{P}_{t}^{(0)} \rangle = \sum_{j,k=1}^{n} \omega_{t}^{j,k} M_{t}^{j,k(0)},$$

$$\operatorname{var}\{\delta \mathcal{P}_{t}^{(0)}\} = \sum_{j_{1},k_{1},j_{2},k_{2}=1}^{n} \omega_{t}^{j_{1},k_{1}} \omega_{t}^{j_{2},k_{2}} V_{t}^{j_{1},k_{1};j_{2},k_{2}(0)},$$
(A.1)

where the superscript 0 highlights the driftless character, and

$$\mathcal{M}_{t}^{j,k(0)} = \mathbf{C}_{\xi}^{j,k} (\mathbf{S}^{k} \mathbf{A}^{k} \mathbf{A}^{j,\dagger})_{t,t}, \tag{A.2}$$

$$V_{t}^{j_{1},k_{1};j_{2},k_{2}(0)} = \mathbf{C}_{\varepsilon}^{j_{1},j_{2}} \mathbf{C}_{\varepsilon}^{k_{1},k_{2}} (\mathbf{S}^{k_{1}} \mathbf{S}^{k_{2},\dagger})_{t,t} + \mathbf{C}_{\varepsilon}^{j_{1},j_{2}} \mathbf{C}_{\xi}^{k_{1},k_{2}} (\mathbf{S}^{k_{1}} \mathbf{A}^{k_{1}} \mathbf{A}^{k_{2},\dagger} \mathbf{S}^{k_{2},\dagger})_{t,t}$$

$$+ \mathbf{C}_{\varepsilon}^{k_{1},k_{2}} \mathbf{C}_{\xi}^{j_{1},j_{2}} (\mathbf{S}^{k_{1}} \mathbf{S}^{k_{2},\dagger})_{t,t} (\mathbf{A}^{j_{1}} \mathbf{A}^{j_{2},\dagger})_{t,t}$$

$$+ \mathbf{C}_{\xi}^{j_{1},j_{2}} (\mathbf{A}^{j_{1}} \mathbf{A}^{j_{2},\dagger})_{t,t} \mathbf{C}_{\xi}^{k_{1},k_{2}} (\mathbf{S}^{k_{1}} \mathbf{A}^{k_{1}} \mathbf{A}^{k_{2},\dagger} \mathbf{S}^{k_{2},\dagger})_{t,t}$$

$$+ \mathbf{C}_{\xi}^{j_{1},k_{2}} \mathbf{C}_{\xi}^{k_{1},j_{2}} (\mathbf{S}^{k_{1}} \mathbf{A}^{k_{1},\dagger} \mathbf{A}^{j_{2}})_{t,t} (\mathbf{S}^{k_{2}} \mathbf{A}^{j_{1},\dagger} \mathbf{A}^{k_{2}})_{t,t}, \tag{A.3}$$

where the matrices \mathbf{C}_{ε} , \mathbf{C}_{ξ} , \mathbf{A}^{j} , and \mathbf{S}^{j} are defined in Sec. 2.1. The structural separation between auto-correlations and inter-asset cross-corrections from Eq. (3) is also reflected in these formulas.

Now we relax the former assumption of zero mean returns by adding constant drifts μ^{j} . Employing the standard tools for averaging Gaussian variables, one can evaluate the mean and variance of this P&L. First, we get

$$\langle \delta \mathcal{P}_t \rangle = \langle \delta \mathcal{P}_t^{(0)} \rangle + \sum_{j,k=1}^n \omega_t^{j,k} \sum_{t'=1}^{t-1} \mathbf{S}_{t,t'}^k \mu^j \mu^k.$$
 (A.4)

Denoting

$$\hat{S}_t^k = \sum_{t'=1}^{t-1} \mathbf{S}_{t,t'}^k, \tag{A.5}$$

one has

$$\langle \delta \mathcal{P}_t \rangle = \sum_{j,k=1}^n \omega_t^{j,k} \, \mathcal{M}_t^{j,k}, \tag{A.6}$$

with

$$\mathcal{M}_t^{j,k} = \mathcal{M}_t^{j,k(0)} + \mu^j \mu^k \hat{S}_t^k, \tag{A.7}$$

in which $\mathcal{M}_t^{j,k(0)}$ is given by Eq. (A.2).

