Outline of the talk

Metric Embeddings into Trees

Arnold Filtser

Bar-Ilan University

July 02, 2024

Metric space

A metric space is an ordered pair (X, d_X), where X is a set and $d_X : X \times X \to \mathbb{R}_{\geq 0}$ is a function such that:

1. **Identity**: $\forall x, y \in X,\ d_X(x, y) = 0 \iff x = y$.
2. **Symmetry**: $\forall x, y \in X,\ d_X(x, y) = d_X(y, x)$.
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Examples:

- Weighted graph $G = (V, E, w)$ with shortest path distance.

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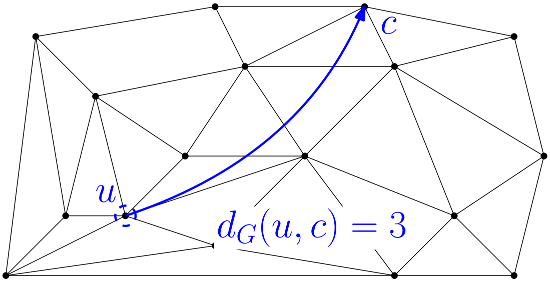

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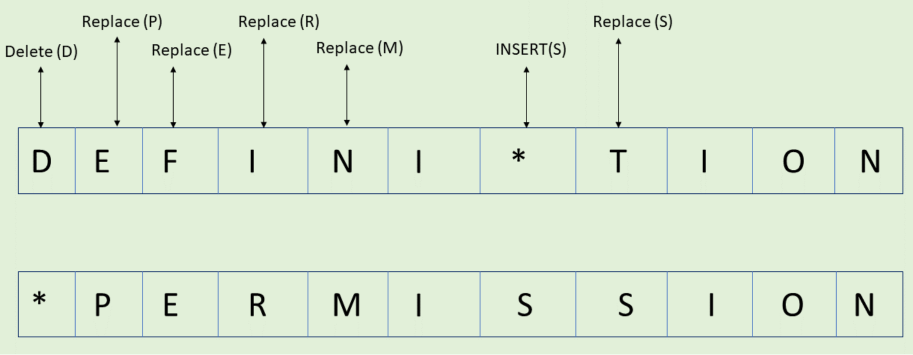

DEFINITION
EFINITION
PFINITION
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PRRNITION
PRRMI-TION
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- Euclidean space ℓ_2 in $\mathbb{R}^d$: $d_{\ell_2}(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|_2 = \sqrt{\sum_{i=1}^{d} (x_i - y_i)^2}$.

$$\left\| \begin{pmatrix} 5 \\ 8 \\ -3 \\ 4 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 10 \\ 1 \\ 1 \\ 3 \end{pmatrix} \right\|_2 = \sqrt{\underbrace{|5-1|^2}_{16} + \underbrace{|8-10|^2}_{4} + \underbrace{|(-3)-1|^2}_{16} + \underbrace{|4-1|^2}_{9} + \underbrace{|1-3|^2}_{4}} = 7$$

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Many problems are defined w.r.t. metric spaces. Examples:

- Metric TSP.
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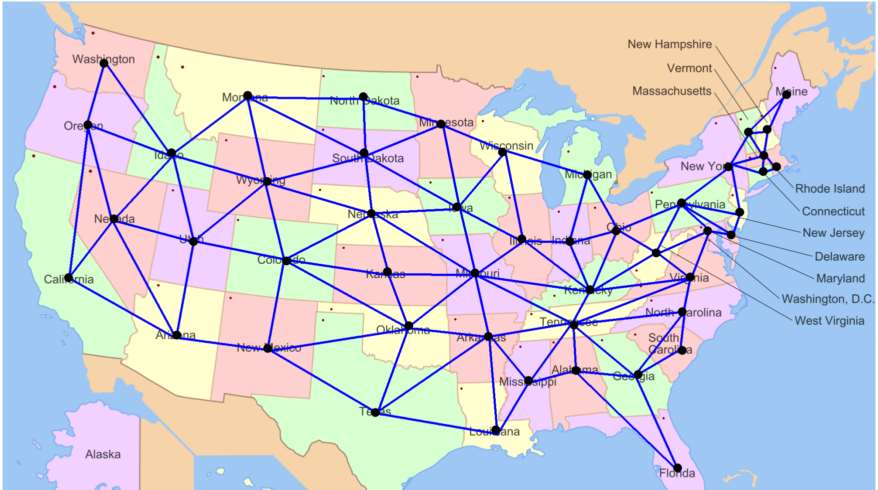
Given a metric space (X, d_X) find a permutation $x_0, x_1, \ldots, x_{n-1}$ of the points in X minimizing $\sum_{i=0}^{n-1} d_X(x_i, x_{i+1})$ (i.e. a cycle of minimum weight).



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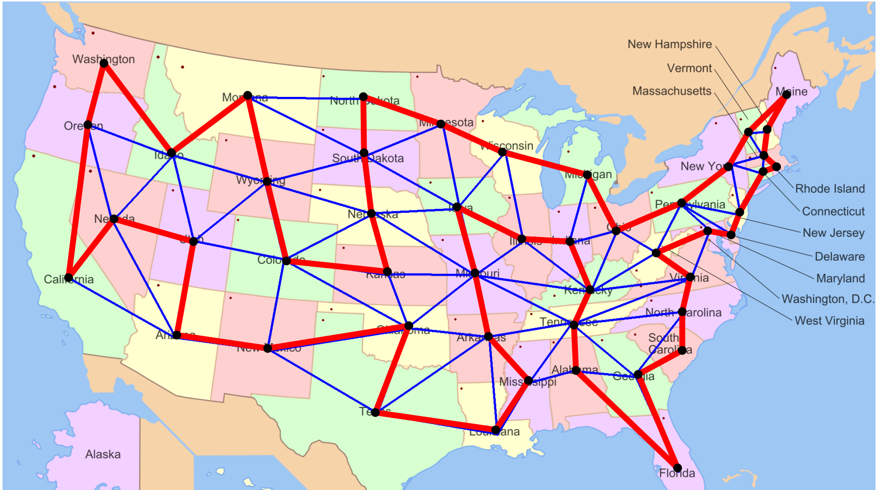

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NP-hard: a large class of equivalent problems (i.e. if you solved one-you solved all) for which we don't know of any efficient algorithms. It is generally believed that there are no efficient algorithms for these problems.

Theorem (Karp's list of 21 problems [Karp72])

The following problems are NP-Complete:

1. *SAT*
2. *0–1 integer programming*
3. *Clique*
4. *Set packing*
5. *Vertex cover*
6. *Set covering*
7. *Feedback node set*
8. *Feedback arc set*
9. *Directed Hamilton circuit*
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11. *3-SAT*
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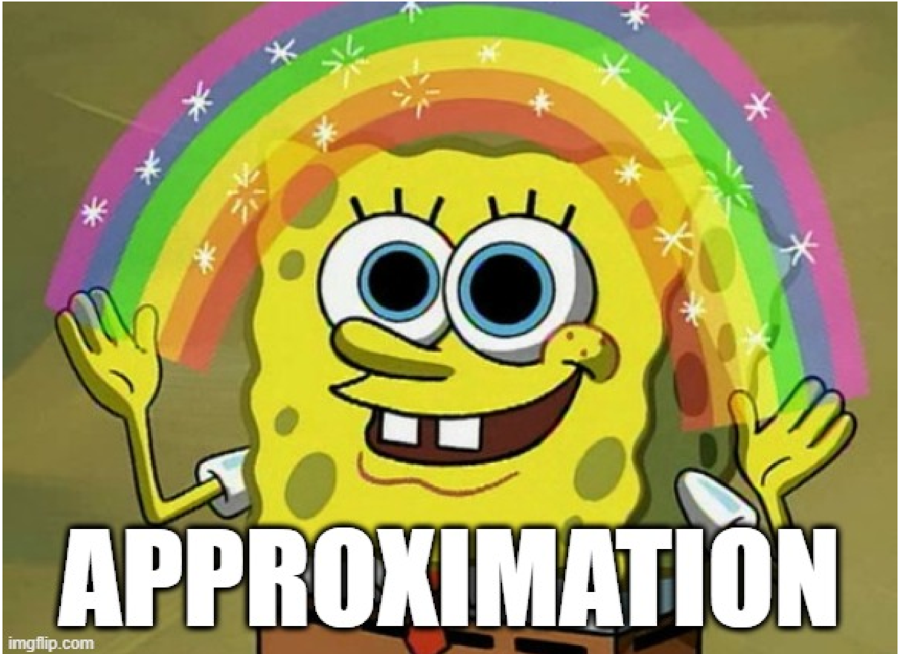
How should we cope with NP-hard problems?


APPROXIMATION
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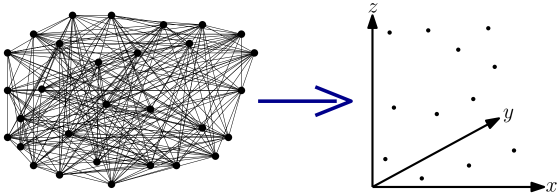
Metric Embeddings

Embedding

$(X, d_X), (Y, d_Y)$ metric spaces. $f : (X, d_X) \to (Y, d_Y)$ is called an **embedding**.

$f :$



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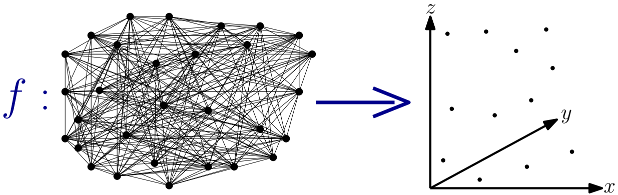

Preserve (approxierly) properties of the original space:

- **Distances**
- Cuts, Flows
- Commute time
- Effective resistance
- Clustering statistics.
- etc.

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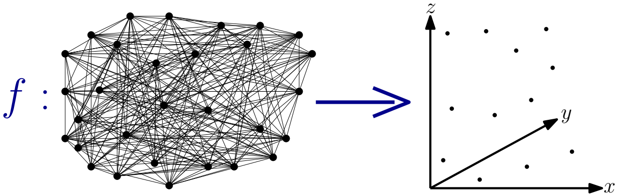

f has **distortion** t if:

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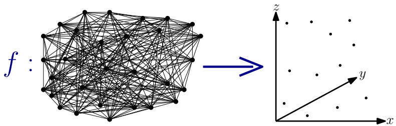

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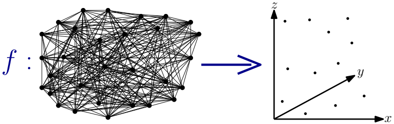

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So that we could run efficient algorithms on it...

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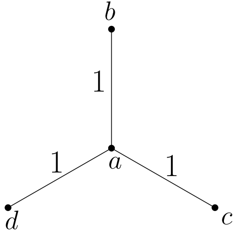

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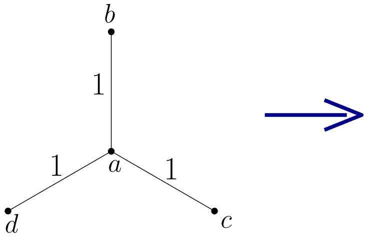

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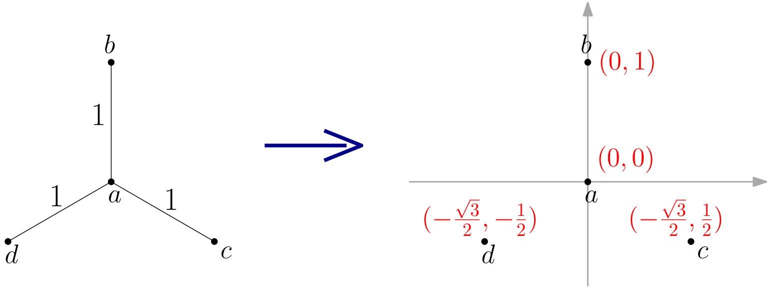

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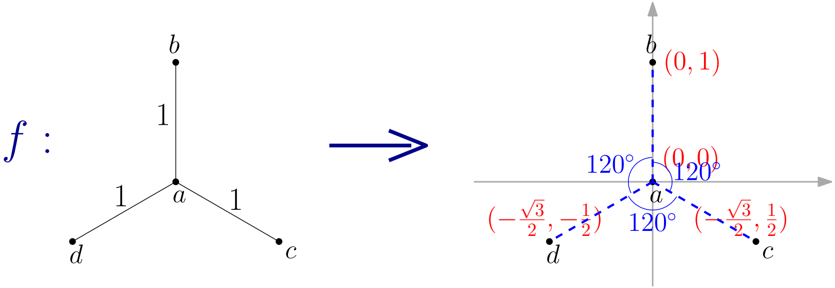

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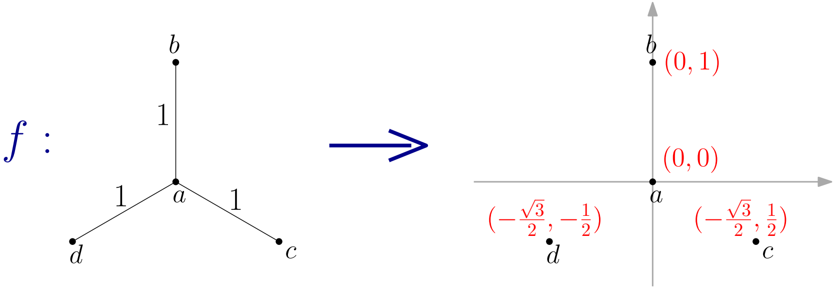

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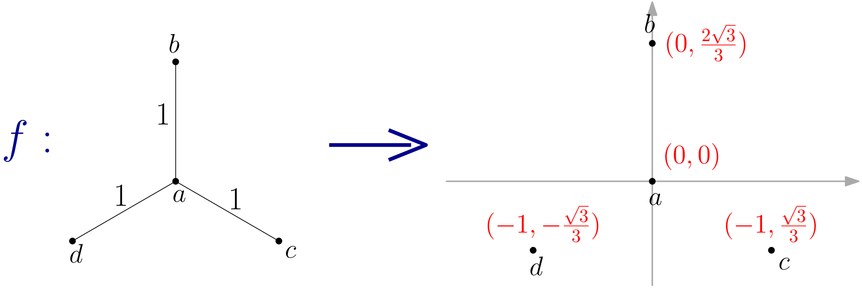

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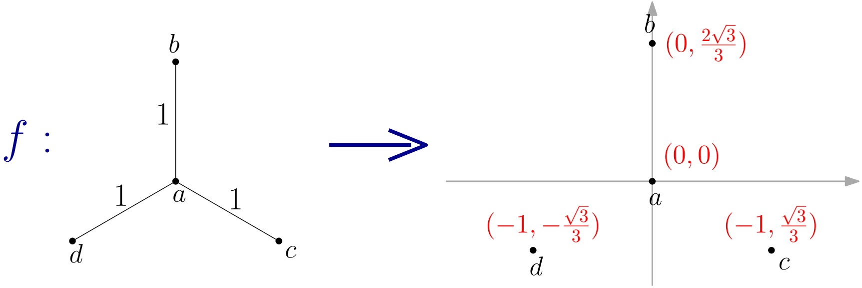

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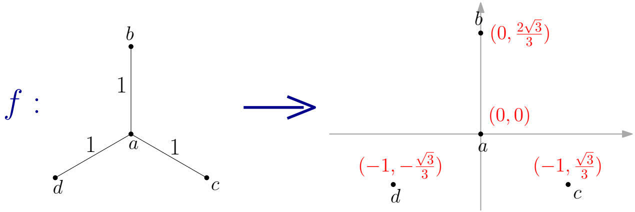

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The distortion of the embedding is $\frac{2}{\sqrt{3}} \approx 1.1547$.

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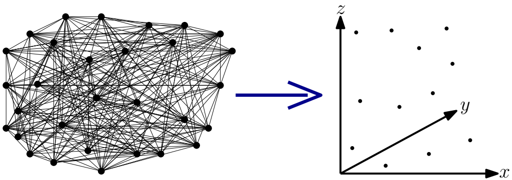

Theorem ([Bourgain 85])

Every n-point metric (X, d_X) is ***embeddable*** *into* <u>*Euclidean*</u> *space $(\mathbb{R}^d, \|\cdot\|_2)$ with* ***distortion*** *$O(\log n)$.*

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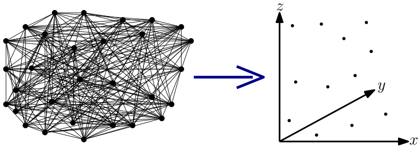
f : 

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Theorem ([Linial, London, Rabinovich 95])

[Bou85] is tight.

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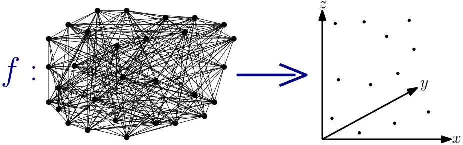
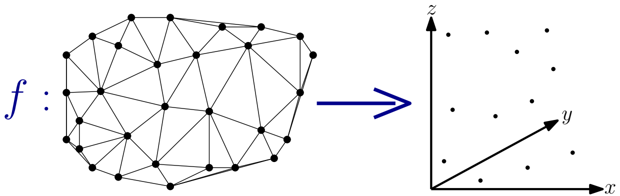


Applications:

- Approximation algorithms (e.g. **sparsest cut**, min graph bandwidth)
- Parallel computation (e.g. SSSP in MPC)
- Computational Biology (e.g. clustering and detecting protein seq.)
- etc.

Theorem ([Rao 99])

*Every n-point **planar** metric (X, d_X) is **embeddable** into Euclidean space $(\mathbb{R}^d, \|\cdot\|_2)$ with **distortion** $O(\sqrt{\log n})$.*



Planar metric- the shortest path metric of a planar graph.

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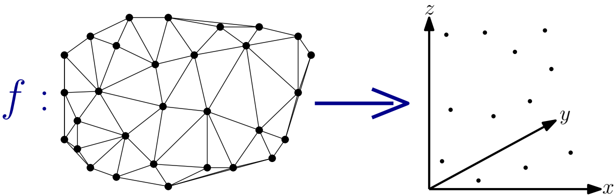

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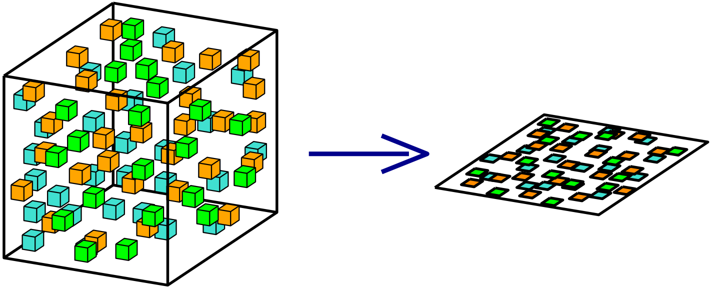
Theorem ([Newman, Rabinovich 03])

[Rao99] is tight.

Embedding

$(X, d_X), (Y, d_Y)$ metric spaces. $f : (X, d_X) \to (Y, d_Y)$ is called an **embedding**.

$f :$ 

Theorem ([Johnson, Lindenstrauss 84], Dimension Reduction)

$X \subset (\mathbb{R}^d, \|\cdot\|_2)$ set of size n. Then X embeds into $O(\log n/\epsilon^2)$ dimensional Euclidean space with distortion $1+\epsilon$.

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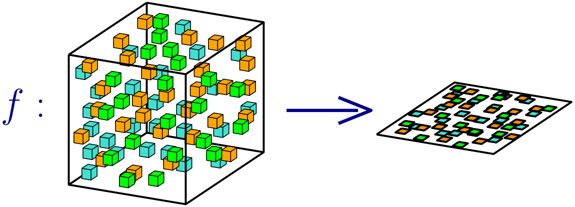

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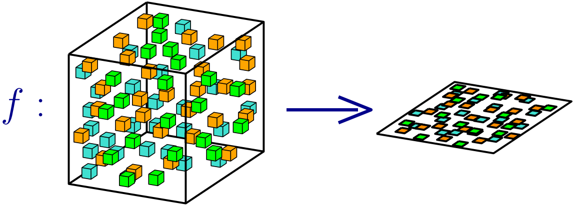


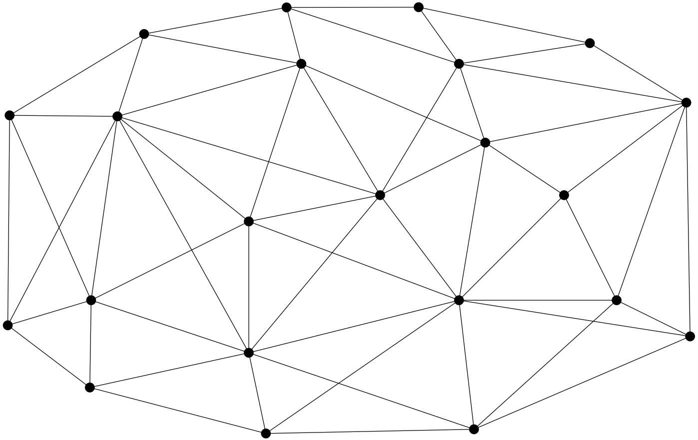
Applications:

- Speeding up-computation
- Clustering
- Nearest Neighbor Search
- Machine Learning
- etc.

Graph Spanners

$G = (V, E, w)$ weighted graph, a t-**spanner** is a **subgraph** $H = (V, E_H)$

s.t. $\forall u, v \in V, \quad d_H(u, v) \leq t \cdot d_G(u, v)$



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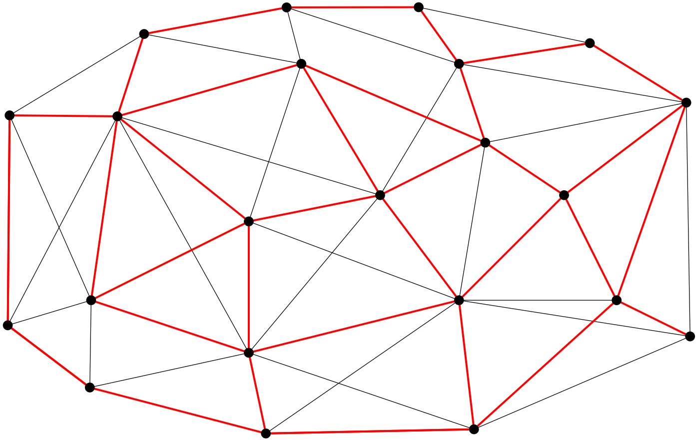

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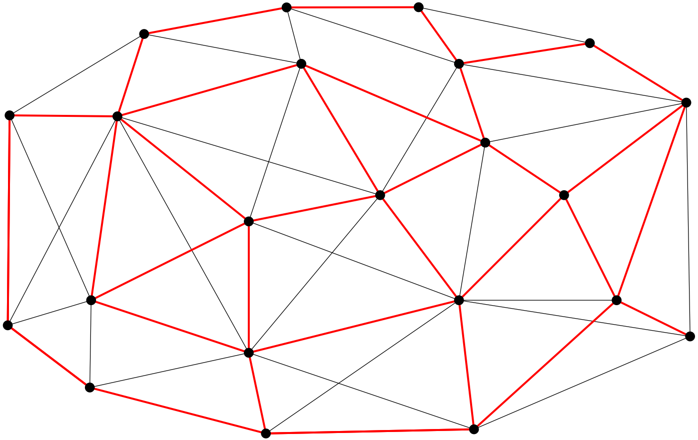

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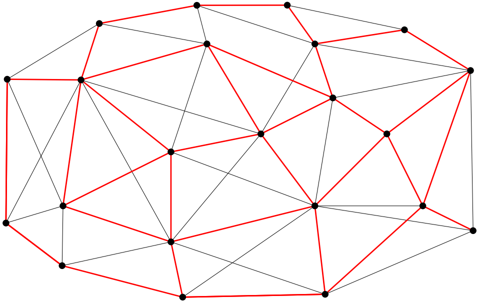

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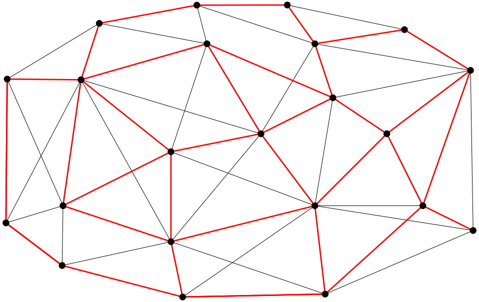

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Applications:

- Approximation Algorithms (e.g. PTAS for TSP)
- Distributed Computing
- Network Routing
- Computational Biology (e.g. measure genetic distance)
- etc.

Outline of the talk

Embedding into Trees

Tree is very **simple** and **desirable** target space.

Many NP-hard problems are easy on trees (using dynamic programming).

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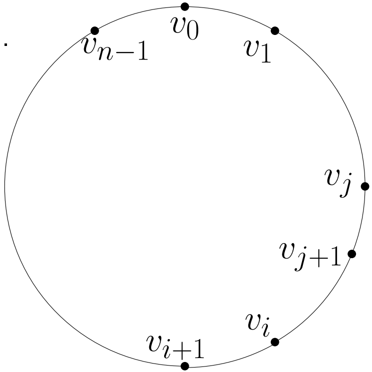

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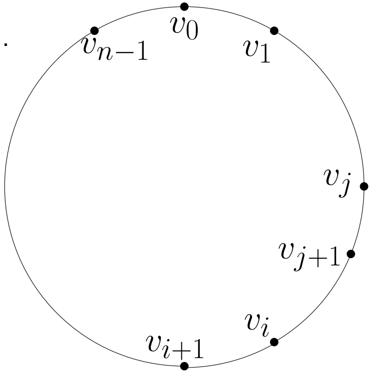

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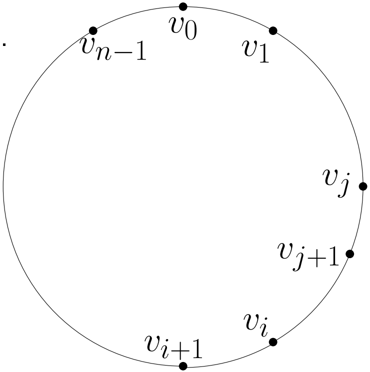

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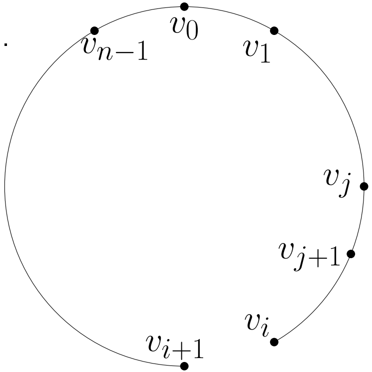

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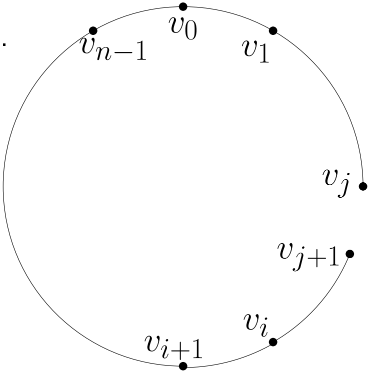

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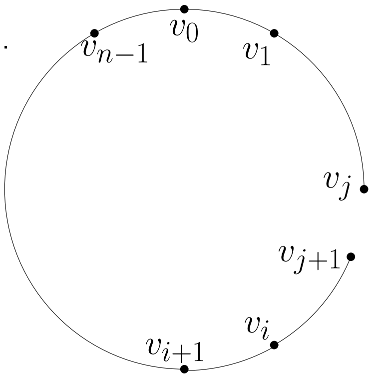

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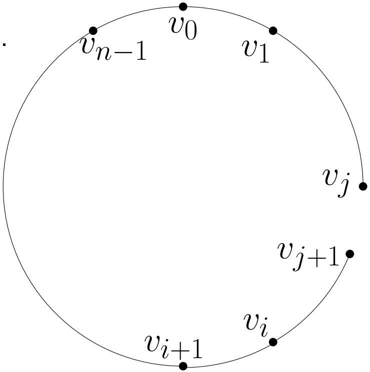

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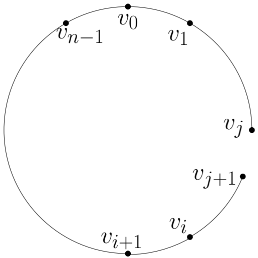

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By triangle inequality and linearity of expectation

$$\forall v_i,v_j,\quad \mathbb{E}_{T\sim\mathcal{D}}[d_T(v_i,v_j)] = \sum_{q=i}^{j-1}\mathbb{E}_{T\sim\mathcal{D}}[d_T(v_q,v_{q+1 \pmod{n}})] \le 2\cdot d_{C_n}(v_i,v_j)\ .$$

Stochastic Embedding into Trees

Theorem ([Fakcharoenphol, Rao, Talwar 04], improving [Bartal 96+98])

Every n-point metric space (X, d) embeds into ***distribution*** $\mathcal{D}$ *over* ***dominating trees*** *with* ***expected distortion*** $O(\log n)$.

