הפקולטה
למדעים מדויקים
אוניברסיטת בר־אילן

הזמנה

אירוע חשיפה
לתואר שני ושלישי

במחלקות מדעי המחשב,
מתמטיקה, כימיה ופיזיקה

יום רביעי 10.7.24 | 16:00-19:00
בניין מדעי המחשב (503)
באוניברסיטת בר-אילן

בואו להכיר את מגוון מסלולי הלימוד
לתואר שני ושלישי (דוקטורט)
בפקולטה למדעים מדוייקים

Metric Embeddings intro Trees

Arnold Filtser

Bar-Ilan University

July 02, 2024

Outline of the talk

1. Introduction
2. Stochastic embedding into trees
3. Ramsey type embeddings
4. Clan embedding
5. Group Steiner Tree
6. Conclusion

Metric space

A metric space is an ordered pair (X, d_X), where X is a set and $d_X : X \times X \to \mathbb{R}_{\geq 0}$ is a function such that:

1. **Identity**: $\forall x, y \in X,\ d_X(x, y) = 0 \iff x = y$.
2. **Symmetry**: $\forall x, y \in X,\ d_X(x, y) = d_X(y, x)$.
3. **Triangle-inequality:** $\forall x, y, z \in X,\ d_X(x, y) \leq d_X(x, z) + d_X(z, y)$.

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Examples:

- Euclidean space ℓ_2 in $\mathbb{R}^d$: $d_{\ell_2}(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|_2 = \sqrt{\sum_{i=1}^{d} (x_i - y_i)^2}$.
- Manhattan distance ℓ_1 in $\mathbb{R}^d$: $d_{\ell_1}(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|_1 = \sum_{i=1}^{d} |x_i - y_i|$.
- Edit distance: given two strings A, B how many edit operations (insert, delete, substitute) are required to transform A to B?
- Weighted graph $G = (V, E, w)$ with shortest path distance.

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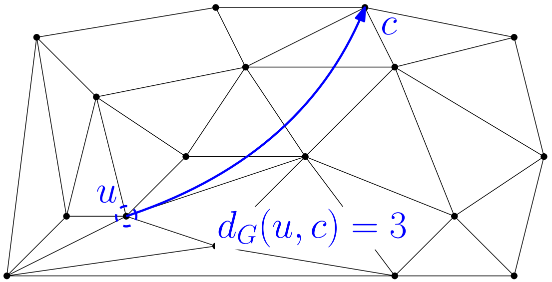

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Many problems are defined w.r.t. metric spaces. Examples:

- k-center.
- Steiner tree.
- Metric TSP.

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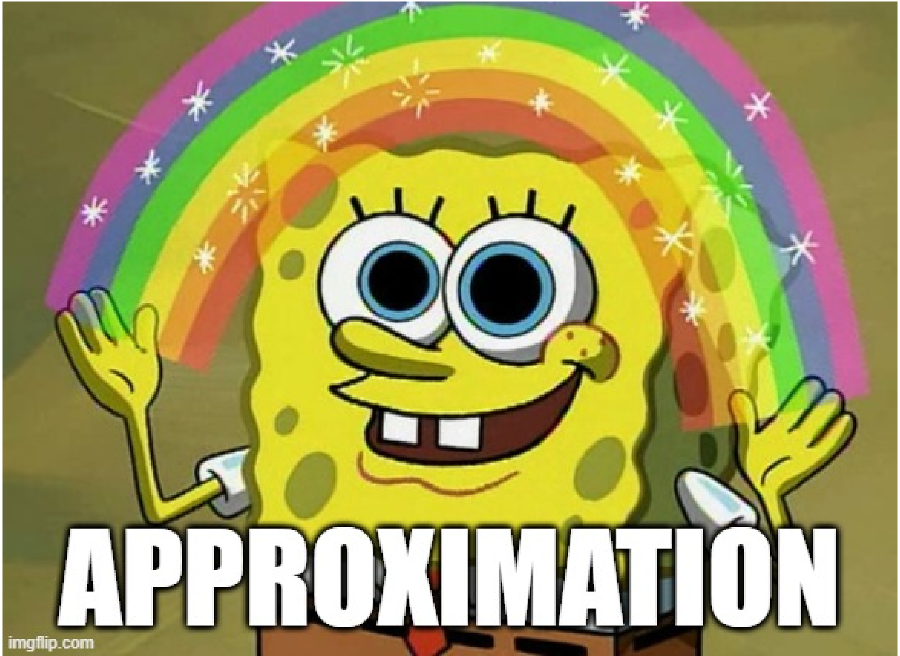
Often these problems are NP-hard.


APPROXIMATION
imgflip.com

Metric Embeddings

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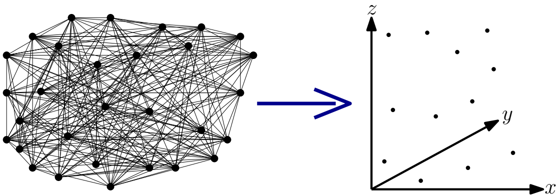

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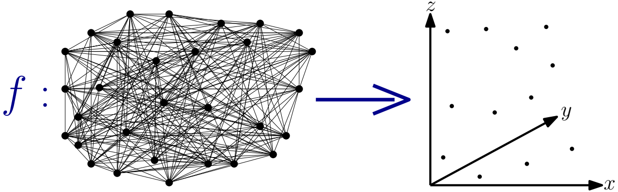

Preserve (approxierly) properties of the original space:

- **Distances**
- Cuts, Flows
- Commute time
- Effective resistance
- Clustering statistics.
- etc.

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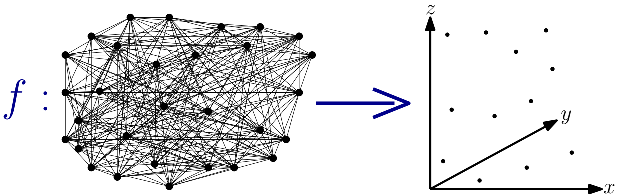

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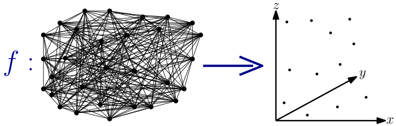

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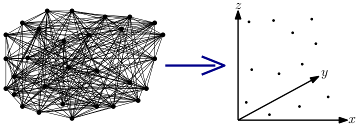

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So that we could run efficient algorithms on it...

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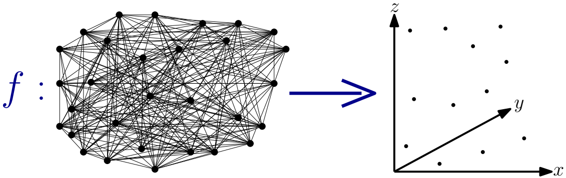

Theorem ([Bourgain 85])

Every n-point metric (X, d_X) is ***embeddable*** *into* <u>*Euclidean*</u> *space* $(\mathbb{R}^d, \|\cdot\|_2)$ *with* ***distortion*** $O(\log n)$.

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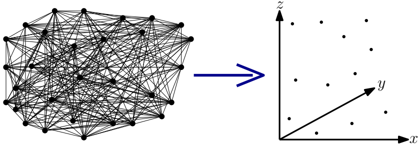
f : 

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Theorem ([Linial, London, Rabinovich 95])

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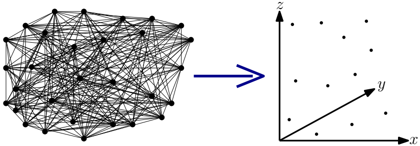
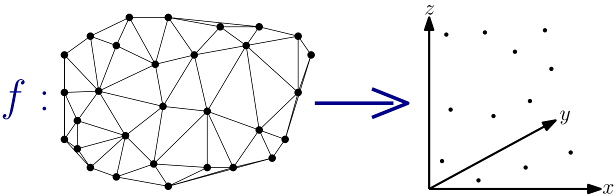
$f:$ 

Applications:

- Approximation algorithms (e.g. **sparsest cut**, min graph bandwidth)
- Parallel computation (e.g. SSSP in MPC)
- Computational Biology (e.g. clustering and detecting protein seq.)
- etc.

Theorem ([Rao 99])

Every n-point ***planar*** *metric* (X, d_X) *is* ***embeddable*** *into* <u>*Euclidean*</u> *space* $(\mathbb{R}^d, \|\cdot\|_2)$ *with* ***distortion*** $O(\sqrt{\log n})$.



Planar metric- the shortest path metric of a planar graph.

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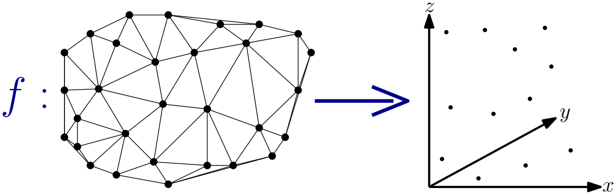

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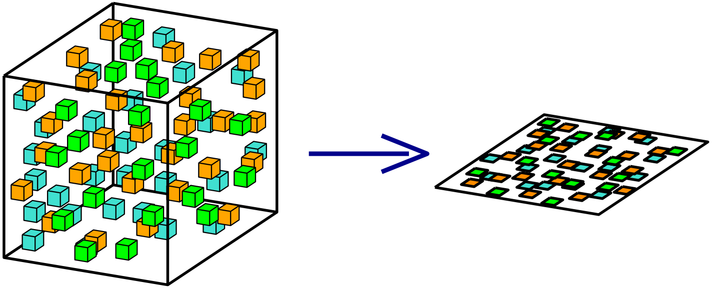
$f :$ 

Theorem ([Johnson, Lindenstrauss 84], Dimension Reduction)

$X \subset (\mathbb{R}^d, \|\cdot\|_2)$ set of size n. Then X embeds into $O(\log n/\epsilon^2)$ dimensional Euclidean space with distortion $1+\epsilon$.

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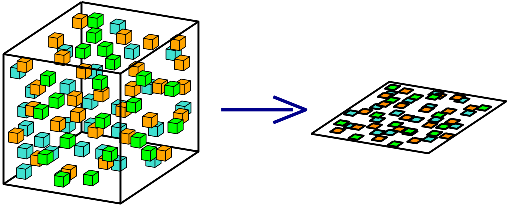
f : 

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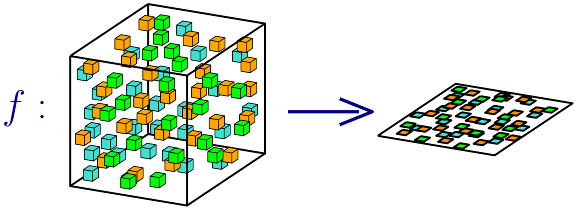


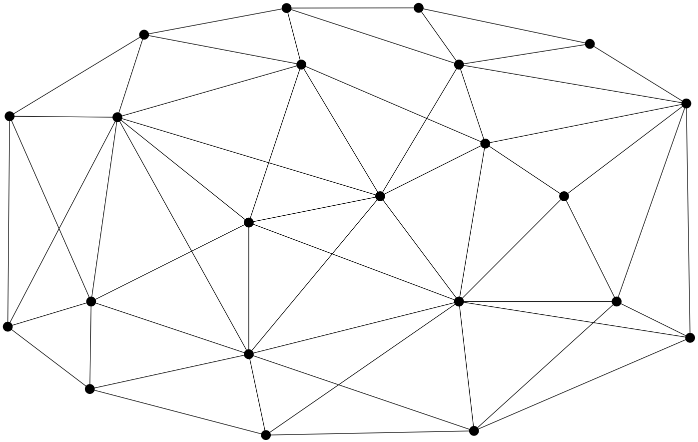
Applications:

- Speeding up-computation
- Clustering
- Nearest Neighbor Search
- Machine Learning
- etc.

Graph Spanners

$G = (V, E, w)$ weighted graph, a t-**spanner** is a **subgraph** $H = (V, E_H)$

s.t. $\forall u, v \in V, \quad d_H(u, v) \leq t \cdot d_G(u, v)$



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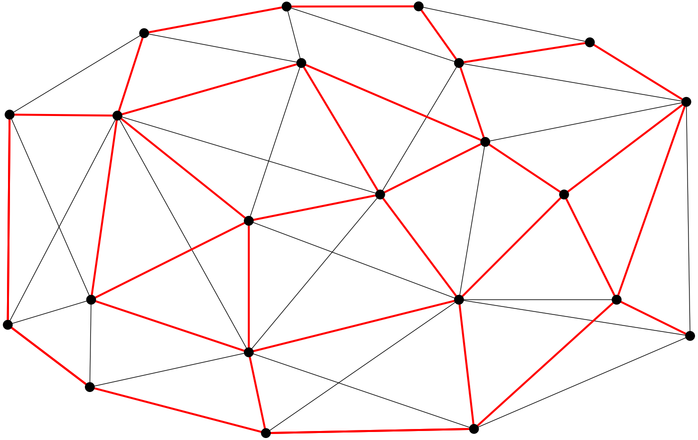

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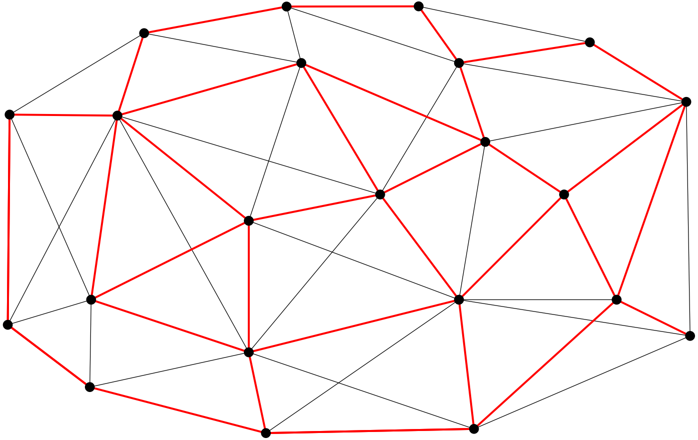

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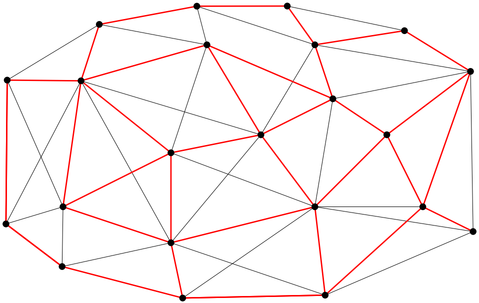

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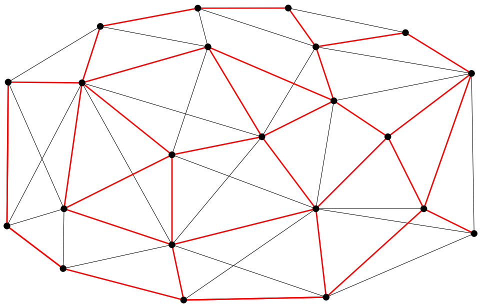

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Applications:

- Approximation Algorithms (e.g. PTAS for TSP)
- Distributed Computing
- Network Routing
- Computational Biology (e.g. measure genetic distance)
- etc.

Outline of the talk

Embedding into Trees

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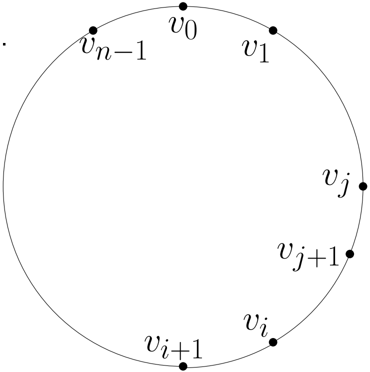

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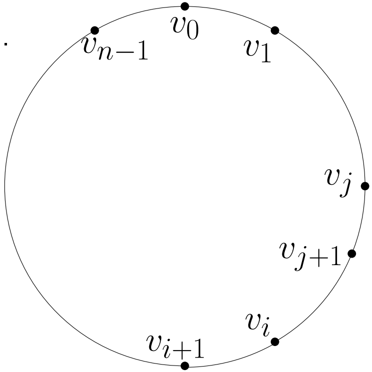

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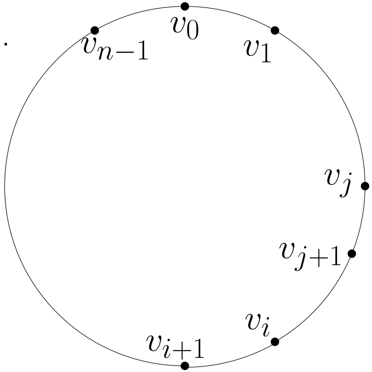

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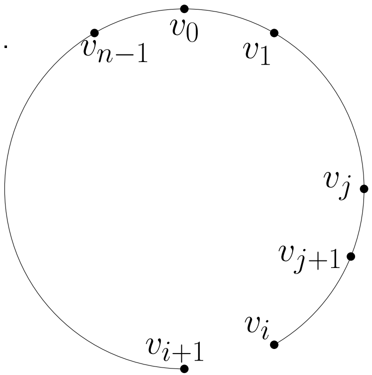

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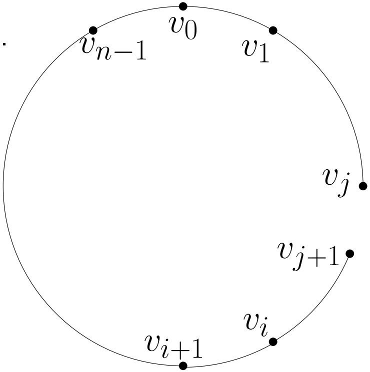

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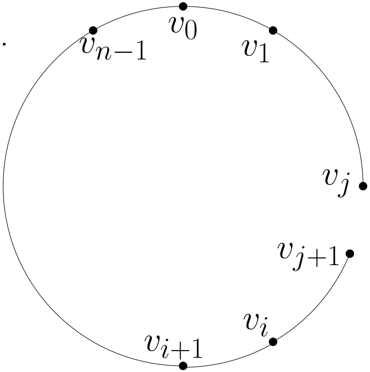

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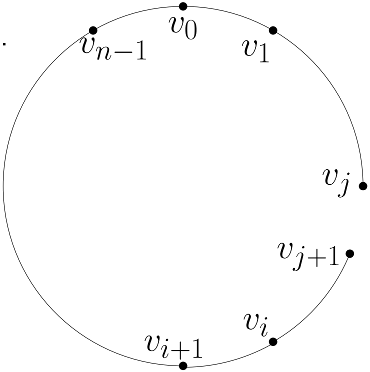

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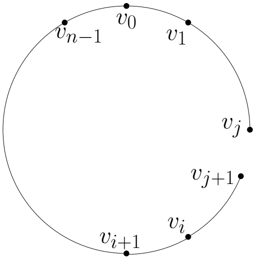

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By triangle inequality and linearity of expectation

$$\forall v_i, v_j, \quad \mathbb{E}_{T\sim\mathcal{D}}[d_T(v_i,v_j)] = \sum_{q=i}^{j-1}\mathbb{E}_{T\sim\mathcal{D}}[d_T(v_q,v_{q+1 \pmod n})] \le 2\cdot d_{C_n}(v_i,v_j)\ .$$

Stochastic Embedding into Trees

Theorem ([Fakcharoenphol, Rao, Talwar 04], improving [Bartal 96+98])

Every n-point metric space (X, d) embeds into ***distribution*** $\mathcal{D}$ *over* ***dominating trees*** *with* ***expected distortion*** $O(\log n)$.

