eDNA

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2024-07-16

Detection of eDNA from tows

Model:

- N organisms located at $(x, y, z) = (0, 0, h)$
- shed S copies per unit time
- plume moves at a constant velocity v towards positive x values
- disperse fully vertically after a short distance
- disperse by a random walk in the y direction such that the dispersion in y after time t is $\sigma(t) = \sqrt{2A_H t}$
- decay over time such that the number density at time t is reduced by a factor of $e^{-\alpha t}$ after release at time 0

Consider time t, following a time interval Δt at $t = 0$,

- $NS\Delta t$ copies are released at $t = 0$
- the number density has decayed by a factor $e^{-\alpha t}$
- the plume has travelled distance $x = vt$
- the $NS\Delta t$ copies are dispersed over a distance $\Delta x = v\Delta t$ in x
- the copies are dispersed uniformly over the depth h
- the plume has dispersed in y with dispersion $\sigma(x) = \sqrt{2A_H t} = \sqrt{aA_H x/v}$

Volume number density at location (x, y, z) is

$$\begin{aligned} C(x,y,z) &= NS(\Delta t) \times e^{-\alpha t} \times \frac{1}{\Delta x} \times \frac{1}{\sqrt{2\pi\sigma^2(x)}} e^{-\frac{1}{2}y^2/\sigma^2(x)} \times \frac{1}{h} \\ &= \frac{NS}{hv} \frac{1}{\sqrt{2\pi\sigma^2(x)}} e^{-\frac{1}{2}y^2/\sigma^2(x) - \alpha x/v} \end{aligned}$$

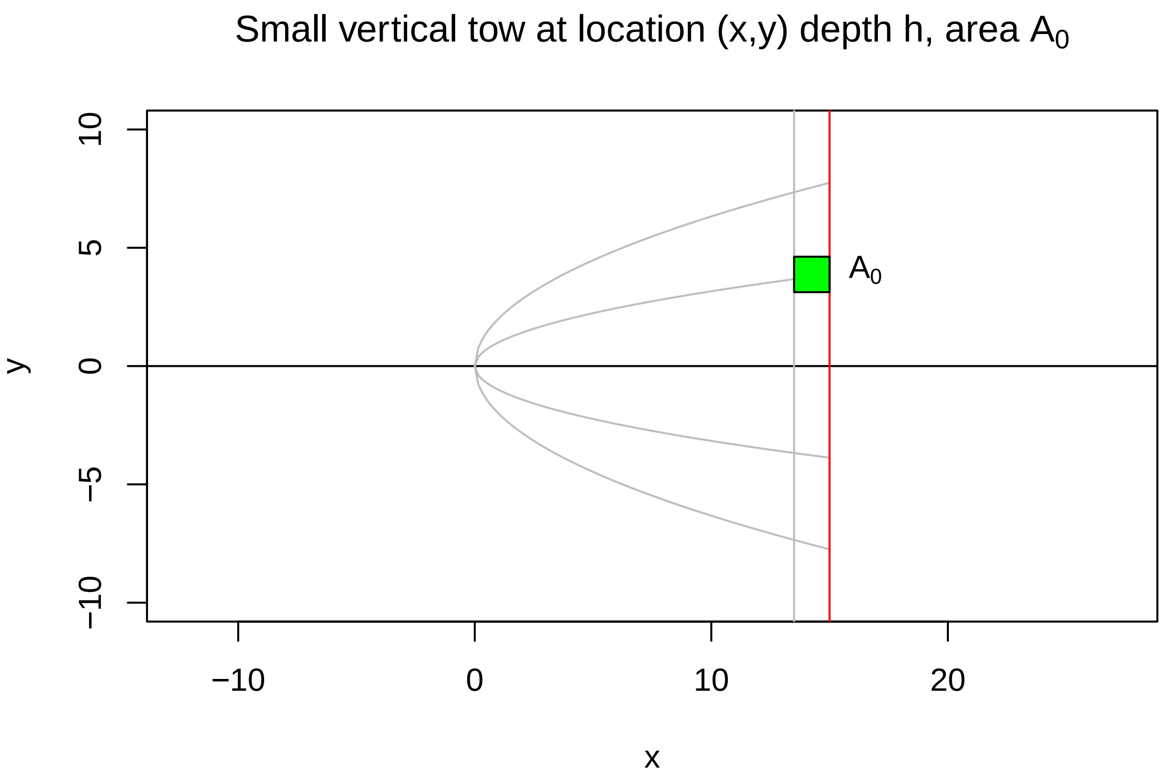
```
xmax <- 15
ymax <- 10
np <- 101
Ah <- 1; v <- 2
xvec <- seq(from=0,to=xmax,length=np)
y1 <- 2*sqrt(2*Ah*xmax/v)
dx <- 1.5
dt <- dx/v
```