Similarly, long but straightforward computations yield

$$\operatorname{var}\{\delta \mathcal{P}_t\} = \sum_{j_1, k_1, j_2, k_2 = 1}^{n} \omega_t^{j_1, k_1} \, \omega_t^{j_2, k_2} \, V_t^{j_1, k_1; j_2, k_2}, \tag{A.8}$$

with

$$V_{t}^{j_{1},k_{1};j_{2},k_{2}} = V_{t}^{j_{1},k_{1};j_{2},k_{2}(0)} + \mu^{j_{1}}\mu^{j_{2}} \left(\mathbf{C}_{\varepsilon}^{k_{1},k_{2}} [\mathbf{S}^{k_{1}}\mathbf{S}^{k_{2},\dagger}]_{t,t} + \mathbf{C}_{\xi}^{k_{1},k_{2}} [\mathbf{S}^{k_{1}}\mathbf{A}^{k_{1}}\mathbf{A}^{k_{2},\dagger}\mathbf{S}^{k_{2},\dagger}]_{t,t} \right) + \mu^{j_{1}}\mu^{k_{2}}\hat{S}_{t}^{k_{2}}\mathbf{C}_{\xi}^{k_{1},j_{2}} [\mathbf{S}^{k_{1}}\mathbf{A}^{k_{1}}\mathbf{A}^{j_{2},\dagger}]_{t,t} + \mu^{j_{2}}\mu^{k_{1}}\hat{S}_{t}^{k_{1}}\mathbf{C}_{\xi}^{k_{2},j_{1}} [\mathbf{S}^{k_{2}}\mathbf{A}^{k_{2}}\mathbf{A}^{j_{1},\dagger}]_{t,t} + \mu^{k_{1}}\mu^{k_{2}}\hat{S}_{t}^{k_{1}}\hat{S}_{t}^{k_{2}} \left(\mathbf{C}_{\varepsilon}^{j_{1},j_{2}} + \mathbf{C}_{\varepsilon}^{j_{1},j_{2}} [\mathbf{A}^{j_{1}}\mathbf{A}^{j_{2},\dagger}]_{t,t} \right),$$
(A.9)

in which $V_t^{j_1,k_1;j_2,k_2(0)}$ is given by Eq. (A.3).

Once the mean and the variance of the incremental P&L are known, the dynamic allocation problem for a portfolio of trend following strategies is reduced to the standard optimization problem for a portfolio composed of n^2 "virtual" assets (indexed by a double index j,k) whose means are $\mathcal{M}_t^{j,k}$ and the covariance is $V_t^{j_1,k_1;j_2,k_2}$. One can therefore search for the weights $\omega_t^{j,k}$ that optimize a chosen criterion (e.g., to minimize the variance under a fixed expected return for the Markowitz theory). Here we aim to maximize the squared Sharpe ratio (or squared risk-adjusted return of the portfolio) in Eq. (7) that reads

$$S^2 = \frac{(\mathcal{M}_t^{\dagger} \omega_t)^2}{(\omega_t^{\dagger} V_t \omega_t)}.$$
 (A.10)

The optimization leads to the following equations on the weights $\omega_t^{j,k}$:

$$\frac{\partial \mathcal{S}^2}{\partial \omega_t^{j,k}} = \frac{2(\mathcal{M}_t^{\dagger} \omega_t)}{(\omega_t^{\dagger} V_t \omega_t)^2} \left[\mathcal{M}_t^{j,k} (\omega_t^{\dagger} V_t \omega_t) - (V_t \omega_t)^{j,k} (\mathcal{M}_t^{\dagger} \omega_t) \right] = 0 \qquad (j, k = 1, \dots, n),$$
(A.11)

or, equivalently,

$$\sum_{j_1,k_1,j_2,k_2=1}^{n} \left[\mathcal{M}_t^{j,k} V_t^{j_1,k_1;j_2,k_2} - V_t^{j,k;j_1,k_1} \mathcal{M}_t^{j_2,k_2} \right] \omega_t^{j_1,k_1} \omega_t^{j_2,k_2} = 0$$
 (A.12)

for all indices j, k = 1, ..., n. This is a set of n^2 quadratic equations onto n^2 unknown weights $\omega_t^{j,k}$. Since \mathcal{M}_t and V_t depend on time due to the dynamic character of TF strategies, the optimal weights need to be re-evaluated at each time step of the TF strategy.

