

הזמנה

airou chshifa latvaar shni v'shliishi b'machlikot mdu'i ha'machshav, matematika, cimia v'fizika

יום רביעי 10.7.24 | 16:00-19:00

בבניין מדעי המחשב (503)
אוניברסיטת בר-אילן

בואו להכיר את מגוון מסלולי הלימוד
لتואר שני ושלישי (דוקטורט)
בפקולטה למדעים מדויקים



Metric Embeddings intro Trees

Arnold Filtser
Bar-Ilan University

July 02, 2024

Outline of the talk

- 1 Introduction
- 2 Stochastic embedding into trees
- 3 Ramsey type embeddings
- 4 Clan embedding
- 5 Group Steiner Tree
- 6 Conclusion

Metric space

A metric space is an ordered pair (X, d_X) , where X is a set and $d_X : X \times X \rightarrow \mathbb{R}_{\geq 0}$ is a function such that:

- ① **Identity:** $\forall x, y \in X, d_X(x, y) = 0 \iff x = y.$
- ② **Symmetry:** $\forall x, y \in X, d_X(x, y) = d_X(y, x).$
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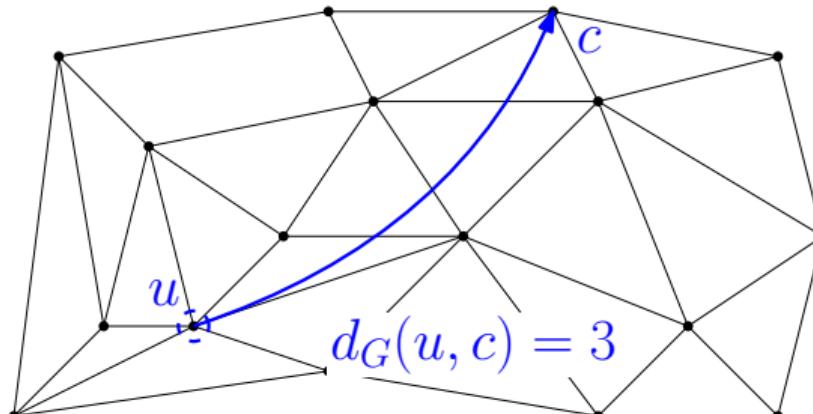
Examples:

- Euclidean space ℓ_2 in \mathbb{R}^d : $d_{\ell_2}(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|_2 = \sqrt{\sum_{i=1}^d (x_i - y_i)^2}.$
- Manhattan distance ℓ_1 in \mathbb{R}^d : $d_{\ell_1}(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|_1 = \sum_{i=1}^d |x_i - y_i|.$
- Edit distance: given two strings A, B how many edit operations
(insert, delete, substitute) are required to transform A to B ?
- Weighted graph $G = (V, E, w)$ with shortest path distance.

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Many problems are defined w.r.t. metric spaces. Examples:

- k -center.
- Steiner tree.
- Metric TSP.

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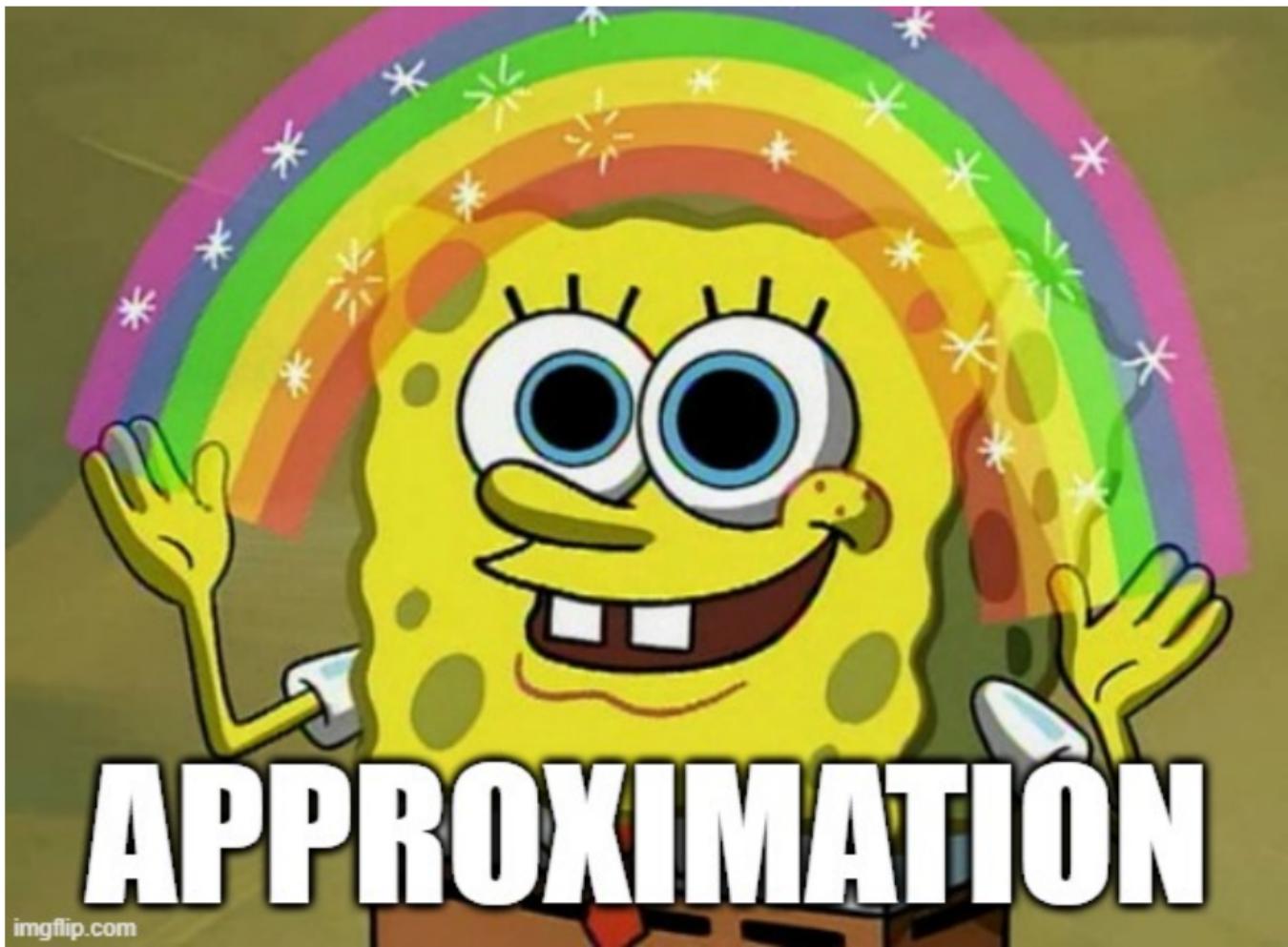
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Often these problems are NP-hard.

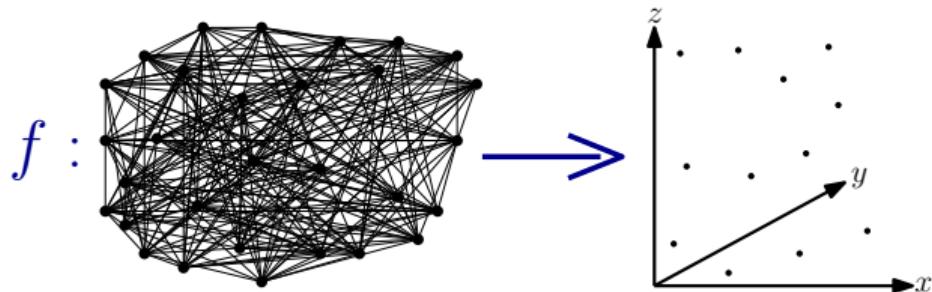


Metric Embeddings

Embedding

$(X, d_X), (Y, d_Y)$ metric spaces.

$f : (X, d_X) \rightarrow (Y, d_Y)$ is called an **embedding**.

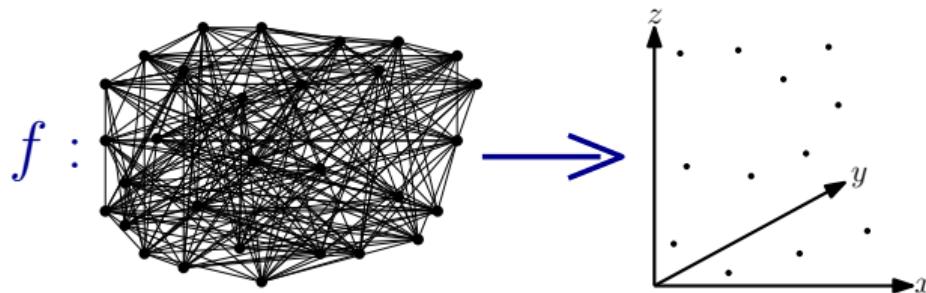


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Preserve (approximately) properties of the original space:

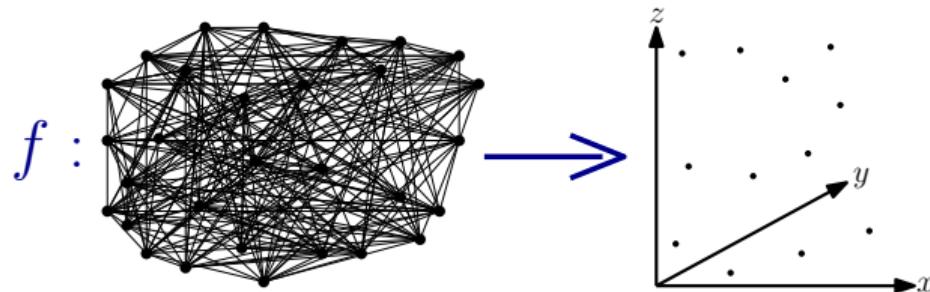
- Distances
- Cuts, Flows
- Commute time
- Effective resistance
- Clustering statistics.
- etc.

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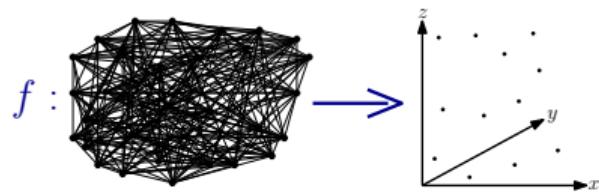
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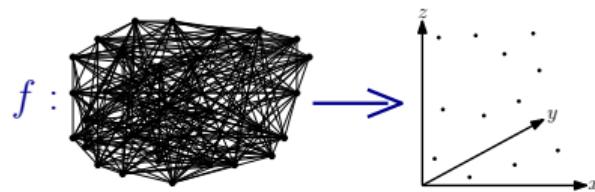
It is highly desirable that the target space Y will have **simple structure**.

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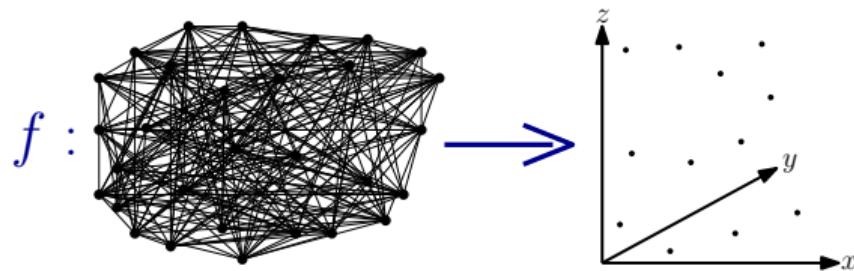
So that we could run efficient algorithms on it...

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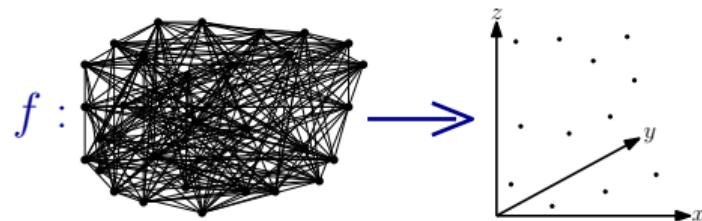
Theorem ([Bourgain 85])

Every n -point metric (X, d_X) is **embeddable** into Euclidean space $(\mathbb{R}^d, \|\cdot\|_2)$ with **distortion** $O(\log n)$.

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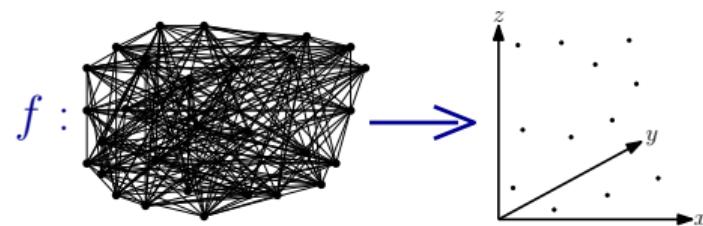
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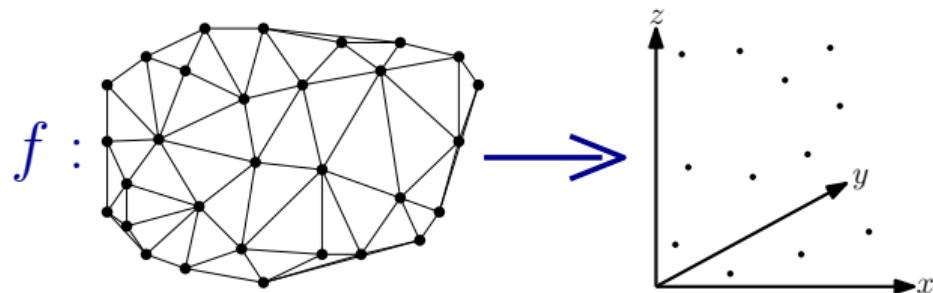


Applications:

- Approximation algorithms (e.g. **sparsest cut**, min graph bandwidth)
- Parallel computation (e.g. SSSP in MPC)
- Computational Biology (e.g. clustering and detecting protein seq.)
- etc.

Theorem ([Rao 99])

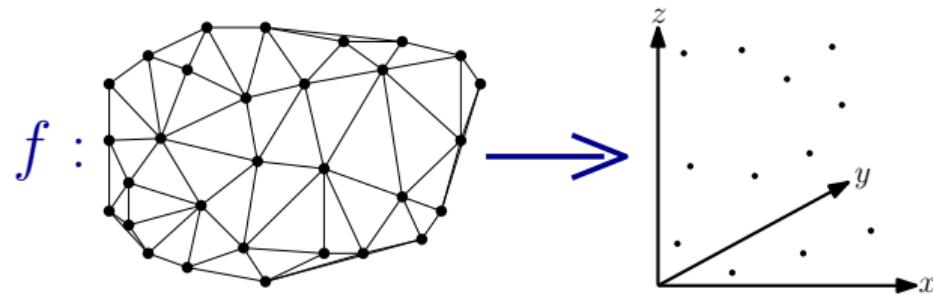
Every n -point **planar metric** (X, d_X) is **embeddable** into Euclidean space $(\mathbb{R}^d, \|\cdot\|_2)$ with **distortion** $O(\sqrt{\log n})$.



Planar metric- the shortest path metric of a planar graph.

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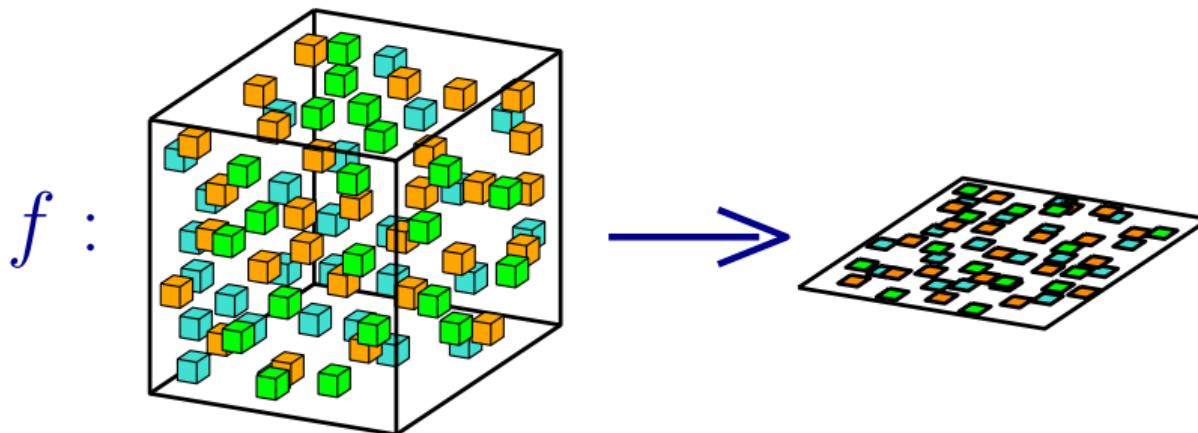
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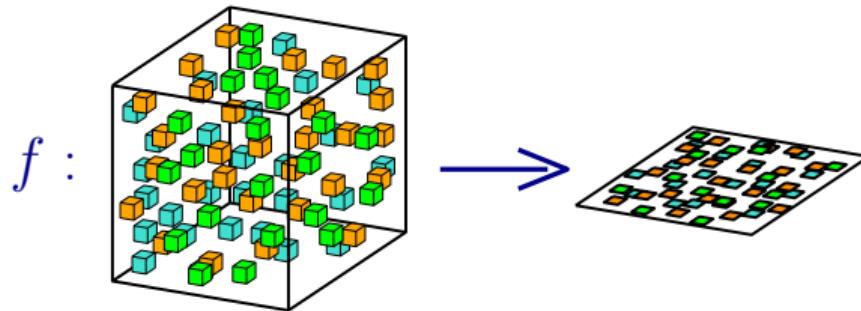
Theorem ([Johnson, Lindenstrauss 84], Dimension Reduction)

$X \subset (\mathbb{R}^d, \|\cdot\|_2)$ set of size n . Then X embeds into $O(\log n/\epsilon^2)$ dimensional Euclidean space with distortion $1 + \epsilon$.

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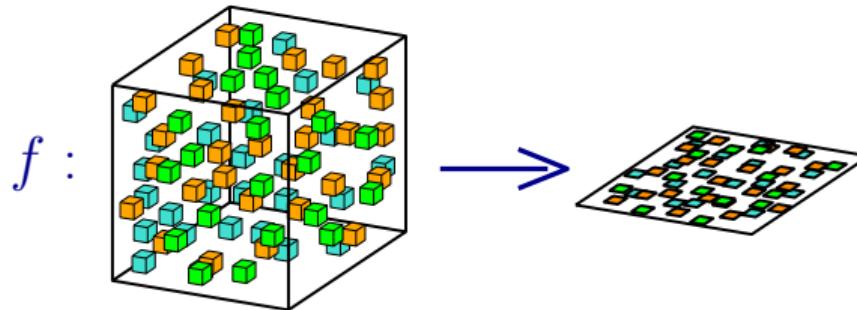
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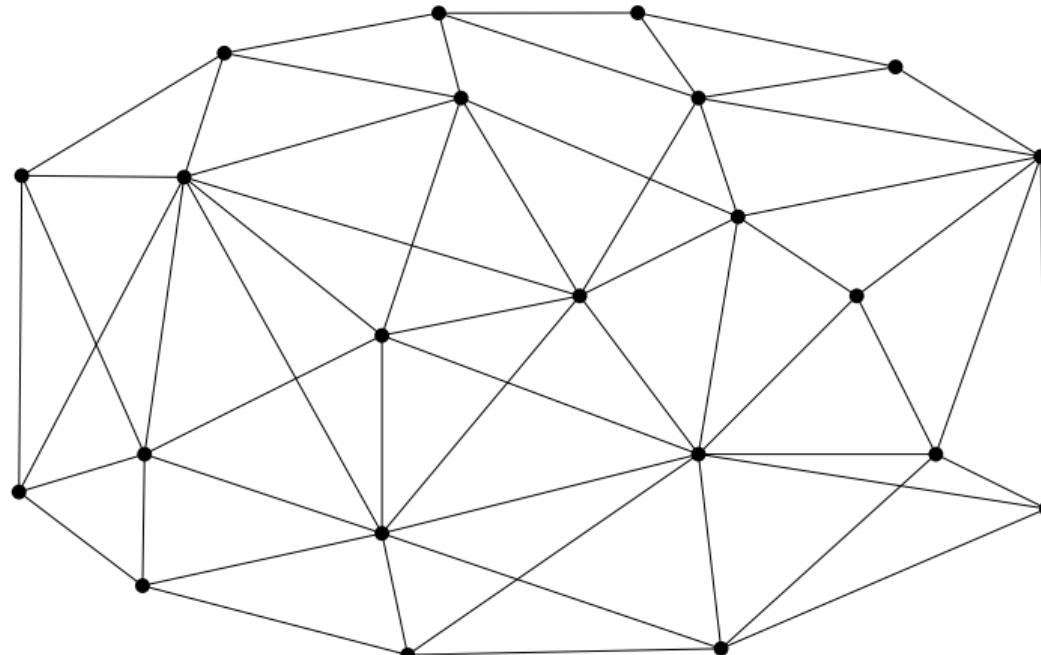
- Speeding up-computation
- Clustering
- Nearest Neighbor Search
- Machine Learning
- etc.

Graph Spanners

$G = (V, E, w)$ weighted graph, a **t -spanner** is a **subgraph** $H = (V, E_H)$

s.t.

$$\forall u, v \in V, \quad d_H(u, v) \leq t \cdot d_G(u, v)$$

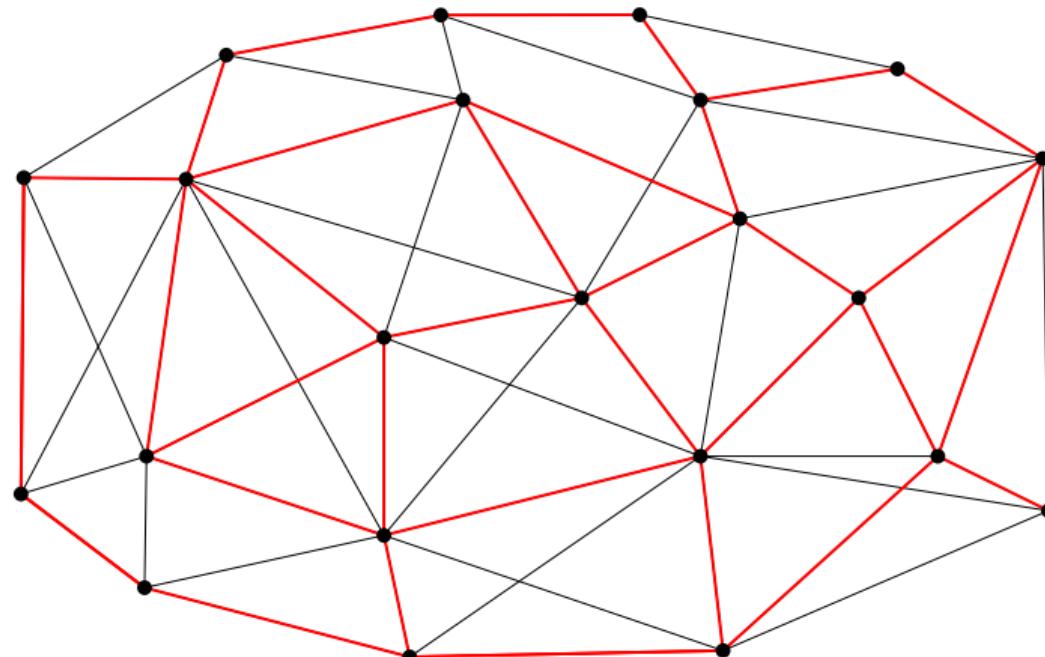


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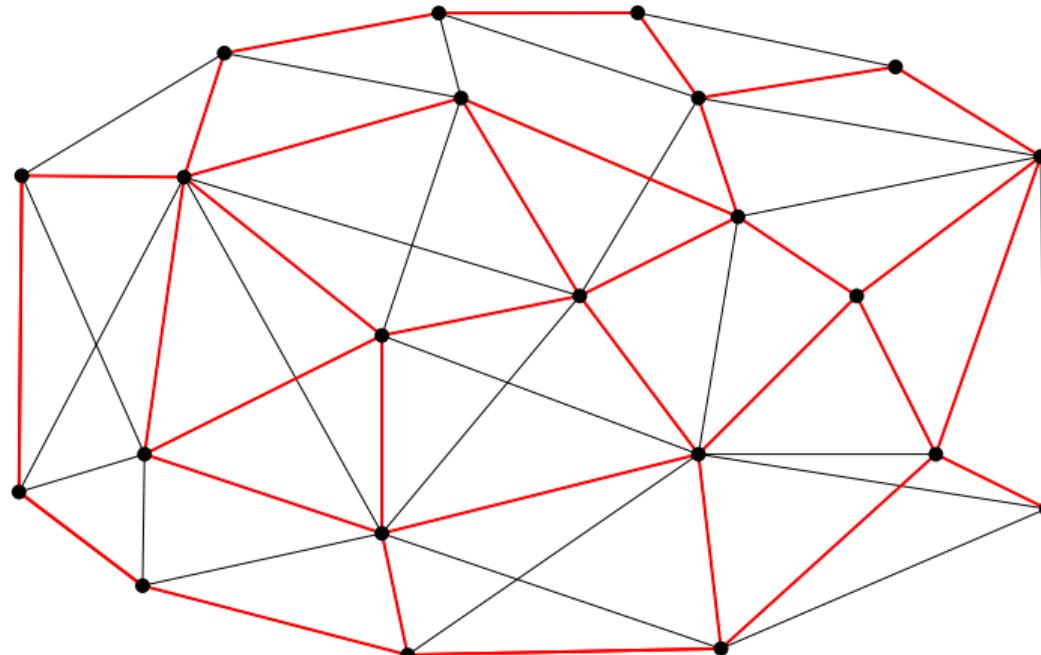


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Stretch

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Sparsity

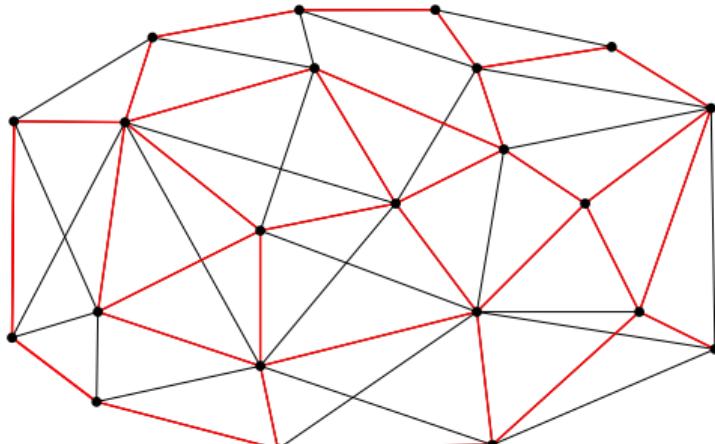
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[Althofer, Das, Dobkin, Joseph, Soares 93]:

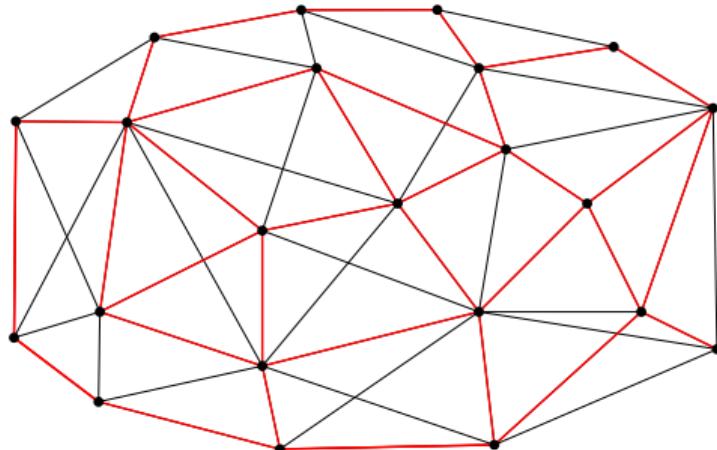
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- Approximation Algorithms (e.g. PTAS for TSP)
- Distributed Computing
- Network Routing
- Computational Biology (e.g. measure genetic distance)
- etc.

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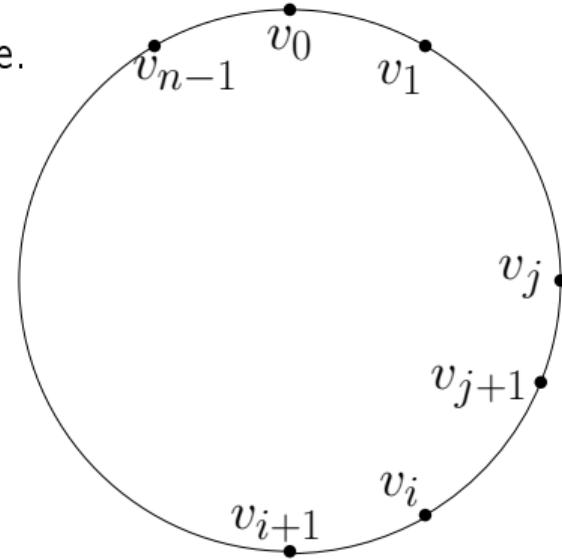
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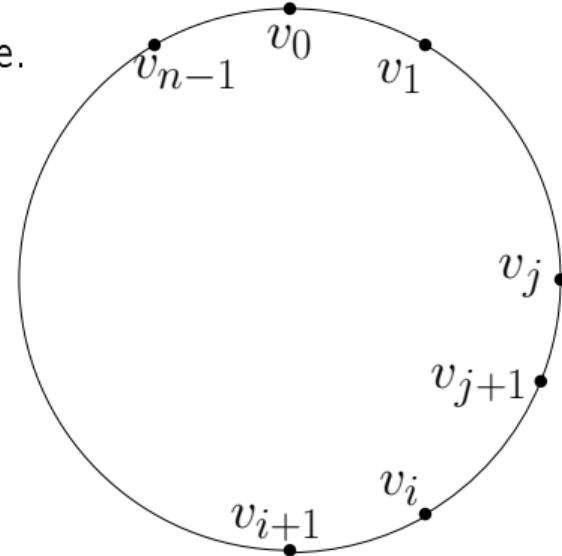


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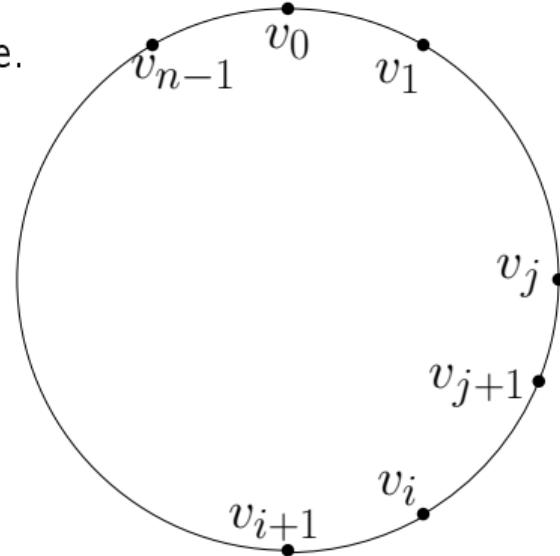
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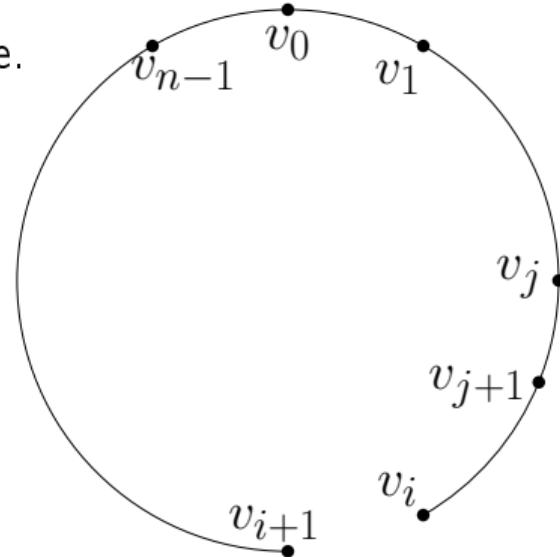


