eDNA

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Particle dispersal

If there is a mass released at time t_0 at location $\mathbf{x}_0$ then it is transported by advection (water flow) and dispersion (turbulence).

Let $p(\mathbf{x}, t|\mathbf{x}_0, t_0, \mathbf{v}(), \mathbf{a}(), \Psi)$ be the proportion of the mass released that is found at location $\mathbf{x}$ at time t, given water flow $\mathbf{v}()$ and dispersion $\mathbf{a}()$. Ψ is a set of additional parameters controlling the flow/dispersion model.

A simple model is a simple Gaussian spread in 2D:

$$
\begin{aligned}
p(\mathbf{x}, t|\mathbf{x}_0, t_0, \mathbf{v}, \mathbf{a}, \Psi) &= \frac{I(t \geq t_0)}{2\pi\sqrt{a_x a_y}(t - t_0 + t_\varepsilon)} \exp\left(-\frac{(x - x_0 - v_x(t - t_0))^2}{2a_x(t - t_0 + t_\varepsilon)} - \frac{(y - y_0 - v_y(t - t_0))^2}{2a_y(t - t_0 + t_\varepsilon)}\right) \quad (1) \\
&= g(\mathbf{x}, t - t_0|\mathbf{x}_0, \mathbf{v}, \mathbf{a}, \Psi) \quad (2)
\end{aligned}
$$

where a small positive temporal offset $\Psi = \{t_\varepsilon\}$ is added to avoid a singularity at $t = t_0$.

We have

$$\iint p(\mathbf{x}, t|\mathbf{x}_0, t_0, \mathbf{v}, \mathbf{a}, \Psi)\, \mathrm{d}\mathbf{x} = 1$$

for all times $t > t_0$.

```
source("funcs.R")
```

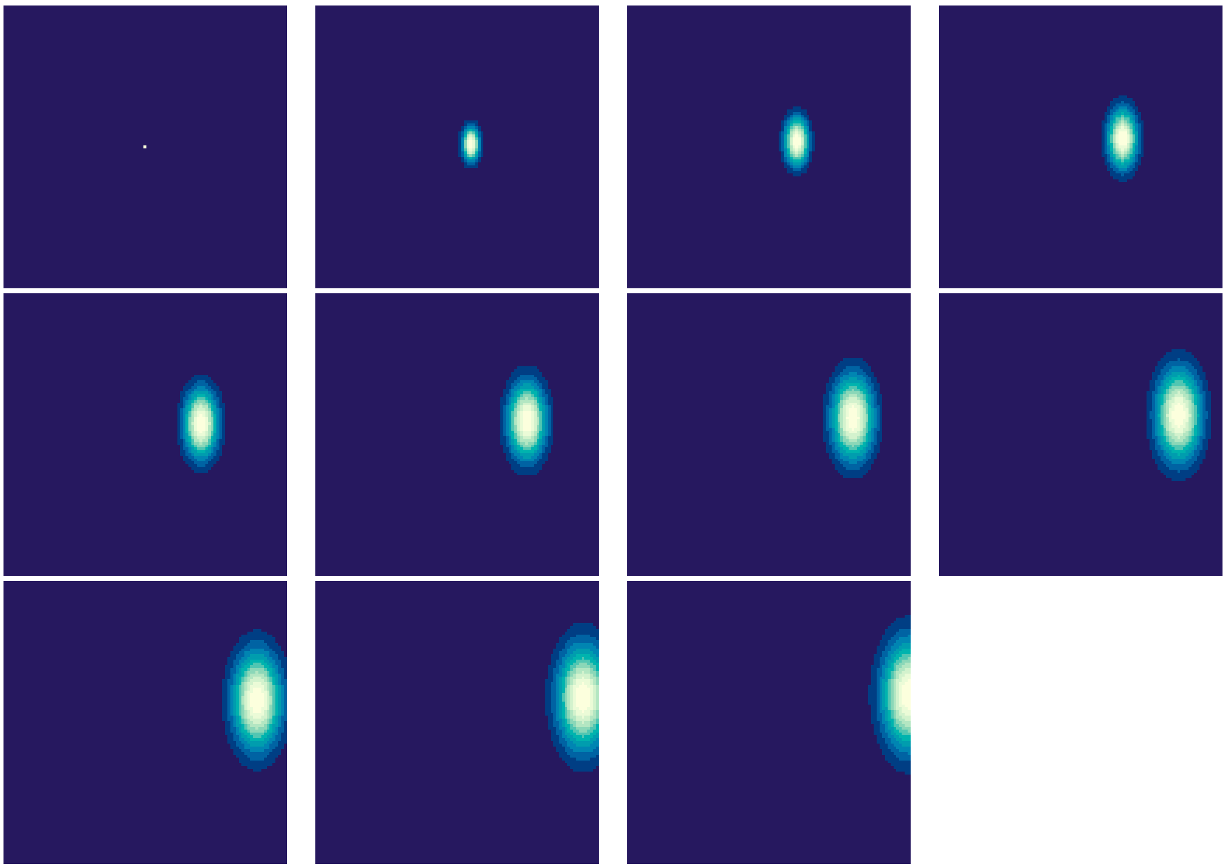
```
nx <- 101
ny <- 101
xmin <- 0; xmax <- 100
ymin <- 0; ymax <- 100
xvec <- seq(from=xmin, to=xmax, length=nx)
yvec <- seq(from=ymin, to=ymax, length=ny)

x0 <- mean(c(xmin,xmax))
y0 <- mean(c(ymin,ymax))
t0 <- 0
vx <- 5; vy <- 1
ax <- 2^2; ay <- 4^2
teps <- 0.01
```