Appendix A.1. Approximate solution of the general problem

The optimal weights $\omega_t^{j,k}$ satisfy Eqs. (A.12), which can be re-written with \mathcal{M}_t and V_t as

$$\mathcal{M}_{t}^{j,k} \left(\sum_{j_{1},k_{1},j_{2},k_{2}=1}^{n} \omega_{j_{1},k_{1}} V_{t}^{j_{1},k_{1};j_{2},k_{2}} \omega_{t}^{j_{2},k_{2}} \right) = \sum_{j_{1},k_{1}=1}^{n} V_{t}^{j,k;j_{1},k_{1}} \omega_{t}^{j_{1},k_{1}}$$

$$\times \left(\sum_{j_{2},k_{2}=1}^{n} \mathcal{M}_{t}^{j_{2},k_{2}} \omega_{t}^{j_{2},k_{2}} \right)$$

for all j, k = 1, ..., n. Treating the two sums in parentheses as (unknown) constants and thinking of $\omega_t^{j,k}$ and $\mathcal{M}_t^{j,k}$ as vectors (with a double index j, k), one might wish writing an explicit solution in the form

$$\omega_t = cV_t^{-1} \mathcal{M}_t, \tag{A.14}$$

where c is an arbibrary normalization constant, and V_t^{-1} is the "inverse" of V_t . Given the sophisticated tensorial structure of V_t in Eqs. (A.3, A.9), the definition of its inverse and thus the meaning of Eq. (A.14) are problematic in general.

Here we discuss two assumptions under which such an explicit solution is possible. First, we assume that the matrices \mathbf{S}^{j} and \mathbf{A}^{j} describing the signal and the autocorrelation structure of asset returns are the same for all stocks, i.e., $\mathbf{S}^{j} = \mathbf{S}$ and $\mathbf{A}^{j} = \mathbf{A}$. The sum over j_{1} and k_{1} in the right-hand side of Eq. (A.13) can be understood as a matrix (with respect to indices j and k) and shortly denoted as $[V_{t}\omega_{t}]_{j,k}$. According to Eqs. (A.3, A.9), this matrix can be written as

$$V_{t}\omega_{t} = f_{\varepsilon\varepsilon}(t)\mathbf{C}_{\varepsilon}\omega_{t}\mathbf{C}_{\varepsilon} + f_{\varepsilon\xi}(t)\mathbf{C}_{\varepsilon}\omega_{t}\mathbf{C}_{\xi} + f_{\xi\varepsilon}(t)\mathbf{C}_{\xi}\omega_{t}\mathbf{C}_{\varepsilon}$$

$$+\mathbf{C}_{\xi}\left(f_{\xi\xi}^{(1)}(t)\omega_{t} + f_{\xi\xi}^{(2)}(t)\omega_{t}^{\dagger}\right)\mathbf{C}_{\xi} + \mathbf{M}\omega_{t}(f_{\varepsilon\varepsilon}(t)\mathbf{C}_{\varepsilon} + f_{\varepsilon\xi}(t)\mathbf{C}_{\xi})$$

$$+[\hat{S}_{t}]^{2}\left(\mathbf{C}_{\varepsilon} + \mathbf{C}_{\xi}[\mathbf{A}\mathbf{A}^{\dagger}]_{t,t}\right)\omega_{t}\mathbf{M} + \hat{S}_{t}(\mathbf{S}\mathbf{A}\mathbf{A}^{\dagger})_{t,t}\left(\mathbf{M}\omega_{t}^{\dagger}\mathbf{C}_{\xi} + \mathbf{C}_{\xi}\omega_{t}^{\dagger}\mathbf{M}\right),$$
(A.15)

where **M** is the matrix of drifts, $\mathbf{M}_{j,k} = \mu^j \mu^k$, and

$$f_{\varepsilon\varepsilon}(t) = (\mathbf{S}\mathbf{S}^{\dagger})_{t,t},$$
 (A.16a)

$$f_{\varepsilon\xi}(t) = (\mathbf{S}\mathbf{A}\mathbf{A}^{\dagger}\mathbf{S}^{\dagger})_{t,t},$$
 (A.16b)

$$f_{\xi\varepsilon}(t) = (\mathbf{S}\mathbf{S}^{\dagger})_{t,t}(\mathbf{A}\mathbf{A}^{\dagger})_{t,t},$$
 (A.16c)

$$f_{\xi\xi}^{(1)}(t) = (\mathbf{S}\mathbf{A}\mathbf{A}^{\dagger}\mathbf{S}^{\dagger})_{t,t}(\mathbf{A}\mathbf{A}^{\dagger})_{t,t},$$
 (A.16d)

$$f_{\xi\xi}^{(2)}(t) = [(\mathbf{S}\mathbf{A}^{\dagger}\mathbf{A})_{t,t}]^2.$$
 (A.16e)