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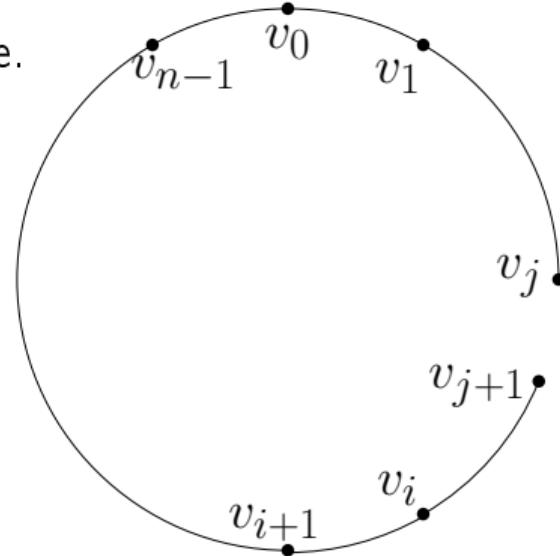
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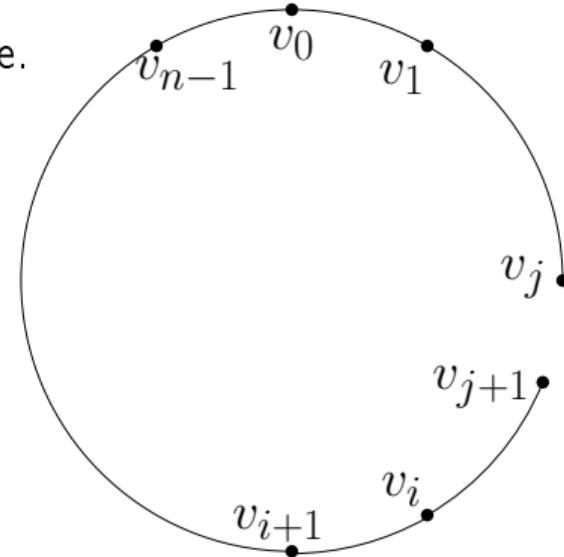
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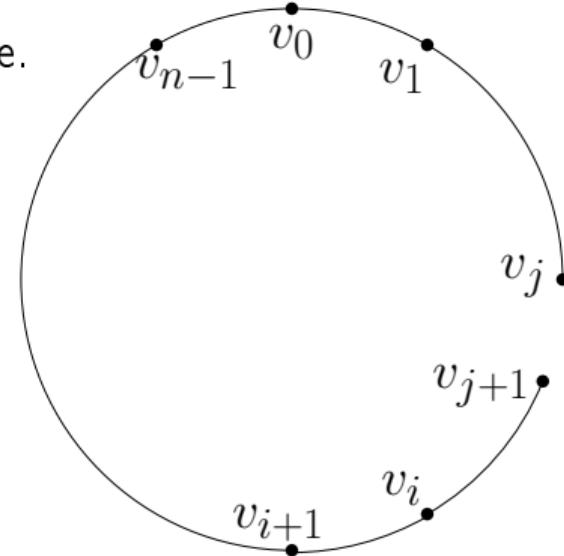
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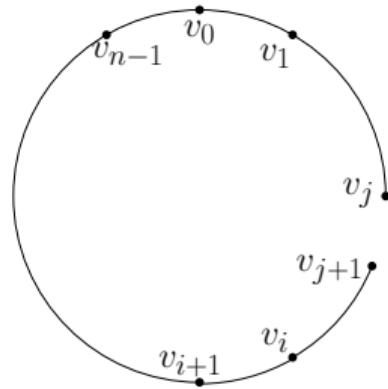
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By triangle inequality and linearity of expectation

$$\forall v_i, v_j, \quad \mathbb{E}_{T \sim \mathcal{D}}[d_T(v_i, v_j)] = \sum_{q=i}^{j-1} \mathbb{E}_{T \sim \mathcal{D}}[d_T(v_q, v_{q+1 \pmod n})] \leq 2 \cdot d_{C_n}(v_i, v_j).$$

Stochastic Embedding into Trees

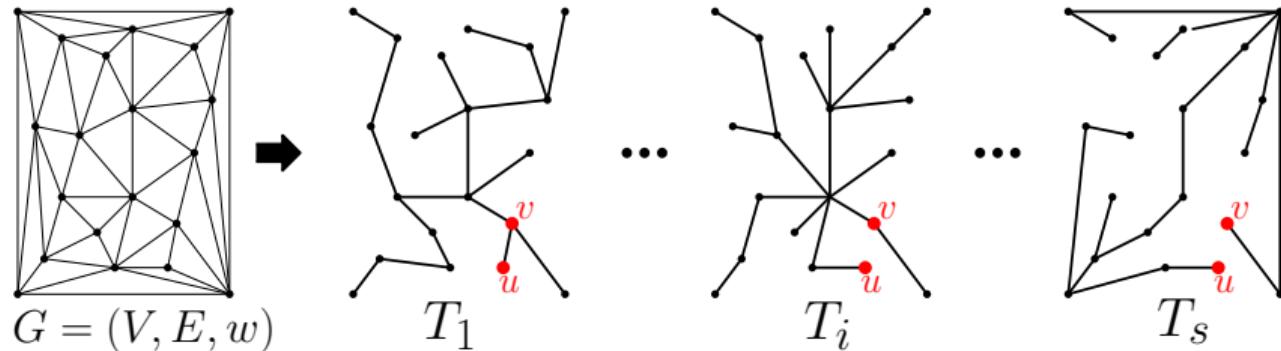
Theorem ([Fakcharoenphol, Rao, Talwar 04], improving [Bartal 96+98])

*Every n -point metric space (X, d) embeds into **distribution \mathcal{D}** over **dominating trees** with **expected distortion $O(\log n)$** .*

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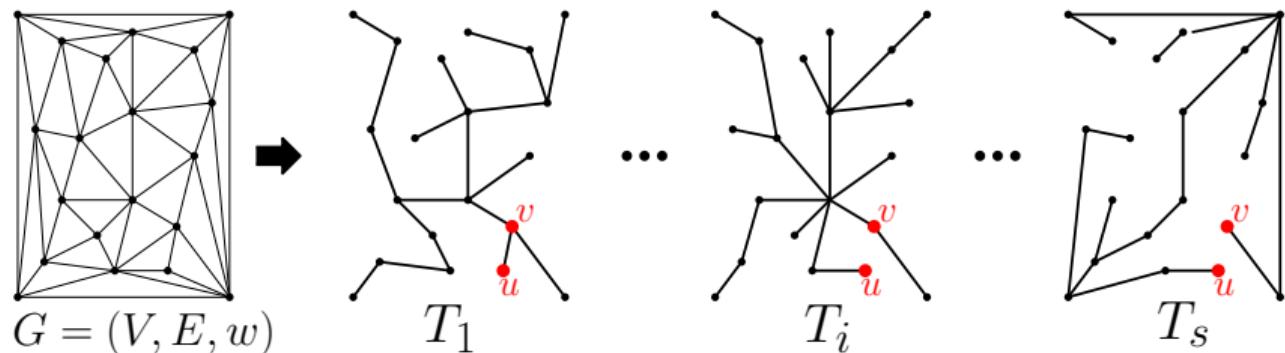
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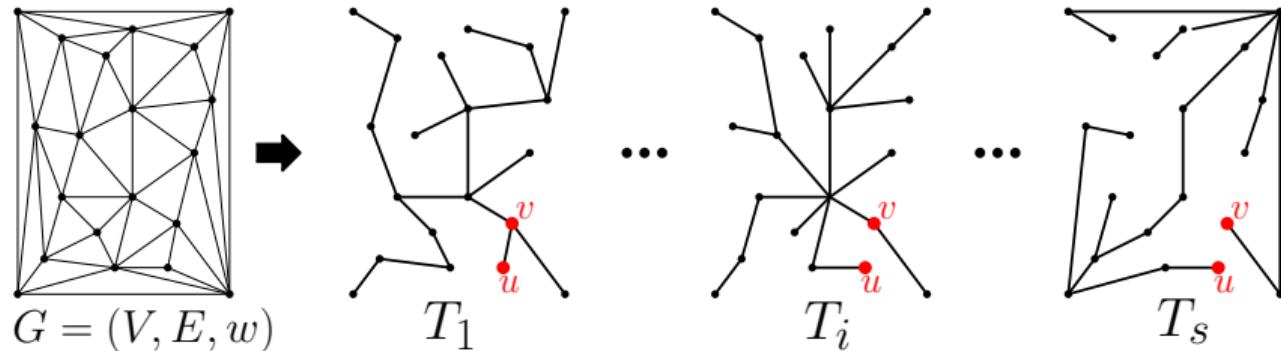


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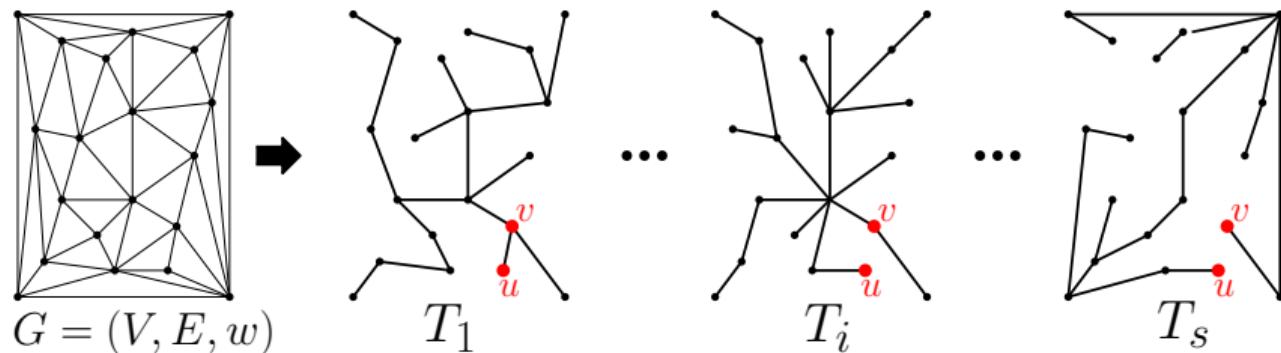
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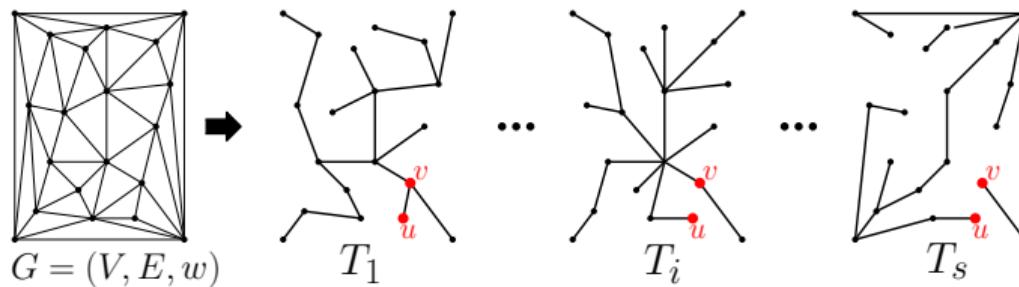
For every $u, v \in X$ $\mathbb{E}_{T \sim \mathcal{D}}[d_T(f(u), f(v))] \leq O(\log n) \cdot d_X(u, v)$.

[Bartal 96]: Tight!

Stochastic Embedding into Trees

Theorem ([Fakcharoenphol, Rao, Talwar 04], improving [Bartal 96+98])

*Every n -point metric space (X, d) embeds into **distribution \mathcal{D}** over **dominating trees** with **expected distortion $O(\log n)$** .*



A useful hammer



Transforms arbitrary metric into a tree!

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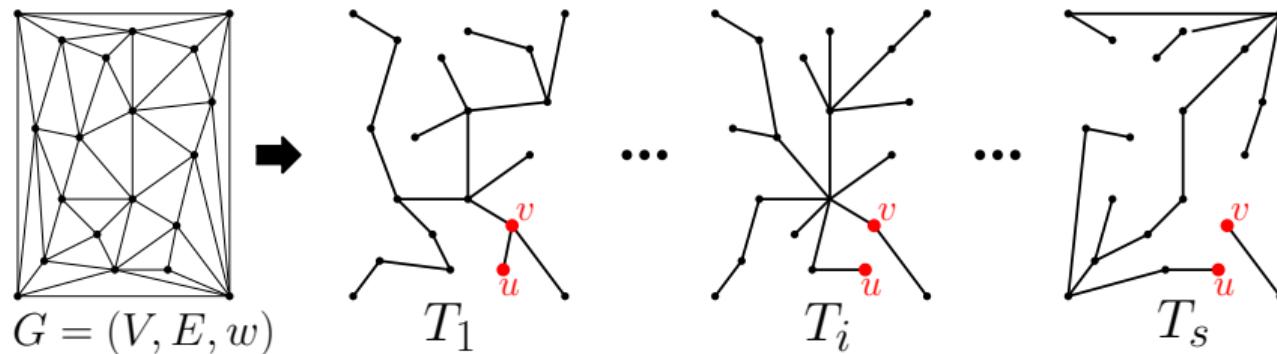
Applications:

- Approximation Algorithms.
- **Online Algorithms.**
- Distributed Computing.
- etc.

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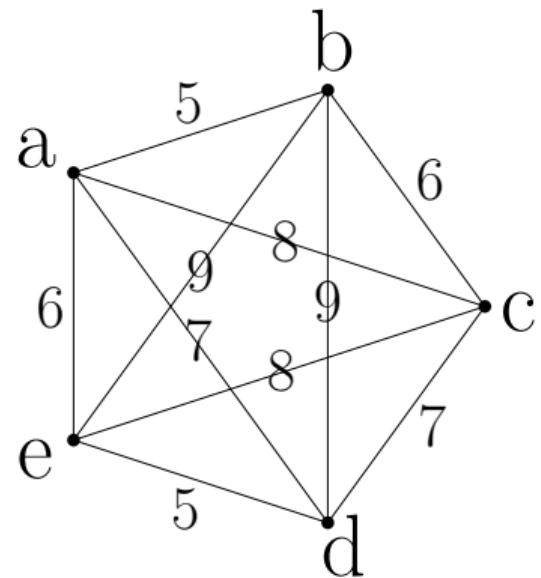


Compromises: Gurantee only in **expectation**



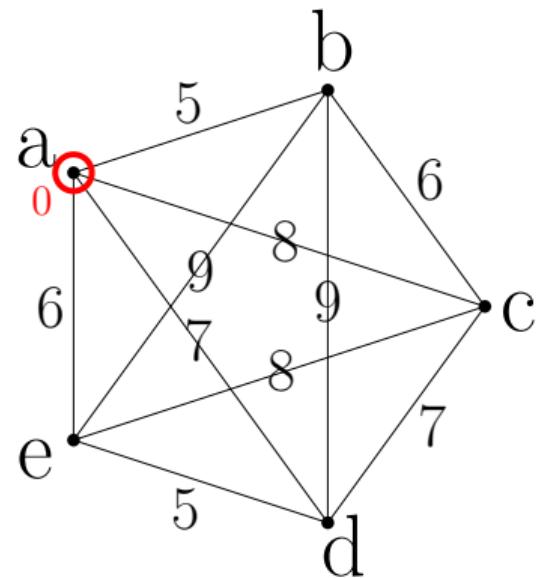
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Input: Metric space (X, d_X) . Initial configuration $x_0 \in X$.



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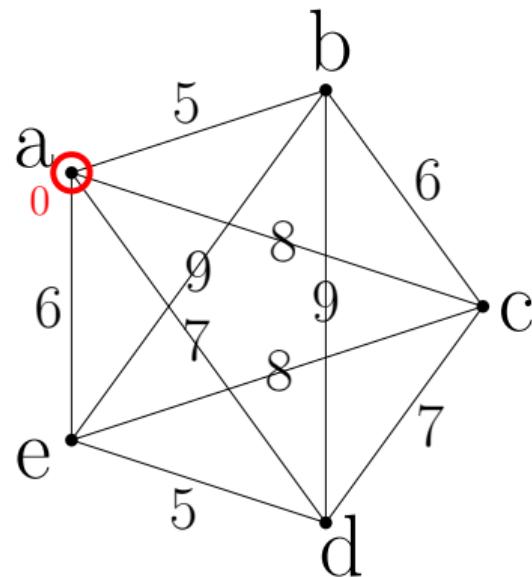
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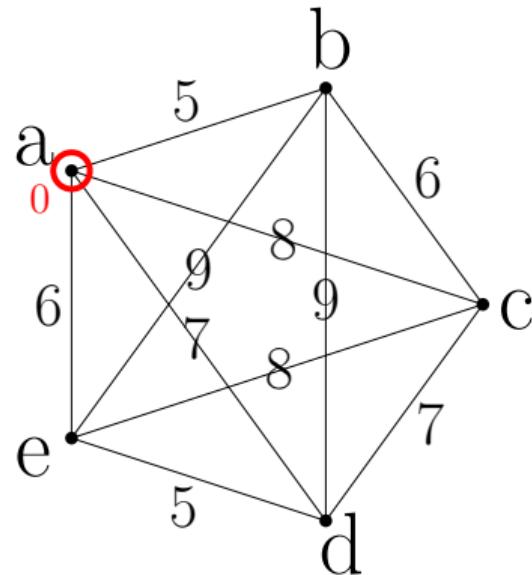
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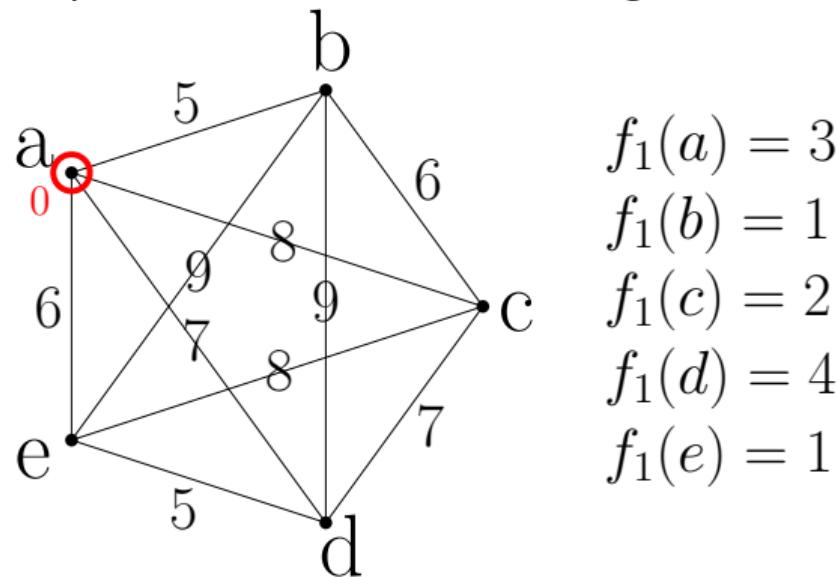
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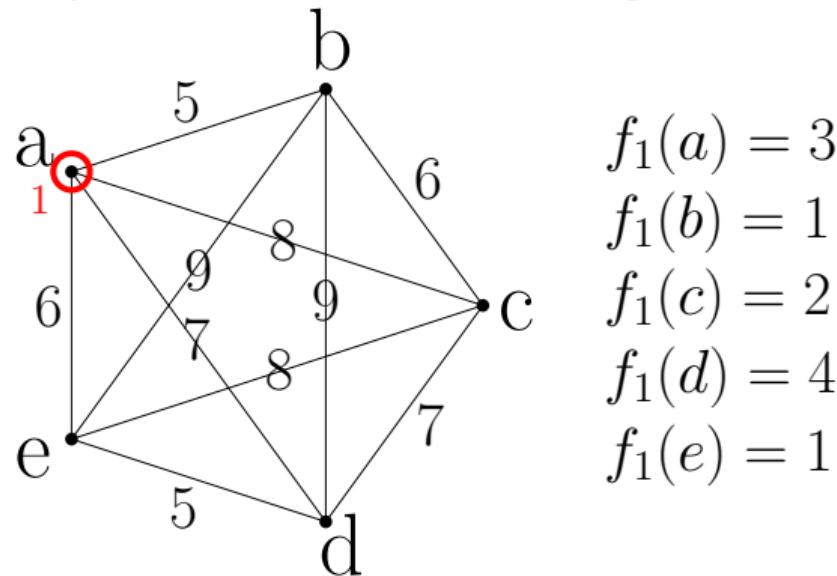
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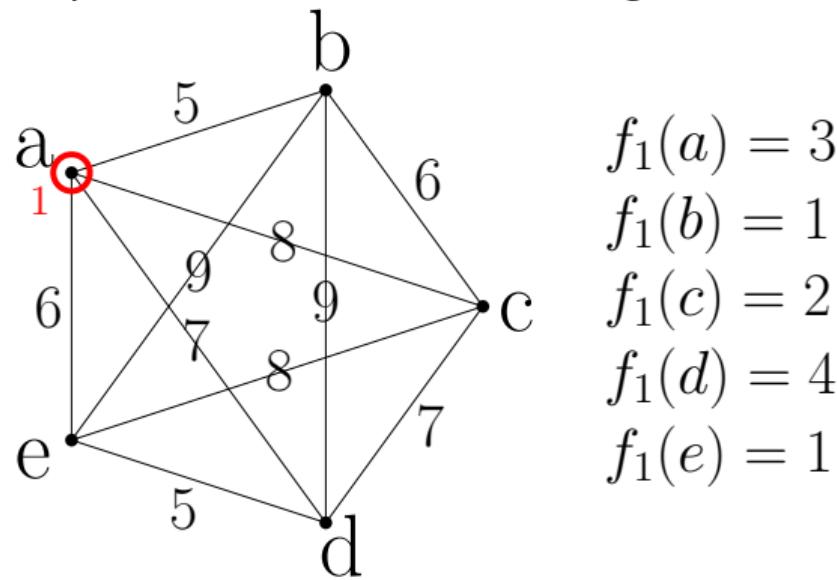
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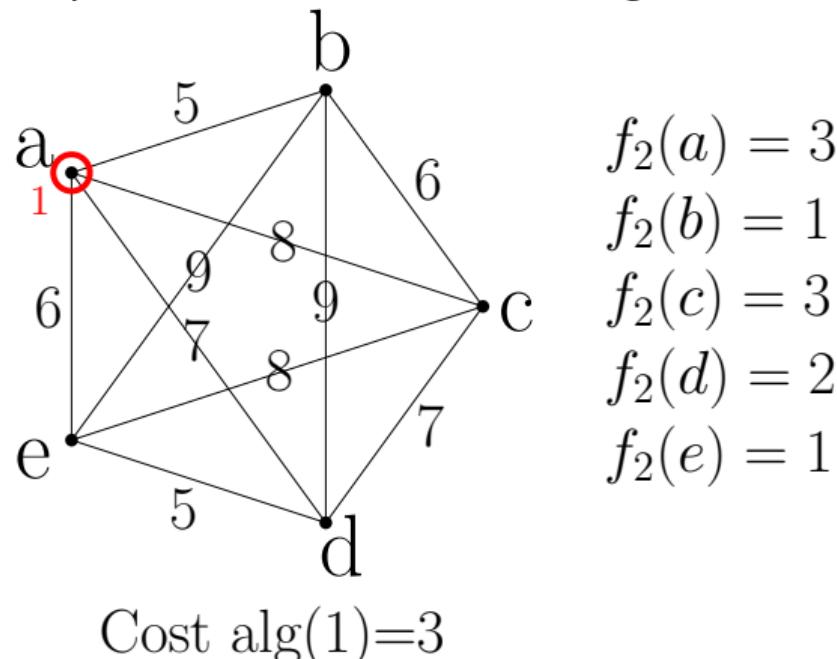
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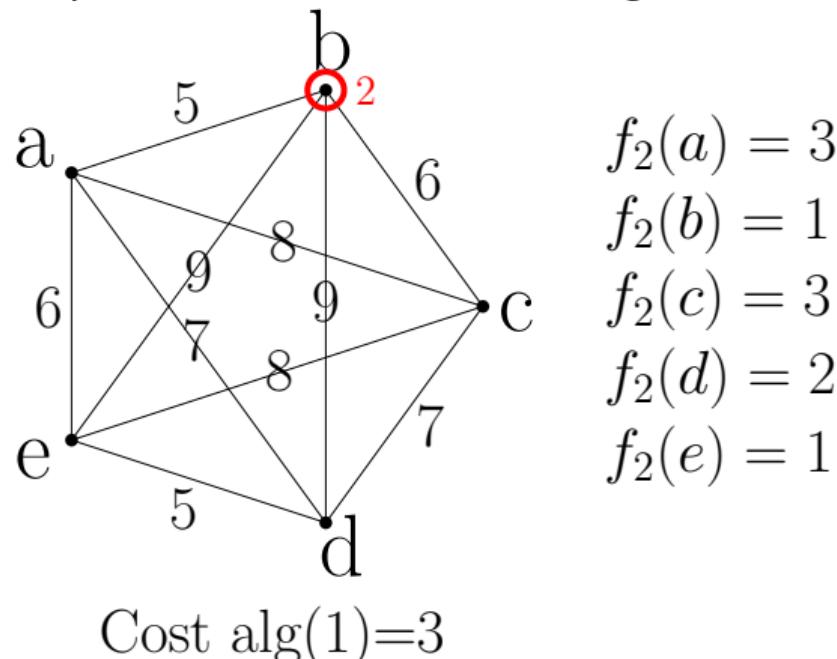
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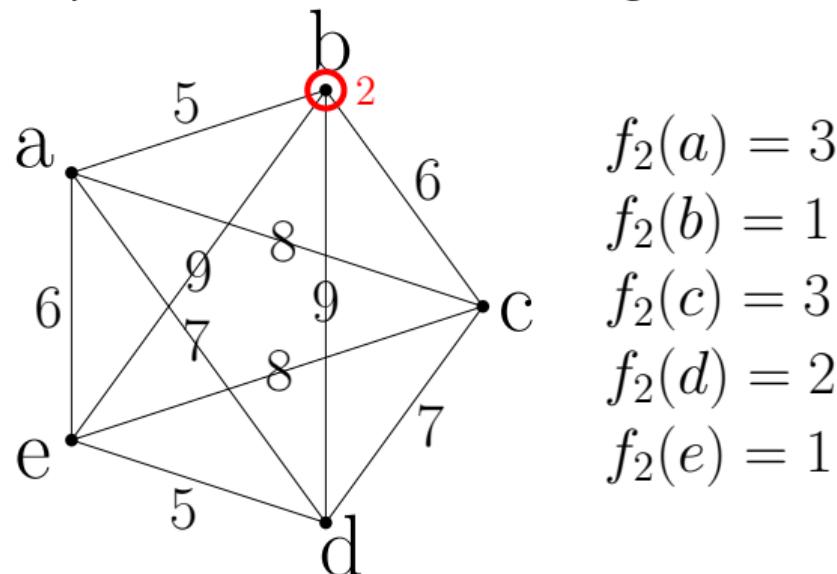
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$$\text{Cost alg}(2) = 3 + 5 + 1 = 9$$

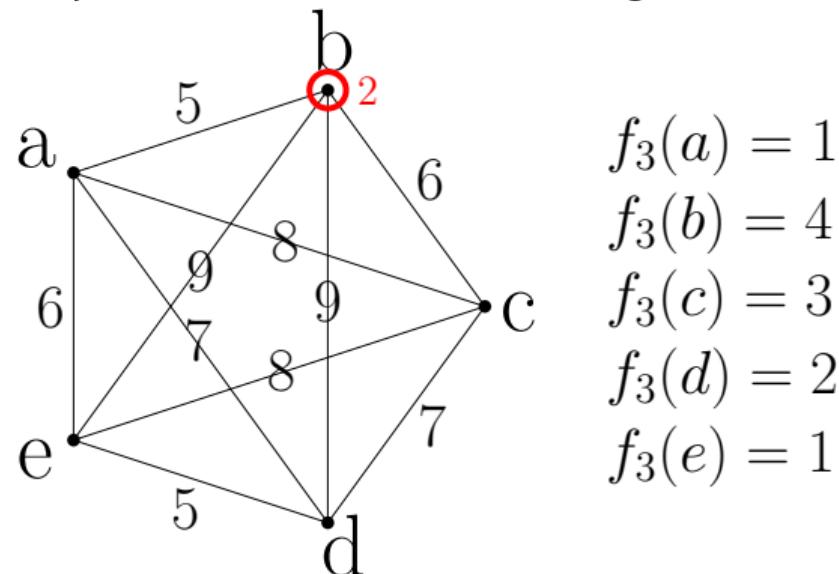
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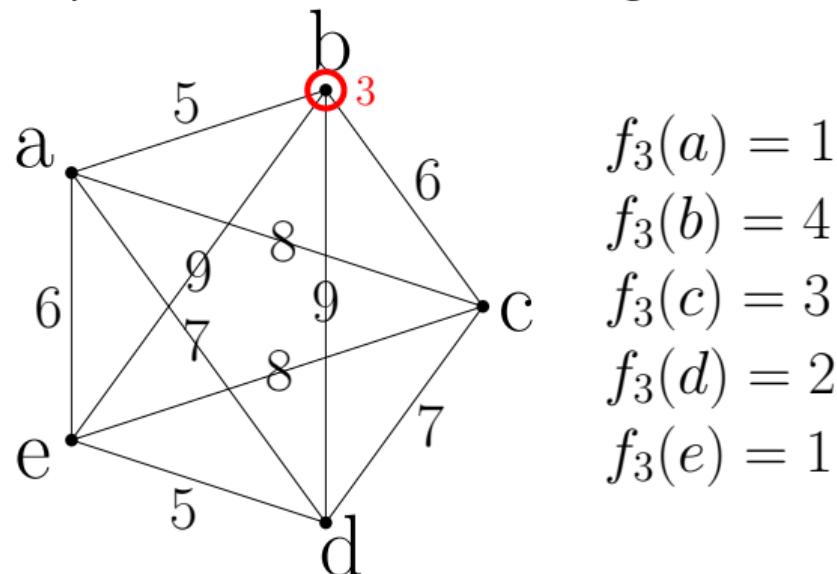
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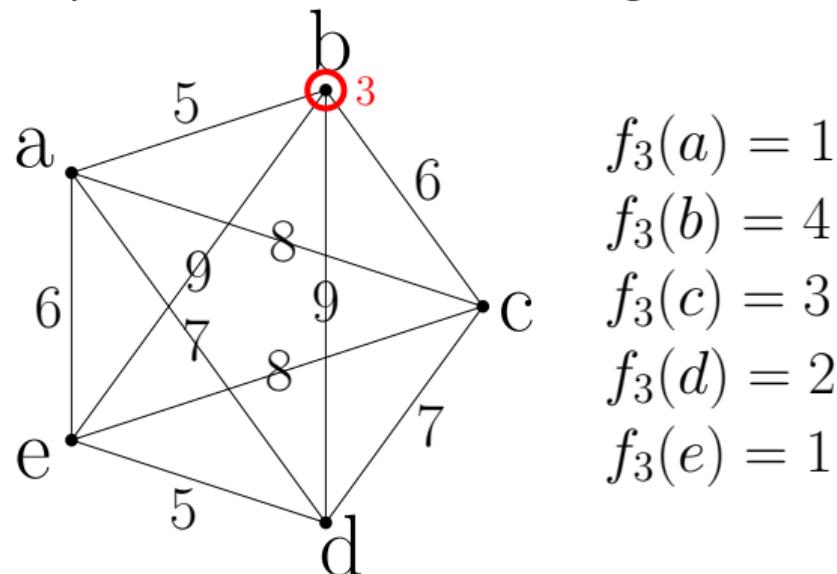
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$$\text{Cost alg}(3) = 9 + 4 = 13$$

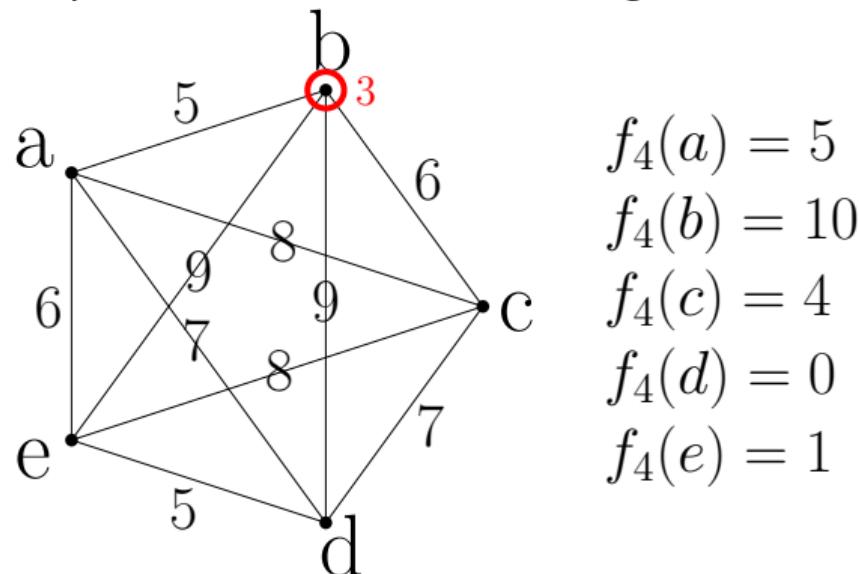
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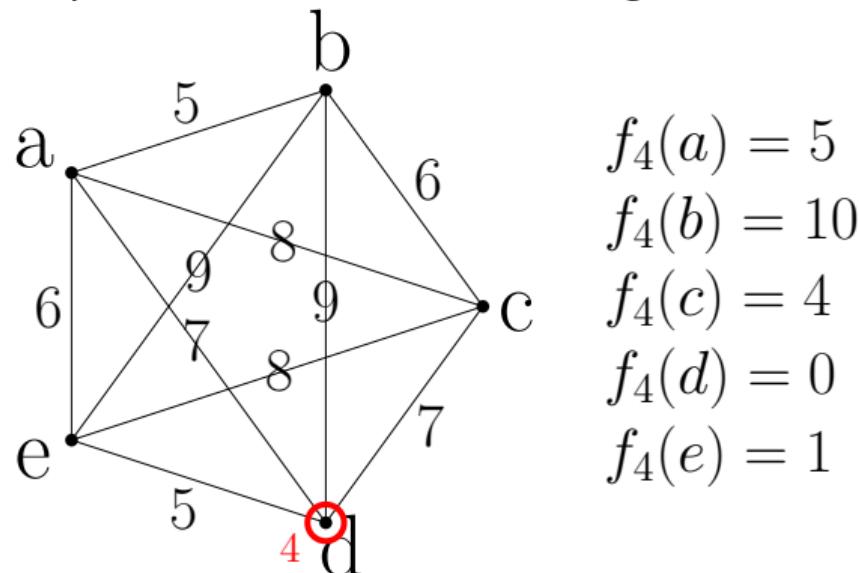
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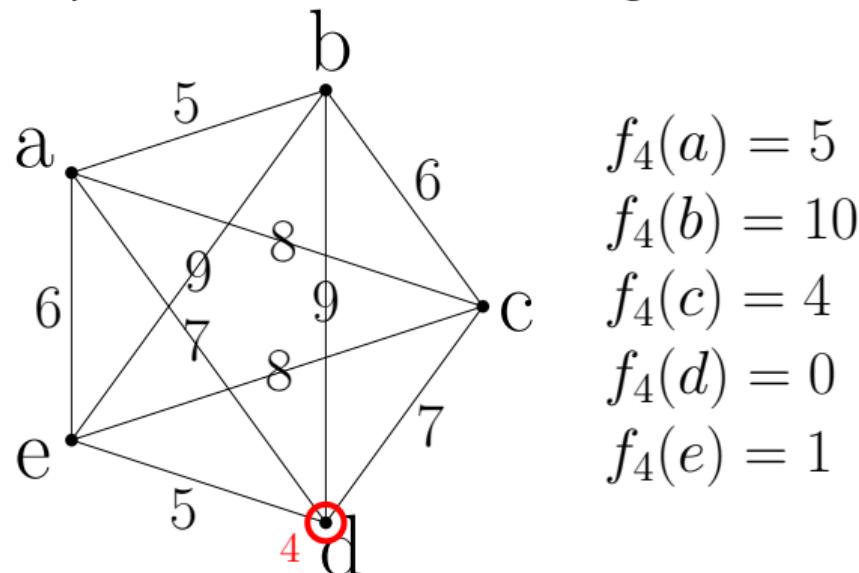
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$$\text{Cost alg}(4) = 13 + 9 + 0 = 22$$