Consider a full depth tow at location (x, y) with area A_0. The expected number of copies captured is

$$\begin{aligned}
K_V(x,y) &= \int\int_{A_0} \mathrm{d}x\mathrm{d}y \int_0^h \mathrm{d}z\; C(x,y,z) \\
&= A_0 C(x,y) \\
&= \frac{NSA_0}{v} \frac{1}{\sqrt{2\pi\sigma^2(x)}} e^{-\frac{1}{2}y^2/\sigma^2(x) - \alpha x/v} \\
&= \frac{NSV_V}{vh} \frac{1}{\sqrt{2\pi\sigma^2(x)}} e^{-\frac{1}{2}y^2/\sigma^2(x) - \alpha x/v}
\end{aligned}$$

where $C(x, y) = hC(x, y, z)$ is the area density (since $C(x, y, z)$ is independent of z), and $V_V = A_0 h$ is the volume of water sampled.

```
plot(NA,NA,xlim=c(0,xmax),ylim=ymax*c(-1,1),xlab="x",ylab="y",asp=1)
abline(h=0)
lines(xvec, +sqrt(2*Ah*xvec/v),col="grey")
lines(xvec, -sqrt(2*Ah*xvec/v),col="grey")
lines(xvec, +2*sqrt(2*Ah*xvec/v),col="grey")
lines(xvec, -2*sqrt(2*Ah*xvec/v),col="grey")
abline(v=xmax,col="red")
abline(v=xmax-dx,col="grey")
text(xmax, y1/2, lab=expression(A[0]), pos=4)
polygon(xmax+c(-dx,0,0,-dx,-dx), (y1/2) + (dx/2)*c(-1,-1,1,1,-1), col="green")
title(expression("Small vertical tow at location (x,y) depth h, area "*A[0]))
```