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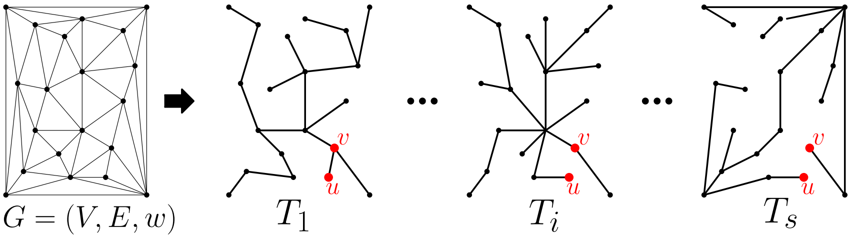

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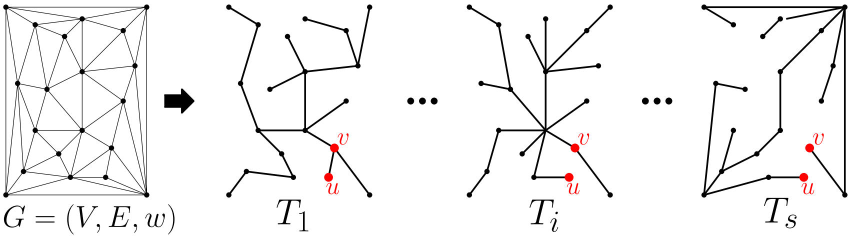
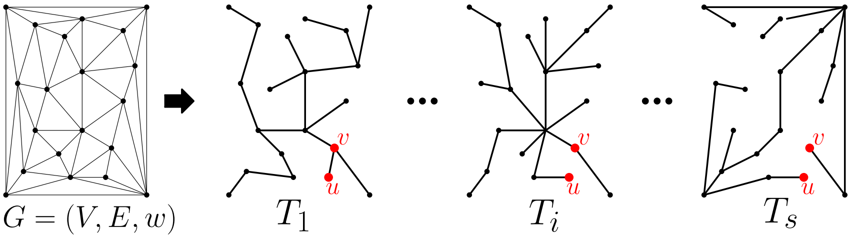


For every $u, v \in X$ and $T \in \mathrm{supp}(\mathcal{D})$, $d_X(u, v) \leq d_T(f(u), f(v))$.

Stochastic Embedding into Trees

Theorem ([Fakcharoenphol, Rao, Talwar 04], improving [Bartal 96+98])

*Every n-point metric space (X, d) embeds into **distribution** $\mathcal{D}$ over **dominating trees** with **expected distortion** $O(\log n)$.*



For every $u, v \in X$ and $T \in \mathrm{supp}(\mathcal{D})$, $d_X(u, v) \leq d_T(f(u), f(v))$.

For every $u, v \in X$ $\quad \mathbb{E}_{T\sim\mathcal{D}}[d_T(f(u), f(v))] \leq O(\log n) \cdot d_X(u, v)$.

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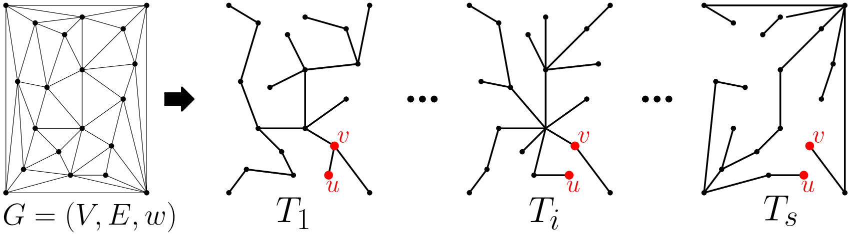

For every $u, v \in X$ and $T \in \mathrm{supp}(\mathcal{D})$, $d_X(u, v) \leq d_T(f(u), f(v))$.

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[Bartal 96]: Tight!

Stochastic Embedding into Trees

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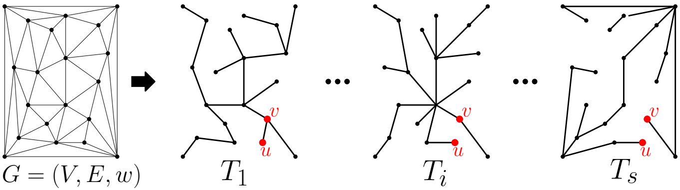
*Every n-point metric space (X, d) embeds into **distribution** $\mathcal{D}$ over **dominating trees** with **expected distortion** $O(\log n)$.*



A useful hammer

Transforms arbitrary metric into a tree!

Stochastic Embedding into Trees

Theorem ([Fakcharoenphol, Rao, Talwar 04], improving [Bartal 96+98])

Every n-point metric space (X, d) embeds into ***distribution*** $\mathcal{D}$ *over* ***dominating trees*** *with* ***expected distortion*** $O(\log n)$.

A useful hammer

Transforms arbitrary metric into a tree!

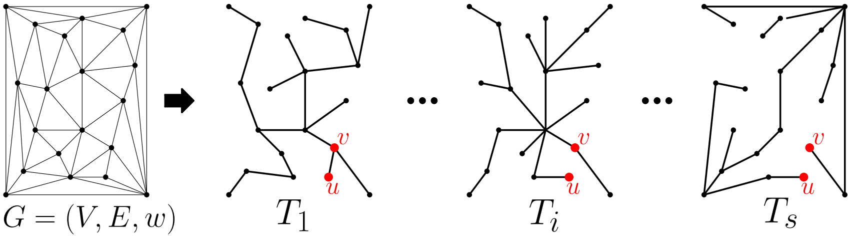
Applications:

- Approximation Algorithms.
- **Online Algorithms**.
- Distributed Computing.
- etc.

Stochastic Embedding into Trees

Theorem ([Fakcharoenphol, Rao, Talwar 04], improving [Bartal 96+98])

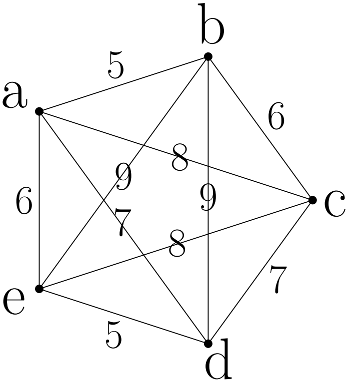
Every n-point metric space (X, d) embeds into ***distribution*** $\mathcal{D}$ *over* ***dominating trees*** *with* ***expected distortion*** $O(\log n)$.



Compromises: Gurantee only in **expectation**

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.



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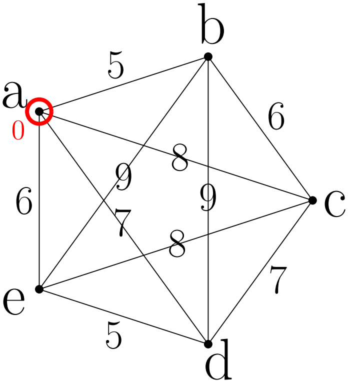
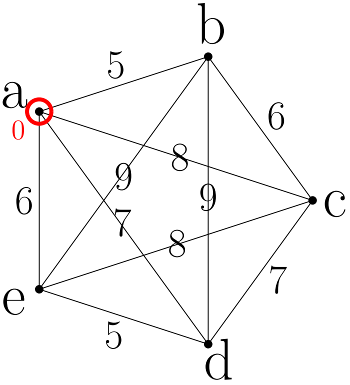


Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.



$$f_1(a) = 3$$
$$f_1(b) = 1$$
$$f_1(c) = 2$$
$$f_1(d) = 4$$
$$f_1(e) = 1$$

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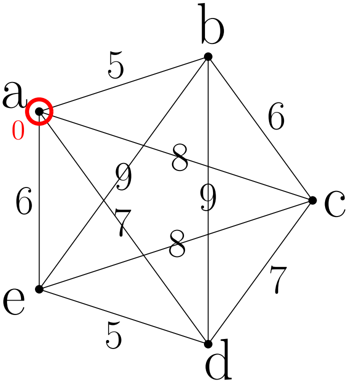

$f_1(a) = 3$
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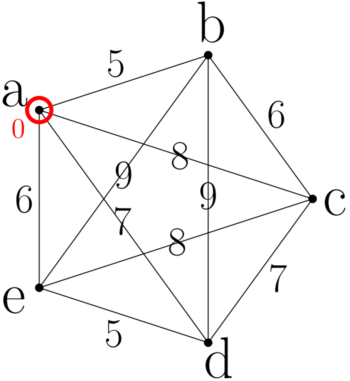
Goal: Minimize the competitive ratio between our algorithm to opt.



$f_1(a) = 3$

$f_1(b) = 1$

$f_1(c) = 2$

$f_1(d) = 4$

$f_1(e) = 1$

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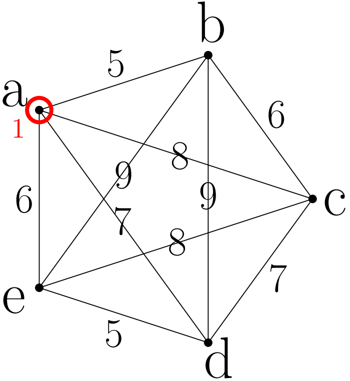

$$f_1(a) = 3$$
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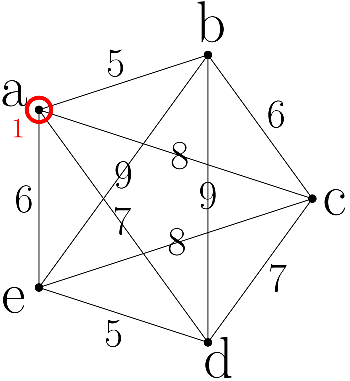

$$f_1(a) = 3$$
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Cost alg(1)=3

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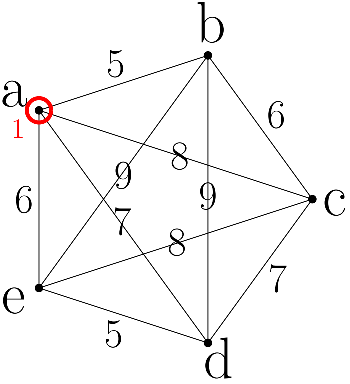

$$f_2(a) = 3$$
$$f_2(b) = 1$$
$$f_2(c) = 3$$
$$f_2(d) = 2$$
$$f_2(e) = 1$$

Cost alg(1)=3

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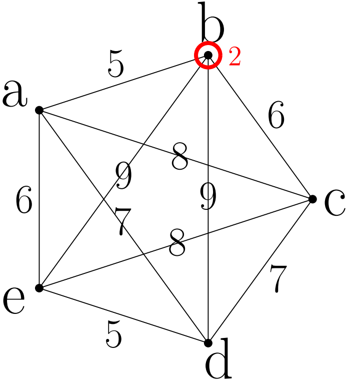

$$f_2(a) = 3$$
$$f_2(b) = 1$$
$$f_2(c) = 3$$
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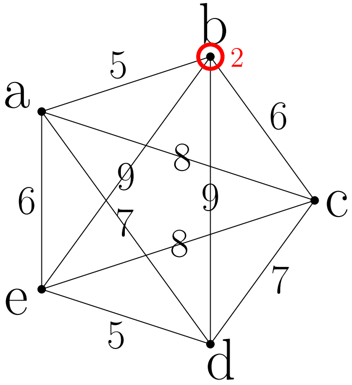

$$f_2(a) = 3$$
$$f_2(b) = 1$$
$$f_2(c) = 3$$
$$f_2(d) = 2$$
$$f_2(e) = 1$$

Cost alg(2)=$3 + 5 + 1 = 9$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

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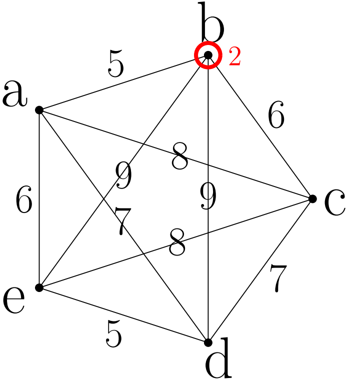

$$f_3(a) = 1$$
$$f_3(b) = 4$$
$$f_3(c) = 3$$
$$f_3(d) = 2$$
$$f_3(e) = 1$$

Cost alg(2)=$3 + 5 + 1 = 9$

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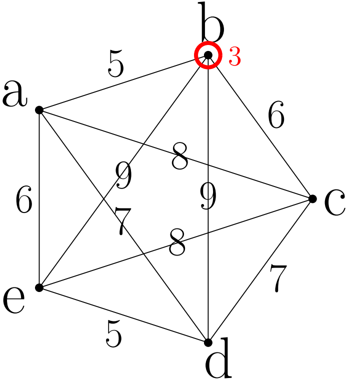

$f_3(a) = 1$

$f_3(b) = 4$

$f_3(c) = 3$

$f_3(d) = 2$

$f_3(e) = 1$

Cost alg(2)$=3 + 5 + 1 = 9$

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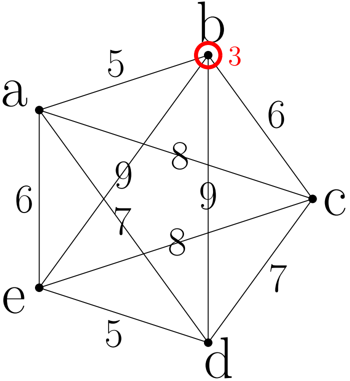

$$f_3(a) = 1$$
$$f_3(b) = 4$$
$$f_3(c) = 3$$
$$f_3(d) = 2$$
$$f_3(e) = 1$$

Cost alg(3)$=9 + 4 = 13$

Online problem - Metrical Task System (MTS)

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At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

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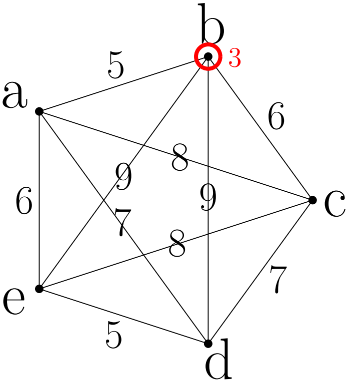

$f_4(a) = 5$
$f_4(b) = 10$
$f_4(c) = 4$
$f_4(d) = 0$
$f_4(e) = 1$

Cost alg(3)$=9 + 4 = 13$

Online problem - Metrical Task System (MTS)

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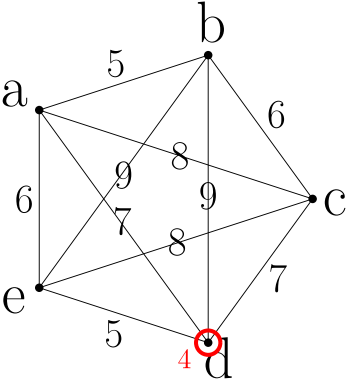

$f_4(a) = 5$
$f_4(b) = 10$
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Cost alg(3)=$9 + 4 = 13$

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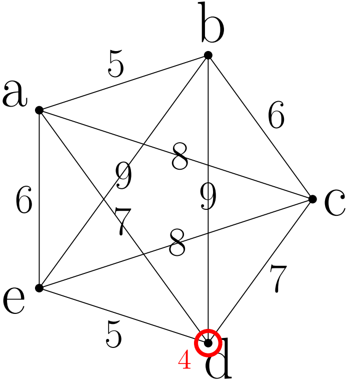


$f_4(a) = 5$
$f_4(b) = 10$
$f_4(c) = 4$
$f_4(d) = 0$
$f_4(e) = 1$

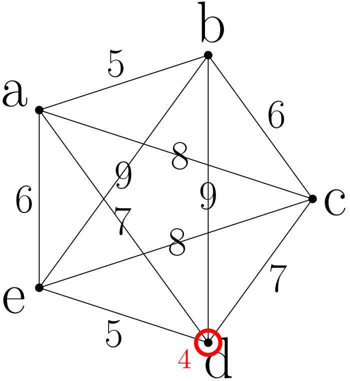
Cost alg(4)$=13 + 9 + 0 = 22$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$$f_5(a) = 3$$
$$f_5(b) = 4$$
$$f_5(c) = 4$$
$$f_5(d) = 5$$
$$f_5(e) = 1$$

Cost alg(4)=$13 + 9 + 0 = 22$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
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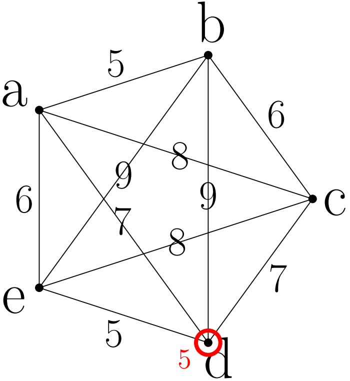

$f_5(a) = 3$
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$f_5(d) = 5$
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Cost alg(4)$=13 + 9 + 0 = 22$

Online problem - Metrical Task System (MTS)

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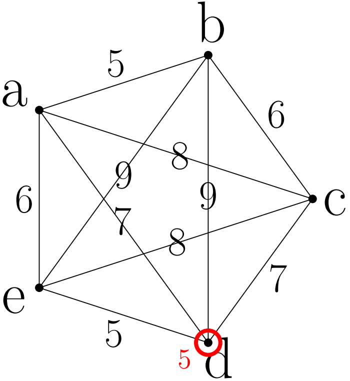

$f_5(a) = 3$
$f_5(b) = 4$
$f_5(c) = 4$
$f_5(d) = 5$
$f_5(e) = 1$

Cost alg(5)=22 + 5 = 27

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

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What about Opt?

Online problem - Metrical Task System (MTS)

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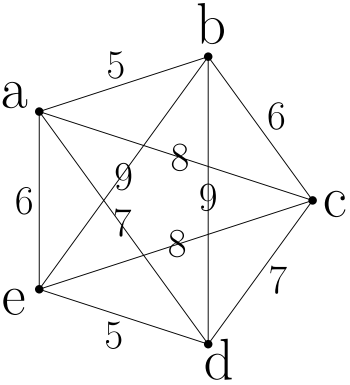
Goal: Minimize the competitive ratio between our algorithm to opt.