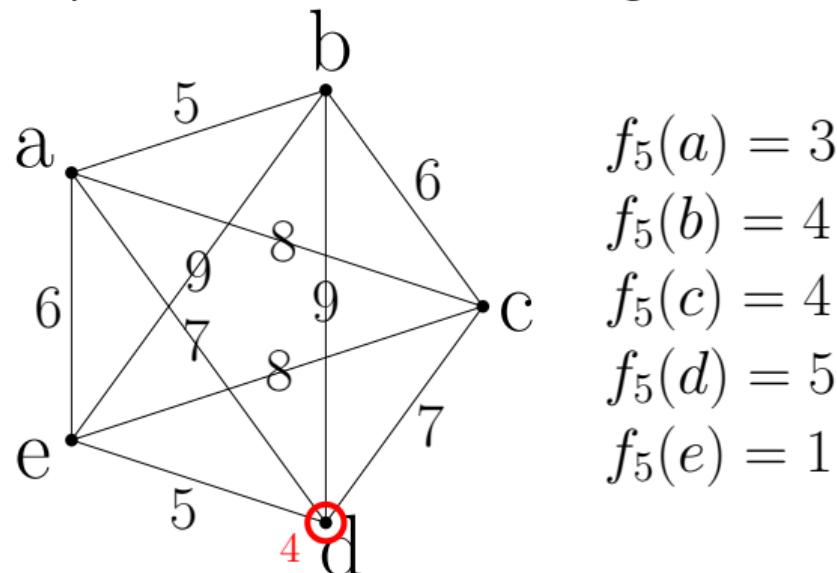
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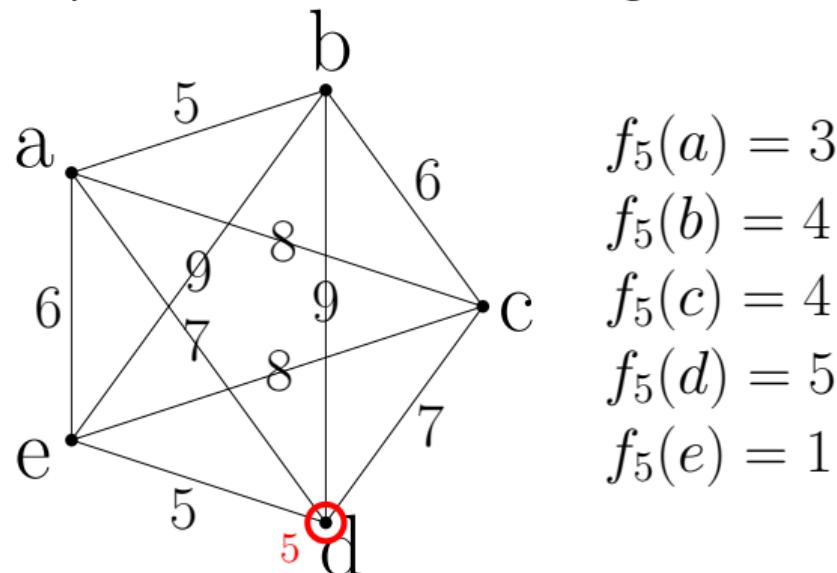
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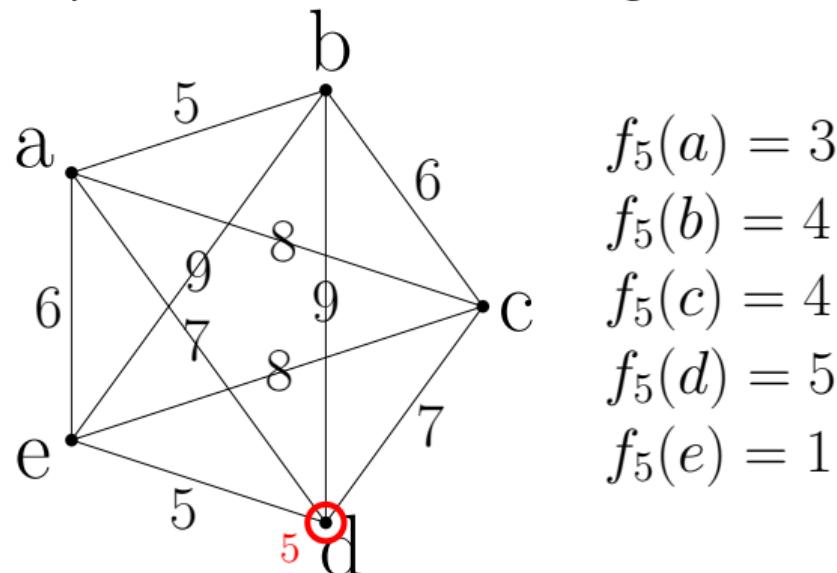
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$$\text{Cost alg}(5) = 22 + 5 = 27$$

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What about Opt?

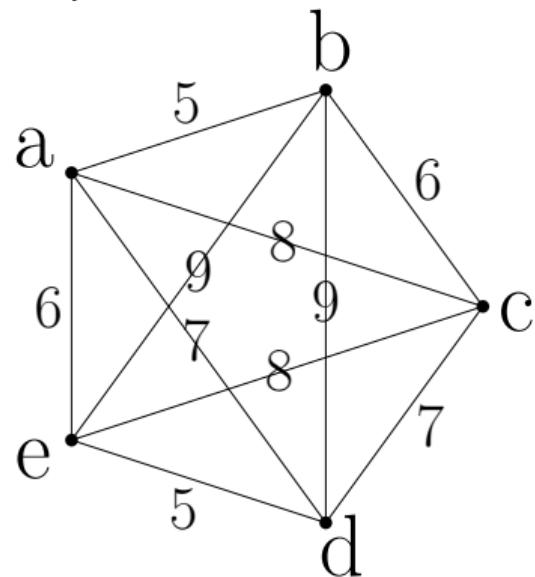
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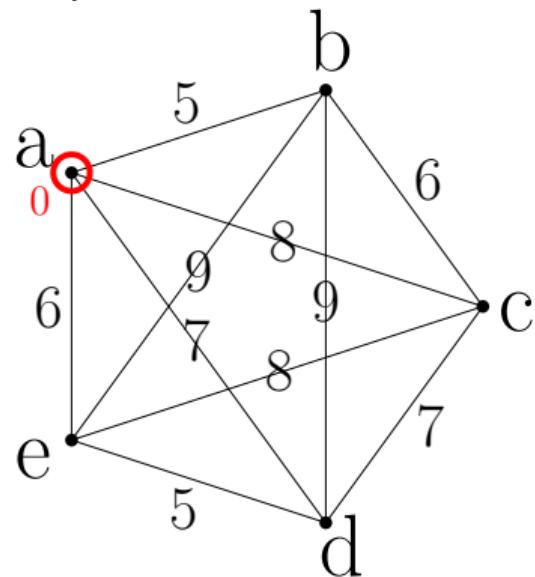
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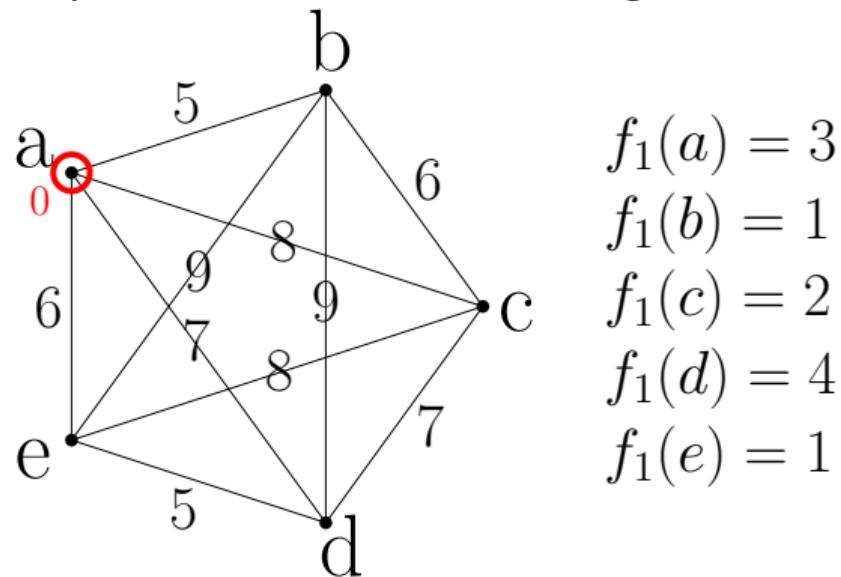
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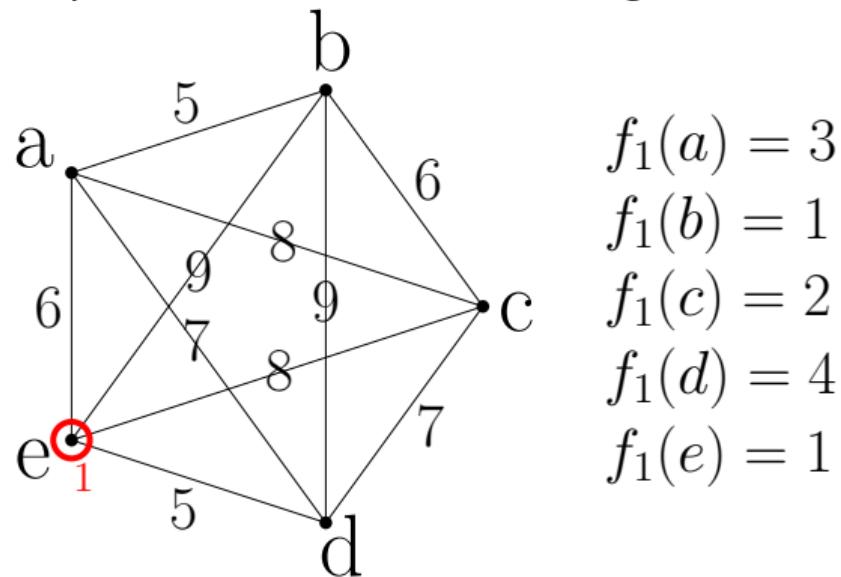
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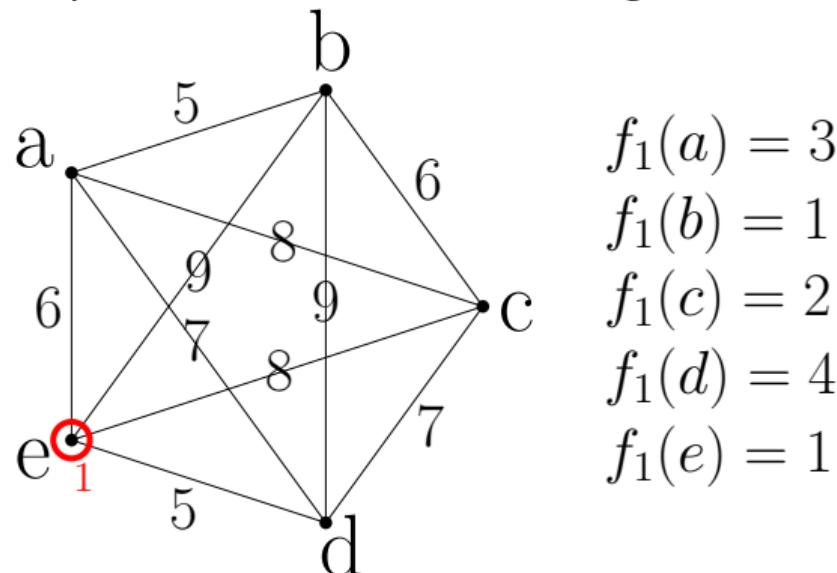
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$$\text{Cost opt}(1) = 6 + 1 = 7$$

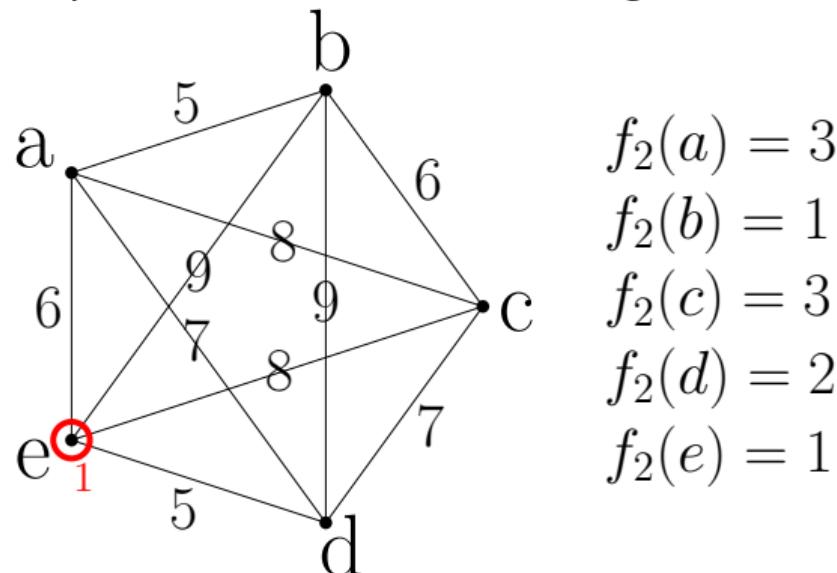
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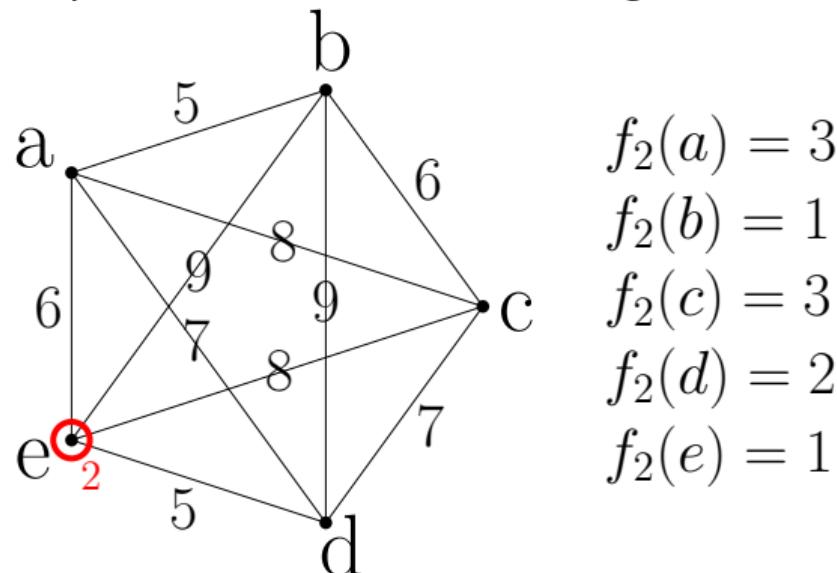
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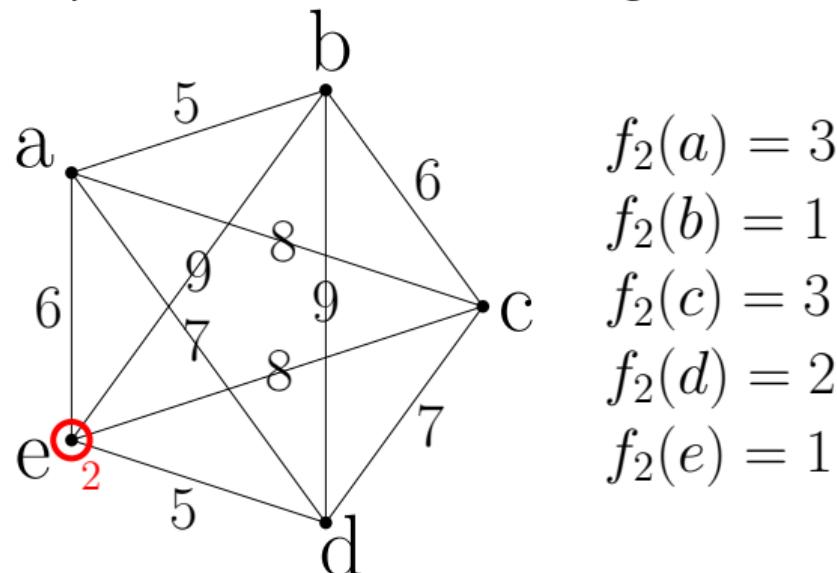
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$$\text{Cost opt}(2) = 7 + 1 = 8$$

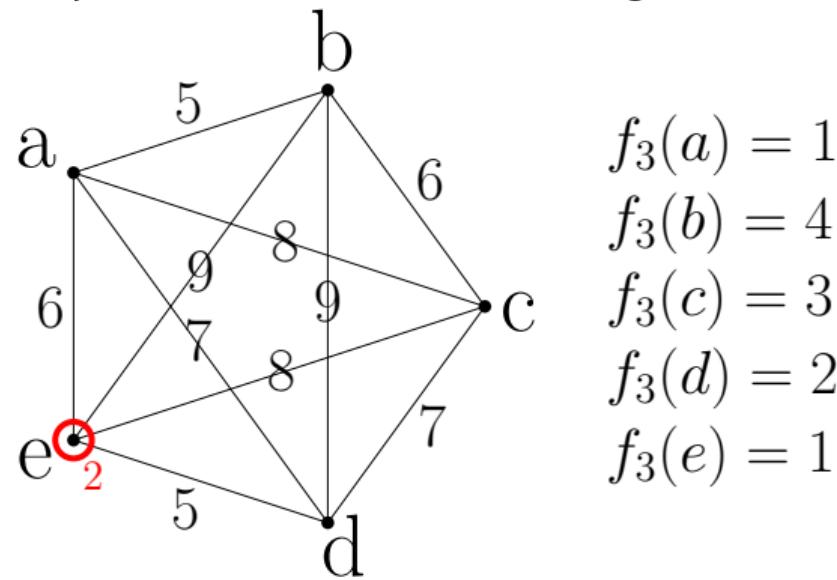
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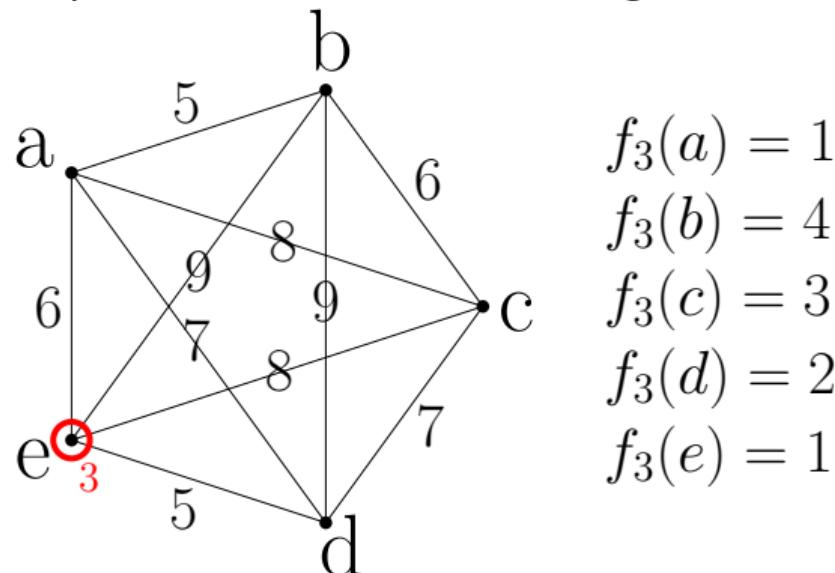
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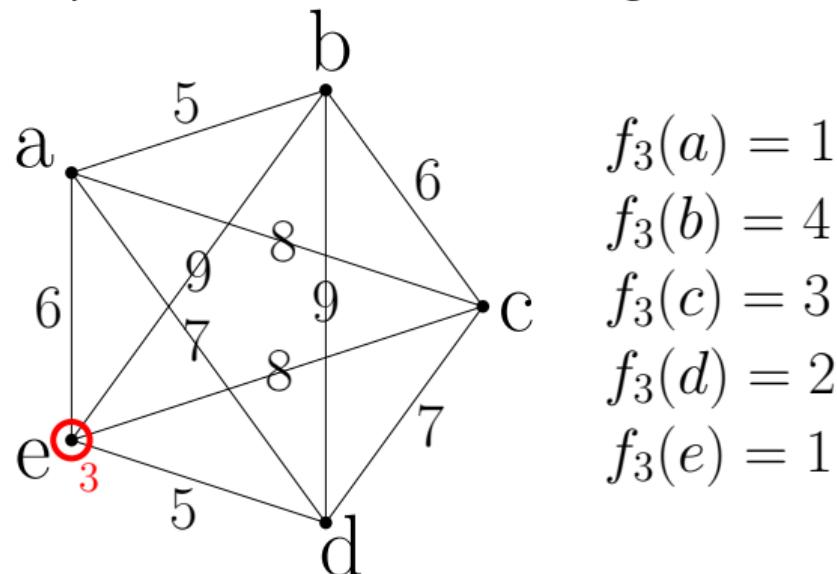
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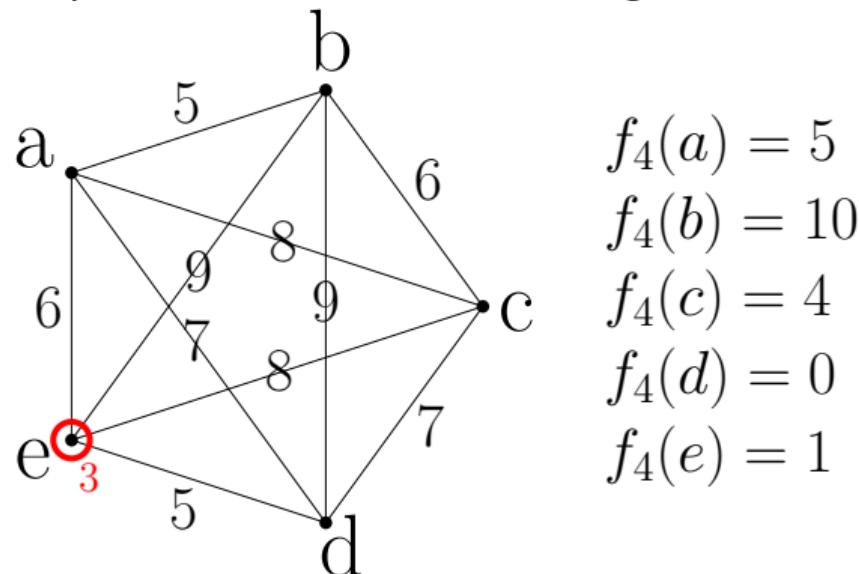
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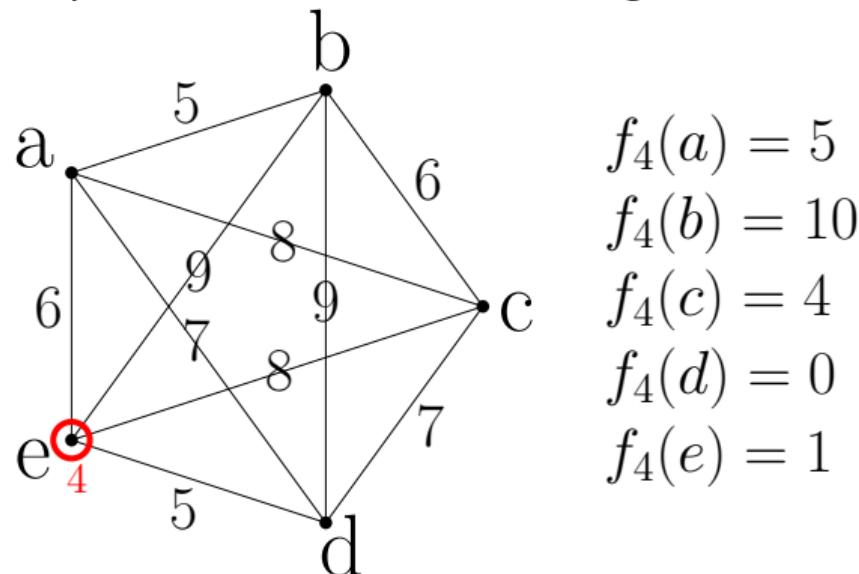
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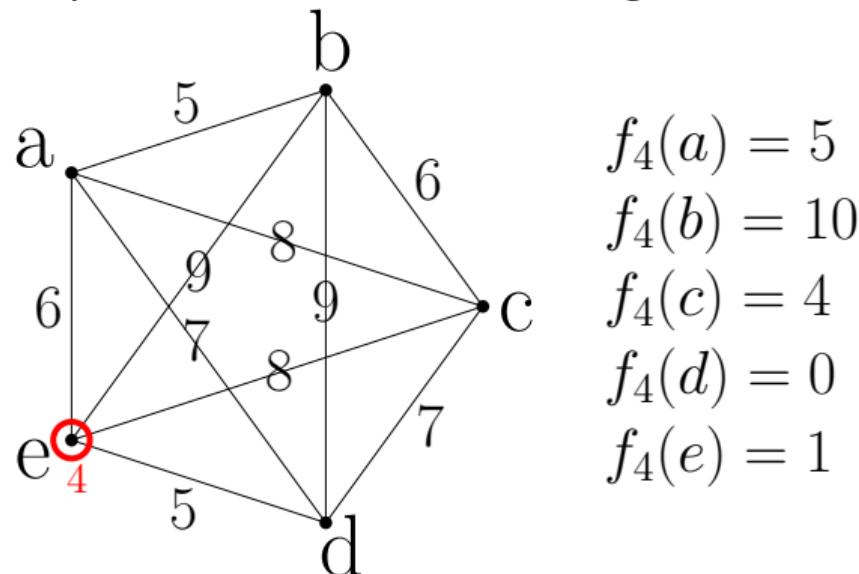
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$$\text{Cost opt}(4) = 9 + 1 = 10$$

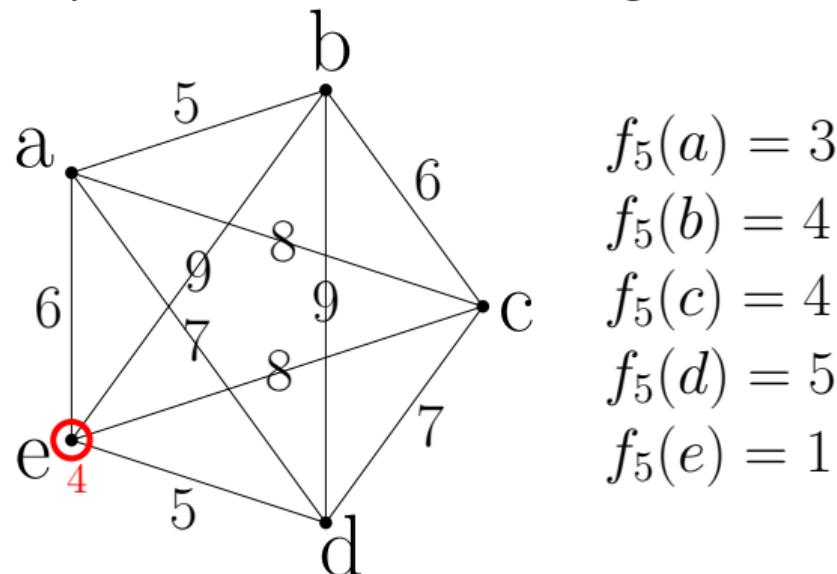
Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X) . Initial configuration $x_0 \in X$.

At step i , a task arrives with cost function $f_i : X \rightarrow \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$$\text{Cost opt}(4) = 9 + 1 = 10$$

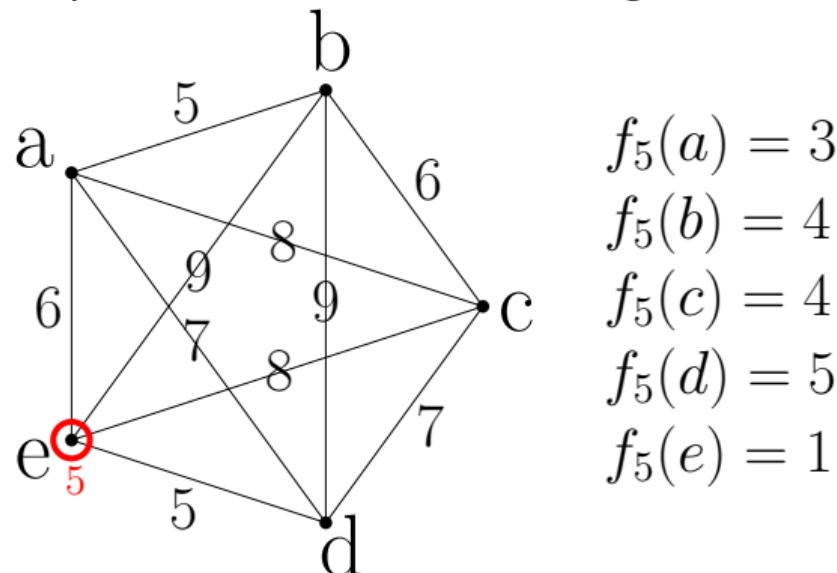
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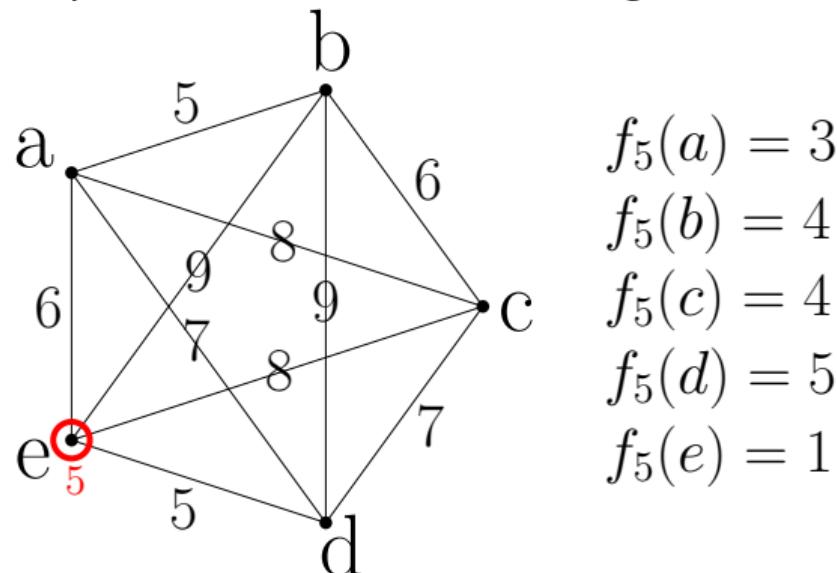
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$$\text{Cost opt}(5) = 10 + 1 = 11$$

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$$\text{Competitive ratio} = \max_{\text{input } I} \frac{\text{Alg}(I)}{\text{opt}(I)}.$$

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$$\text{Competitive ratio} = \max_{\text{input } I} \frac{\text{Alg}(I)}{\text{opt}(I)}.$$

Here $\frac{\text{Alg}(I)}{\text{opt}(I)} = \frac{27}{11}$.

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$$\text{Here } \frac{\text{Alg}(I)}{\text{opt}(I)} = \frac{27}{11}.$$

Competitive ratio against oblivious adversary is

$$\max_{\text{input } I} \frac{\mathbb{E}[\text{Alg}(I)]}{\text{opt}(I)}.$$

Online problem - Metrical Task System (MTS)

Approach: embed into a tree, and then make all the decisions based on the tree.

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Theorem ([Fiat, Mendel 2000])

Given an n point tree T , there is an online algorithm for MTS with competitive ratio $O(\log n \cdot \log \log n)$ against oblivious adversary.*

* Actually on an HST, which is a special kind of tree. [FRT04] is into HST's.