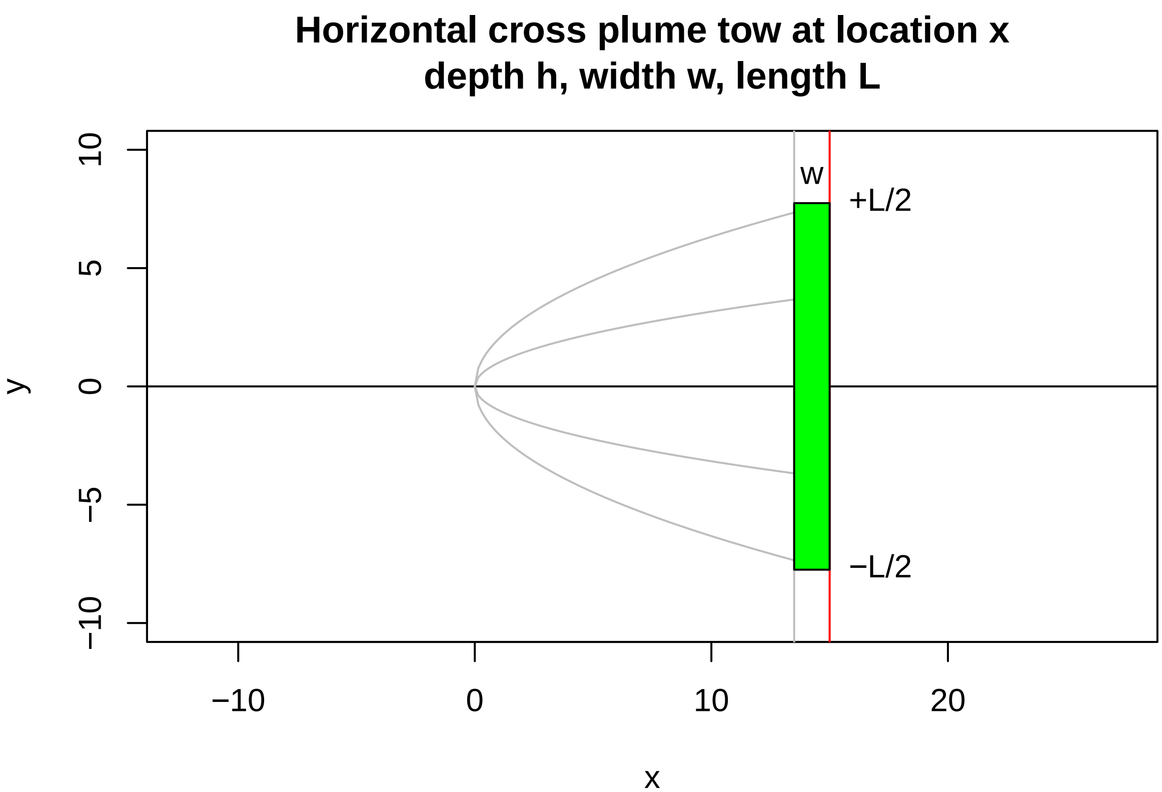


Consider a full depth tow of length L and width w at constant x across the plume, with L being sufficiently large that the full plume is captured (i.e. $\frac{1}{2}L > 3\sigma(x)$. The expected number of copies captured is

$$
\begin{aligned}
K_H(x) &= \int_{x-w/2}^{x+w/2} \mathrm{d}x \int_{-L/2}^{L/2} \mathrm{d}y \int_0^h \mathrm{d}z \; C(x,y,z) \\
&= w\frac{NS}{hv}e^{-\alpha x/v}h \\
&= \frac{NSw}{v}e^{-\alpha x/v} \\
&= \frac{NSV_H}{hLv}e^{-\alpha x/v}
\end{aligned}
$$

where $V_H = wLh$ is the volume of water sampled.

```
plot(NA,NA,xlim=c(0,xmax),ylim=ymax*c(-1,1),xlab="x",ylab="y",asp=1)
abline(h=0)
lines(xvec, +sqrt(2*Ah*xvec/v),col="grey")
lines(xvec, -sqrt(2*Ah*xvec/v),col="grey")
lines(xvec, +2*sqrt(2*Ah*xvec/v),col="grey")
lines(xvec, -2*sqrt(2*Ah*xvec/v),col="grey")
abline(v=xmax,col="red")
abline(v=xmax-dx,col="grey")
text(xmax-dx/2, y1, lab="w", pos=3)
text(xmax, y1, lab="+L/2", pos=4)
text(xmax, -y1, lab="-L/2", pos=4)
polygon(xmax+c(-dx,0,0,-dx,-dx), y1*c(-1,-1,1,1,-1), col="green")
title("Horizontal cross plume tow at location x\ndepth h, width w, length L")
```



Detection

In both of these scenarios if K is the expected number of copies captured, then the actual number of copies captured follows a Poisson distribution

$$Y|K \sim \text{Poisson}(K)$$

i.e.

$$\Pr(Y = y|K) = e^{-K}\frac{K^y}{y!}$$

Assume that the probability of detection ($Z = 1$) given the presence of y copies is $d(y)$, i.e.

$$\Pr(Z = 1|y) = d(y)$$

then the probability of detection, marginalising out the statistical variation, is

$$\begin{aligned}
\Pr(Z = 1) &= \sum_{y=0}^{\infty} \Pr(Z = 1|y)\Pr(Y = y) \\
&= \sum_{y=0}^{\infty} d(y)e^{-K}\frac{K^y}{y!}
\end{aligned}$$

If $d(y)$ is a simple threshold function $d(y) = I(y \geq y_t)$ (i.e. detection is certain at or above a threshold number y_t, and impossible below that threshold) then

$$\begin{aligned}
\Pr(Z = 1) &= \sum_{y=y_t}^{\infty} e^{-K}\frac{K^y}{y!} \\
&= 1 - \sum_{y=0}^{y_t-1} e^{-K}\frac{K^y}{y!}
\end{aligned}$$

```
knitr::knit_exit()
```