Metric Embeddings intro Trees

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Outline of the talk

1. Introduction
2. Stochastic embedding into trees
3. Ramsey type embeddings
4. Clan embedding
5. Group Steiner Tree
6. Conclusion

Metric space

A metric space is an ordered pair (X, d_X), where X is a set and $d_X : X \times X \to \mathbb{R}_{\geq 0}$ is a function such that:

1. **Identity**: $\forall x, y \in X,\ d_X(x, y) = 0 \iff x = y$.
2. **Symmetry**: $\forall x, y \in X,\ d_X(x, y) = d_X(y, x)$.
3. **Triangle-inequality:** $\forall x, y, z \in X,\ d_X(x, y) \leq d_X(x, z) + d_X(z, y)$.

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Examples:

- Euclidean space ℓ_2 in $\mathbb{R}^d$: $d_{\ell_2}(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|_2 = \sqrt{\sum_{i=1}^{d}(x_i - y_i)^2}$.
- Manhattan distance ℓ_1 in $\mathbb{R}^d$: $d_{\ell_1}(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|_1 = \sum_{i=1}^{d} |x_i - y_i|$.
- Edit distance: given two strings A, B how many edit operations (insert, delete, substitute) are required to transform A to B?
- Weighted graph $G = (V, E, w)$ with shortest path distance.

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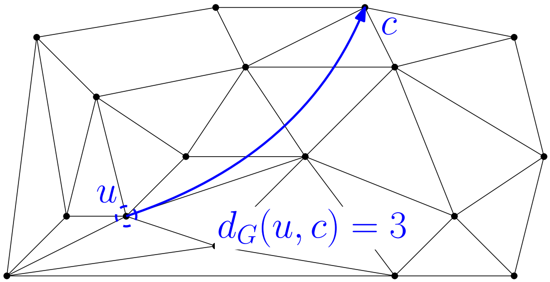

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Many problems are defined w.r.t. metric spaces. Examples:

- k-center.
- Steiner tree.
- Metric TSP.

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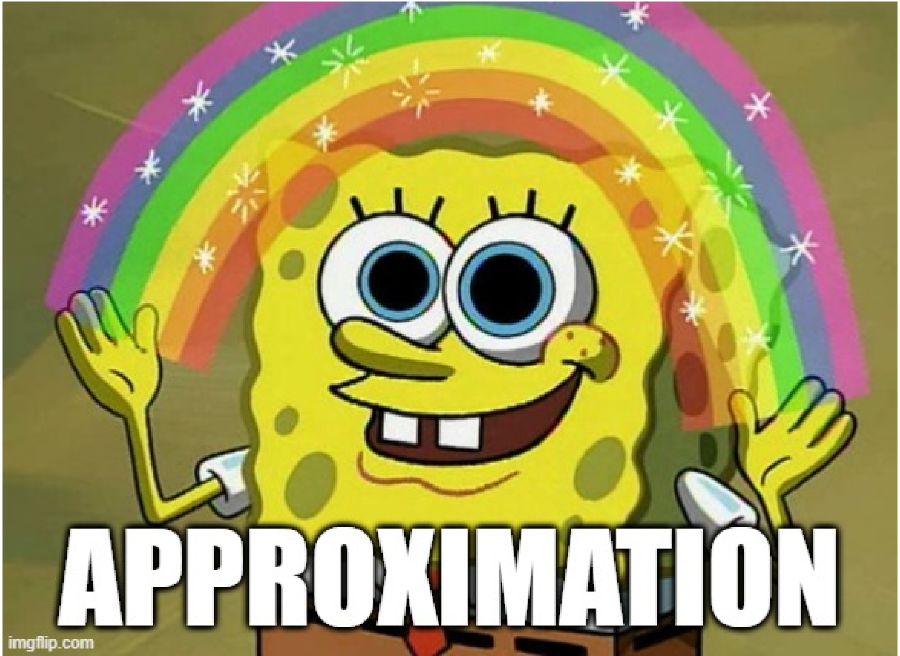
Often these problems are NP-hard.


APPROXIMATION
imgflip.com

Metric Embeddings

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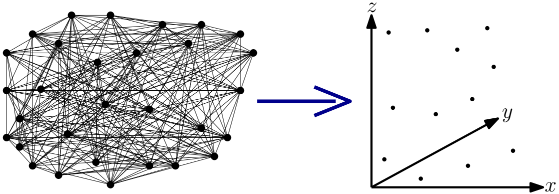
$f :$ 

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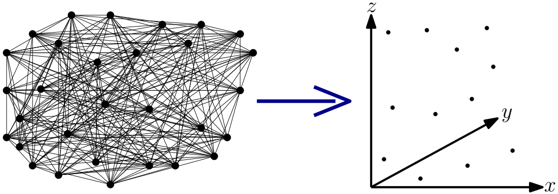

Preserve (approxierly) properties of the original space:

- **Distances**
- Cuts, Flows
- Commute time
- Effective resistance
- Clustering statistics.
- etc.

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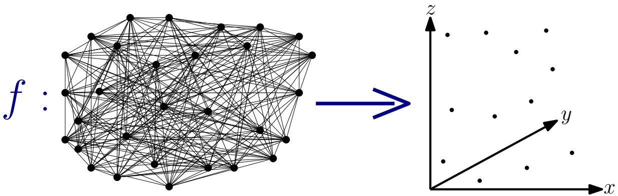

f has **distortion** t if:

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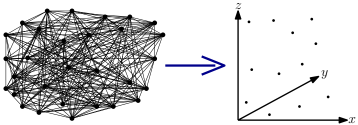
f : 

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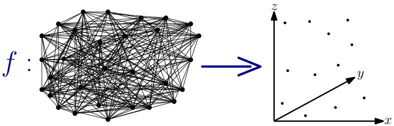

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So that we could run efficient algorithms on it...

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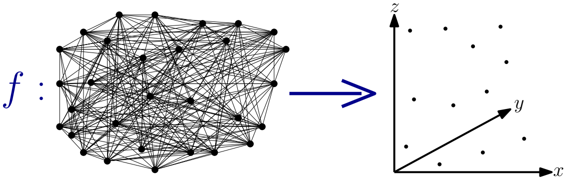

Theorem ([Bourgain 85])

Every n-point metric (X, d_X) is ***embeddable*** *into* <u>*Euclidean*</u> *space $(\mathbb{R}^d, \|\cdot\|_2)$ with* ***distortion*** *$O(\log n)$.*

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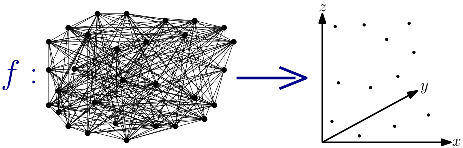

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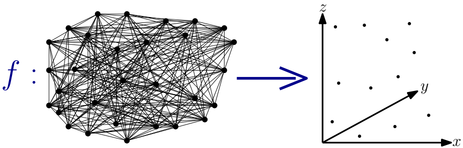
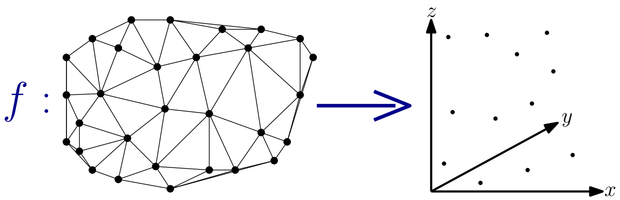


Applications:

- Approximation algorithms (e.g. **sparsest cut**, min graph bandwidth)
- Parallel computation (e.g. SSSP in MPC)
- Computational Biology (e.g. clustering and detecting protein seq.)
- etc.

Theorem ([Rao 99])

Every n-point ***planar*** *metric* (X, d_X) *is* ***embeddable*** *into* *Euclidean* *space* $(\mathbb{R}^d, \|\cdot\|_2)$ *with* ***distortion*** $O(\sqrt{\log n})$.



Planar metric- the shortest path metric of a planar graph.

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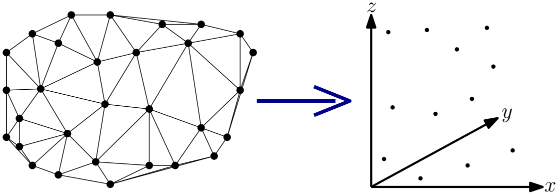
$f:$ 

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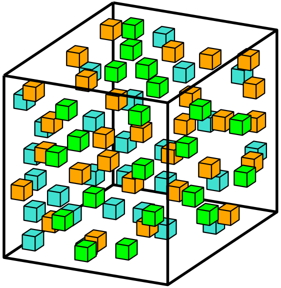
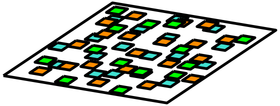
$f :$  $\longrightarrow$ 

Theorem ([Johnson, Lindenstrauss 84], Dimension Reduction)

$X \subset (\mathbb{R}^d, \|\cdot\|_2)$ set of size n. Then X embeds into $O(\log n/\epsilon^2)$ dimensional Euclidean space with distortion $1+\epsilon$.

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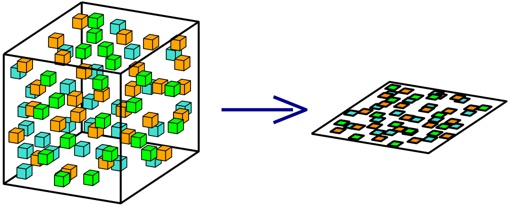
$f :$ 

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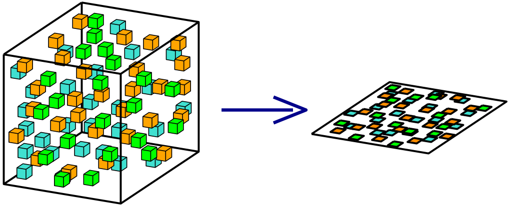
$f:$ 

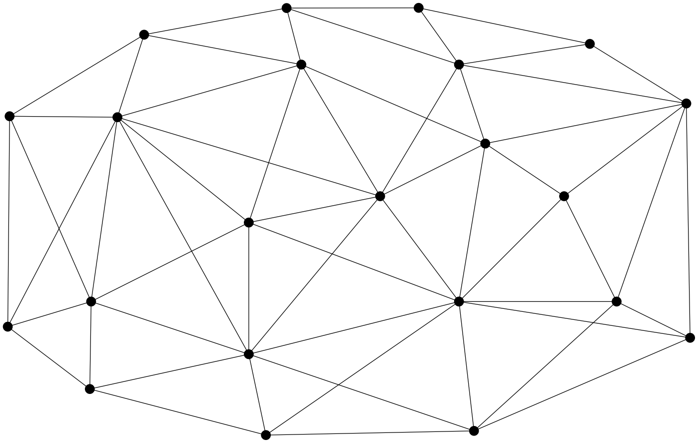
Applications:

- Speeding up-computation
- Clustering
- Nearest Neighbor Search
- Machine Learning
- etc.

Graph Spanners

$G = (V, E, w)$ weighted graph, a t-**spanner** is a **subgraph** $H = (V, E_H)$

s.t. $\forall u, v \in V, \quad d_H(u, v) \leq t \cdot d_G(u, v)$



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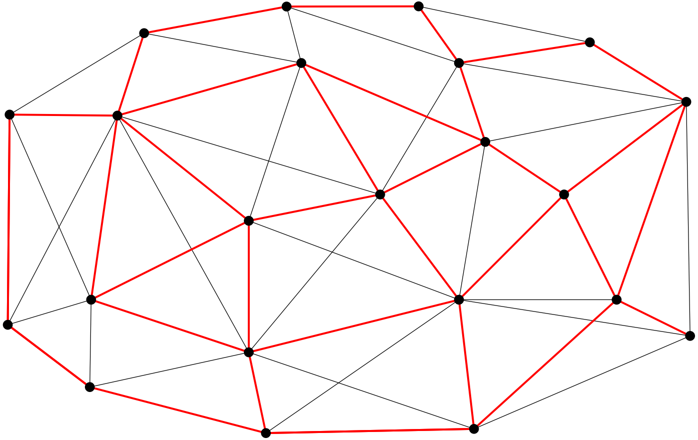

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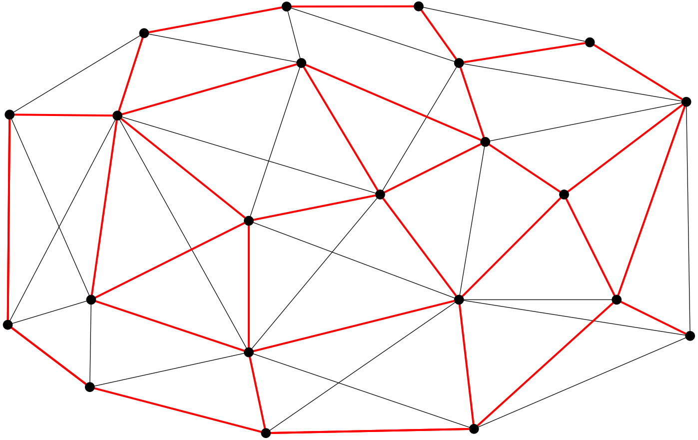

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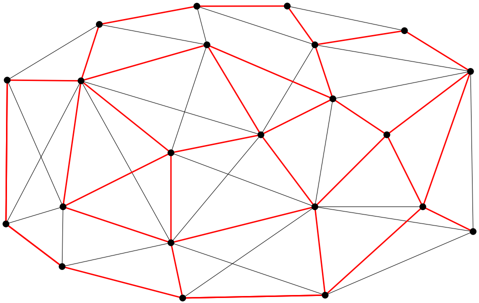

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For $k \geq 1$, every graph admits $2k - 1$ spanner with $O(n^{1+\frac{1}{k}})$ edges.

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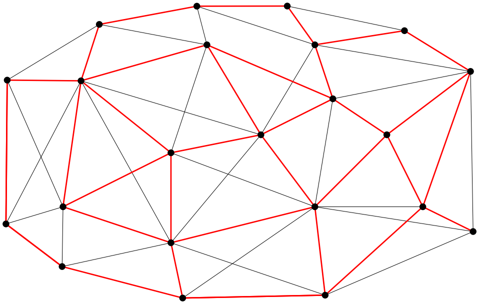

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Applications:

- Approximation Algorithms (e.g. PTAS for TSP)
- Distributed Computing
- Network Routing
- Computational Biology (e.g. measure genetic distance)
- etc.

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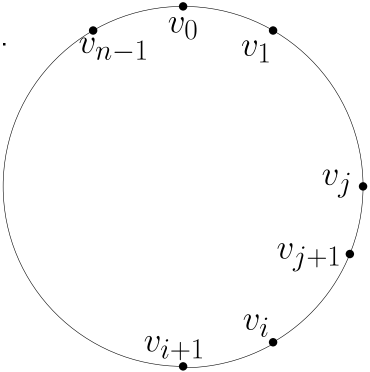

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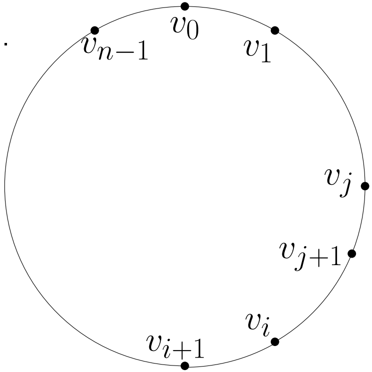

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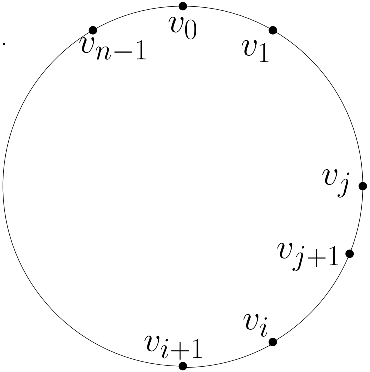

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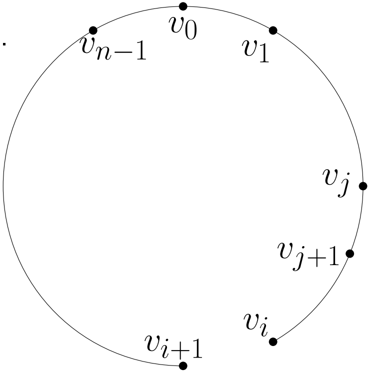

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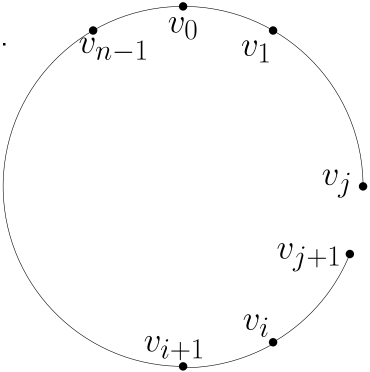

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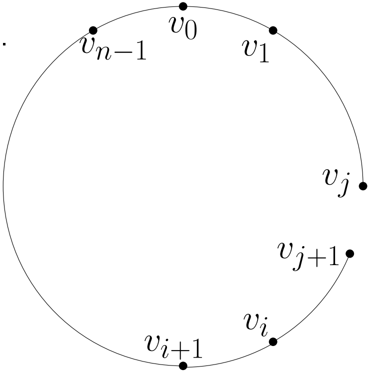

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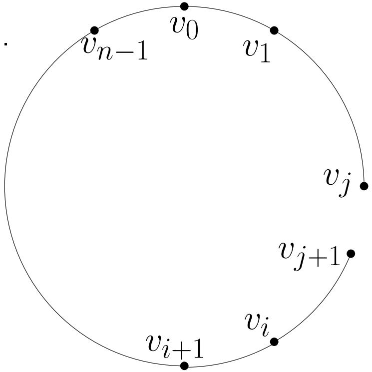

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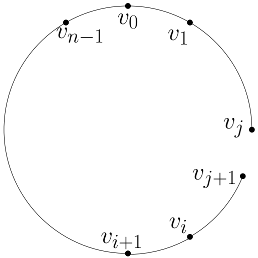

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By triangle inequality and linearity of expectation

$$\forall v_i, v_j, \quad \mathbb{E}_{T\sim\mathcal{D}}[d_T(v_i,v_j)] = \sum_{q=i}^{j-1}\mathbb{E}_{T\sim\mathcal{D}}[d_T(v_q,v_{q+1 \pmod{n}})] \leq 2\cdot d_{C_n}(v_i,v_j) \ .$$

Stochastic Embedding into Trees

Theorem ([Fakcharoenphol, Rao, Talwar 04], improving [Bartal 96+98])

Every n-point metric space (X, d) embeds into ***distribution*** $\mathcal{D}$ *over* ***dominating trees*** *with* ***expected distortion*** $O(\log n)$.