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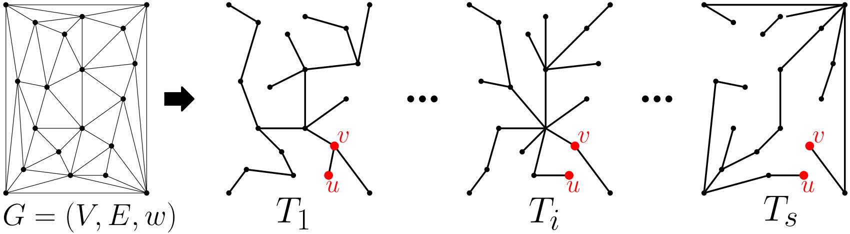

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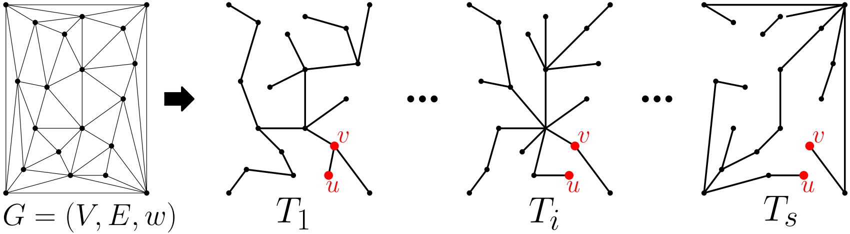

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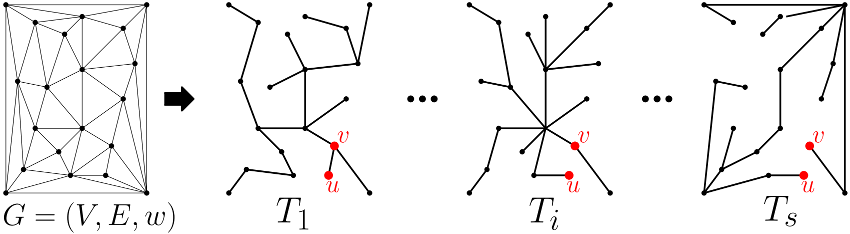

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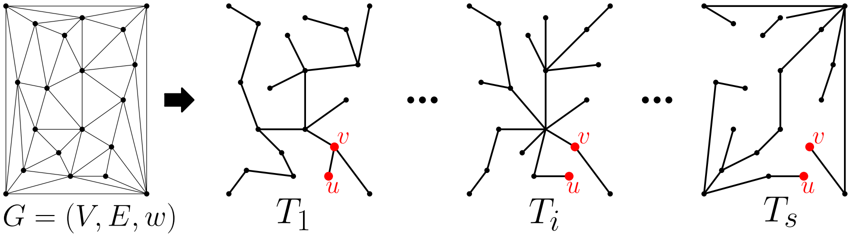

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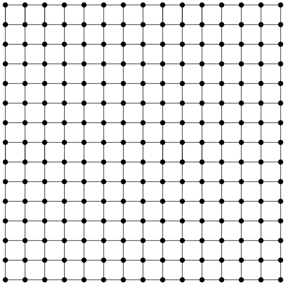
In fact, tight already for the $n \times n$ grid graph!



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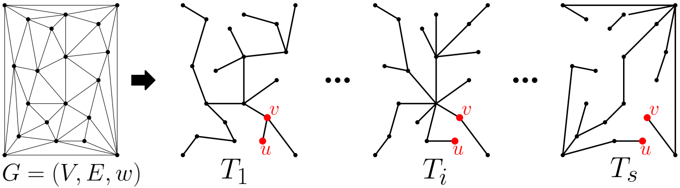

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Transforms arbitrary metric into a tree!

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Applications:

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Outline of the talk

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Could we do better by allowing the oracle to returned approximated distances?

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Sample $s = 4 \log n$ trees $T_1, \ldots, T_s$. Given x, y return $\mathrm{DO}(x, y) = \min_{i\in[1,s]} d_{T_i}(x, y)$.

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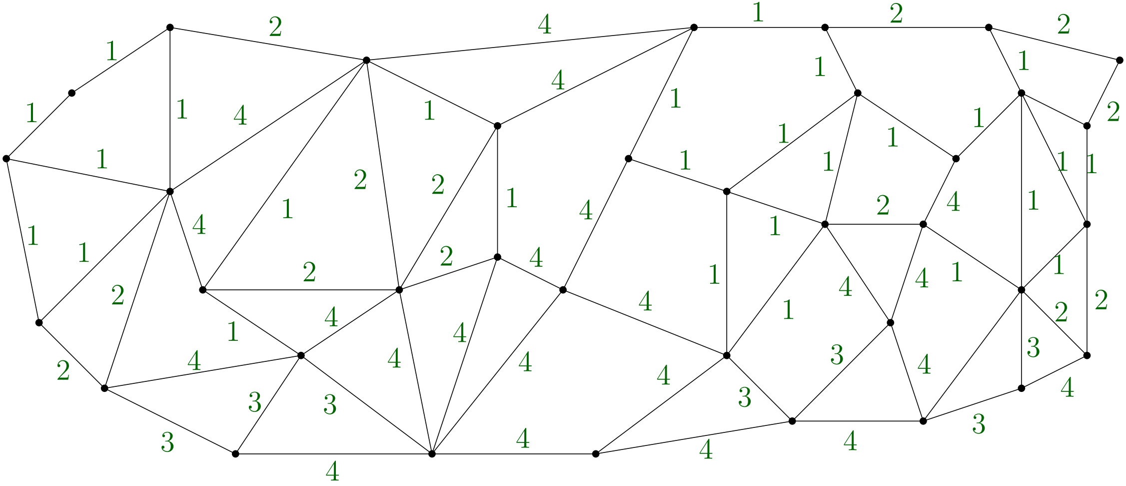
Outline of the talk

Minimum Spanning Tree (MST)

Given a weighted graph $G = (V, E, w)$ find a spanning tree if minimum total weight.



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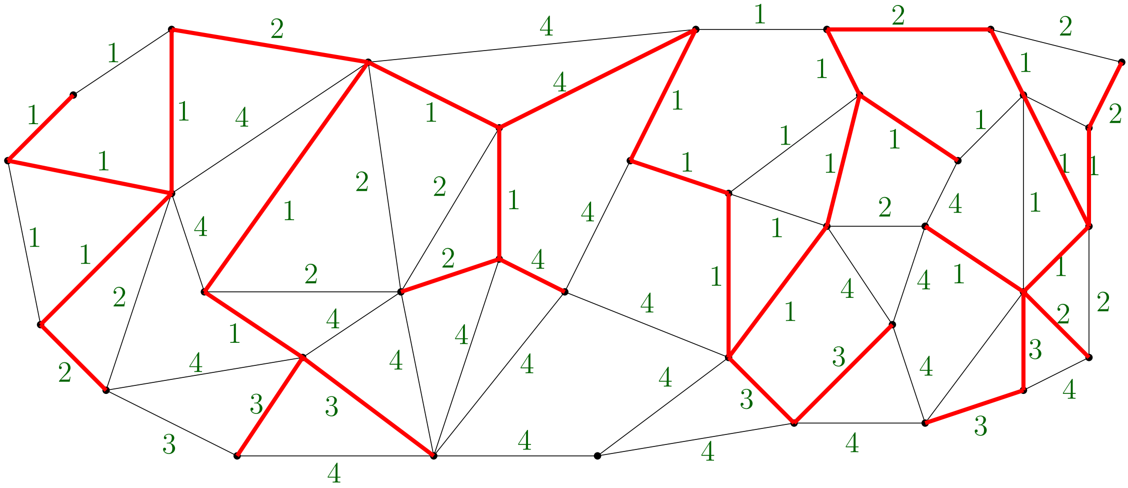

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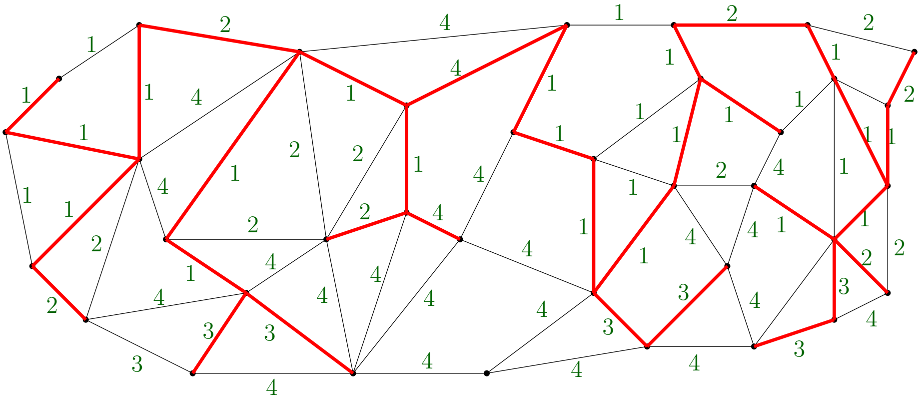
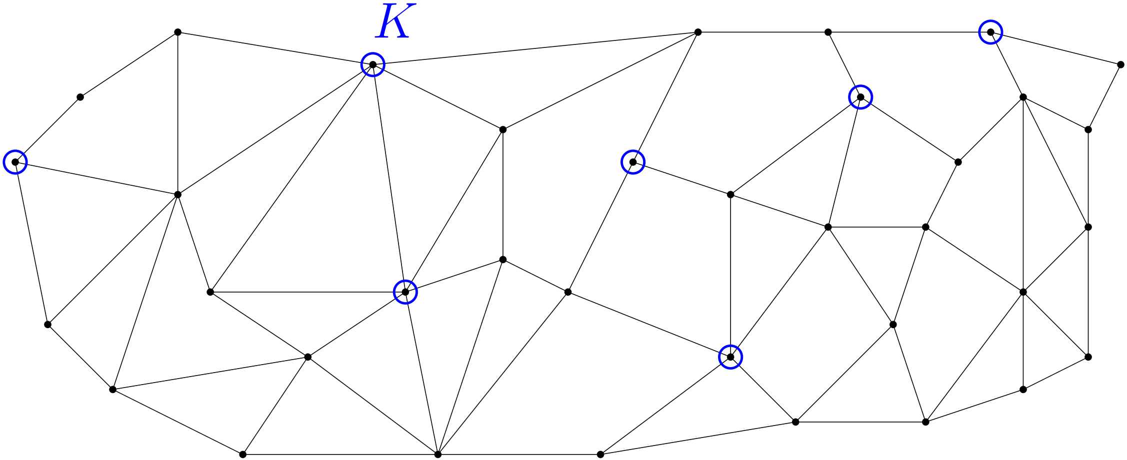


Classic problem, admits efficient poly-time solution.

Steiner Tree

Given set of terminals K, find minimum weight tree T spanning K



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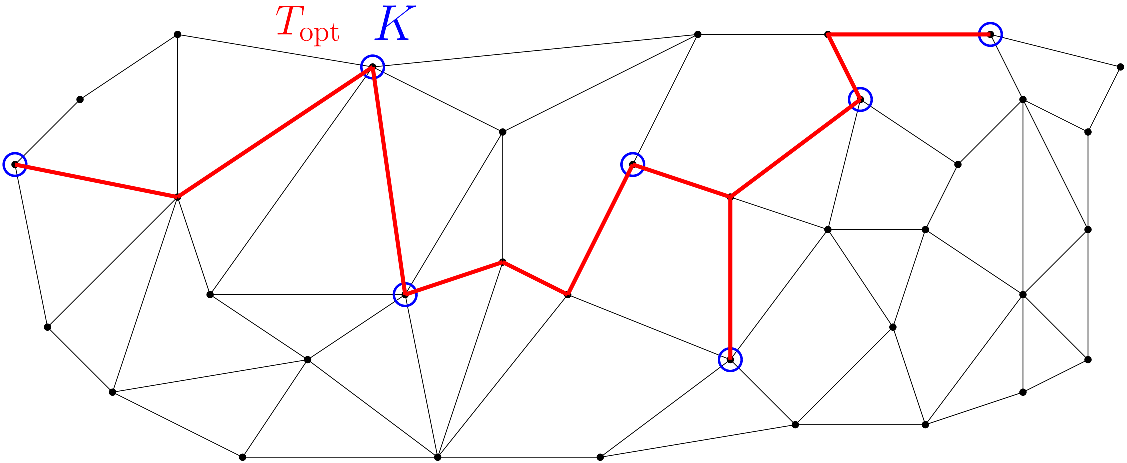

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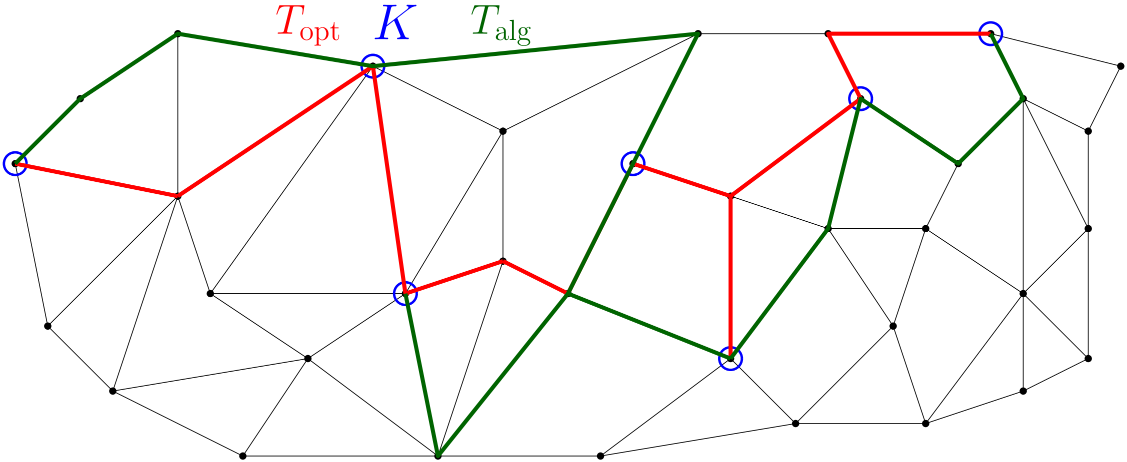

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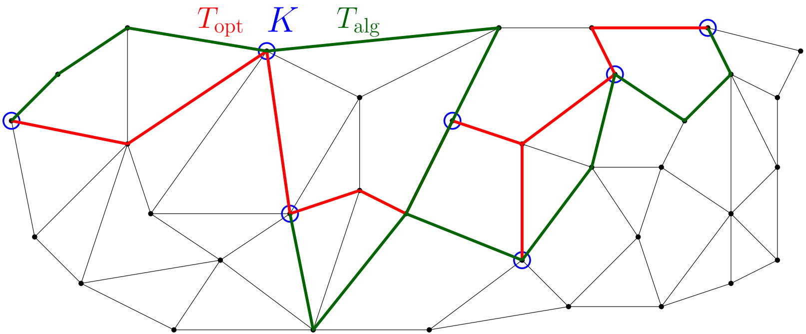

NP-hard. There is a simple 2-approximation algorithm for the Steiner tree problem.

That is, there is a polynomial time algorithm that returns a tree T_{alg} of weight at most $w(T_{\mathrm{alg}}) \leq 2 \cdot w(T_{\mathrm{opt}})$.

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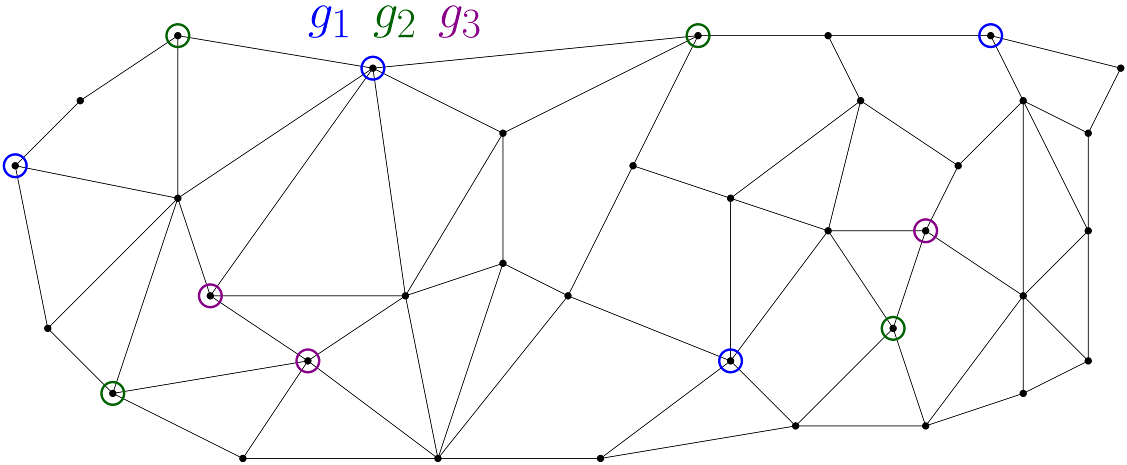
Given subsets $g_1, g_2, \ldots, g_k \subseteq V$, find minimum weight tree T spanning at least one vertex from each g_i



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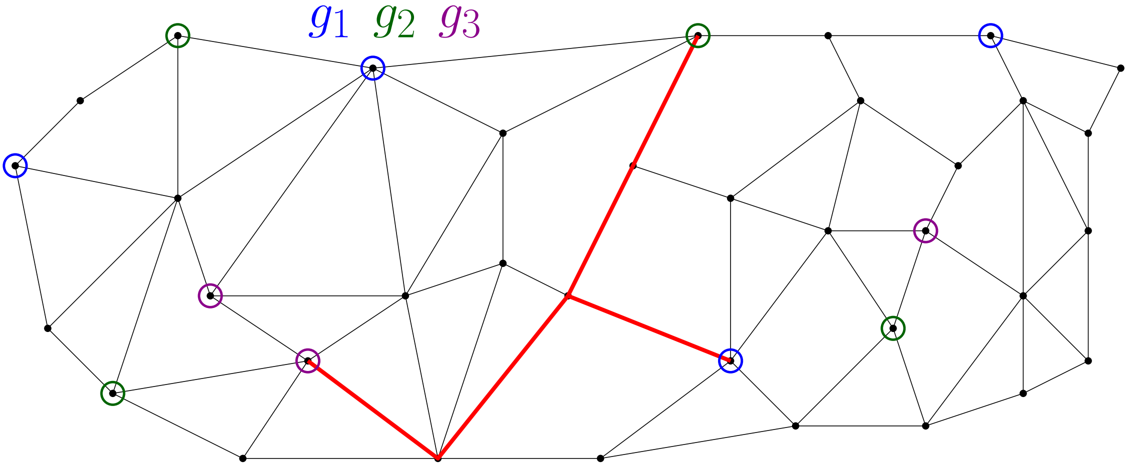

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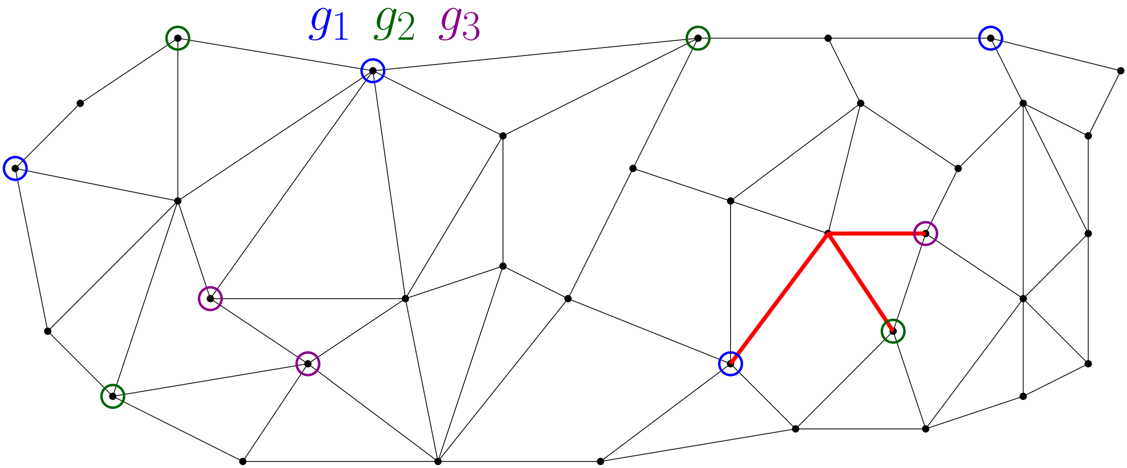

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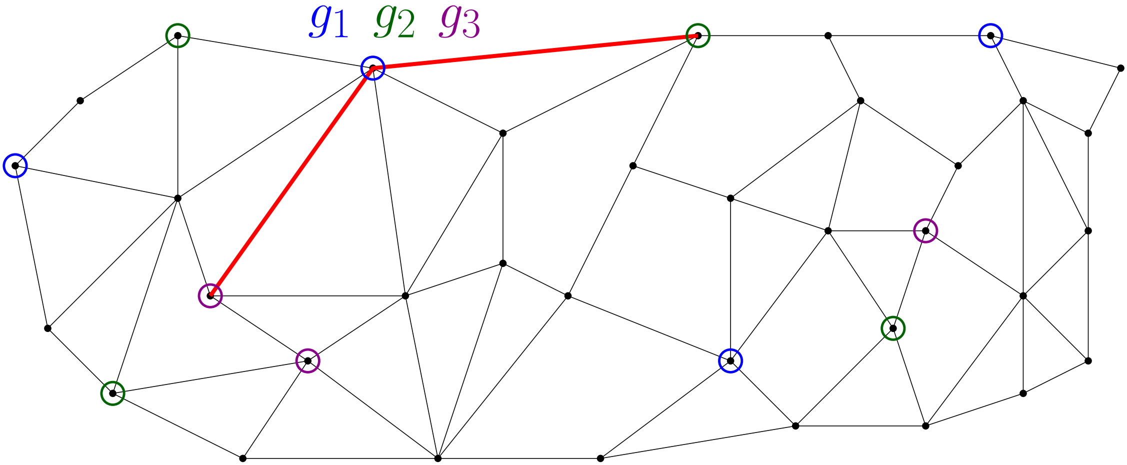

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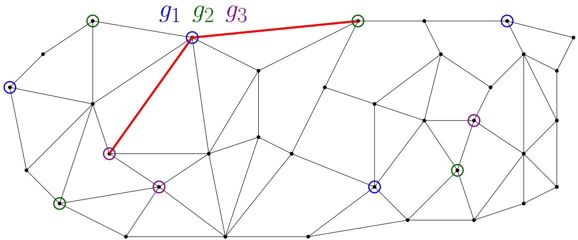

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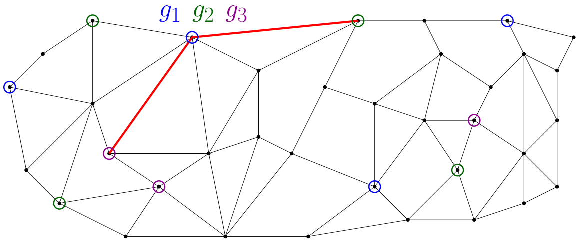

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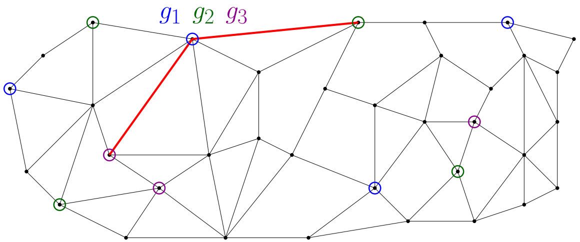

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We will use stochastic tree embeddings to generalize [GKR00] to general graphs.

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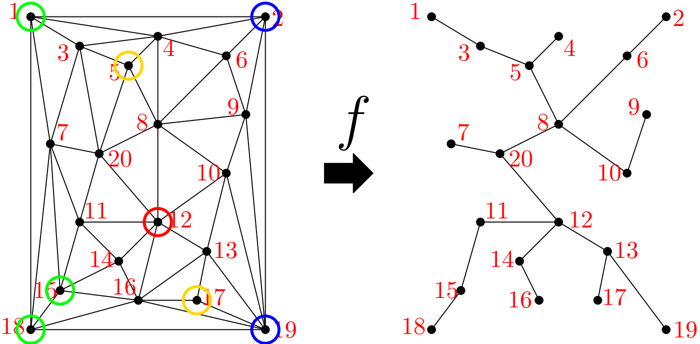
Embedding f with expected distortion $O(\log n)$.