Online problem - Metrical Task System (MTS)

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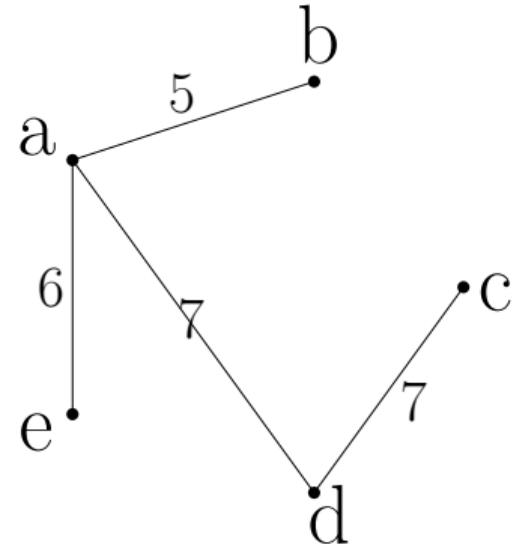
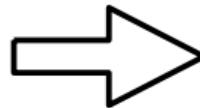
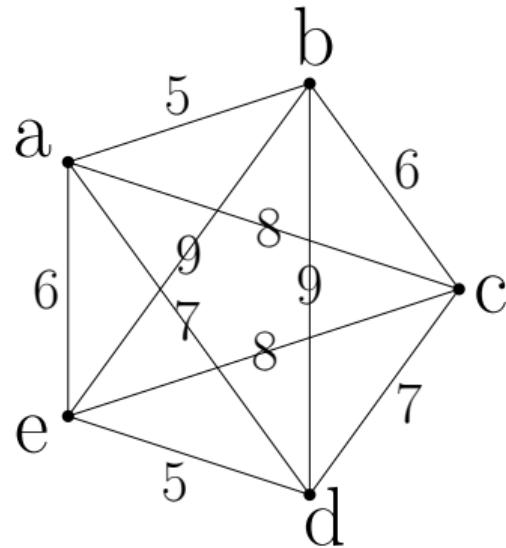
Algorithm: 1. Sample a tree T over (X, d_X) using [FRT04].

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Make the same decisions as [FM00].

Online problem - Metrical Task System (MTS)

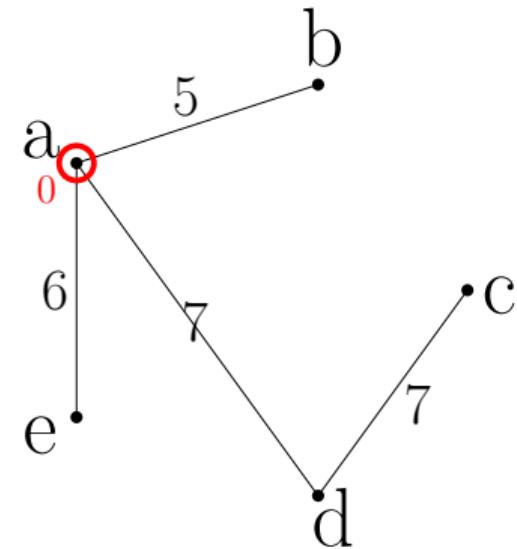
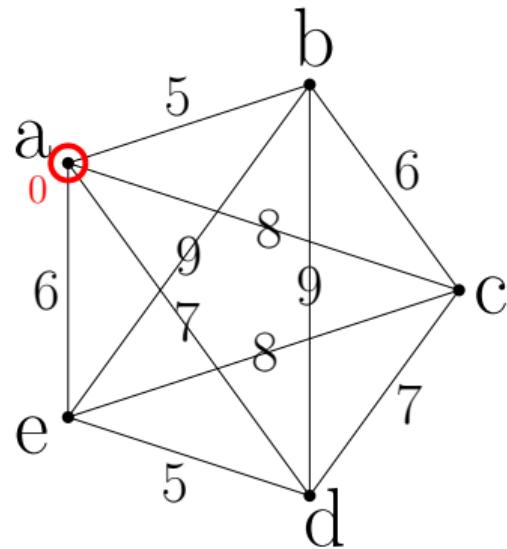
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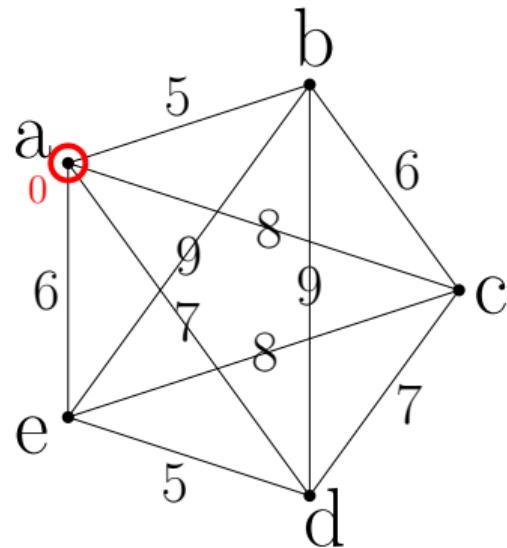
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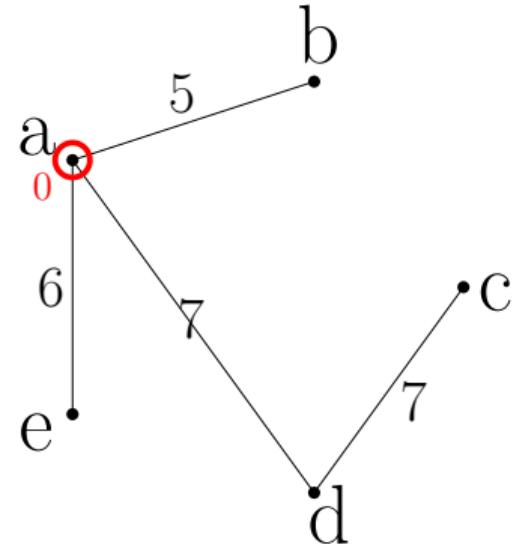
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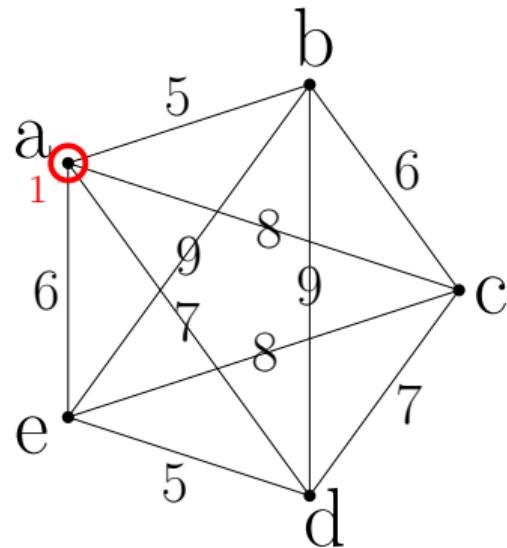
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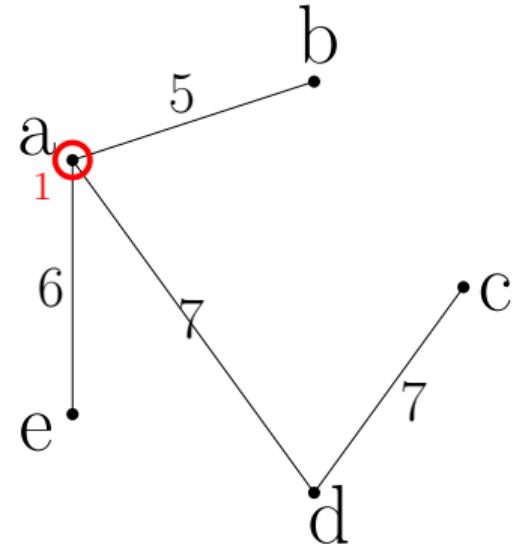
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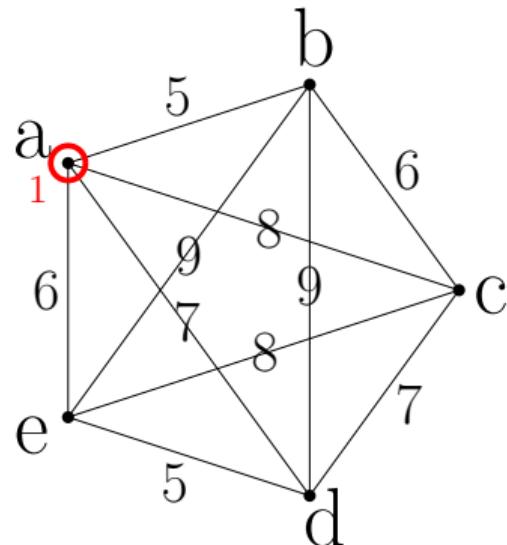


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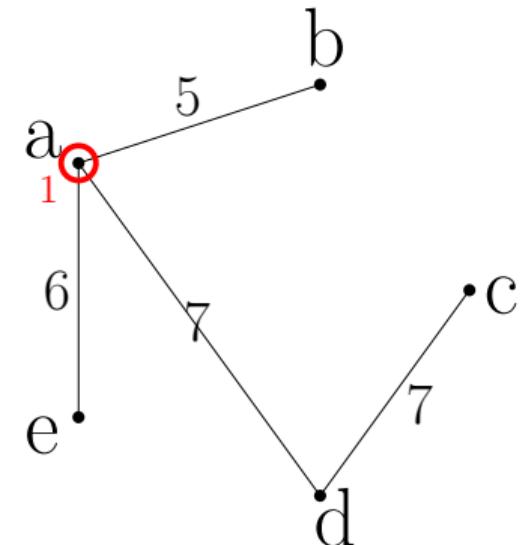
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Cost alg(1)=3

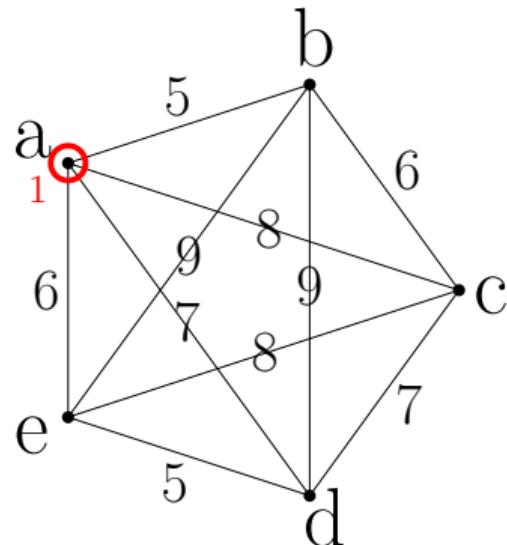
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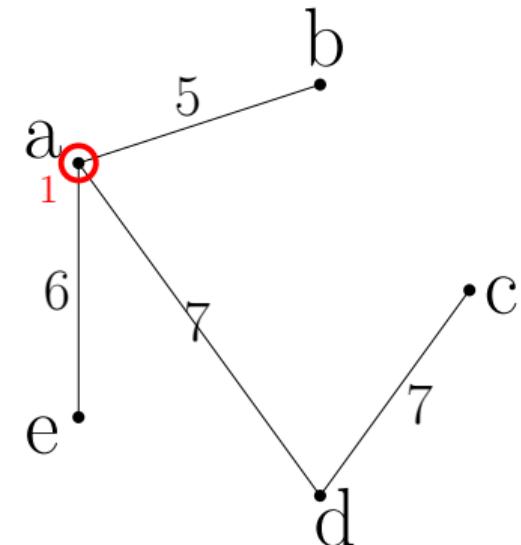
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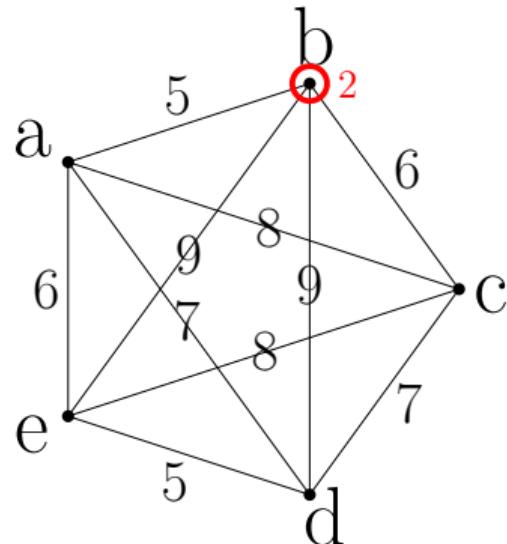


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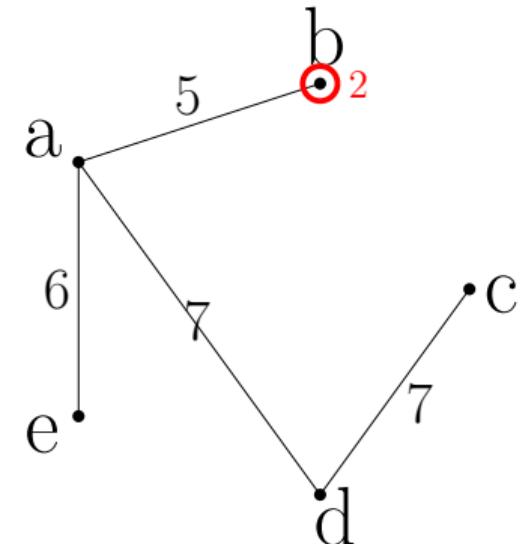
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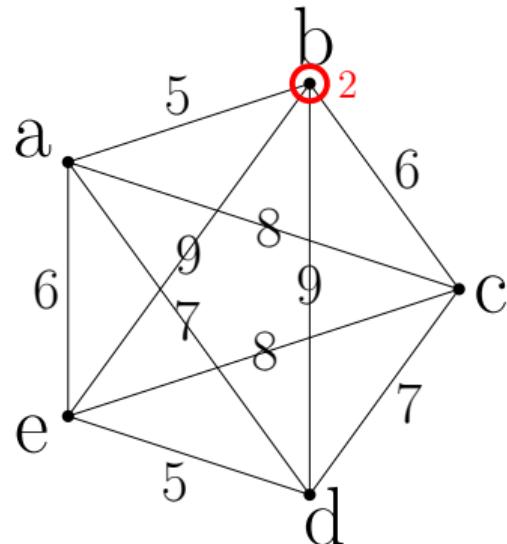


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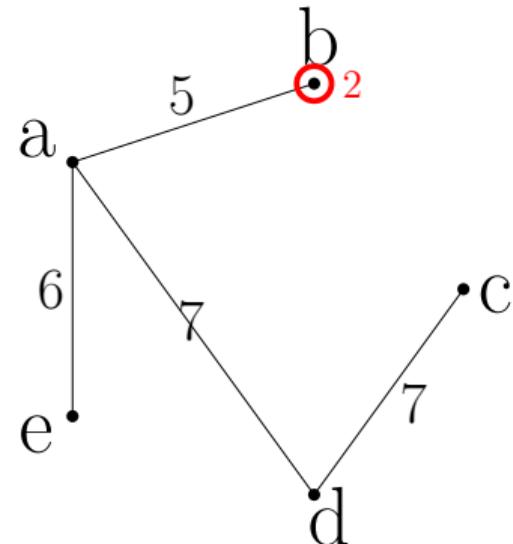
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$$\text{Cost alg}(2) = 3 + 5 + 1 = 9$$

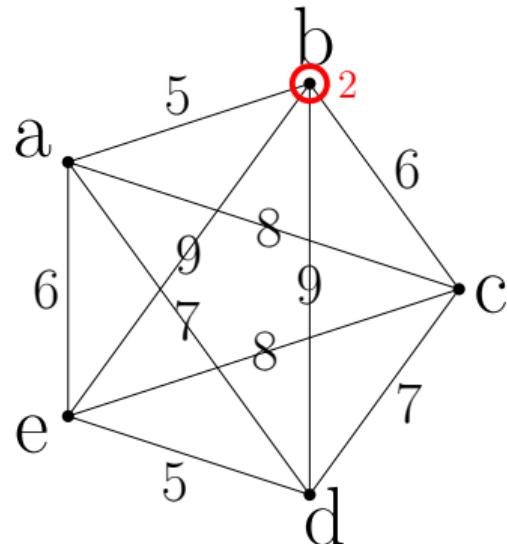


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Online problem - Metrical Task System (MTS)

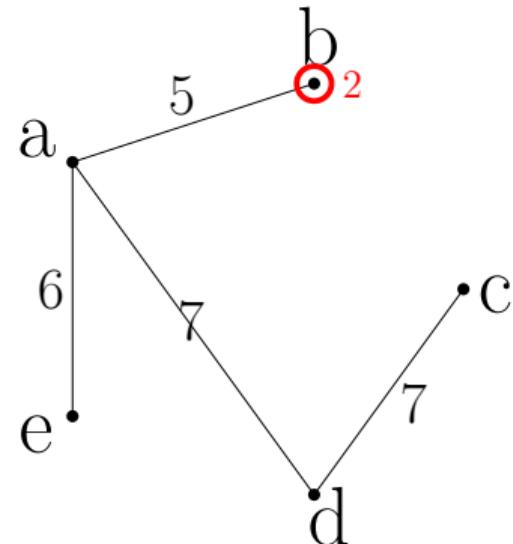
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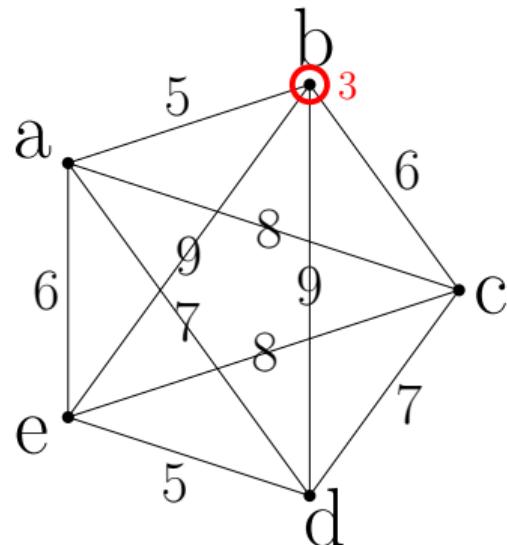


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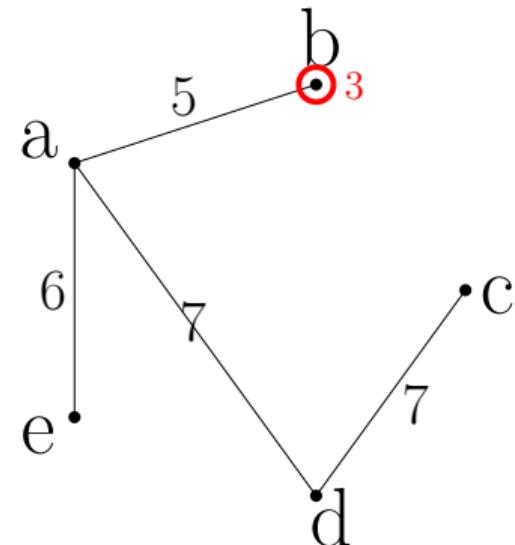
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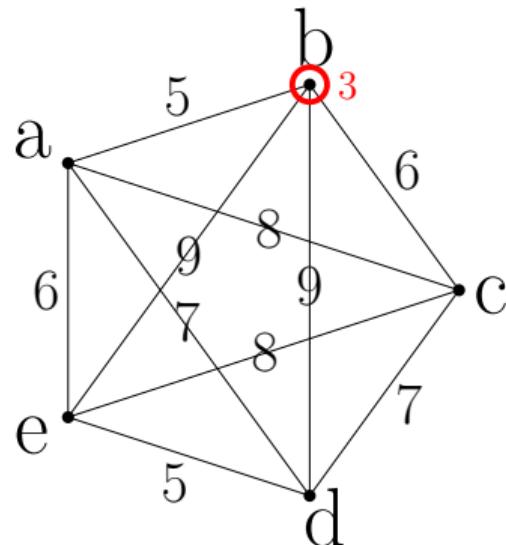
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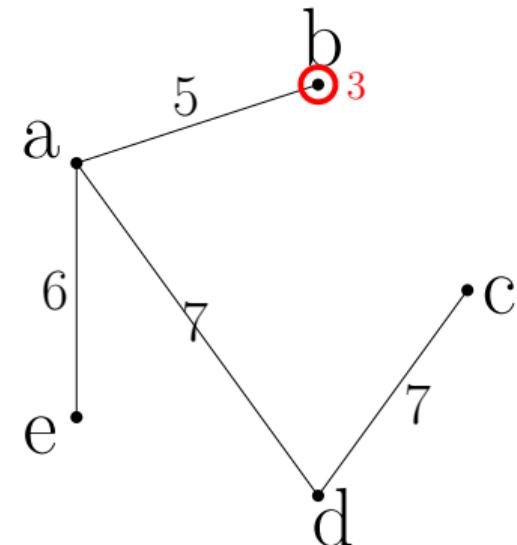
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$$\text{Cost alg}(3)=9 + 4 = 13$$

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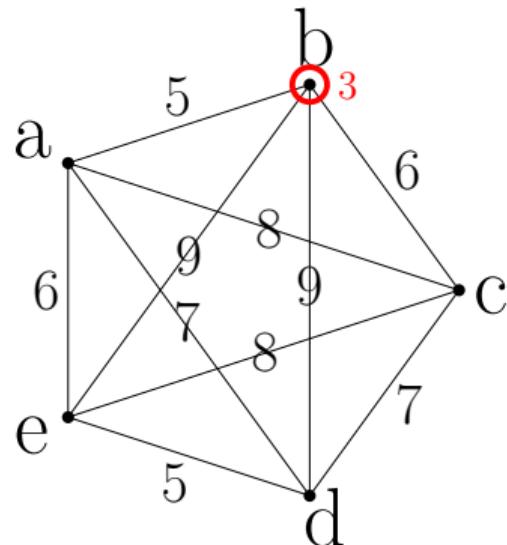


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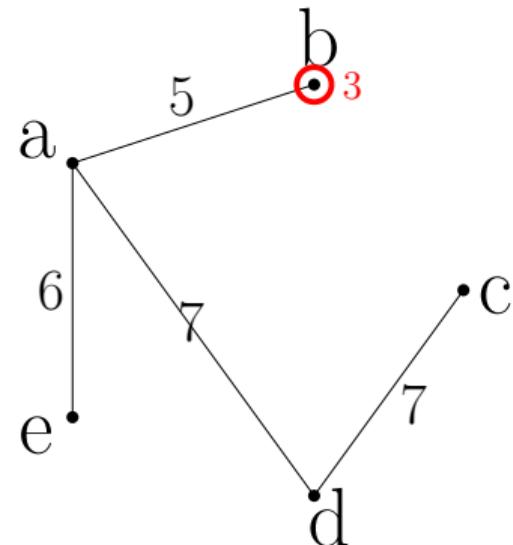
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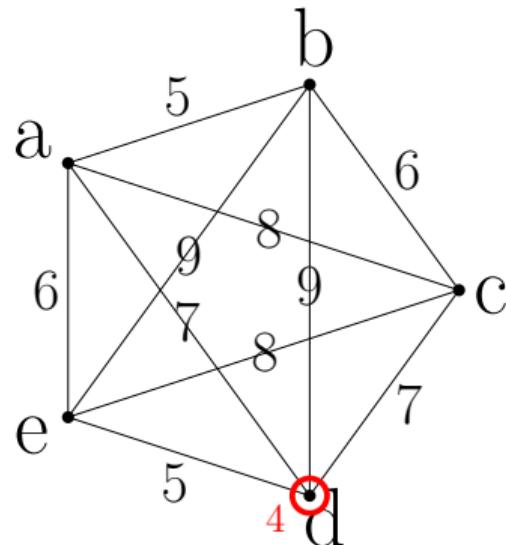


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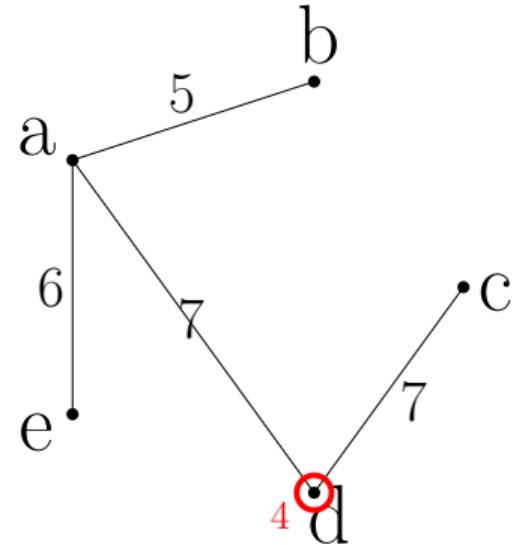
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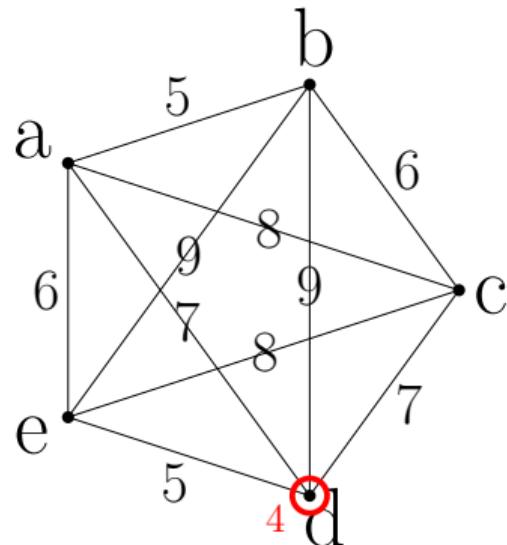


$$\text{Cost alg}_T(3) = 9 + 4 = 13$$

Online problem - Metrical Task System (MTS)

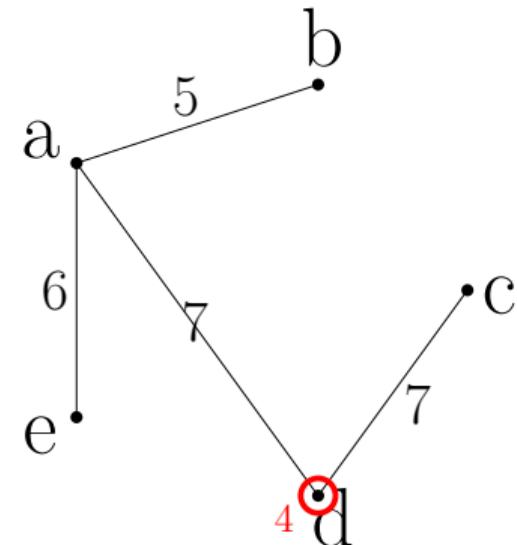
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1. Sample a tree T over (X, d_X) using [FRT04].
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Make the same decisions as [FM00].



$$\text{Cost alg}(4) = 13 + 9 + 0 = 22$$

$$\begin{aligned}f_4(a) &= 5 \\f_4(b) &= 10 \\f_4(c) &= 4 \\f_4(d) &= 0 \\f_4(e) &= 1\end{aligned}$$

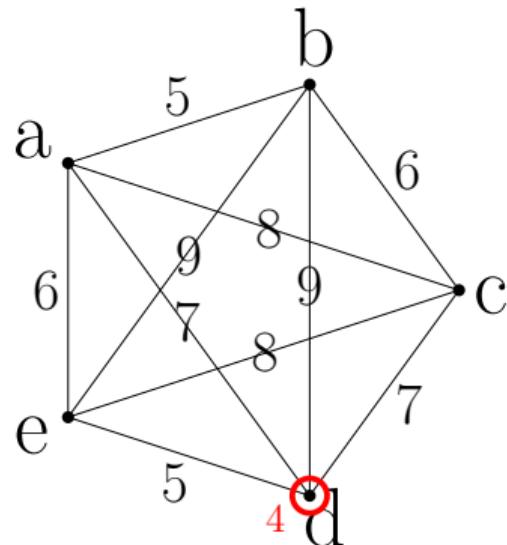


$$\text{Cost alg}_T(4) = 13 + 12 + 0 = 25$$

Online problem - Metrical Task System (MTS)

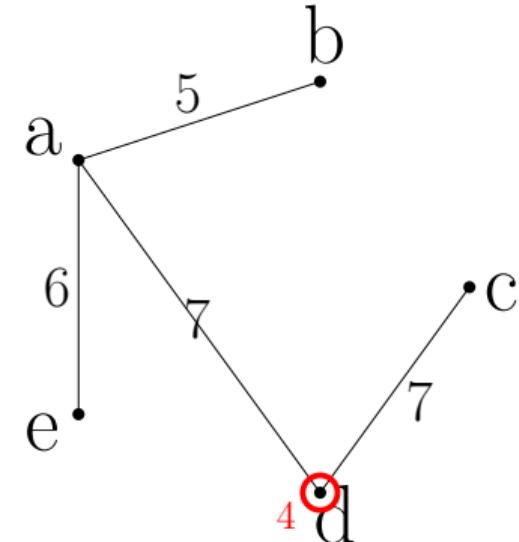
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$$\text{Cost alg}(4) = 13 + 9 + 0 = 22$$

$$\begin{aligned}f_5(a) &= 3 \\f_5(b) &= 4 \\f_5(c) &= 4 \\f_5(d) &= 5 \\f_5(e) &= 1\end{aligned}$$

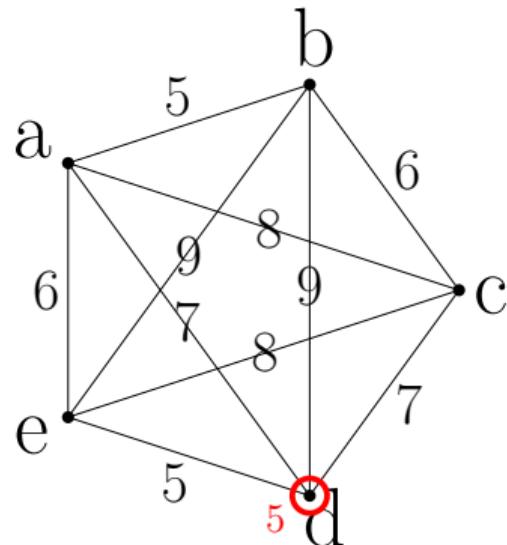


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Online problem - Metrical Task System (MTS)

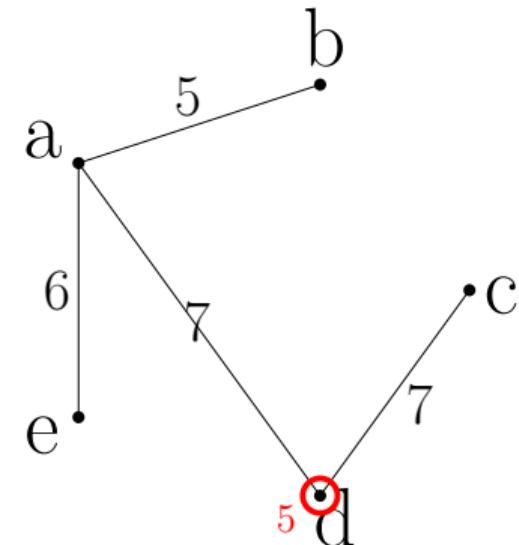
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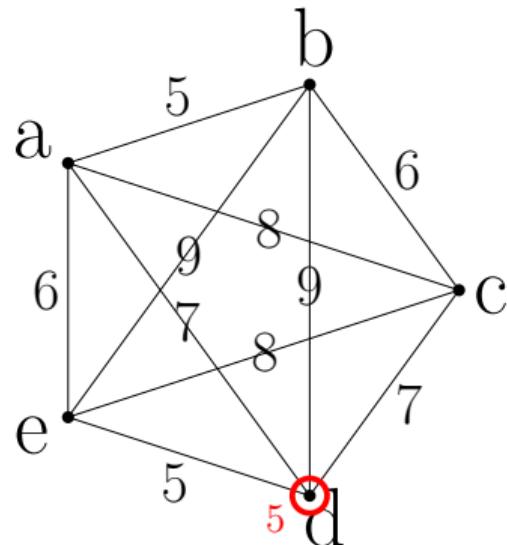


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Online problem - Metrical Task System (MTS)

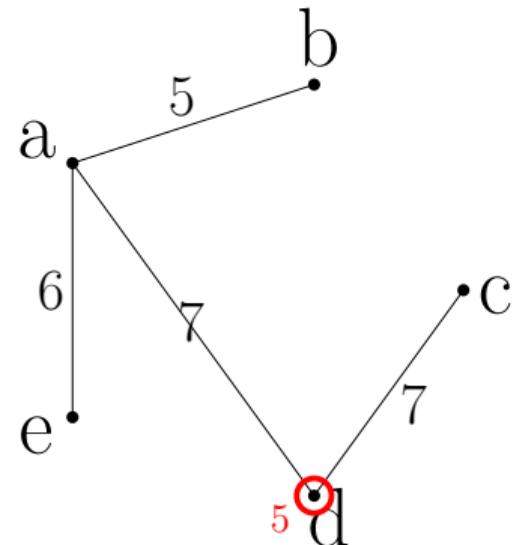
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$$\text{Cost alg}(5) = 22 + 5 = 27$$

$$\begin{aligned}f_5(a) &= 3 \\f_5(b) &= 4 \\f_5(c) &= 4 \\f_5(d) &= 5 \\f_5(e) &= 1\end{aligned}$$



$$\text{Cost alg}_T(5) = 25 + 5 = 30$$

Online problem - Metrical Task System (MTS)

Algorithm:

1. Sample a tree T over (X, d_X) using [FRT04].
2. Run [FM00] on T with the same cost functions.
Make the same decisions as [FM00].

Analysis. Let x_1, x_2, \dots, x_k be the decisions of opt. Thus
 $\text{opt} = \sum_{i=1}^k f_i(x_i) + \sum_{i=1}^k d_X(x_{i-1}, x_i).$

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We've sampled a tree T , x_1, x_2, \dots, x_k is also a valid decisions for T . Hence
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[FM00] is $O(\log n \cdot \log \log n)$ -competitive on T . Hence it choose points y_1, \dots, y_k such that $\mathbb{E}[\text{alg}_T] = \mathbb{E}[\sum_{i=1}^k f_i(y_i) + \sum_{i=1}^k d_T(y_{i-1}, y_i)] \leq O(\log n \cdot \log \log n) \cdot \text{opt}_T.$

We made the same decisions, so overall:

$$\mathbb{E} [\text{alg}] = \mathbb{E} \left[\sum_{i=1}^k f_i(y_i) + \sum_{i=1}^k d_X(y_{i-1}, y_i) \right]$$

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We made the same decisions, so overall:

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Theorem

MTS has an $O(\log^2 n \cdot \log \log n)$ competitive algorithm against oblivious adversary.