For example here is the spread over time of mass released at $\mathbf{x}_0 = (50, 50)^T$ at time $t = 0$ with constant flow field $\mathbf{v} = (5, 1)^T$ and dispersion $\mathbf{a} = (4, 16)^T$ and with $t_\varepsilon = 0.01$.

```
par(mfrow=c(3,4))
par(mar=0.1*c(1,1,1,1))
colvec <- hcl.colors(12, "YlOrRd", rev = TRUE)
colvec <- hcl.colors(12, "YlGnBu")#, rev = TRUE)
for(t in 0:10) {
   amat <- outer(xvec,yvec, dispfunc, t=t,
```

```
                 x0=x0,y0=y0,t0=0, vx=vx,vy=vy, ax=ax,ay=ay, teps=teps)
  #if(t==0) amax <- max(amat)
  amax <- max(amat)
  breaks <- seq(from=0, to=amax, length=length(colvec)+1)
  image(xvec, yvec, amat, asp=1, col=colvec, breaks=breaks, axes=FALSE)
  #image(xvec, yvec, amat,asp=1, main=bquote("Forwards:" ~ t==.(t)), xlab="x", ylab="y",
  #       col=colvec, breaks=breaks)
}
```



A simple backtracking model simply reverses the flow field. If we are interested in the likely origin of mass observed at location $\mathbf{x}_1$ at time t_1 we can backtrack by reversing $\mathbf{v}()$.