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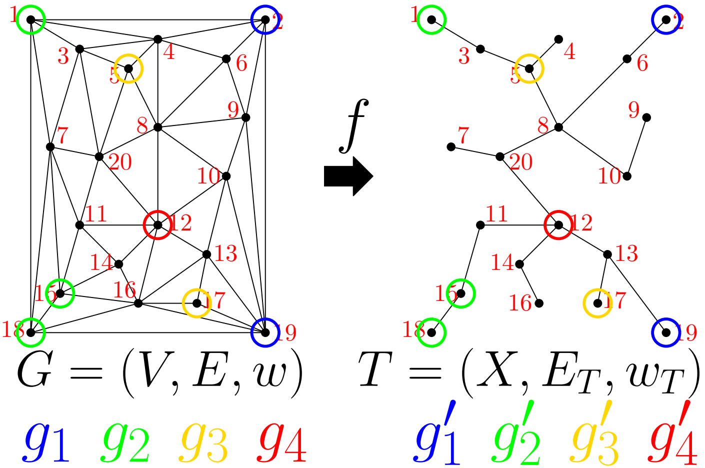

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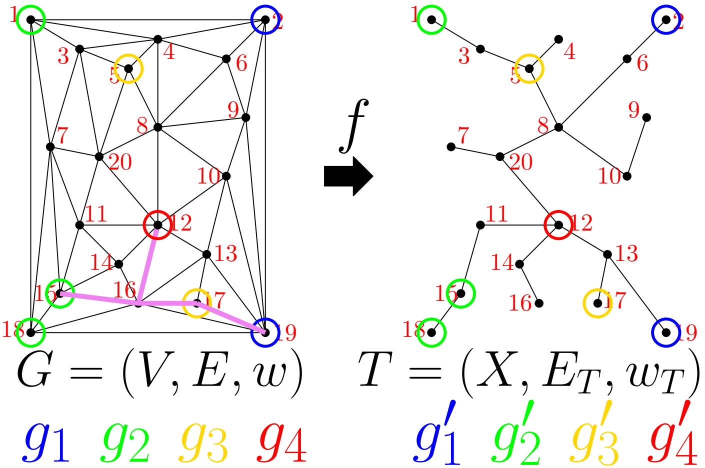

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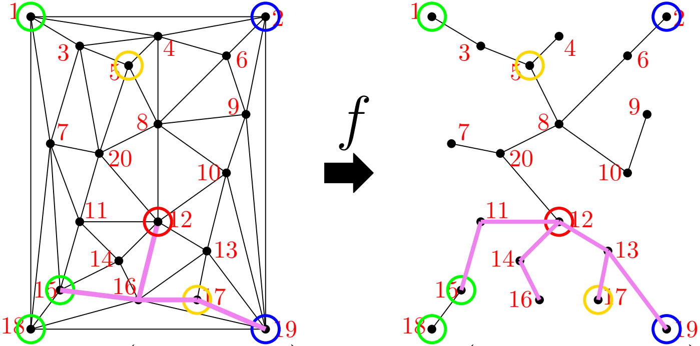
$S^\star_T$: $\forall (u,v) \in S^\star$ add the path from $f(u)$ to $f(v)$.
(valid solution)



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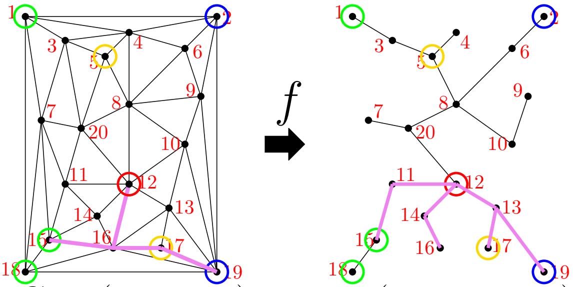

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$$\mathbb{E}[w_T(S^\star_T)] \le \sum_{(u,v)\in S^\star} \mathbb{E}\left[d_T(f(u), f(v))\right] = O(\log n) \cdot \sum_{(u,v)\in S^\star} d_T(f(u), f(v))$$

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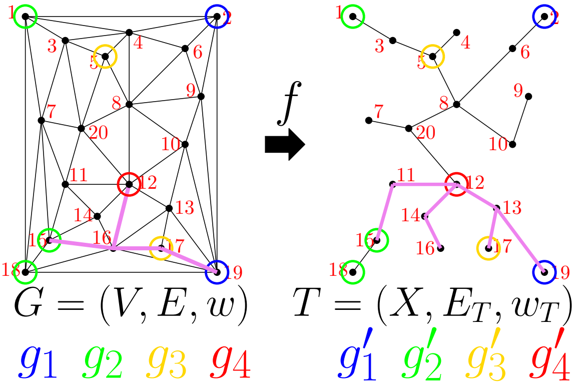

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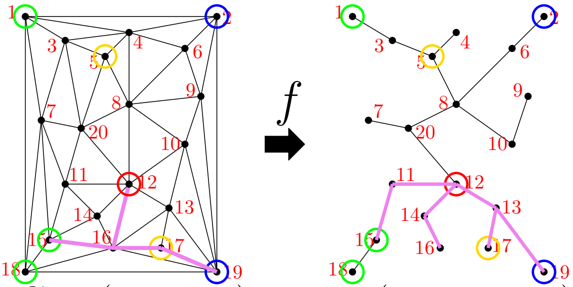

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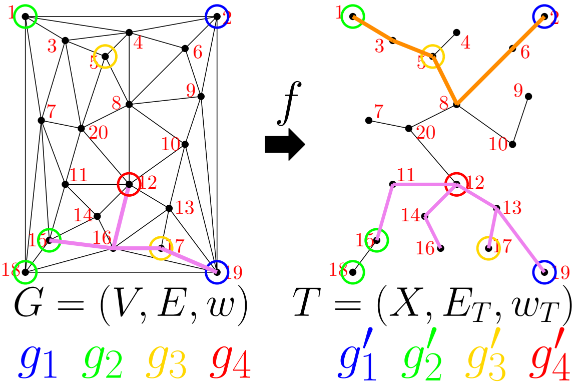

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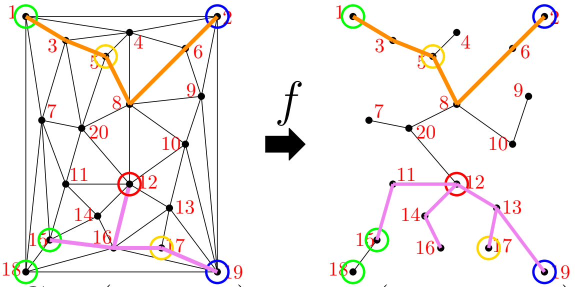

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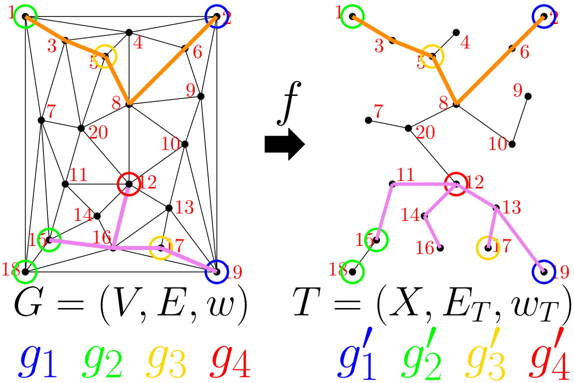

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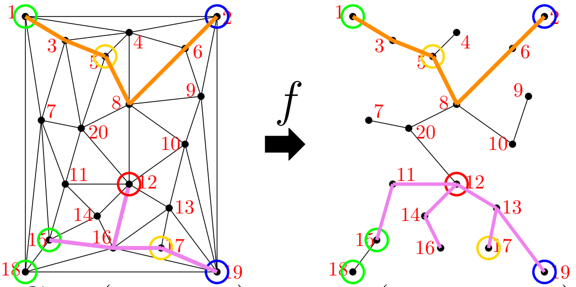

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We got an $O(\log^2 n \cdot \log k)$ **approximation** (in expectation)

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How to get the approximation guarantee with high probability?

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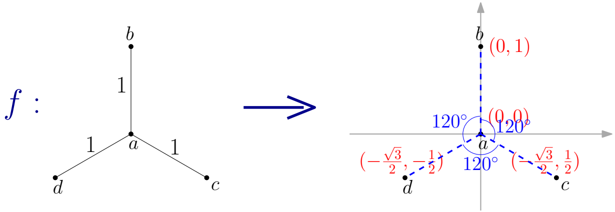
How to get the approximation guarantee with high probability?

Repeat the process $O(\log n)$ times, and return the observed solution of minimum weight.

Outline of the talk

$f : (X, d_X) \to (Y, d_Y)$ has **distortion** t if:

$$\forall x, y \in X, d_X(x, y) \leq d_Y(f(x), f(y)) \leq t \cdot d_X(x, y) \ .$$



	a	b	c	d
a		1	1	1
b	1		2	2
c	1	2		2
d	1	2	2	

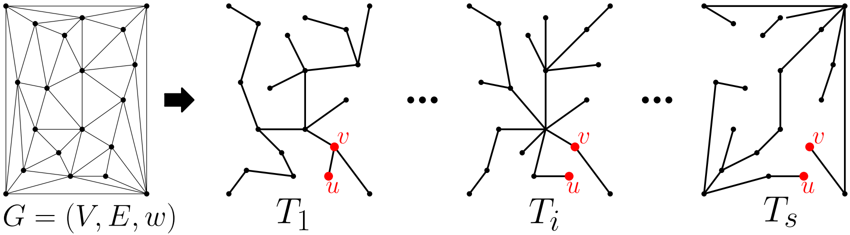
	a	b	c	d
a		$2/\sqrt{3}$	$2/\sqrt{3}$	$2/\sqrt{3}$
b	$2/\sqrt{3}$		2	2
c	$2/\sqrt{3}$	2		2
d	$2/\sqrt{3}$	2	2	

The distortion of the embedding is $\frac{2}{\sqrt{3}} \approx 1.1547$.

Stochastic Embedding into Trees

Theorem ([Fakcharoenphol, Rao, Talwar 04], improving [Bartal 96+98])

Every n-point metric space (X, d) embeds into ***distribution*** $\mathcal{D}$ *over* ***dominating trees*** *with* ***expected distortion*** $O(\log n)$.



For every $u, v \in X$ and $T \in \mathrm{supp}(\mathcal{D})$, $d_X(u, v) \leq d_T(f(u), f(v))$.

For every $u, v \in X$ $\quad \mathbb{E}_{T\sim\mathcal{D}}[d_T(f(u), f(v))] \leq O(\log n) \cdot d_X(u, v)$.

[Alon, Karp, Peleg, West 95]: Tight!

Distance Oracle construction

Sample $s = 4\log n$ trees $T_1, \ldots, T_s$. Given x, y return $\mathrm{DO}(x, y) = \min_{i\in[1,s]} d_{T_i}(x, y)$.

Clearly, as the trees are **dominating**, $\mathrm{DO}(x, y) = \min_{i\in[1,s]} d_{T_i}(x, y) \geq d_X(x, y)$.

Thus with high probability, for every $x, y \in X$

$$\mathrm{DO}(x, y) < 2 \cdot \mathbb{E}_{T\sim\mathcal{D}}[d_T(x, y)] = O(\log n) \cdot d_G(x, y) \ .$$

Space: storing $O(\log n)$ trees. Total space is $O(n \log n)$ (machine words).

Query time: computing $d_{T_i}(x, y)$ for $i \in [1, s]$.

There is a data structure computing distance in (some) trees in $O(1)$ time.

Overall $O(\log n)$ query time.

Overall we obtained distance approximation $O(\log n)$ with $O(\log n)$ query time and $O(n \log n)$ space.

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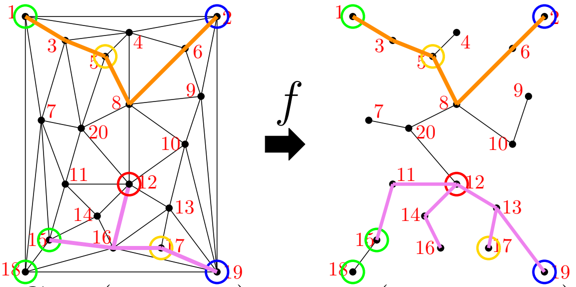

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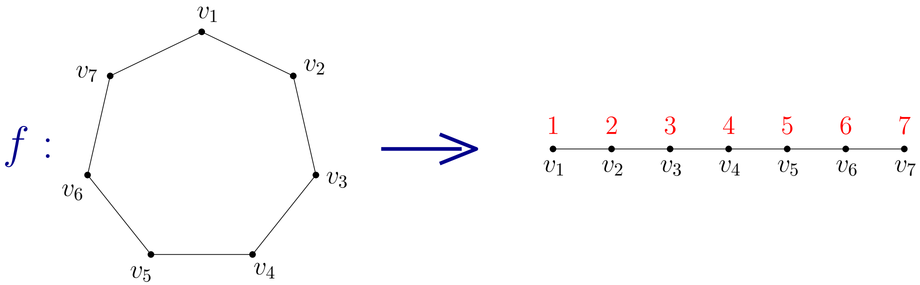
Come and talk to me, there are fun and challenging problems (and a generous scholarship) waiting for you.

You can learn about my research from many different videos in my home-page.

Quiz.

Q0: Consider an embedding of the circle graph C_7 into the line, such that the vertices $v_1, v_2, v_3, v_4, v_5, v_6, v_7$ are mapped to $\{1,2,3,4,5,6,7\}$ respectively.
What is the distortion?

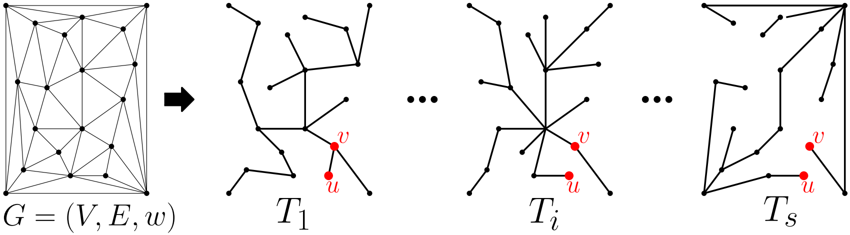


Quiz. Consider a graph family called *Graphica Prime*, such that every n-graph G in the family embeds into **distribution** $\mathcal{D}$ over **dominating trees** with **expected distortion** $\boldsymbol{t}$.

Stochastic Embedding into Trees

Theorem (Stochastic embedding for graph family $\mathcal{G}$)

*Every graph $G = (V, E, w)$ in Graphica Prime embeds into **distribution** $\mathcal{D}$ over **dominating trees** with **expected distortion** $\boldsymbol{t}$.*



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Quiz.

Consider a graph family called *Graphica Prime*, such that every n-graph G in the family embeds into **distribution** $\mathcal{D}$ over **dominating trees** with **expected distortion** $\boldsymbol{t}$. Using similar techniques to what we did in class:

Q1: What **distance oracle** can you achieve for graphs in *Graphica Prime*?

Q2: What **approximation factor** can you obtain for graphs in *Graphica Prime* for the group Steiner tree problem?

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Link to quiz:

Can also be found in my homepage:

arnold.filtser.com

Or just google Arnold Filtser.

Link to slides:

Outline of the talk - Appendix

We will begin our tour of metric embeddings into trees with the classics: [Bartal 96]

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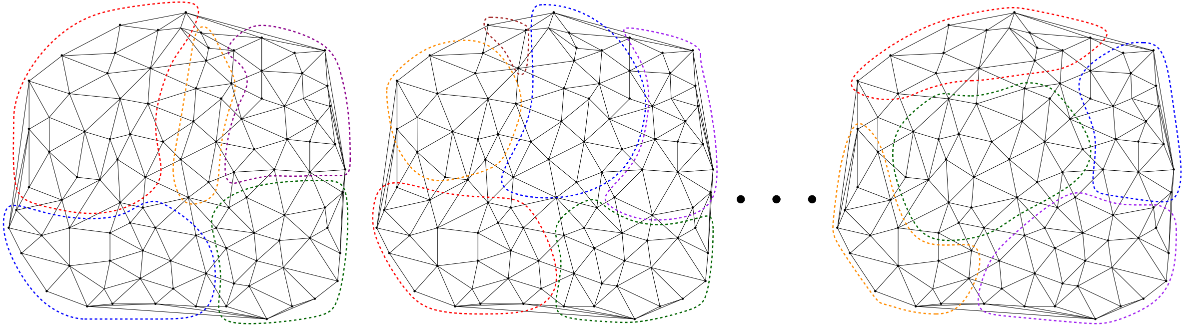
This one is based on random partitions of metric spaces.

Definition (Padded Decomposition)

Given a metric space (X, d_X) (or a weight graph $G = (V, E, w)$).
Distribution $\mathcal{D}$ over partitions of G is (β, Δ)-padded decomposition if:



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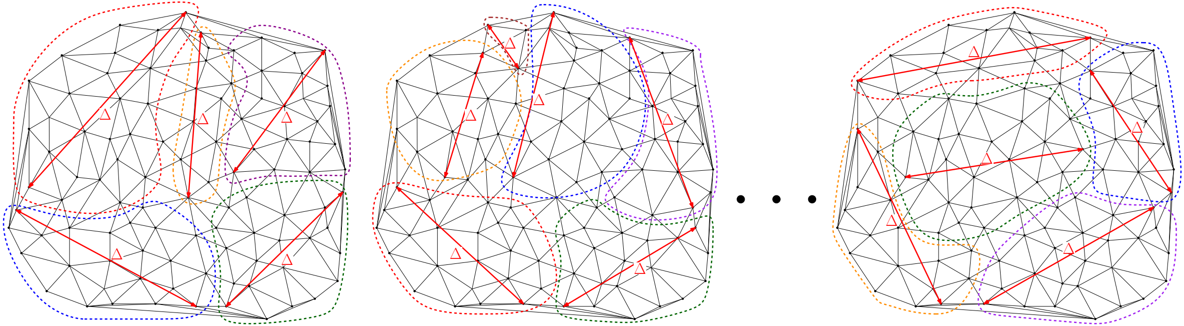

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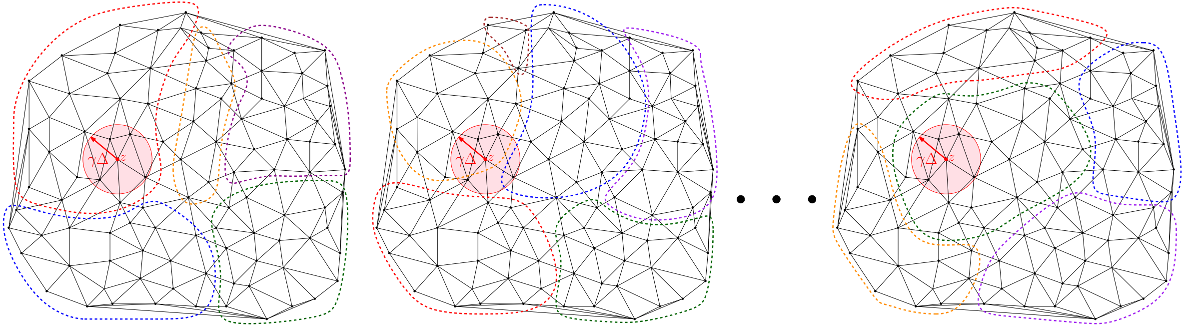

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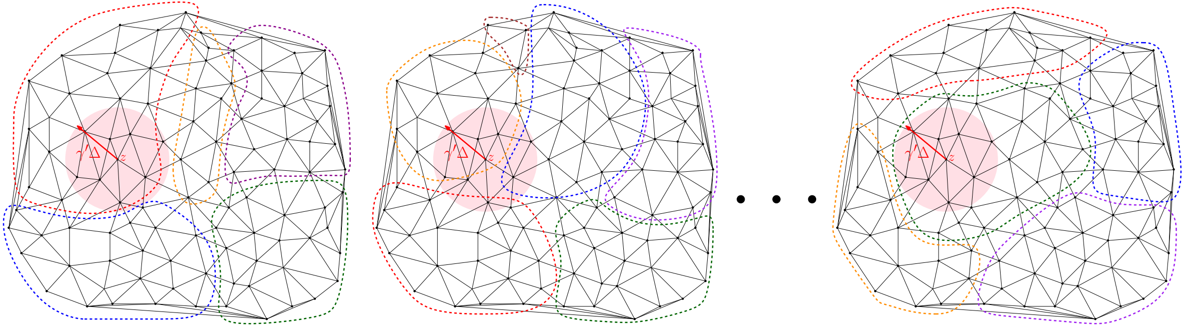

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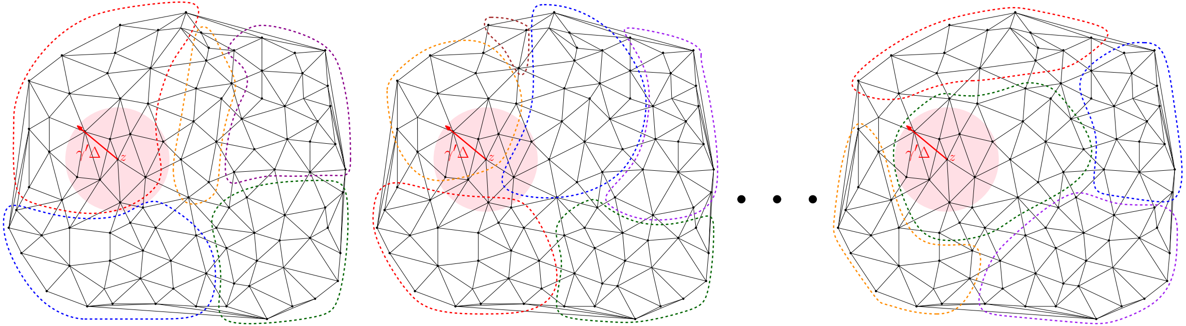

G admits a β-padded decomposition **scheme**:

$\forall \Delta > 0$, G admits (β, Δ)-padded decomposition.

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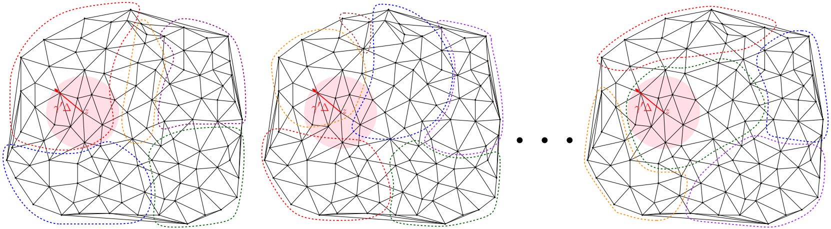

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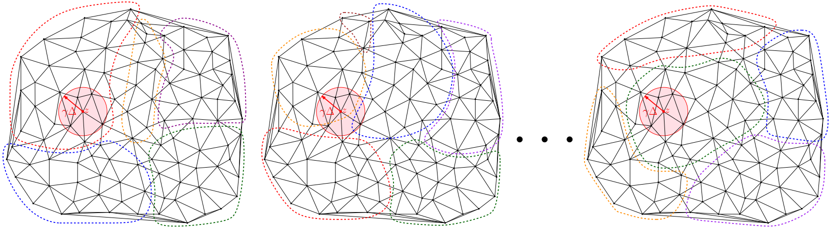

Note: $\Pr[B(z, \frac{1}{\beta} \cdot \Delta) \subseteq P(z)] \geq \Omega(1)$.

For small enough γ, cut probability: $\Pr[B(z, \gamma\Delta) \not\subseteq P(z)] \leq 1 - e^{-\beta\gamma} \approx \beta\gamma$.

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This is also tight! [Bartal 96]

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Every n-point metric space (X, d_X) admits an $O(\log n)$-padded decomposition scheme.

Algorithm:

1. Arbitrarily order X: $x_1, x_2, \ldots, x_n$.
2. For $i = 1$ to n
   1. Sample $r_i \sim \mathrm{Exp}(1)$.
   2. $C_i = B\left(x_i, \tilde{r}_i = r_i \cdot \frac{\Delta}{c \cdot \log n}\right) \setminus \cup_{j<i} C_j$
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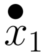

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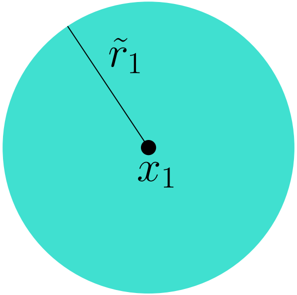

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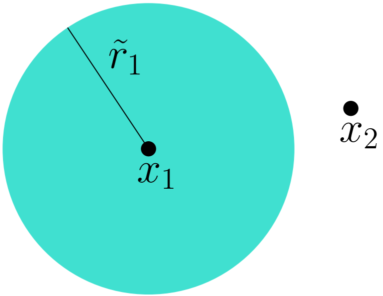

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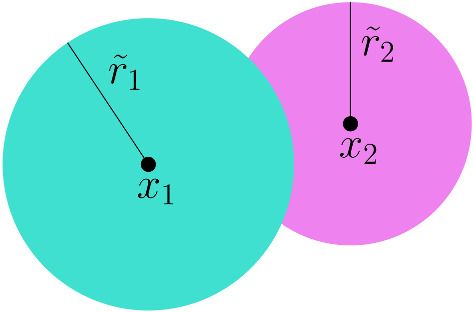

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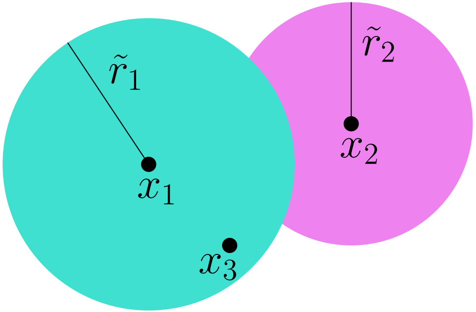

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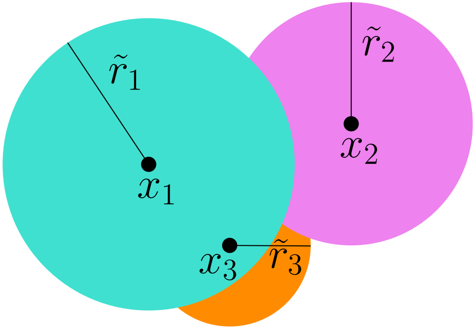

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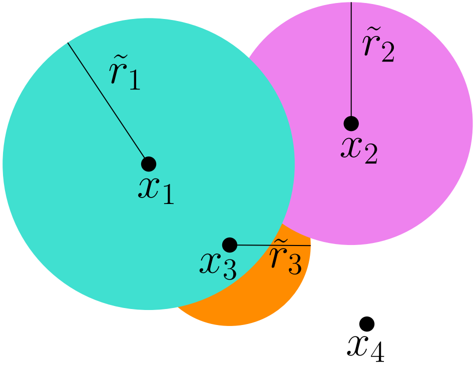

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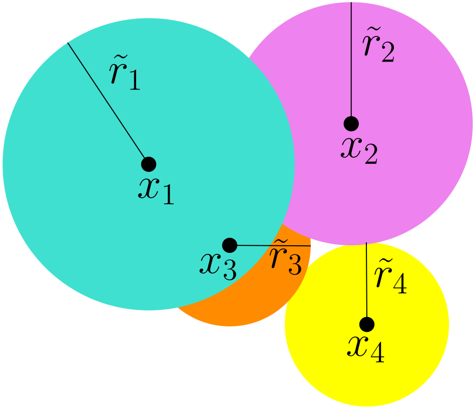

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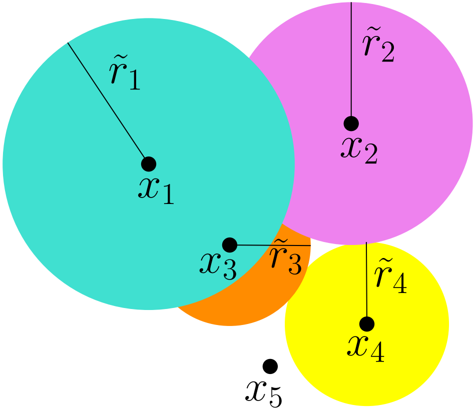

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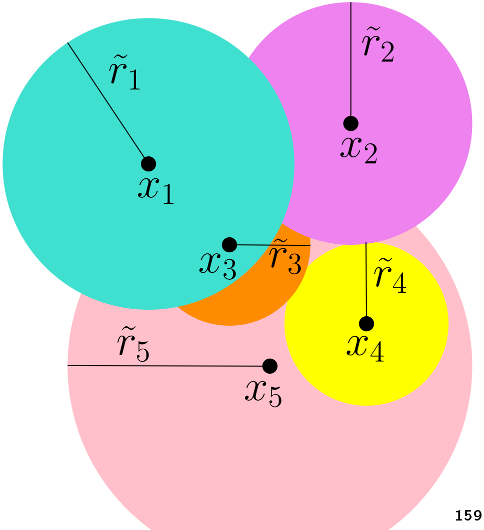

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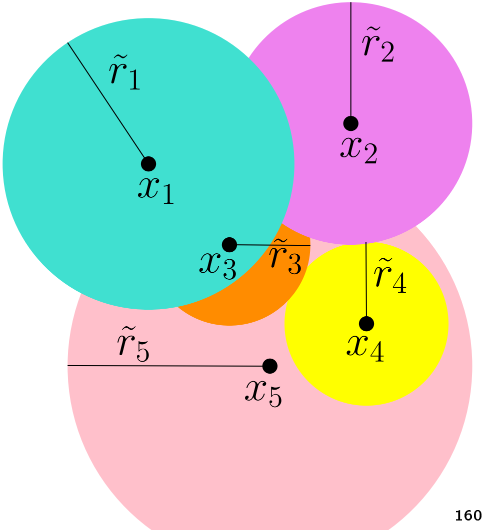

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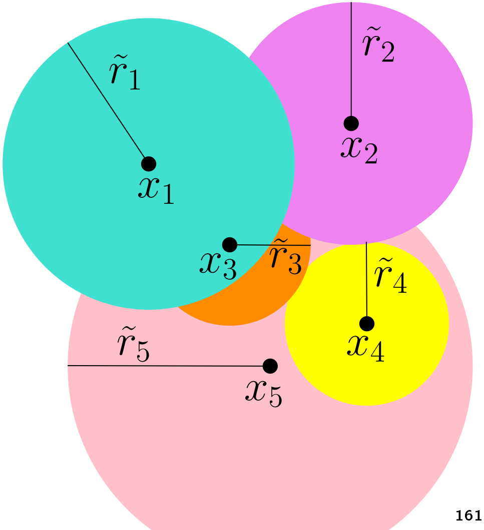

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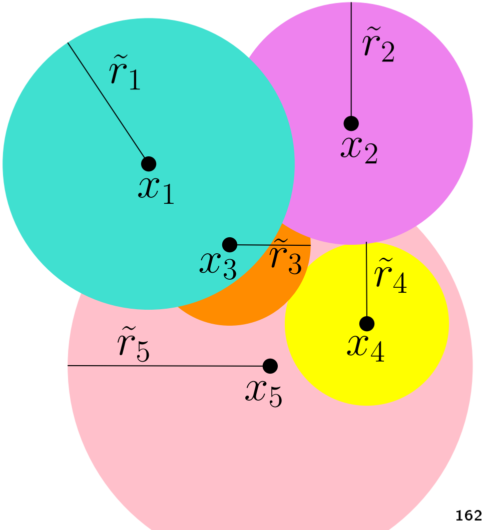
$\Rightarrow$ all clusters have diameter $\leq \Delta$.



