Imperial College London

COURSEWORK 1

IMPERIAL COLLEGE LONDON

DEPARTMENT OF COMPUTING

Computational Finance

Author:
Arnold Cheung (CID: 01184493)

Date: November 11, 2019

1 Exercise 1. Dividend Discount Model

a) The price of stock is given by:

$$\begin{aligned}
PV &= \sum_{t=1}^{\infty} \frac{D_t}{(1+r)^{t-1}} \\
&= \sum_{t=1}^{\infty} \frac{D_1(1+g)^{t-1}}{(1+r)^{t-1}} \\
&= \sum_{t=0}^{\infty} \frac{D_1(1+g)^{t}}{(1+r)^{t}} \\
&= D_1\left(\frac{1}{1-\frac{(1+g)}{(1+r)}}\right) \\
&= D_1\left(\frac{(1+r)}{(1+r)-(1+g)}\right) \\
&= D_1\left(\frac{1+r}{r-g}\right) \qquad (1)
\end{aligned}$$

b) The price of stock is given by:

$$\begin{aligned}
PV &= \sum_{t=1}^{\infty} \frac{D_t}{(1+r)^{t-1}} \\
&= \sum_{t=1}^{\infty} \frac{D_1+(t-1)I}{(1+r)^{t-1}} \\
&= \sum_{t=0}^{\infty} \frac{D_1+tI}{(1+r)^{t}} \\
&= \sum_{t=0}^{\infty} \frac{D_1}{(1+r)^{t}} + \sum_{t=0}^{\infty} \frac{tI}{(1+r)^{t}} \\
&= \sum_{t=0}^{\infty} \frac{D_1}{(1+r)^{t}} + \sum_{t=0}^{\infty} tI(1+r)^{-t} \\
&= \sum_{t=0}^{\infty} \frac{D_1}{(1+r)^{t}} - \sum_{t=0}^{\infty} I(1+r)\frac{d}{d(1+r)}(1+r)^{-t} \\
&= \sum_{t=0}^{\infty} \frac{D_1}{(1+r)^{t}} - I(1+r)\frac{d}{d(1+r)}\left(\sum_{t=0}^{\infty}(1+r)^{-t}\right) \\
&= \frac{D_1(r+1)}{r} - I(1+r)\frac{d}{d(1+r)}\left(\frac{r+1}{r}\right) \\
&= \frac{D_1(r+1)}{r} + \frac{I(r+1)}{r^2} \qquad (2)
\end{aligned}$$

2 Exercise 2. Fixed-Income Securities

a) The price of this bond is given by its present value:

$$P = \frac{F}{[1+s_n]^n} + \sum_{k=1}^{n} \frac{C}{[1+s_k]^k} \tag{3}$$

For $F = 100$, $C = 10$, $n = 2$, and the given spot rates, the price, P of the bond is:

$$\begin{aligned} P &= \frac{100}{[1+s_2]^2} + \frac{10}{[1+s_1]} + \frac{10}{[1+s_2]^2} \\ P &= 107.5 \end{aligned} \tag{4}$$

b) The yield of the bond is such that:

$$P = \frac{F}{[1+(\frac{\lambda}{m})]^n} + \sum_{k=1}^{n} \frac{\frac{C}{m}}{[1+(\frac{\lambda}{m})]^k} \tag{5}$$

Specifically for this bond:

$$\begin{gathered} P = \frac{F}{(1+\lambda)^2} + \frac{C}{(1+\lambda)^2} + \frac{C}{(1+\lambda)} \\ P(1+\lambda)^2 = F + C + C + C\lambda \\ P\lambda^2 + (2P - C)\lambda + (P - F - 2C) = 0 \\ \lambda = \frac{-(2P-C) \pm \sqrt{(2P-C)^2 - 4P(P-F-2C)}}{2P} \end{gathered} \tag{6}$$

Which when evaluated is:

$$\begin{aligned} \lambda &= 0.059 \\ &= 5.9\% \end{aligned} \tag{7}$$

c) The bond's Macauley duration is given by:

$$D = \frac{1+\frac{\lambda}{m}}{\lambda} - \frac{1+\frac{\lambda}{m}+n(\frac{c}{m}-\frac{\lambda}{m})}{c[(1+\frac{\lambda}{m})^n - 1] + \lambda} \tag{8}$$

$$D = \frac{1+\lambda}{\lambda} - \frac{1+\lambda+n(c-\lambda)}{c[(1+\lambda)^n - 1] + \lambda} \tag{9}$$

$$D = \frac{1+\lambda}{\lambda} - \frac{1+\lambda+n(c-\lambda)}{c[(1+\lambda)^n - 1] + \lambda} \tag{10}$$