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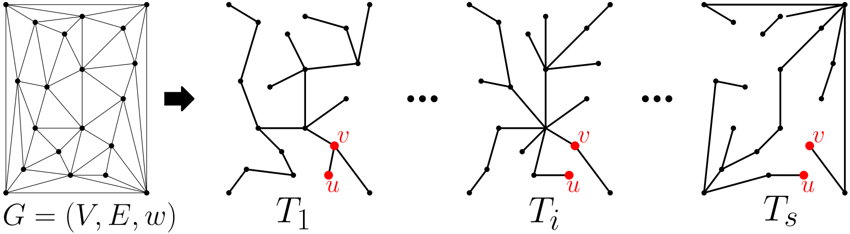

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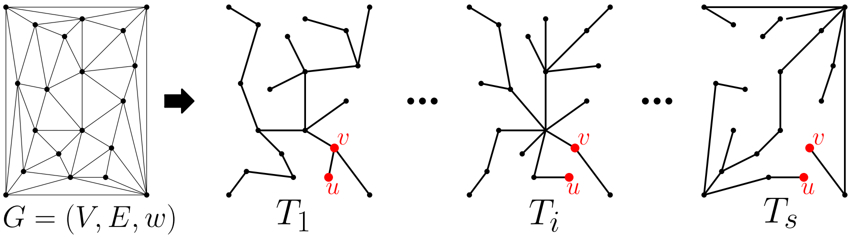

For every $u, v \in X$ and $T \in \mathrm{supp}(\mathcal{D})$, $d_X(u, v) \leq d_T(f(u), f(v))$.

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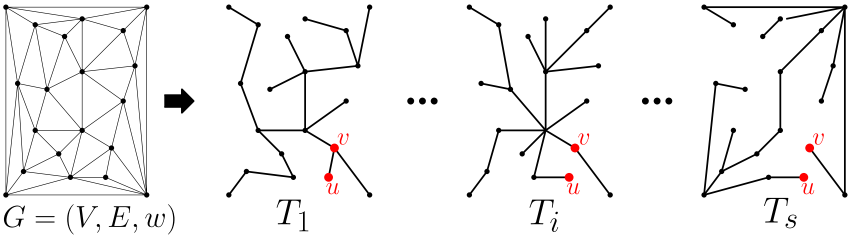
*Every n-point metric space (X, d) embeds into **distribution** $\mathcal{D}$ over **dominating trees** with **expected distortion** $O(\log n)$.*



For every $u, v \in X$ and $T \in \mathrm{supp}(\mathcal{D})$, $d_X(u, v) \leq d_T(f(u), f(v))$.

For every $u, v \in X$ $\quad \mathbb{E}_{T\sim\mathcal{D}}[d_T(f(u), f(v))] \leq O(\log n) \cdot d_X(u, v)$.

Stochastic Embedding into Trees

Theorem ([Fakcharoenphol, Rao, Talwar 04], improving [Bartal 96+98])

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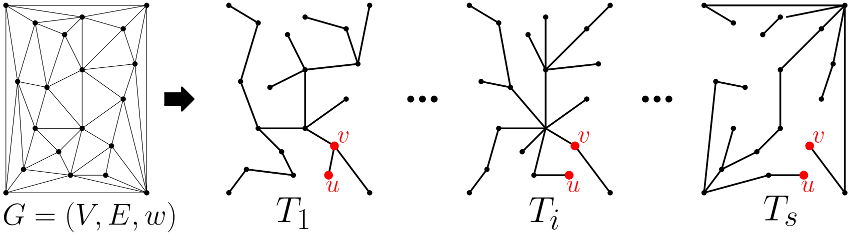

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[Bartal 96]: Tight!

Stochastic Embedding into Trees

Theorem ([Fakcharoenphol, Rao, Talwar 04], improving [Bartal 96+98])

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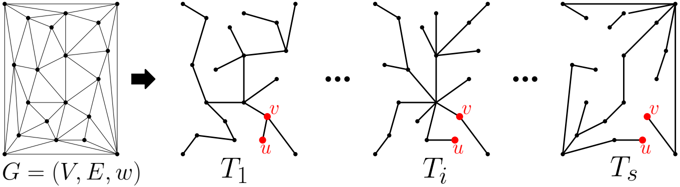


A useful hammer

Transforms arbitrary metric into a tree!

Stochastic Embedding into Trees

Theorem ([Fakcharoenphol, Rao, Talwar 04], improving [Bartal 96+98])

Every n-point metric space (X, d) embeds into ***distribution*** $\mathcal{D}$ *over* ***dominating trees*** *with* ***expected distortion*** $O(\log n)$.

A useful hammer

Transforms arbitrary metric into a tree!

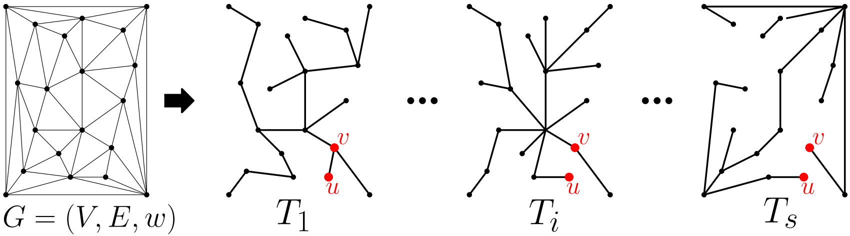
Applications:

- Approximation Algorithms.
- **Online Algorithms**.
- Distributed Computing.
- etc.

Stochastic Embedding into Trees

Theorem ([Fakcharoenphol, Rao, Talwar 04], improving [Bartal 96+98])

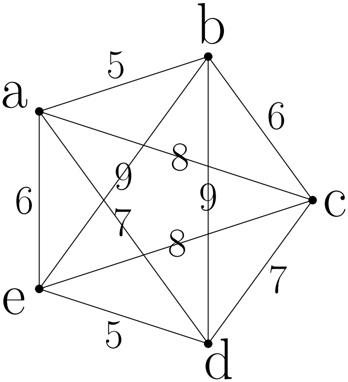
Every n-point metric space (X, d) embeds into ***distribution*** $\mathcal{D}$ *over* ***dominating trees*** *with* ***expected distortion*** $O(\log n)$.



Compromises: Gurantee only in **expectation**

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.



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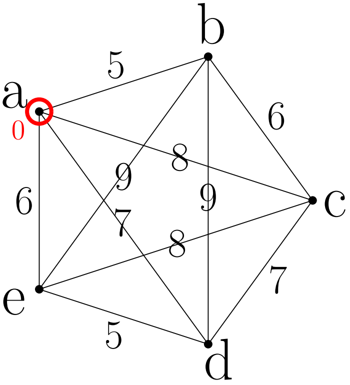

Online problem - Metrical Task System (MTS)

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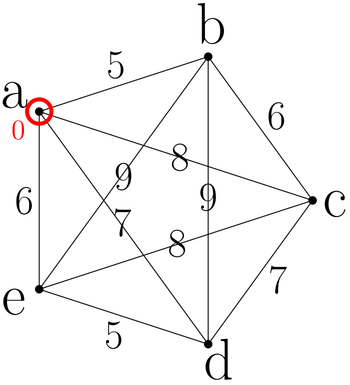

$$f_1(a) = 3$$
$$f_1(b) = 1$$
$$f_1(c) = 2$$
$$f_1(d) = 4$$
$$f_1(e) = 1$$

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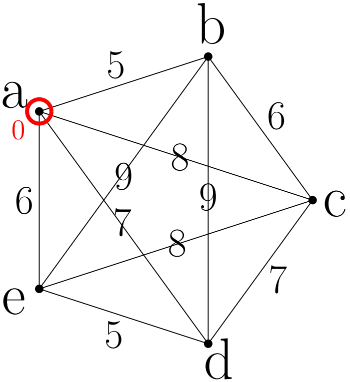
Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.



$$f_1(a) = 3$$
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$$f_1(d) = 4$$
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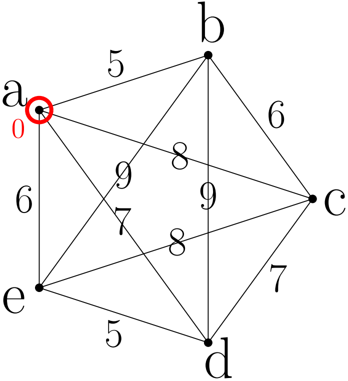
Goal: Minimize the competitive ratio between our algorithm to opt.



$$f_1(a) = 3$$
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$$f_1(c) = 2$$
$$f_1(d) = 4$$
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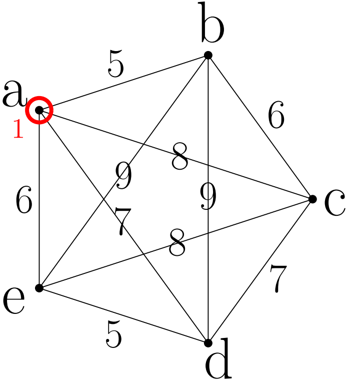

$$f_1(a) = 3$$
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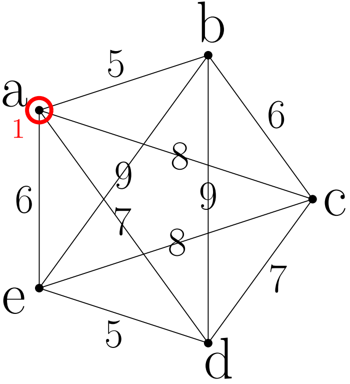

$f_1(a) = 3$
$f_1(b) = 1$
$f_1(c) = 2$
$f_1(d) = 4$
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Cost alg(1)=3

Online problem - Metrical Task System (MTS)

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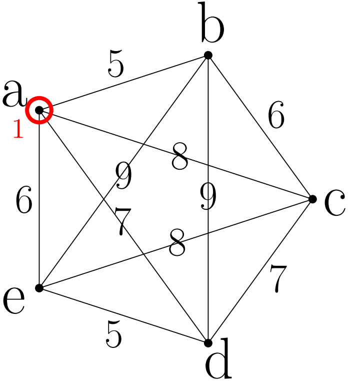

$f_2(a) = 3$
$f_2(b) = 1$
$f_2(c) = 3$
$f_2(d) = 2$
$f_2(e) = 1$

Cost alg(1)=3

Online problem - Metrical Task System (MTS)

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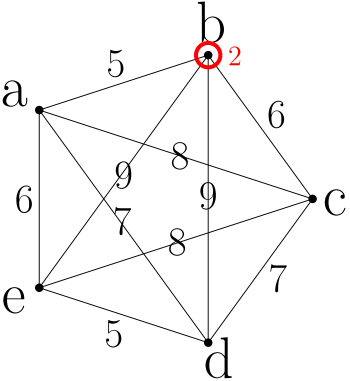

$$f_2(a) = 3$$
$$f_2(b) = 1$$
$$f_2(c) = 3$$
$$f_2(d) = 2$$
$$f_2(e) = 1$$

Cost alg(1)=3

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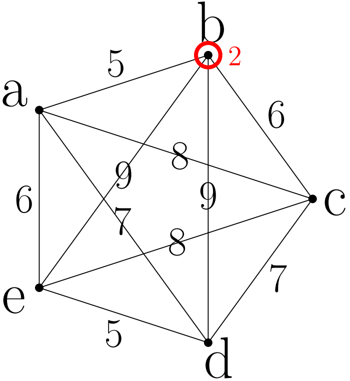


$$f_2(a) = 3$$
$$f_2(b) = 1$$
$$f_2(c) = 3$$
$$f_2(d) = 2$$
$$f_2(e) = 1$$

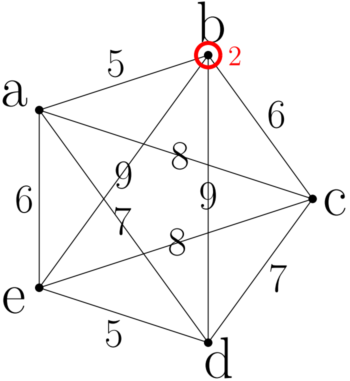
Cost alg(2)=$3 + 5 + 1 = 9$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$$f_3(a) = 1$$
$$f_3(b) = 4$$
$$f_3(c) = 3$$
$$f_3(d) = 2$$
$$f_3(e) = 1$$

$$\text{Cost alg}(2) = 3 + 5 + 1 = 9$$

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Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
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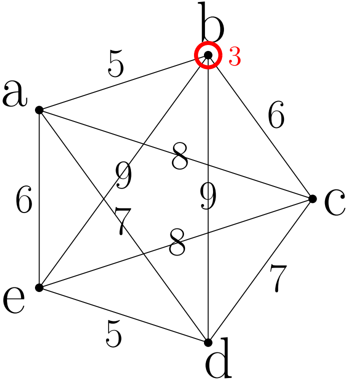

$$f_3(a) = 1$$
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Cost alg(2)$=3 + 5 + 1 = 9$

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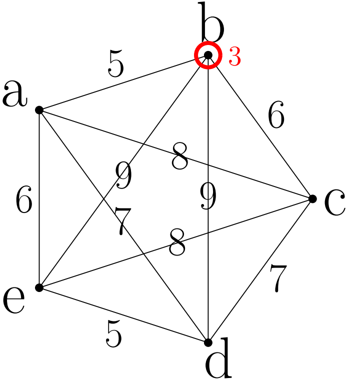

$$f_3(a) = 1$$
$$f_3(b) = 4$$
$$f_3(c) = 3$$
$$f_3(d) = 2$$
$$f_3(e) = 1$$

Cost alg(3)$=9 + 4 = 13$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

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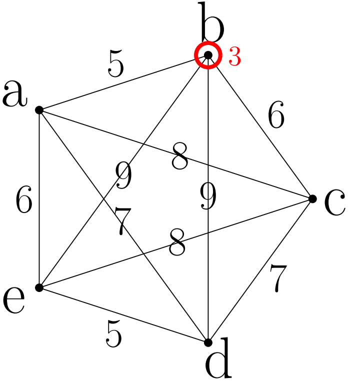

$$f_4(a) = 5$$
$$f_4(b) = 10$$
$$f_4(c) = 4$$
$$f_4(d) = 0$$
$$f_4(e) = 1$$

Cost alg(3)$=9+4=13$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

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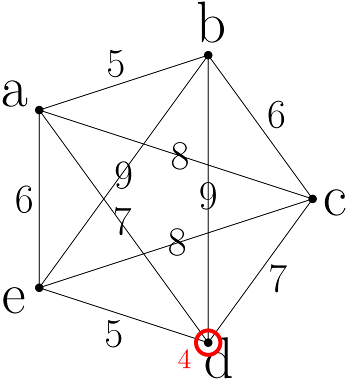

$$f_4(a) = 5$$
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Cost alg(3)$=9+4=13$

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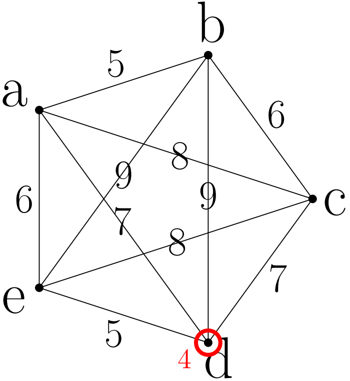


$f_4(a) = 5$

$f_4(b) = 10$

$f_4(c) = 4$

$f_4(d) = 0$

$f_4(e) = 1$

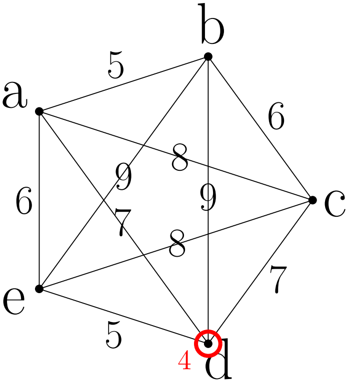
Cost alg(4)$=13 + 9 + 0 = 22$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$$f_5(a) = 3$$
$$f_5(b) = 4$$
$$f_5(c) = 4$$
$$f_5(d) = 5$$
$$f_5(e) = 1$$

Cost alg(4)$=13 + 9 + 0 = 22$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

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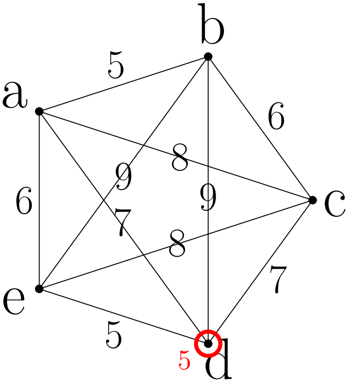

$f_5(a) = 3$
$f_5(b) = 4$
$f_5(c) = 4$
$f_5(d) = 5$
$f_5(e) = 1$

Cost alg(4)=$13 + 9 + 0 = 22$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

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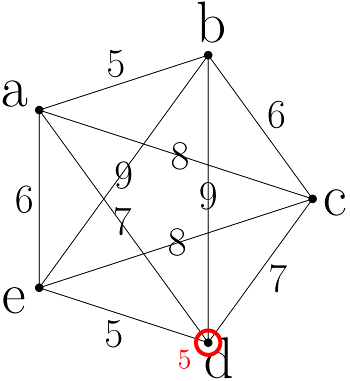

$$f_5(a) = 3$$
$$f_5(b) = 4$$
$$f_5(c) = 4$$
$$f_5(d) = 5$$
$$f_5(e) = 1$$

Cost alg(5)=$22 + 5 = 27$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.