Second, we assume that autocorrelations are weak so that one can neglect terms which are of the second order in the matrix \mathbf{C}_{ξ} . Similarly, we neglect terms containing both \mathbf{M} and \mathbf{C}_{ξ} as drifts are as well small. In this case, the above expression can be approximated as

$$V_t \omega_t \approx f_{\varepsilon\varepsilon}(t) \left(\mathbf{C}_{\varepsilon} + g_1(t) \mathbf{C}_{\xi} + \mathbf{M} \right) \omega_t \left(\mathbf{C}_{\varepsilon} + g_2(t) \mathbf{C}_{\xi} + g_3(t) \mathbf{M} \right), \quad (A.17)$$

where

$$g_1(t) = \frac{f_{\xi\varepsilon}(t)}{f_{\varepsilon\varepsilon}(t)}, \qquad g_2(t) = \frac{f_{\varepsilon\xi}(t)}{f_{\varepsilon\varepsilon}(t)}, \qquad g_3(t) = \frac{[\hat{S}_t]^2}{f_{\varepsilon\varepsilon}(t)}.$$
 (A.18)

Rewriting Eqs. (A.13) in a matrix form as $V_t\omega_t = c\mathcal{M}_t$ with an unknown constant c, one can finally invert this matrix relation to get

$$\omega_t \approx \left(\mathbf{C}_{\varepsilon} + g_1(t)\mathbf{C}_{\xi} + \mathbf{M}\right)^{-1} \left(h_{\xi}(t)\mathbf{C}_{\xi} + h_{\mu}(t)\mathbf{M}\right) \left(\mathbf{C}_{\varepsilon} + g_2(t)\mathbf{C}_{\xi} + g_3(t)\mathbf{M}\right)^{-1},$$
(A.19)

with

$$h_{\xi}(t) = c \frac{(\mathbf{S} \mathbf{A} \mathbf{A}^{\dagger})_{t,t}}{f_{\varepsilon\varepsilon}(t)}, \qquad h_{\mu}(t) = c \frac{\hat{S}_t}{f_{\varepsilon\varepsilon}(t)}.$$
 (A.20)

Here, the unknown constant c is included into functions $h_{\xi}(t)$ and $h_{\mu}(t)$. This is an approximate optimal solution for the matrix of weights $\omega_t^{j,k}$. Its explicit, easily computable matrix form is one of the main theoretical results of the paper. In this solution, the matrices \mathbf{A} and \mathbf{S} determining assets auto-correlation and TF signals, induce time dependence via the functions $g_1(t)$, $g_2(t)$, $g_3(t)$, $h_{\xi}(t)$ and $h_{\mu}(t)$. We emphasize that the impact of time dependence is in general highly nontrivial given that the functions $g_i(t)$ stand in front of matrices in a linear combination which is inverted.

Neglecting again the contribution of small matrices \mathbf{C}_{ξ} and \mathbf{M} (as compared to \mathbf{C}_{ε}), we get a practical approximation of the optimal solution:

$$\omega_t \approx \mathbf{C}_{\varepsilon}^{-1} (h_{\xi}(t) \mathbf{C}_{\xi} + h_{\mu}(t) \mathbf{M}) \mathbf{C}_{\varepsilon}^{-1}.$$
 (A.21)

In this approximation, the impact of time dependence is explicit: functions $h_{\xi}(t)$ and $h_{\mu}(t)$ determine relative contributions of auto-correlation induced stochastic trends and net returns, respectively. We recall that these functions are determined up to an arbitrary multiplicative factor so that an additional constraint on the optimal portfolio will be needed to fix the weights (e.g., the targeted variance of the portfolio). In the stationary regime, the functions $h_{\xi}(t)$ and $h_{\mu}(t)$ reach their limits, denoted h_{ξ} and h_{μ} . Finally, the covariance matrix of instantaneous fluctuations of returns, \mathbf{C}_{ε} , is close, in the leading order, to the covariance matrix of returns, \mathbf{C} . We can thus replace \mathbf{C}_{ε} by \mathbf{C} to rewrite Eq. (A.21) in the form (8) presented in the text.