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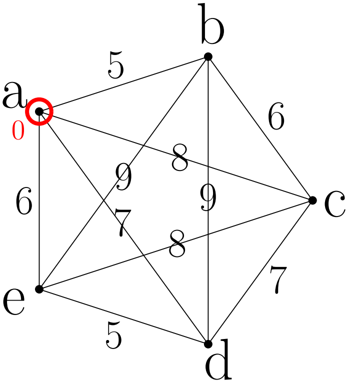

Online problem - Metrical Task System (MTS)

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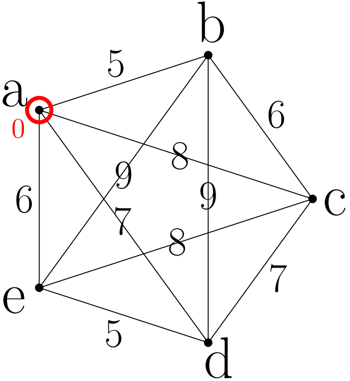

$$f_1(a) = 3$$
$$f_1(b) = 1$$
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$$f_1(d) = 4$$
$$f_1(e) = 1$$

Online problem - Metrical Task System (MTS)

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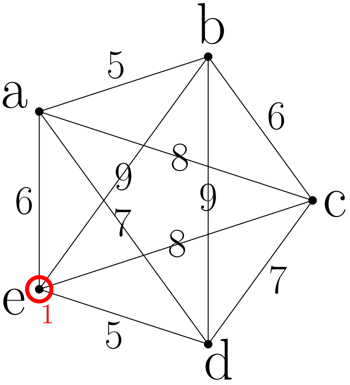

$f_1(a) = 3$

$f_1(b) = 1$

$f_1(c) = 2$

$f_1(d) = 4$

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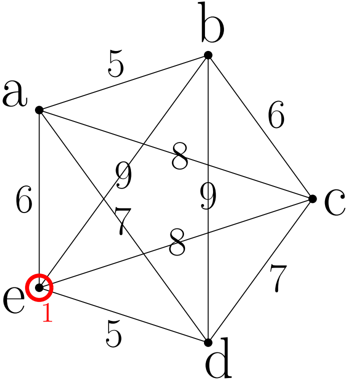
Goal: Minimize the competitive ratio between our algorithm to opt.



$$f_1(a) = 3$$
$$f_1(b) = 1$$
$$f_1(c) = 2$$
$$f_1(d) = 4$$
$$f_1(e) = 1$$

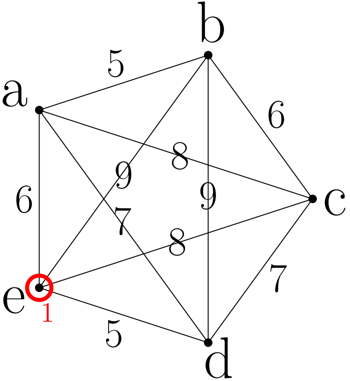
Cost opt(1)=$6 + 1 = 7$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$$f_2(a) = 3$$
$$f_2(b) = 1$$
$$f_2(c) = 3$$
$$f_2(d) = 2$$
$$f_2(e) = 1$$

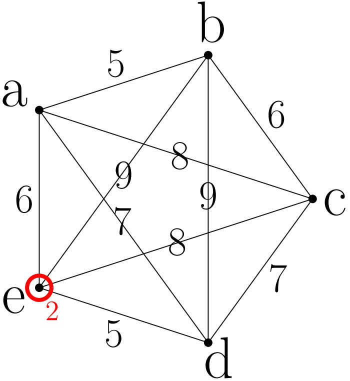
Cost opt(1)$=6+1=7$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$$f_2(a) = 3$$
$$f_2(b) = 1$$
$$f_2(c) = 3$$
$$f_2(d) = 2$$
$$f_2(e) = 1$$

Cost opt(1)=$6 + 1 = 7$

Online problem - Metrical Task System (MTS)

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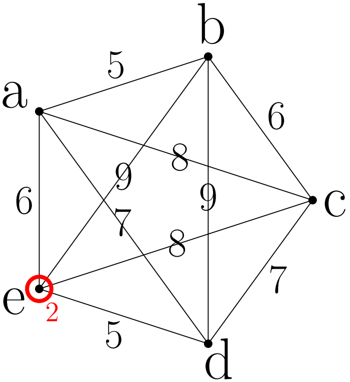
Goal: Minimize the competitive ratio between our algorithm to opt.



$f_2(a) = 3$
$f_2(b) = 1$
$f_2(c) = 3$
$f_2(d) = 2$
$f_2(e) = 1$

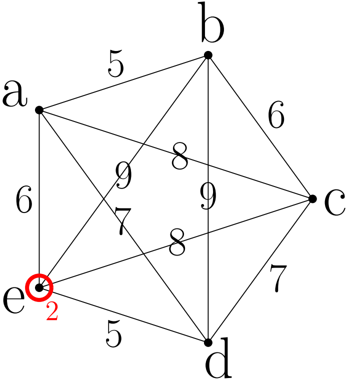
Cost opt(2)$=7 + 1 = 8$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$$f_3(a) = 1$$
$$f_3(b) = 4$$
$$f_3(c) = 3$$
$$f_3(d) = 2$$
$$f_3(e) = 1$$

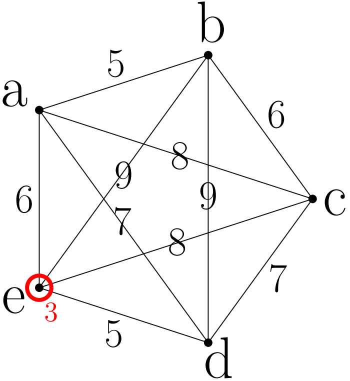
Cost opt(2)$=7+1=8$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$f_3(a) = 1$
$f_3(b) = 4$
$f_3(c) = 3$
$f_3(d) = 2$
$f_3(e) = 1$

Cost opt(2)$=7 + 1 = 8$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

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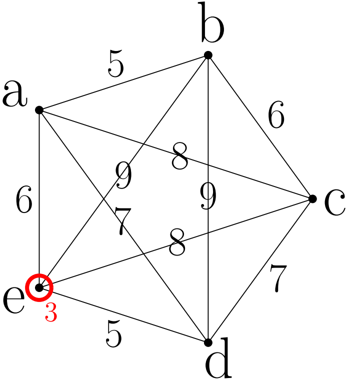
Goal: Minimize the competitive ratio between our algorithm to opt.



$f_3(a) = 1$
$f_3(b) = 4$
$f_3(c) = 3$
$f_3(d) = 2$
$f_3(e) = 1$

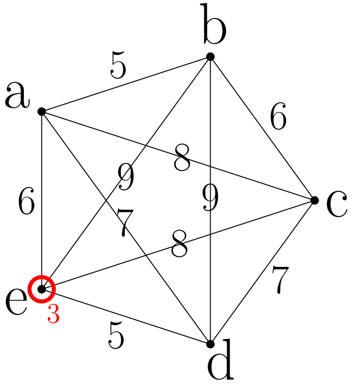
Cost opt(3)$=8 + 1 = 9$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$f_4(a) = 5$

$f_4(b) = 10$

$f_4(c) = 4$

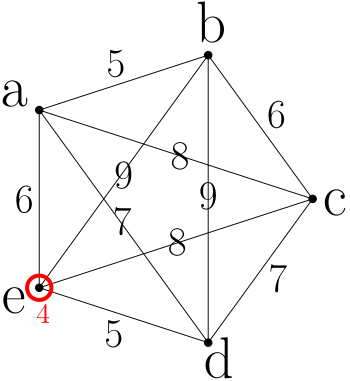
$f_4(d) = 0$

$f_4(e) = 1$

Cost opt(3)$=8 + 1 = 9$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.
Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.
Goal: Minimize the competitive ratio between our algorithm to opt.



$f_4(a) = 5$
$f_4(b) = 10$
$f_4(c) = 4$
$f_4(d) = 0$
$f_4(e) = 1$

Cost opt(3)=$8 + 1 = 9$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

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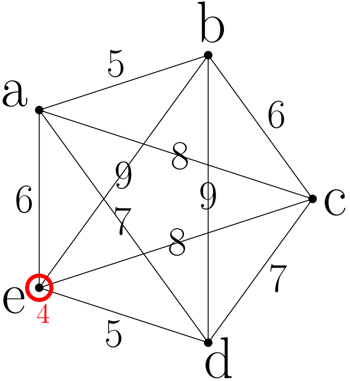


$f_4(a) = 5$
$f_4(b) = 10$
$f_4(c) = 4$
$f_4(d) = 0$
$f_4(e) = 1$

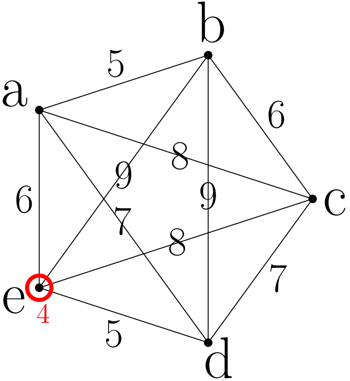
Cost opt(4)$=9 + 1 = 10$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$f_5(a) = 3$
$f_5(b) = 4$
$f_5(c) = 4$
$f_5(d) = 5$
$f_5(e) = 1$

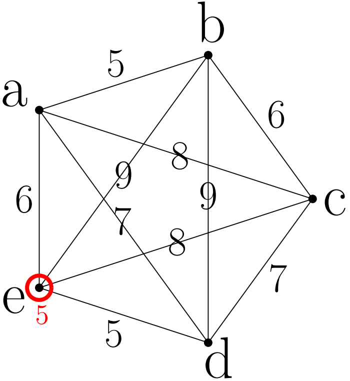
Cost opt(4)$=9 + 1 = 10$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$f_5(a) = 3$
$f_5(b) = 4$
$f_5(c) = 4$
$f_5(d) = 5$
$f_5(e) = 1$

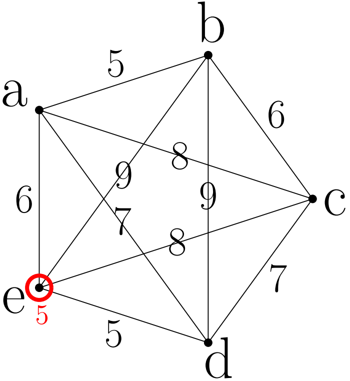
Cost opt(4)=$9 + 1 = 10$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$$f_5(a) = 3$$
$$f_5(b) = 4$$
$$f_5(c) = 4$$
$$f_5(d) = 5$$
$$f_5(e) = 1$$

Cost opt(5)=$10 + 1 = 11$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.

$$\text{Competitive ratio} = \max_{\text{input } I} \frac{\mathrm{Alg}(I)}{\mathrm{opt}(I)}.$$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
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Goal: Minimize the competitive ratio between our algorithm to opt.

$$\text{Competitive ratio} = \max_{\text{input } I} \frac{\mathrm{Alg}(I)}{\mathrm{opt}(I)}.$$

Here $\frac{\mathrm{Alg}(I)}{\mathrm{opt}(I)} = \frac{27}{11}$.

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.

$$\text{Competitive ratio} = \max_{\text{input } I} \frac{\mathrm{Alg}(I)}{\mathrm{opt}(I)}.$$

Here $\frac{\mathrm{Alg}(I)}{\mathrm{opt}(I)} = \frac{27}{11}$.

Competitive ratio against oblivious adversary is

$$\max_{\text{input } I} \frac{\mathbb{E}[\mathrm{Alg}(I)]}{\mathrm{opt}(I)}.$$

Online problem - Metrical Task System (MTS)

Approach: embed into a tree, and then make all the decisions based on the tree.

Online problem - Metrical Task System (MTS)

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Theorem ([Fiat, Mendel 2000])

Given an n point tree T, there is an online algorithm for MTS with competitive ratio $O(\log n \cdot \log \log n)$ against oblivious adversary.*

* Actually on an HST, which is a special kind of tree. [FRT04] is into HST's.

Online problem - Metrical Task System (MTS)

Approach: embed into a tree, and then make all the decisions based on the tree.

Theorem ([Fiat, Mendel 2000])

Given an n point tree* T, there is an online algorithm for MTS with competitive ratio $O(\log n \cdot \log\log n)$ against oblivious adversary.

Algorithm: 1. Sample a tree T over (X, d_X) using [FRT04].

Online problem - Metrical Task System (MTS)

Approach: embed into a tree, and then make all the decisions based on the tree.

Theorem ([Fiat, Mendel 2000])

Given an n point tree* T, there is an online algorithm for MTS with competitive ratio $O(\log n \cdot \log\log n)$ against oblivious adversary.

Algorithm: 1. Sample a tree T over (X, d_X) using [FRT04].

2. Run [FM00] on T with the same cost functions.

Make the same decisions as [FM00].

Online problem - Metrical Task System (MTS)

Algorithm: 1. Sample a tree T over (X, d_X) using [FRT04].
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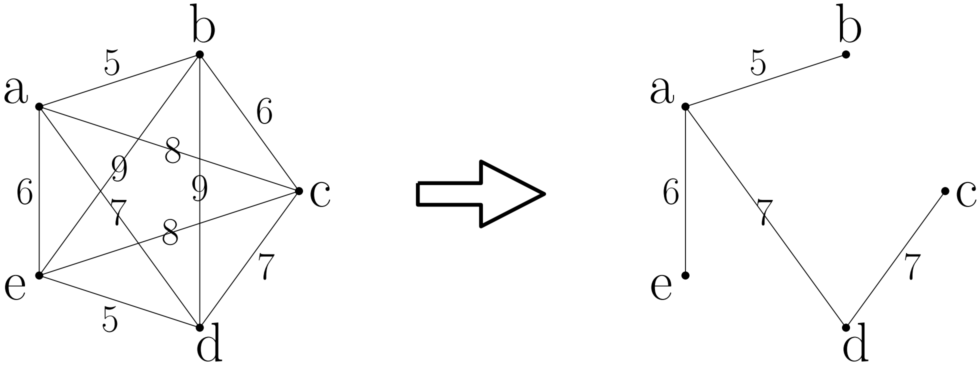

Online problem - Metrical Task System (MTS)

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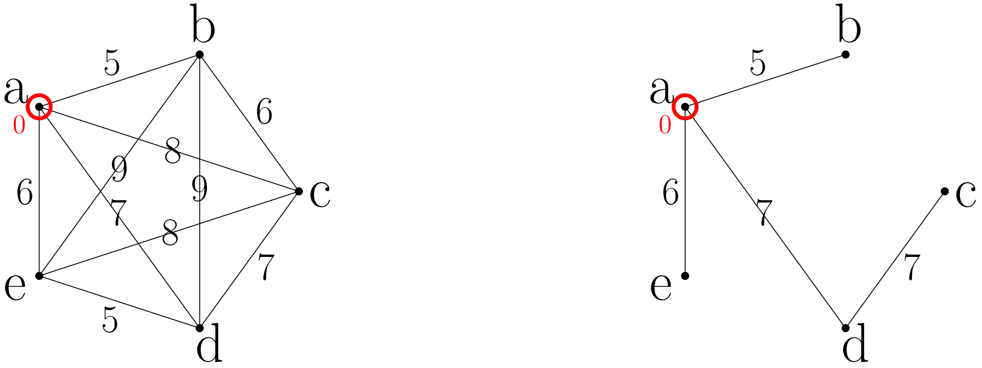

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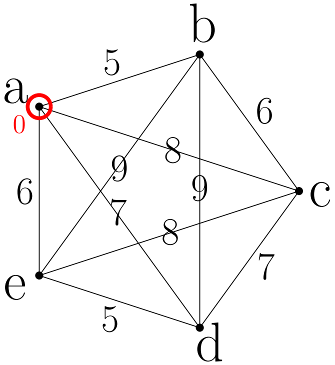
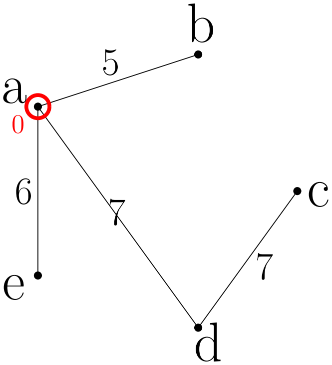


$$f_1(a) = 3$$
$$f_1(b) = 1$$
$$f_1(c) = 2$$
$$f_1(d) = 4$$
$$f_1(e) = 1$$



Online problem - Metrical Task System (MTS)

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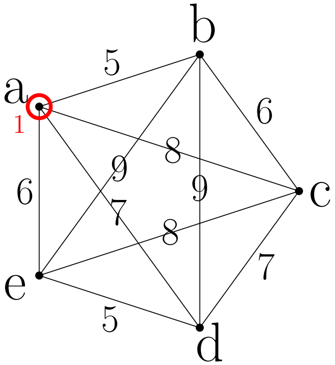
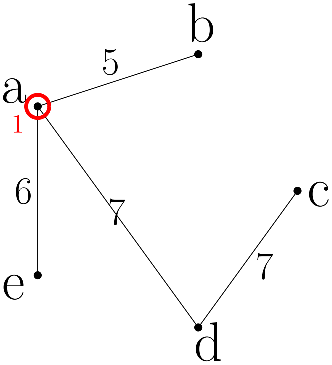


$f_1(a) = 3$
$f_1(b) = 1$
$f_1(c) = 2$
$f_1(d) = 4$
$f_1(e) = 1$



Online problem - Metrical Task System (MTS)

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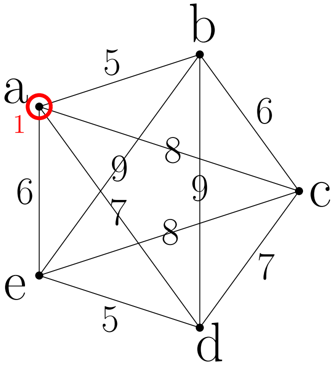
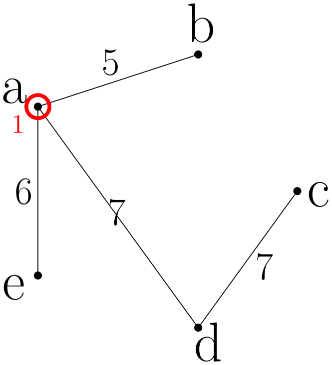


Cost alg(1)=3

$f_1(a) = 3$
$f_1(b) = 1$
$f_1(c) = 2$
$f_1(d) = 4$
$f_1(e) = 1$



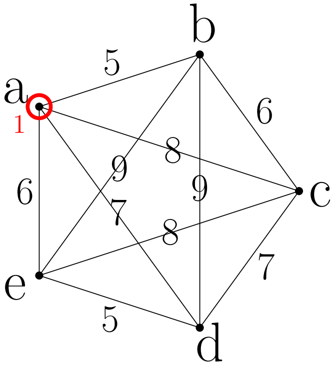
Cost $\text{alg}_T(1)$=3

Online problem - Metrical Task System (MTS)

Algorithm: 1. Sample a tree T over (X, d_X) using [FRT04].

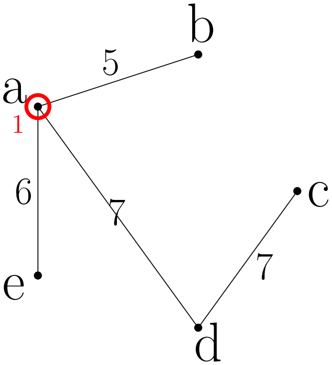
2. Run [FM00] on T with the same cost functions.

Make the same decisions as [FM00].