What about Opt?

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Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
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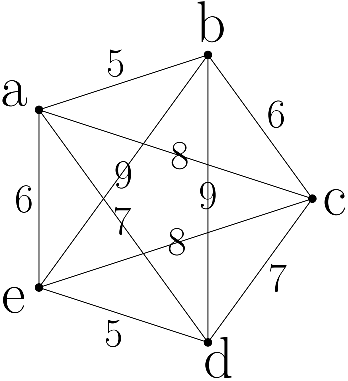
Goal: Minimize the competitive ratio between our algorithm to opt.



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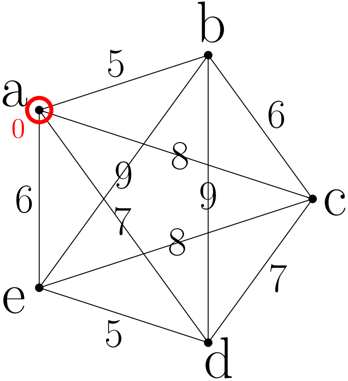

Online problem - Metrical Task System (MTS)

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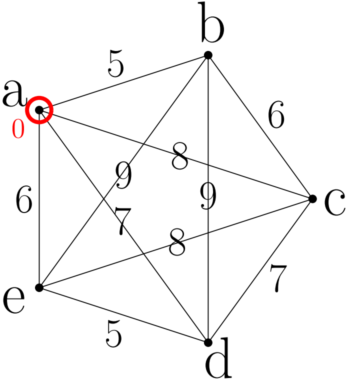
Goal: Minimize the competitive ratio between our algorithm to opt.



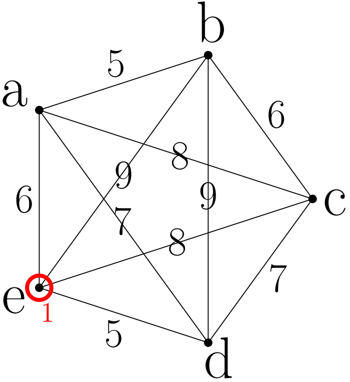
$f_1(a) = 3$
$f_1(b) = 1$
$f_1(c) = 2$
$f_1(d) = 4$
$f_1(e) = 1$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$f_1(a) = 3$
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$f_1(d) = 4$
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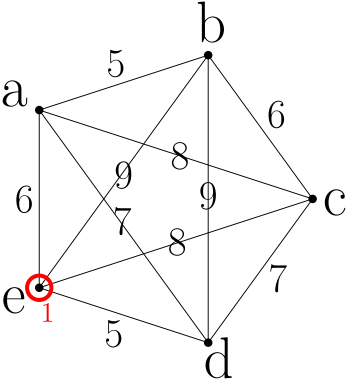


$$f_1(a) = 3$$
$$f_1(b) = 1$$
$$f_1(c) = 2$$
$$f_1(d) = 4$$
$$f_1(e) = 1$$

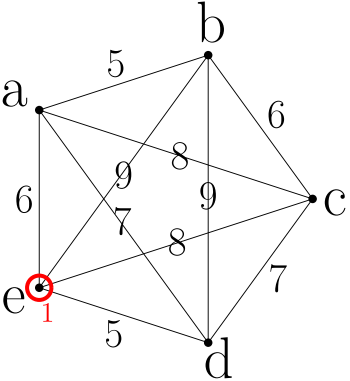
Cost opt(1)$=6 + 1 = 7$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$$f_2(a) = 3$$
$$f_2(b) = 1$$
$$f_2(c) = 3$$
$$f_2(d) = 2$$
$$f_2(e) = 1$$

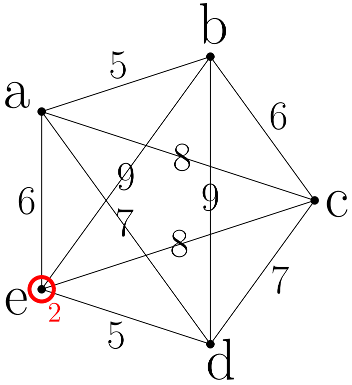
Cost opt(1)$=6 + 1 = 7$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$$f_2(a) = 3$$
$$f_2(b) = 1$$
$$f_2(c) = 3$$
$$f_2(d) = 2$$
$$f_2(e) = 1$$

Cost opt(1)$=6+1=7$

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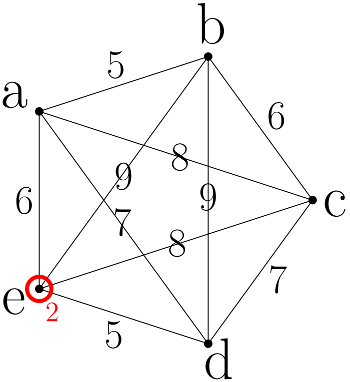


$$f_2(a) = 3$$
$$f_2(b) = 1$$
$$f_2(c) = 3$$
$$f_2(d) = 2$$
$$f_2(e) = 1$$

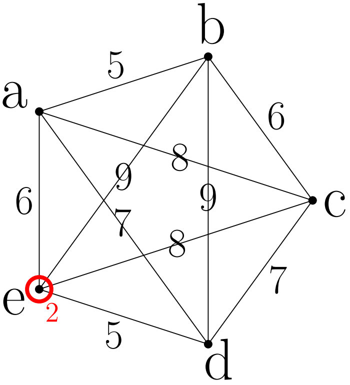
Cost opt(2)$=7 + 1 = 8$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$$f_3(a) = 1$$
$$f_3(b) = 4$$
$$f_3(c) = 3$$
$$f_3(d) = 2$$
$$f_3(e) = 1$$

Cost opt(2)$=7 + 1 = 8$

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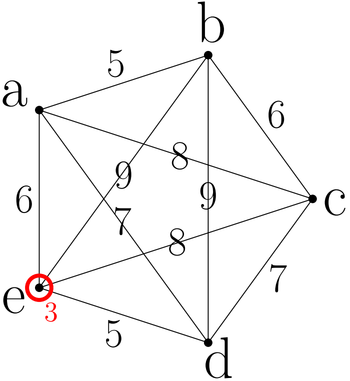
Goal: Minimize the competitive ratio between our algorithm to opt.



$f_3(a) = 1$
$f_3(b) = 4$
$f_3(c) = 3$
$f_3(d) = 2$
$f_3(e) = 1$

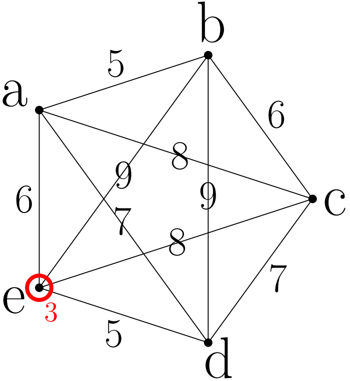
Cost opt(2)=$7 + 1 = 8$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$f_3(a) = 1$
$f_3(b) = 4$
$f_3(c) = 3$
$f_3(d) = 2$
$f_3(e) = 1$

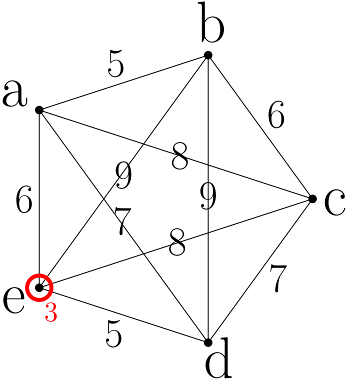
Cost opt(3)$=8+1=9$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$f_4(a) = 5$
$f_4(b) = 10$
$f_4(c) = 4$
$f_4(d) = 0$
$f_4(e) = 1$

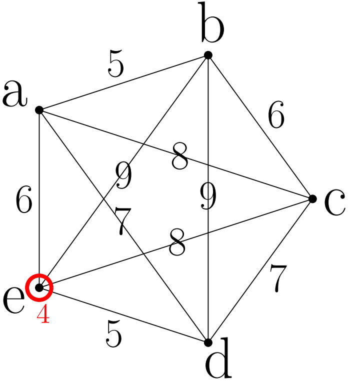
Cost opt(3)$=8 + 1 = 9$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$f_4(a) = 5$
$f_4(b) = 10$
$f_4(c) = 4$
$f_4(d) = 0$
$f_4(e) = 1$

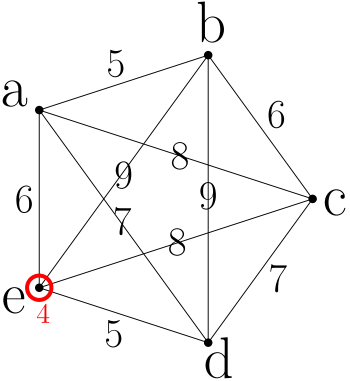
Cost opt(3)$=8+1=9$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$$f_4(a) = 5$$
$$f_4(b) = 10$$
$$f_4(c) = 4$$
$$f_4(d) = 0$$
$$f_4(e) = 1$$

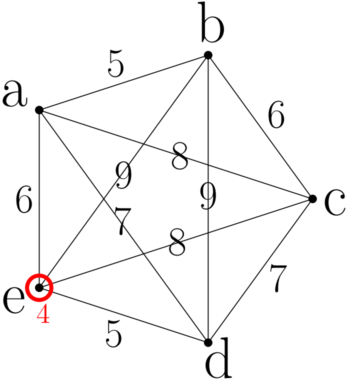
Cost opt(4)=$9 + 1 = 10$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$f_5(a) = 3$

$f_5(b) = 4$

$f_5(c) = 4$

$f_5(d) = 5$

$f_5(e) = 1$

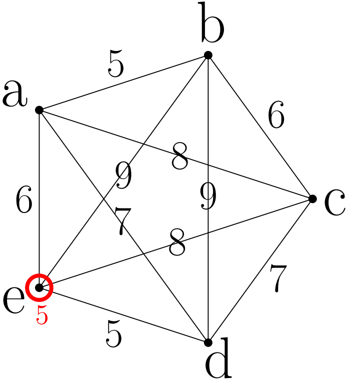
Cost opt(4)=$9 + 1 = 10$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$$f_5(a) = 3$$
$$f_5(b) = 4$$
$$f_5(c) = 4$$
$$f_5(d) = 5$$
$$f_5(e) = 1$$

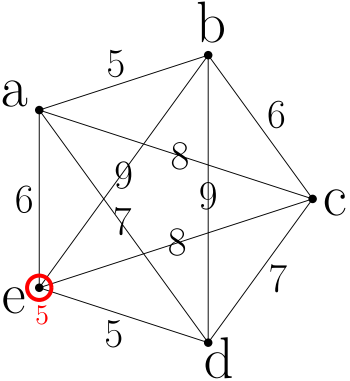
Cost opt(4)=$9 + 1 = 10$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$$f_5(a) = 3$$
$$f_5(b) = 4$$
$$f_5(c) = 4$$
$$f_5(d) = 5$$
$$f_5(e) = 1$$

Cost opt(5)$=10 + 1 = 11$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.
Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.
Goal: Minimize the competitive ratio between our algorithm to opt.

$$\text{Competitive ratio} = \max_{\text{input } I} \frac{\mathrm{Alg}(I)}{\mathrm{opt}(I)}.$$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.

$$\text{Competitive ratio} = \max_{\text{input } I} \frac{\mathrm{Alg}(I)}{\mathrm{opt}(I)}.$$

Here $\frac{\mathrm{Alg}(I)}{\mathrm{opt}(I)} = \frac{27}{11}$.

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.

$$\text{Competitive ratio} = \max_{\text{input } I} \frac{\mathrm{Alg}(I)}{\mathrm{opt}(I)}.$$

Here $\frac{\mathrm{Alg}(I)}{\mathrm{opt}(I)} = \frac{27}{11}$.

Competitive ratio against oblivious adversary is

$$\max_{\text{input } I} \frac{\mathbb{E}[\mathrm{Alg}(I)]}{\mathrm{opt}(I)}.$$

Online problem - Metrical Task System (MTS)

Approach: embed into a tree, and then make all the decisions based on the tree.

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Theorem ([Fiat, Mendel 2000])

Given an n point tree T, there is an online algorithm for MTS with competitive ratio $O(\log n \cdot \log \log n)$.*

* Actually on an HST, which is a special kind of tree. [FRT04] is into HST's.

Online problem - Metrical Task System (MTS)

Approach: embed into a tree, and then make all the decisions based on the tree.

Theorem ([Fiat, Mendel 2000])

Given an n point tree T, there is an online algorithm for MTS with competitive ratio $O(\log n \cdot \log\log n)$.*

Algorithm: 1. Sample a tree T over (X, d_X) using [FRT04].

Online problem - Metrical Task System (MTS)

Approach: embed into a tree, and then make all the decisions based on the tree.

Theorem ([Fiat, Mendel 2000])

Given an n point tree * T, *there is an online algorithm for MTS with competitive ratio* $O(\log n \cdot \log \log n)$.

Algorithm: 1. Sample a tree T over (X, d_X) using [FRT04].

2. Run [FM00] on T with the same cost functions.

Make the same decisions as [FM00].

Online problem - Metrical Task System (MTS)

Algorithm: 1. Sample a tree T over (X, d_X) using [FRT04].

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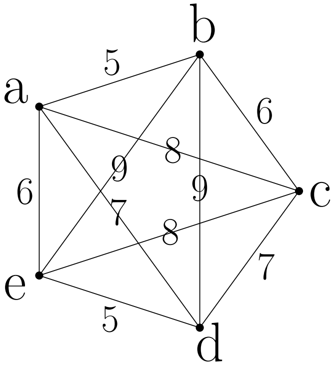

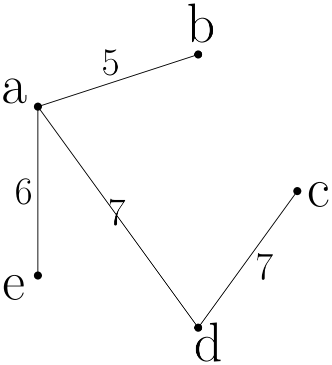

Online problem - Metrical Task System (MTS)

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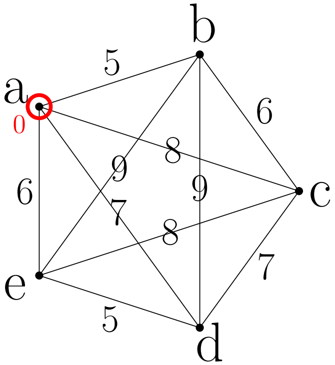

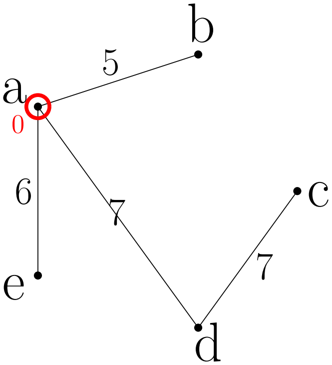

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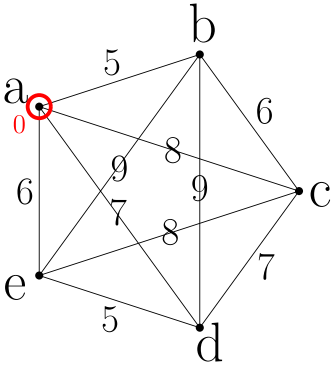
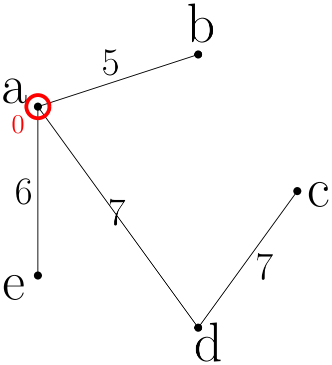


$$f_1(a) = 3$$
$$f_1(b) = 1$$
$$f_1(c) = 2$$
$$f_1(d) = 4$$
$$f_1(e) = 1$$



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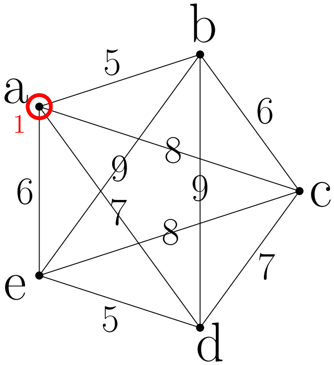
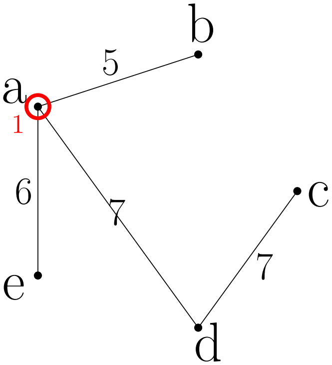


$$f_1(a) = 3$$
$$f_1(b) = 1$$
$$f_1(c) = 2$$
$$f_1(d) = 4$$
$$f_1(e) = 1$$



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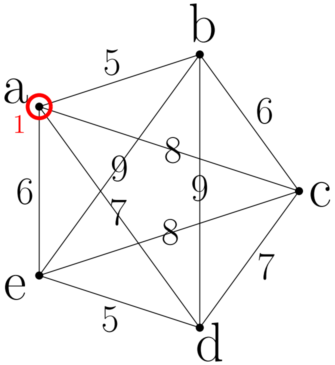
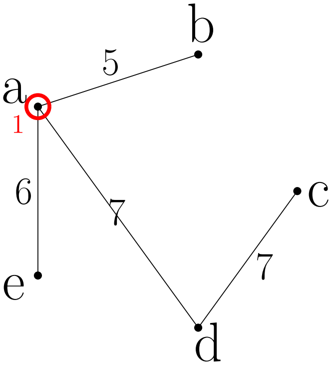