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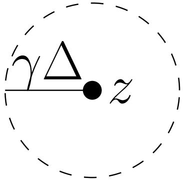

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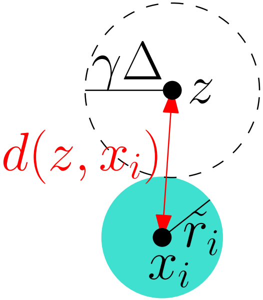

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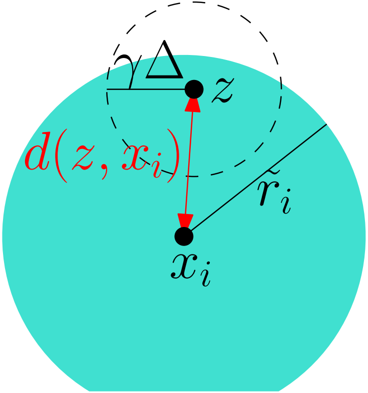

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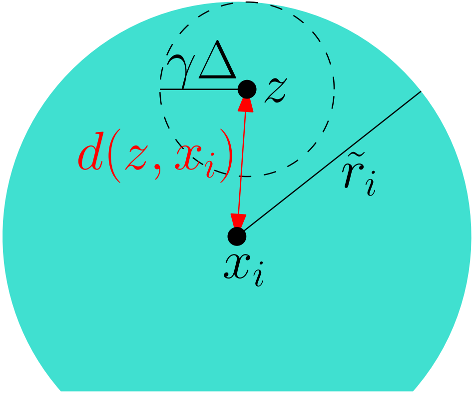

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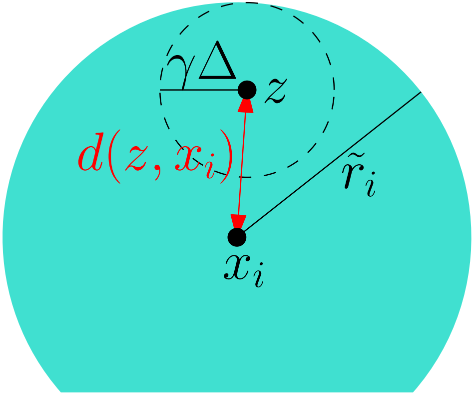

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By **Memorylessness**,

$$\Pr\left[B(z, \gamma\Delta) \subseteq C_i \mid B(z, \gamma\Delta) \cap C_i \neq \emptyset\right] \geq \Pr\left[\tilde{r}_i \geq 2\gamma\Delta\right] = e^{-\gamma \cdot 2c \log n}$$

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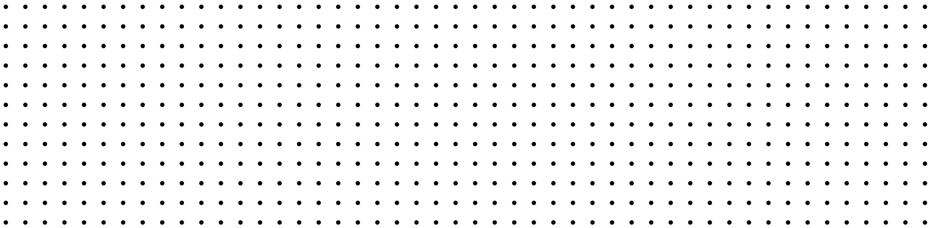

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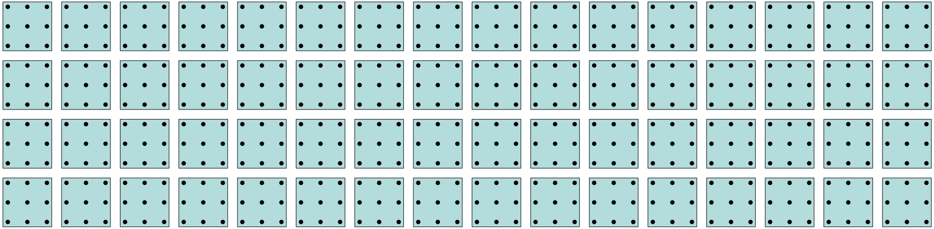

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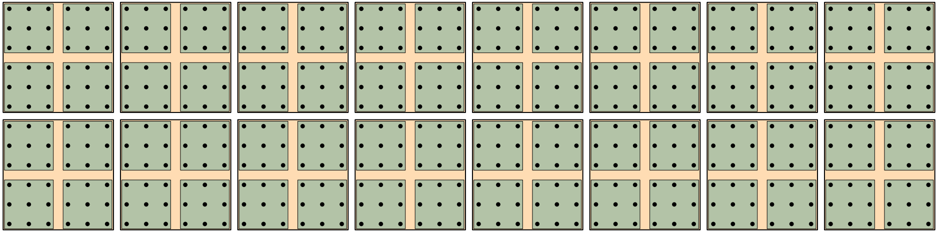

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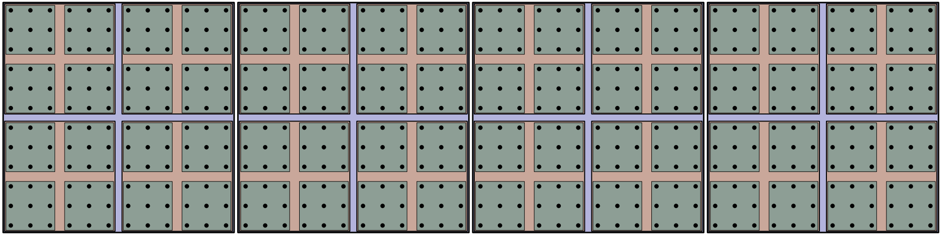

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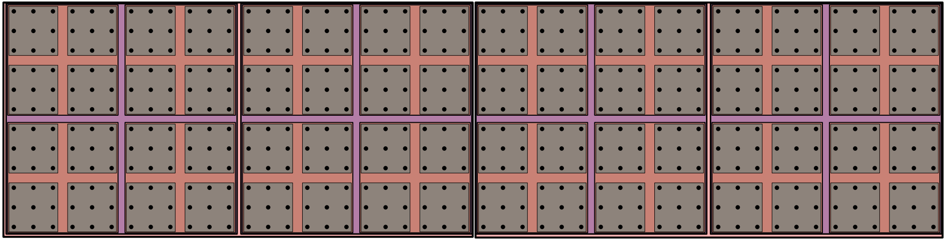

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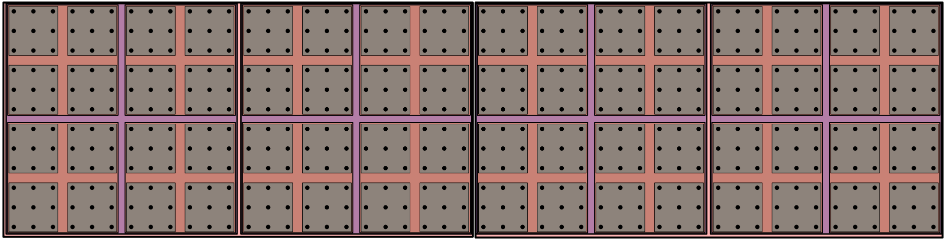

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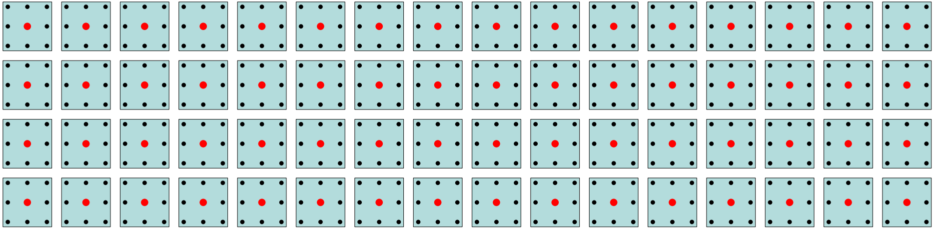

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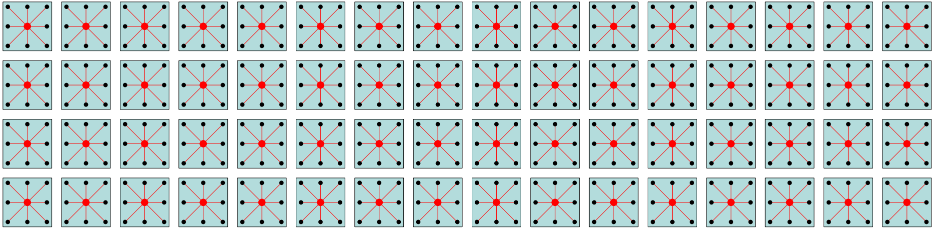

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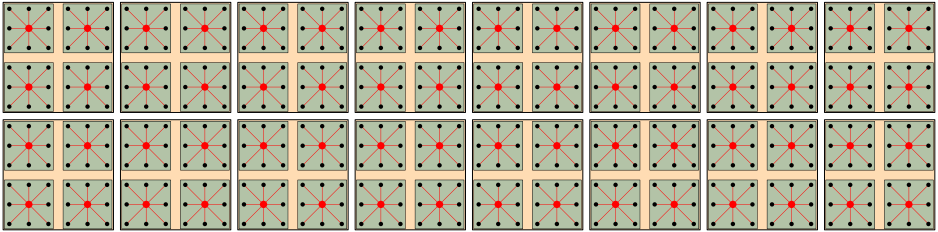

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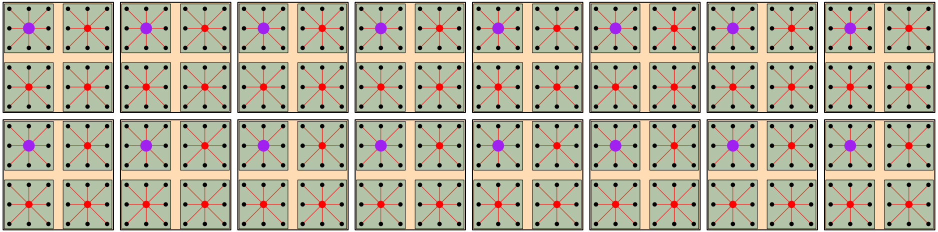

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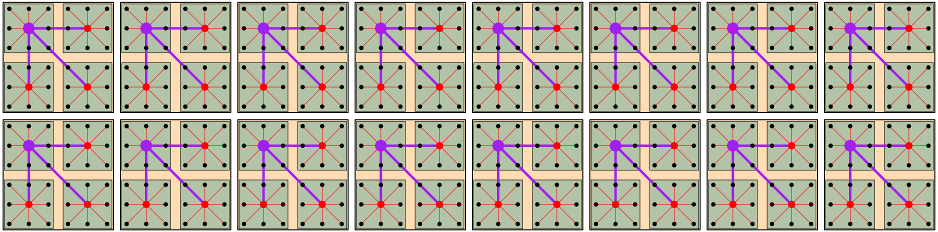

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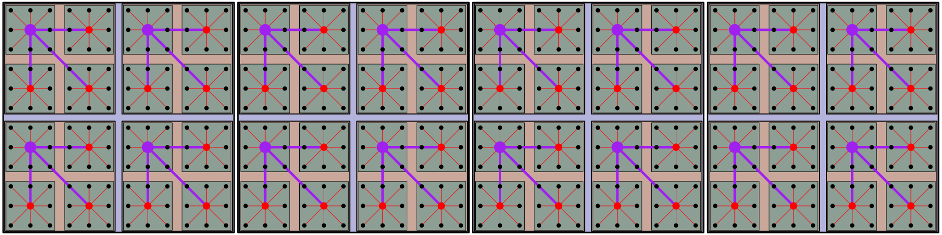

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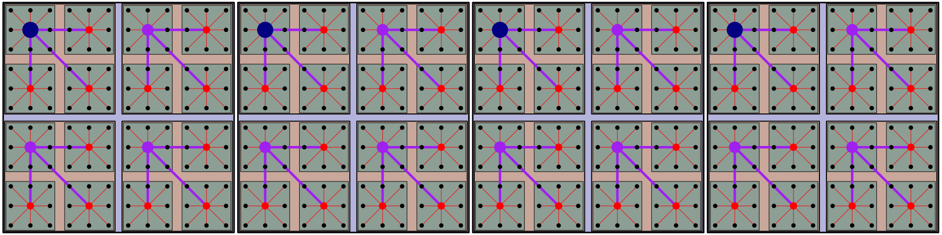

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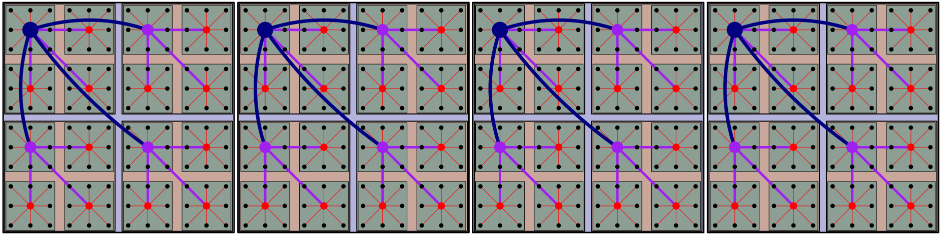

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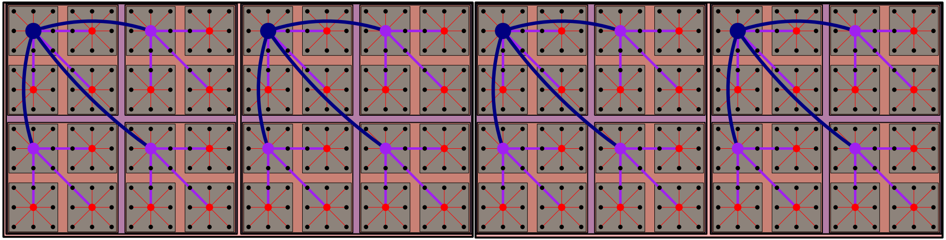

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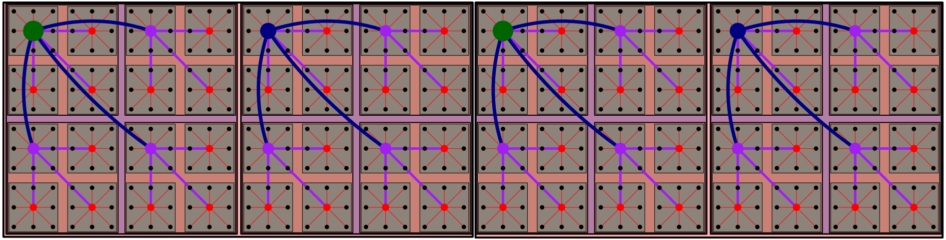

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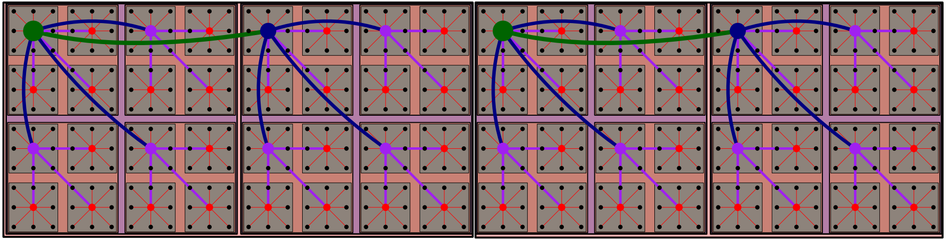

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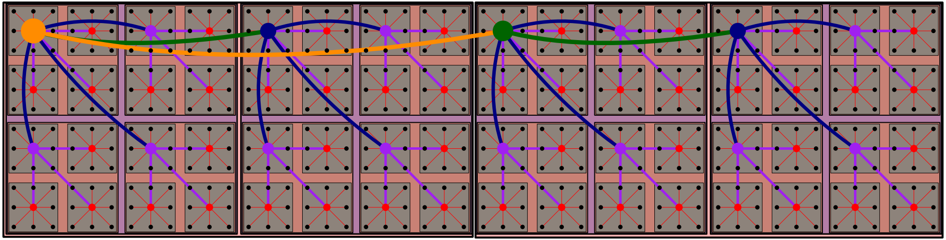

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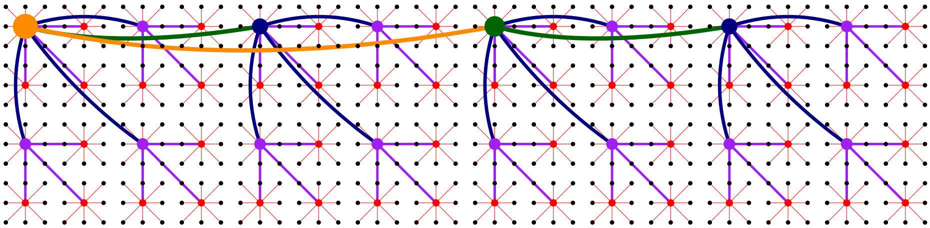

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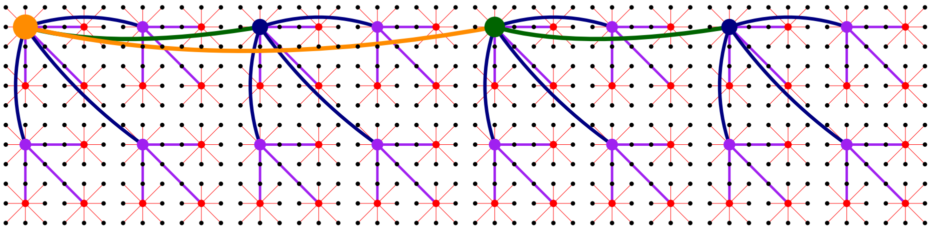

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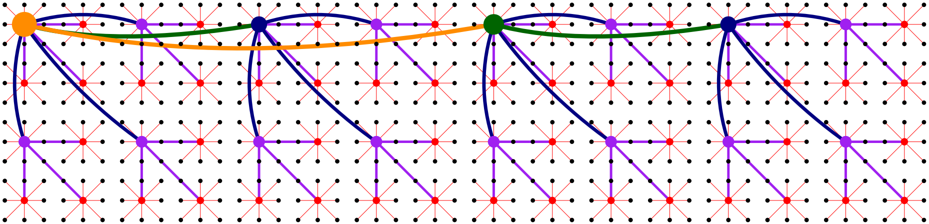

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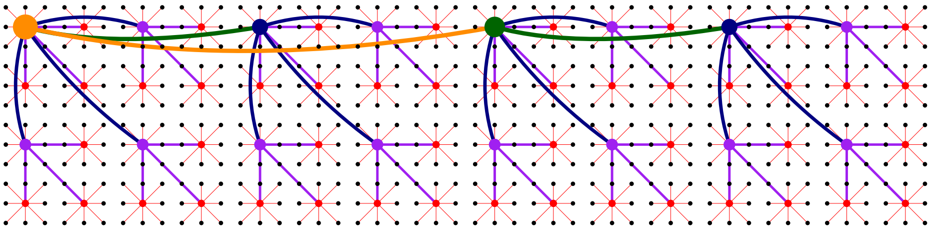

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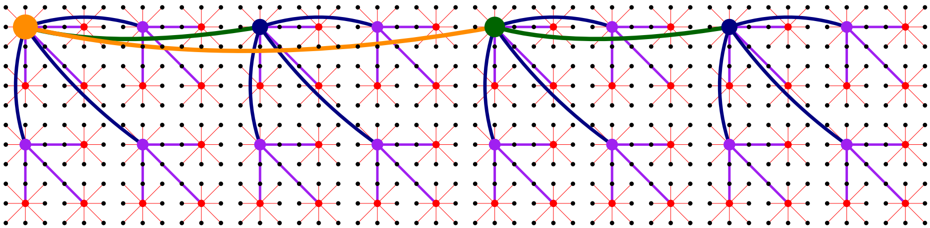

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Specifically, the probability to cut x, y at scale Δ is

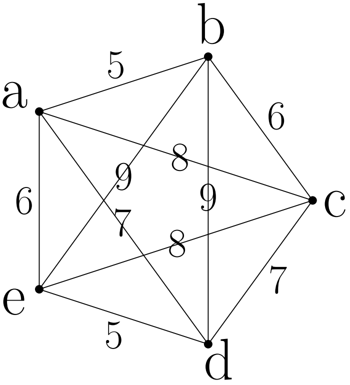
$$\approx \frac{d_X(x,y)}{\Delta} \cdot \log \frac{|B(x, c \cdot 2^i)|}{|B(x, 2^i/c)|}$$

for some constant c, instead of $\approx \frac{d_X(x,y)}{\Delta} \cdot \log n$. Then the sum "telescopes".

Outline of the talk - Appendix

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.



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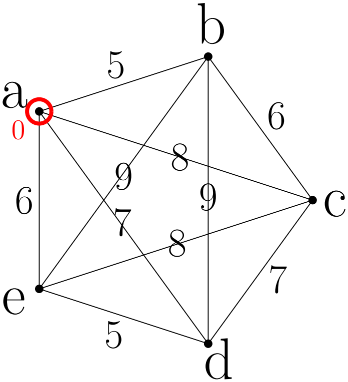

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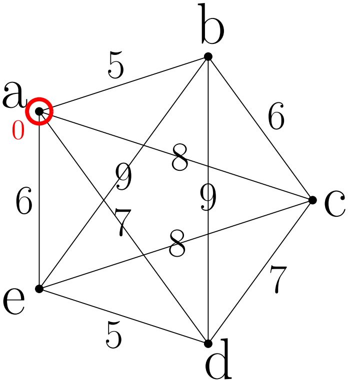

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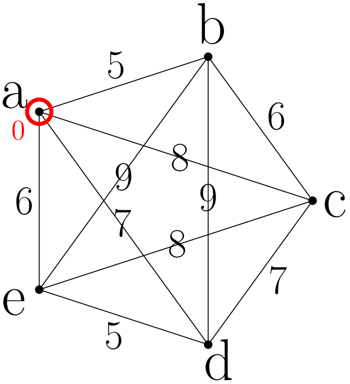

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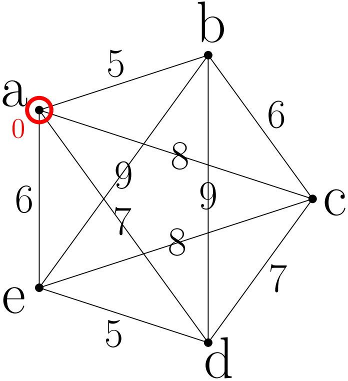

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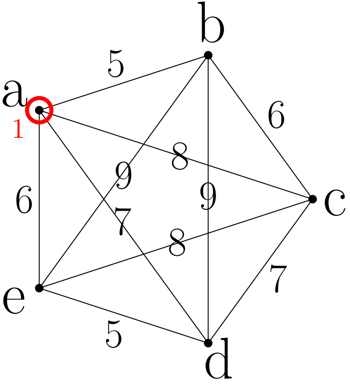


$f_1(a) = 3$

$f_1(b) = 1$

$f_1(c) = 2$

$f_1(d) = 4$

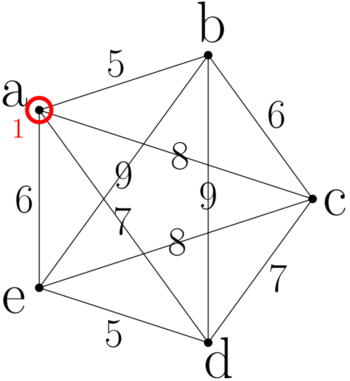
$f_1(e) = 1$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$f_1(a) = 3$

$f_1(b) = 1$

$f_1(c) = 2$

$f_1(d) = 4$

$f_1(e) = 1$

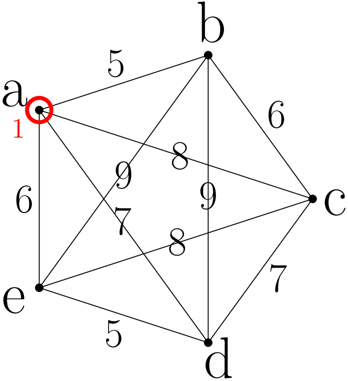
Cost alg(1)=3

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.