Appendix B. Emergence of the dominant factor: an interacting agents model

In this Appendix, we discuss a simple model of interacting agents to rationalize the emergence of the dominant factor. This model is inspired by studies of collective opinions shifts and other models of statistical physics [52, 53]. We emphasize that this model is fully unrelated to our model from Sec. 2 and serves exclusively to provide complementary support to empirical evidences of the dominant factor.

We suppose that there are A interacting trading agents. At each moment of time t, each agent adopts one of N available trading strategies (that could correspond to N portfolios based on N eigenvectors of the covariance matrix). To describe this choice, we introduce a matrix S(t) of size $A \times N$ whose element $S_{a,n}(t)$ is equal to 1 if the agent a adopts the n-th strategy at time t, and 0 otherwise: $S_{a,k}(t) = \delta_{k,n}$. At the next time step t+1, each agent re-evaluates his strategy in the following way: first, one computes the "preference matrix" $X_{a,k}$ of the a-th agent to the strategy k,

$$X_{a,k} = \varepsilon_{a,k} + \sum_{a'=1}^{A} J_{a,a'} S_{a',k}(t),$$
 (B.1)

where $\varepsilon_{a,k}$ is the individual a-th agent's preference to the strategy k, while the matrix $J_{a,a'}$ characterizes to which extent the agent a is influenced by another agent a'; second, for each agent a, one selects the strategy k_{max} with the maximal preference $X_{a,k_{\text{max}}}$ among all $X_{a,1}, X_{a,2}, \ldots, X_{a,N}$, and sets $S_{a,k}(t +$ 1) = $\delta_{k,k_{\text{max}}}$. In other words, the agent a adopts for time t+1 the strategy that was most preferred for him at time t. If there was no interaction with other agents (i.e., $J_{a,a'} = 0$ for all a, a'), each agent would keep its preferred strategy that corresponds to the maximum of $\varepsilon_{a,1}, \varepsilon_{a,2}, \ldots, \varepsilon_{a,N}$. In the presence of interactions, the agent selects his strategy as a compromise between his own individual preferences (characterized by $\varepsilon_{a,k}$) and the influence of other agents and their preferred strategies. In the ultimate limit when the interactions are all equal and very high, $J_{a,a'} = J \to \infty$, if there is a single strategy adopted by the largest number of agents at the beginning, then this strategy will provide the maximum of $X_{a,k}$ for all agents and thus will be adopted by all agents at the next step. Clearly, one can expect a transition between the no interaction limit (when each agent keeps using its preferred strategy) and the strong interaction limit (when all agents use the same strategy).

The emergence of the dominant mode (i.e., a single strategy adopted by all agents) depends on the matrices $J_{a,a'}$ and $\varepsilon_{a,k}$ governing the dynamics. In statistical physics, it is common that fine details of the model parameters do not matter in the limit of a large number of particles (here, agents). The same kind of universality is expected for the present model. We perform simulations to illustrate that the overall amplitude of interactions (as compared

to individual preferences $\varepsilon_{a,k}$) is the major parameter that determines the transition.

Without dwelling on the analysis of this model, we make a simple choice of the parameters: the interaction matrix $J_{a,a'}$ is considered to be constant, $J_{a,a'} = J$, i.e., all agents have the same level of influence on each other. In turn, the elements of the matrix ε are independent centered normally distributed numbers with unit variance. The initial state $S_{a,k}(0)$ is also set randomly, by selecting for each agent a one preferred strategy among N available with a uniform law. In this setting, the level of interactions J (as compared to the unit level of individual preferences) is the major parameter, along with the number of agents A and the number of strategies N. Each simulation is performed for T steps, with T being chosen to allow for convergence to a steady state, resulting in the matrix $S_{a,k}(t)$ at all time steps $t = 0, 1, 2, \ldots, T$. From this basic quantity, we compute the empirical average over all agents,

$$\overline{S}_k(t) = \frac{1}{A} \sum_{a=1}^{A} S_{a,k}(t),$$
 (B.2)

which represents the overall interest of agents into the k-th strategy at time t. Given that all $\overline{S}_k(t)$ are still random variables, we repeat simulations M times to approximate the expectation $I_k(t) = \mathbf{E}\{\overline{S}_k(t)\}$ by averaging out random fluctuations among simulated results.