Outline of the talk

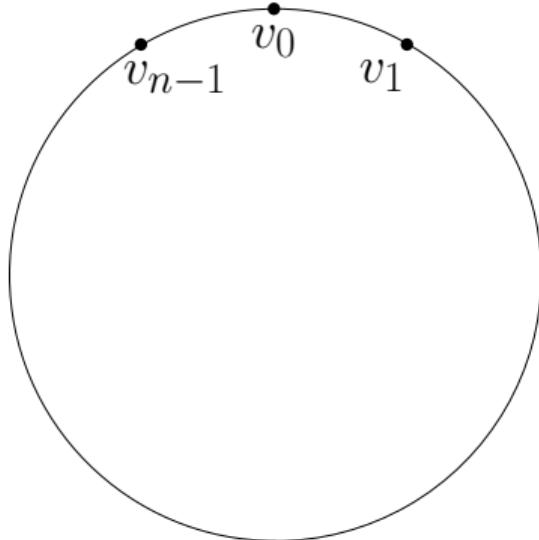
- 1 Introduction
- 2 Stochastic embedding into trees
- 3 Ramsey type embeddings
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Ramsey type Embeddings

Ramsey type theorem: Every **big** enough object, contains a **structured subset**.

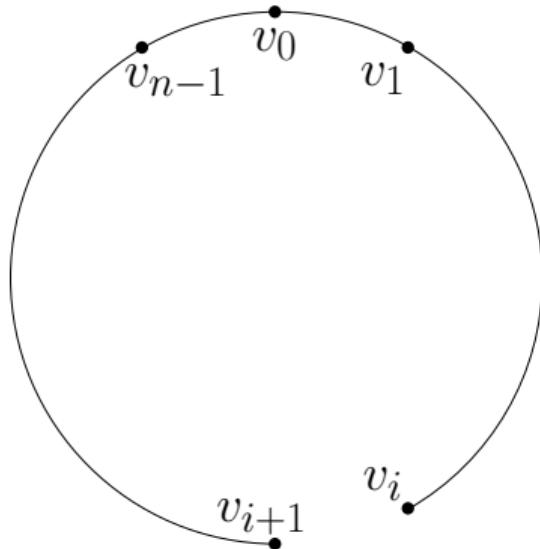
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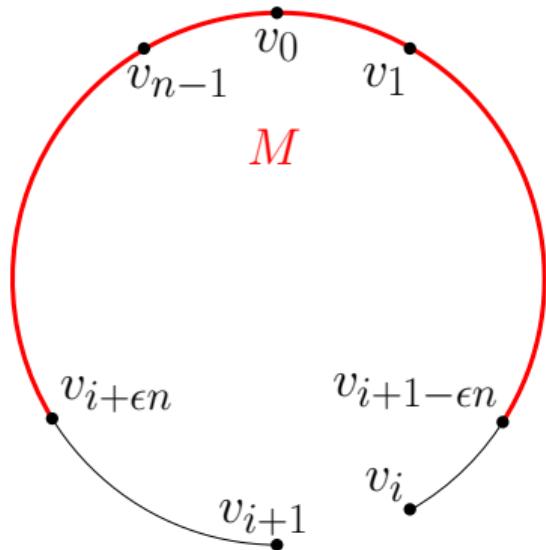
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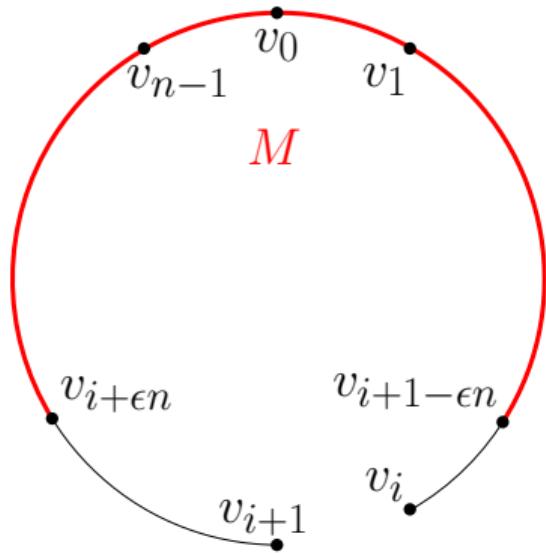


Suppose we delete $\{v_i, v_{i+1}\}$.

Set $M = \{v_{i+1-\epsilon n}, v_{i+2-\epsilon n}, \dots, v_{i+\epsilon n}\}$.

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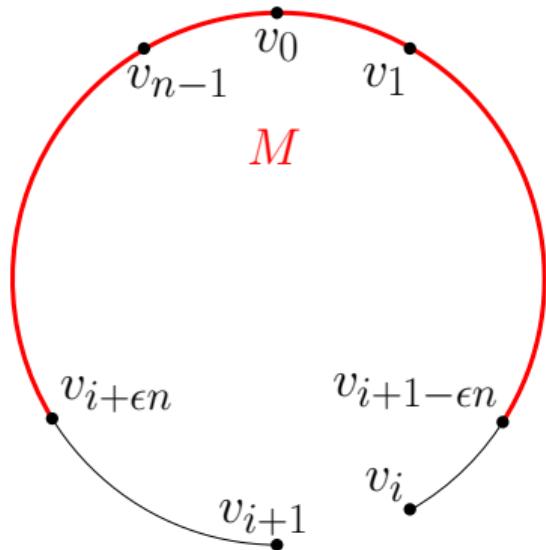
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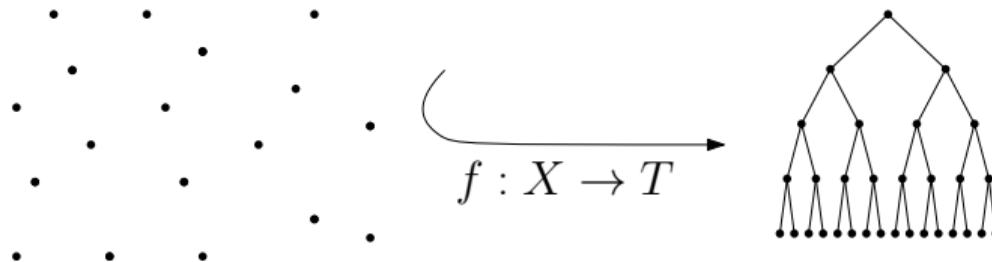
Choose i u.a.r., then $\Pr[v \in M] = 1 - 2\epsilon$.

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Fix $k > 1$, what is the largest subset $M \subset X$,

s.t. (M, d_X) embeds into a tree with **distortion** k ?

$$(X, d_X)$$

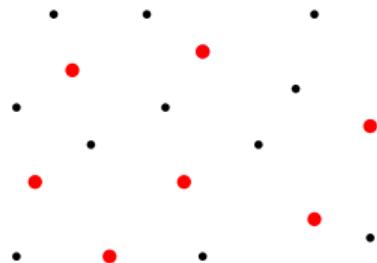


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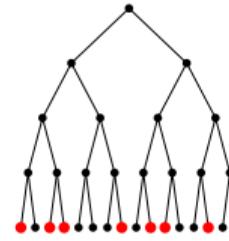
Fix $k > 1$, what is the largest subset $\textcolor{red}{M} \subset X$,

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$\textcolor{red}{M}$ (X, d_X)



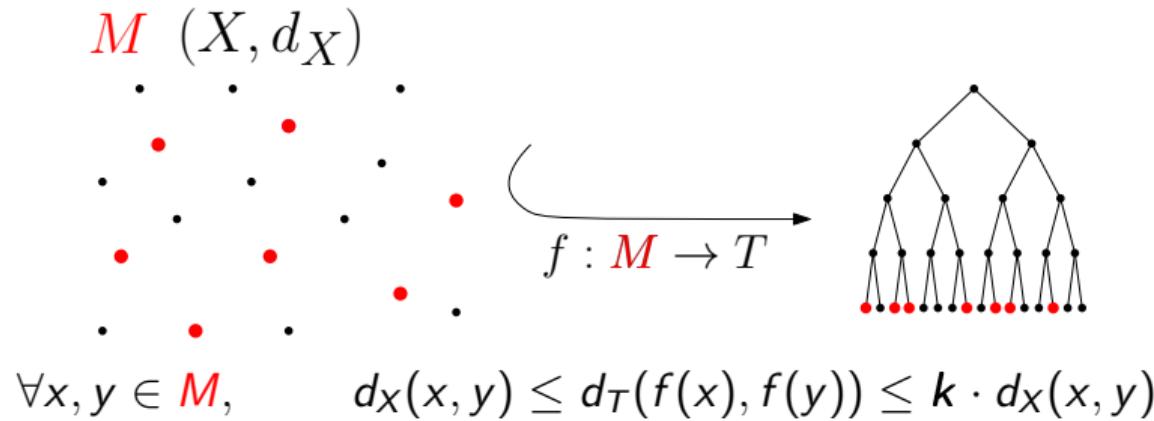
$$f : \textcolor{red}{M} \rightarrow T$$



$$\forall x, y \in \textcolor{red}{M}, \quad d_X(x, y) \leq d_T(f(x), f(y)) \leq k \cdot d_X(x, y)$$

Ramsey type Embeddings

Fix $k > 1$, what is the largest subset $M \subset X$,
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Theorem ([Mendel, Naor 07], following [BFM86, BLMN05])

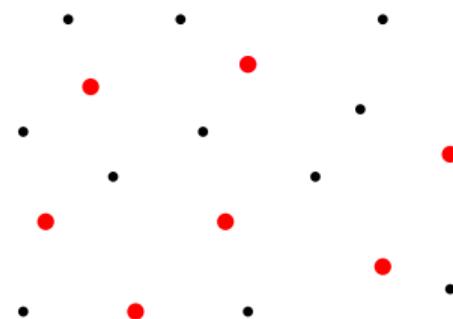
∀ n -point metric space and $k \geq 1$, ∃ subset M of size $n^{1-1/k}$
that embeds into a tree with distortion $O(k)$.

Ramsey type Embeddings

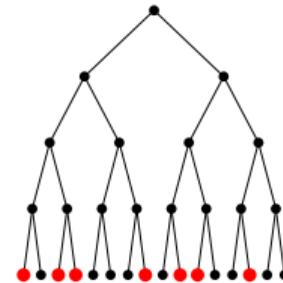
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$$M \quad (X, d_X)$$



$$f : M \rightarrow T$$

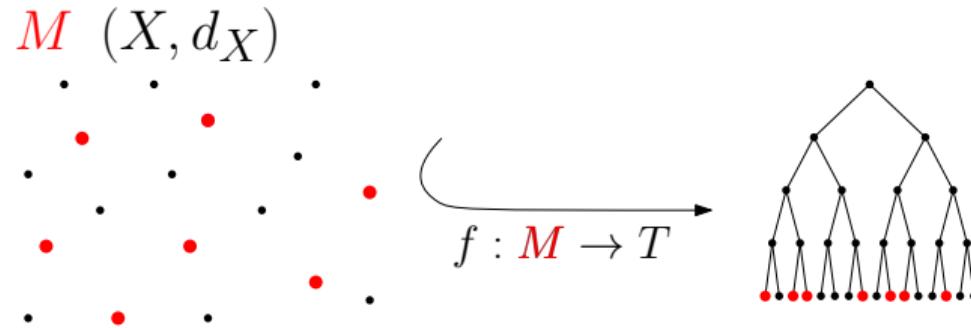


Asymptotically tight.

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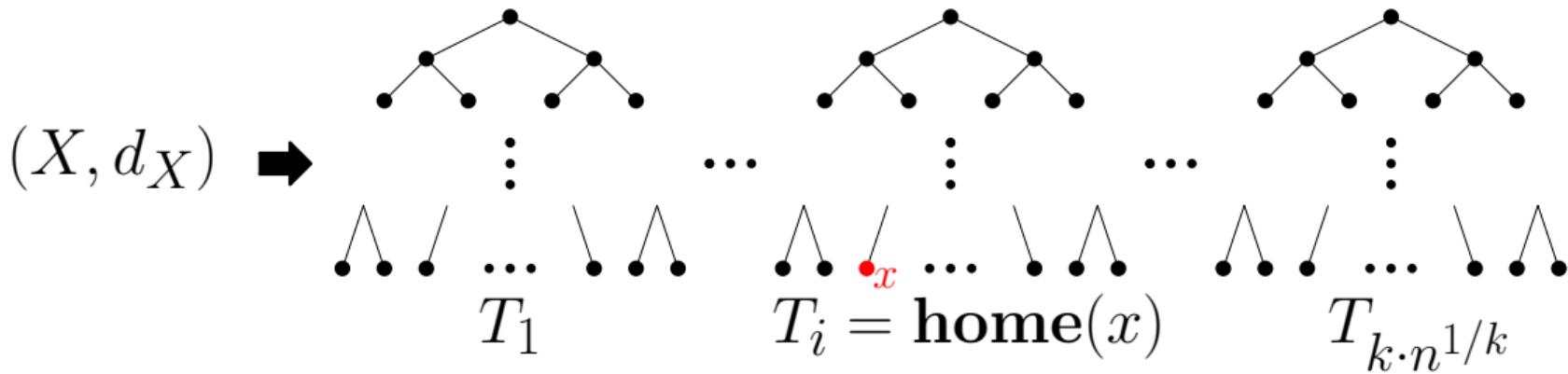
[Naor, Tao 12]: distortion $2e \cdot k$.

Ramsey type Embeddings

Corollary

For every n -point metric space and $k \geq 1$, there is a set \mathcal{T} of $k \cdot n^{\frac{1}{k}}$ trees and a mapping $\text{home} : X \rightarrow \mathcal{T}$, such that for every $x, y \in X$,

$$d_{\text{home}(x)}(y) \leq O(k) \cdot d_X(x, y)$$



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Applications:

- **Distance oracle**
- Compact routing scheme
- Online algorithms
- Approximate ranking
- etc.

Ramsey type Embeddings

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Compromises: only partial guarantees



Distance Oracle

A **succinct** data structure that **approximately** answers distance queries.

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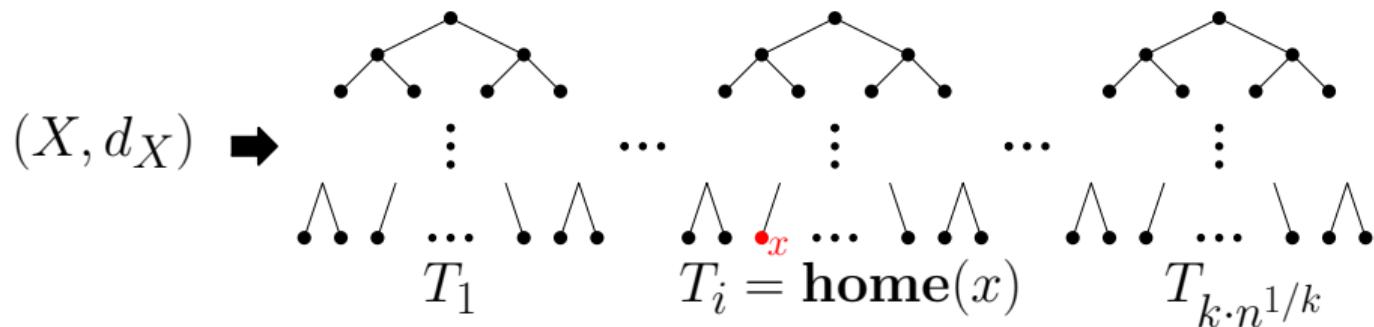
The properties of interest are size, distortion and query time.

Distance Oracles

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For every n -point metric space and $k \geq 1$, there is a set \mathcal{T} of $k \cdot n^{\frac{1}{k}}$ trees and a mapping $\text{home} : X \rightarrow \mathcal{T}$, such that for every $x, y \in X$,

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Theorem (Tree Distance Oracle [Harel, Tarjan 84], [Bender, Farach-Colton 00])

For every tree metric*, there is an exact distance oracle of linear size and constant query time.

Distance Oracles

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For every tree metric*, there is an exact distance oracle of linear size and constant query time.

Theorem (Ramsey based Deterministic Distance Oracle)

For any n -point metric space, there is a distance oracle with :

Distortion	Size	Query time
$O(k)$	$O(k \cdot n^{1+1/k})$	$O(1)$

Outline of the talk

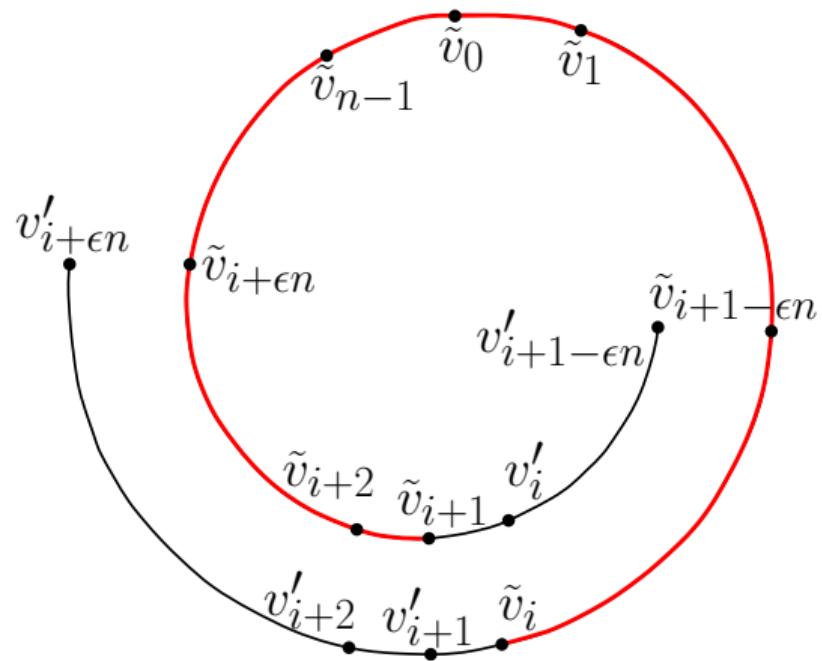
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Clan Embedding

Idea: **duplicate** vertices to meet all guarantees!

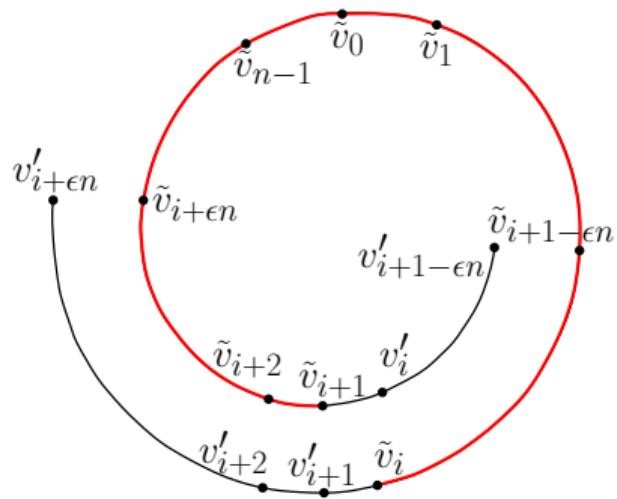
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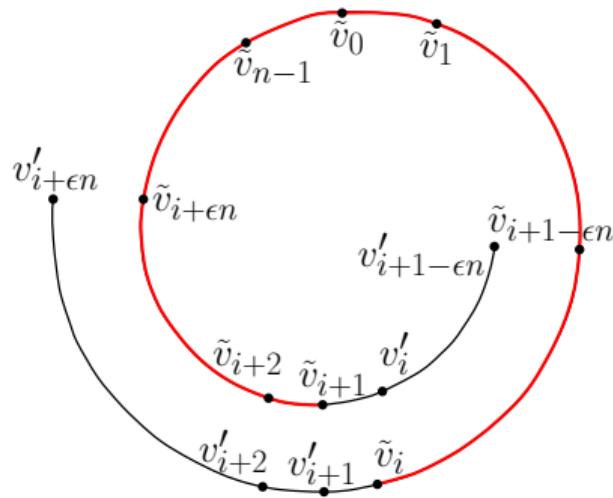


One-to-many embedding from (X, d_X) to (Y, d_Y) : A map $f : X \rightarrow 2^Y$ where:

- 1) $\forall x, f(x) \neq \emptyset$.
- 2) $\forall x, y, f(x) \cap f(y) = \emptyset$.

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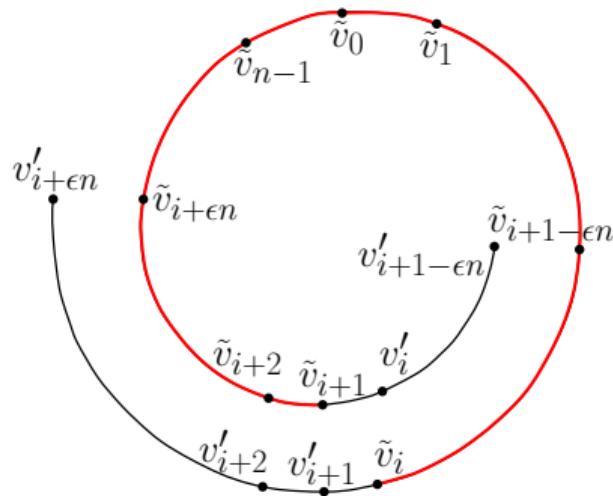
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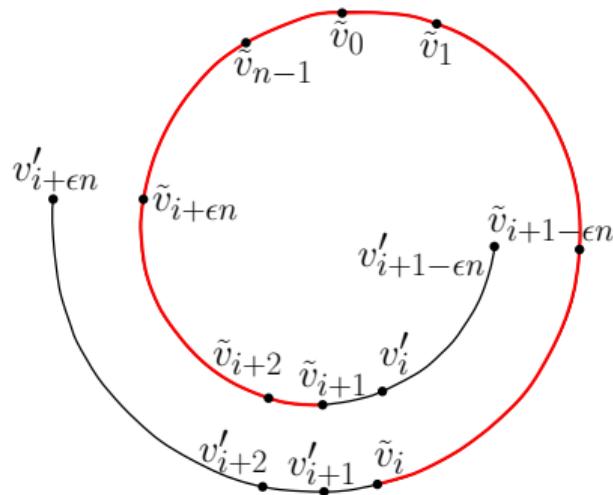
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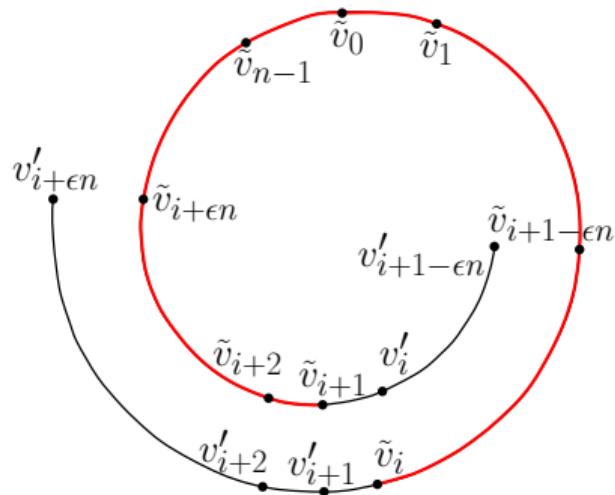
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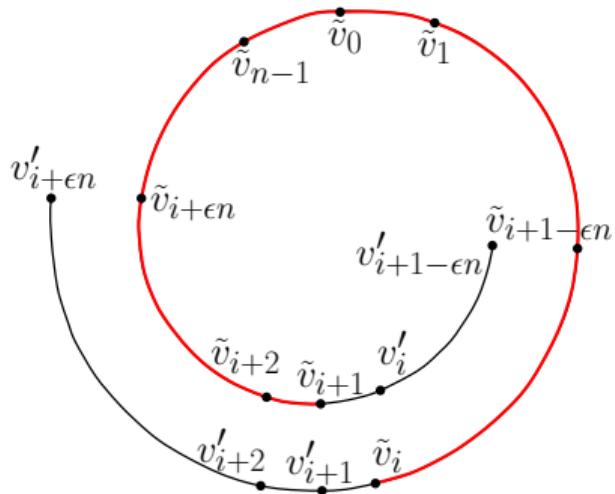
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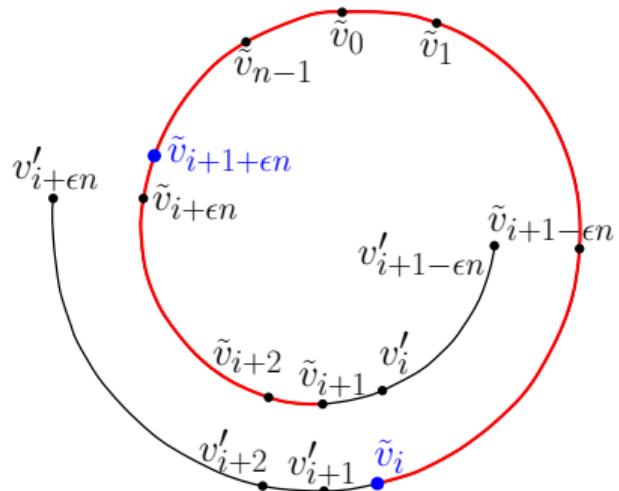
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Clan embedding is a pair (f, χ) , where f is **dominating** one-to-many embedding.
 (f, χ) has **distortion** t , if $\forall x, y \in X, \min_{y' \in f(y)} d_Y(\chi(x), y') \leq t \cdot d_X(x, y)$

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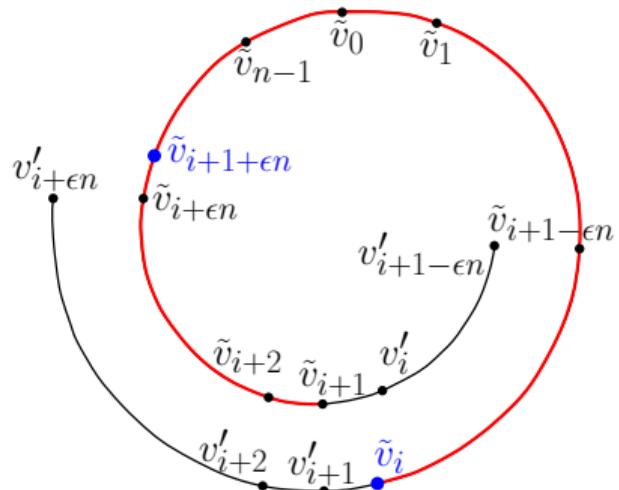
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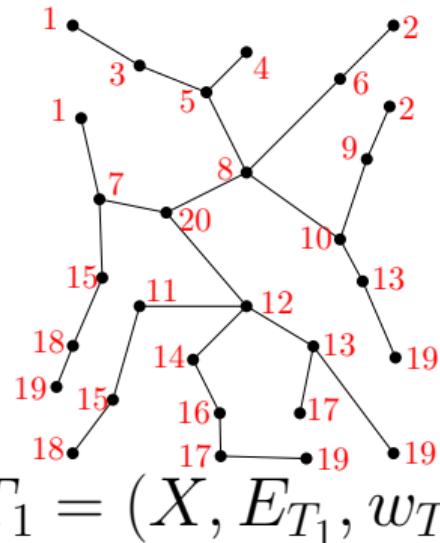
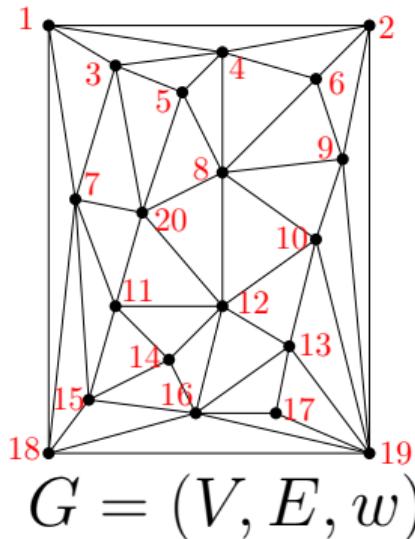
Choose i u.a.r., then $\mathbb{E}[|f(v_i)|] = 1 + 2\epsilon$.

Theorem (Clan embedding into trees, [Filtser, Le 21])

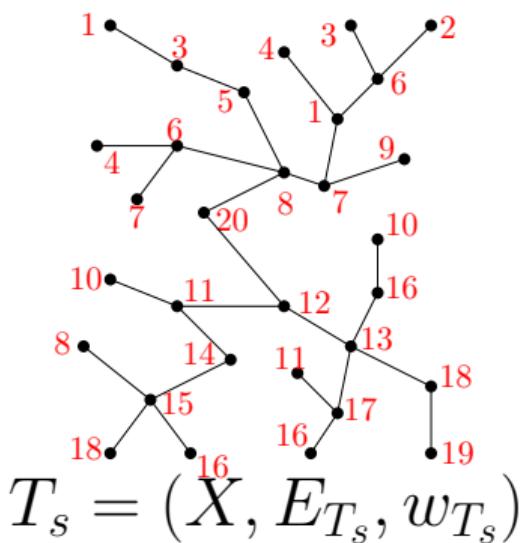
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distribution \mathcal{D} over dominating clan embeddings into trees such that:

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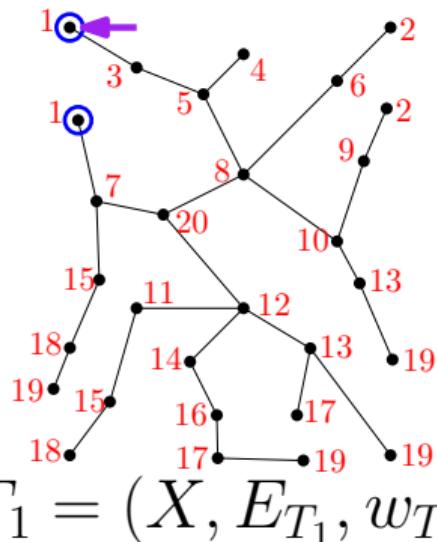
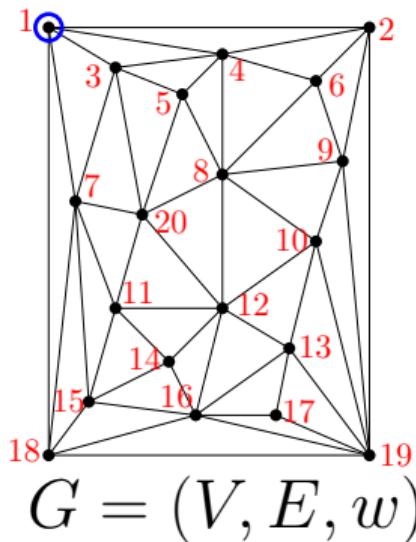


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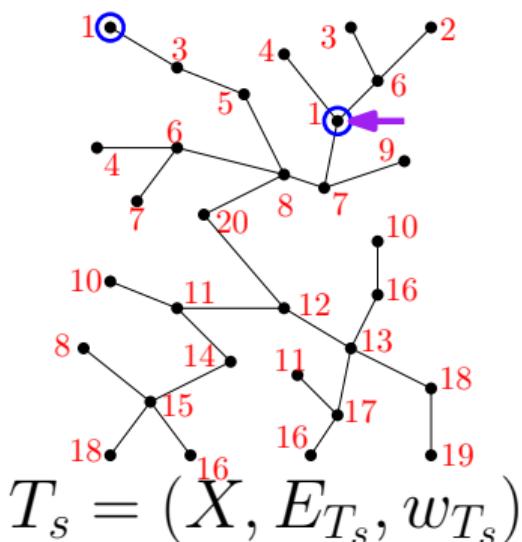
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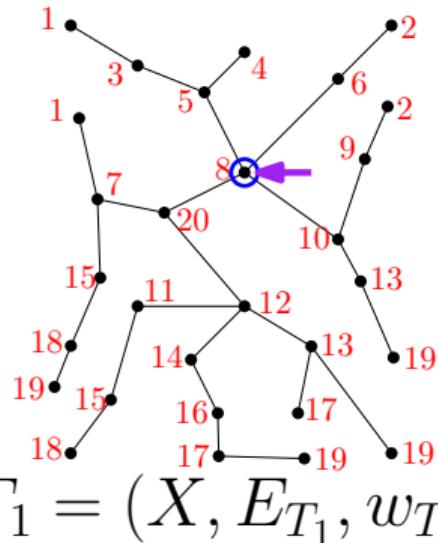
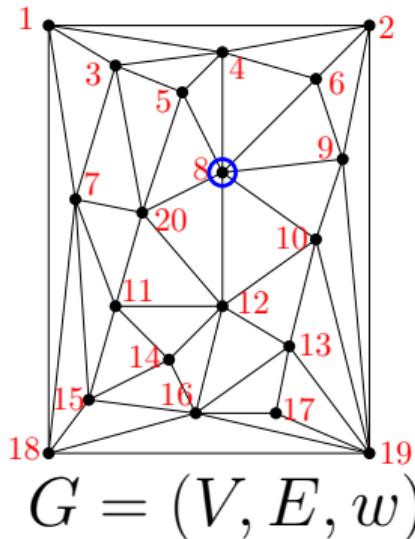


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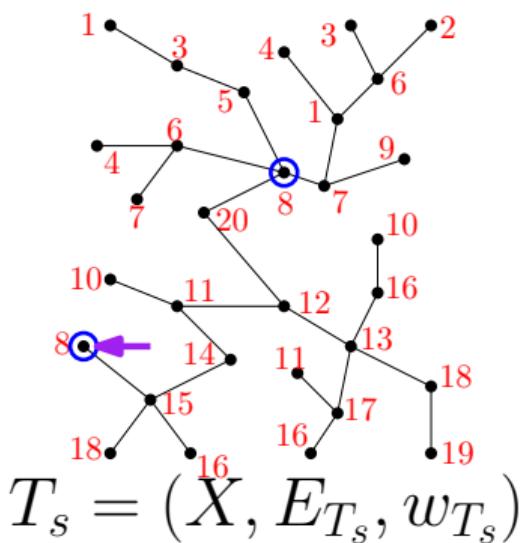
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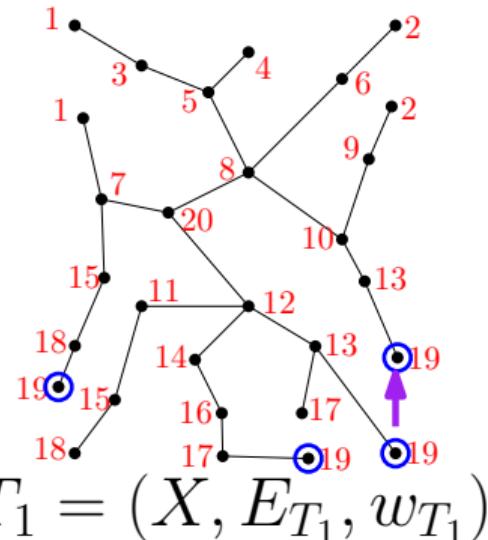
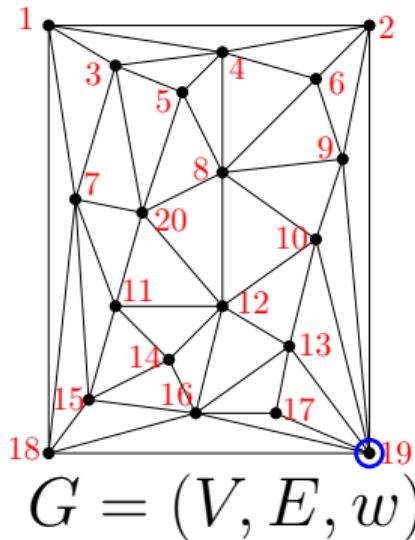


Theorem (Clan embedding into trees, [Filtser, Le 21])

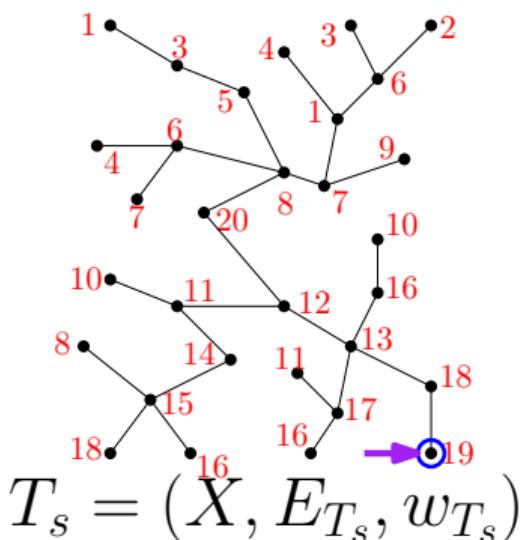
(X, d_X) n point metric space, $\forall \epsilon \in (0, 1)$, there is

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(both) Tight!