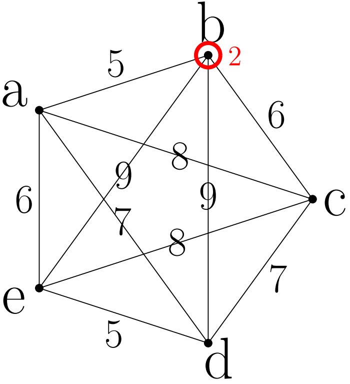


$f_2(a) = 3$
$f_2(b) = 1$
$f_2(c) = 3$
$f_2(d) = 2$
$f_2(e) = 1$

Cost alg(1)=3

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.
Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.
Goal: Minimize the competitive ratio between our algorithm to opt.



$$f_2(a) = 3$$
$$f_2(b) = 1$$
$$f_2(c) = 3$$
$$f_2(d) = 2$$
$$f_2(e) = 1$$

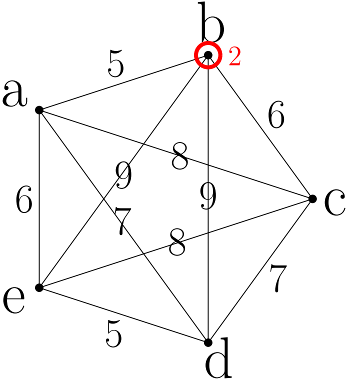
Cost alg(1)=3

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$f_2(a) = 3$
$f_2(b) = 1$
$f_2(c) = 3$
$f_2(d) = 2$
$f_2(e) = 1$

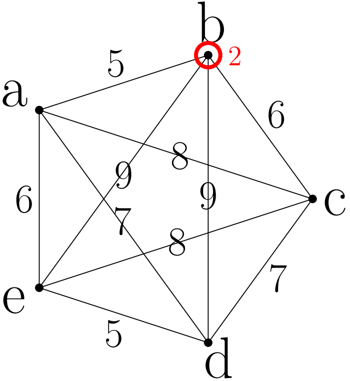
Cost alg(2)$=3+5+1=9$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$f_3(a) = 1$
$f_3(b) = 4$
$f_3(c) = 3$
$f_3(d) = 2$
$f_3(e) = 1$

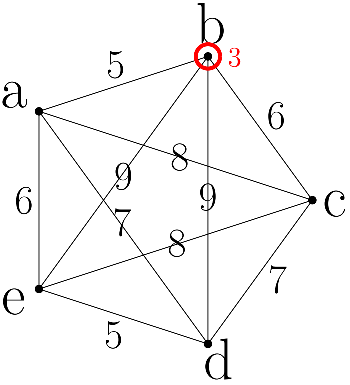
Cost alg(2)$=3 + 5 + 1 = 9$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$$f_3(a) = 1$$
$$f_3(b) = 4$$
$$f_3(c) = 3$$
$$f_3(d) = 2$$
$$f_3(e) = 1$$

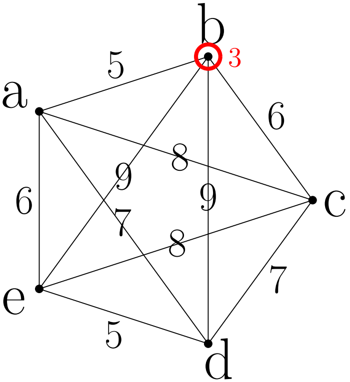
Cost alg(2)$=3 + 5 + 1 = 9$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$$f_3(a) = 1$$
$$f_3(b) = 4$$
$$f_3(c) = 3$$
$$f_3(d) = 2$$
$$f_3(e) = 1$$

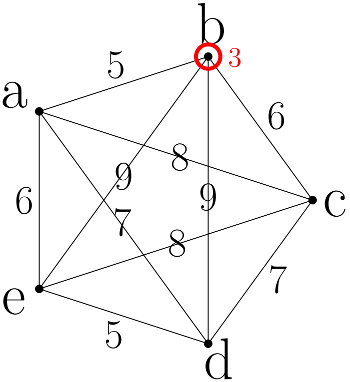
Cost alg(3)$=9 + 4 = 13$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$f_4(a) = 5$
$f_4(b) = 10$
$f_4(c) = 4$
$f_4(d) = 0$
$f_4(e) = 1$

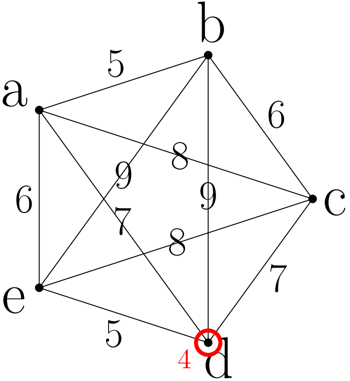
Cost alg(3)$=9 + 4 = 13$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$f_4(a) = 5$
$f_4(b) = 10$
$f_4(c) = 4$
$f_4(d) = 0$
$f_4(e) = 1$

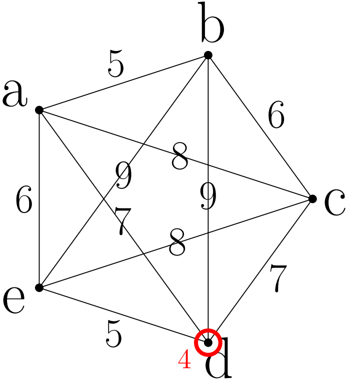
Cost alg(3)$=9 + 4 = 13$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$f_4(a) = 5$

$f_4(b) = 10$

$f_4(c) = 4$

$f_4(d) = 0$

$f_4(e) = 1$

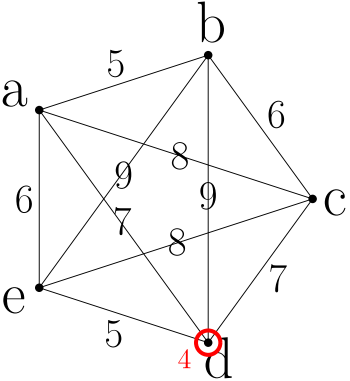
Cost alg(4)$=13 + 9 + 0 = 22$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$f_5(a) = 3$
$f_5(b) = 4$
$f_5(c) = 4$
$f_5(d) = 5$
$f_5(e) = 1$

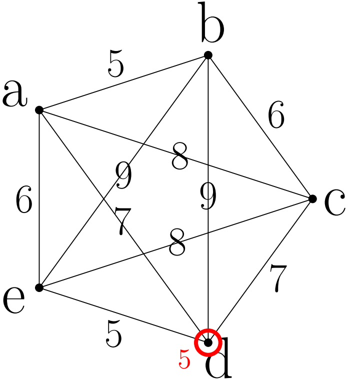
Cost alg(4)=$13 + 9 + 0 = 22$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$$f_5(a) = 3$$
$$f_5(b) = 4$$
$$f_5(c) = 4$$
$$f_5(d) = 5$$
$$f_5(e) = 1$$

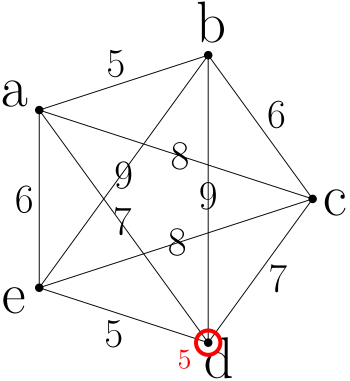
Cost alg(4)$=13 + 9 + 0 = 22$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$f_5(a) = 3$
$f_5(b) = 4$
$f_5(c) = 4$
$f_5(d) = 5$
$f_5(e) = 1$

Cost alg(5)=$22 + 5 = 27$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.

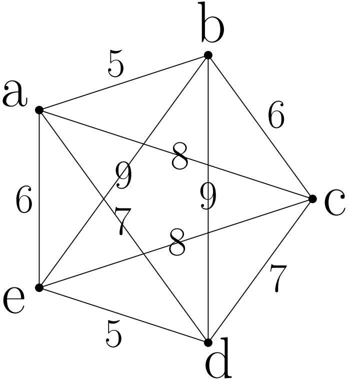
What about Opt?

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.

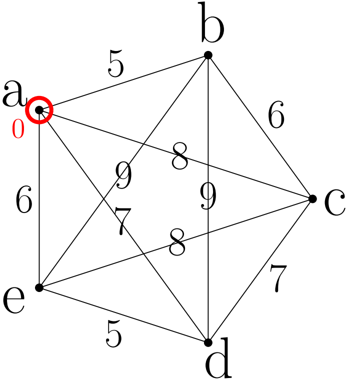


Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.

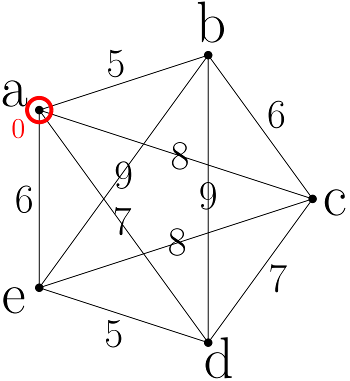


Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



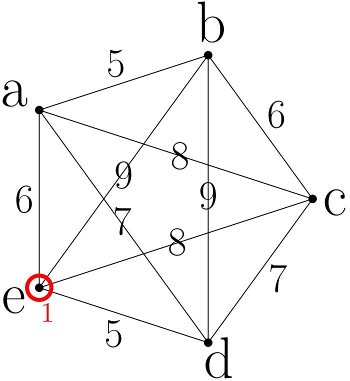
$$f_1(a) = 3$$
$$f_1(b) = 1$$
$$f_1(c) = 2$$
$$f_1(d) = 4$$
$$f_1(e) = 1$$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



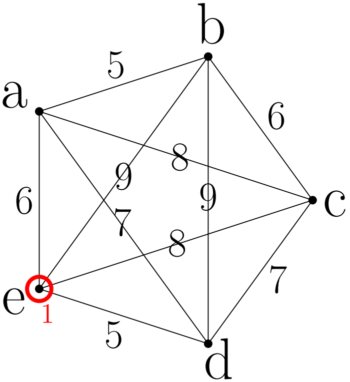
$$f_1(a) = 3$$
$$f_1(b) = 1$$
$$f_1(c) = 2$$
$$f_1(d) = 4$$
$$f_1(e) = 1$$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$$f_1(a) = 3$$
$$f_1(b) = 1$$
$$f_1(c) = 2$$
$$f_1(d) = 4$$
$$f_1(e) = 1$$

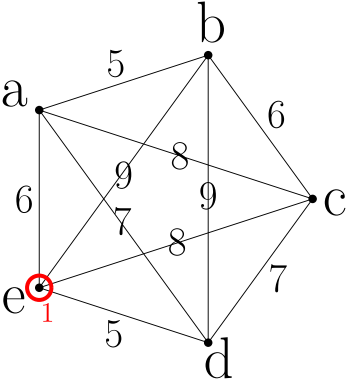
Cost opt(1)$=6+1=7$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$f_2(a) = 3$
$f_2(b) = 1$
$f_2(c) = 3$
$f_2(d) = 2$
$f_2(e) = 1$

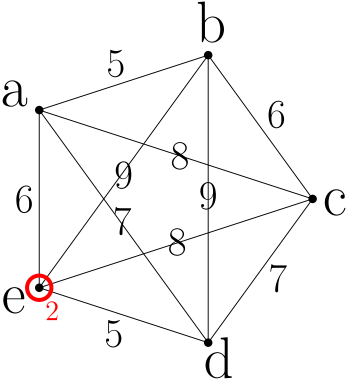
Cost opt(1)$=6+1=7$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$$f_2(a) = 3$$
$$f_2(b) = 1$$
$$f_2(c) = 3$$
$$f_2(d) = 2$$
$$f_2(e) = 1$$

Cost opt(1)$=6 + 1 = 7$

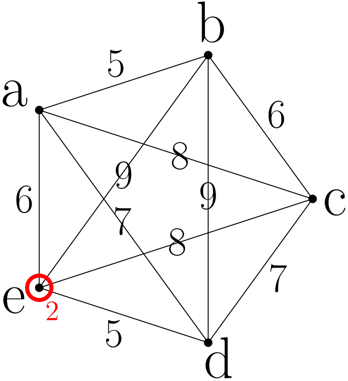
Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.

At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$f_2(a) = 3$

$f_2(b) = 1$

$f_2(c) = 3$

$f_2(d) = 2$

$f_2(e) = 1$

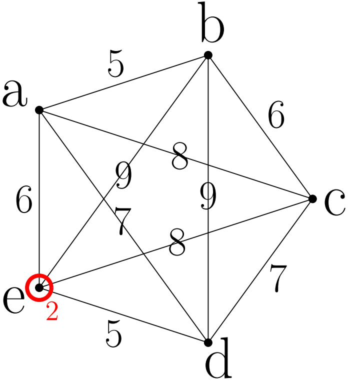
Cost opt(2)$=7+1=8$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$f_3(a) = 1$
$f_3(b) = 4$
$f_3(c) = 3$
$f_3(d) = 2$
$f_3(e) = 1$

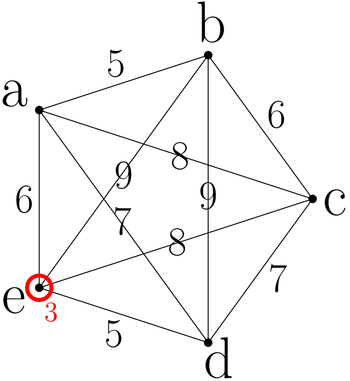
Cost opt(2)$=7 + 1 = 8$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$$f_3(a) = 1$$
$$f_3(b) = 4$$
$$f_3(c) = 3$$
$$f_3(d) = 2$$
$$f_3(e) = 1$$

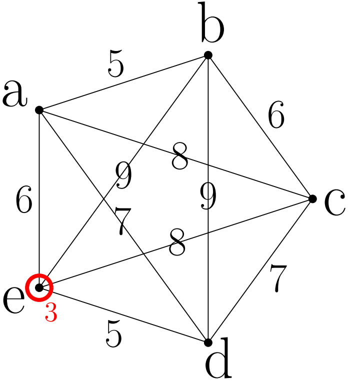
Cost opt(2)$=7+1=8$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.

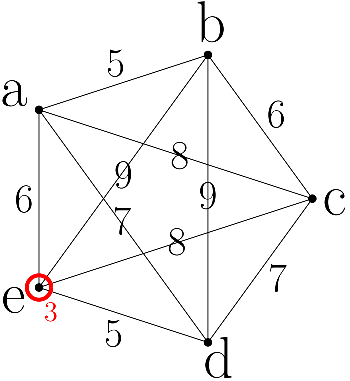


$f_3(a) = 1$
$f_3(b) = 4$
$f_3(c) = 3$
$f_3(d) = 2$
$f_3(e) = 1$

Cost opt(3)$=8 + 1 = 9$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.
Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.
Goal: Minimize the competitive ratio between our algorithm to opt.



$$f_4(a) = 5$$
$$f_4(b) = 10$$
$$f_4(c) = 4$$
$$f_4(d) = 0$$
$$f_4(e) = 1$$

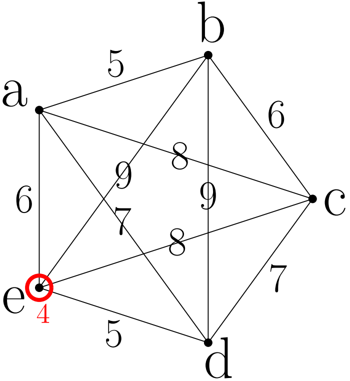
Cost opt(3)$=8+1=9$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$f_4(a) = 5$
$f_4(b) = 10$
$f_4(c) = 4$
$f_4(d) = 0$
$f_4(e) = 1$

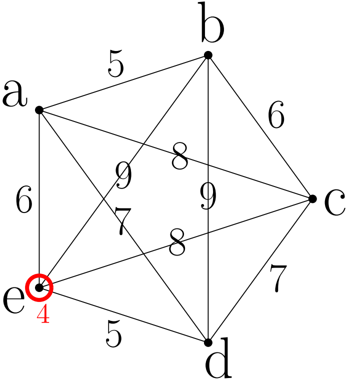
Cost opt(3)$=8 + 1 = 9$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$f_4(a) = 5$
$f_4(b) = 10$
$f_4(c) = 4$
$f_4(d) = 0$
$f_4(e) = 1$

Cost opt(4)=$9 + 1 = 10$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

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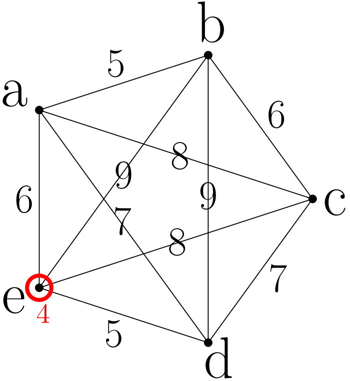

$$f_5(a) = 3$$
$$f_5(b) = 4$$
$$f_5(c) = 4$$
$$f_5(d) = 5$$
$$f_5(e) = 1$$

Cost opt(4)=$9 + 1 = 10$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

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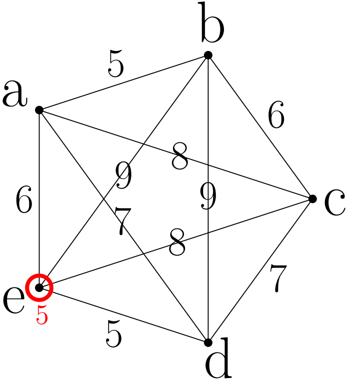


$$f_5(a) = 3$$
$$f_5(b) = 4$$
$$f_5(c) = 4$$
$$f_5(d) = 5$$
$$f_5(e) = 1$$

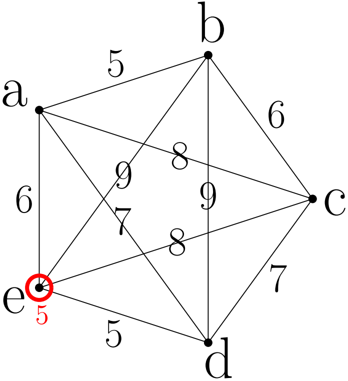
Cost opt(4)$=9 + 1 = 10$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.

Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.

Goal: Minimize the competitive ratio between our algorithm to opt.



$$f_5(a) = 3$$
$$f_5(b) = 4$$
$$f_5(c) = 4$$
$$f_5(d) = 5$$
$$f_5(e) = 1$$

Cost opt(5)=$10 + 1 = 11$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.
Output: Point $x_i \in X$ s.t. the task performed at x_i at cost $d_X(x_{i-1}, x_i) + f_i(x_i)$.
Goal: Minimize the competitive ratio between our algorithm to opt.

$$\text{Competitive ratio} = \max_{\text{input } I} \frac{\mathrm{Alg}(I)}{\mathrm{opt}(I)}.$$

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
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$$\text{Competitive ratio} = \max_{\text{input } I} \frac{\mathrm{Alg}(I)}{\mathrm{opt}(I)}.$$

Here $\frac{\mathrm{Alg}(I)}{\mathrm{opt}(I)} = \frac{27}{11}$.

Online problem - Metrical Task System (MTS)

Input: Metric space (X, d_X). Initial configuration $x_0 \in X$.
At step i, a task arrives with cost function $f_i : X \to \mathbb{R}_{\geq 0}$.
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$$\text{Competitive ratio} = \max_{\text{input } I} \frac{\mathrm{Alg}(I)}{\mathrm{opt}(I)}.$$

Here $\frac{\mathrm{Alg}(I)}{\mathrm{opt}(I)} = \frac{27}{11}$.

Competitive ratio against oblivious adversary is

$$\max_{\text{input } I} \frac{\mathbb{E}[\mathrm{Alg}(I)]}{\mathrm{opt}(I)}.$$

Online problem - Metrical Task System (MTS)

Approach: embed into a tree, and then make all the decisions based on the tree.

Online problem - Metrical Task System (MTS)

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Theorem ([Fiat, Mendel 2000])

Given an n point tree * T*, there is an online algorithm for MTS with competitive ratio* $O(\log n \cdot \log\log n)$ *against oblivious adversary.*

* Actually on an HST, which is a special kind of tree. [FRT04] is into HST's.

Online problem - Metrical Task System (MTS)

Approach: embed into a tree, and then make all the decisions based on the tree.

Theorem ([Fiat, Mendel 2000])

Given an n point tree * T*, there is an online algorithm for MTS with competitive ratio* $O(\log n \cdot \log\log n)$ *against oblivious adversary.*

Algorithm: 1. Sample a tree T over (X, d_X) using [FRT04].

Online problem - Metrical Task System (MTS)

Approach: embed into a tree, and then make all the decisions based on the tree.

Theorem ([Fiat, Mendel 2000])

Given an n point tree * T, *there is an online algorithm for MTS with competitive ratio* $O(\log n \cdot \log\log n)$ *against oblivious adversary.*

Algorithm: 1. Sample a tree T over (X, d_X) using [FRT04].
2. Run [FM00] on T with the same cost functions.
Make the same decisions as [FM00].

Online problem - Metrical Task System (MTS)

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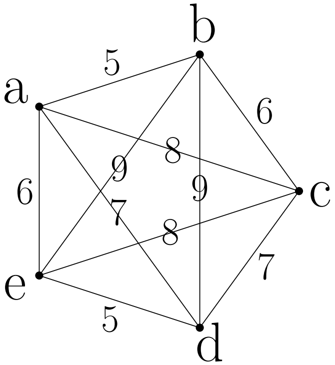

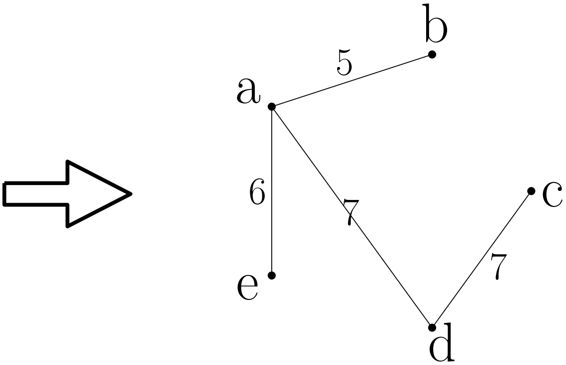

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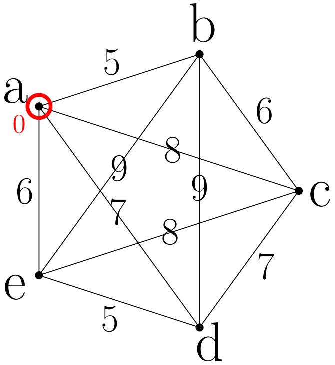
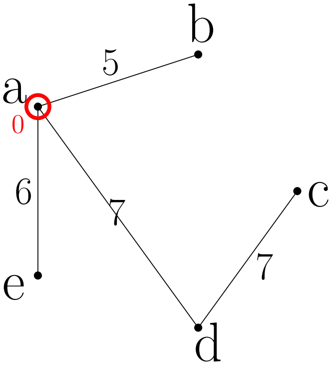
Make the same decisions as [FM00].





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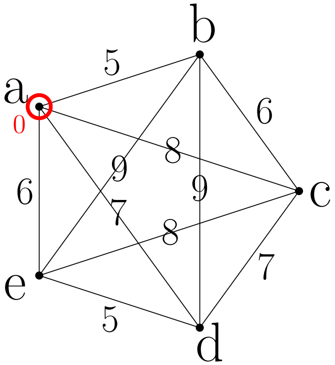


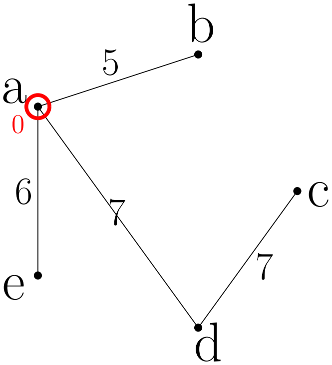
$f_1(a) = 3$

$f_1(b) = 1$

$f_1(c) = 2$

$f_1(d) = 4$

$f_1(e) = 1$



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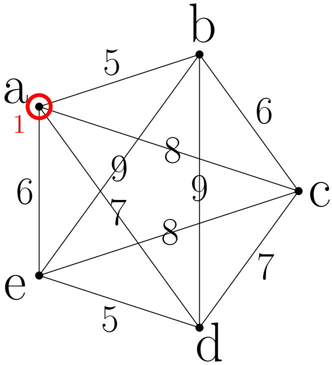
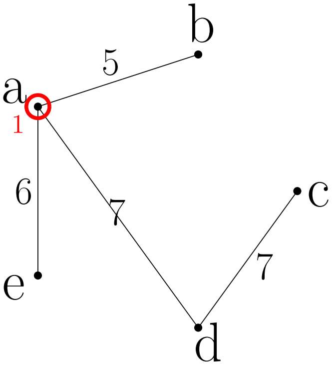


$$f_1(a) = 3$$
$$f_1(b) = 1$$
$$f_1(c) = 2$$
$$f_1(d) = 4$$
$$f_1(e) = 1$$



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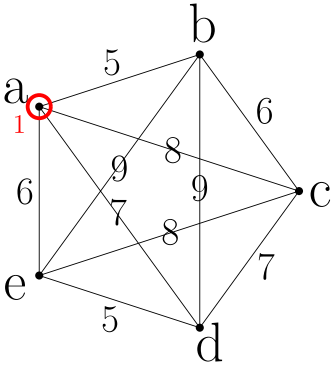
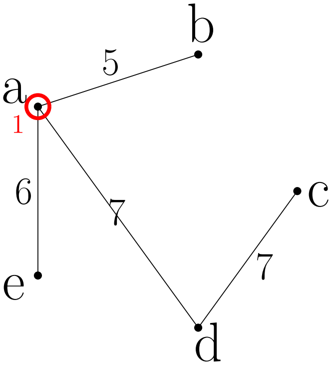


Cost alg(1)=3

$$f_1(a) = 3$$
$$f_1(b) = 1$$
$$f_1(c) = 2$$
$$f_1(d) = 4$$
$$f_1(e) = 1$$



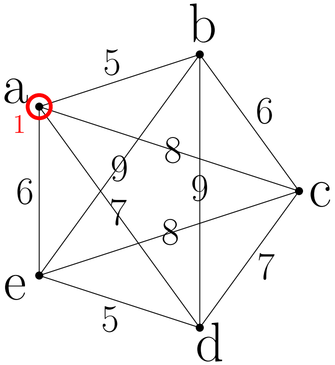
Cost $\text{alg}_T(1)$=3

Online problem - Metrical Task System (MTS)

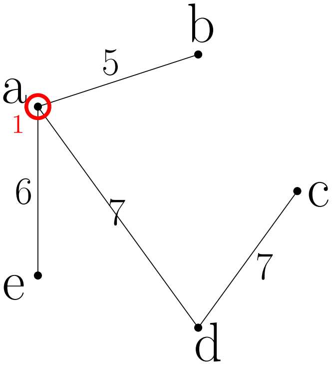
Algorithm: 1. Sample a tree T over (X, d_X) using [FRT04].

2. Run [FM00] on T with the same cost functions.

Make the same decisions as [FM00].



$$f_2(a) = 3$$
$$f_2(b) = 1$$
$$f_2(c) = 3$$
$$f_2(d) = 2$$
$$f_2(e) = 1$$



Cost alg(1)=3

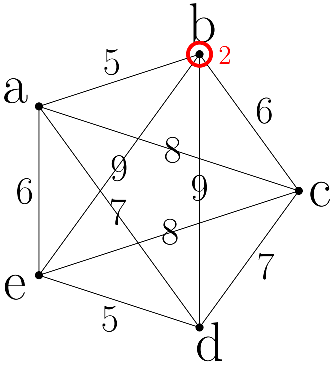
Cost $\mathrm{alg}_T(1)$=3

Online problem - Metrical Task System (MTS)

Algorithm: 1. Sample a tree T over (X, d_X) using [FRT04].

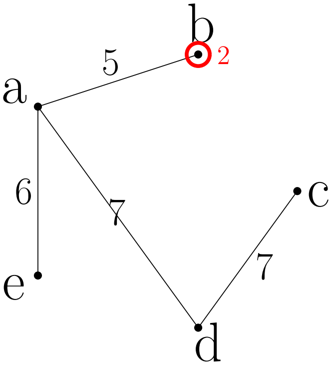
2. Run [FM00] on T with the same cost functions.

Make the same decisions as [FM00].



$f_2(a) = 3$
$f_2(b) = 1$
$f_2(c) = 3$
$f_2(d) = 2$
$f_2(e) = 1$

Cost alg(1)=3



Cost $\mathrm{alg}_T(1)$=3

Online problem - Metrical Task System (MTS)

Algorithm: 1. Sample a tree T over (X, d_X) using [FRT04].