By definition, each $I_k(t)$ is a number from 0 to 1 such that their sum over k is equal to 1. In other words, $I_k(t)$ can be interpreted as the average fraction of agents interested in the k strategy. At the beginning, the uniform assignment of preferred strategies among the agents yields $I_k(0) = 1/N$, i.e., all strategies are equally preferred. As time goes on, interactions between agents can spontaneously break the initial symmetry between all strategies and lead to the emergence of a dominant strategy preferred by the majority of agents. In the following, we will illustrate the behavior of $I_k(t)$ for different choices of the parameters. We will also look at the dynamics of the largest fraction, i.e., how $\max_k\{I_k(t)\}$ evolves with time. In particular, we will see how the steady-state value of this maximum depends on the level of interactions J.

To reduce the dependence on A and N, we set $J = j\sqrt{N}/A$, where j is some intrinsic amplitude of the interactions that is then rescaled by A and \sqrt{N} . Figure B.4 shows the steady-state value of the maximum $\max_k\{I_k(t)\}$ as a function of the interaction amplitude j.

Figure B.5 shows the dynamics of the fraction of agents, $\overline{S}_k(t)$, for two simulations. The choice of an intermediate level of interactions, j=1.5, leads to two sorts of outcomes: either there is no dominant strategy (Fig. B.5a), i.e., all strategies remain more or less equally adopted by the agents; or one dominant strategy emerges (Fig. B.5b), while the remaining strategies are abandonned. We emphasize that, as all strategies are equivalent at the beginning, the choice of the "winner" strategy is random and realized due to a spontaneous symmetry breaking among the strategies. For a smaller level of interactions (say, j=1 or less), almost all outcomes of simulations appear without the dominant strategy (not shown). In contrast, when interactions are stronger (say, j=2 or higher), almost all outcomes appear with the dominant strategy.

Finally, Fig. B.6 shows the dynamics of the maximal fraction $\max_k \{I_k(t)\}$. For weak interactions with j=1.25, this fraction remains constant, showing that each agent mainly keeps using its preferred strategy, irrespectively of the others. At intermediate interactions (j=1.5), the maximal fraction $\max_k \{I_k(t)\}$ grows at first time steps and then reaches a steady-state value which is larger than in the case j=1.25 but still relatively small. This value reflects the fact that some outcomes do not show a dominant strategy, whereas some other outcomes do. As j is further increased, the number of outcomes with the dominant strategy is getting significantly larger.

In summary, the proposed simplistic model illustrates how interactions between agents may lead to the emergence of a single dominant strategy adopted by all agents. While this model does not aim to mimic or capture the real mechanisms of decision making in financial trading, it simply checks that such mechanisms may potentially rationalize the emergence of a dominant strategy.

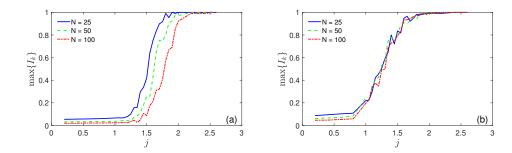


Figure B.4: Steady-state value of the maximum $\max_{k} \{I_k(t)\}$ as a function of the interaction amplitude j. This quantity was computed by setting M = 100, T = 50, three values of N (as indicated in the legend), and two values of A: A = 1000 (a) and A = 100 (b).

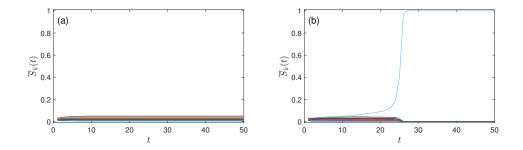


Figure B.5: Two random realizations of the dynamics of the fraction of agents, $\overline{S}_k(t)$, computed for $A=1000,\ N=50,\ T=50,$ and j=1.5, without (a) and with (b) a dominant strategy. Fifty curves show time evolution of the fraction $\overline{S}_k(t)$ of each strategy, with $k=1,2,\ldots,50$.

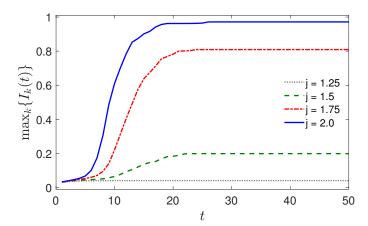


Figure B.6: Time evolution of the average maximal fraction of agents interested in one strategy, $\max_k\{I_k(t)\}$, computed for $A=1000,\ N=50,\ T=50,\ M=100$, and four values of the level of interactions j as indicated in the plot.