Cost alg(1)=3

$$f_1(a) = 3$$
$$f_1(b) = 1$$
$$f_1(c) = 2$$
$$f_1(d) = 4$$
$$f_1(e) = 1$$



Cost $\mathrm{alg}_T(1)$=3

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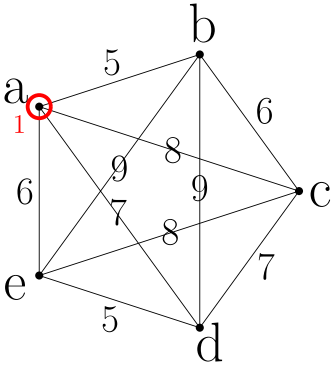
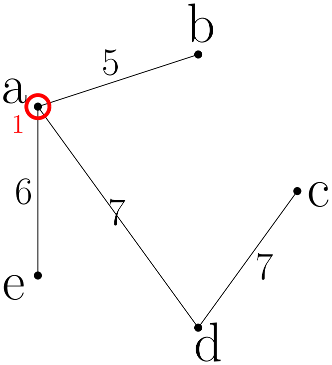


Cost alg(1)=3

$$f_2(a) = 3$$
$$f_2(b) = 1$$
$$f_2(c) = 3$$
$$f_2(d) = 2$$
$$f_2(e) = 1$$



Cost $\text{alg}_T(1)$=3

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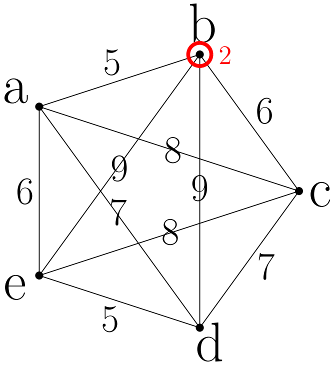
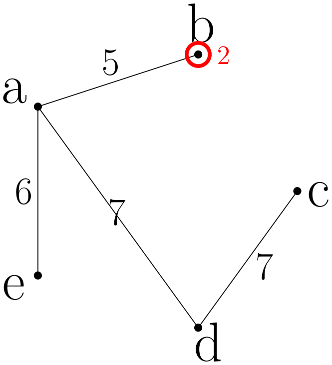


Cost alg(1)=3

$f_2(a) = 3$
$f_2(b) = 1$
$f_2(c) = 3$
$f_2(d) = 2$
$f_2(e) = 1$



Cost $\mathrm{alg}_T(1)$=3

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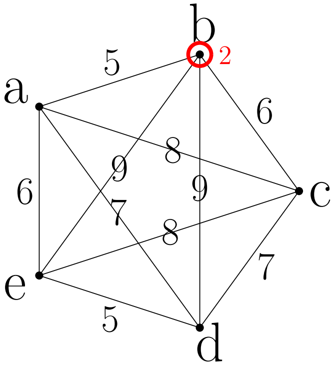
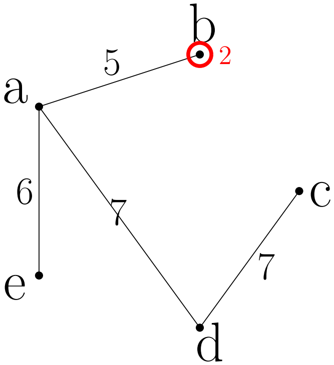


$$f_2(a) = 3$$
$$f_2(b) = 1$$
$$f_2(c) = 3$$
$$f_2(d) = 2$$
$$f_2(e) = 1$$



Cost $\mathrm{alg}(2) = 3 + 5 + 1 = 9$

Cost $\mathrm{alg}_T(2) = 3 + 5 + 1 = 9$

Online problem - Metrical Task System (MTS)

Algorithm: 1. Sample a tree T over (X, d_X) using [FRT04].

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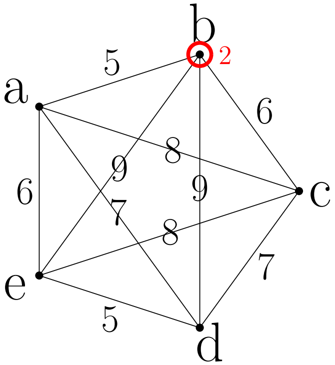
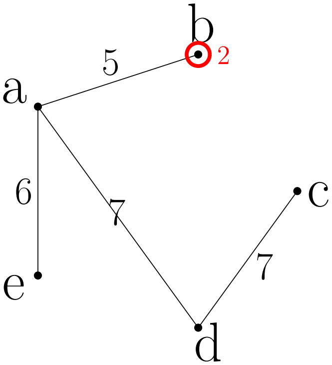


$f_3(a) = 1$
$f_3(b) = 4$
$f_3(c) = 3$
$f_3(d) = 2$
$f_3(e) = 1$



Cost alg(2)$=3 + 5 + 1 = 9$

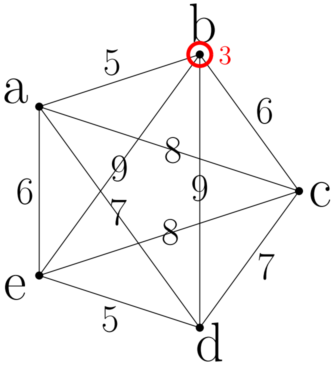
Cost $\mathrm{alg}_T(2)$$=3 + 5 + 1 = 9$

Online problem - Metrical Task System (MTS)

Algorithm: 1. Sample a tree T over (X, d_X) using [FRT04].

2. Run [FM00] on T with the same cost functions.

Make the same decisions as [FM00].



$$f_3(a) = 1$$
$$f_3(b) = 4$$
$$f_3(c) = 3$$
$$f_3(d) = 2$$
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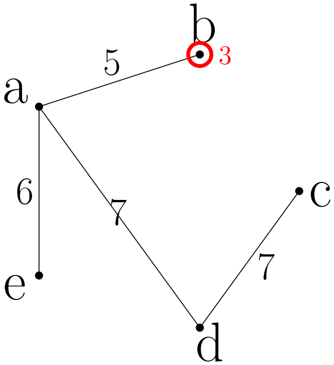

Cost alg(2)=$3 + 5 + 1 = 9$

Cost $\text{alg}_T(2)$=$3 + 5 + 1 = 9$

Online problem - Metrical Task System (MTS)

Algorithm: 1. Sample a tree T over (X, d_X) using [FRT04].
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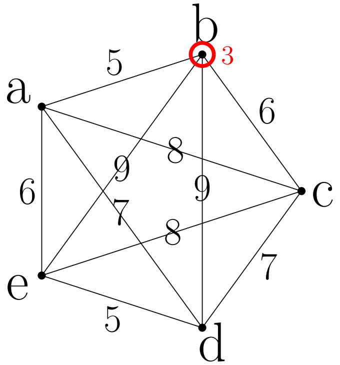
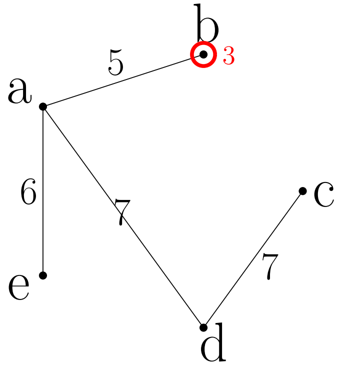


$$f_3(a) = 1$$
$$f_3(b) = 4$$
$$f_3(c) = 3$$
$$f_3(d) = 2$$
$$f_3(e) = 1$$

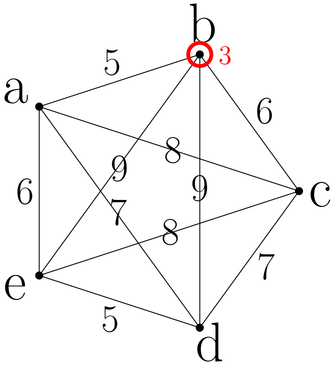


Cost alg(3)$=9 + 4 = 13$

Cost $\text{alg}_T(3)$$=9 + 4 = 13$

Online problem - Metrical Task System (MTS)

Algorithm: 1. Sample a tree T over (X, d_X) using [FRT04].
2. Run [FM00] on T with the same cost functions.
Make the same decisions as [FM00].



$f_4(a) = 5$
$f_4(b) = 10$
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$f_4(d) = 0$
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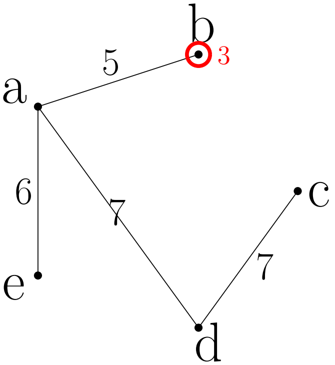

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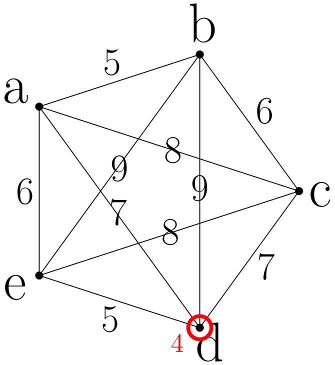

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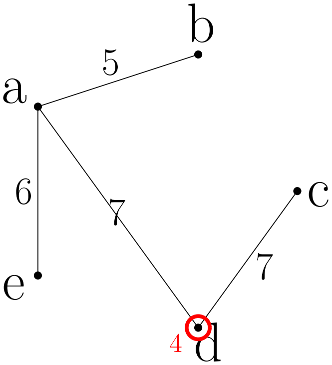

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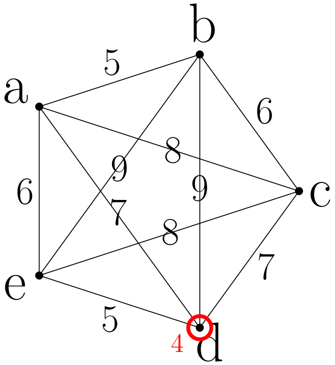

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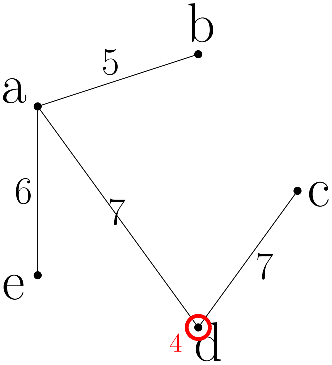

Cost alg(4)$=13 + 9 + 0 = 22$

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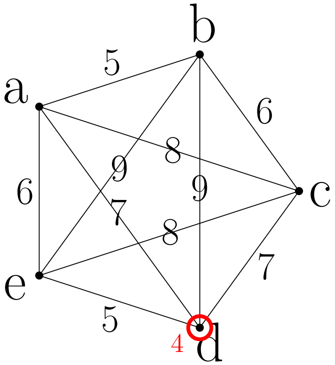

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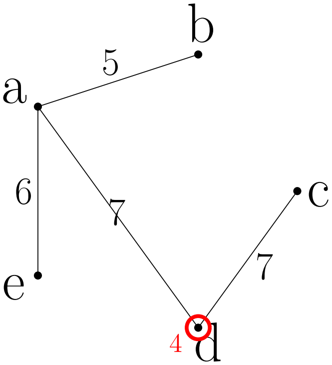

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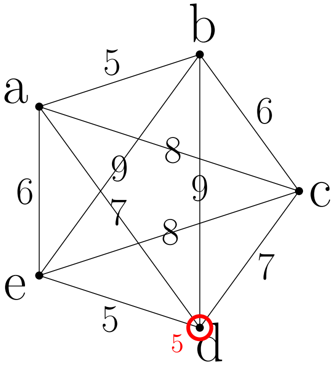

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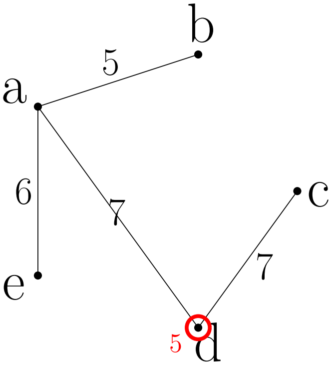

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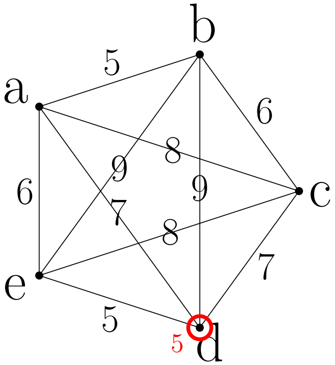

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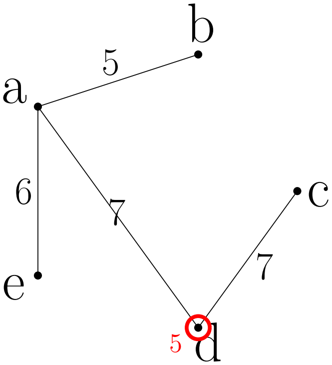

Cost alg(5)$=22+5=27$

Cost $\text{alg}_T(5)$$=25+5=30$

Online problem - Metrical Task System (MTS)

Algorithm: 1. Sample a tree T over (X, d_X) using [FRT04].
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Make the same decisions as [FM00].

Analysis. Let $x_1, x_2, \ldots, x_k$ be the decisions of opt. Thus $\mathrm{opt} = \sum_{i=1}^{k} f_i(x_i) + \sum_{i=1}^{k} d_X(x_{i-1}, x_i)$.

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We've sampled a tree T, $x_1, x_2, \ldots, x_k$ is also a valid decisions for T. Hence
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[FM00] is $O(\log n \cdot \log\log n)$-competitive on T. Hence it choose points $y_1, \ldots, y_k$ such that $\mathrm{alg}_T = \sum_{i=1}^{k} f_i(y_i) + \sum_{i=1}^{k} d_T(y_{i-1}, y_i) \leq O(\log n \cdot \log\log n) \cdot \mathrm{opt}_T$.

We made the same decisions, so overall:

$$\mathbb{E}[\mathrm{alg}] = \mathbb{E}\left[\sum_{i=1}^{k} f_i(y_i) + \sum_{i=1}^{k} d_X(y_{i-1}, y_i)\right]$$

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&\leq O(\log n \cdot \log\log n) \cdot \mathbb{E}\left[\sum_{i=1}^{k} f_i(x_i) + \sum_{i=1}^{k} d_T(x_{i-1}, x_i)\right]
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$$

Theorem

MTS has an $O(\log^2 n \cdot \log\log n)$ competitive algorithm against oblivious adversary.

Outline of the talk

Ramsey type Embeddings

Ramsey type theorem: Every **big** enough object, contains a **structured subset**.

Ramsey type Embeddings

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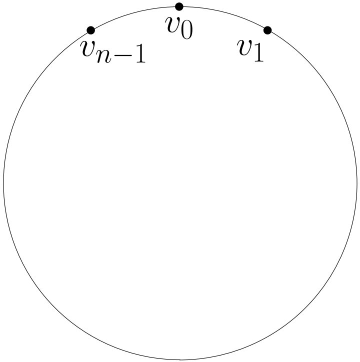

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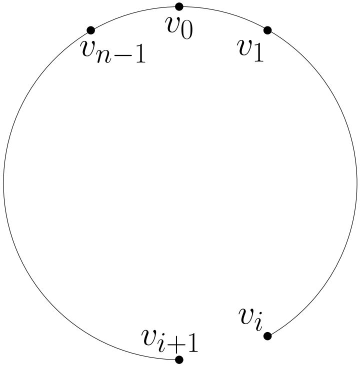

Suppose we delete $\{v_i, v_{i+1}\}$.

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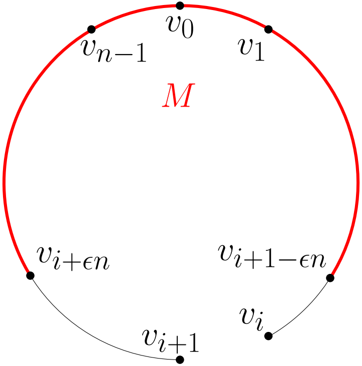

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Set $M = \{v_{i+1-\epsilon n}, v_{i+2-\epsilon n}, \ldots, v_{i+\epsilon n}\}$.

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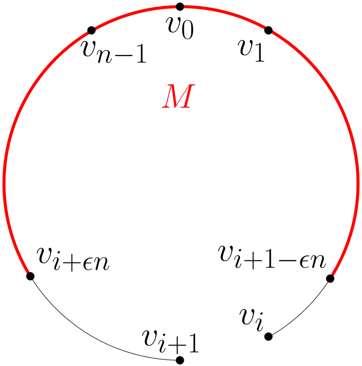

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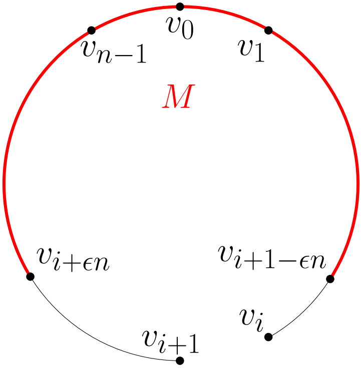

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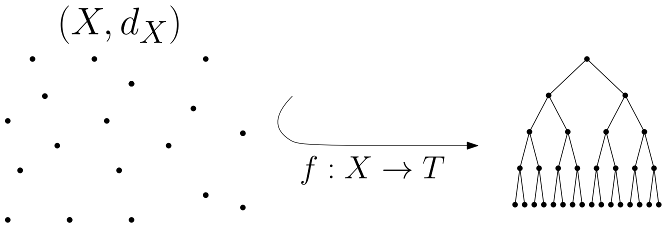
Choose i u.a.r., then $\qquad \Pr[v \in M] = 1 - 2\epsilon.$

Ramsey type Embeddings

Fix $\boldsymbol{k} > 1$, what is the **largest subset** $M \subset X$,
s.t. (M, d_X) embeds into a tree with **distortion** $\boldsymbol{k}$?