Cost alg(1)=3

$$f_2(a) = 3$$
$$f_2(b) = 1$$
$$f_2(c) = 3$$
$$f_2(d) = 2$$
$$f_2(e) = 1$$



Cost $\mathrm{alg}_T(1)$=3

Online problem - Metrical Task System (MTS)

Algorithm: 1. Sample a tree T over (X, d_X) using [FRT04].
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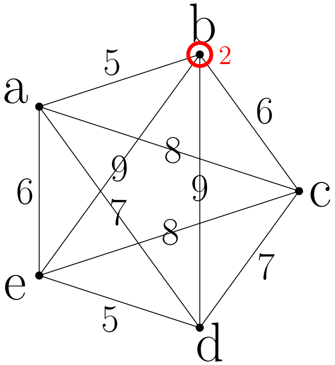
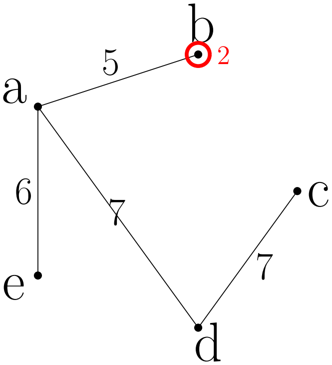
Make the same decisions as [FM00].



Cost alg(1)=3

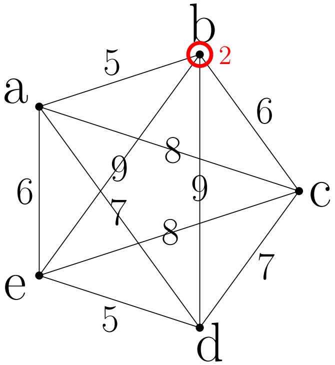
$f_2(a) = 3$
$f_2(b) = 1$
$f_2(c) = 3$
$f_2(d) = 2$
$f_2(e) = 1$



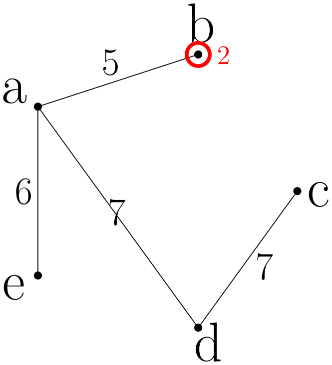
Cost $\text{alg}_T(1)$=3

Online problem - Metrical Task System (MTS)

Algorithm: 1. Sample a tree T over (X, d_X) using [FRT04].
2. Run [FM00] on T with the same cost functions.
Make the same decisions as [FM00].



$f_2(a) = 3$
$f_2(b) = 1$
$f_2(c) = 3$
$f_2(d) = 2$
$f_2(e) = 1$



Cost alg(2)$=3 + 5 + 1 = 9$

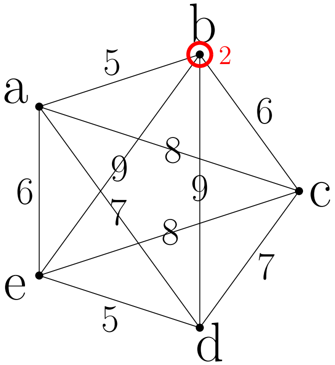
Cost $\text{alg}_T(2)$$=3 + 5 + 1 = 9$

Online problem - Metrical Task System (MTS)

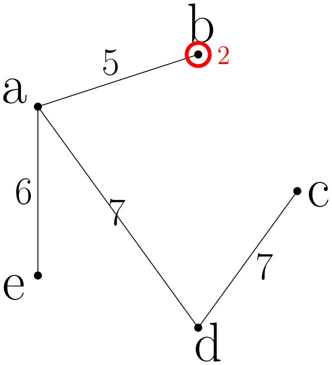
Algorithm: 1. Sample a tree T over (X, d_X) using [FRT04].

2. Run [FM00] on T with the same cost functions.

Make the same decisions as [FM00].



$$f_3(a) = 1$$
$$f_3(b) = 4$$
$$f_3(c) = 3$$
$$f_3(d) = 2$$
$$f_3(e) = 1$$



Cost alg(2)$=3+5+1=9$

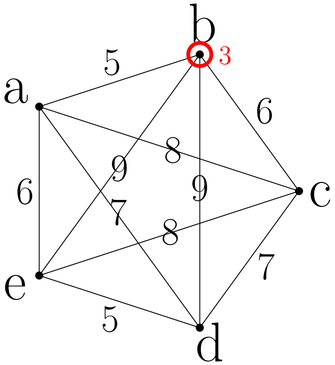
Cost alg$_T$(2)$=3+5+1=9$

Online problem - Metrical Task System (MTS)

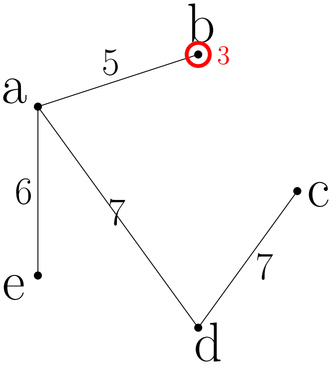
Algorithm: 1. Sample a tree T over (X, d_X) using [FRT04].

2. Run [FM00] on T with the same cost functions.

Make the same decisions as [FM00].



$$f_3(a) = 1$$
$$f_3(b) = 4$$
$$f_3(c) = 3$$
$$f_3(d) = 2$$
$$f_3(e) = 1$$

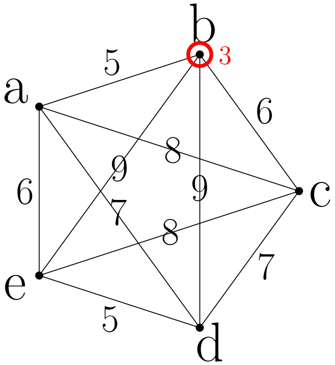


Cost alg(2)=$3+5+1=9$

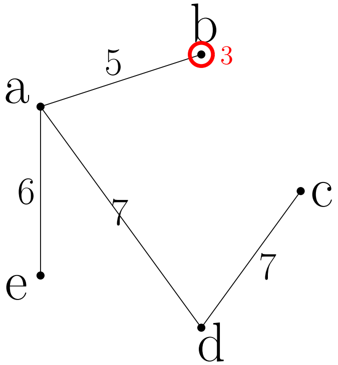
Cost $\mathrm{alg}_T(2)$=$3+5+1=9$

Online problem - Metrical Task System (MTS)

Algorithm: 1. Sample a tree T over (X, d_X) using [FRT04].
2. Run [FM00] on T with the same cost functions.
Make the same decisions as [FM00].



$f_3(a) = 1$
$f_3(b) = 4$
$f_3(c) = 3$
$f_3(d) = 2$
$f_3(e) = 1$



Cost alg(3)=$9 + 4 = 13$

Cost $\mathrm{alg}_T(3)$=$9 + 4 = 13$

Online problem - Metrical Task System (MTS)

Algorithm: 1. Sample a tree T over (X, d_X) using [FRT04].

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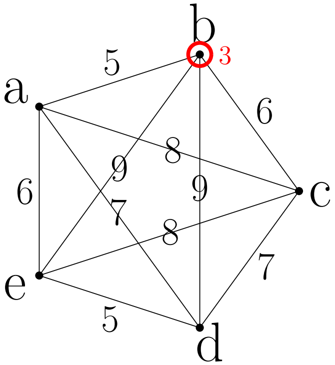

Cost alg(3)$=9+4=13$

$$f_4(a) = 5$$
$$f_4(b) = 10$$
$$f_4(c) = 4$$
$$f_4(d) = 0$$
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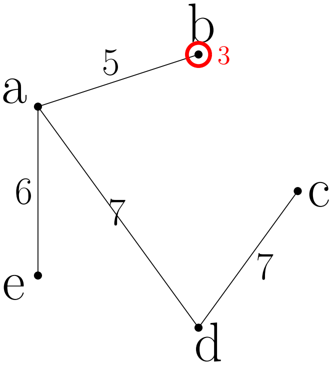

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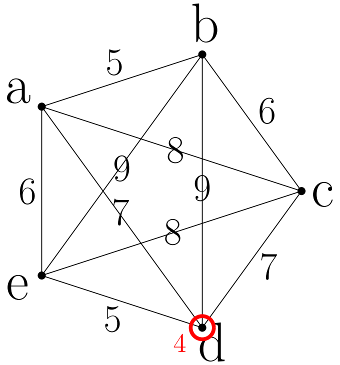

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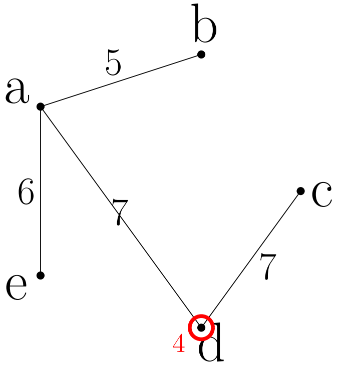

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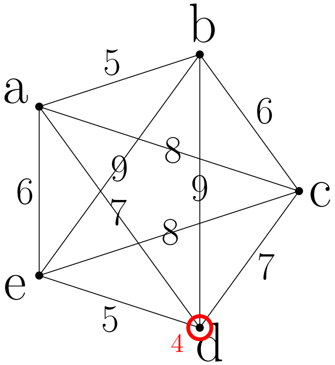

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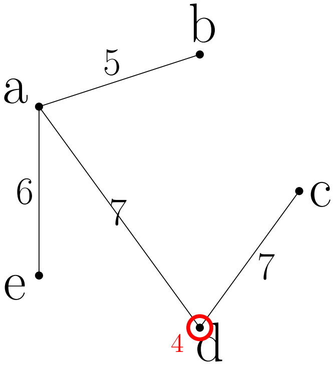

Cost alg(4)=$13 + 9 + 0 = 22$

Cost $\text{alg}_T(4)$=$13 + 12 + 0 = 25$

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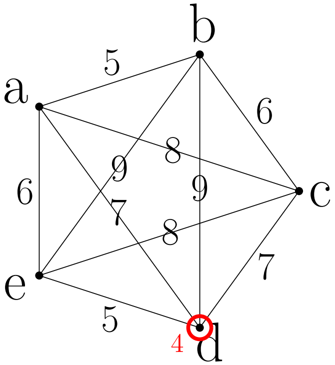

$$f_5(a) = 3$$
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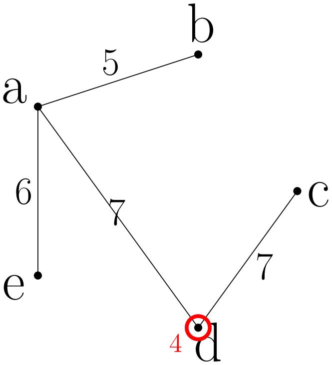

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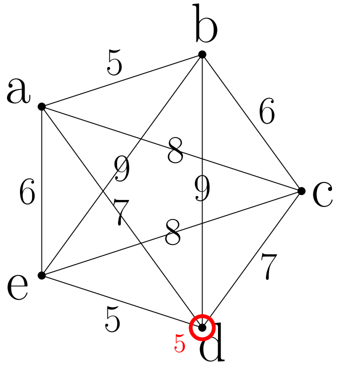

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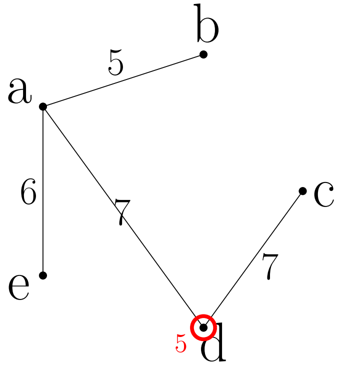

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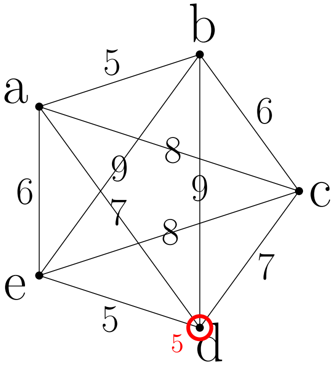

Cost alg(5)=$22 + 5 = 27$

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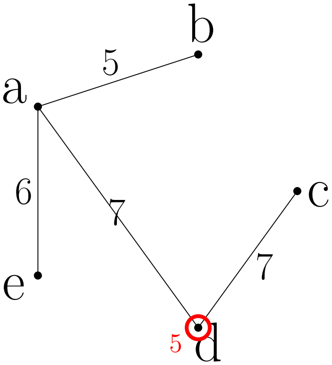

Cost $\text{alg}_T(5)$=$25 + 5 = 30$

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Algorithm: 1. Sample a tree T over (X, d_X) using [FRT04].
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Make the same decisions as [FM00].

Analysis. Let $x_1, x_2, \ldots, x_k$ be the decisions of opt. Thus $\mathrm{opt} = \sum_{i=1}^{k} f_i(x_i) + \sum_{i=1}^{k} d_X(x_{i-1}, x_i)$.

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We've sampled a tree T, $x_1, x_2, \ldots, x_k$ is also a valid decisions for T. Hence
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[FM00] is $O(\log n \cdot \log\log n)$-competitive on T. Hence it choose points $y_1, \ldots, y_k$ such that $\mathbb{E}[\mathrm{alg}_T] = \mathbb{E}[\sum_{i=1}^{k} f_i(y_i) + \sum_{i=1}^{k} d_T(y_{i-1}, y_i)] \leq O(\log n \cdot \log\log n) \cdot \mathrm{opt}_T$.

We made the same decisions, so overall:

$$\mathbb{E}\left[\mathrm{alg}\right] = \mathbb{E}\left[\sum_{i=1}^{k} f_i(y_i) + \sum_{i=1}^{k} d_X(y_{i-1}, y_i)\right]$$

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\mathbb{E}\left[\text{alg}\right] &= \mathbb{E}\left[\sum_{i=1}^{k} f_i(y_i) + \sum_{i=1}^{k} d_X(y_{i-1}, y_i)\right] \\
&\leq \mathbb{E}\left[\sum_{i=1}^{k} f_i(y_i) + \sum_{i=1}^{k} d_T(y_{i-1}, y_i)\right]
\end{aligned}$$

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\end{aligned}$$

Theorem

MTS has an $O(\log^2 n \cdot \log\log n)$ competitive algorithm against oblivious adversary.

Outline of the talk

Ramsey type Embeddings

Ramsey type theorem: Every **big** enough object, contains a **structured subset**.

Ramsey type Embeddings

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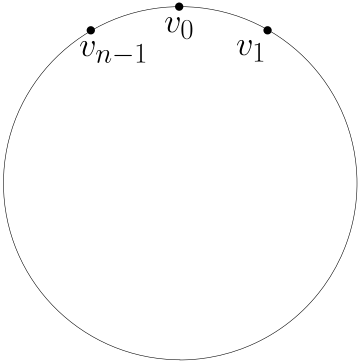

Ramsey type Embeddings

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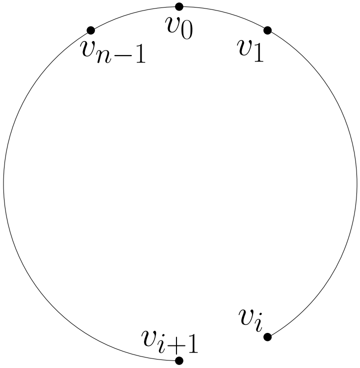

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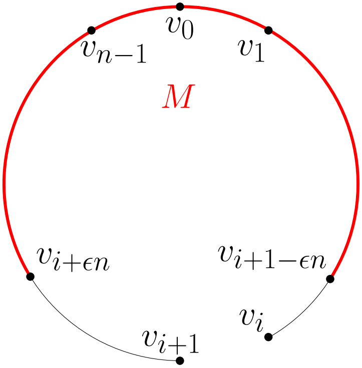

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Set $M = \{v_{i+1-\epsilon n}, v_{i+2-\epsilon n}, \ldots, v_{i+\epsilon n}\}$.

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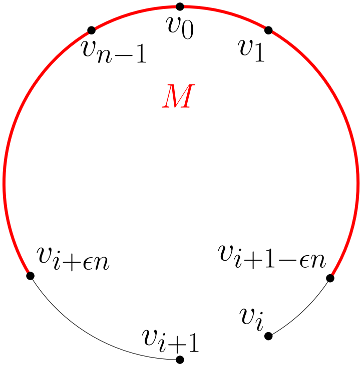

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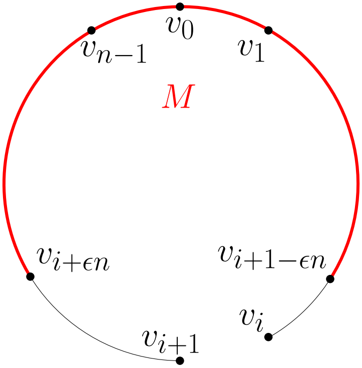

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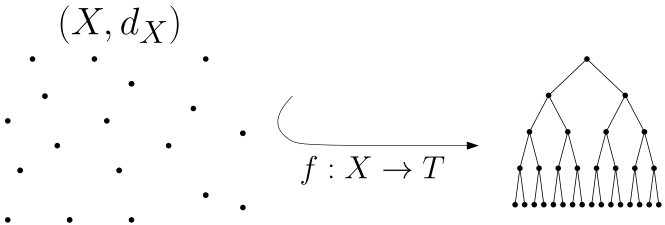
Choose i u.a.r., then $\Pr[v \in M] = 1 - 2\epsilon$.

Ramsey type Embeddings

Fix $k > 1$, what is the **largest subset** $M \subset X$,
s.t. (M, d_X) embeds into a tree with **distortion** k?



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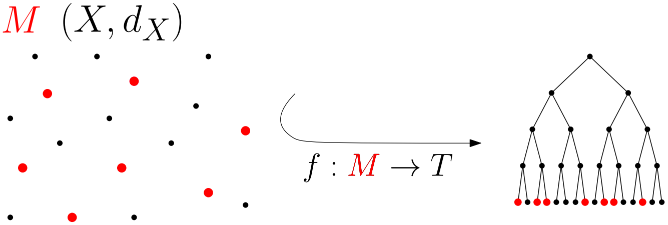

$$\forall x, y \in M, \qquad d_X(x,y) \leq d_T(f(x), f(y)) \leq \boldsymbol{k} \cdot d_X(x,y)$$

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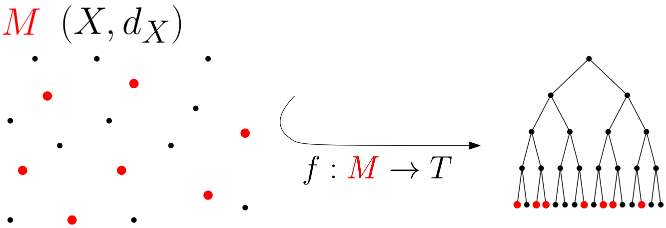

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Theorem ([Mendel, Naor 07], following [BFM86, BLMN05])

*$\forall$ n-point metric space and $\boldsymbol{k} \geq 1$, $\exists$ **subset** M of size $n^{1-1/k}$ that **embeds** into a **tree** with distortion $O(\boldsymbol{k})$.*

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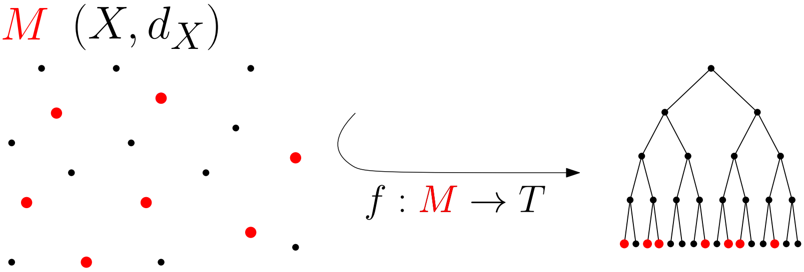

Asymptotically tight.