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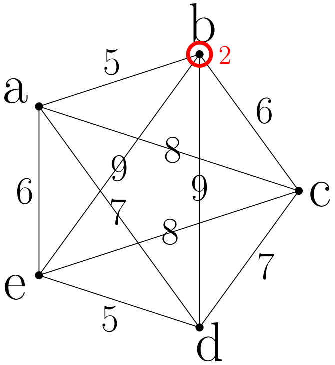
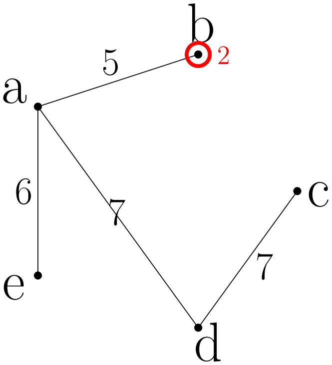
Make the same decisions as [FM00].



$$f_2(a) = 3$$
$$f_2(b) = 1$$
$$f_2(c) = 3$$
$$f_2(d) = 2$$
$$f_2(e) = 1$$



Cost $\mathrm{alg}(2){=}3 + 5 + 1 = 9$

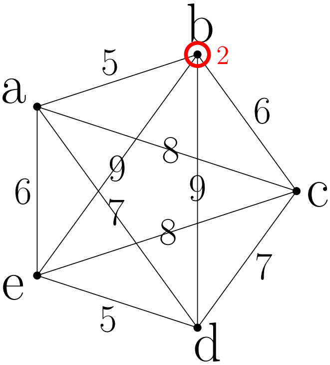
Cost $\mathrm{alg}_T(2){=}3 + 5 + 1 = 9$

Online problem - Metrical Task System (MTS)

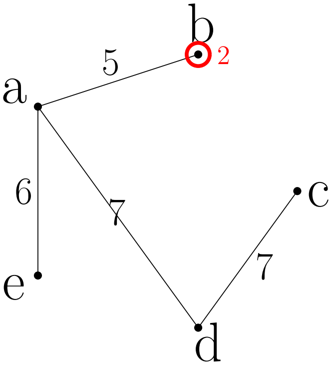
Algorithm: 1. Sample a tree T over (X, d_X) using [FRT04].

2. Run [FM00] on T with the same cost functions.

Make the same decisions as [FM00].



$$f_3(a) = 1$$
$$f_3(b) = 4$$
$$f_3(c) = 3$$
$$f_3(d) = 2$$
$$f_3(e) = 1$$



Cost alg(2)=$3 + 5 + 1 = 9$

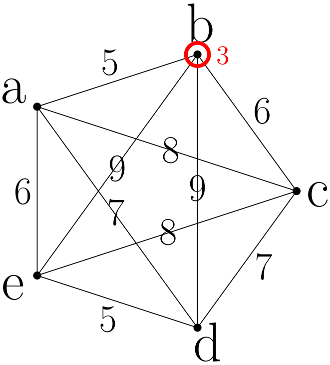
Cost $\mathrm{alg}_T(2)$=$3 + 5 + 1 = 9$

Online problem - Metrical Task System (MTS)

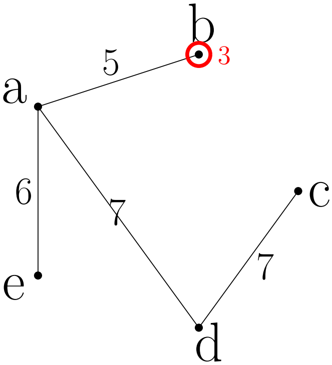
Algorithm: 1. Sample a tree T over (X, d_X) using [FRT04].

2. Run [FM00] on T with the same cost functions.

Make the same decisions as [FM00].



$$f_3(a) = 1$$
$$f_3(b) = 4$$
$$f_3(c) = 3$$
$$f_3(d) = 2$$
$$f_3(e) = 1$$



Cost alg(2)=$3 + 5 + 1 = 9$

Cost $\mathrm{alg}_T(2)$=$3 + 5 + 1 = 9$

Online problem - Metrical Task System (MTS)

Algorithm: 1. Sample a tree T over (X, d_X) using [FRT04].
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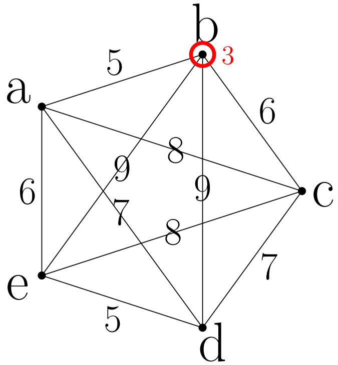
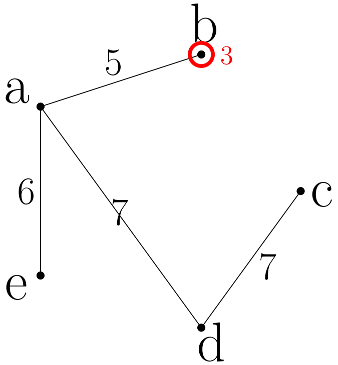
Make the same decisions as [FM00].



Cost alg(3)=$9 + 4 = 13$

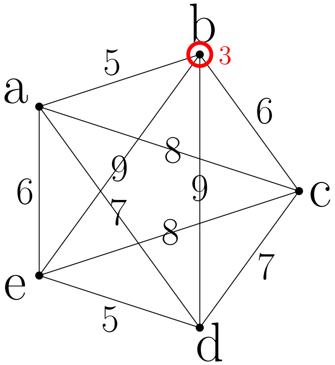
$f_3(a) = 1$
$f_3(b) = 4$
$f_3(c) = 3$
$f_3(d) = 2$
$f_3(e) = 1$



Cost $\text{alg}_T(3)$=$9 + 4 = 13$

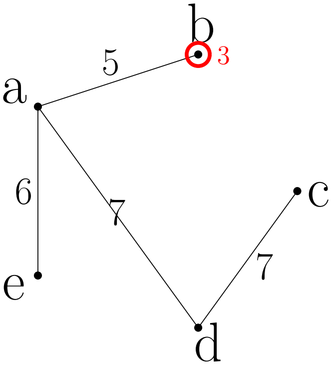
Online problem - Metrical Task System (MTS)

Algorithm: 1. Sample a tree T over (X, d_X) using [FRT04].
2. Run [FM00] on T with the same cost functions.
Make the same decisions as [FM00].



Cost alg(3)=$9 + 4 = 13$

$f_4(a) = 5$
$f_4(b) = 10$
$f_4(c) = 4$
$f_4(d) = 0$
$f_4(e) = 1$



Cost $\text{alg}_T(3)$=$9 + 4 = 13$

Online problem - Metrical Task System (MTS)

Algorithm: 1. Sample a tree T over (X, d_X) using [FRT04].

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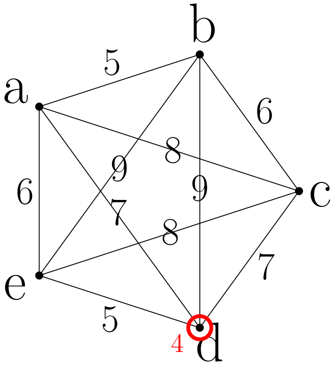
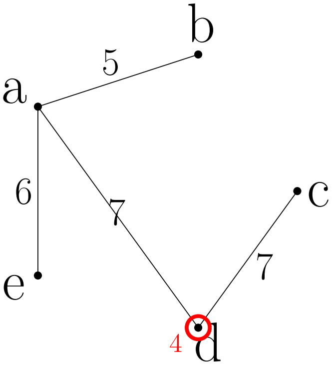


$$f_4(a) = 5$$
$$f_4(b) = 10$$
$$f_4(c) = 4$$
$$f_4(d) = 0$$
$$f_4(e) = 1$$



Cost alg(3)=$9 + 4 = 13$

Cost $\text{alg}_T(3)$=$9 + 4 = 13$

Online problem - Metrical Task System (MTS)

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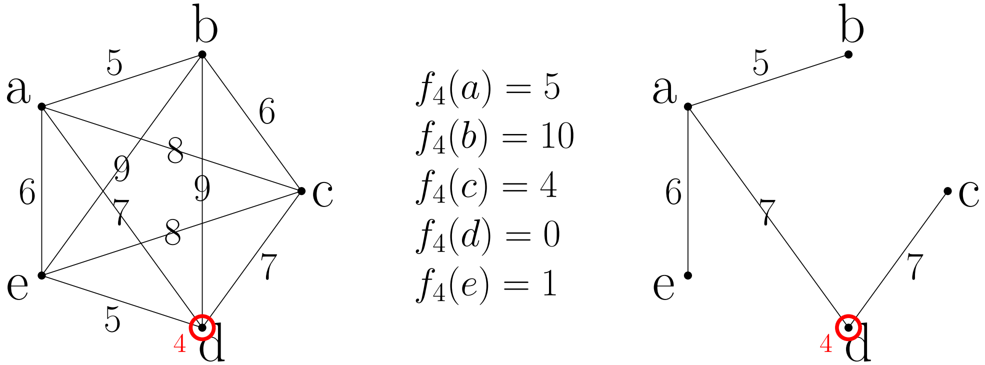
Make the same decisions as [FM00].



$$f_4(a) = 5$$
$$f_4(b) = 10$$
$$f_4(c) = 4$$
$$f_4(d) = 0$$
$$f_4(e) = 1$$

Cost $\mathrm{alg}(4) = 13 + 9 + 0 = 22$

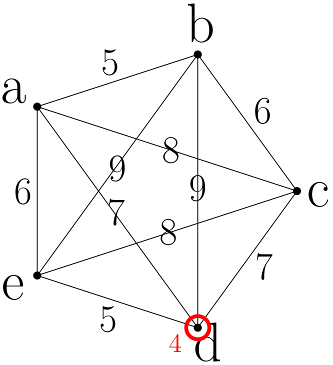
Cost $\mathrm{alg}_T(4) = 13 + 12 + 0 = 25$

Online problem - Metrical Task System (MTS)

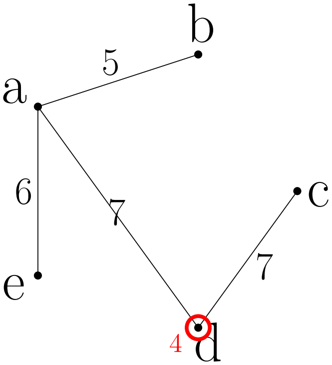
Algorithm: 1. Sample a tree T over (X, d_X) using [FRT04].

2. Run [FM00] on T with the same cost functions.

Make the same decisions as [FM00].



$$f_5(a) = 3$$
$$f_5(b) = 4$$
$$f_5(c) = 4$$
$$f_5(d) = 5$$
$$f_5(e) = 1$$



Cost alg(4)$=13 + 9 + 0 = 22$

Cost $\text{alg}_T(4)$$=13 + 12 + 0 = 25$

Online problem - Metrical Task System (MTS)

Algorithm: 1. Sample a tree T over (X, d_X) using [FRT04].
2. Run [FM00] on T with the same cost functions.
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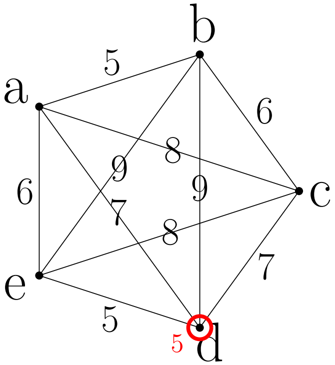
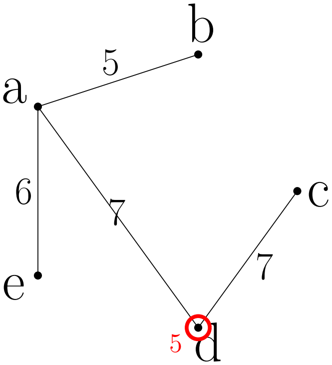


$f_5(a) = 3$
$f_5(b) = 4$
$f_5(c) = 4$
$f_5(d) = 5$
$f_5(e) = 1$



Cost $\mathrm{alg}(4)$=$13 + 9 + 0 = 22$

Cost $\mathrm{alg}_T(4)$=$13 + 12 + 0 = 25$

Online problem - Metrical Task System (MTS)

Algorithm: 1. Sample a tree T over (X, d_X) using [FRT04].
2. Run [FM00] on T with the same cost functions.
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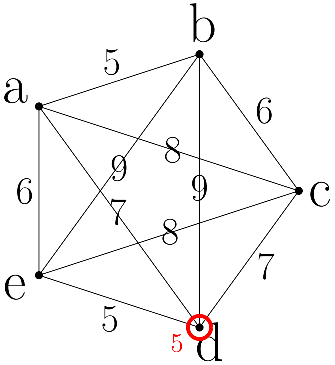
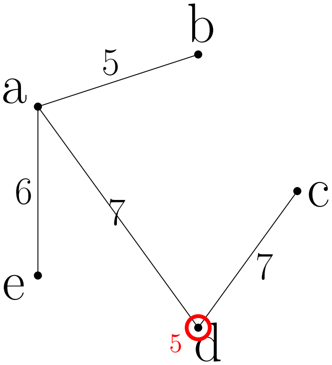


$f_5(a) = 3$
$f_5(b) = 4$
$f_5(c) = 4$
$f_5(d) = 5$
$f_5(e) = 1$



Cost alg(5)=$22 + 5 = 27$

Cost $\mathrm{alg}_T(5)$=$25 + 5 = 30$

Online problem - Metrical Task System (MTS)

Algorithm: 1. Sample a tree T over (X, d_X) using [FRT04].
2. Run [FM00] on T with the same cost functions.
Make the same decisions as [FM00].

Analysis. Let $x_1, x_2, \ldots, x_k$ be the decisions of opt. Thus
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[FM00] is $O(\log n \cdot \log\log n)$-competitive on T. Hence it choose points $y_1, \ldots, y_k$ such that $\mathbb{E}[\mathrm{alg}_T] = \mathbb{E}[\sum_{i=1}^{k} f_i(y_i) + \sum_{i=1}^{k} d_T(y_{i-1}, y_i)] \leq O(\log n \cdot \log\log n) \cdot \mathrm{opt}_T$.

We made the same decisions, so overall:

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Theorem

MTS has an $O(\log^2 n \cdot \log\log n)$ competitive algorithm against oblivious adversary.

Outline of the talk - Appendix

Ramsey type Embeddings

Ramsey type theorem: Every **big** enough object, contains a **structured subset**.

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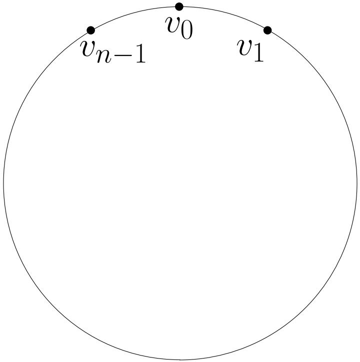

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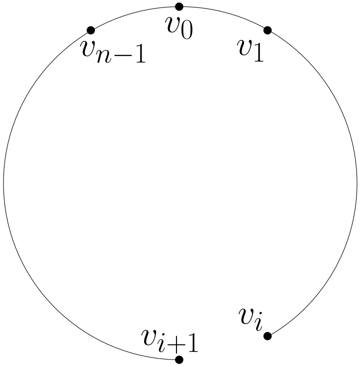

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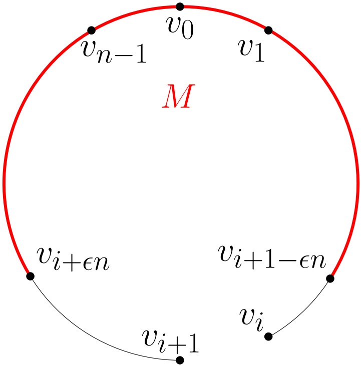

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Set $M = \{v_{i+1-\epsilon n}, v_{i+2-\epsilon n}, \ldots, v_{i+\epsilon n}\}$.

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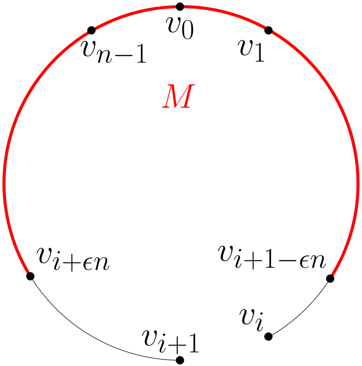

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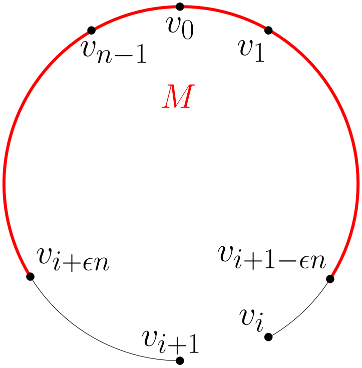

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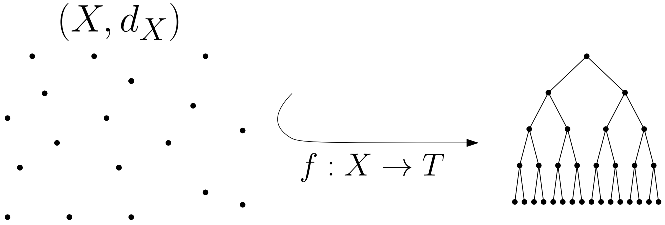
Choose i u.a.r., then $\Pr[v \in M] = 1 - 2\epsilon$.

Ramsey type Embeddings

Fix $k > 1$, what is the **largest subset** $M \subset X$, s.t. (M, d_X) embeds into a tree with **distortion** k?



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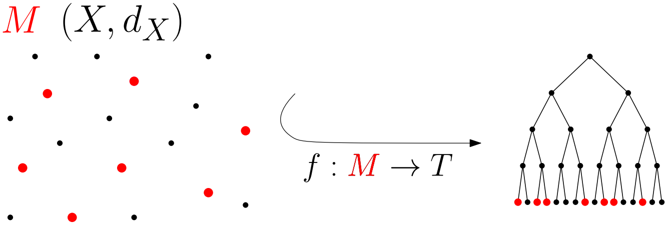

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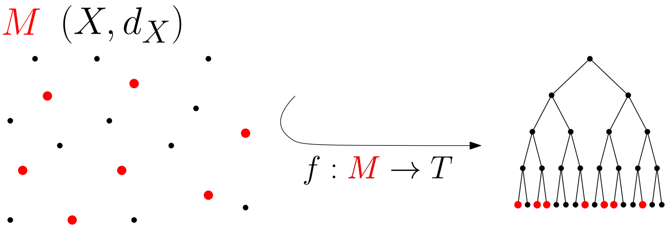

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$\forall$ *n-point metric space and* $\boldsymbol{k} \ge 1$, $\exists$ ***subset*** M *of size* $n^{1-1/k}$
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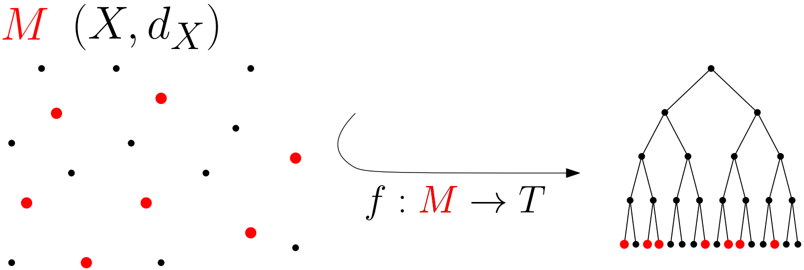

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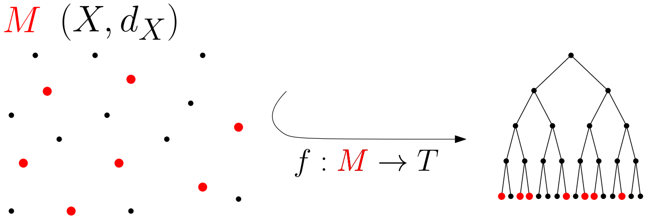

Asymptotically tight.

[Naor, Tao 12]: distortion $2e \cdot k$.

Ramsey type Embeddings

Corollary

For every n-point metric space and $\boldsymbol{k} \geq 1$, *there is a **set*** $\mathcal{T}$ *of* $k \cdot n^{\frac{1}{k}}$ ***trees** and a* *mapping* **home** : $X \to \mathcal{T}$, *such that for every* $x, y \in X$,

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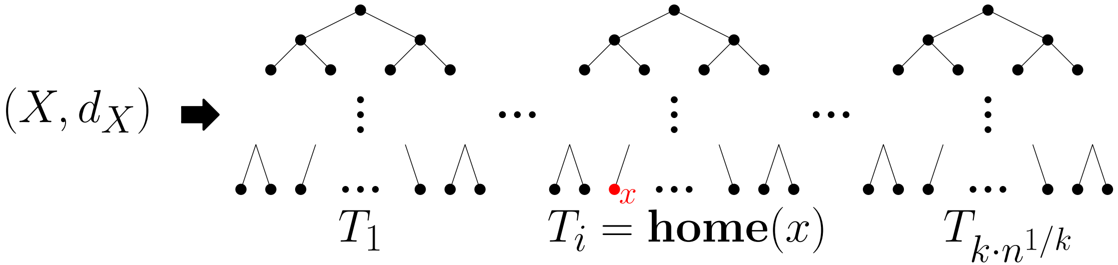

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Applications:

- **Distance oracle**
- Compact routing scheme
- Online algorithms
- Approximate ranking
- etc.

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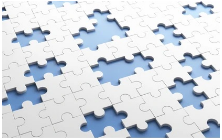
Compromises: only partial guarantees



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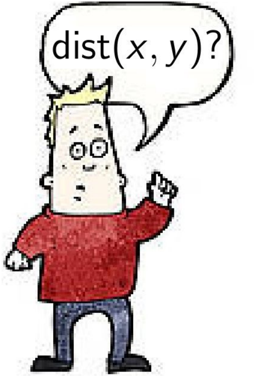

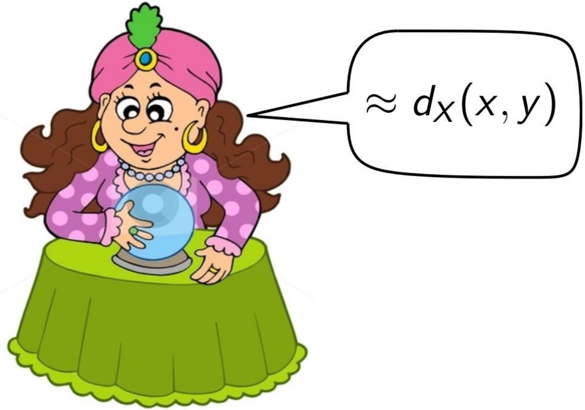

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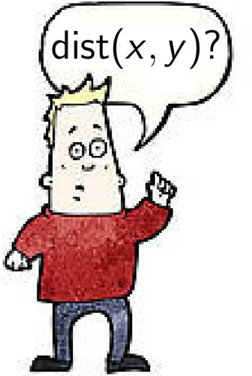

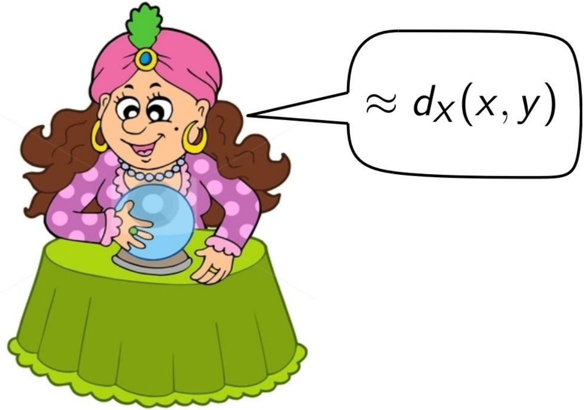

The properties of interest are size, distortion and query time.

Distance Oracles

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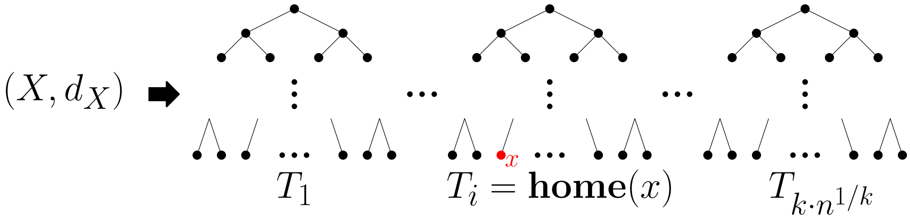

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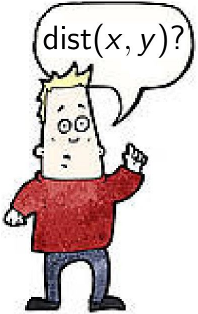

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For every ***tree metric****, there is an exact distance oracle of* ***linear size*** *and* ***constant*** *query* ***time****.*

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Theorem (Ramsey based Deterministic Distance Oracle)

For any n-point metric space, there is a distance oracle with :

Distortion	***Size***	***Query time***
$O(k)$	$O(k \cdot n^{1+1/k})$	$O(1)$

Outline of the talk - Appendix

Clan Embedding

Idea: **duplicate** vertices to meet all guarantees!