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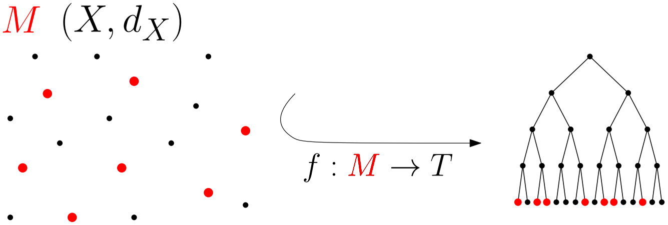

$\forall x, y \in M, \qquad d_X(x,y) \leq d_T(f(x), f(y)) \leq \boldsymbol{k} \cdot d_X(x,y)$

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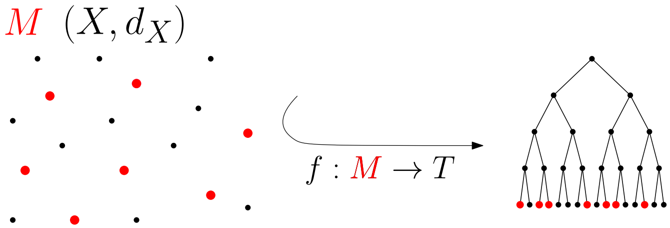

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Theorem ([Mendel, Naor 07], following [BFM86, BLMN05])

*$\forall$ n-point metric space and $k \geq 1$, $\exists$ **subset** M of size $n^{1-1/k}$ that **embeds** into a **tree** with distortion $O(k)$.*

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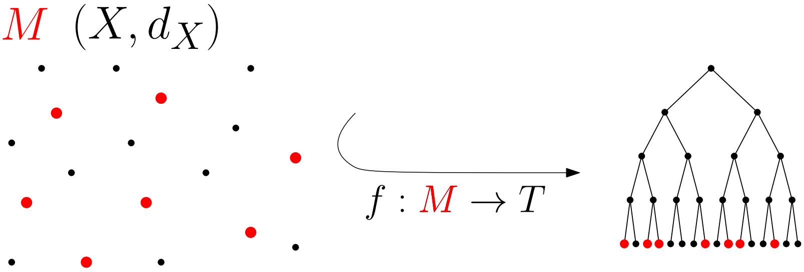

Asymptotically tight.

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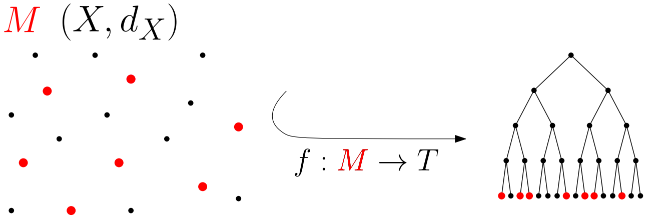

Asymptotically tight.

[Naor, Tao 12]: distortion $2e \cdot k$.

Ramsey type Embeddings

Corollary

For every n-point metric space and $\boldsymbol{k} \geq 1$*, there is a* ***set*** $\color{red}{\mathcal{T}}$ *of* $\color{red}{k \cdot n^{\frac{1}{k}}}$ ***trees*** *and a* *mapping* **home** : $X \to \mathcal{T}$, *such that for every* $x, y \in X$,

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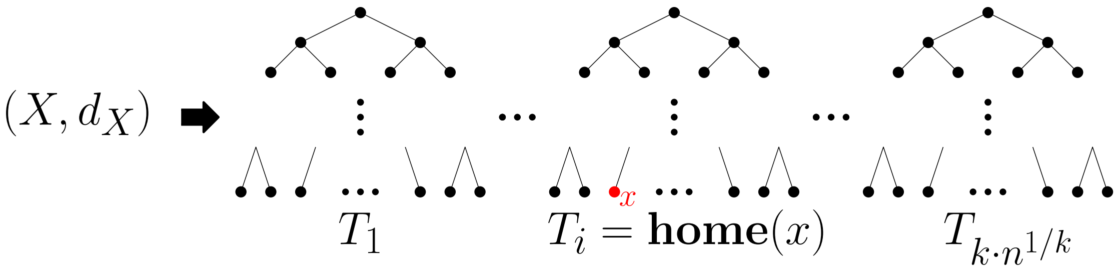

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Applications:

- **Distance oracle**
- Compact routing scheme
- Online algorithms
- Approximate ranking
- etc.

Ramsey type Embeddings

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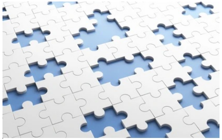
Compromises: only partial guarantees



Distance Oracle

A **succinct** data structure that **approximately** answers distance queries.

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The properties of interest are size, distortion and query time.

Distance Oracles

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For every n-point metric space and $\boldsymbol{k} \geq 1$*, there is a* ***set*** $\mathcal{T}$ *of* $k \cdot n^{\frac{1}{k}}$ ***trees*** *and a* *mapping* **home** : $X \to \mathcal{T}$*, such that for every* $x, y \in X$,

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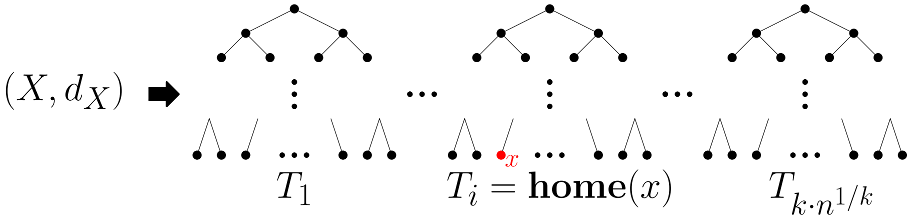

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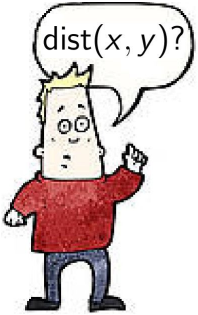

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Theorem (Tree Distance Oracle [Harel, Tarjan 84], [Bender, Farach-Colton 00])

For every ***tree metric****, there is an exact distance oracle of* ***linear size*** *and* ***constant*** *query* ***time****.*

Distance Oracles

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For every ***tree metric****, there is an exact distance oracle of* ***linear size*** *and* ***constant*** *query* ***time****.*

Theorem (Ramsey based Deterministic Distance Oracle)

For any n-point metric space, there is a distance oracle with :

Distortion	***Size***	***Query time***
$O(k)$	$O(k \cdot n^{1+1/k})$	$O(1)$

Outline of the talk

Clan Embedding

Idea: **duplicate** vertices to meet all guarantees!

Clan Embedding

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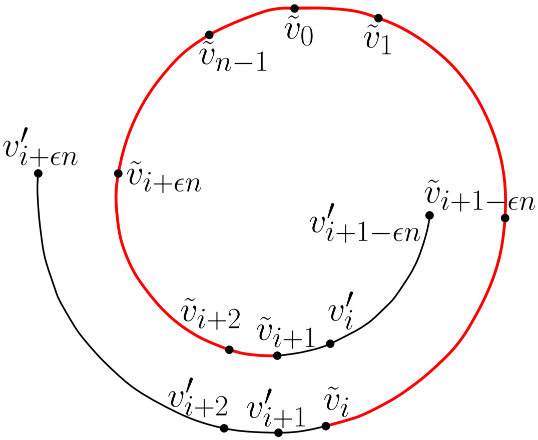

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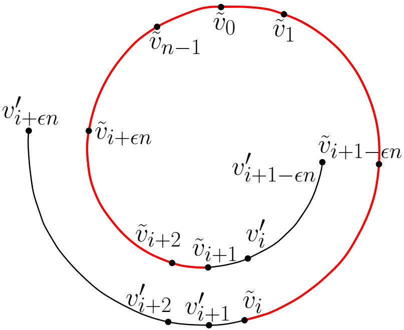

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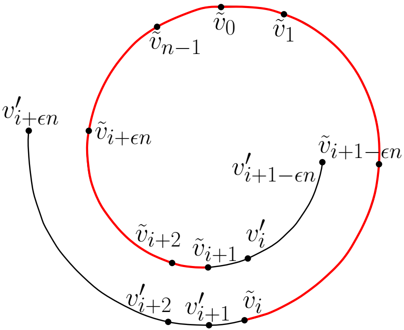

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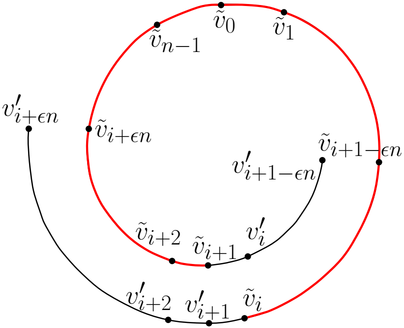

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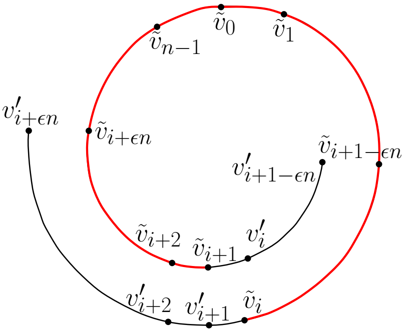

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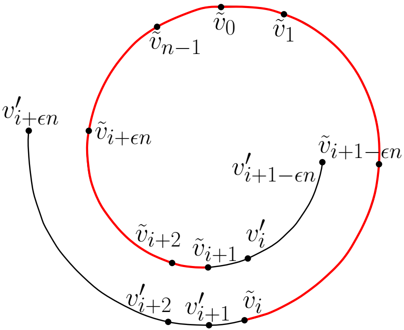

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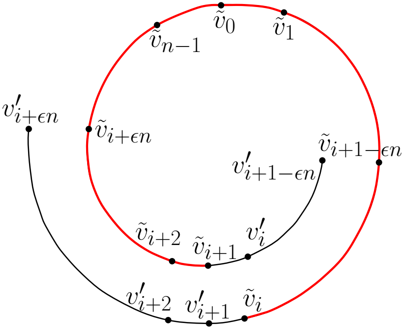

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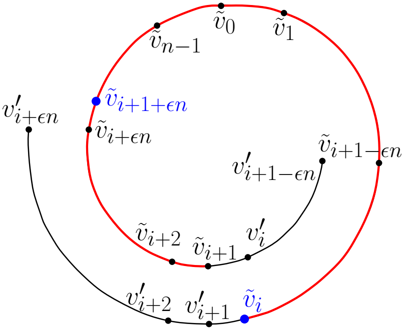

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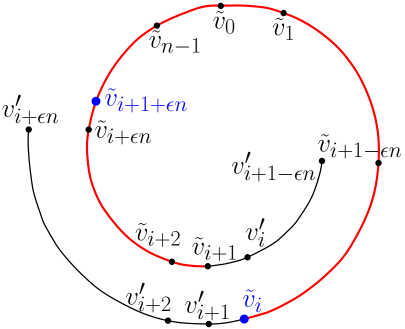

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Choose i u.a.r., then $\qquad \mathbb{E}[|f(v_i)|]=1+2\epsilon.$

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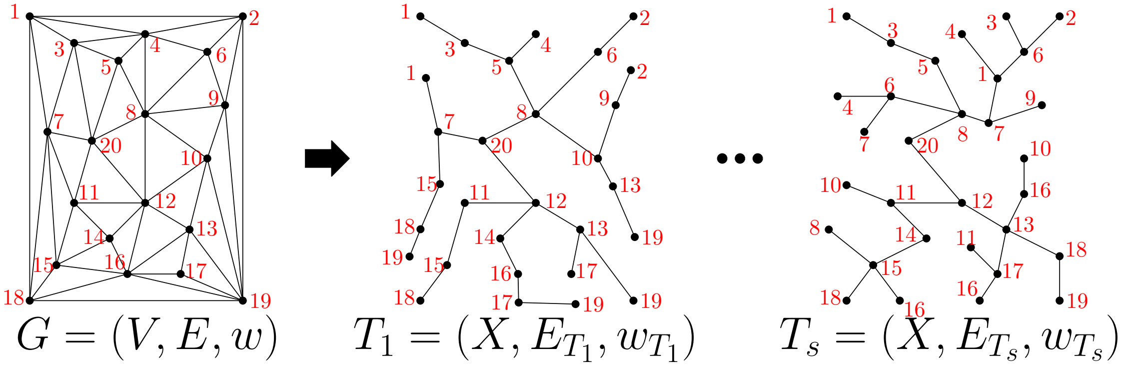

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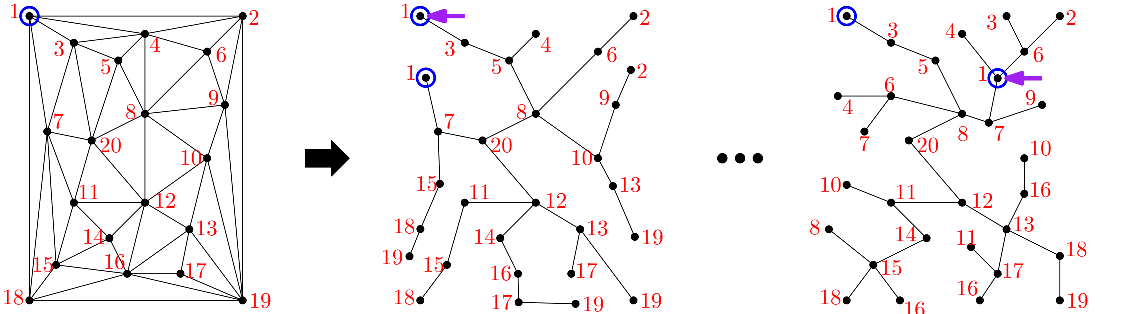

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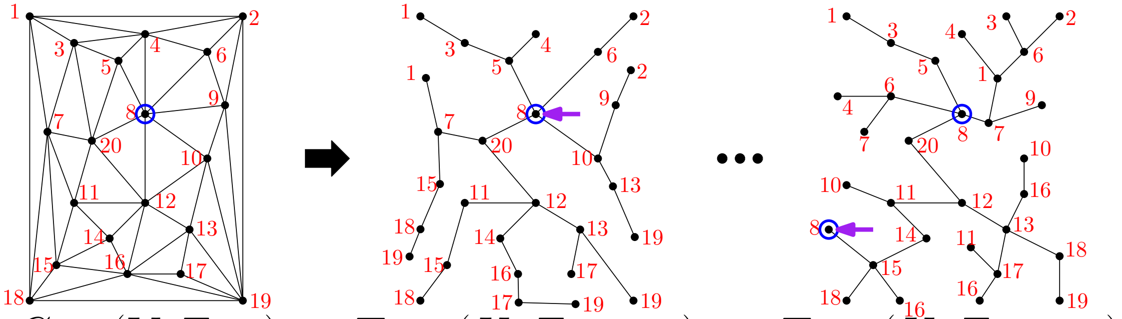

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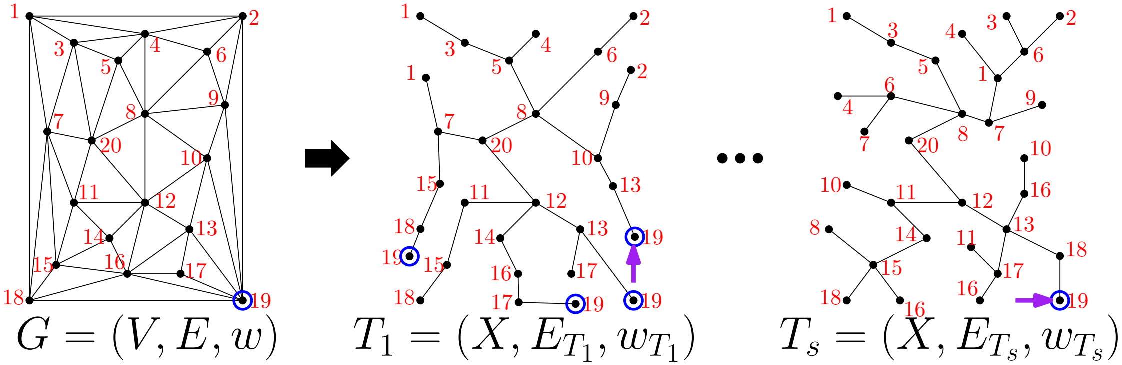

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(both) Tight!

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Compromises: Not a real classic embedding

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Originally appeared in [Bartal, Mendel 04] in the context of multi-embeddings.