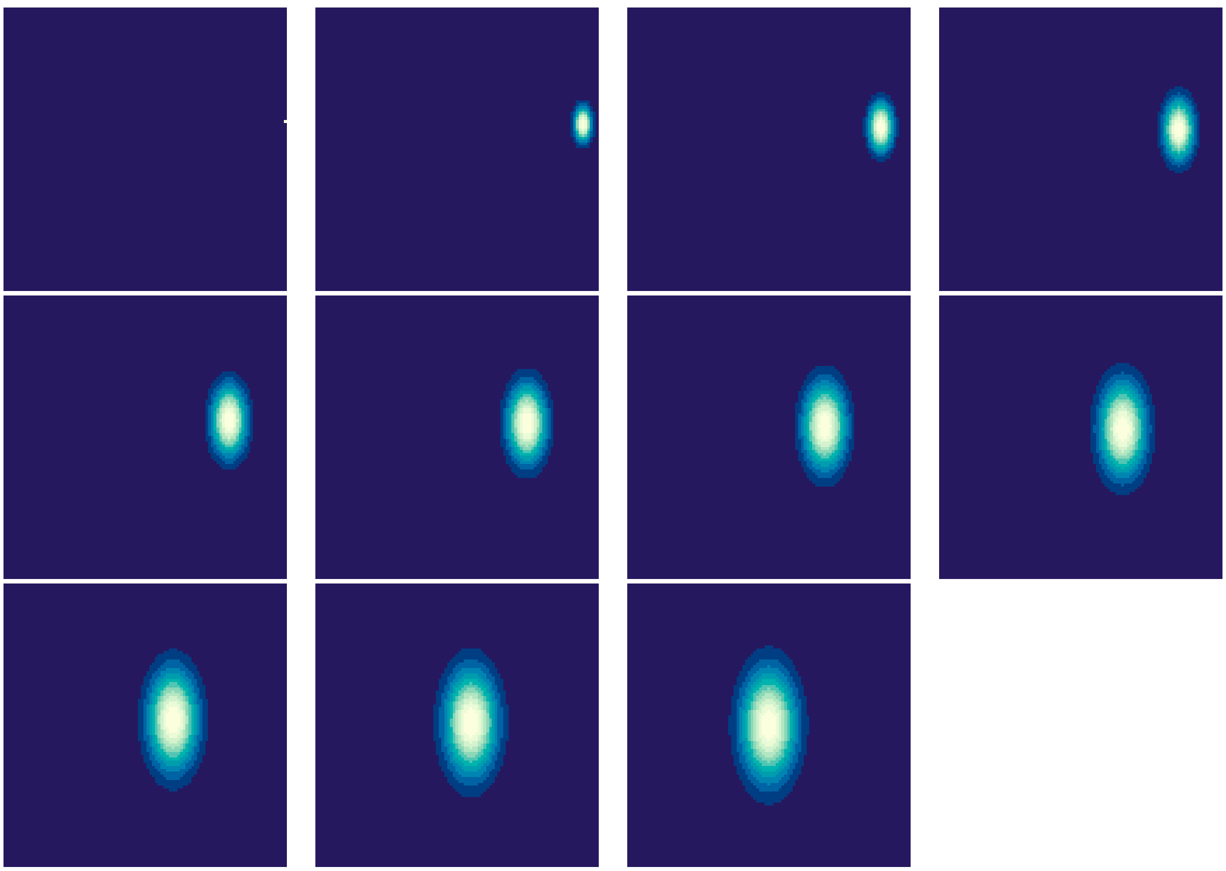
We might assume that the likelihood that the observed mass was released at location $\mathbf{x}$ at time $t < t_1$ is given by

$$\begin{aligned}
\tilde{p}(\mathbf{x}, t|\mathbf{x}_1, t_1, \mathbf{v}(), \mathbf{a}(), \Psi) &= g(\mathbf{x}, t_1 - t|\mathbf{x}_1, -\mathbf{v}(), \mathbf{a}(), \Psi) & (3)\\
&= p(\mathbf{x}, -t|\mathbf{x}_1, -t_1, -\mathbf{v}(), \mathbf{a}(), \Psi) & (4)
\end{aligned}$$