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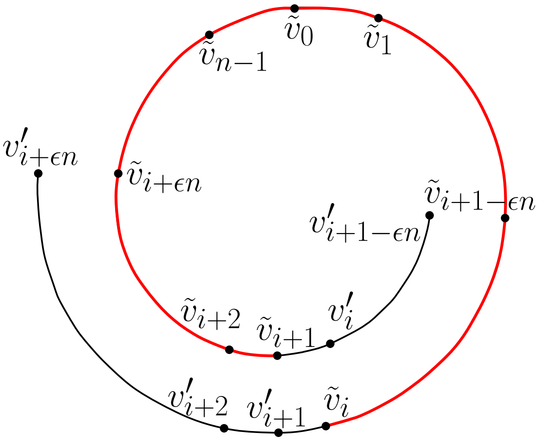

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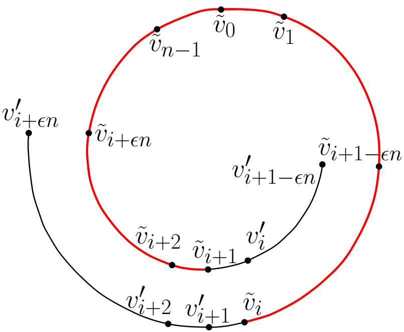

One-to-many embedding from (X, d_X) to (Y, d_Y): A map $f : X \to 2^Y$ where:

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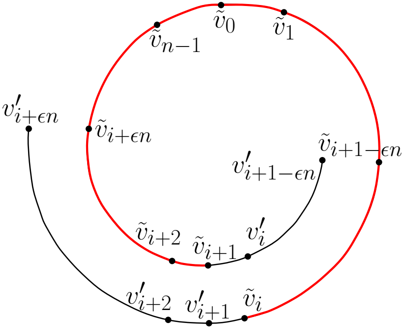

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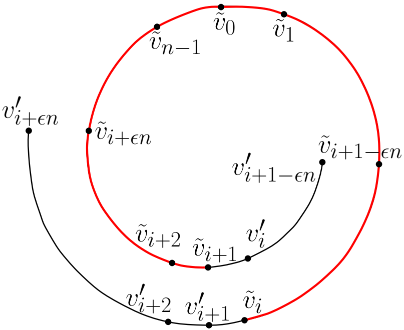

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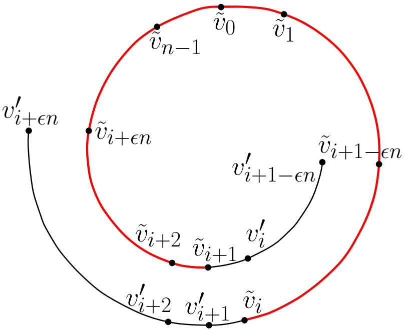

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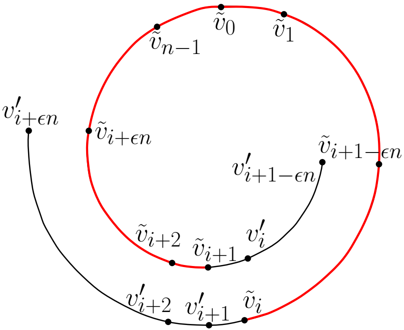

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$f : C_n \to 2^{P_{2n}}$, $\qquad f(v_i) \subseteq \{\tilde{v}_i, v'_i\}$, $\qquad \chi(v_i) = \tilde{v}_i$

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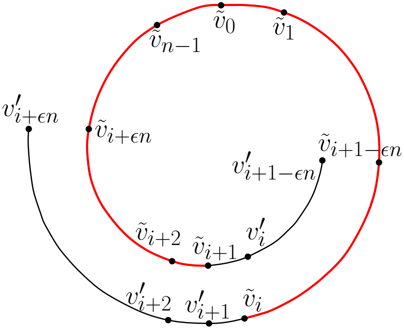

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Dominating: $d_X(x, y) \leq \min_{x' \in f(x), y' \in f(y)} d_Y(x', y')$

Each clan $f(x)$ has a **chief**: $\chi(x) \in f(x)$

$f : C_n \to 2^{P_{2n}}$, $\quad f(v_i) \subseteq \{\tilde{v}_i, v'_i\}$, $\quad \chi(v_i) = \tilde{v}_i$

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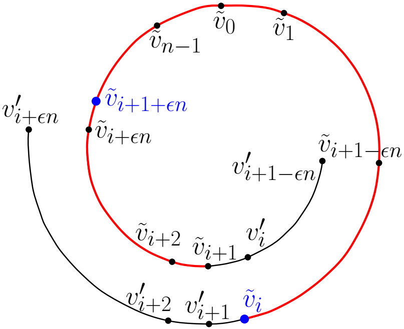

$$f : C_n \to 2^{P_{2n}}, \qquad f(v_i)\subseteq\{\tilde{v}_i, v'_i\}, \qquad \chi(v_i)=\tilde{v}_i$$

$$\min_{y'\in f(v_{i+1+\epsilon n})} d_{P_{2n}}(\chi(v_i),y') \le (1-\epsilon)n$$

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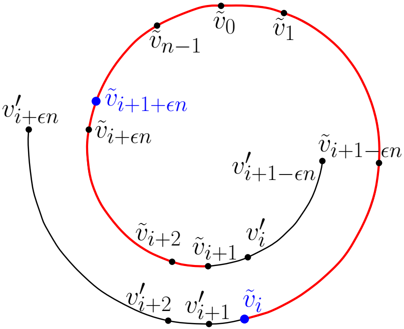

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Choose i u.a.r., then $\qquad \mathbb{E}[|f(v_i)|] = 1 + 2\epsilon.$

Theorem (Clan embedding into trees, [Filtser, Le 21])

(X, d_X) *n point metric space,* $\forall \epsilon \in (0,1)$*, there is* ***distribution*** $\mathcal{D}$ *over* ***dominating clan embeddings*** *into trees such that:*

- $\forall\ (f, \chi) \in \mathrm{supp}(\mathcal{D})$ *has distortion* $O(\frac{\log n}{\epsilon})$.
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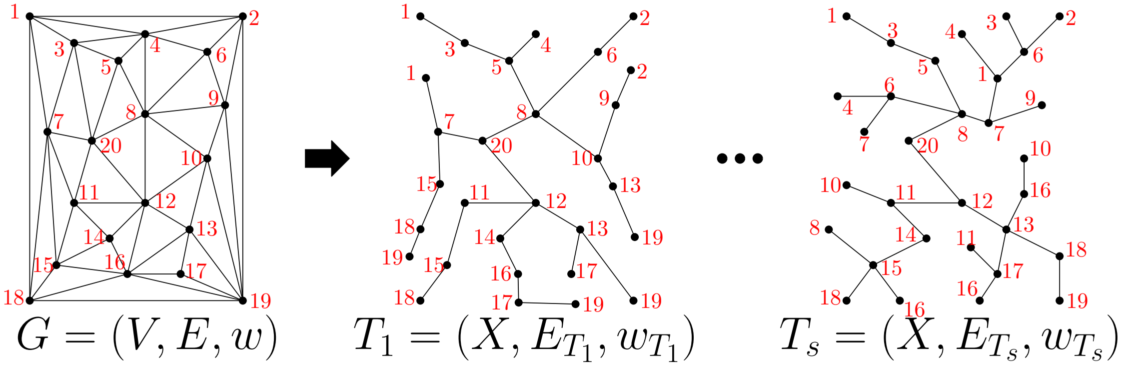

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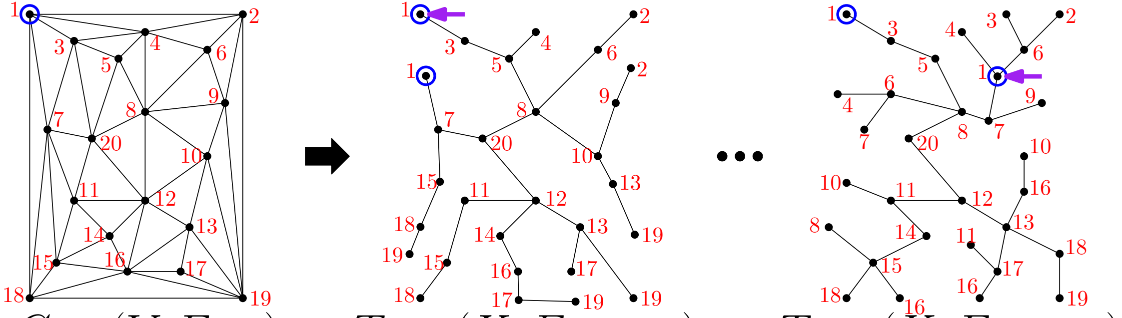

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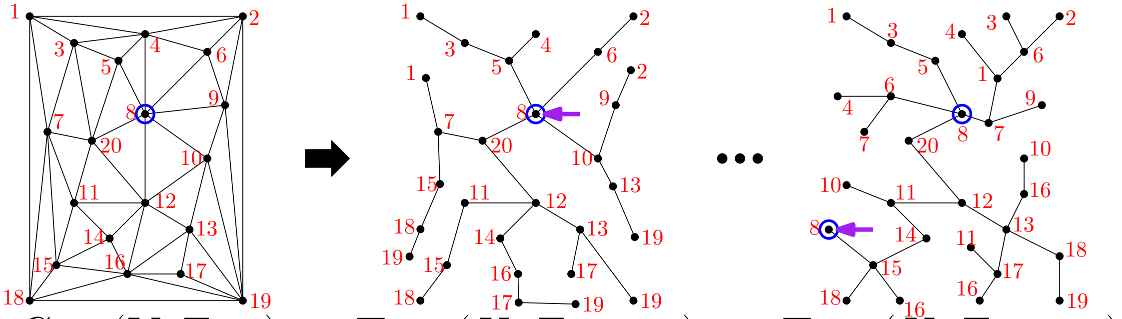

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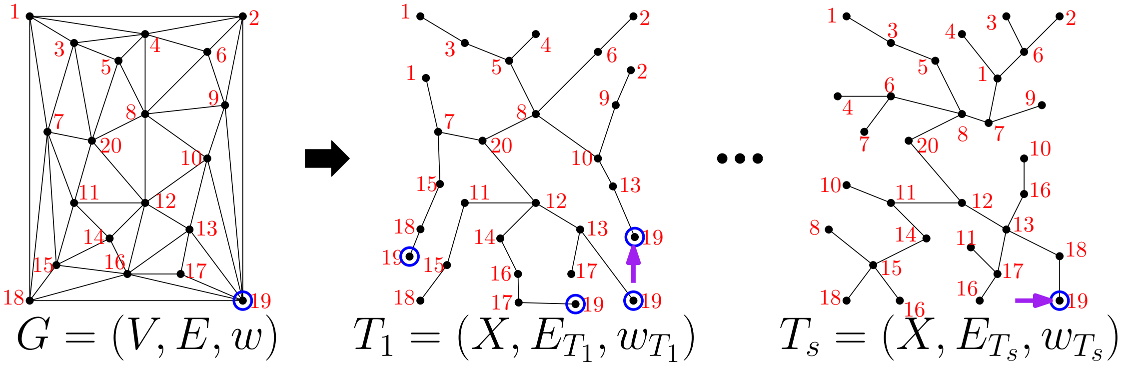

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(both) Tight!

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Compromises: Not a real classic embedding

Path Distortion

Originally appeared in [Bartal, Mendel 04] in the context of multi-embeddings.

Definition

One-to-many embedding $f : X \to 2^Y$ has *path-distortion* t if for every sequence $(x_0, x_1, \ldots, x_m)$ in X there is a sequence $y_0, \ldots, y_m$ where $y_i \in f(x_i)$, and

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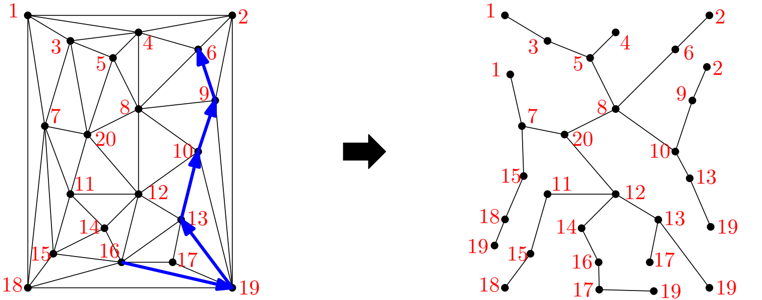

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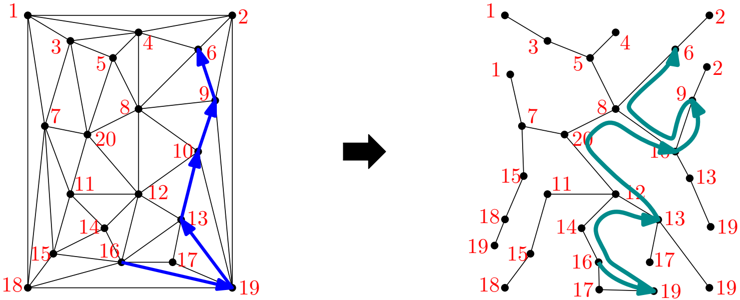

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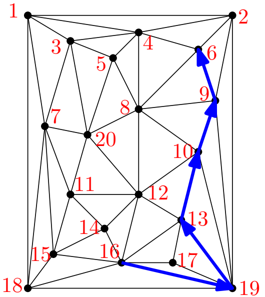

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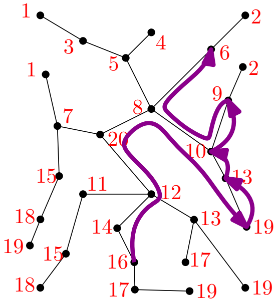

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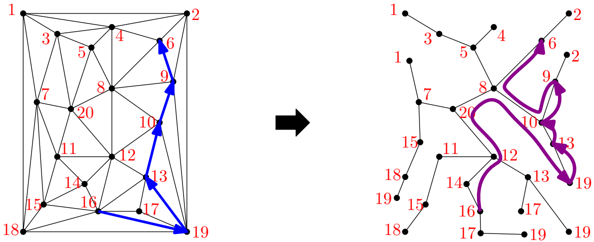

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Our clan embeddings with distortion $O(k)$ have path distortion $O(k \cdot \log n)$.

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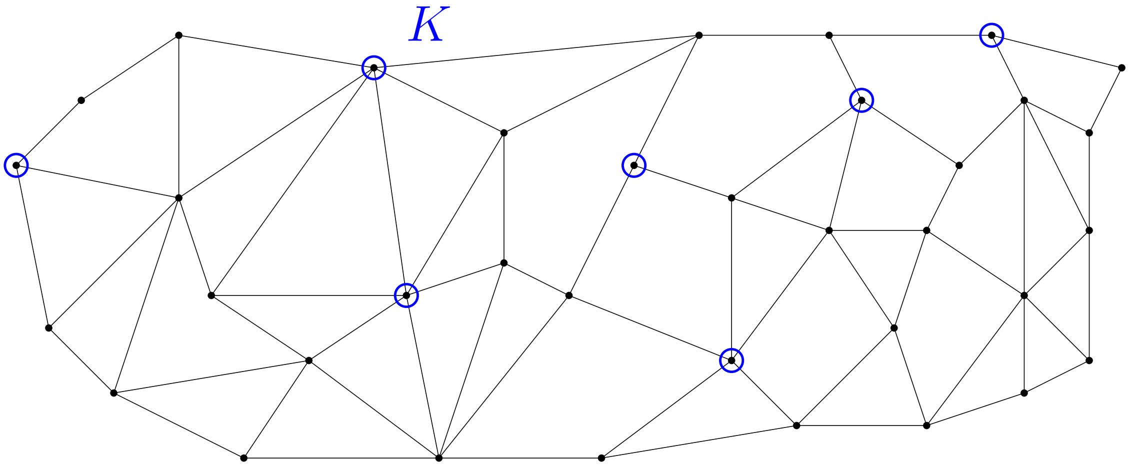
Or a total of $O(n^{1+\frac{1}{2}})$ copies and path distortion $O(\log n)$.

Outline of the talk - Appendix

7. Bartal 96 and Padded decompositions
8. Metrical Task System
9. Ramsey type embeddings
10. Clan embedding
11. Group Steiner Tree (using clan embedding)

Steiner Tree

Given set of terminals K, find minimum weight tree T spanning K



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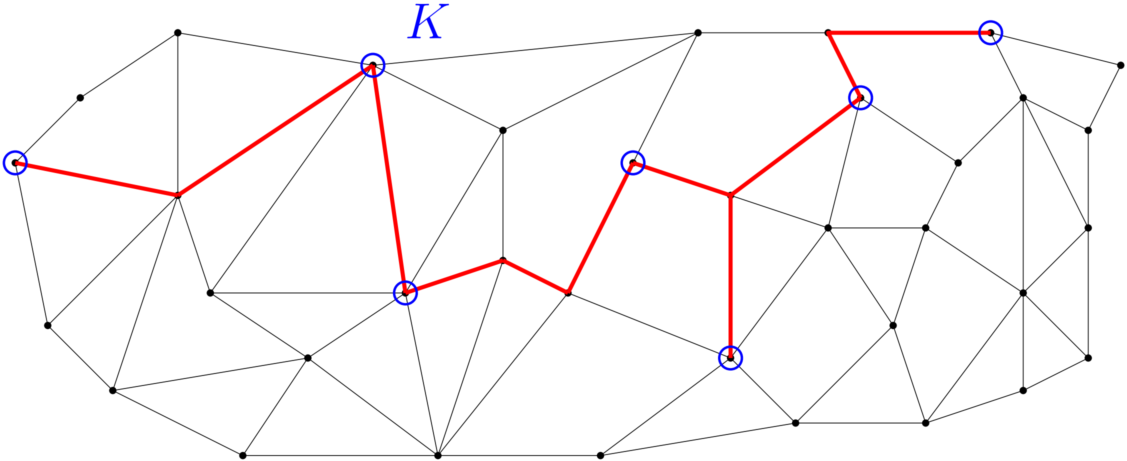

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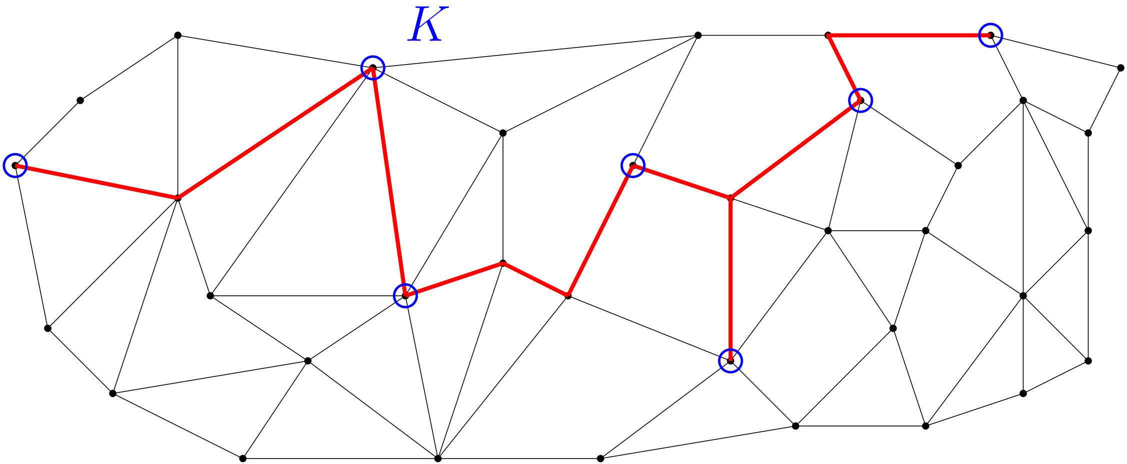
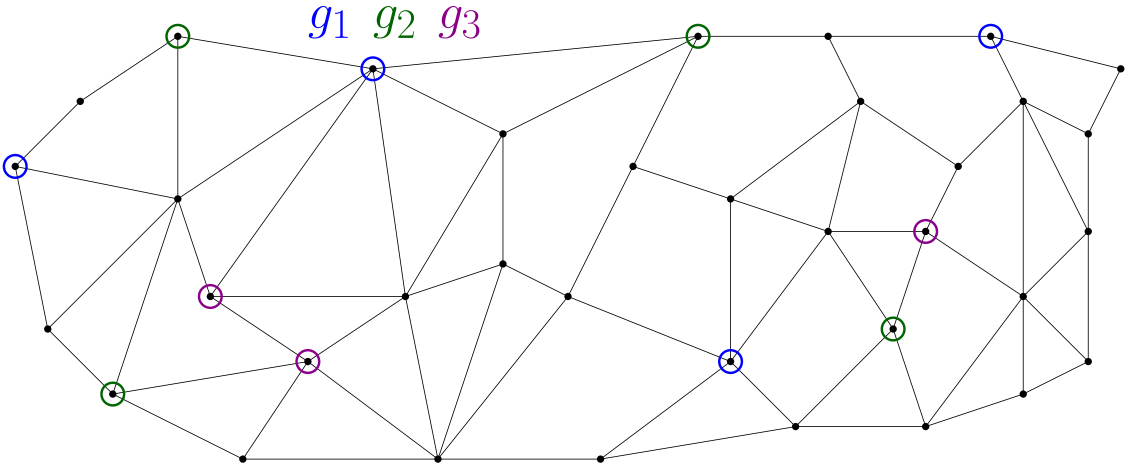


In class we saw a 2-approximation algorithm for the Steiner tree problem.

Group Steiner Tree

Given subsets $g_1, g_2, \ldots, g_k \subseteq V$, find minimum weight tree T spanning at least one vertex from each g_i



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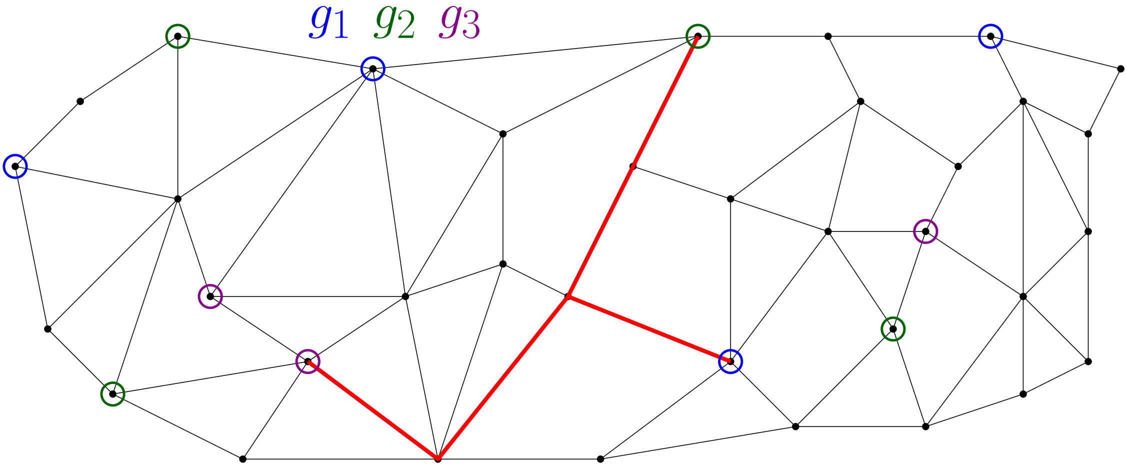

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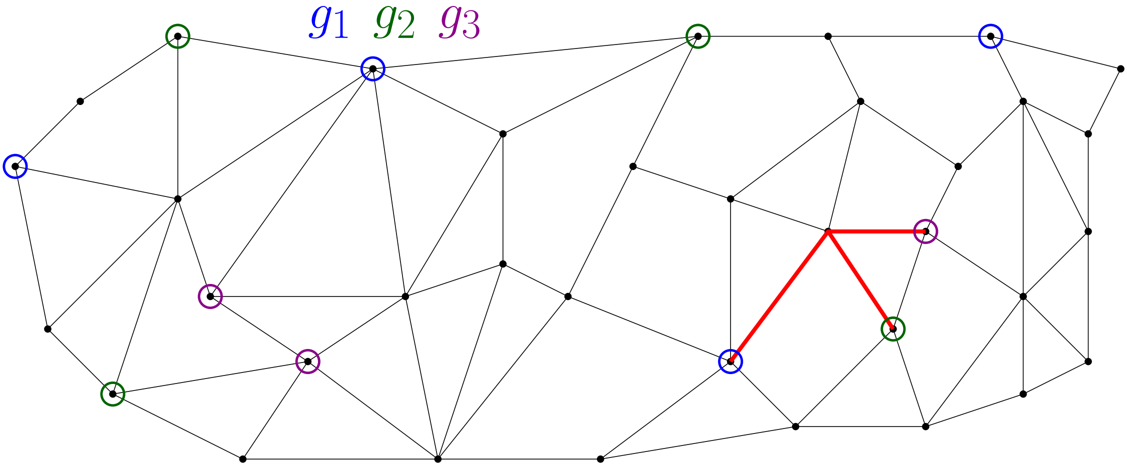

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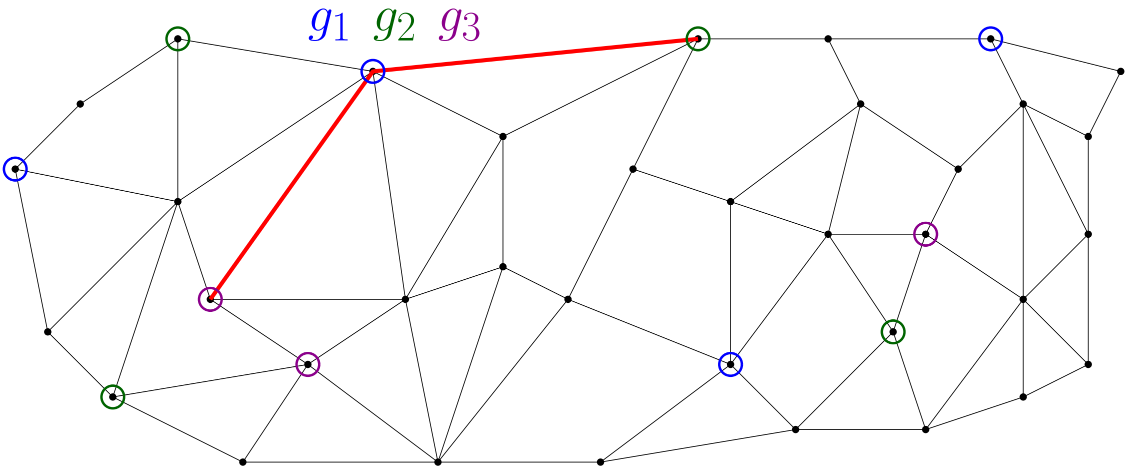

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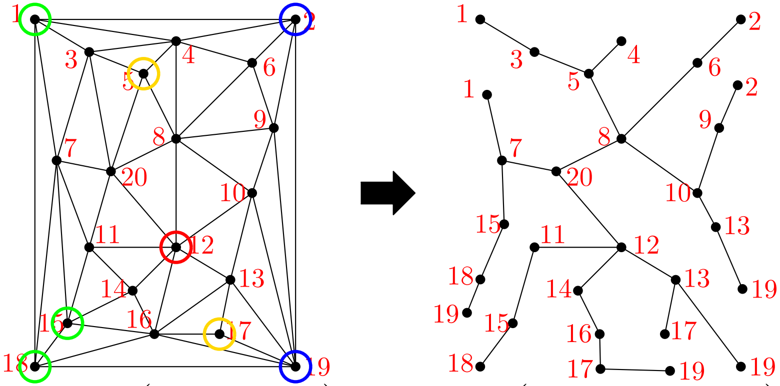

$$G = (V, E, w) \qquad T = (X, E_T, w_T)$$

$g_1 \quad g_2 \quad g_3 \quad g_4$

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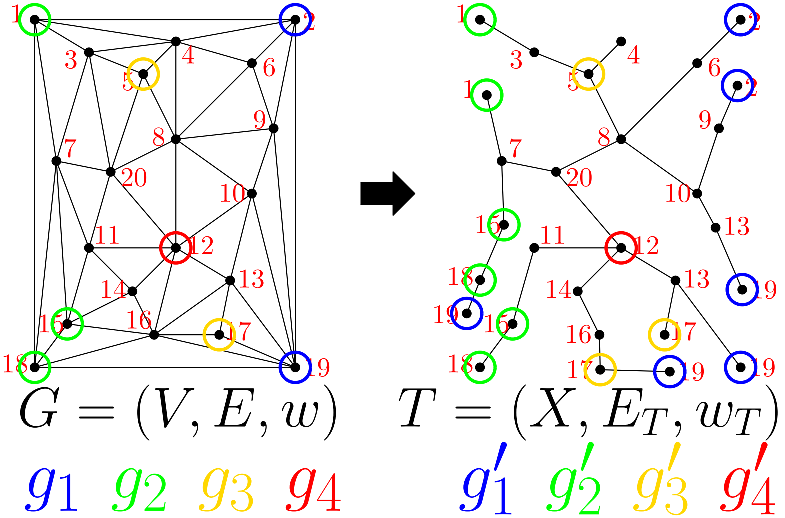
$g_i' = f(g_i)$



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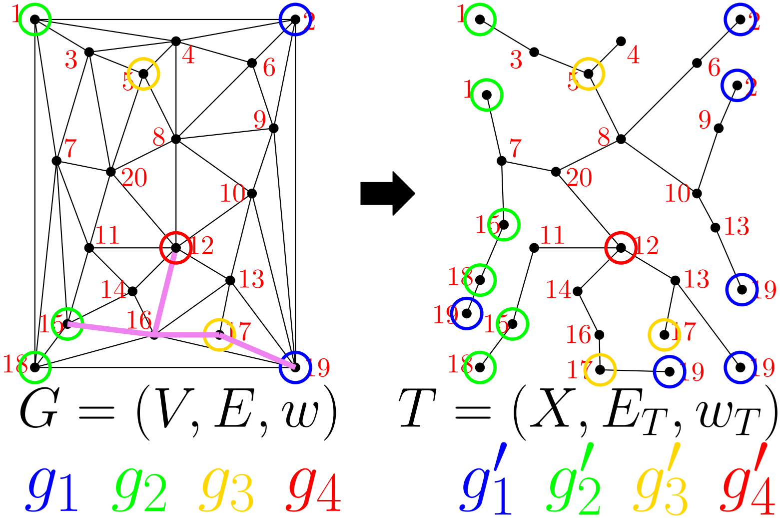

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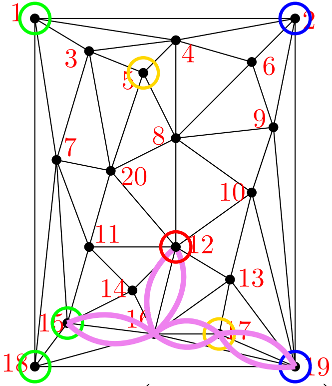
Double each edge: $2S^\star$



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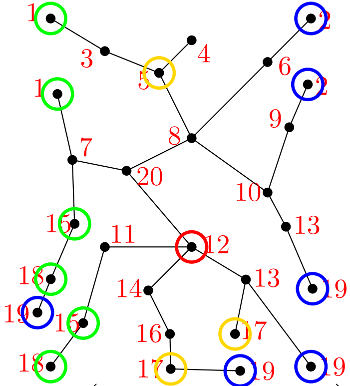

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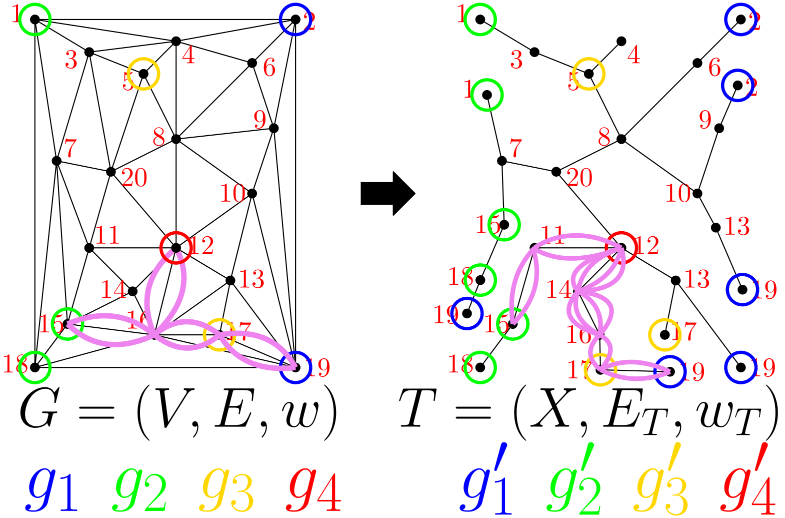
Double each edge: $2S^\star$

Guaranteed path: $S_T^\star$ (valid solution), $w(S_T^\star) \le O(\log n) \cdot w(S^\star)$



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Clan embedding f with path distortion $O(\log n)$.