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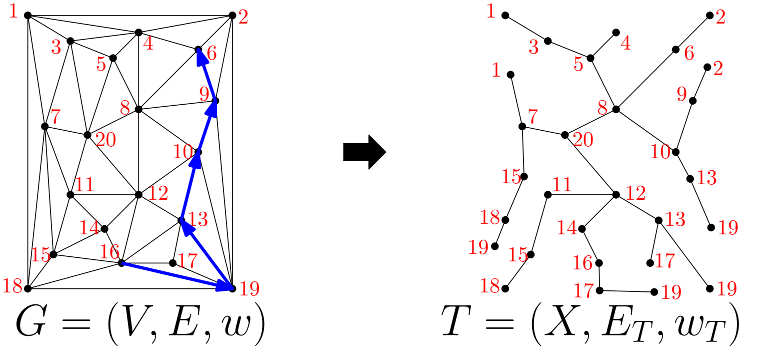

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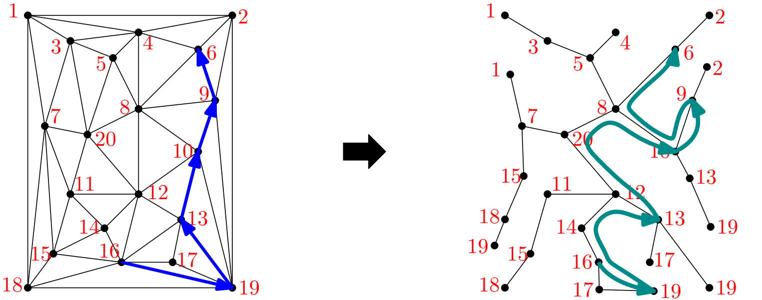

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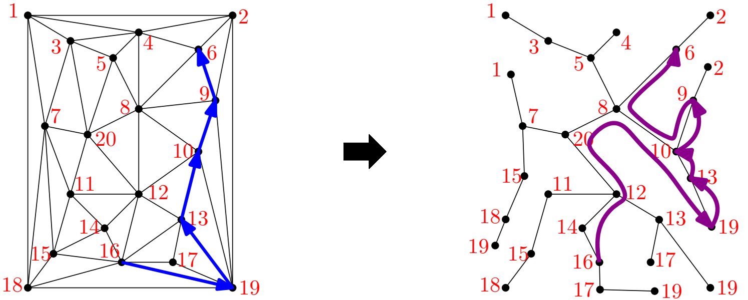

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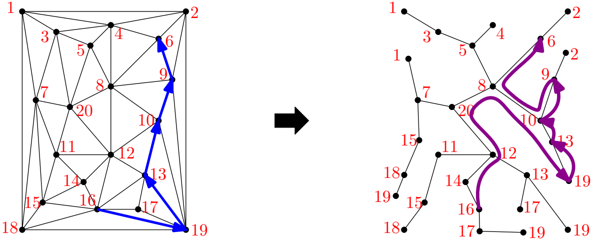

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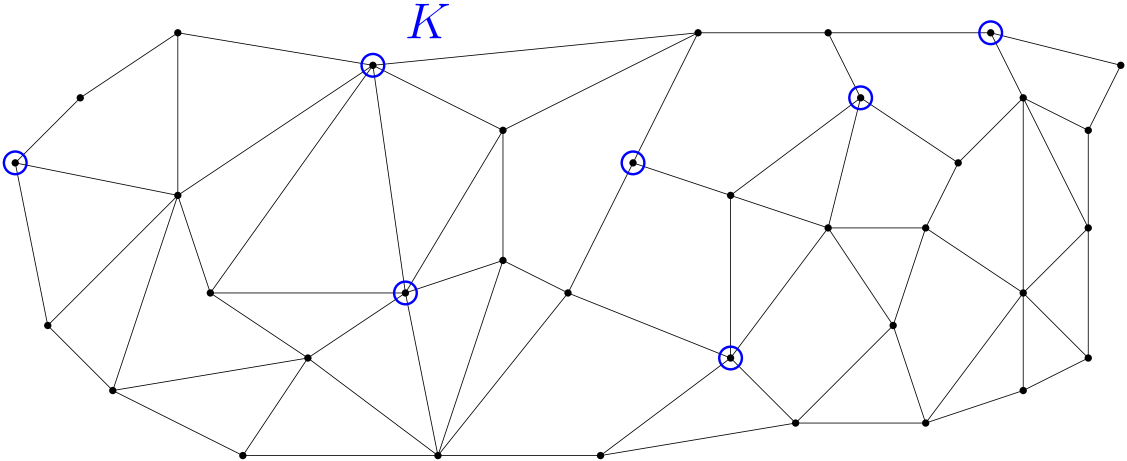
Or a total of $O(n^{1+\frac{1}{2}})$ copies and path distortion $O(\log n)$.

Steiner Tree

Given set of terminals K, find minimum weight tree T spanning K



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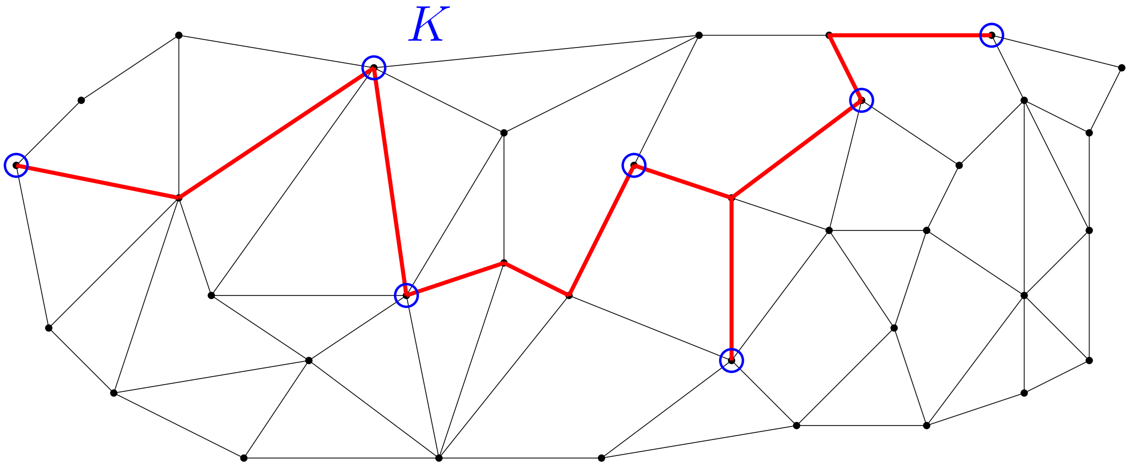

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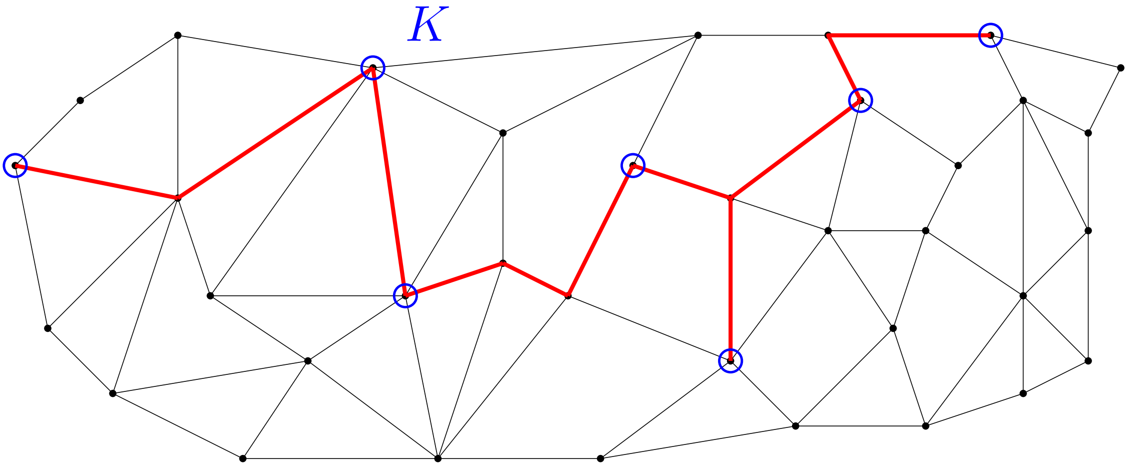
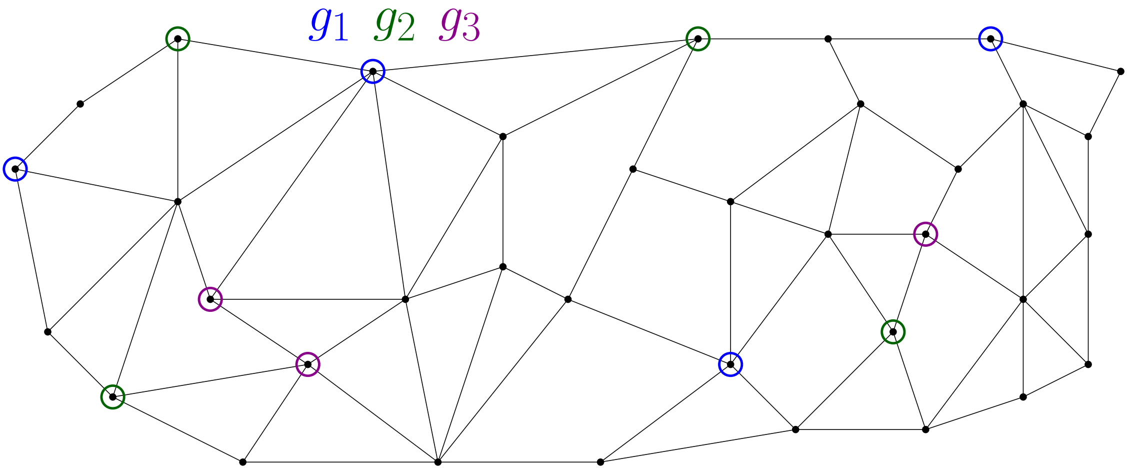


In class we saw a 2-approximation algorithm for the Steiner tree problem.

Group Steiner Tree

Given subsets $g_1, g_2, \ldots, g_k \subseteq V$, find minimum weight tree T spanning at least one vertex from each g_i



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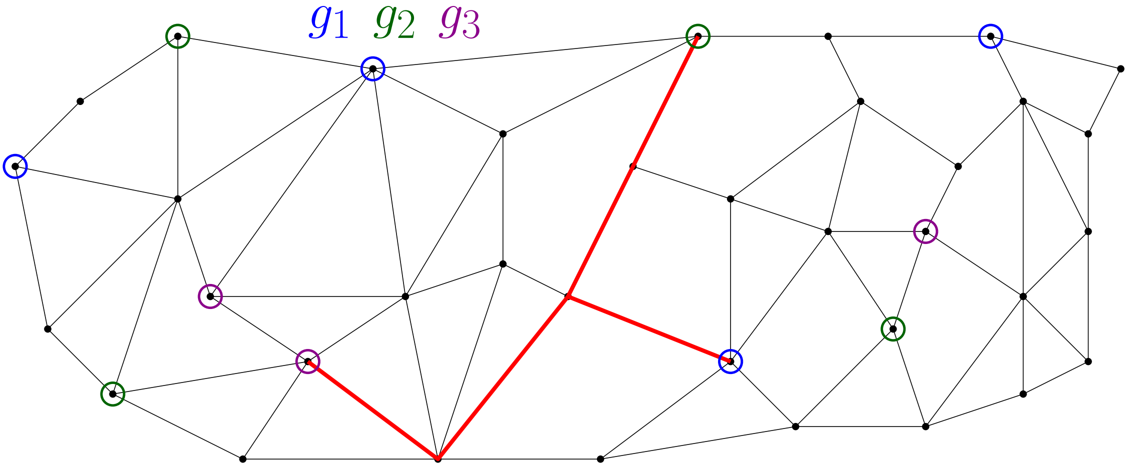

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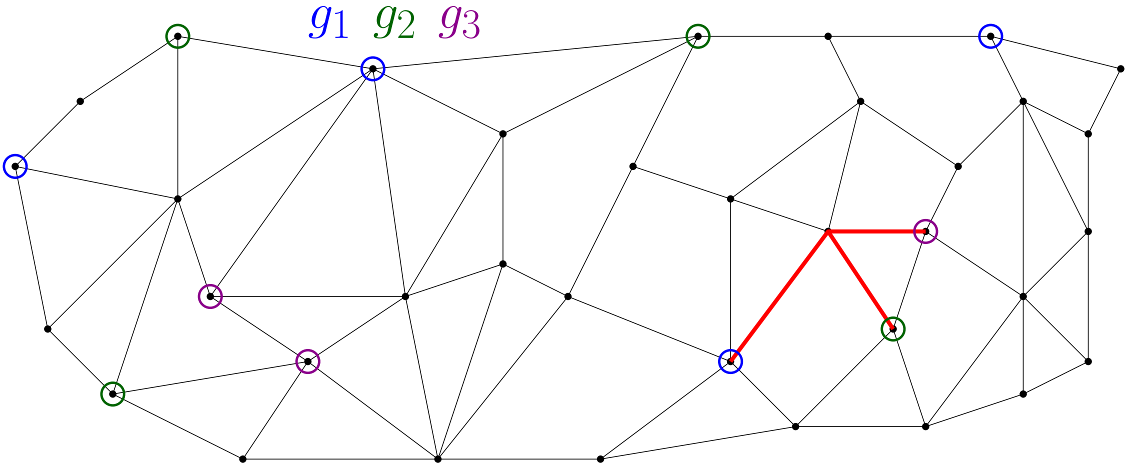

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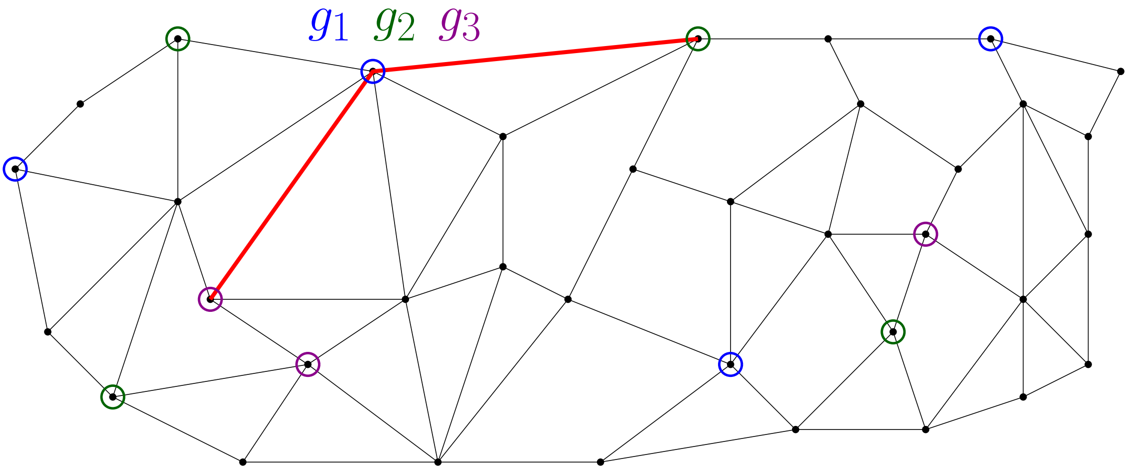

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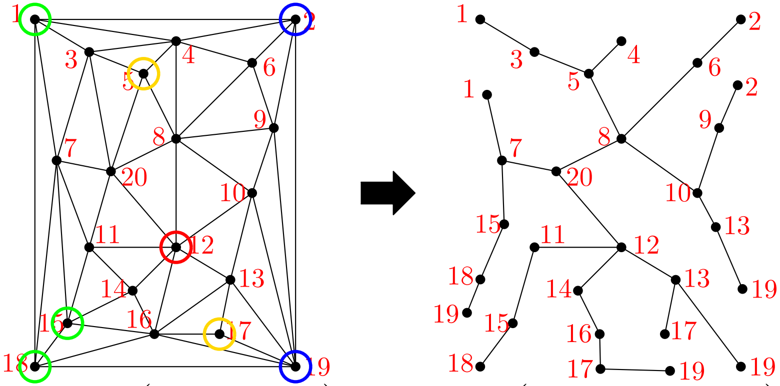

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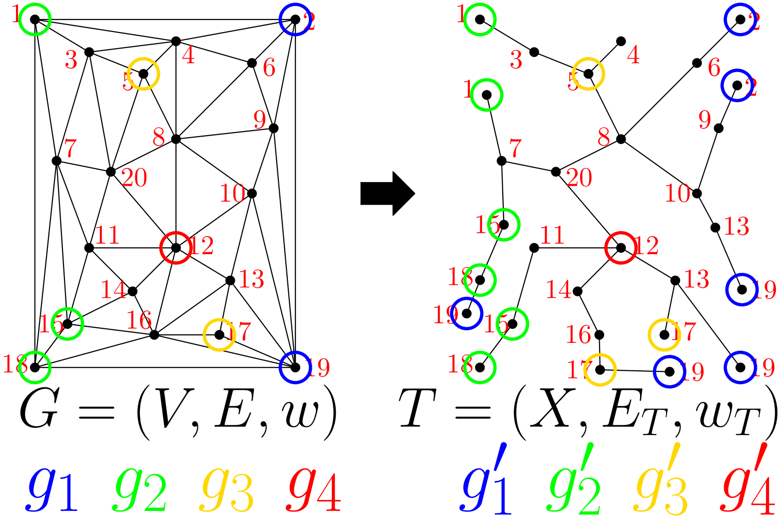

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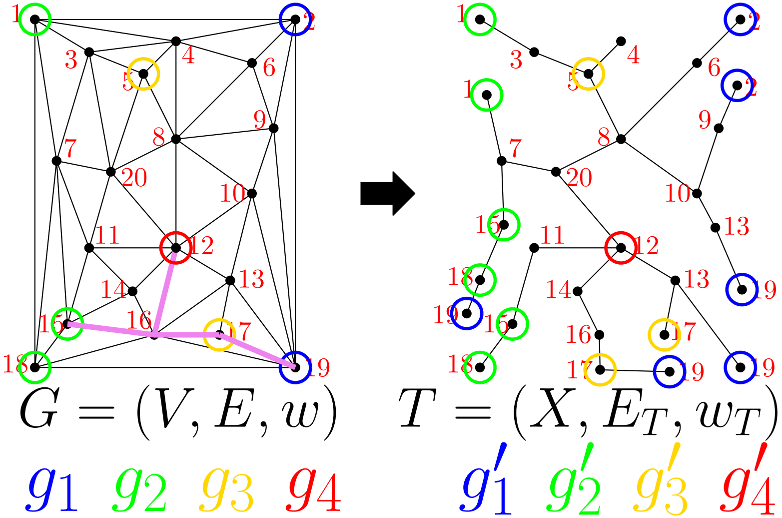

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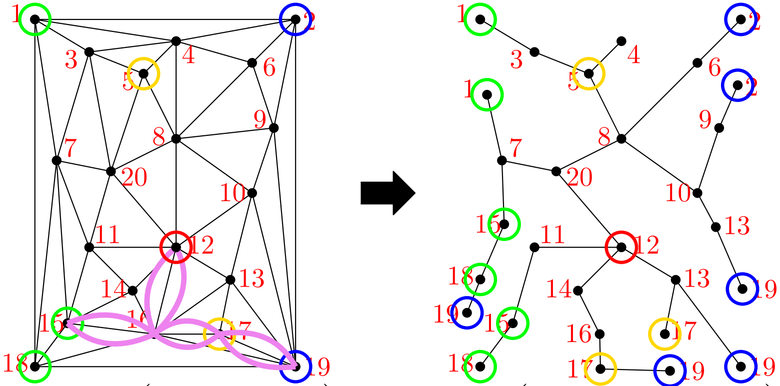
Double each edge: $2S^\star$



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Given subsets $g_1, g_2, \ldots, g_k \subseteq V$, find minimum weight tree T spanning at least one vertex from each g_i.