In the simple Gaussian model above this is

$$\begin{aligned}
&\tilde{p}(\mathbf{x}, t|\mathbf{x}_1, t_1, \mathbf{v}(), \mathbf{a}(), \Psi) & (5)\\
&\quad = p(\mathbf{x}, t_1|\mathbf{x}_1, t, -\mathbf{v}(), \mathbf{a}(), \Psi) & (6)\\
&\quad = p(\mathbf{x}, -t|\mathbf{x}_1, -t_1, -\mathbf{v}(), \mathbf{a}(), \Psi) & (7)\\
&\quad = g(\mathbf{x}, t_1 - t|\mathbf{x}_1, -\mathbf{v}(), \mathbf{a}(), \Psi) & (8)\\
&\quad = \frac{I(t \le t_1)}{2\pi\sqrt{a_x a_y}(t_1 - t + t_\varepsilon)} \exp\left(-\frac{(x - x_1 + v_x(t_1 - t))^2}{2a_x(t_1 - t + t_\varepsilon)} - \frac{(y - y_1 + v_y(t_1 - t))^2}{2a_y(t_1 - t + t_\varepsilon)}\right) & (9)
\end{aligned}$$

```r
par(mfrow=c(3,4))
par(mar=0.1*c(1,1,1,1))
x1 <- 100; y1 <- 60
t1 <- 10
for(t in 10:0) {
   amat <- outer(xvec,yvec, dispfunc, t=-t,
                 x0=x1,y0=y1,t0=-t1, vx=-vx,vy=-vy, ax=ax,ay=ay, teps=teps)
   #if(t==10) amax <- max(amat)
   amax <- max(amat)
   breaks <- seq(from=0, to=amax, length=length(colvec)+1)
   image(xvec, yvec, amat,asp=1, col=colvec, breaks=breaks, axes=FALSE)
   #image(xvec, yvec, amat,asp=1, main=bquote("Backwards: " ~ t==.(t)), xlab="x", ylab="y",
   #      col=colvec, breaks=breaks)
}
```



Continuous release with decay

Assume that at location $\mathbf{x}_0$ there is a continuous release of mass at a rate N_0 kg/s. Further assume that the mass decays at a rate

$$d(t|T_h, T_m) = 2^{-t/T_h} I(0 \le t \le T_m)$$

where T_h is the decay half life and T_m is the maximum age at which all particles have disintegrated. If we set $\lambda = (1/T_h)\log 2$ then

$$d(t|T_h, T_m) = e^{-\lambda t} I(0 \le t \le T_m)$$

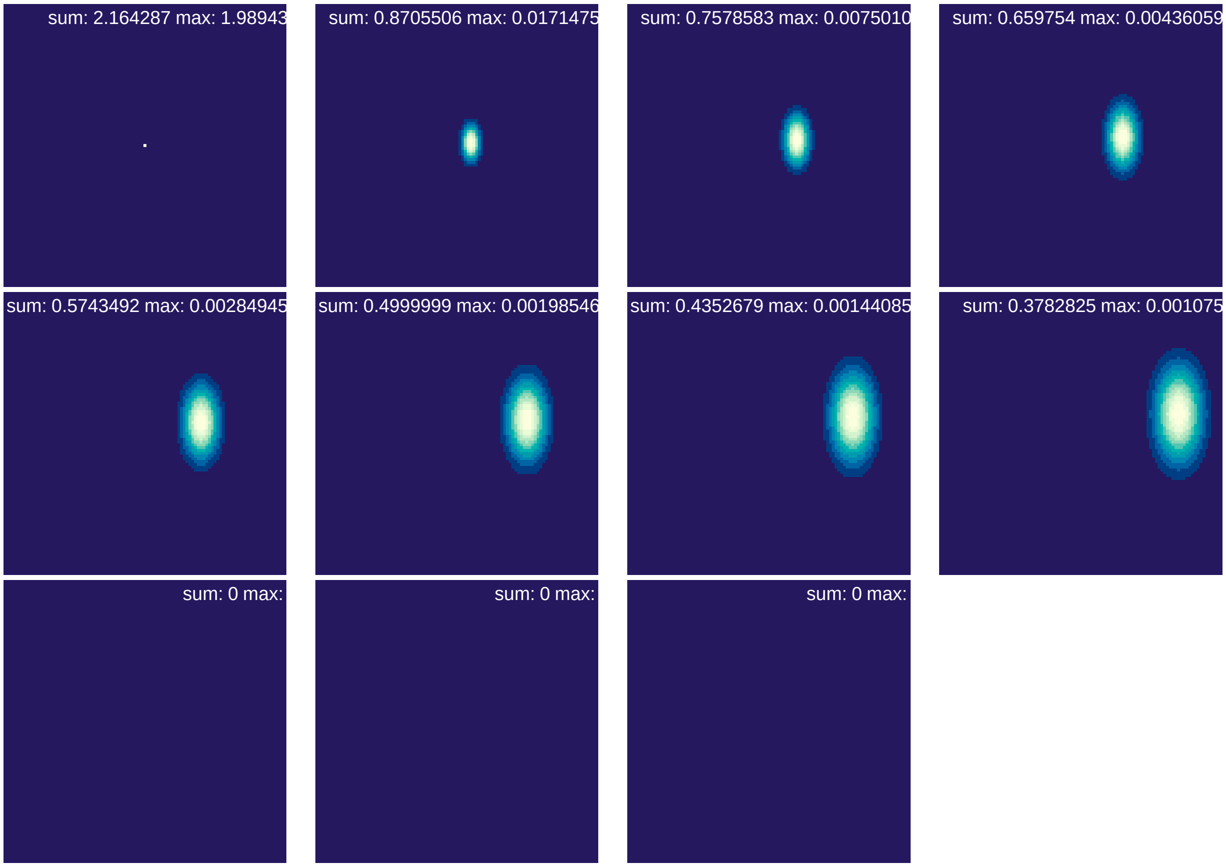
The steady state mass in the system is then