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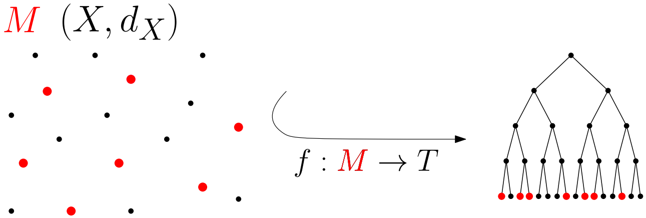


Asymptotically tight.

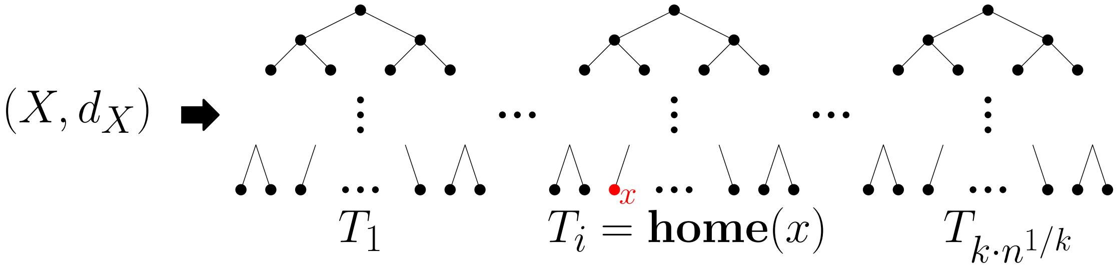
[Naor, Tao 12]: distortion $2e \cdot k$.

Ramsey type Embeddings

Corollary

For every n-point metric space and $\boldsymbol{k} \geq 1$*, there is a* ***set*** $\mathcal{T}$ *of* $k \cdot n^{\frac{1}{k}}$ ***trees*** *and a* *mapping* **home** : $X \to \mathcal{T}$*, such that for every* $x, y \in X$*,*

$$d_{\mathbf{home}(x)}(x, y) \leq O(\boldsymbol{k}) \cdot d_X(x, y)$$



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Applications:

- **Distance oracle**
- Compact routing scheme
- Online algorithms
- Approximate ranking
- etc.

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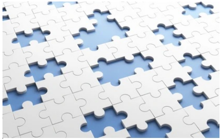
Compromises: only partial guarantees



Distance Oracle

A **succinct** data structure that **approximately** answers distance queries.

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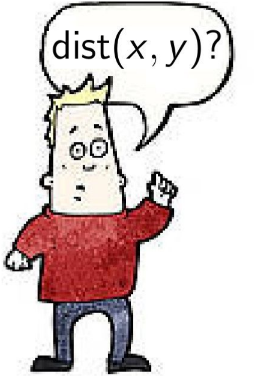

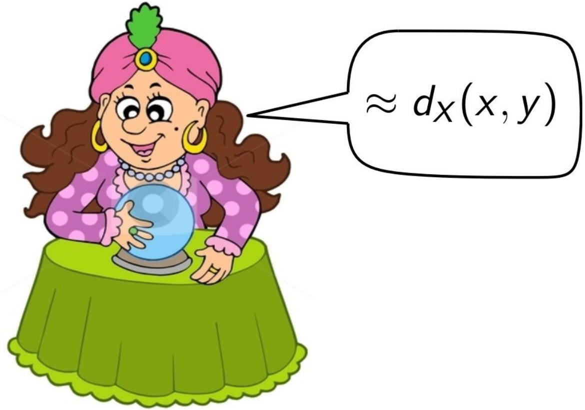

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A **succinct** data structure that **approximately** answers distance queries.

The properties of interest are size, distortion and query time.

Distance Oracles

Corollary

For every n-point metric space and $\boldsymbol{k} \geq 1$*, there is a* ***set*** $\mathcal{T}$ *of* $k \cdot n^{\frac{1}{k}}$ ***trees*** *and a* *mapping* **home** $: X \to \mathcal{T}$*, such that for every* $x, y \in X$*,*

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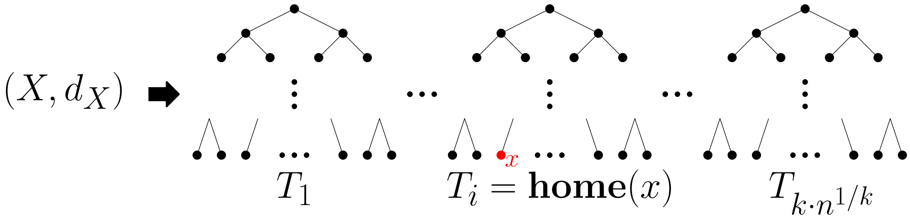

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Theorem (Tree Distance Oracle [Harel, Tarjan 84], [Bender, Farach-Colton 00])

For every ***tree metric****, there is an exact distance oracle of* ***linear size*** *and* ***constant*** *query* ***time****.*

Distance Oracles

Corollary

For every n-point metric space and $\boldsymbol{k} \geq 1$, there is a ***set*** $\mathcal{T}$ *of* $k \cdot n^{\frac{1}{k}}$ ***trees*** *and a mapping* **home** $: X \to \mathcal{T}$*, such that for every $x, y \in X$,*

$$d_{\mathbf{home}(x)}(x, y) \leq O(\boldsymbol{k}) \cdot d_X(x, y)$$

Theorem (Tree Distance Oracle [Harel, Tarjan 84], [Bender, Farach-Colton 00])

For every ***tree metric****, there is an exact distance oracle of* ***linear size*** *and* ***constant*** *query* ***time****.*

Theorem (Ramsey based Deterministic Distance Oracle)

For any n-point metric space, there is a distance oracle with :

Distortion	***Size***	***Query time***
$O(k)$	$O(k \cdot n^{1+1/k})$	$O(1)$

Outline of the talk

Clan Embedding

Idea: **duplicate** vertices to meet all guarantees!

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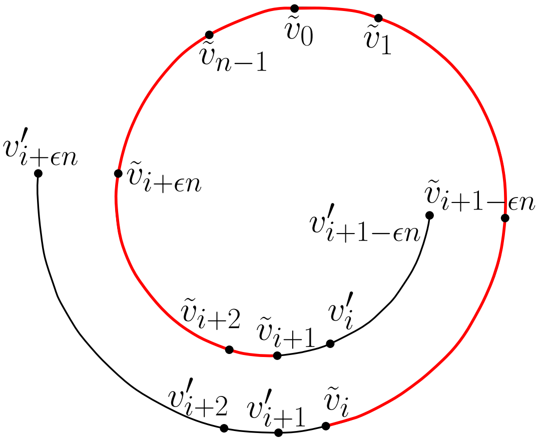

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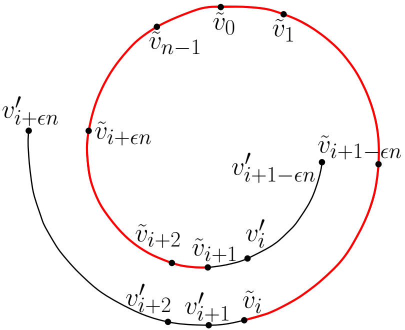

One-to-many embedding from (X, d_X) to (Y, d_Y): A map $f : X \to 2^Y$ where:

1) $\forall x,\ f(x) \neq \emptyset$.

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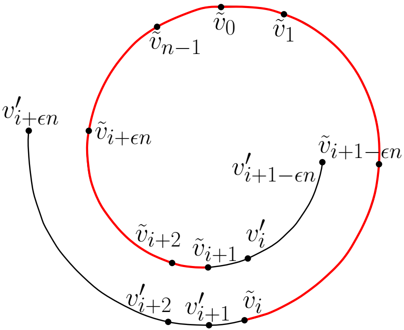

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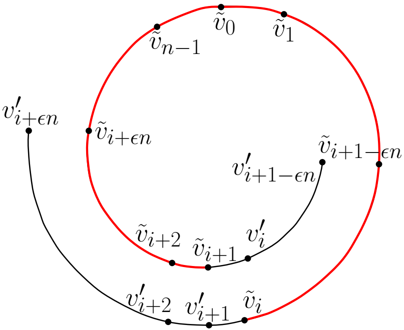

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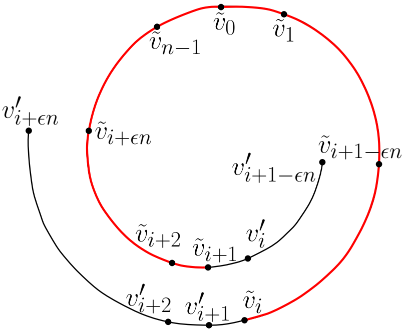

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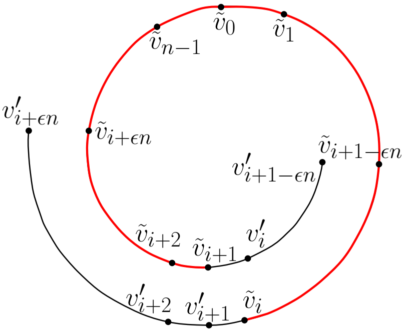

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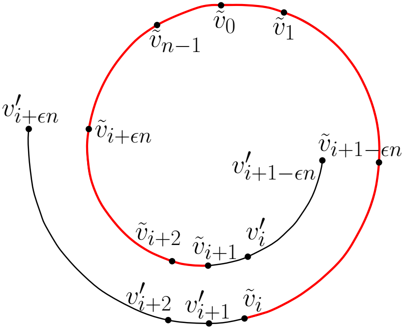

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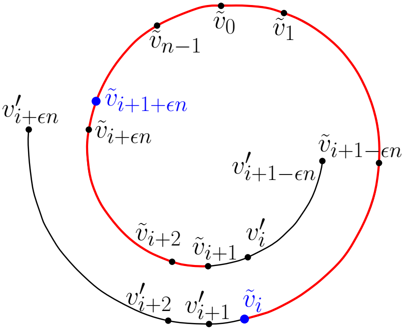

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$$\min_{y' \in f(v_{i+1+\epsilon n})} d_{P_{2n}}(\chi(v_i), y') \le (1 - \epsilon)n$$
$$< \frac{1}{\epsilon} \cdot d_{C_n}(v_i, v_{i+\epsilon n+1})$$

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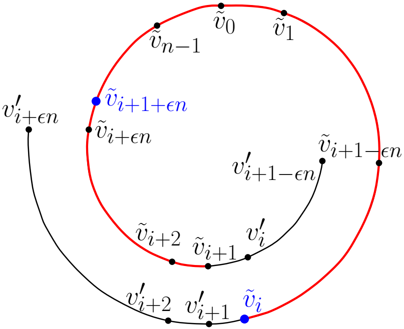

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Choose i u.a.r., then $\qquad \mathbb{E}[|f(v_i)|] = 1 + 2\epsilon.$

Theorem (Clan embedding into trees, [Filtser, Le 21])

(X, d_X) n point metric space, $\forall\epsilon \in (0, 1)$, there is ***distribution*** *$\mathcal{D}$ over* ***dominating clan embeddings*** *into trees such that:*

- *$\forall\ (f, \chi) \in \mathrm{supp}(\mathcal{D})$ has distortion $O(\frac{\log n}{\epsilon})$.*
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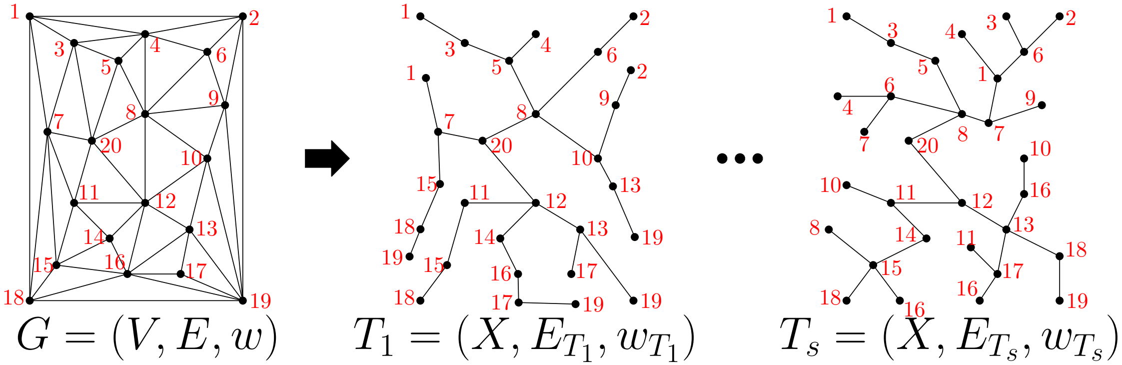

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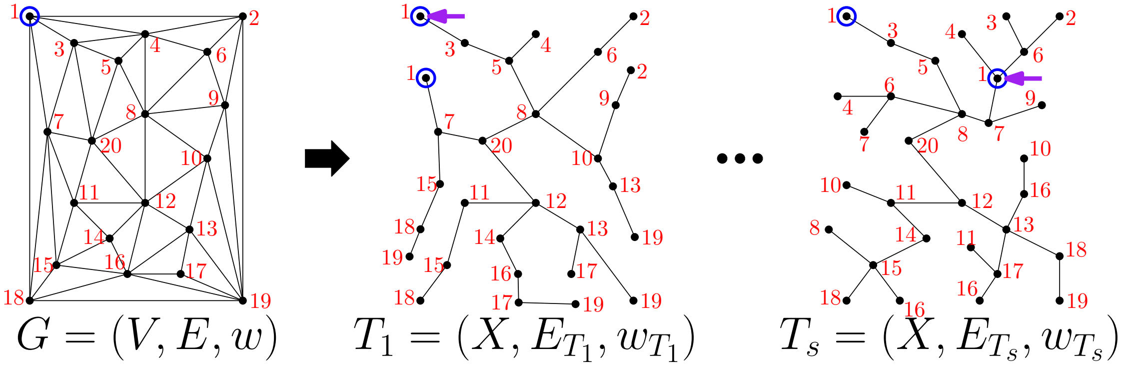

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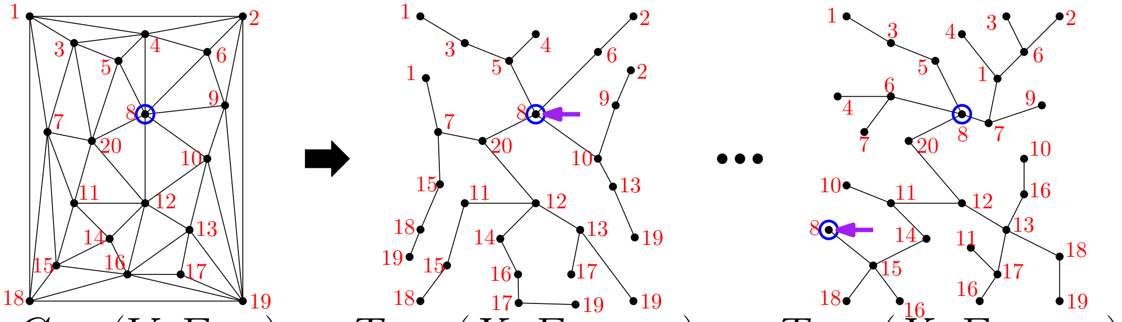

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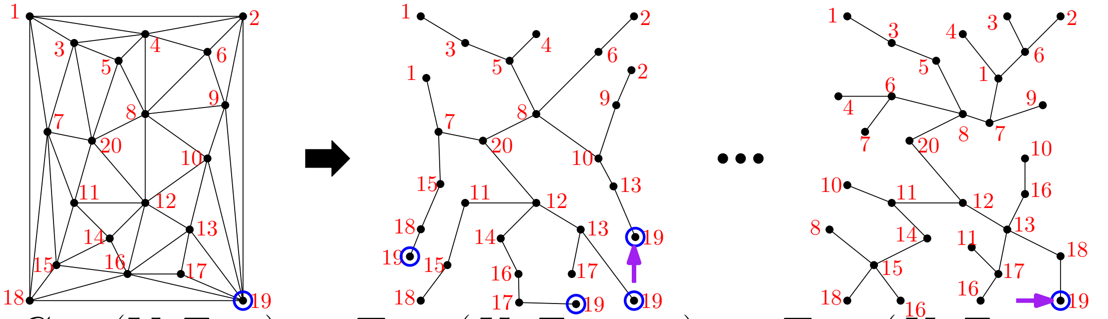

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(both) Tight!

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Compromises: Not a real classic embedding

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Originally appeared in [Bartal, Mendel 04] in the context of multi-embeddings.