$g'_i = f(g_i)$

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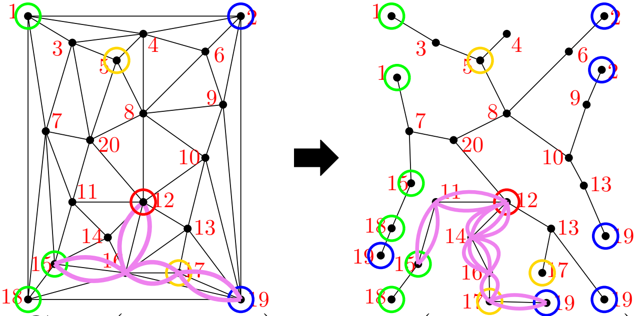

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g_1 g_2 g_3 g_4 $\quad$ g'_1 g'_2 g'_3 g'_4

Theorem ([Garg, Konjevod, Ravi 00])

$O(\log n \cdot \log k)$-approximation algorithm for the GST problem on trees.

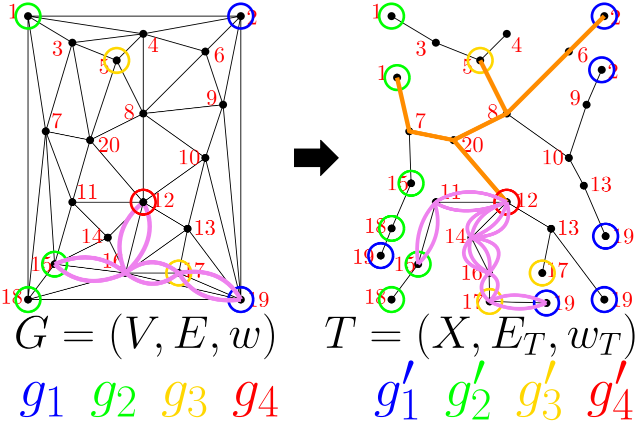
Given subsets $g_1, g_2, \ldots, g_k \subseteq V$, find minimum weight tree T spanning at least one vertex from each g_i.

$S^\star$ optimal solution.

Double each edge: $2S^\star$

Guaranteed path: $S^\star_T$, $w(S^\star_T) \leq O(\log n) \cdot w(S^\star)$

$\tilde{S}_T$ solution, $w(\tilde{S}_T) \leq O(\log n \cdot \log k) \cdot w(S^\star_T) \leq O(\log^2 n \cdot \log k) \cdot w(S^\star)$



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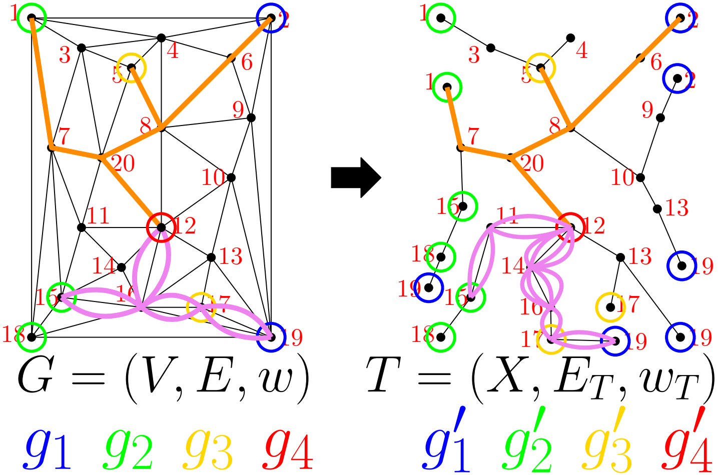
Return: $\tilde{S} = f^{-1}(\tilde{S}_T)$



$\tilde{S} = \cup_{\{v',u'\} \in \tilde{S}_T} P^T_{v',u'}$ is a union of shortest paths in G

Given subsets $g_1, g_2, \ldots, g_k \subseteq V$, find minimum weight tree T spanning at least one vertex from each g_i.

$S^\star$ optimal solution.

Double each edge: $2S^\star$

Guaranted path: $S^\star_T$,
$w(S^\star_T) \leq O(\log n) \cdot w(S^\star)$

$\tilde{S}_T$ solution, $w(\tilde{S}_T) \leq O(\log n \cdot \log k) \cdot w(S^\star_T) \leq O(\log^2 n \cdot \log k) \cdot w(S^\star)$

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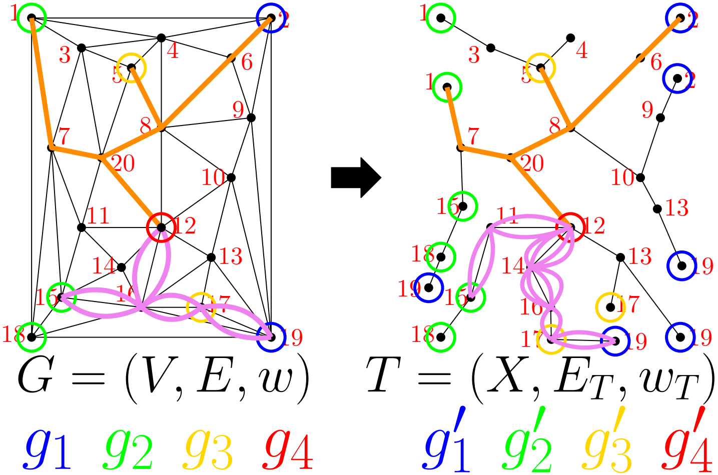

$\tilde{S} = \cup_{\{v', u'\} \in \tilde{S}_T} P^T_{v', u'}$ is a union of shortest paths in G

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Given subsets $g_1, g_2, \ldots, g_k \subseteq V$, find minimum weight tree T spanning at least one vertex from each g_i.

$S^\star$ optimal solution.

Double each edge: $2S^\star$

Guaranteed path: $S^\star_T$, $w(S^\star_T) \leq O(\log n) \cdot w(S^\star)$

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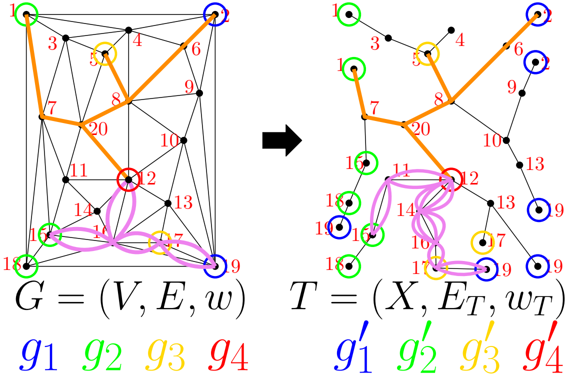

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We got an $O(\log^2 n \cdot \log k)$ **approximation**.

Clan Embeddings construction

Theorem (Clan embedding into trees, [Filtser, Le 21])

(X, d_X) n point metric space, $\forall k \in \mathbb{N}$, there is ***distribution*** *$\mathcal{D}$ over* ***dominating clan embeddings*** *into trees such that:*

- $\forall\ (f, \chi) \in \mathrm{supp}(\mathcal{D})$ *has distortion* $O(k)$.
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Definition (Ultrametric)

Ultrametric (X, d) is a metric space satisfying the **strong** triangle inequality:

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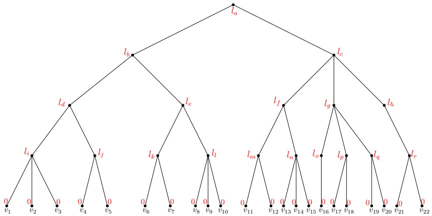
(X, d_X) is a HST if X is mapped (by ϕ) to **leaves** of a rooted tree T where:

- The nodes of T associated with **monotone** labels l_v.
- $d_X(x, y) = \Gamma_{\mathrm{lca}(\varphi(x), \varphi(y))}$.



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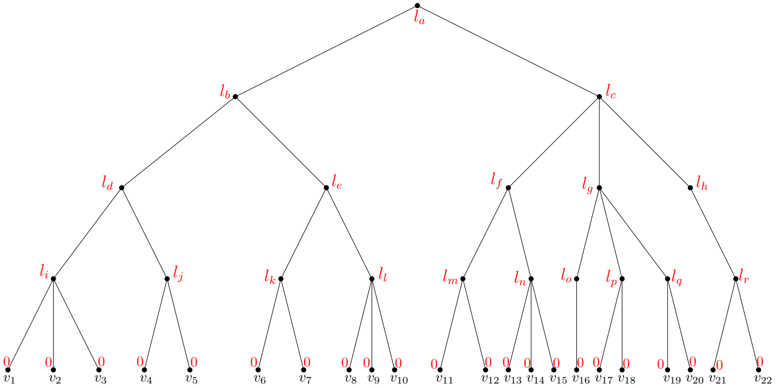

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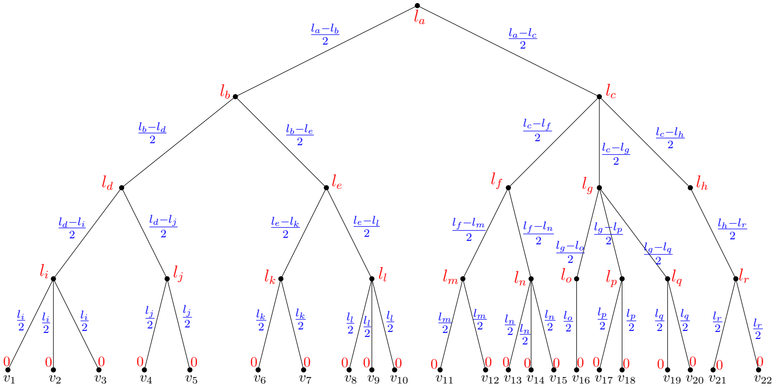
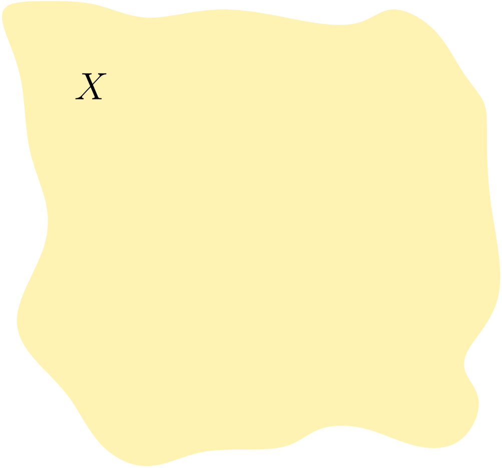


Construction



$$\ell(r_U) = \mathrm{diam}(X) = D$$

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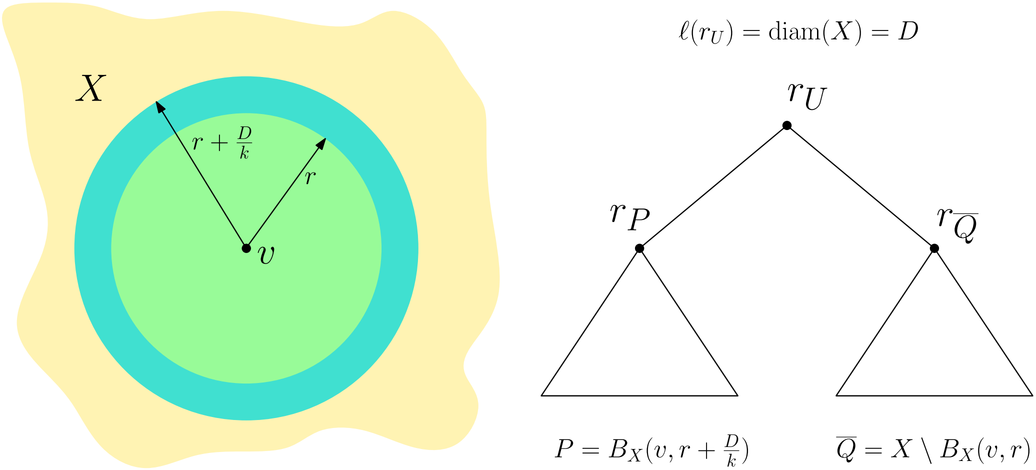

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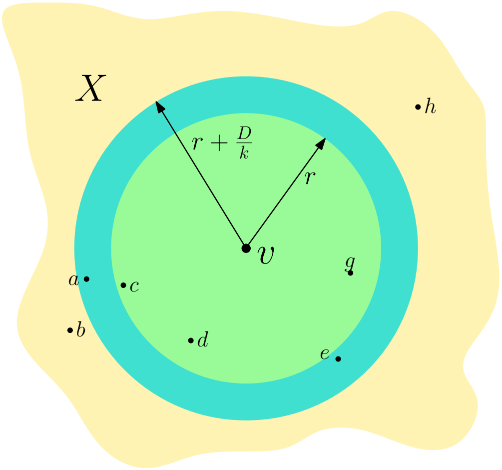

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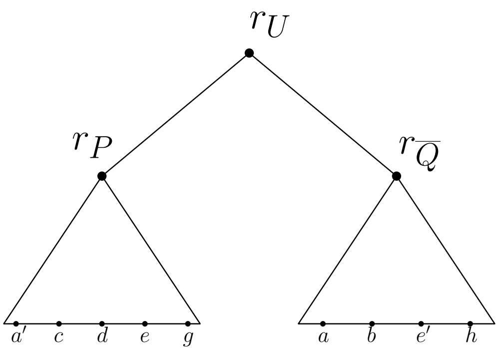
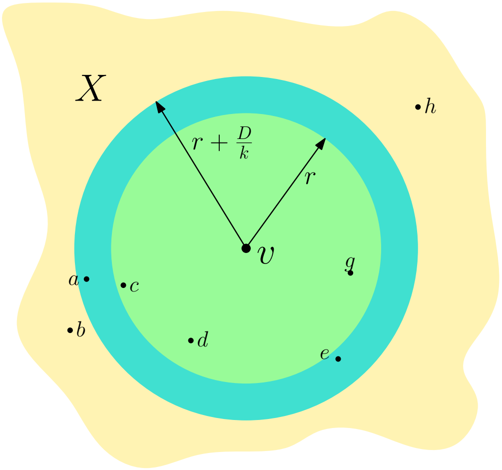
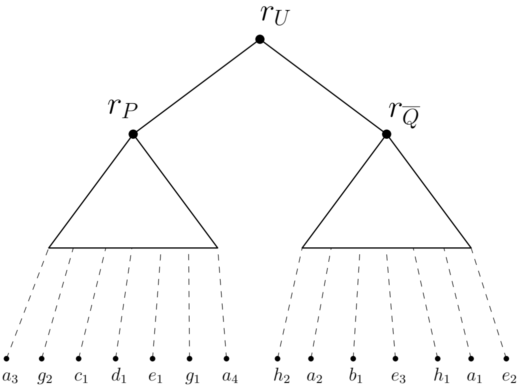


$P = B_X(v, r + \frac{D}{k})$ $\overline{Q} = X \setminus B_X(v, r)$

Construction





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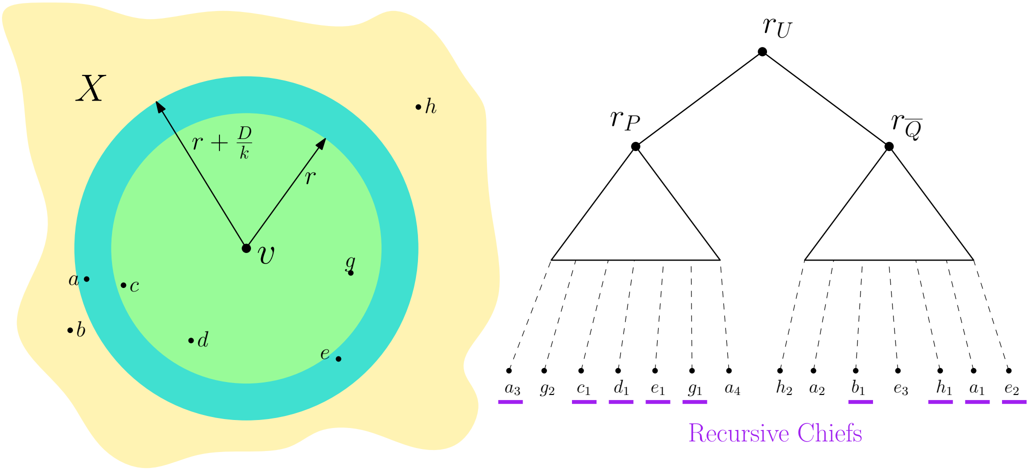

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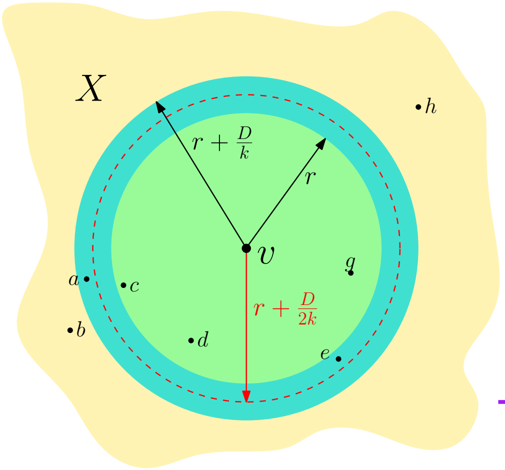
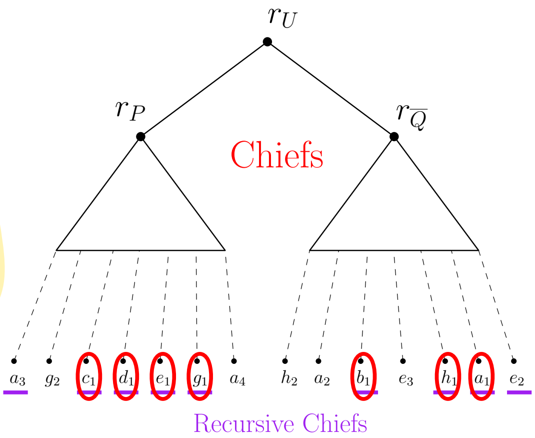
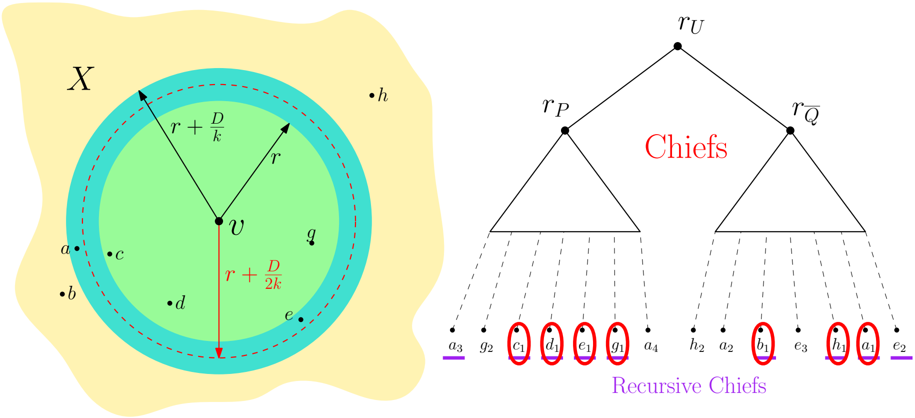


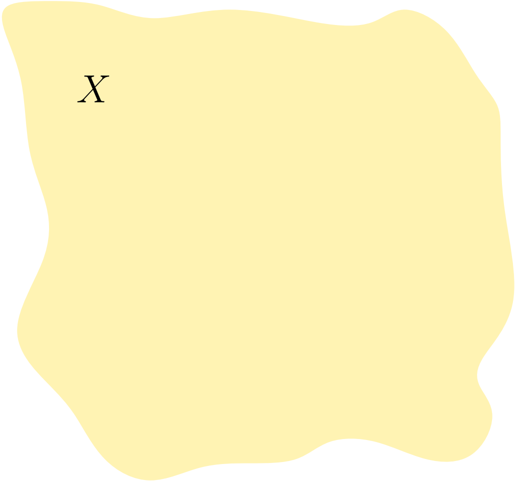


Construction - distortion bound



$$\min_{c' \in f(c)} d_U(c', \chi(a)) = D \le 2k \cdot d_X(c, a) \ .$$

Construction - cardinality bound



$$D = \operatorname{diam}(X)$$

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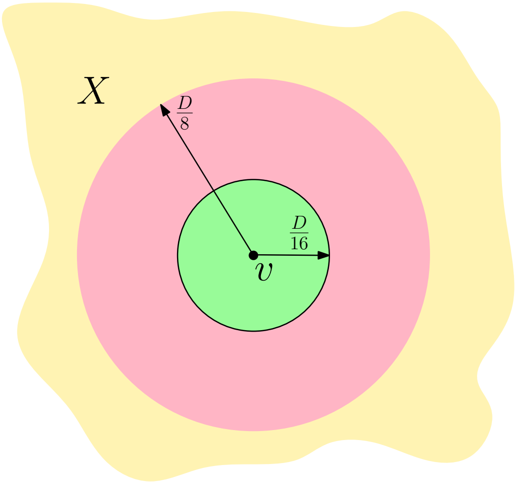

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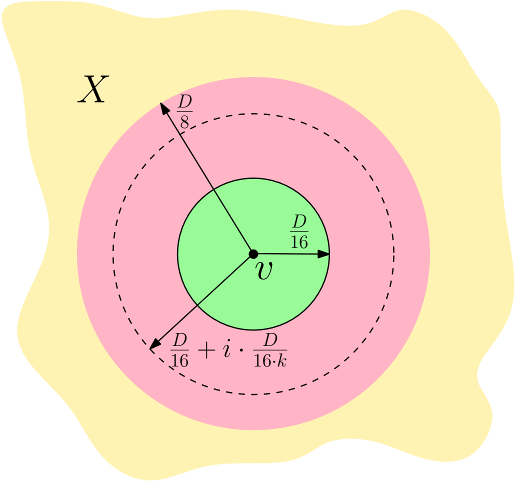

$$D = \operatorname{diam}(X)$$

$$A_i = B_X(v, \tfrac{D}{16} + i \cdot \tfrac{D}{16 \cdot k})$$
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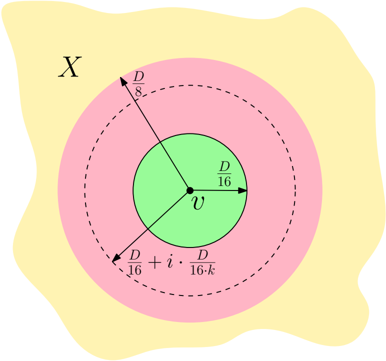

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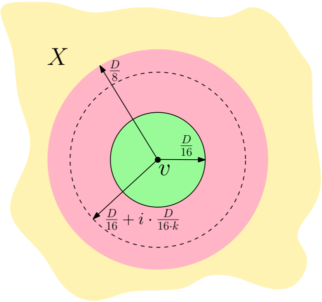

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$$|A_k| > |A_{k-1}| \cdot \left(\frac{|A_k|}{|A_0|}\right)^{\frac{1}{k}} > |A_{k-2}| \cdot \left(\frac{|A_k|}{|A_0|}\right)^{\frac{2}{k}} > \cdots > |A_0| \cdot \left(\frac{|A_k|}{|A_0|}\right)^{\frac{k}{k}} = |A_k|$$

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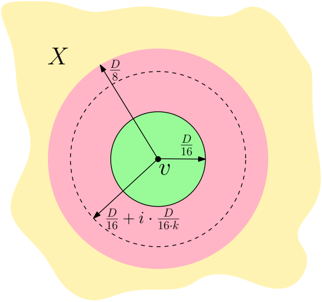

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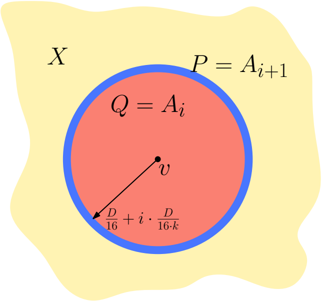

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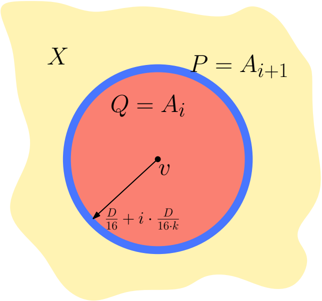

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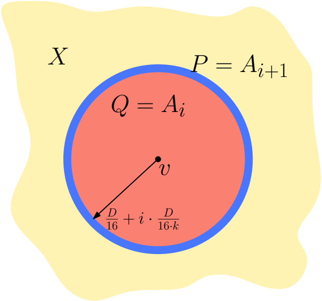

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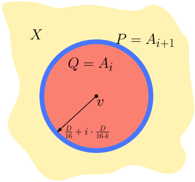

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\begin{aligned}
|f(X)| &= |f(P)| + |f(\overline{Q})| \\
&\le |P| \cdot \mu^*(P)^{\frac{1}{k}} + \left|\overline{Q}\right| \cdot \mu^*(\overline{Q})^{\frac{1}{k}} \\
&\le |Q| \cdot \mu^*(X)^{\frac{1}{k}} + \left|\overline{Q}\right| \cdot \mu^*(X)^{\frac{1}{k}} \\
&= |X| \cdot \mu^*(X)^{\frac{1}{k}} \;.
\end{aligned}
$$