$$
\begin{aligned}
M_0 &= \int_{-\infty}^{t} N_0 d(t-t_0|T_h,T_m)\,\mathrm{d}t_0 && (10)\\
&= N_0 \int_{-\infty}^{t} e^{-\lambda(t-t_0)} I(t_0 > t-T_m)\,\mathrm{d}t_0 && (11)\\
&= N_0 \int_{t-T_m}^{t} e^{-\lambda(t-t_0)}\,\mathrm{d}t_0 && (12)\\
&= N_0 \int_{0}^{T_m} e^{-\lambda u}\,\mathrm{d}u && (13)\\
&= \frac{N_0}{\lambda}\left[1-e^{-\lambda T_m}\right] && (14)\\
&= \frac{N_0 T_h}{\log 2}\left[1-e^{-\lambda T_m}\right] && (15)
\end{aligned}
$$

The mass density transiting per unit time at location $\mathbf{x}$ at time t from a point source at $\mathbf{x}_0$ is at time t_0 is

$$\frac{\mathrm{d}\rho(\mathbf{x},t|N_0,\mathbf{x}_0,t_0,\mathbf{v}(),\mathbf{a}(),\Psi)}{\mathrm{d}t} = N_0 d(t-t_0|T_h,T_m)p(\mathbf{x},t|\mathbf{x}_0,t_0,\mathbf{v}(),\mathbf{a}(),\Psi)$$

```
par(mfrow=c(3,4))
par(mar=0.1*c(1,1,1,1))
n0 <- 1
th <- 5
tm <- 7
for(t in 0:10) {
   amat <- outer(xvec,yvec, denratefunc, t=t,
                 x0=x0,y0=y0,t0=0, vx=vx,vy=vy, ax=ax,ay=ay, teps=teps, n0=n0,th=th,tm=tm)
   #amat <- log(1+amat)
   #if(t==0) {
   #   amin <- min(amat)
   #   amax <- max(amat)
   #}
   amin <- min(amat)
   amax <- max(amat)
   breaks <- seq(from=amin, to=amax, length=length(colvec)+1)
   image(xvec, yvec, amat, asp=1, col=colvec, breaks=breaks, axes=FALSE)
   #image(xvec, yvec, amat,asp=1, main=bquote("Forwards with decay:" ~ t==.(t)), xlab="x", ylab="y",
   #      col=colvec, breaks=breaks)
   mtext(bquote("sum:"~.(sum(amat))~"max:"~.(max(amat))), side=3, adj=1, cex=0.6, line=-1, col="white")
}
```