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One-to-many embedding $f : X \to 2^Y$ has *path-distortion* t if for every sequence $(x_0, x_1, \ldots, x_m)$ in X there is a sequence $y_0, \ldots, y_m$ where $y_i \in f(x_i)$, and

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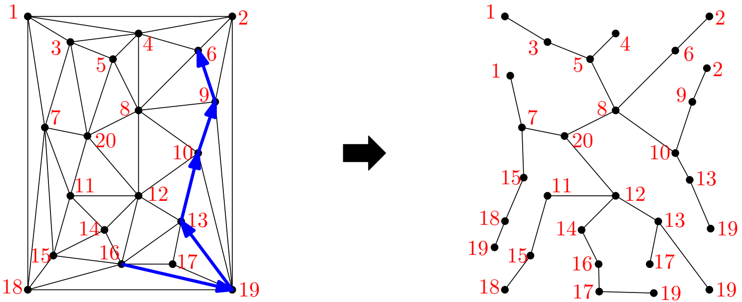

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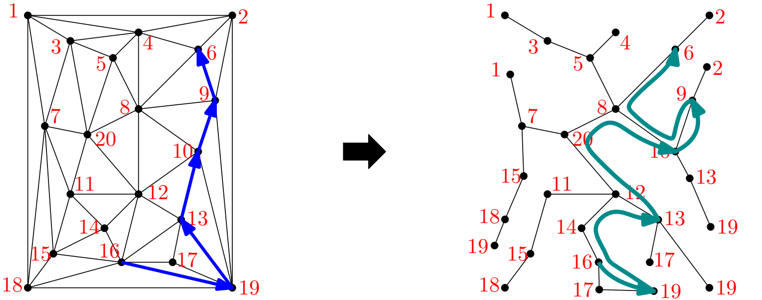

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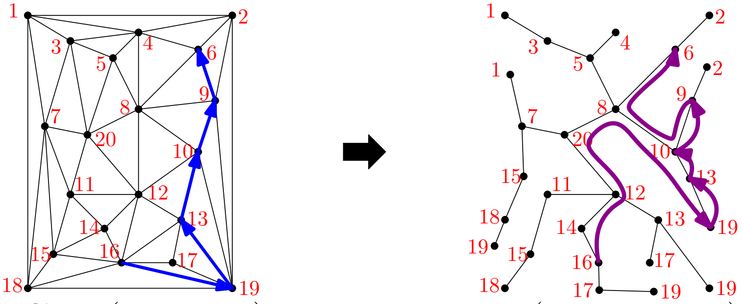

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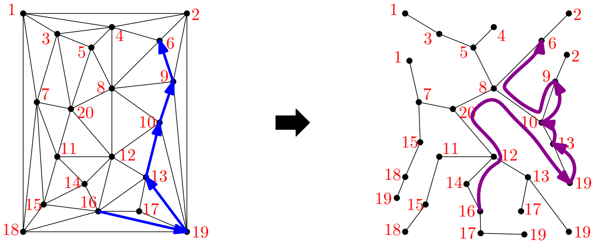

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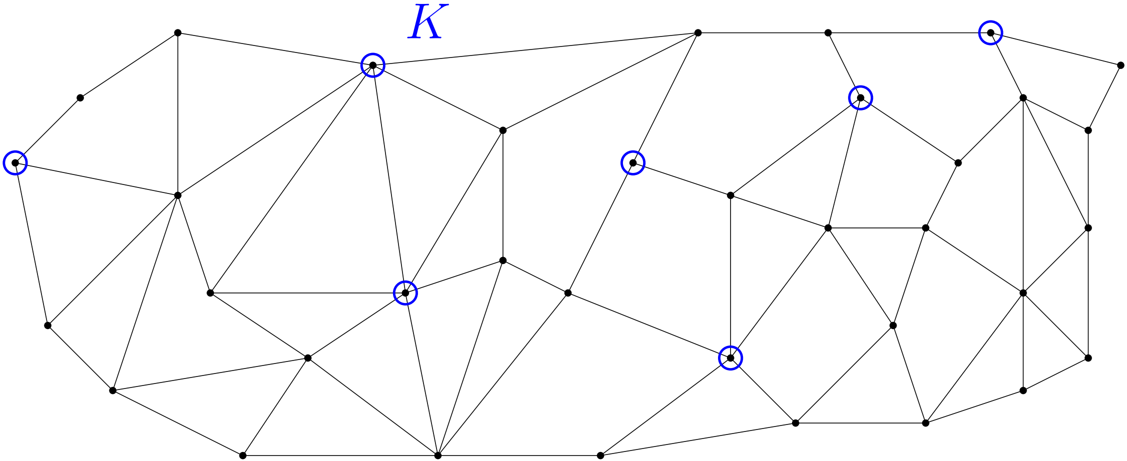
Or a total of $O(n^{1+\frac{1}{2}})$ copies and path distortion $O(\log n)$.

Steiner Tree

Given set of terminals K, find minimum weight tree T spanning K



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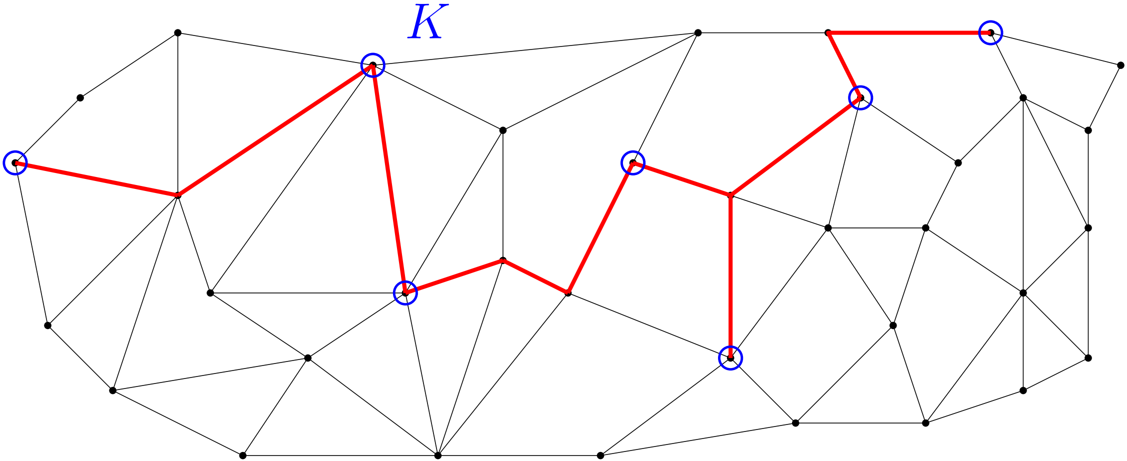

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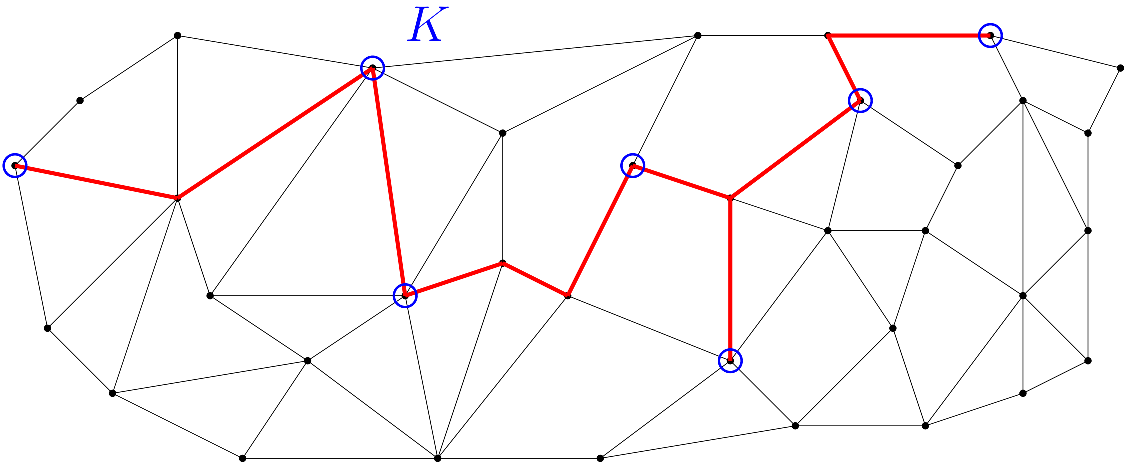
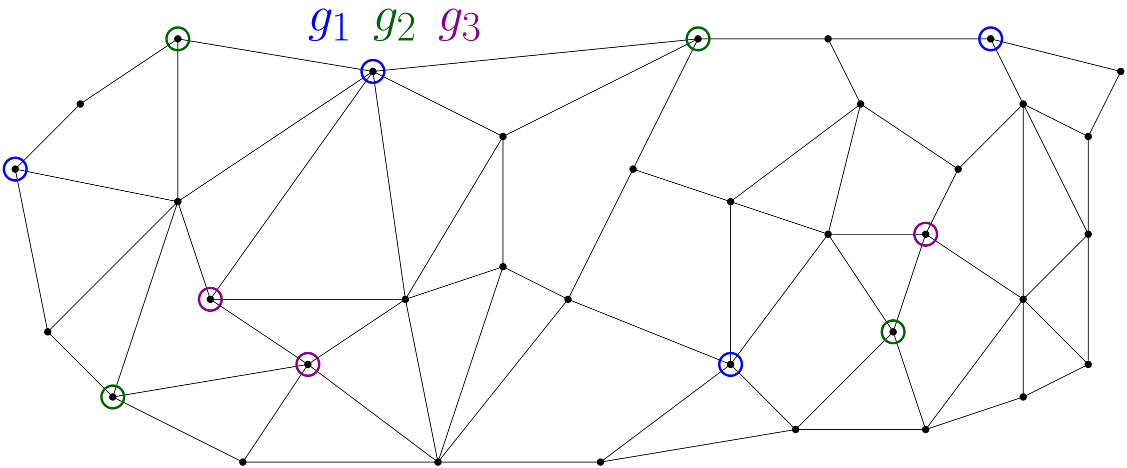


In class we saw a 2-approximation algorithm for the Steiner tree problem.

Group Steiner Tree

Given subsets $g_1, g_2, \ldots, g_k \subseteq V$, find minimum weight tree T spanning at least one vertex from each g_i



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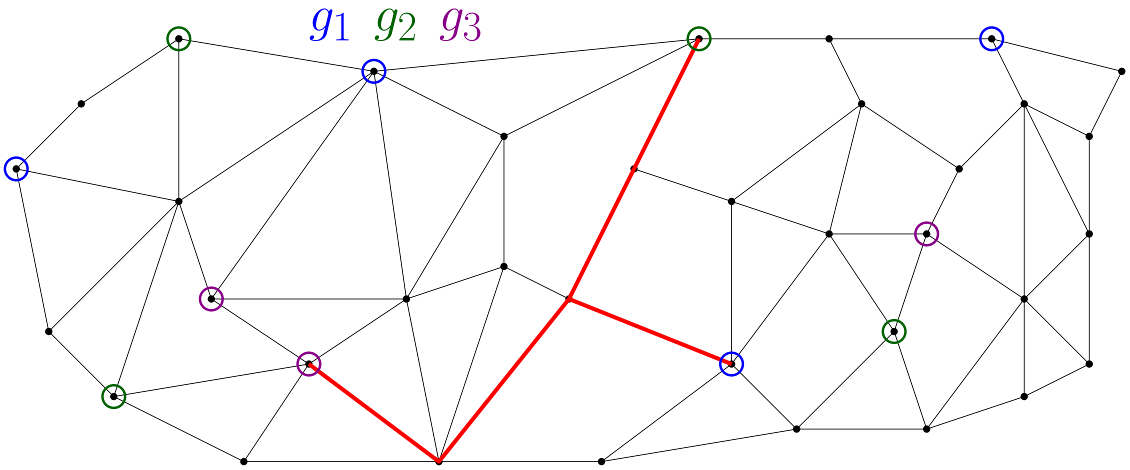

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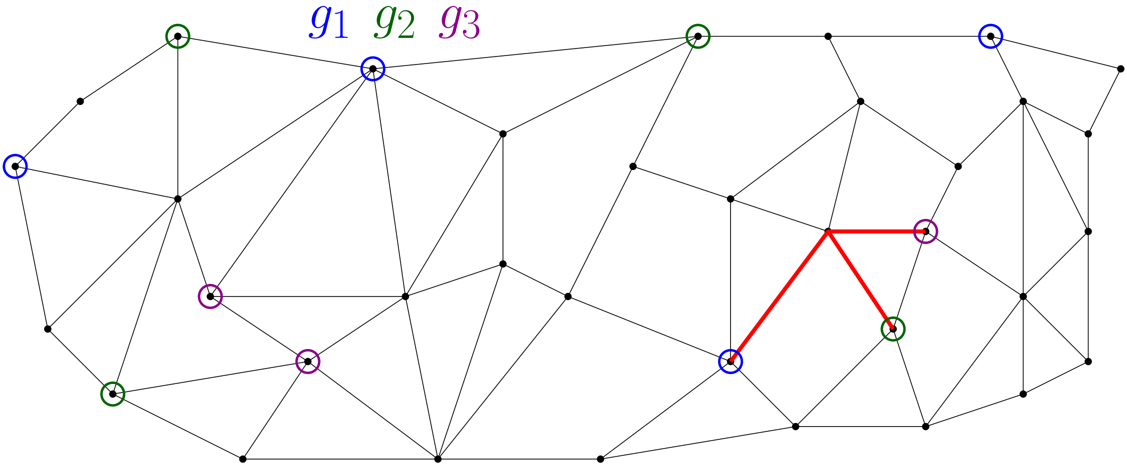

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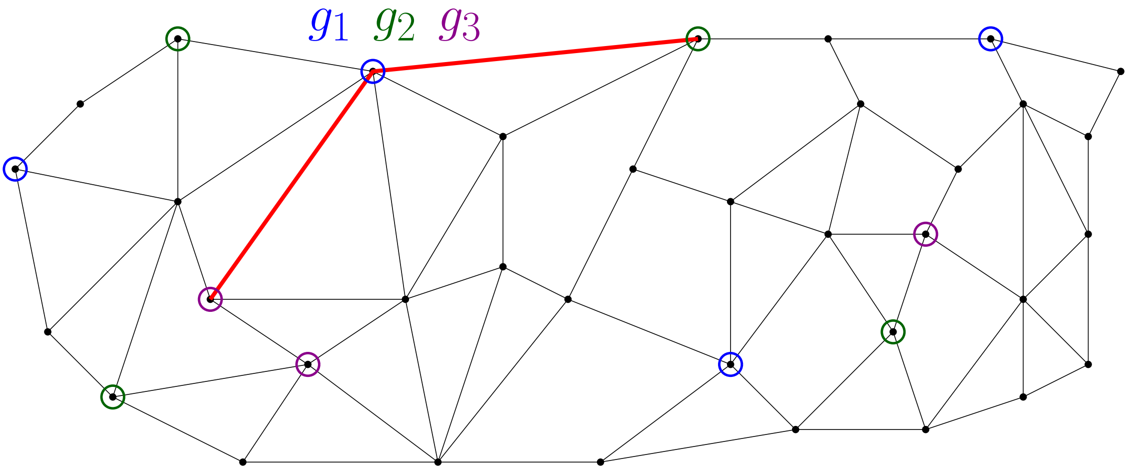

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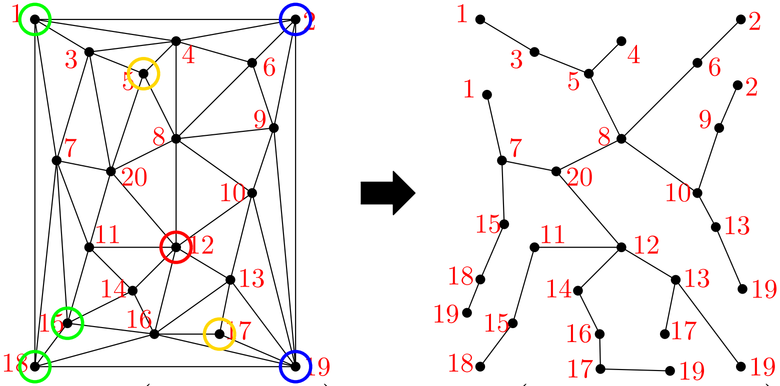

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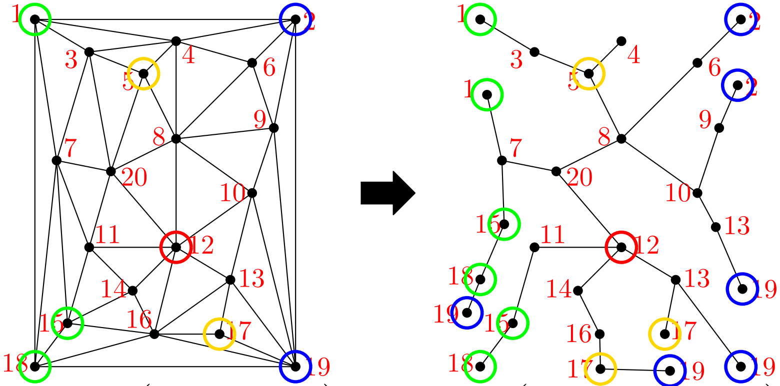

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Given subsets $g_1, g_2, \ldots, g_k \subseteq V$, find minimum weight tree T spanning at least one vertex from each g_i.

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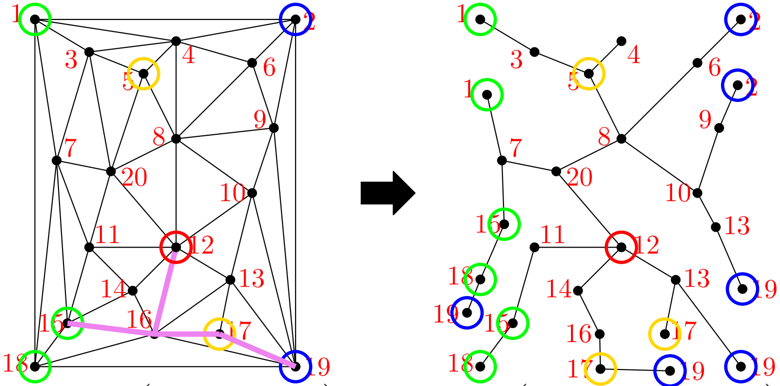

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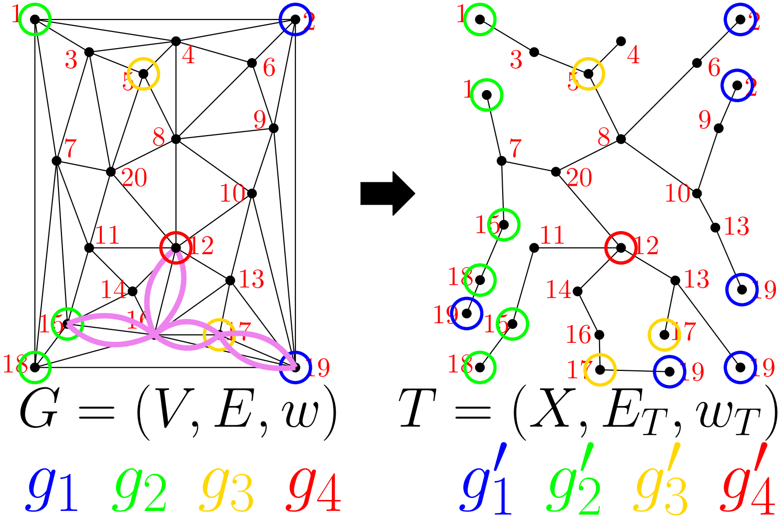

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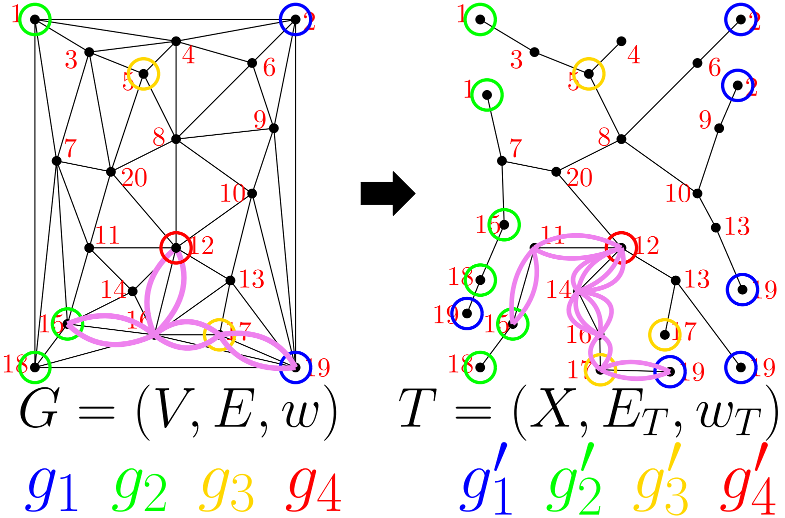

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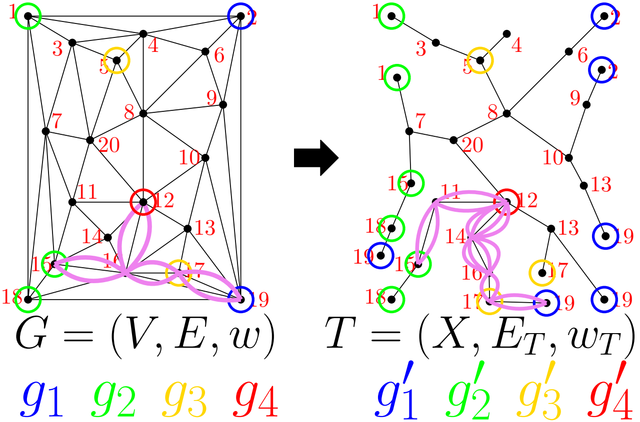

Theorem ([Garg, Konjevod, Ravi 00])

$O(\log n \cdot \log k)$-approximation algorithm for the GST problem on trees.