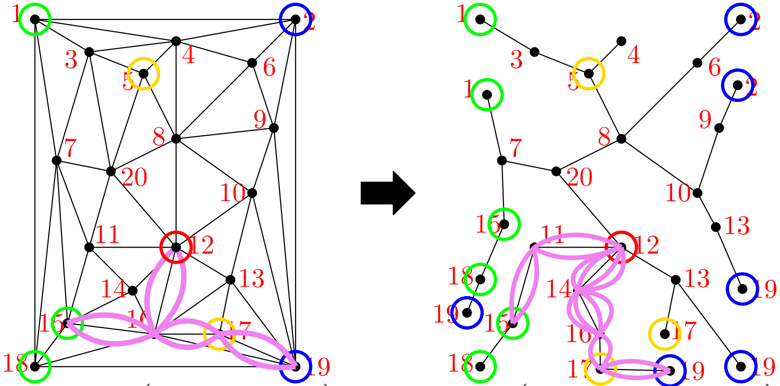
Clan embedding f with path distortion $O(\log n)$.

$g'_i = f(g_i)$

$S^\star$ optimal solution.

Double each edge: $2S^\star$

Guaranteed path: $S^\star_T$ (valid solution), $w(S^\star_T) \leq O(\log n) \cdot w(S^\star)$



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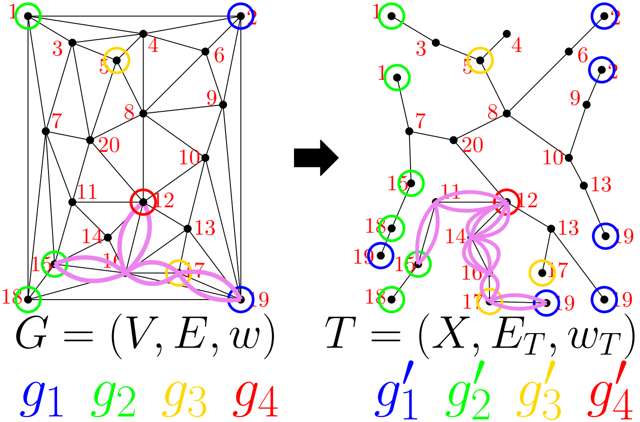

Theorem ([Garg, Konjevod, Ravi 00])

$O(\log n \cdot \log k)$-approximation algorithm for the GST problem on trees.

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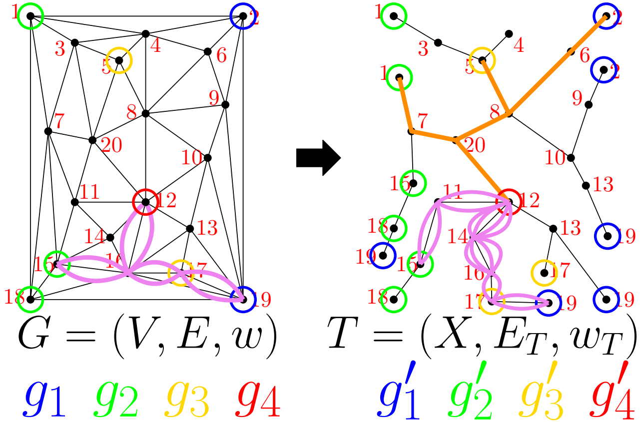

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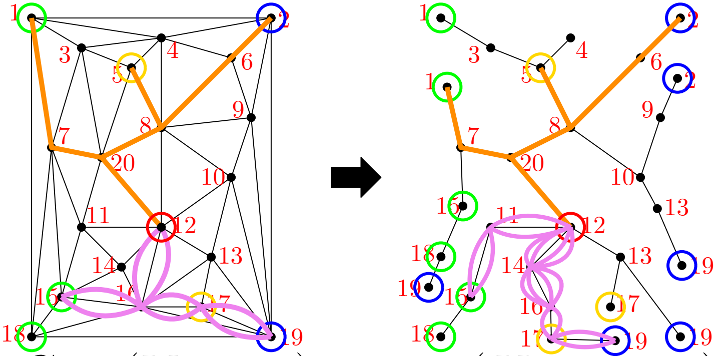
Return: $\tilde{S} = f^{-1}(\tilde{S}_T)$



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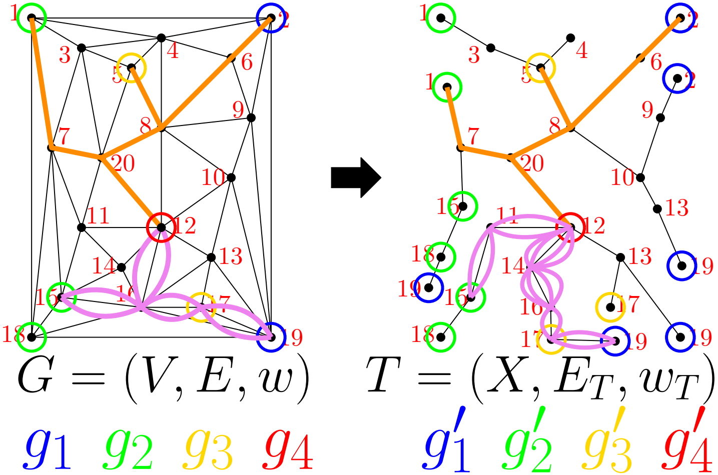

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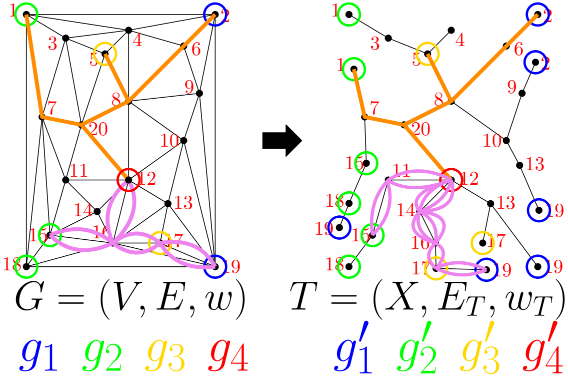
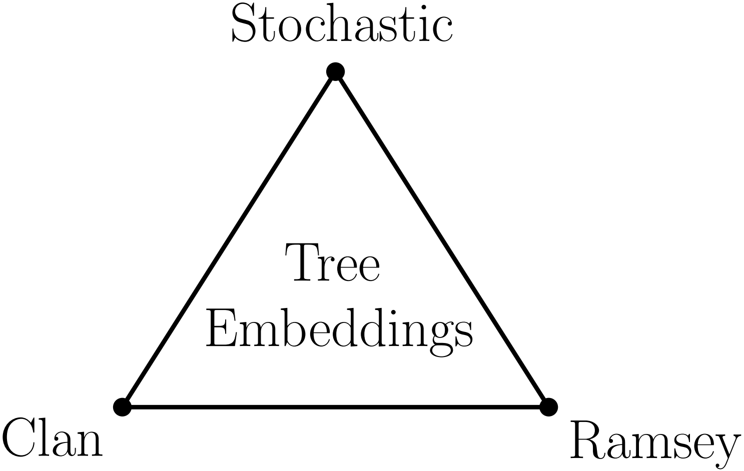


$$w(\tilde{S}) \leq w(\tilde{S}_T) \leq O(\log^2 n \cdot \log k) \cdot w(S^\star).$$

We got an $O(\log^2 n \cdot \log k)$ **approximation**.

Outline of the talk

Trinity



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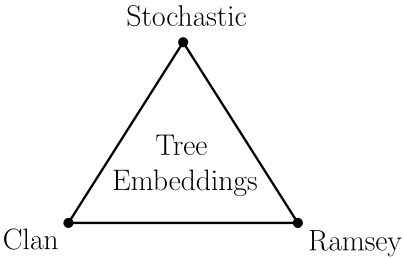


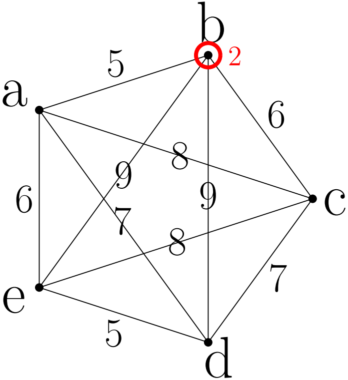
	Worst case gurantte	Satisfy all vertices	one-to-one embedding
Stochastic	✗	✓	✓
Ramsey	✓	✗	✓
Clan	✓	✓	✗

Online problem - Metrical Task System

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$$f_3(a) = 1$$
$$f_3(b) = 4$$
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Cost alg(2)$=3 + 5 + 1 = 9$

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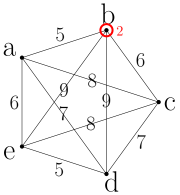

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We showed competitive ratio against oblivious adversary:

$$\text{Competitive ratio} = \max_{\text{input } I} \frac{\mathbb{E}[\mathrm{Alg}(I)]}{\mathrm{opt}(I)} = O(\log^2 n \cdot \log \log n).$$

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Theorem ([Fakcharoenphol, Rao, Talwar 04], improving [Bartal 96+98])

*Every n-point metric space (X, d) embeds into **distribution** $\mathcal{D}$ over **dominating trees** with **expected distortion** $O(\log n)$.*

Theorem ([Fiat Mendel 2000])

Given an n point tree * *T, there is an online algorithm for MTS with competitive ratio $O(\log n \cdot \log\log n)$.*

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Problem: Propose a **deterministic** algorithm for **MTS** with the same competitive ratio.

Link to quiz:

Can also be found my homepage:

arnold.filtser.com

Or just google Arnold Filtser.

Link to slides:

MTS Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
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Goal: Propose a **deterministic** algorithm for **MTS** with the same competitive ratio.

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Clan Embeddings construction

Theorem (Clan embedding into trees, [Filtser, Le 21])

(X, d_X) *n point metric space,* $\forall k \in \mathbb{N}$, *there is* ***distribution*** $\mathcal{D}$ *over* ***dominating clan embeddings*** *into trees such that:*

- $\forall\ (f, \chi) \in \mathrm{supp}(\mathcal{D})$ *has distortion* $O(k)$.
- $\forall\ x \in X$, $\mathbb{E}_{(f,\chi)\sim\mathcal{D}}[|f(x)|] \le O(n^{\frac{1}{k}})$

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Lemma

(X, d_X) *n point metric, there is* ***dominating clan embedding*** *into a tree s.t.:*

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Replace cardinality with weights

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Definition (Ultrametric)

Ultrametric (X, d) is a metric space satisfying the **strong** triangle inequality:

$$\forall x, y, z \in X, \qquad d(x, z) \leq \max\{d(x, y), d(y, z)\} \ .$$

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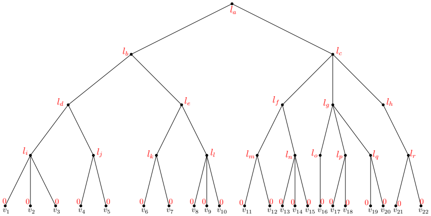
Definition (Hierarchical well-separated tree (HST))

(X, d_X) is a HST if X is mapped (by ϕ) to **leaves** of a rooted tree T where:

- The nodes of T associated with **monotone** labels l_v.
- $d_X(x, y) = \Gamma_{\mathrm{lca}(\varphi(x),\varphi(y))}$.



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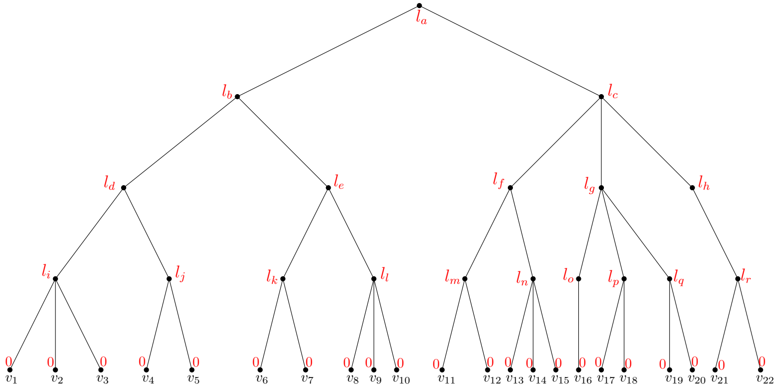

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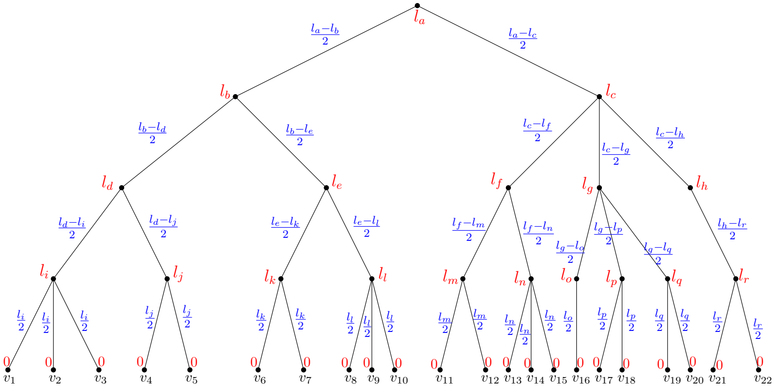
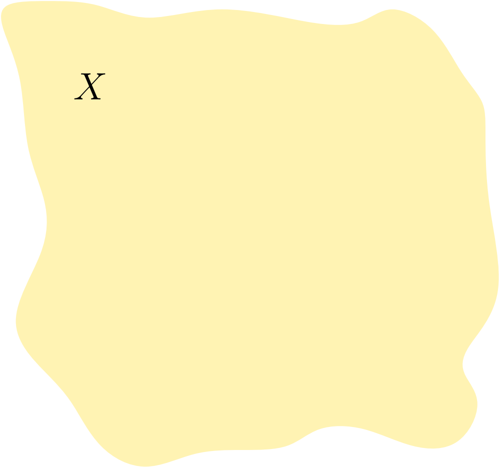


Construction



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r_U

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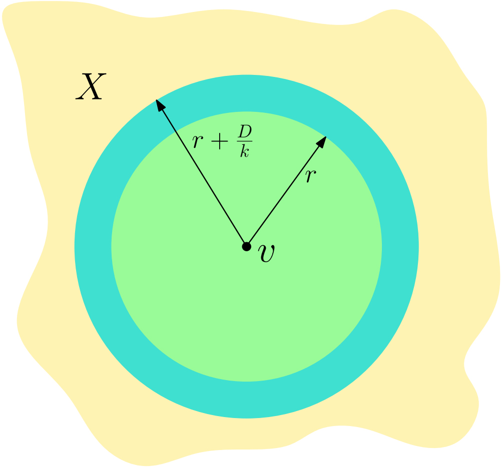

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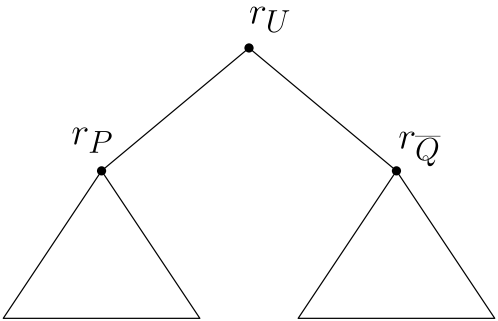

$P = B_X(v, r + \frac{D}{k})$

$\overline{Q} = X \setminus B_X(v, r)$

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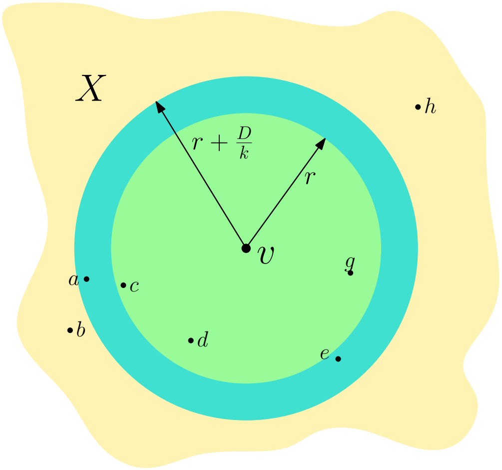