Clan Embedding

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Compromises: Not a real classic embedding



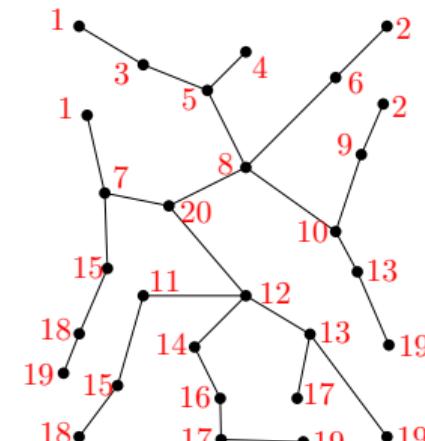
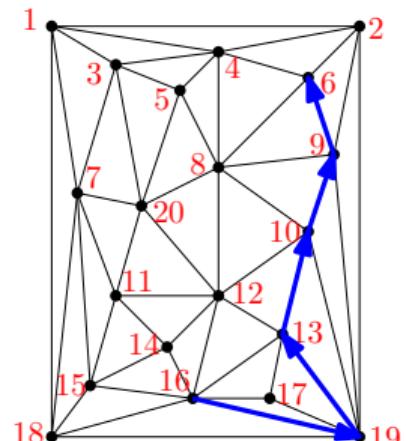
Path Distortion

Originally appeared in [Bartal, Mendel 04] in the context of multi-embeddings.

Definition

One-to-many embedding $f : X \rightarrow 2^Y$ has *path-distortion* t if for every sequence (x_0, x_1, \dots, x_m) in X there is a sequence y_0, \dots, y_m where $y_i \in f(x_i)$, and

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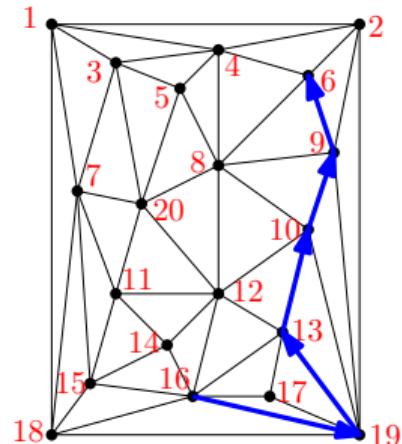
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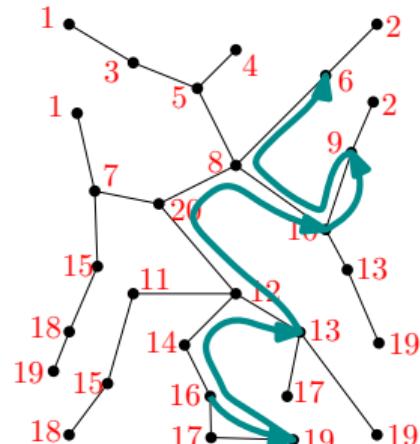
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$$G = (V, E, w)$$



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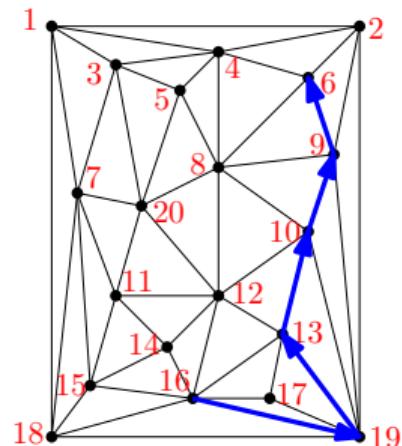
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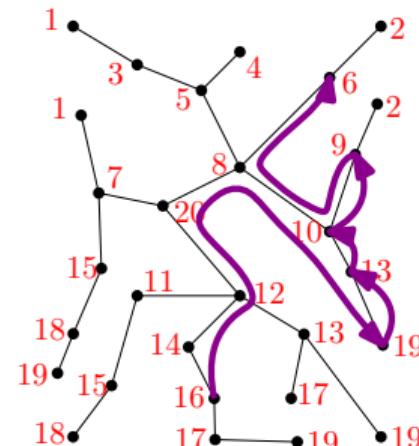
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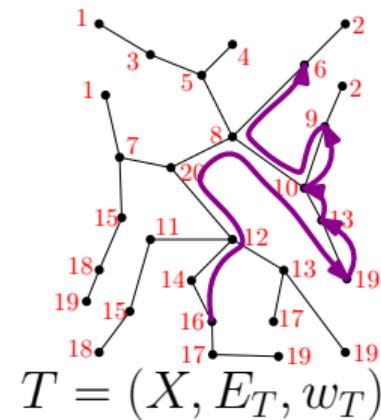
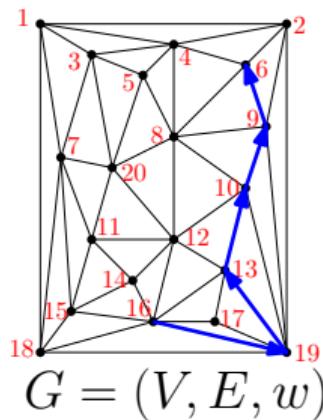


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Our clan embeddings with distortion $O(k)$ have path distortion $O(k \cdot \log n)$.

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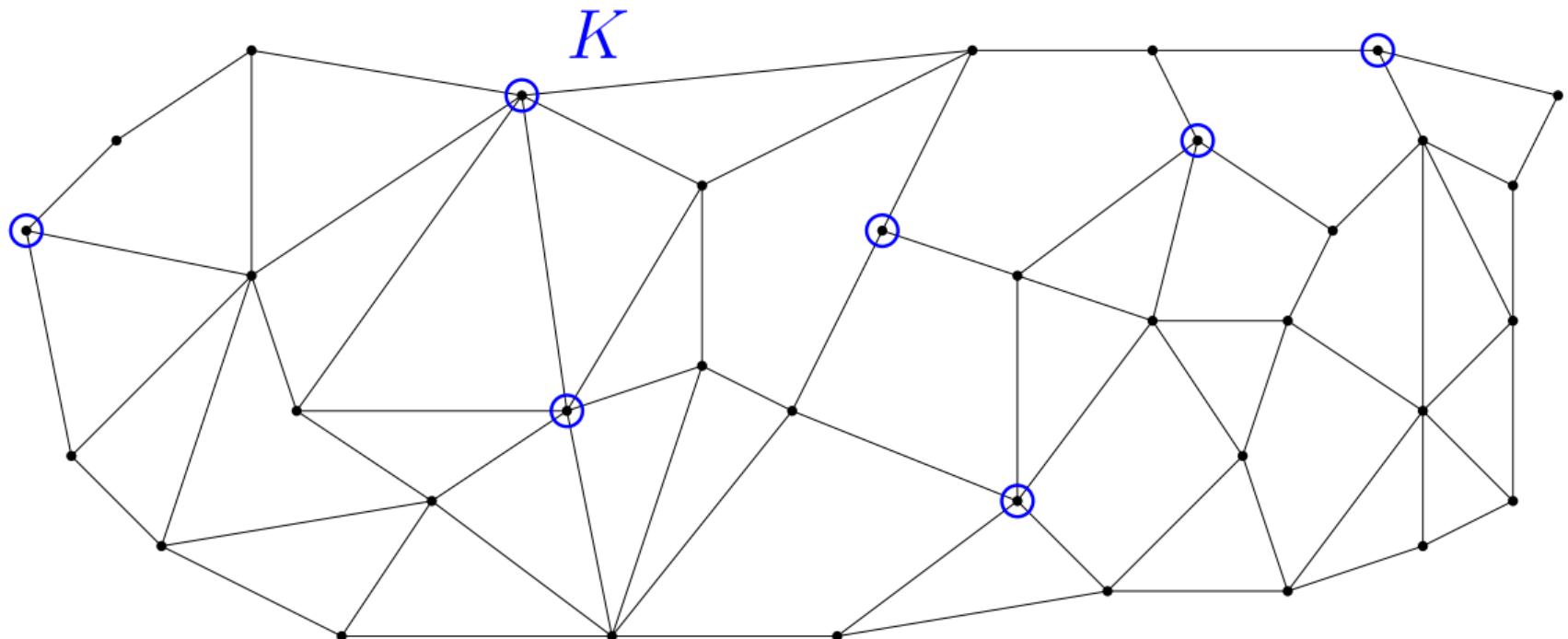
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Or a total of $O(n^{1+\frac{1}{2}})$ copies and path distortion $O(\log n)$.

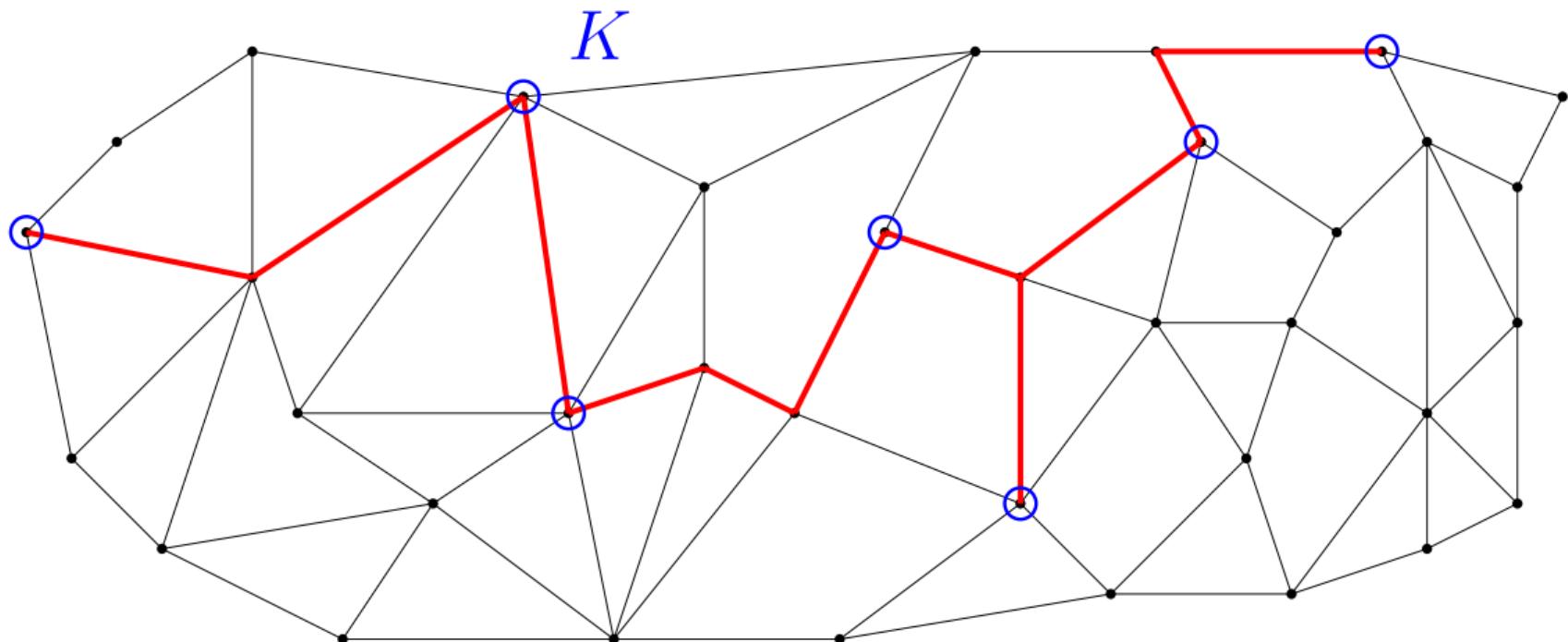
Steiner Tree

Given set of terminals K , find minimum weight tree T spanning K



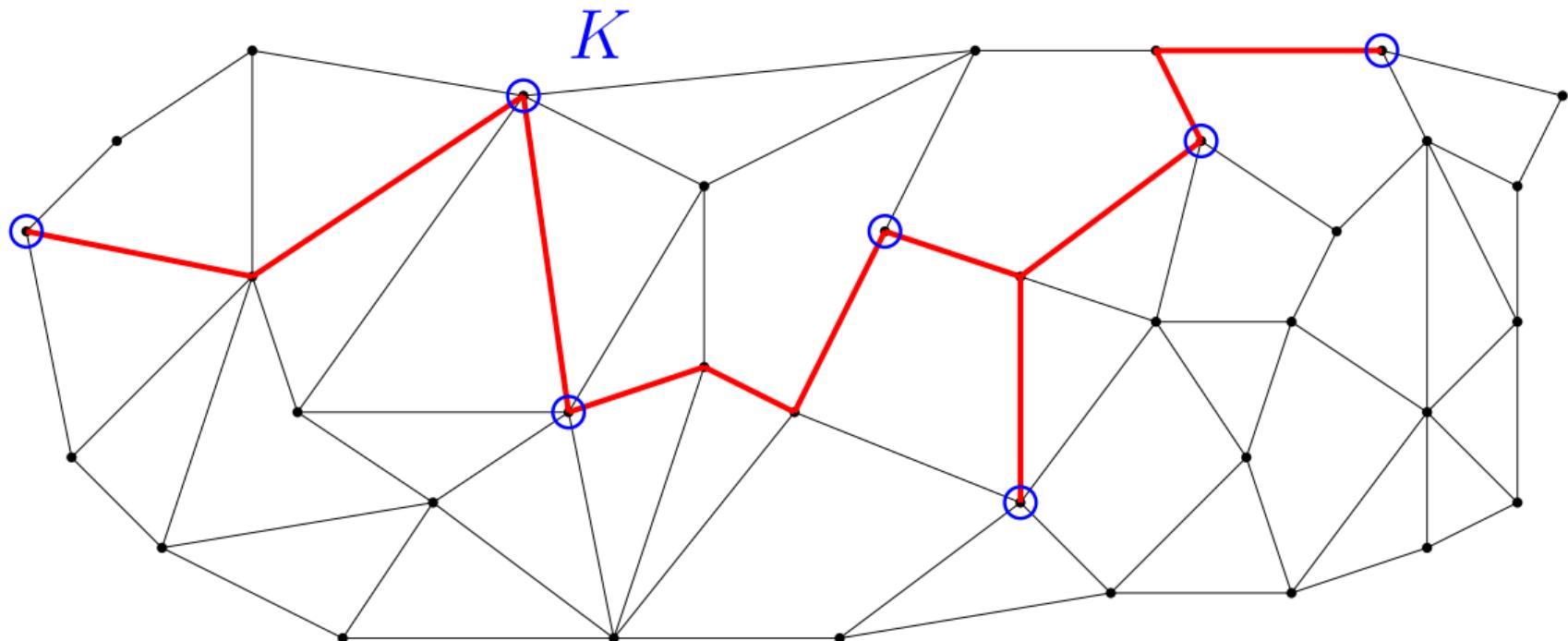
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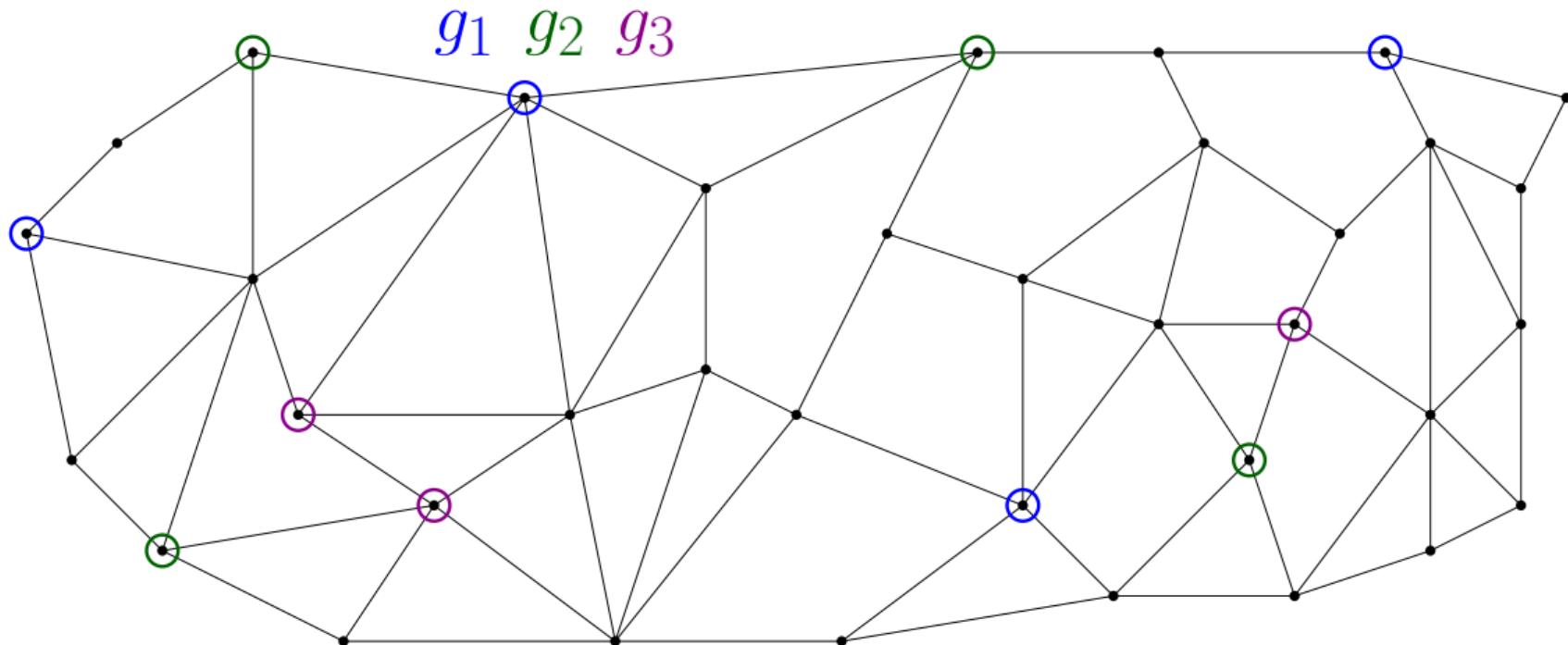
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In class we saw a 2-approximation algorithm for the Steiner tree problem.

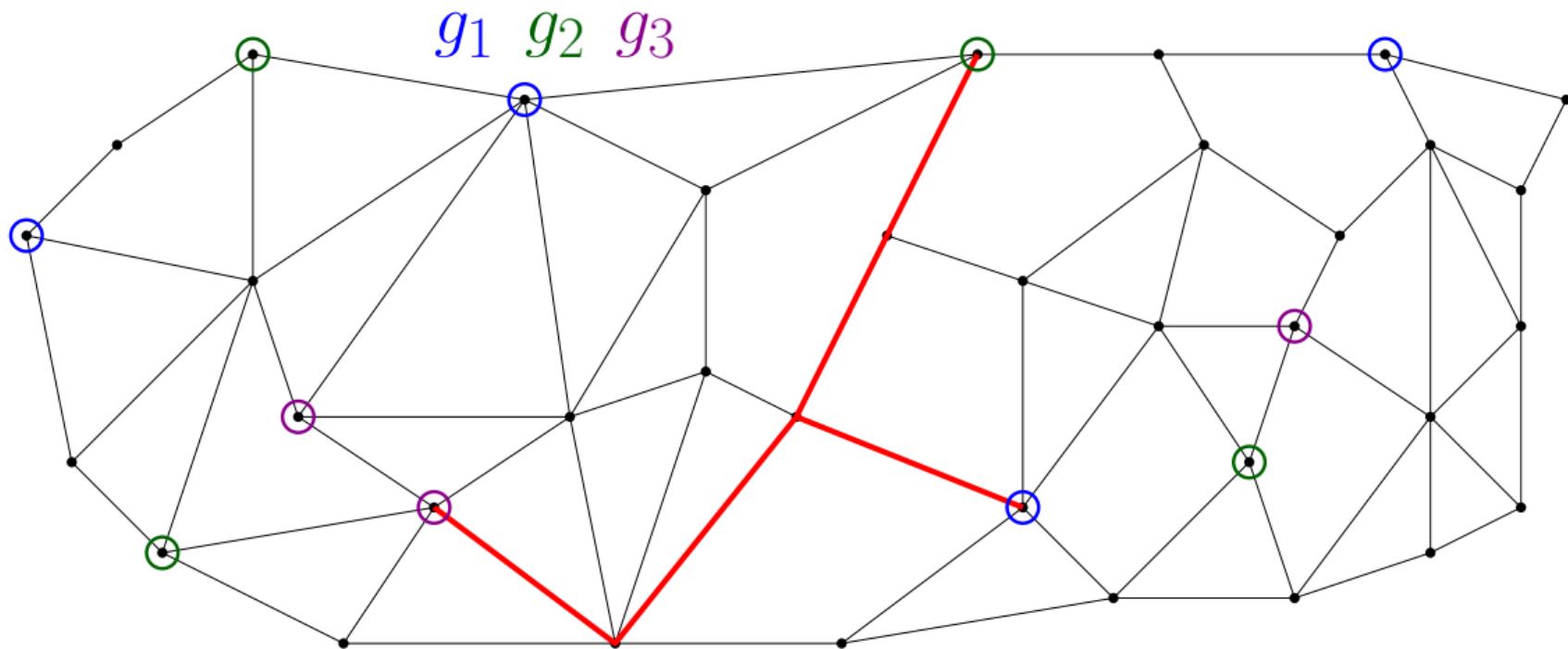
Group Steiner Tree

Given subsets $g_1, g_2, \dots, g_k \subseteq V$, find minimum weight tree T spanning at least one vertex from each g_i ;



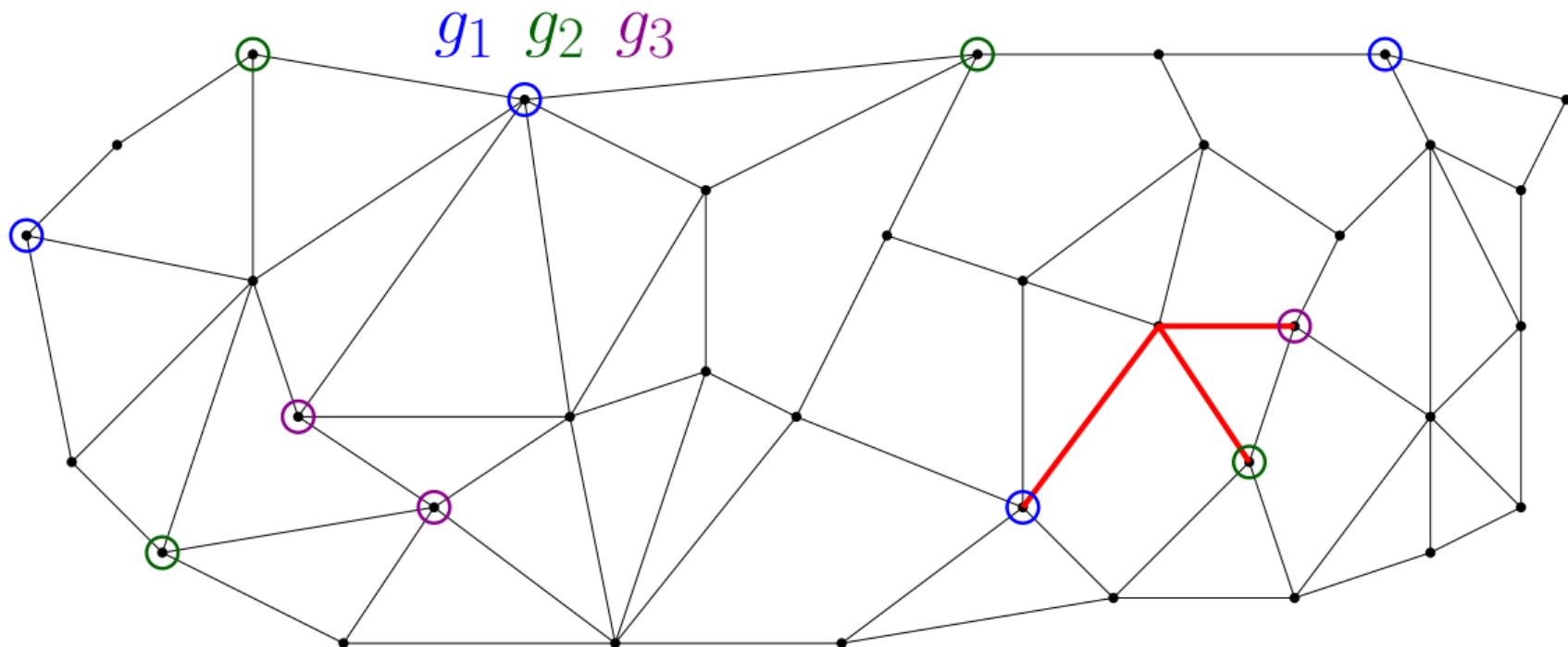
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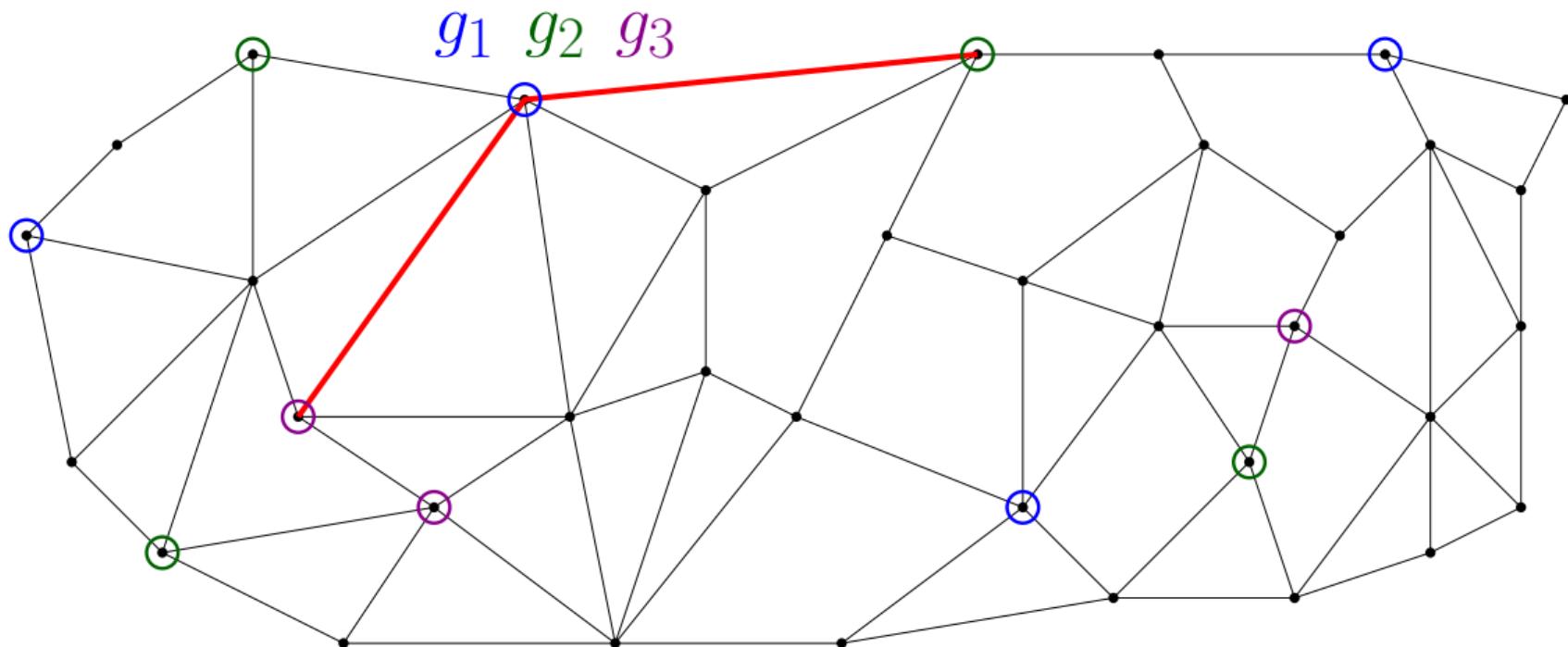
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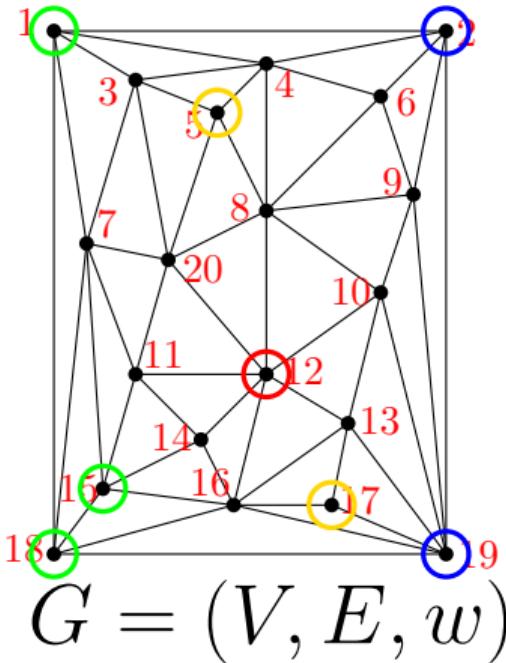
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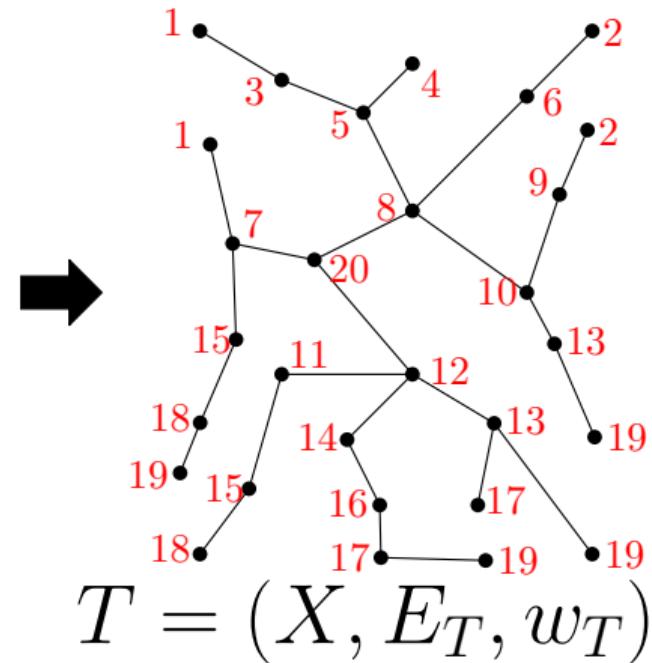


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Clan embedding f with path distortion $O(\log n)$.