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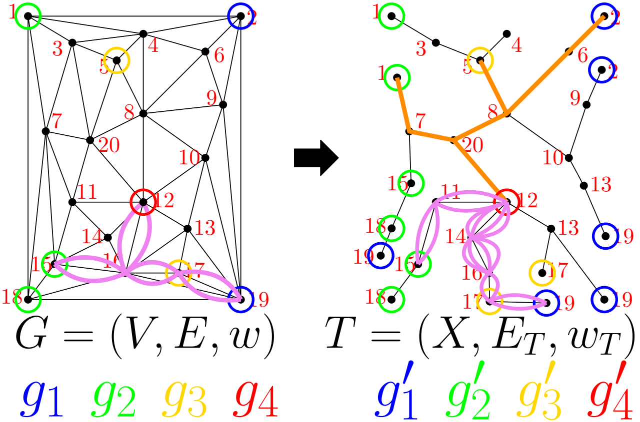

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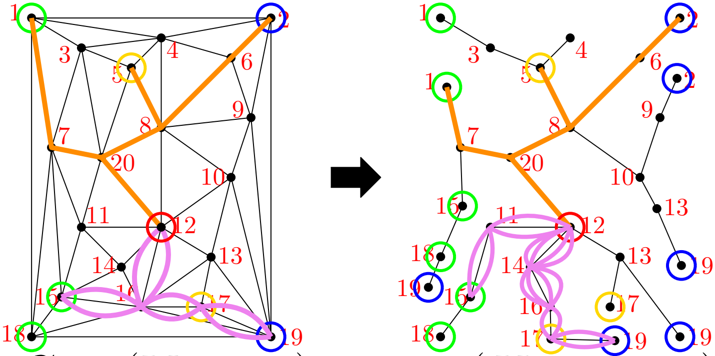

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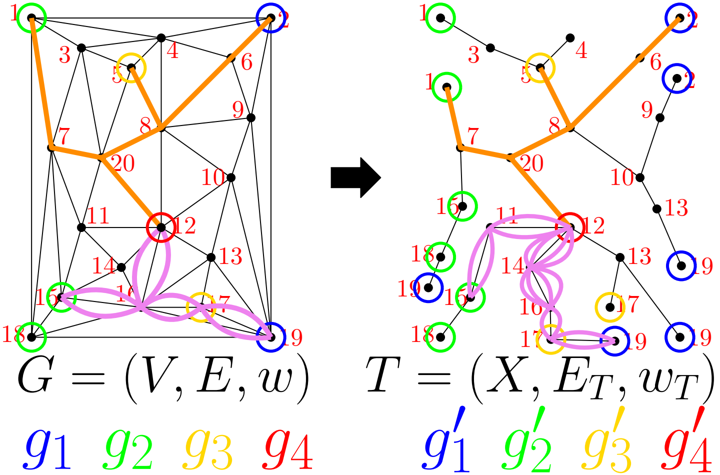

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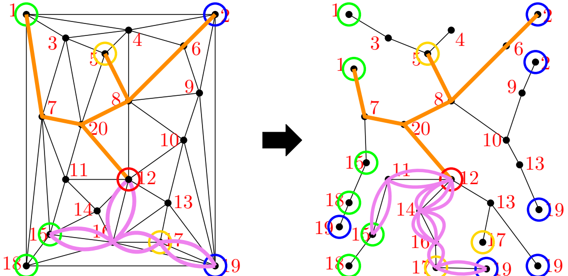

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$g_1 \quad g_2 \quad g_3 \quad g_4$ $\quad$ $g'_1 \quad g'_2 \quad g'_3 \quad g'_4$

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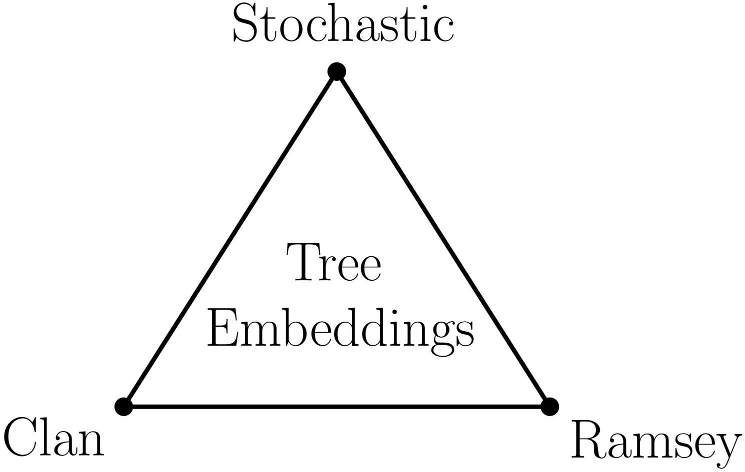
We got an $O(\log^2 n \cdot \log k)$ **approximation**.

Outline of the talk

1. Introduction
2. Stochastic embedding into trees
3. Ramsey type embeddings
4. Clan embedding
5. Group Steiner Tree
6. Conclusion

Trinity



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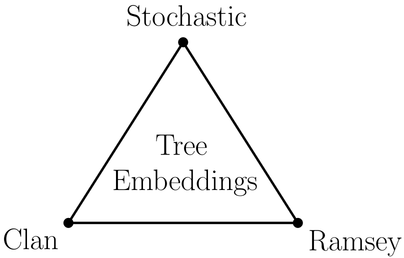
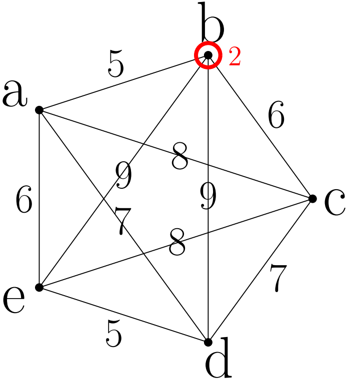


	Worst case gurantte	Satisfy all vertices	one-to-one embedding
Stochastic	✗	✓	✓
Ramsey	✓	✗	✓
Clan	✓	✓	✗

Online problem - Metrical Task System

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.
Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.
Goal: Minimize the competitive ratio between our algorithm to opt.



$$f_3(a) = 1$$
$$f_3(b) = 4$$
$$f_3(c) = 3$$
$$f_3(d) = 2$$
$$f_3(e) = 1$$

Cost alg(2)$=3+5+1=9$

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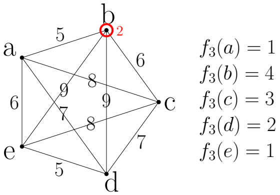

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We showed competitive ratio against oblivious adversary:

$$\text{Competitive ratio} = \max_{\text{input } I} \frac{\mathbb{E}[\mathrm{Alg}(I)]}{\mathrm{opt}(I)} = O(\log^2 n \cdot \log\log n).$$

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Theorem ([Fakcharoenphol, Rao, Talwar 04], improving [Bartal 96+98])

Every n-point metric space (X, d) embeds into ***distribution*** *$\mathcal{D}$ over* ***dominating trees*** *with* ***expected distortion*** *$O(\log n)$.*

Theorem ([Fiat, Mendel 2000])

Given an n point tree T, there is an online algorithm for MTS with competitive ratio $O(\log n \cdot \log\log n)$ against oblivious adversary.*

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Suppose that you are given a **deterministic** α-competitive algorithm for MTS on trees.

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Suppose that you are given a **deterministic** α-competitive algorithm for MTS on trees.

Problem: Propose a **deterministic** algorithm for **MTS** for a general n-point metric.
What is the competitive ratio?

Link to quiz:

Can also be found in my homepage:

arnold.filtser.com

Or just google Arnold Filtser.

Link to slides:

MTS Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
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Clan Embeddings construction

Theorem (Clan embedding into trees, [Filtser, Le 21])

(X, d_X) n point metric space, $\forall k \in \mathbb{N}$, there is ***distribution*** *$\mathcal{D}$ over* ***dominating clan embeddings*** *into trees such that:*

- $\forall\ (f, \chi) \in \mathrm{supp}(\mathcal{D})$ *has distortion* $O(k)$.
- $\forall\ x \in X$, $\mathbb{E}_{(f,\chi)\sim\mathcal{D}}[|f(x)|] \le O(n^{\frac{1}{k}})$

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Lemma

(X, d_X) n point metric, there is ***dominating clan embedding*** *into a tree s.t.:*

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Replace cardinality with weights

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(X, d_X) *n point metric, and weights* $\mu : X \to \mathbb{R}_{\ge 1}$
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Definition (Ultrametric)

Ultrametric (X, d) is a metric space satisfying the **strong** triangle inequality:

$$\forall x, y, z \in X, \qquad d(x, z) \leq \max\{d(x, y), d(y, z)\} \ .$$

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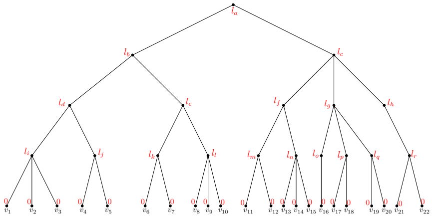
Definition (Hierarchical well-separated tree (HST))

(X, d_X) is a HST if X is mapped (by ϕ) to **leaves** of a rooted tree T where:

- The nodes of T associated with **monotone** labels l_v.
- $d_X(x, y) = \Gamma_{\mathrm{lca}(\varphi(x),\varphi(y))}$.



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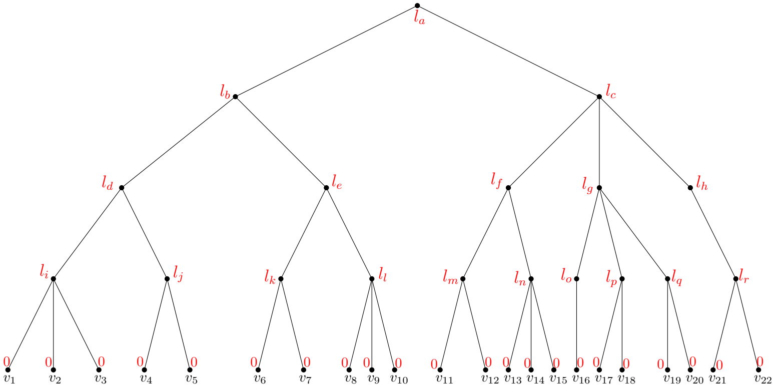

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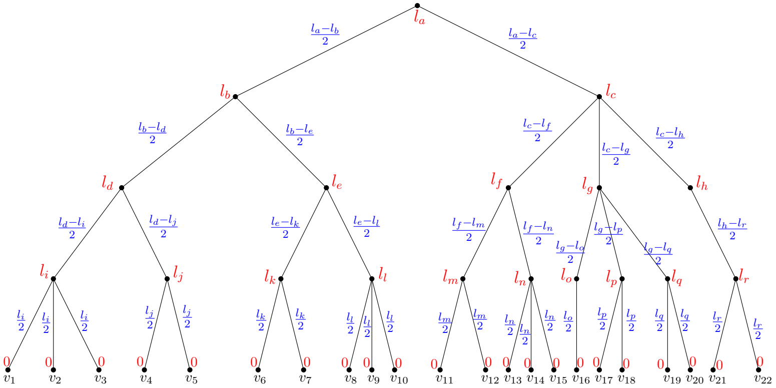
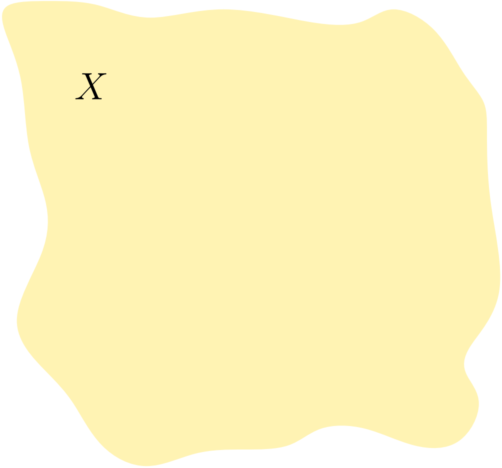
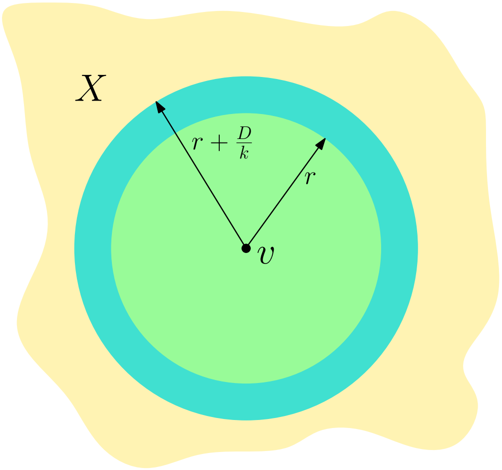


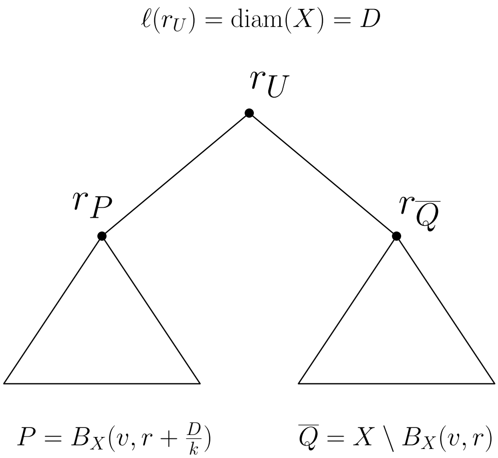
Construction



$$\ell(r_U) = \mathrm{diam}(X) = D$$

Construction





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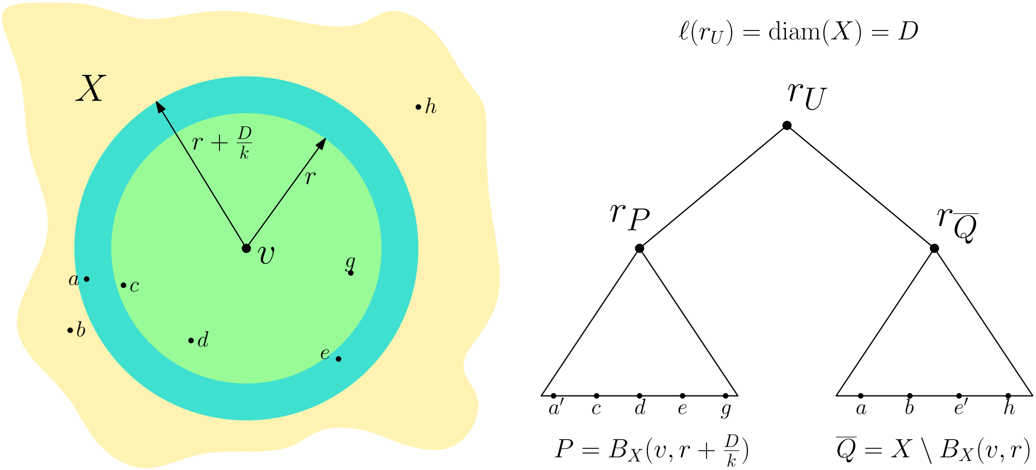

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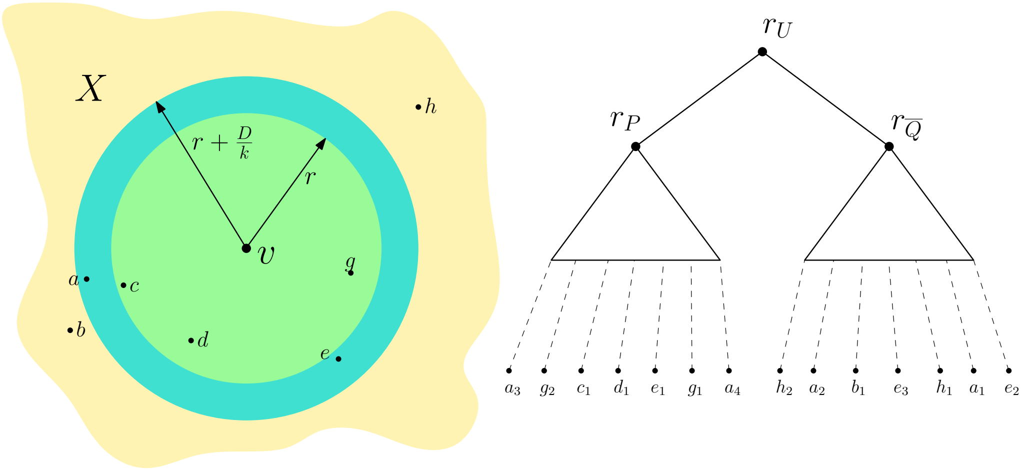

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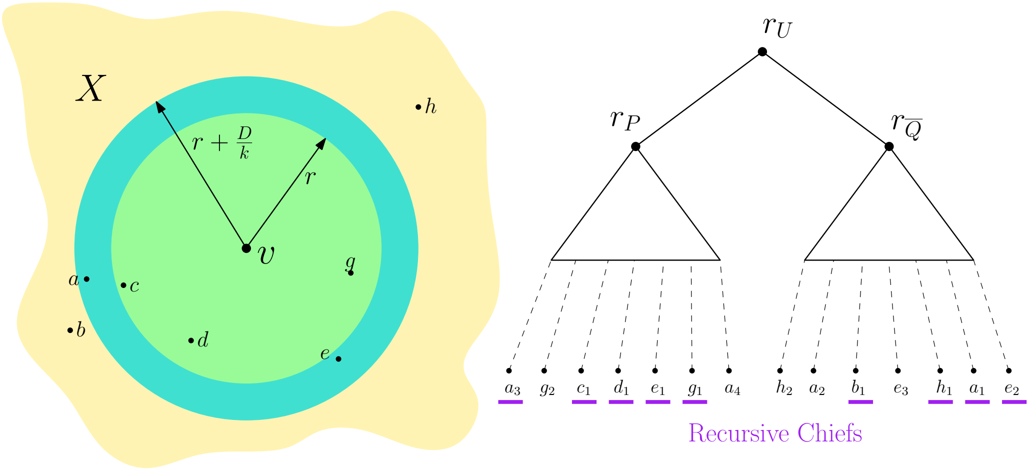