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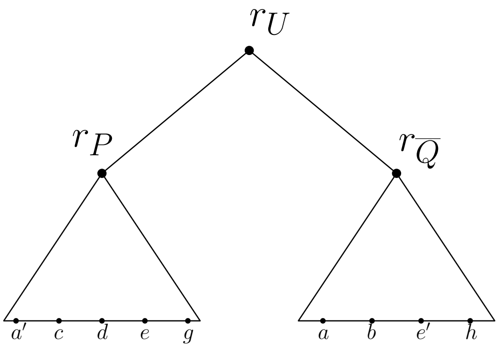

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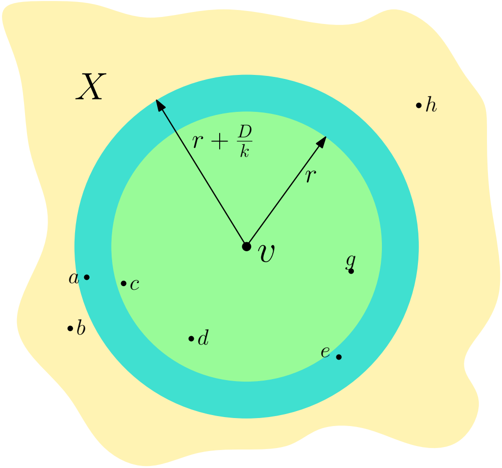

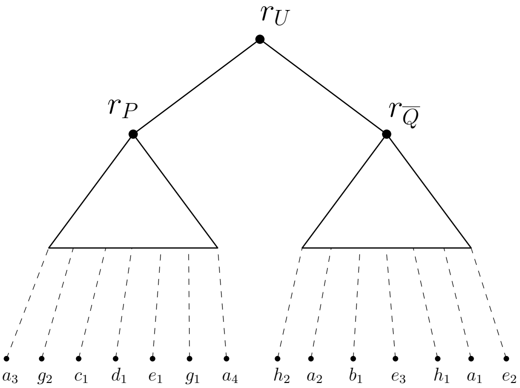

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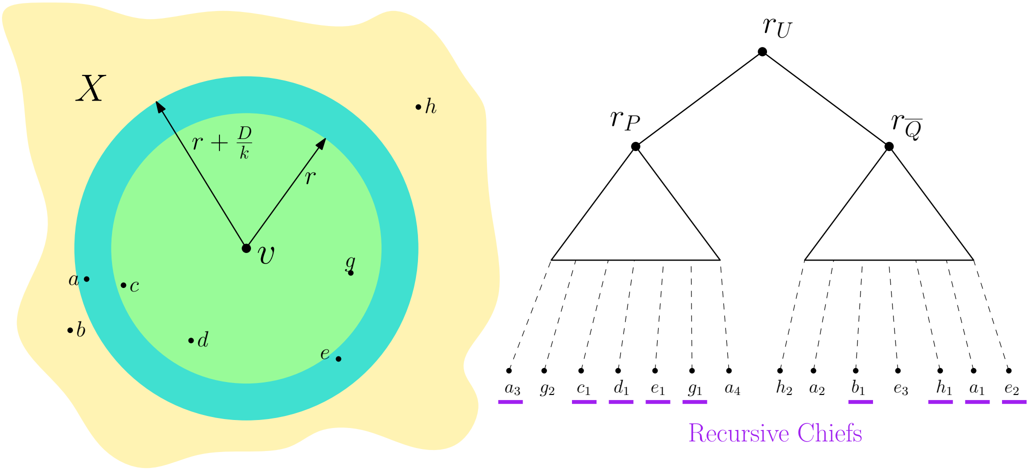

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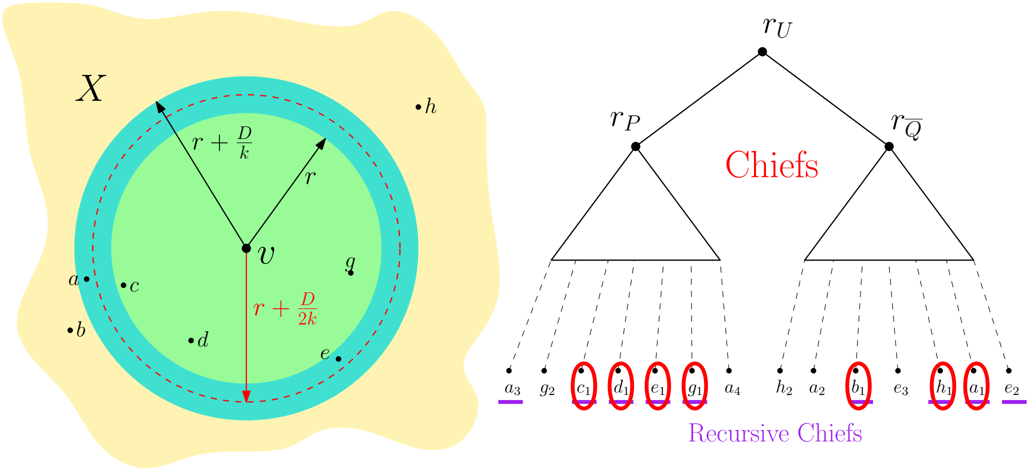
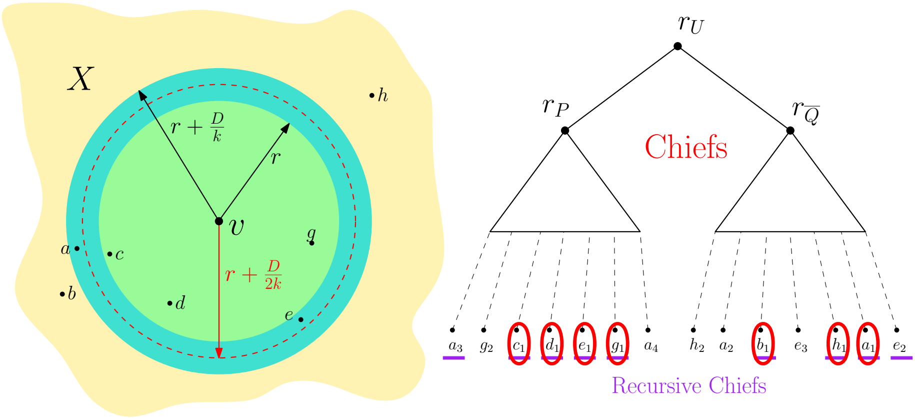
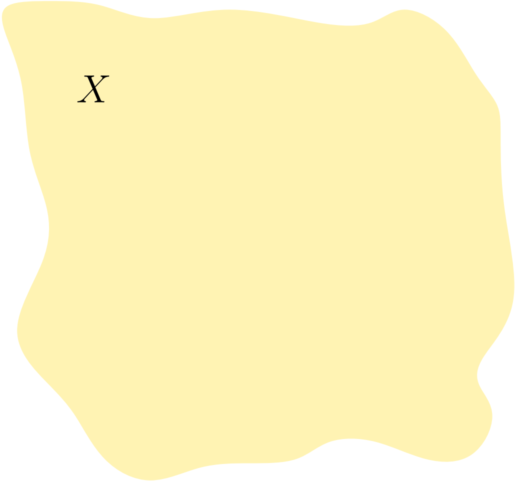


Construction - distortion bound



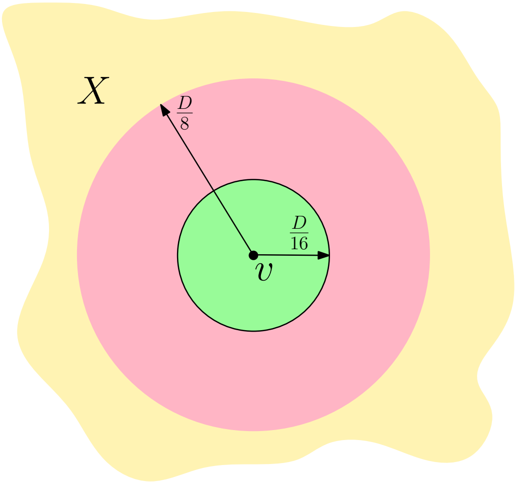
$$\min_{c' \in f(c)} d_U(c', \chi(a)) = D \leq 2k \cdot d_X(c, a) \ .$$

Construction - cardinality bound



$$D = \mathrm{diam}(X)$$

Construction - cardinality bound



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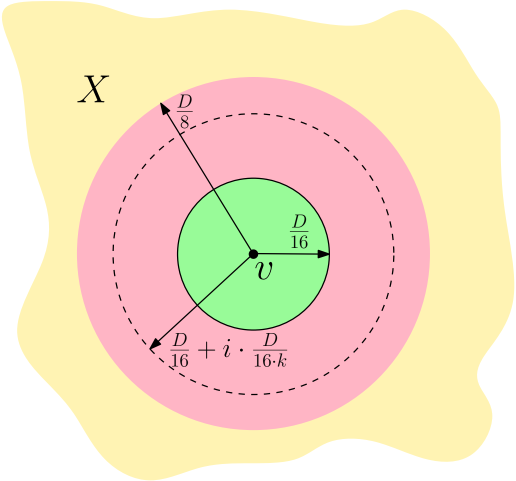

$$D = \mathrm{diam}(X)$$

$$A_i = B_X(v, \frac{D}{16} + i \cdot \frac{D}{16 \cdot k})$$
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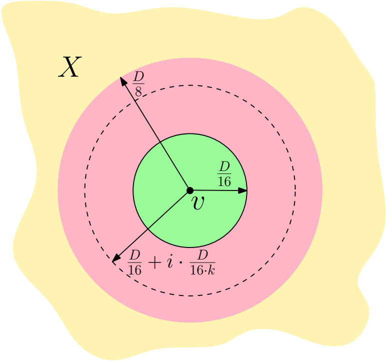

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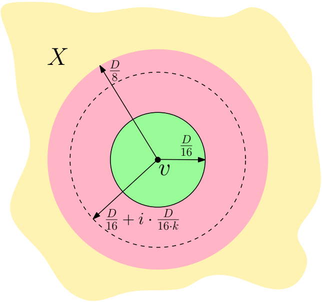

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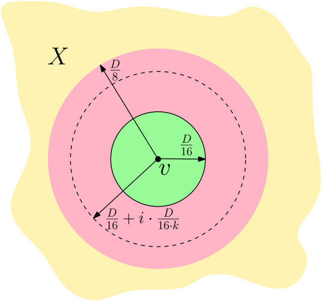
$$|A_k| > |A_{k-1}| \cdot \left(\frac{|A_k|}{|A_0|}\right)^{\frac{1}{k}} > |A_{k-2}| \cdot \left(\frac{|A_k|}{|A_0|}\right)^{\frac{2}{k}} > \cdots > |A_0| \cdot \left(\frac{|A_k|}{|A_0|}\right)^{\frac{k}{k}} = |A_k|$$

Construction - cardinality bound



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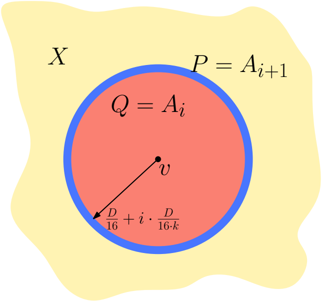
$A_i = B_X(v, \frac{D}{16} + i \cdot \frac{D}{16 \cdot k})$
$i \in \{0, \dots, k-1\}$

There is some i s.t. $\frac{|A_{i+1}|}{|A_i|} \leq \left(\frac{|A_k|}{|A_0|}\right)^{\frac{1}{k}}$.

Claim

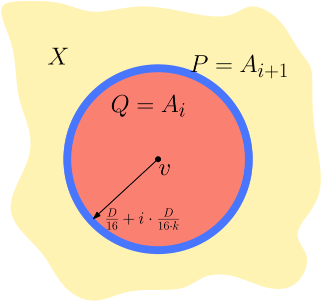
There is $v \in X$, and i, s.t. $\frac{|A_{i+1}|}{|A_i|} \leq \left(\frac{\mu^*(X)}{\mu^*(A_{i+1})}\right)^{1/k} = \left(\frac{\max_{x \in X} \left|B_X(x, \frac{\mathrm{diam(X)}}{4})\right|}{\max_{x \in A_{i+1}} \left|B_{A_{i+1}}(x, \frac{\mathrm{diam(A_{i+1})}}{4})\right|}\right)^{1/k}$.



Claim

There is $v \in X$*, and* i*, s.t.* $\frac{|A_{i+1}|}{|A_i|} \leq \left(\frac{\mu^*(X)}{\mu^*(A_{i+1})}\right)^{1/k} = \left(\frac{\max_{x \in X} \left|B_X(x, \frac{\mathrm{diam(X)}}{4})\right|}{\max_{x \in A_{i+1}} \left|B_{A_{i+1}}(x, \frac{\mathrm{diam(A_{i+1})}}{4})\right|}\right)^{1/k}$.

Set $P = A_{i+1}$, $Q = A_i$, and $\overline{Q} = X \setminus A_i$.



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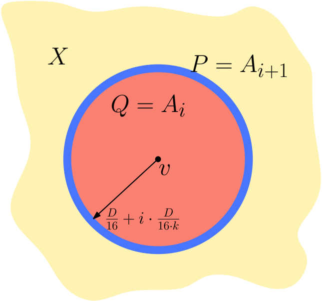
Set $P = A_{i+1}$, $Q = A_i$, and $\overline{Q} = X \setminus A_i$.
By the claim: $|P| \cdot \mu^*(P)^{\frac{1}{k}} \leq |Q| \cdot \mu^*(X)^{\frac{1}{k}}$.



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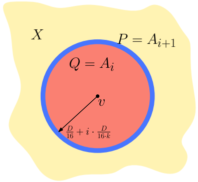
By the claim: $|P| \cdot \mu^*(P)^{\frac{1}{k}} \leq |Q| \cdot \mu^*(X)^{\frac{1}{k}}$.

Recurse on P and $\overline{Q}$.



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Recurse on P and $\overline{Q}$.

We argue by induction: $|f(X)| \leq |X| \cdot \mu^*(X)^{\frac{1}{k}} \leq |X|^{1+\frac{1}{k}}$.

Here $|f(X)| = \sum_{x \in X} |f(x)|$.

Claim

There is $v \in X$, *and* i, *s.t.* $\frac{|A_{i+1}|}{|A_i|} \leq \left(\frac{\mu^*(X)}{\mu^*(A_{i+1})}\right)^{1/k} = \left(\frac{\max_{x \in X}\left|B_X(x, \frac{\mathrm{diam(X)}}{4})\right|}{\max_{x \in A_{i+1}}\left|B_{A_{i+1}}(x, \frac{\mathrm{diam(A_{i+1})}}{4})\right|}\right)^{1/k}$.

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By the claim: $|P| \cdot \mu^*(P)^{\frac{1}{k}} \leq |Q| \cdot \mu^*(X)^{\frac{1}{k}}$. Recurse on P and $\overline{Q}$.

We argue by induction: $|f(X)| \leq |X| \cdot \mu^*(X)^{\frac{1}{k}} \leq |X|^{1+\frac{1}{k}}$.

$$|f(X)| = |f(P)| + |f(\overline{Q})|$$

Claim

There is $v \in X$, and i, s.t. $\frac{|A_{i+1}|}{|A_i|} \leq \left(\frac{\mu^*(X)}{\mu^*(A_{i+1})}\right)^{1/k} = \left(\frac{\max_{x\in X}\left|B_X(x, \frac{\mathrm{diam(X)}}{4})\right|}{\max_{x\in A_{i+1}}\left|B_{A_{i+1}}(x, \frac{\mathrm{diam(A_{i+1})}}{4})\right|}\right)^{1/k}$.

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$$\begin{aligned} |f(X)| &= |f(P)| + |f(\overline{Q})| \\ &\leq |P| \cdot \mu^*(P)^{\frac{1}{k}} + \left|\overline{Q}\right| \cdot \mu^*(\overline{Q})^{\frac{1}{k}} \end{aligned}$$

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There is $v \in X$*, and* i*, s.t.* $\frac{|A_{i+1}|}{|A_i|} \leq \left(\frac{\mu^*(X)}{\mu^*(A_{i+1})}\right)^{1/k} = \left(\frac{\max_{x \in X}\left|B_X(x, \frac{\mathrm{diam(X)}}{4})\right|}{\max_{x \in A_{i+1}}\left|B_{A_{i+1}}(x, \frac{\mathrm{diam(A_{i+1})}}{4})\right|}\right)^{1/k}$.

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|f(X)| &= |f(P)| + |f(\overline{Q})| \\
&\leq |P| \cdot \mu^*(P)^{\frac{1}{k}} + \left|\overline{Q}\right| \cdot \mu^*(\overline{Q})^{\frac{1}{k}} \\
&\leq |Q| \cdot \mu^*(X)^{\frac{1}{k}} + \left|\overline{Q}\right| \cdot \mu^*(X)^{\frac{1}{k}}
\end{aligned}$$

Claim

There is $v \in X$, and i, s.t. $\frac{|A_{i+1}|}{|A_i|} \leq \left(\frac{\mu^*(X)}{\mu^*(A_{i+1})}\right)^{1/k} = \left(\frac{\max_{x \in X}\left|B_X(x, \frac{\mathrm{diam(X)}}{4})\right|}{\max_{x \in A_{i+1}}\left|B_{A_{i+1}}(x, \frac{\mathrm{diam(A_{i+1})}}{4})\right|}\right)^{1/k}$.

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$$
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&\leq |Q| \cdot \mu^*(X)^{\frac{1}{k}} + \left|\overline{Q}\right| \cdot \mu^*(X)^{\frac{1}{k}} \\
&= |X| \cdot \mu^*(X)^{\frac{1}{k}} \ .
\end{aligned}
$$

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&\leq |Q| \cdot \mu^*(X)^{\frac{1}{k}} + \left|\overline{Q}\right| \cdot \mu^*(X)^{\frac{1}{k}} \\
&= |X| \cdot \mu^*(X)^{\frac{1}{k}} \ .
\end{aligned}$$