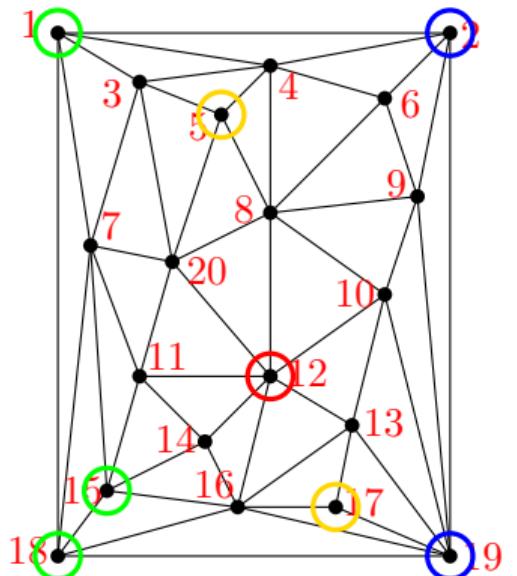
$g_1 \ g_2 \ g_3 \ g_4$



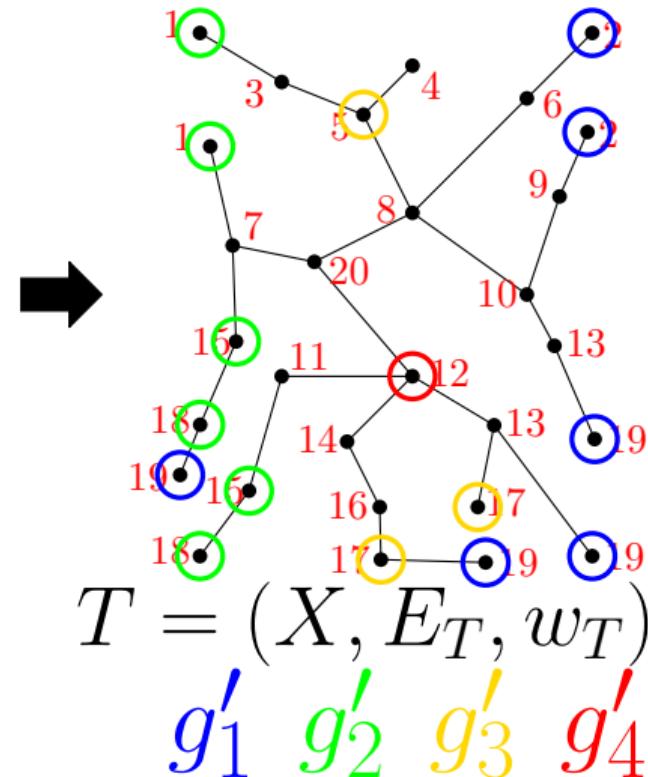
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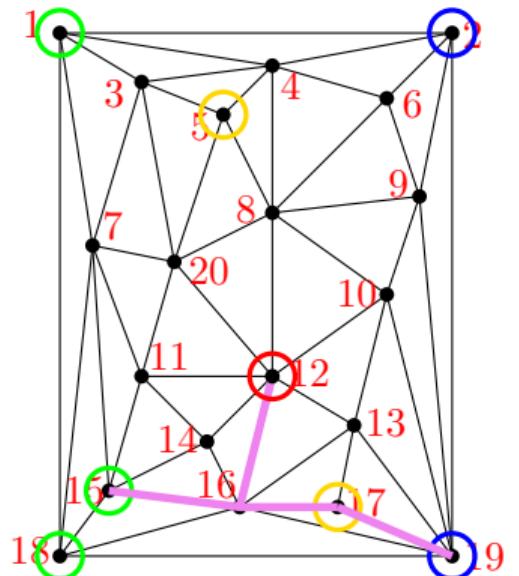


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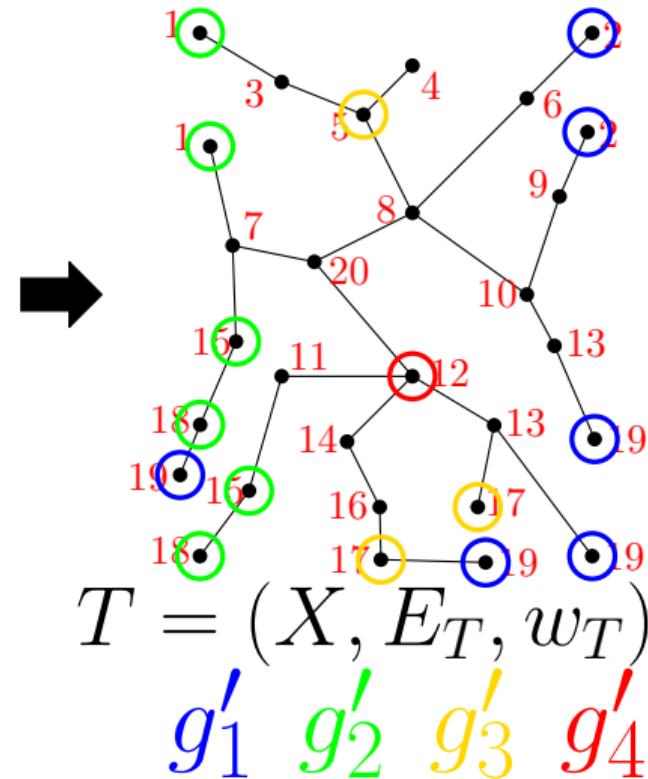
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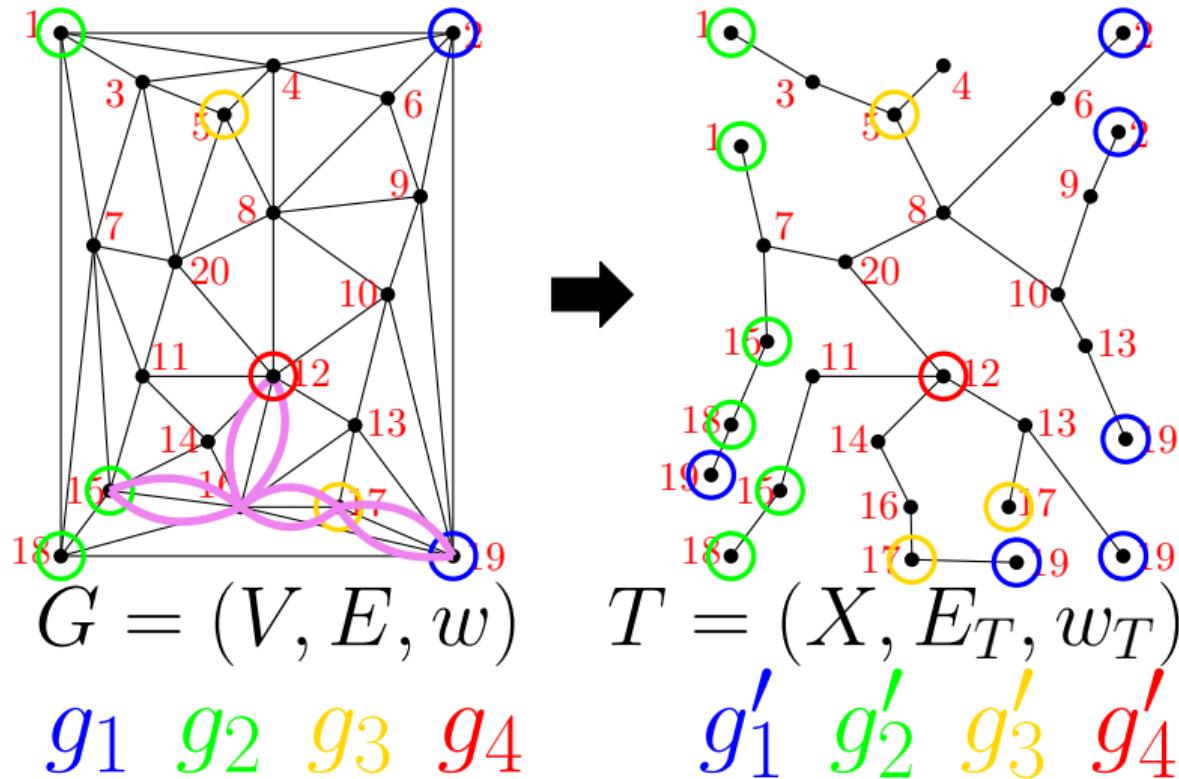
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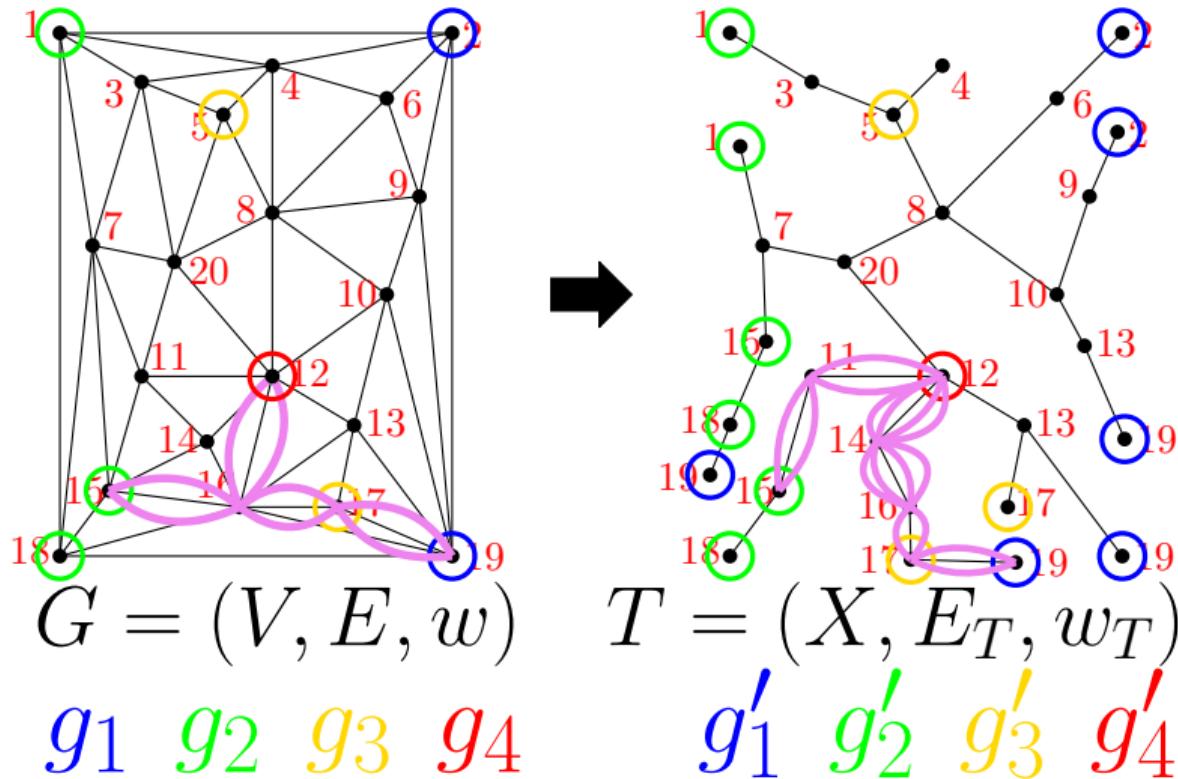
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Guaranteed path: S_T^*
(valid solution),
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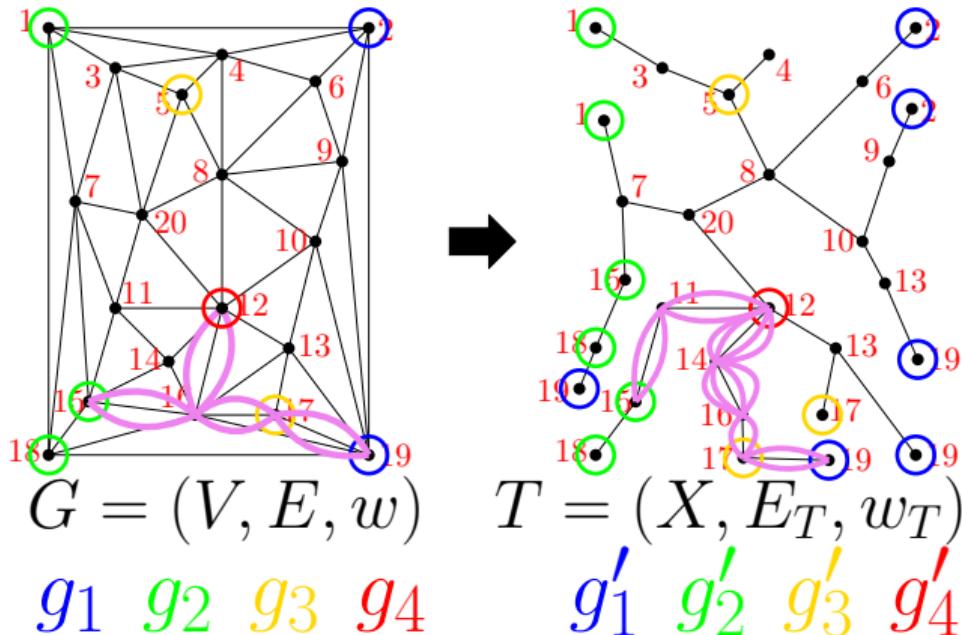
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Theorem ([Garg, Konjevod, Ravi 00])

$O(\log n \cdot \log k)$ -approximation algorithm for the GST problem on trees.

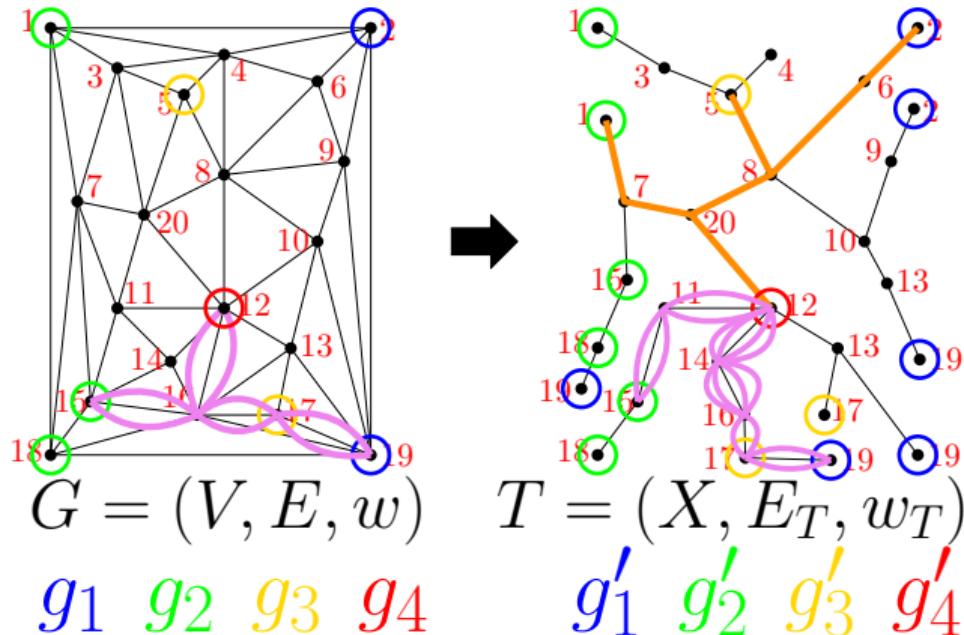
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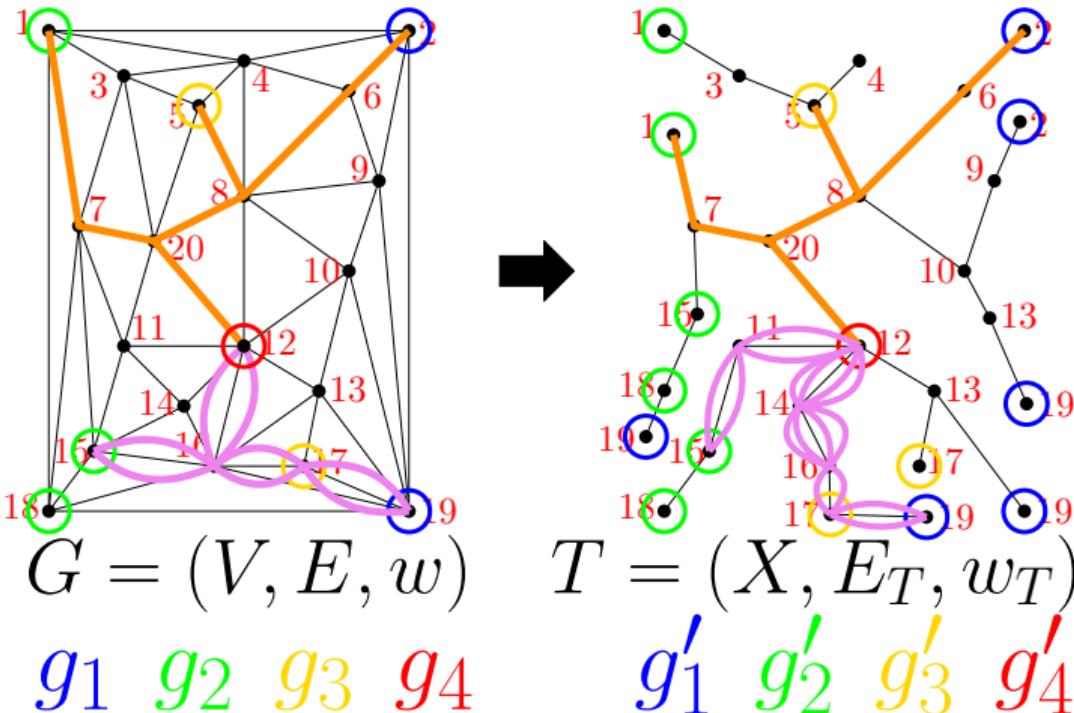
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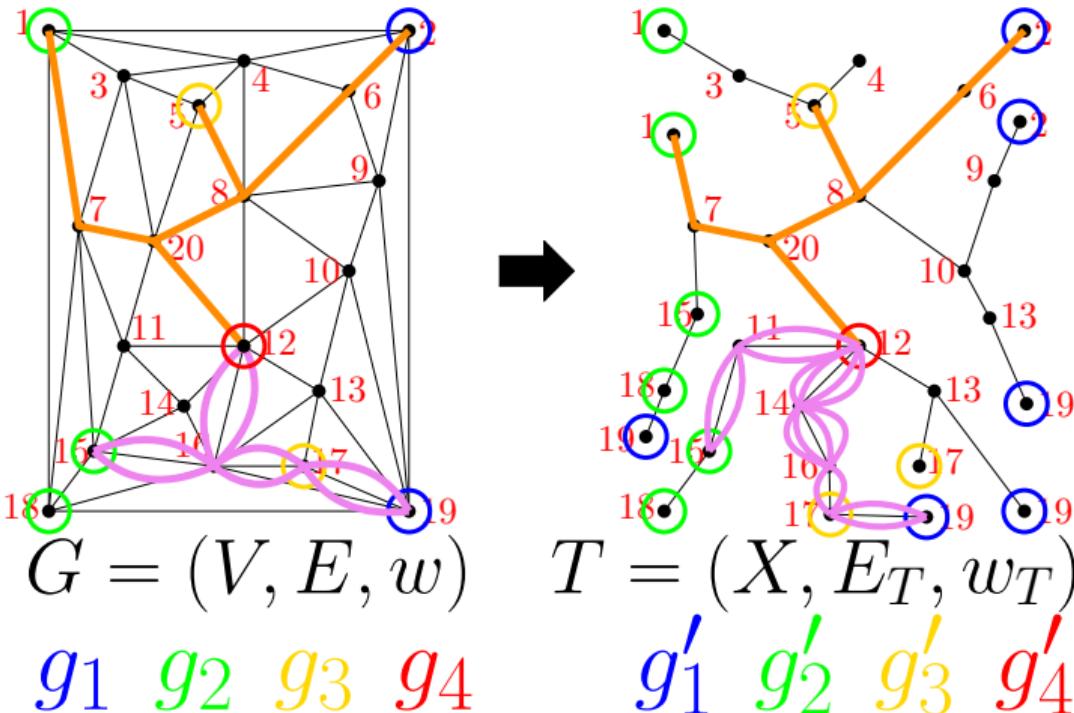
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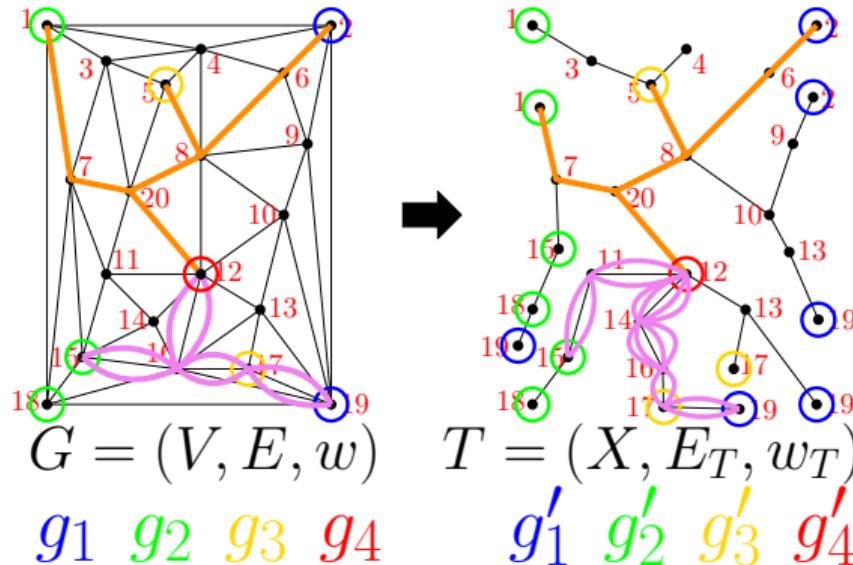
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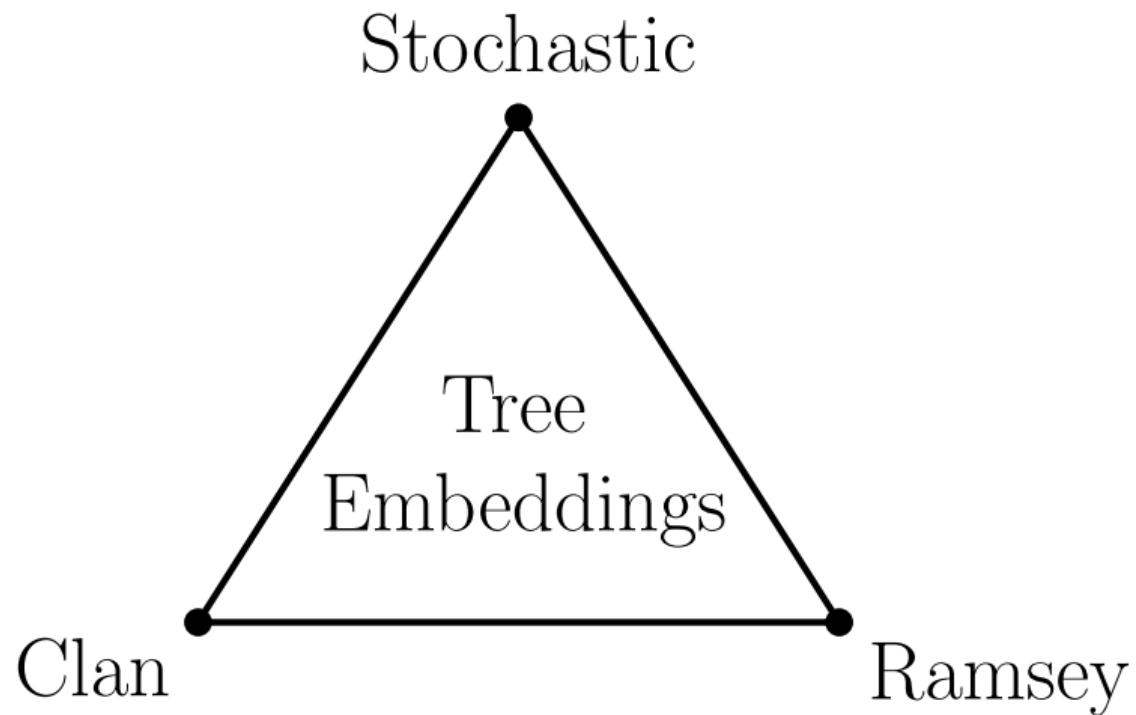
$$w(\tilde{S}) \leq w(\tilde{S}_T) \leq O(\log^2 n \cdot \log k) \cdot w(S^*).$$

We got an $O(\log^2 n \cdot \log k)$ **approximation**.

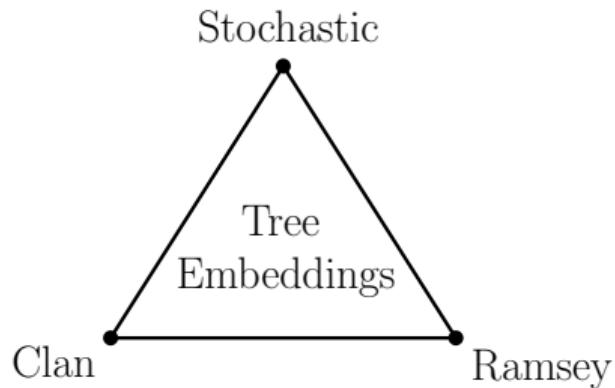


Outline of the talk

- 1 Introduction
- 2 Stochastic embedding into trees
- 3 Ramsey type embeddings
- 4 Clan embedding
- 5 Group Steiner Tree
- 6 Conclusion



Trinity



	Worst case guarantee	Satisfy all vertices	one-to-one embedding
Stochastic	✗	✓	✓
Ramsey	✓	✗	✓
Clan	✓	✓	✗

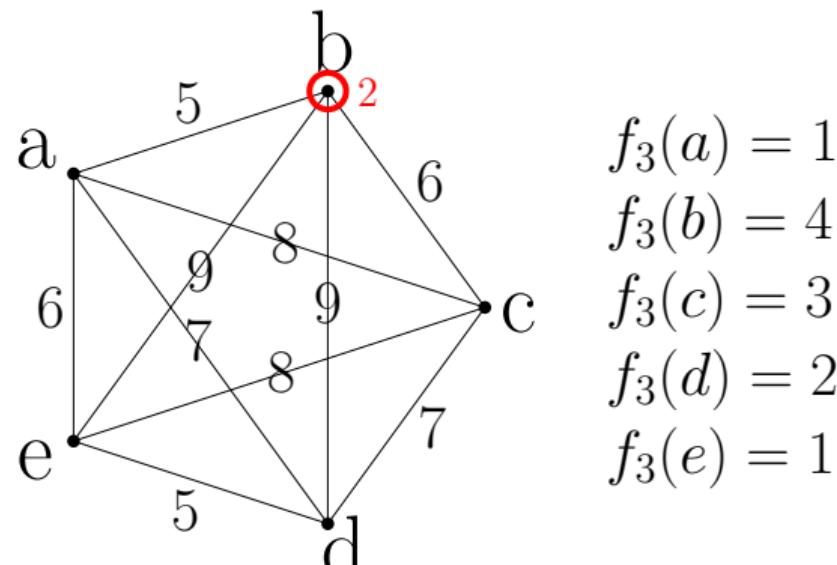
Online problem - Metrical Task System

Input: Metric space (X, d_X) . Initial configuration $x_0 \in X$.

At step i , a task arrives with cost function $f_i : X \rightarrow \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$$\text{Cost alg}(2) = 3 + 5 + 1 = 9$$

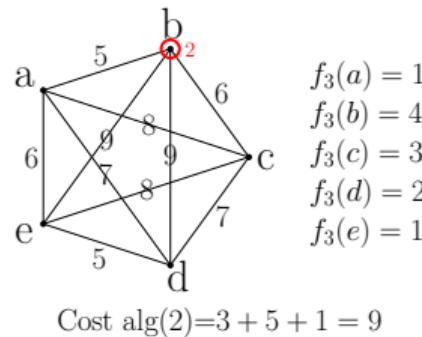
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We showed competitive ratio against oblivious adversary:

$$\text{Competitive ratio} = \max_{\text{input } I} \frac{\mathbb{E}[\text{Alg}(I)]}{\text{opt}(I)} = O(\log^2 n \cdot \log \log n).$$

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Theorem ([Fakcharoenphol, Rao, Talwar 04], improving [Bartal 96+98])

Every n -point metric space (X, d) embeds into **distribution \mathcal{D}** over **dominating trees** with **expected distortion** $O(\log n)$.

Theorem ([Fiat, Mendel 2000])

Given an n point tree* T , there is an online algorithm for MTS with competitive ratio $O(\log n \cdot \log \log n)$ against oblivious adversary.

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Suppose that you are given a **deterministic** k -competitive algorithm for MTS on trees.

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We showed competitive ratio against oblivious adversary:

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Suppose that you are given a **deterministic** k -competitive algorithm for MTS on trees.

Problem: Propose a **deterministic** algorithm for **MTS** for a general n -point metric.

What is the competitive ratio?

Link to quiz:



Can also be found in my homepage:

arnold.filtser.com

Or just google Arnold Filtser.

Link to slides:



MTS Input: Metric space (X, d_X) . Initial configuration $x_0 \in X$.

At step i , a task arrives with cost function $f_i : X \rightarrow \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Assumption: there is a **deterministic** k -competitive algorithm for MTS on trees.

Problem: Propose a **deterministic** algorithm for **MTS** for a general n -point metric.
What is the competitive ratio?