Using the result for λ, and c = 0.1:

$$D = 1.91 \tag{11}$$

d) The price of the bond next year can be calculated by first calculating the forward rates from year 1:

$$f_{i,j} = \left[\frac{(1+s_j)^j}{(1+s_i)^i}\right]^{\frac{1}{j-i}} - 1 \tag{12}$$

$$\begin{aligned} f_{1,2} &= \left[\frac{(1+s_2)^2}{(1+s_1)}\right] - 1 \\ &= 0.08 \\ &= 8\% \end{aligned} \tag{13}$$

$$\begin{aligned} f_{1,3} &= \left[\frac{(1+s_3)^3}{(1+s_1)}\right]^{\frac{1}{2}} - 1 \\ &= 0.10 \\ &= 1 \end{aligned} \tag{14}$$

$$P = \frac{100}{[1+f_{1,3}]^2} + \frac{10}{[1+f_{1,2}]} + \frac{10}{[1+f_{1,3}]^2} \tag{15}$$

$$P = 100.1 \tag{16}$$

3 Exercise 3. Portfolio Optimisation

a) The expected return, $\overline{r}_P$, and the variance, $\overline{\sigma}_P^2$ is given as follows:

$$\begin{aligned} \overline{r}_P &= \mathbb{E}[\boldsymbol{r_P}] \\ &= \mathbb{E}\left[w_0 r_f + \sum_{i=1}^{n} w_i r_i\right] \\ &= w_0 r_f + \sum_{i=1}^{n} w_i \overline{r}_i \\ &= w_0 r_f + \boldsymbol{w}^T \overline{\boldsymbol{r}} \end{aligned} \tag{17}$$

$$\begin{aligned} \overline{\sigma}_P^2 &= var(r_P) \\ &= \mathbb{E}\left[\left(\sum_{i=1}^{n} w_i r_i - \sum_{i=1}^{n} w_i \overline{r}_i\right)^2\right] \\ &= \mathbb{E}\left[\left(\sum_{i=1}^{n} w_i (r_i - \overline{r}_i)\right)\left(\sum_{j=1}^{n} w_j (r_j - \overline{r}_j)\right)\right] \\ &= \mathbb{E}\left[\left(\sum_{i=1}^{n} w_i (r_i - \overline{r}_i)\right)\left(\sum_{j=1}^{n} w_j (r_j - \overline{r}_j)\right)\right] \end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\left[\sum_{i,j=1}^{n} w_i w_j (r_i - \overline{r}_i)(r_j - \overline{r}_j)\right] \\
&= \sum_{i,j=1}^{n} w_i \sigma_{ij} w_j \\
&= \boldsymbol{w}^T \Sigma \boldsymbol{w} \qquad (18)
\end{aligned}
$$

b) To maximise $\overline{r}_P - \frac{a}{2}\sigma_P^2$, the problem can be formulated into a quadratic program:

$$
\begin{aligned}
Maximise &: \overline{r}_P - \frac{a}{2}\sigma_P^2 \\
Where &: w_0 r_f + \boldsymbol{w}^T \overline{\boldsymbol{r}} = \overline{r}_P \\
& w_0 + \boldsymbol{w}^T \boldsymbol{e} = 1
\end{aligned}
$$

$$
\begin{aligned}
Maximise : f(\boldsymbol{w}) &= -\frac{a}{2}\boldsymbol{w}^T \Sigma \boldsymbol{w} + \boldsymbol{w}^T \overline{\boldsymbol{r}} + (1 - \boldsymbol{w}^T \boldsymbol{e}) r_f \\
f(\boldsymbol{w}) &= -\frac{a}{2}\boldsymbol{w}^T \Sigma \boldsymbol{w} + \boldsymbol{w}^T (\overline{\boldsymbol{r}} - r_f \boldsymbol{e}) + r_f \qquad (19)
\end{aligned}
$$