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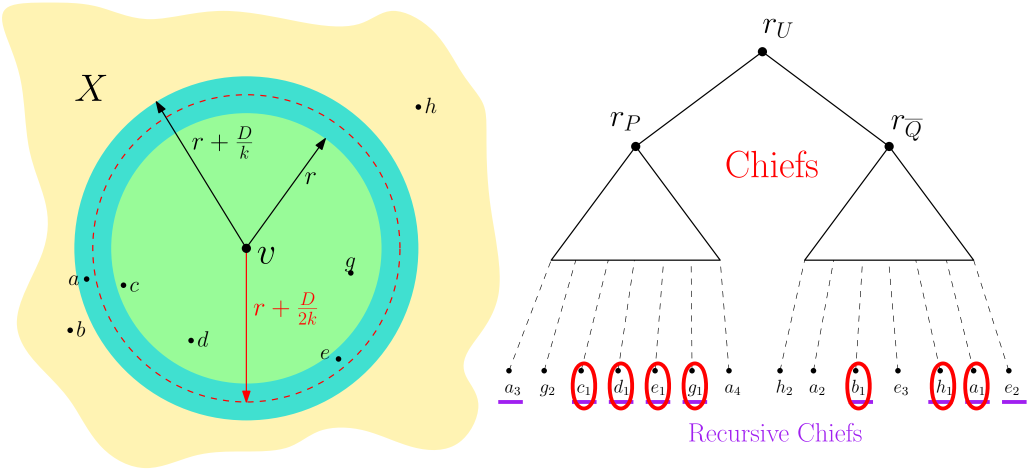
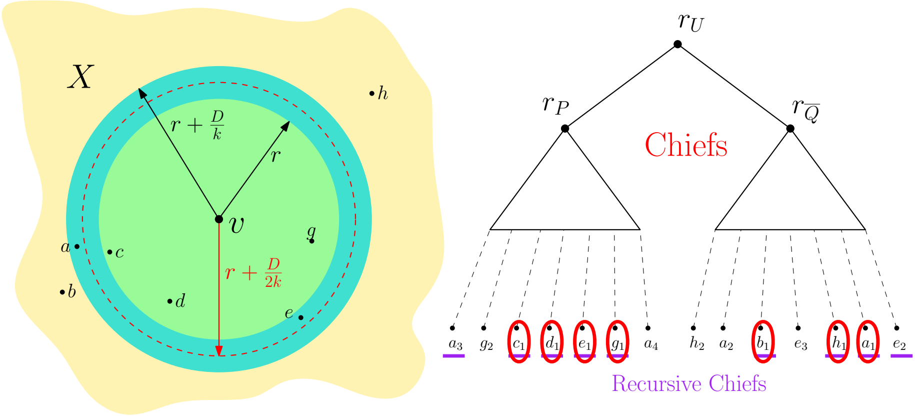
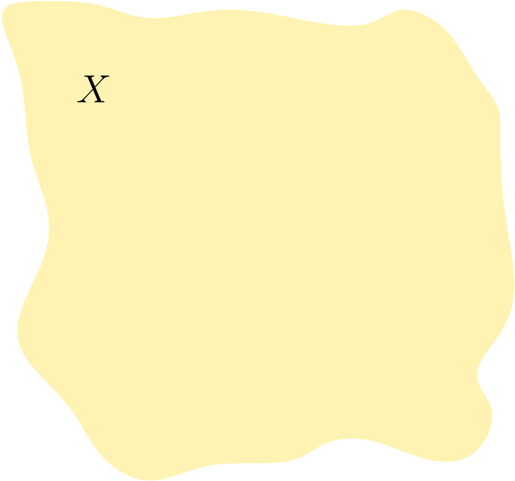


Construction - distortion bound



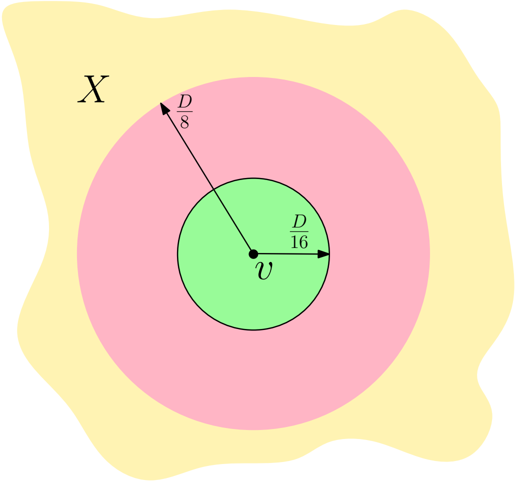
$$\min_{c' \in f(c)} d_U(c', \chi(a)) = D \leq 2k \cdot d_X(c, a) \ .$$

Construction - cardinality bound



$D = \mathrm{diam}(X)$

Construction - cardinality bound



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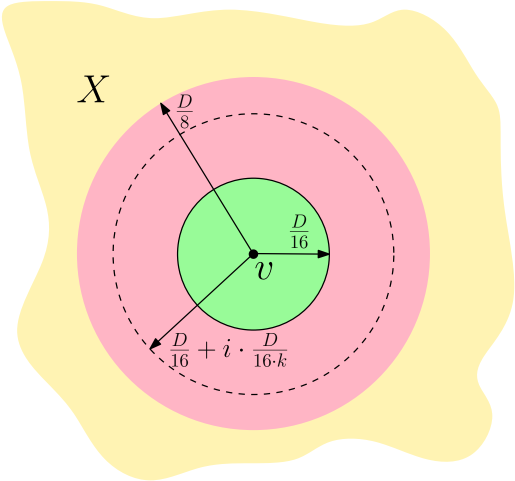

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$$i \in \{0, \dots, k-1\}$$

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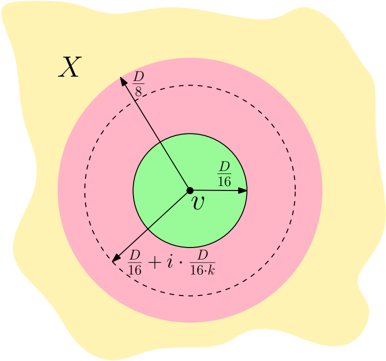

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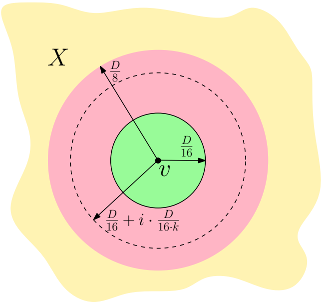

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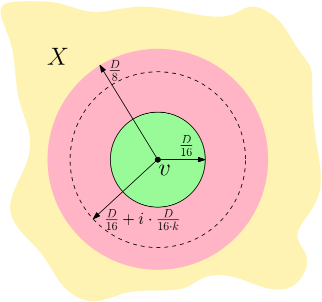
There is some i s.t. $\frac{|A_{i+1}|}{|A_i|} \leq \left(\frac{|A_k|}{|A_0|}\right)^{\frac{1}{k}}$. Otherwise

$$|A_k| > |A_{k-1}| \cdot \left(\frac{|A_k|}{|A_0|}\right)^{\frac{1}{k}} > |A_{k-2}| \cdot \left(\frac{|A_k|}{|A_0|}\right)^{\frac{2}{k}} > \cdots > |A_0| \cdot \left(\frac{|A_k|}{|A_0|}\right)^{\frac{k}{k}} = |A_k|$$

Construction - cardinality bound



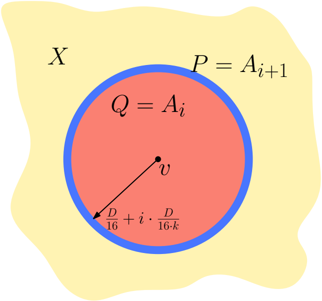
$D = \mathrm{diam}(X)$

$A_i = B_X(v, \frac{D}{16} + i \cdot \frac{D}{16 \cdot k})$
$i \in \{0, \ldots, k-1\}$

There is some i s.t. $\frac{|A_{i+1}|}{|A_i|} \leq \left(\frac{|A_k|}{|A_0|}\right)^{\frac{1}{k}}$.

Claim

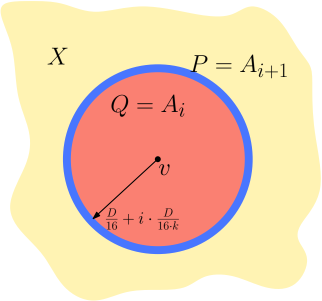
There is $v \in X$, and i, s.t. $\frac{|A_{i+1}|}{|A_i|} \leq \left(\frac{\mu^*(X)}{\mu^*(A_{i+1})}\right)^{1/k} = \left(\frac{\max_{x \in X} \left|B_X(x, \frac{\mathrm{diam(X)}}{4})\right|}{\max_{x \in A_{i+1}} \left|B_{A_{i+1}}(x, \frac{\mathrm{diam(A_{i+1})}}{4})\right|}\right)^{1/k}$.



Claim

There is $v \in X$*, and* i*, s.t.* $\frac{|A_{i+1}|}{|A_i|} \leq \left(\frac{\mu^*(X)}{\mu^*(A_{i+1})}\right)^{1/k} = \left(\frac{\max_{x\in X}\left|B_X(x, \frac{\mathrm{diam(X)}}{4})\right|}{\max_{x\in A_{i+1}}\left|B_{A_{i+1}}(x, \frac{\mathrm{diam(A_{i+1})}}{4})\right|}\right)^{1/k}$.

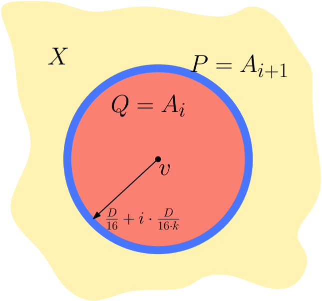
Set $P = A_{i+1}$, $Q = A_i$, and $\overline{Q} = X \setminus A_i$.



Claim

There is $v \in X$, *and* i, *s.t.* $\frac{|A_{i+1}|}{|A_i|} \leq \left(\frac{\mu^*(X)}{\mu^*(A_{i+1})}\right)^{1/k} = \left(\frac{\max_{x \in X}\left|B_X(x, \frac{\mathrm{diam(X)}}{4})\right|}{\max_{x \in A_{i+1}}\left|B_{A_{i+1}}(x, \frac{\mathrm{diam(A_{i+1})}}{4})\right|}\right)^{1/k}.$

Set $P = A_{i+1}$, $Q = A_i$, and $\overline{Q} = X \setminus A_i$.
By the claim: $|P| \cdot \mu^*(P)^{\frac{1}{k}} \leq |Q| \cdot \mu^*(X)^{\frac{1}{k}}$.



Claim

There is $v \in X$, and i, s.t. $\frac{|A_{i+1}|}{|A_i|} \leq \left(\frac{\mu^*(X)}{\mu^*(A_{i+1})}\right)^{1/k} = \left(\frac{\max_{x\in X}\left|B_X(x, \frac{\mathrm{diam(X)}}{4})\right|}{\max_{x\in A_{i+1}}\left|B_{A_{i+1}}(x, \frac{\mathrm{diam(A_{i+1})}}{4})\right|}\right)^{1/k}$.

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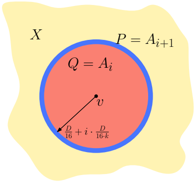
By the claim: $|P| \cdot \mu^*(P)^{\frac{1}{k}} \leq |Q| \cdot \mu^*(X)^{\frac{1}{k}}$.

Recurse on P and $\overline{Q}$.



Claim

There is $v \in X$, and i, s.t. $\frac{|A_{i+1}|}{|A_i|} \leq \left(\frac{\mu^*(X)}{\mu^*(A_{i+1})}\right)^{1/k} = \left(\frac{\max_{x \in X} \left|B_X(x, \frac{\mathrm{diam(X)}}{4})\right|}{\max_{x \in A_{i+1}} \left|B_{A_{i+1}}(x, \frac{\mathrm{diam(A_{i+1})}}{4})\right|}\right)^{1/k}$.

Set $P = A_{i+1}$, $Q = A_i$, and $\overline{Q} = X \setminus A_i$.

By the claim: $|P| \cdot \mu^*(P)^{\frac{1}{k}} \leq |Q| \cdot \mu^*(X)^{\frac{1}{k}}$. Recurse on P and $\overline{Q}$.

We argue by induction: $|f(X)| \leq |X| \cdot \mu^*(X)^{\frac{1}{k}} \leq |X|^{1+\frac{1}{k}}$.

Here $|f(X)| = \sum_{x \in X} |f(x)|$.

Claim

There is $v \in X$*, and* i*, s.t.* $\frac{|A_{i+1}|}{|A_i|} \le \left(\frac{\mu^*(X)}{\mu^*(A_{i+1})}\right)^{1/k} = \left(\frac{\max_{x\in X}\left|B_X(x, \frac{\mathrm{diam(X)}}{4})\right|}{\max_{x\in A_{i+1}}\left|B_{A_{i+1}}(x, \frac{\mathrm{diam(A_{i+1})}}{4})\right|}\right)^{1/k}$.

Set $P = A_{i+1}$, $Q = A_i$, and $\overline{Q} = X \setminus A_i$.
By the claim: $|P| \cdot \mu^*(P)^{\frac{1}{k}} \le |Q| \cdot \mu^*(X)^{\frac{1}{k}}$. Recurse on P and $\overline{Q}$.

We argue by induction: $|f(X)| \le |X| \cdot \mu^*(X)^{\frac{1}{k}} \le |X|^{1+\frac{1}{k}}$.

$$|f(X)| = |f(P)| + |f(\overline{Q})|$$

Claim

There is $v \in X$*, and* i*, s.t.* $\frac{|A_{i+1}|}{|A_i|} \leq \left(\frac{\mu^*(X)}{\mu^*(A_{i+1})}\right)^{1/k} = \left(\frac{\max_{x\in X}\left|B_X(x, \frac{\mathrm{diam(X)}}{4})\right|}{\max_{x\in A_{i+1}}\left|B_{A_{i+1}}(x, \frac{\mathrm{diam(A_{i+1})}}{4})\right|}\right)^{1/k}$.

Set $P = A_{i+1}$, $Q = A_i$, and $\overline{Q} = X \setminus A_i$.

By the claim: $|P| \cdot \mu^*(P)^{\frac{1}{k}} \leq |Q| \cdot \mu^*(X)^{\frac{1}{k}}$.

Recurse on P and $\overline{Q}$.

We argue by induction: $|f(X)| \leq |X| \cdot \mu^*(X)^{\frac{1}{k}} \leq |X|^{1+\frac{1}{k}}$.

$$
\begin{aligned}
|f(X)| &= |f(P)| + |f(\overline{Q})| \\
&\leq |P| \cdot \mu^*(P)^{\frac{1}{k}} + \left|\overline{Q}\right| \cdot \mu^*(\overline{Q})^{\frac{1}{k}}
\end{aligned}
$$

Claim

There is $v \in X$, *and* i, *s.t.* $\frac{|A_{i+1}|}{|A_i|} \leq \left(\frac{\mu^*(X)}{\mu^*(A_{i+1})}\right)^{1/k} = \left(\frac{\max_{x \in X} \left|B_X(x, \frac{\mathrm{diam(X)}}{4})\right|}{\max_{x \in A_{i+1}} \left|B_{A_{i+1}}(x, \frac{\mathrm{diam(A_{i+1})}}{4})\right|}\right)^{1/k}$.

Set $P = A_{i+1}$, $Q = A_i$, and $\overline{Q} = X \setminus A_i$.

By the claim: $|P| \cdot \mu^*(P)^{\frac{1}{k}} \leq |Q| \cdot \mu^*(X)^{\frac{1}{k}}$. Recurse on P and $\overline{Q}$.

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$$\begin{aligned}
|f(X)| &= |f(P)| + |f(\overline{Q})| \\
&\leq |P| \cdot \mu^*(P)^{\frac{1}{k}} + \left|\overline{Q}\right| \cdot \mu^*(\overline{Q})^{\frac{1}{k}} \\
&\leq |Q| \cdot \mu^*(X)^{\frac{1}{k}} + \left|\overline{Q}\right| \cdot \mu^*(X)^{\frac{1}{k}}
\end{aligned}$$

Claim

There is $v \in X$, and i, s.t. $\frac{|A_{i+1}|}{|A_i|} \leq \left(\frac{\mu^*(X)}{\mu^*(A_{i+1})}\right)^{1/k} = \left(\frac{\max_{x\in X}\left|B_X(x, \frac{\mathrm{diam(X)}}{4})\right|}{\max_{x\in A_{i+1}}\left|B_{A_{i+1}}(x, \frac{\mathrm{diam(A_{i+1})}}{4})\right|}\right)^{1/k}$.

Set $P = A_{i+1}$, $Q = A_i$, and $\overline{Q} = X \setminus A_i$.
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|f(X)| &= |f(P)| + |f(\overline{Q})| \\
&\leq |P| \cdot \mu^*(P)^{\frac{1}{k}} + \left|\overline{Q}\right| \cdot \mu^*(\overline{Q})^{\frac{1}{k}} \\
&\leq |Q| \cdot \mu^*(X)^{\frac{1}{k}} + \left|\overline{Q}\right| \cdot \mu^*(X)^{\frac{1}{k}} \\
&= |X| \cdot \mu^*(X)^{\frac{1}{k}} \ .
\end{aligned}$$

Claim

There is $v \in X$, and i, s.t. $\frac{|A_{i+1}|}{|A_i|} \leq \left(\frac{\mu^*(X)}{\mu^*(A_{i+1})}\right)^{1/k} = \left(\frac{\max_{x\in X}\left|B_X(x, \frac{\mathrm{diam(X)}}{4})\right|}{\max_{x\in A_{i+1}}\left|B_{A_{i+1}}(x, \frac{\mathrm{diam(A_{i+1})}}{4})\right|}\right)^{1/k}$.

Set $P = A_{i+1}$, $Q = A_i$, and $\overline{Q} = X \setminus A_i$.

By the claim: $|P| \cdot \mu^*(P)^{\frac{1}{k}} \leq |Q| \cdot \mu^*(X)^{\frac{1}{k}}$. Recurse on P and $\overline{Q}$.

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|f(X)| &= |f(P)| + |f(\overline{Q})| \\
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&\leq |Q| \cdot \mu^*(X)^{\frac{1}{k}} + \left|\overline{Q}\right| \cdot \mu^*(X)^{\frac{1}{k}} \\
&= |X| \cdot \mu^*(X)^{\frac{1}{k}} \ .
\end{aligned}
$$