Clan Embeddings construction

Theorem (Clan embedding into trees, [Filtser, Le 21])

(X, d_X) n point metric space, $\forall k \in \mathbb{N}$, there is

distribution \mathcal{D} over dominating clan embeddings into trees such that:

- $\forall (f, \chi) \in \text{supp}(\mathcal{D})$ has distortion $O(k)$.
- $\forall x \in X, \mathbb{E}_{(f,\chi) \sim \mathcal{D}}[|f(x)|] \leq O(n^{\frac{1}{k}})$

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Definition (Ultrametric)

Ultrametric (X, d) is a metric space satisfying the **strong** triangle inequality:

$$\forall x, y, z \in X, \quad d(x, z) \leq \max \{d(x, y), d(y, z)\} .$$

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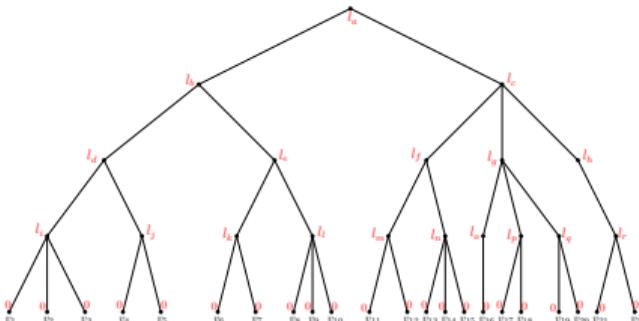
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Definition (Hierarchical well-separated tree (HST))

(X, d_X) is a HST if X is mapped (by ϕ) to **leaves** of a rooted tree T where:

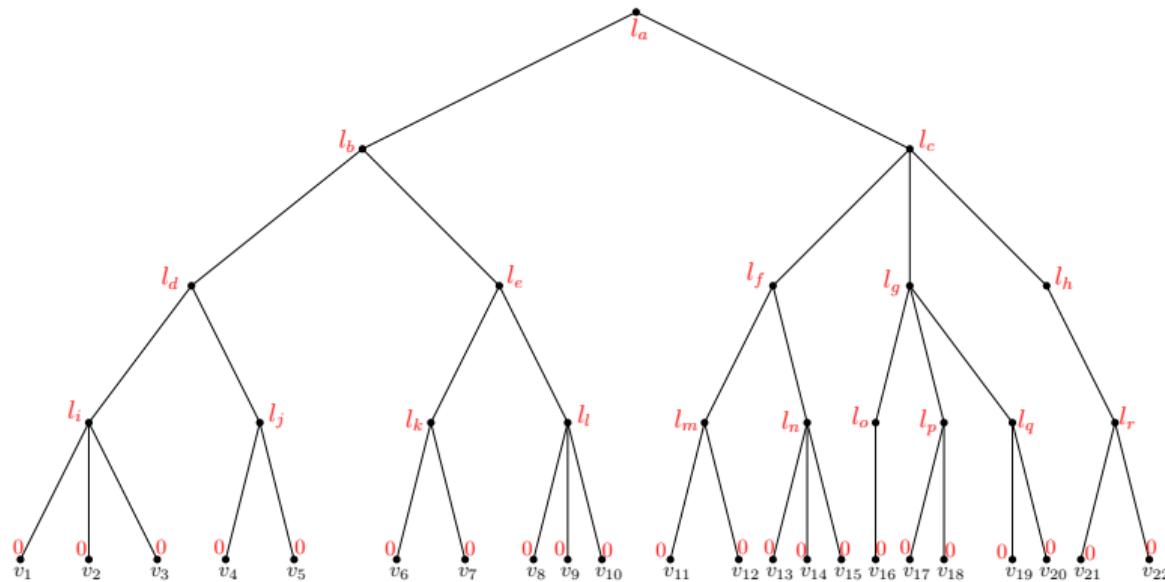
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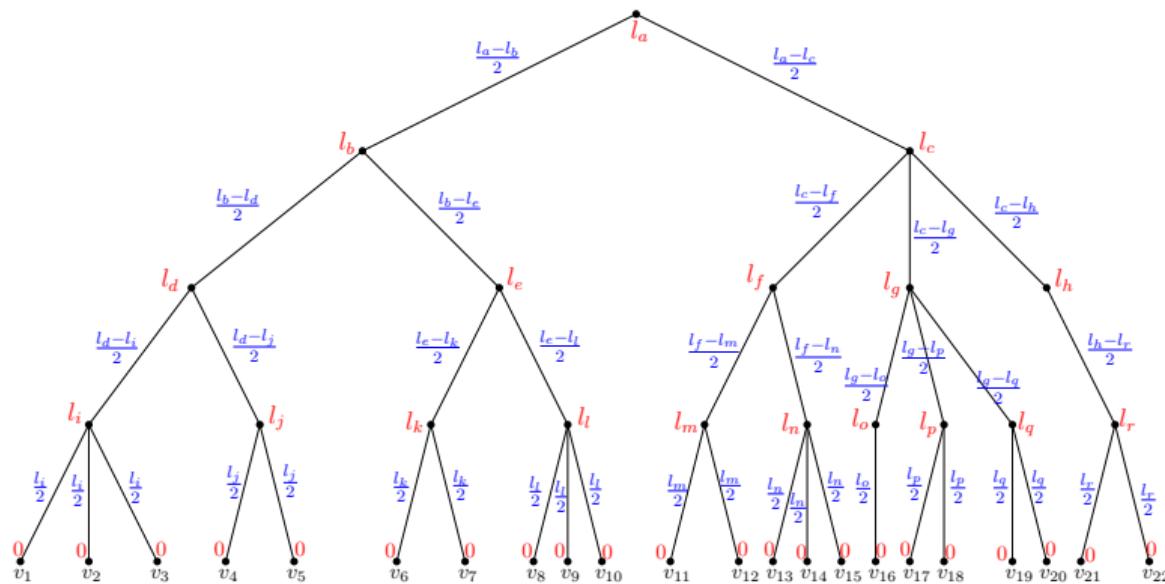
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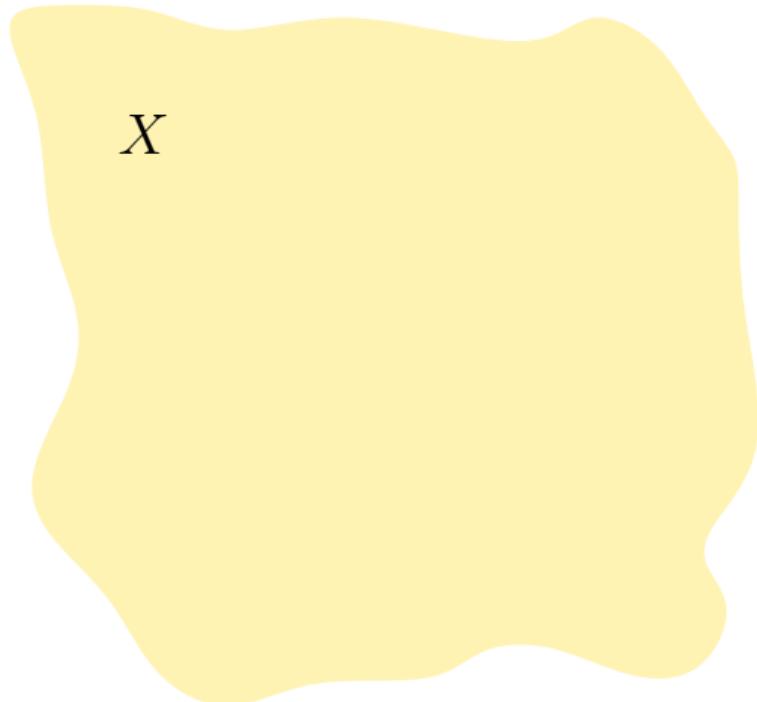
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Construction

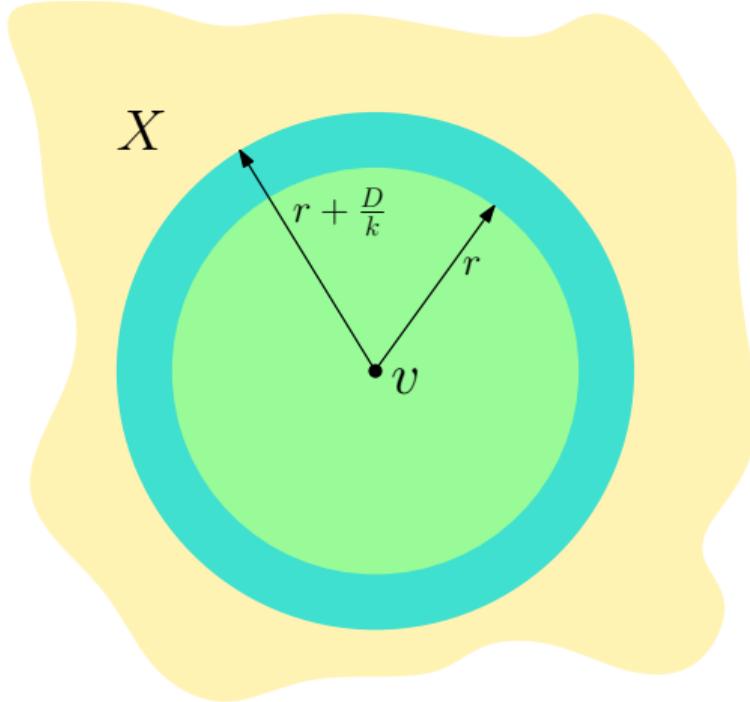


X

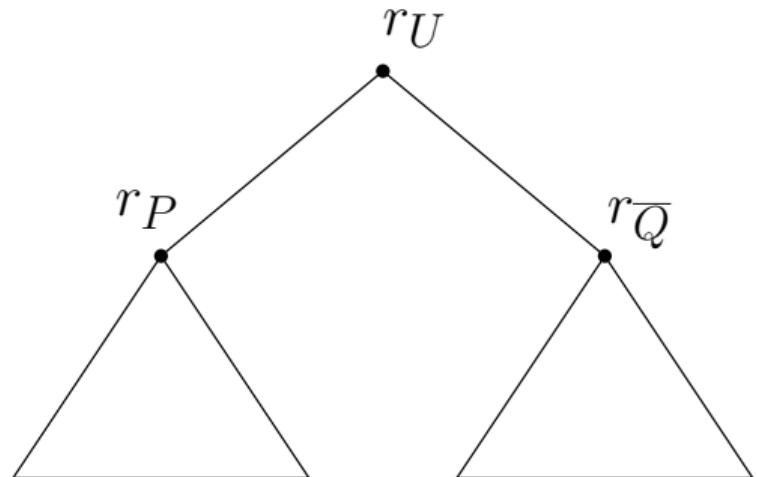
$$\ell(r_U) = \text{diam}(X) = D$$

r_U
•

Construction



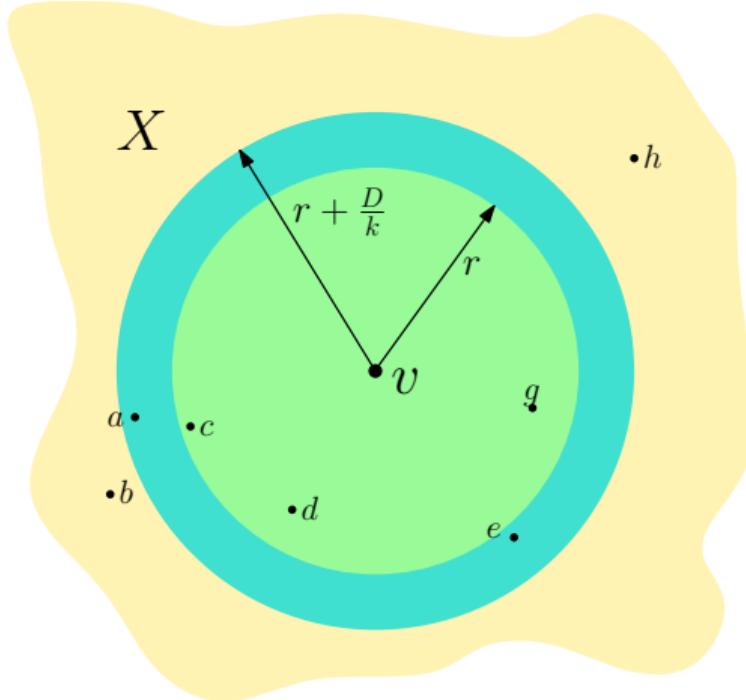
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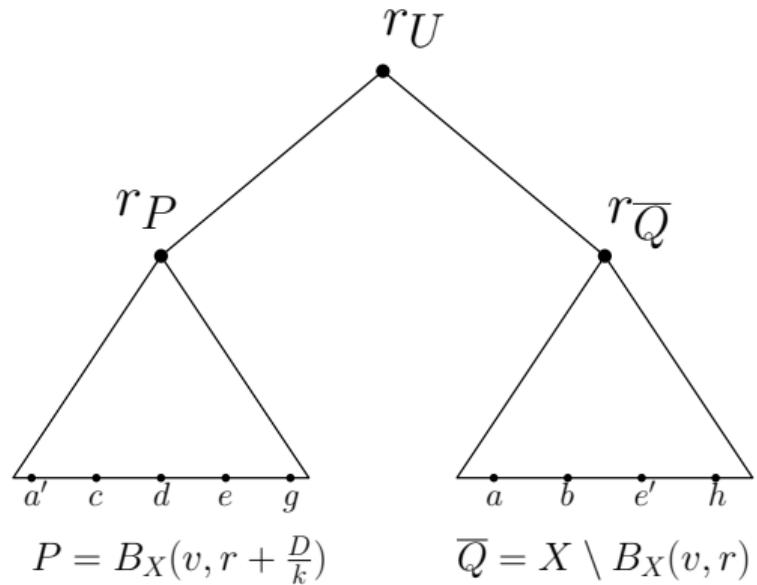
$$P = B_X(v, r + \frac{D}{k})$$

$$\overline{Q} = X \setminus B_X(v, r)$$

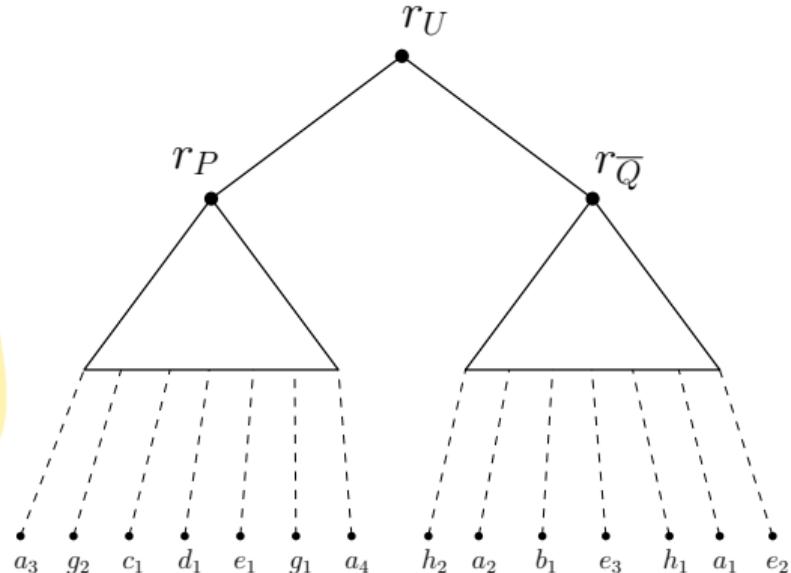
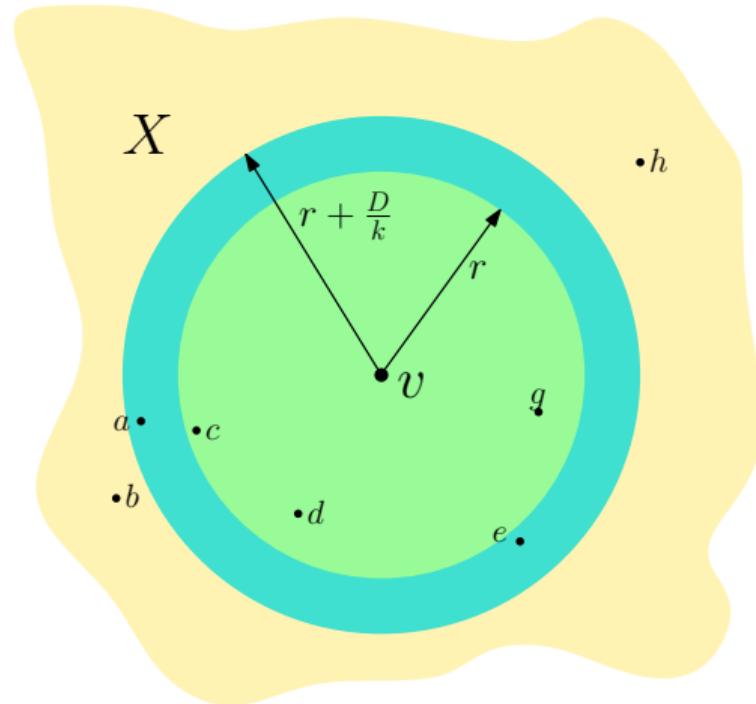
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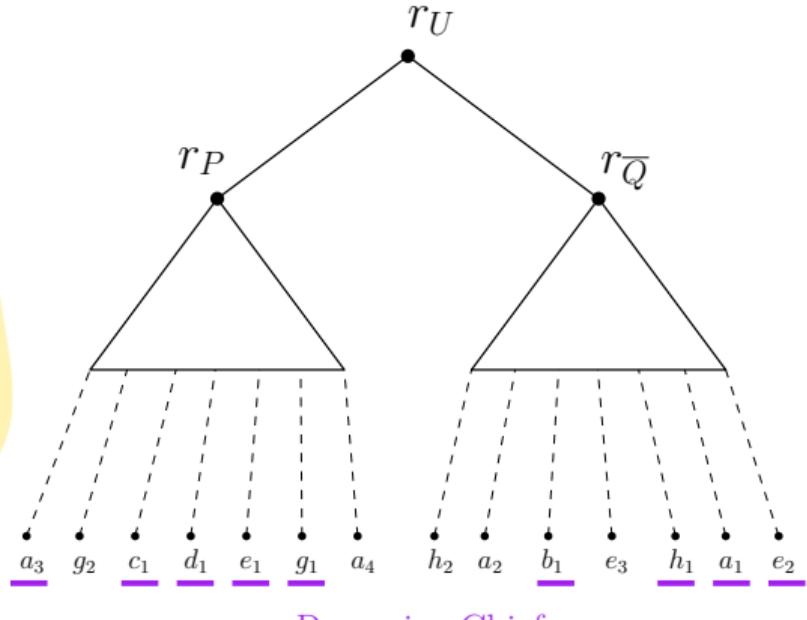
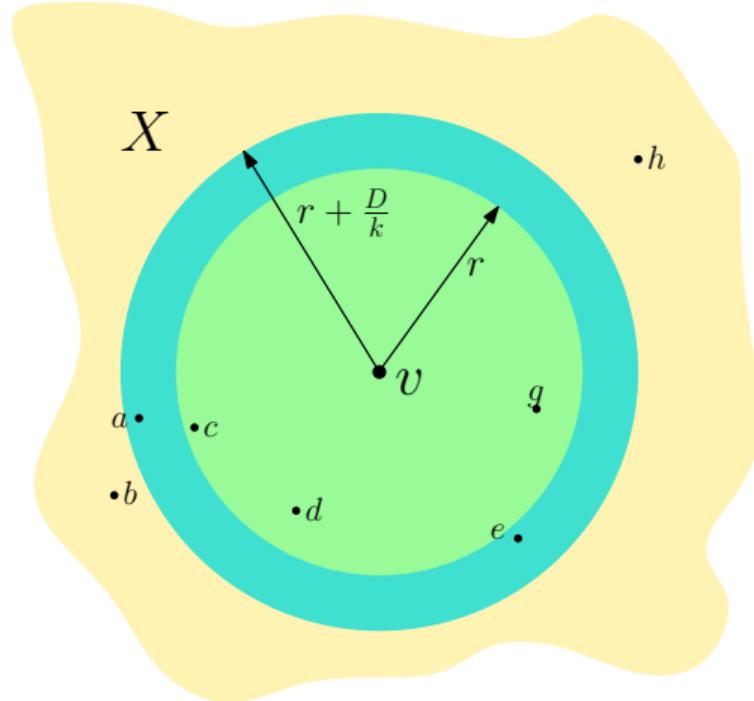
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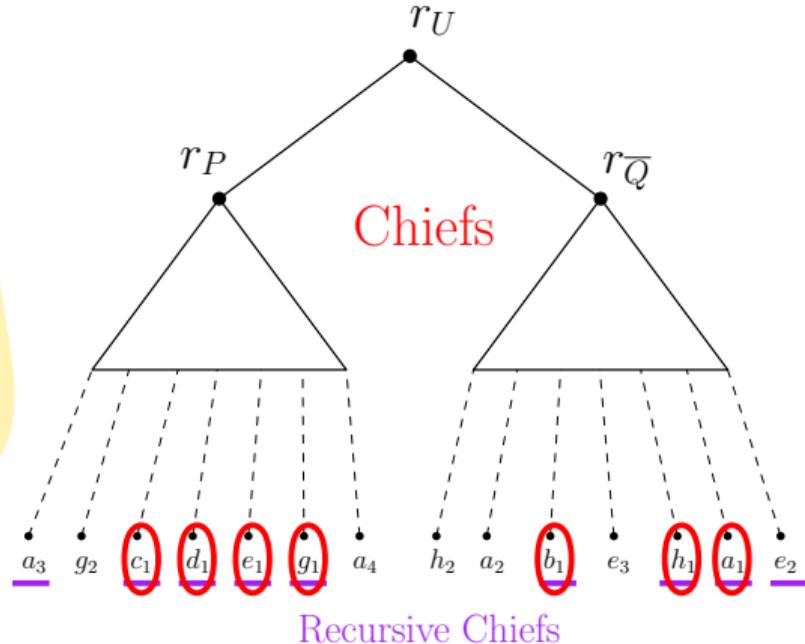
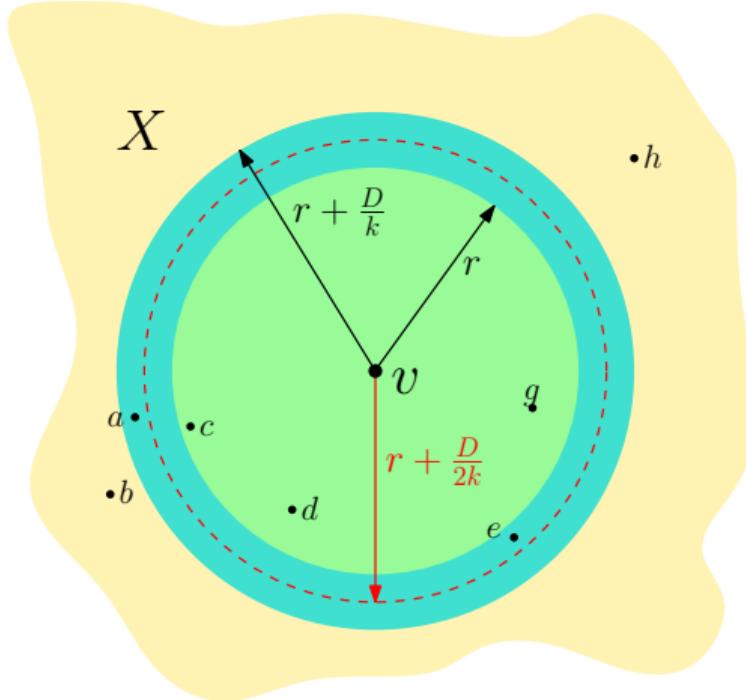
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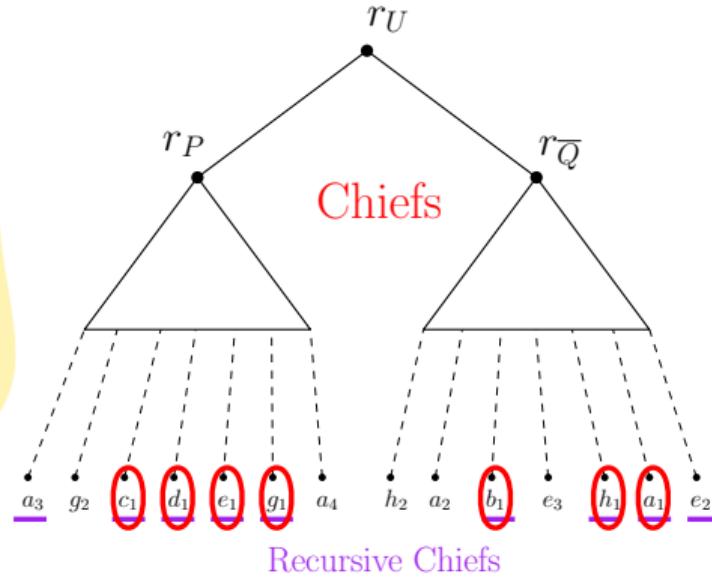
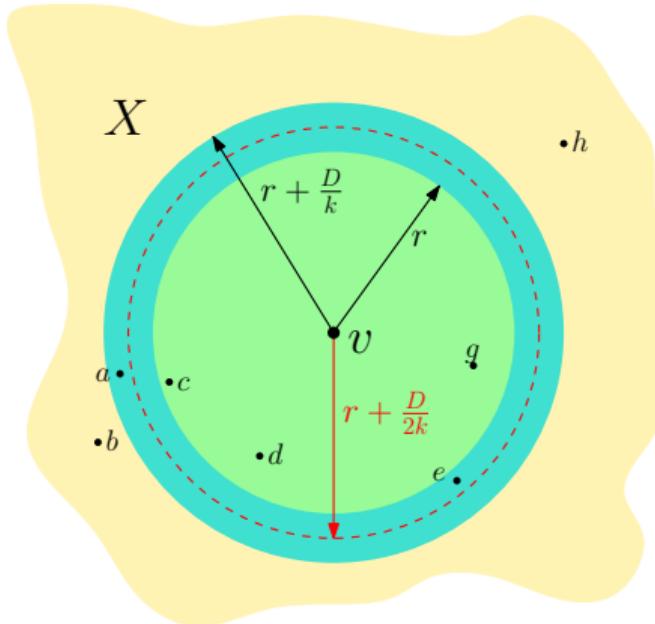
Construction



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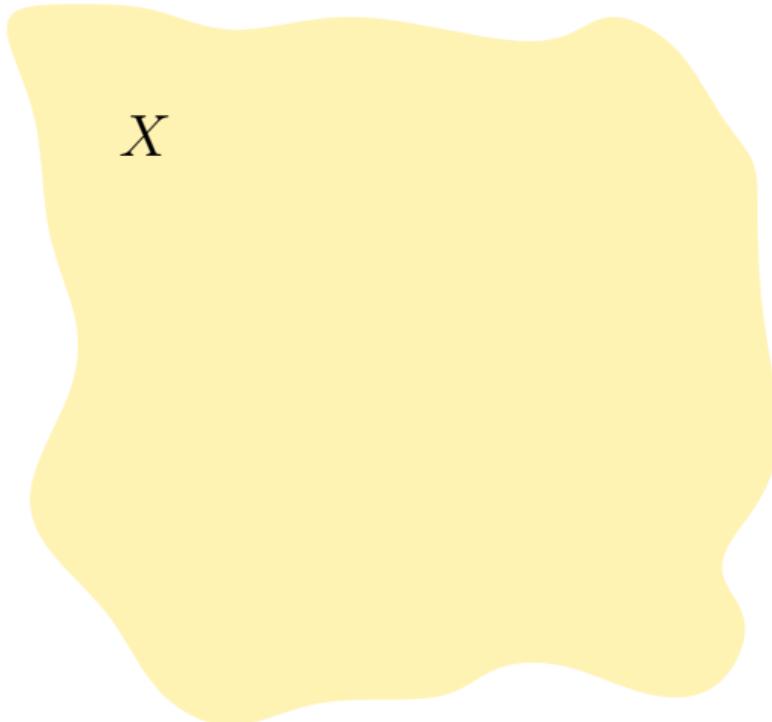


Construction - distortion bound



$$\min_{c' \in f(c)} d_U(c', \chi(a)) = D \leq 2k \cdot d_X(c, a) .$$

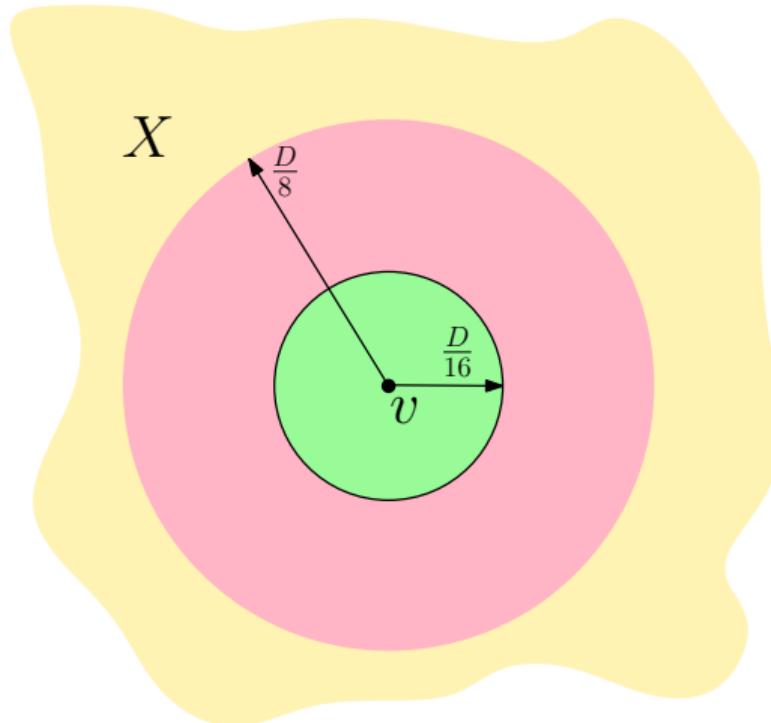
Construction - cardinality bound



X

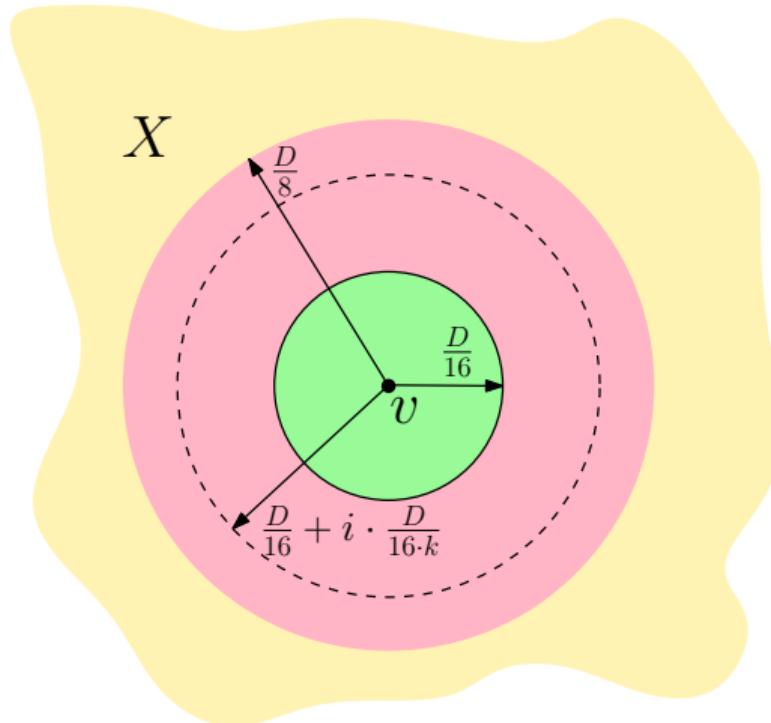
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Construction - cardinality bound



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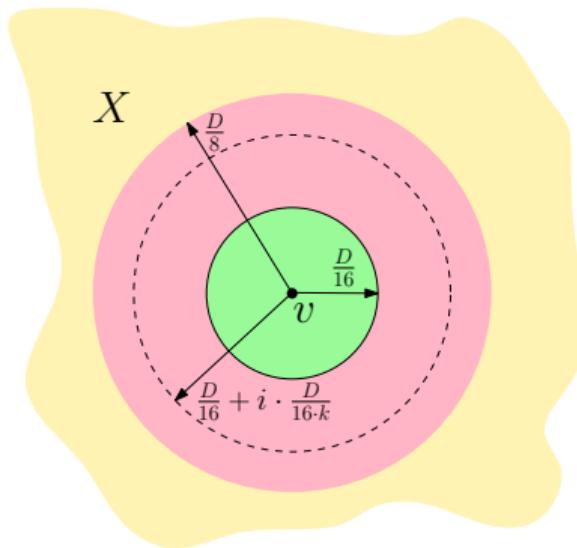
Construction - cardinality bound



$$D = \text{diam}(X)$$

$$A_i = B_X(v, \frac{D}{16} + i \cdot \frac{D}{16 \cdot k})$$
$$i \in \{0, \dots, k-1\}$$

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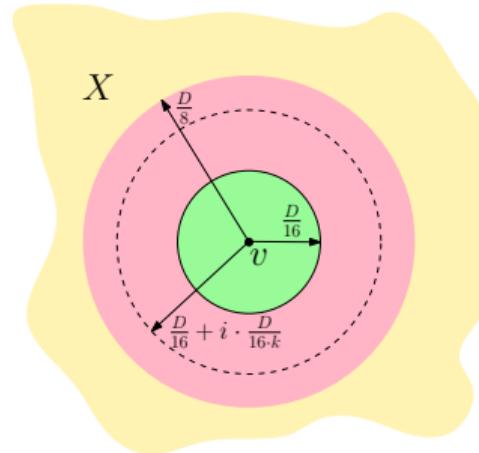


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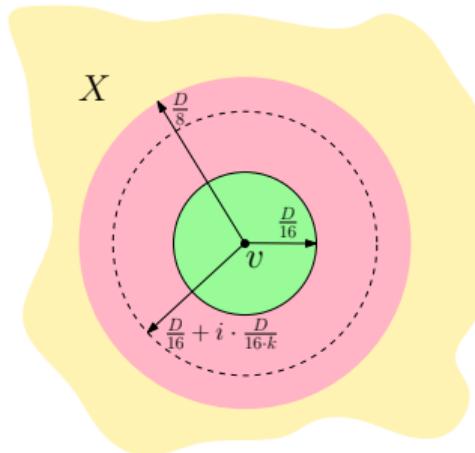
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$$|A_k| > |A_{k-1}| \cdot \left(\frac{|A_k|}{|A_0|}\right)^{\frac{1}{k}} > |A_{k-2}| \cdot \left(\frac{|A_k|}{|A_0|}\right)^{\frac{2}{k}} > \dots > |A_0| \cdot \left(\frac{|A_k|}{|A_0|}\right)^{\frac{k}{k}} = |A_k|$$

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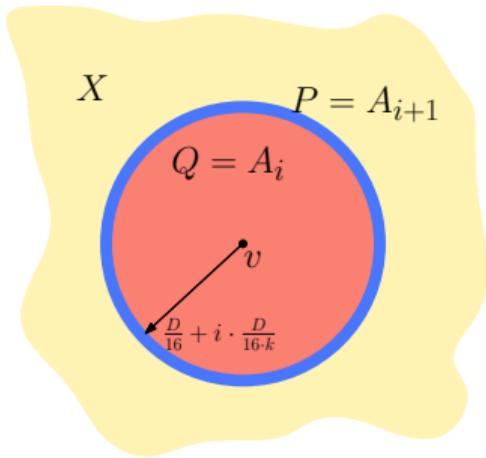
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Claim

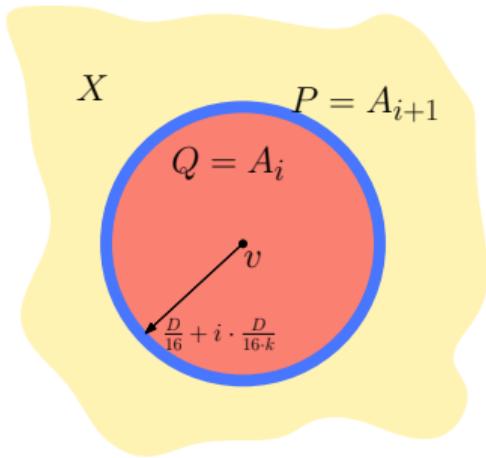
There is $v \in X$, and i , s.t. $\frac{|A_{i+1}|}{|A_i|} \leq \left(\frac{\mu^*(X)}{\mu^*(A_{i+1})}\right)^{1/k} = \left(\frac{\max_{x \in X} |B_X(x, \frac{\text{diam}(X)}{4})|}{\max_{x \in A_{i+1}} |B_{A_{i+1}}(x, \frac{\text{diam}(A_{i+1})}{4})|}\right)^{1/k}$.



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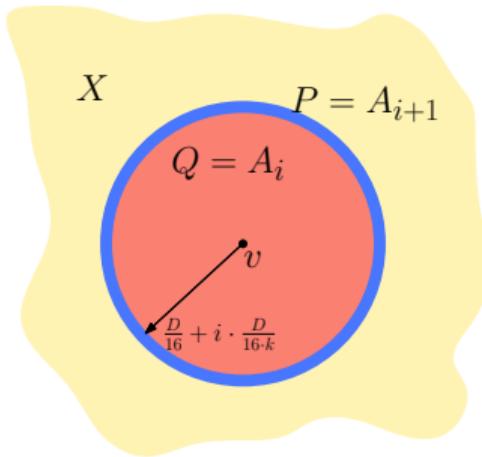


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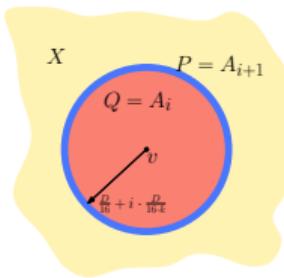
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