Set derivative = 0, and solve for $\boldsymbol{w}$:

$$
f'(\boldsymbol{w}) = -2\boldsymbol{w}^T \Sigma + \overline{\boldsymbol{r}}^T - r_f \boldsymbol{e}^T = 0 \qquad (20)
$$

c) The problem can also be solved by deriving the optimality conditions. The Lagrangian function, L can be obtained by substituting the constraints into the objective function:

$$
L = -\frac{a}{2}\boldsymbol{w}^T \Sigma \boldsymbol{w} + \boldsymbol{w}^T (\overline{\boldsymbol{r}} - r_f \boldsymbol{e}) + r_f \qquad (21)
$$

The optimality condition can be obtained by differentiating the Lagrangian w.r.t. $\boldsymbol{w}$ set derivative to 0:

$$
\begin{aligned}
\frac{dL}{d\boldsymbol{w}} &= -a\Sigma \boldsymbol{w} - r_f \boldsymbol{e} + \overline{\boldsymbol{r}} = 0 \\
&\Rightarrow a\Sigma \boldsymbol{w} + r_f \boldsymbol{e} - \overline{\boldsymbol{r}} = 0 \\
\boldsymbol{w} &= \frac{1}{a}\Sigma^{-1}(\overline{\boldsymbol{r}} - r_f \boldsymbol{e}) \qquad (22)
\end{aligned}
$$

Maximum value found by substituting into the quadratic problem:

$$
\begin{aligned}
&-\frac{a}{2}\left(\frac{1}{a}\Sigma^{-1}(\overline{\boldsymbol{r}} - r_f \boldsymbol{e})\right)^T \Sigma\left(\frac{1}{a}\Sigma^{-1}(\overline{\boldsymbol{r}} - r_f \boldsymbol{e})\right) + \left(\frac{1}{a}\Sigma^{-1}(\overline{\boldsymbol{r}} - r_f \boldsymbol{e})\right)^T (\overline{\boldsymbol{r}} - r_f \boldsymbol{e}) + r_f \\
&= -\frac{1}{2a}(\overline{\boldsymbol{r}} - r_f \boldsymbol{e})^T \Sigma^{-1}(\overline{\boldsymbol{r}} - r_f \boldsymbol{e}) + \frac{1}{a}(\overline{\boldsymbol{r}} - r_f \boldsymbol{e})^T \Sigma^{-1}(\overline{\boldsymbol{r}} - r_f \boldsymbol{e}) + r_f \\
&= \frac{1}{2a}(\overline{\boldsymbol{r}} - r_f \boldsymbol{e})^T \Sigma^{-1}(\overline{\boldsymbol{r}} - r_f \boldsymbol{e}) + r_f \qquad (23)
\end{aligned}
$$

d) As a tends to $+\infty$:

$$\lim_{a\to\infty} \frac{1}{2a}(\overline{\boldsymbol{r}} - r_f\boldsymbol{e})^T \Sigma^{-1}(\overline{\boldsymbol{r}} - r_f\boldsymbol{e}) + r_f = r_f$$

e) Let f denote the optimal value function (expression 23):

$$\begin{aligned} f(\overline{\boldsymbol{r}}) &= \frac{1}{2a}(\overline{\boldsymbol{r}} - r_f\boldsymbol{e})^T \Sigma^{-1}(\overline{\boldsymbol{r}} - r_f\boldsymbol{e}) + r_f \\ \Rightarrow f(\mathbb{E}[\hat{\overline{\boldsymbol{r}}}]) &= \frac{1}{2a}(\mathbb{E}[\hat{\overline{\boldsymbol{r}}}] - r_f\boldsymbol{e})^T \Sigma^{-1}(\mathbb{E}[\hat{\overline{\boldsymbol{r}}}] - r_f\boldsymbol{e}) + r_f \end{aligned} \tag{24}$$

The average estimation of the optimal value of the investor is:

$$\mathbb{E}[f(\hat{\overline{\boldsymbol{r}}})] = \mathbb{E}[\frac{1}{2a}(\hat{\overline{\boldsymbol{r}}} - r_f\boldsymbol{e})^T \Sigma^{-1}(\hat{\overline{\boldsymbol{r}}} - r_f\boldsymbol{e}) + r_f] \tag{25}$$

Jenson's inequality states that:

$$\mathbb{E}[f(x)] \geq f(\mathbb{E}[x]) \tag{26}$$

Therefore, it can be shown that the investor will on average overestimate the optimal value of the portfolio:

$$\mathbb{E}[f(\hat{\overline{\boldsymbol{r}}})] \geq f(\mathbb{E}[\hat{\overline{\boldsymbol{r}}}]) \tag{27}$$