The total (equilibrium) mass density present is

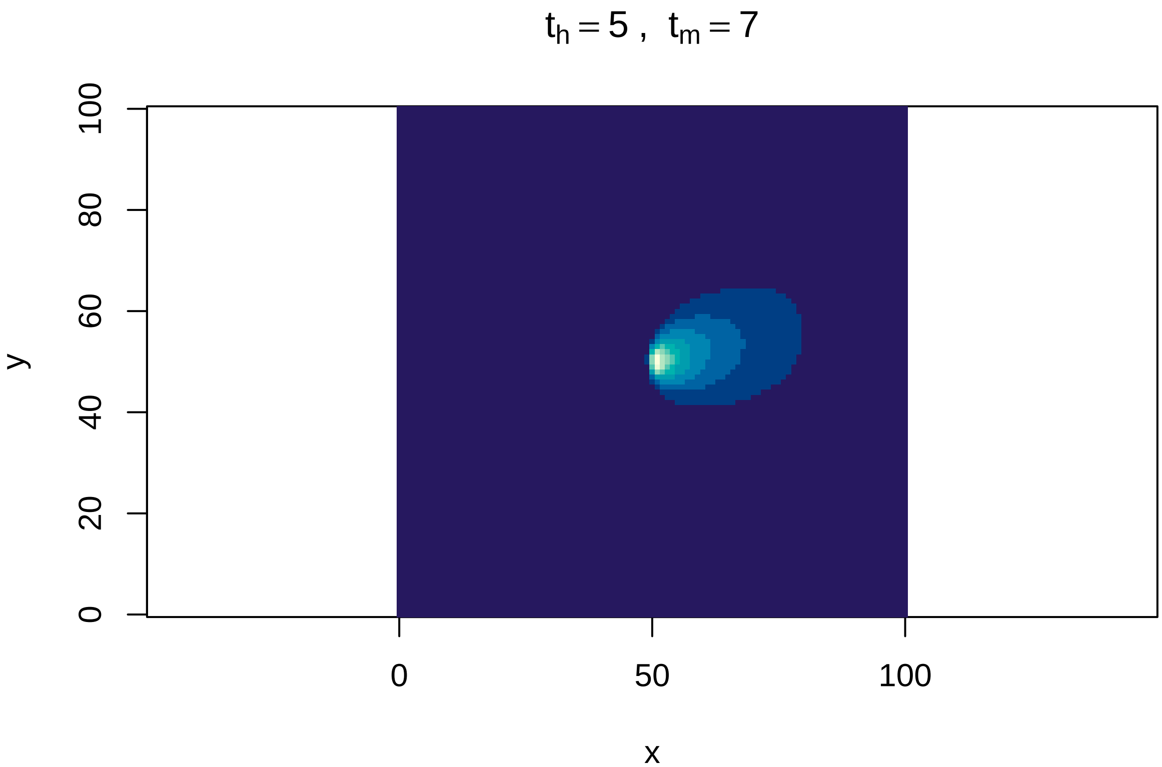
$$
\begin{aligned}
&\rho(\mathbf{x}, t|N_0, \mathbf{x}_0, \mathbf{v}(), \mathbf{a}(), \Psi) && (16)\\
&= \rho(\mathbf{x}|N_0, \mathbf{x}_0, \mathbf{v}(), \mathbf{a}(), \Psi) && (17)\\
&= \int_{-\infty}^{t} N_0 d(t - t_0|T_h, T_m) p(\mathbf{x}, t|\mathbf{x}_0, t_0, \mathbf{v}(), \mathbf{a}(), \Psi)\, \mathrm{d}t_0 && (18)\\
&= N_0 \int_{t-T_m}^{t} e^{-\lambda(t-t_0)} g(\mathbf{x}, t - t_0|\mathbf{x}_0, \mathbf{v}(), \mathbf{a}(), \Psi)\, \mathrm{d}t_0 && (19)\\
&= N_0 \int_{0}^{T_m} e^{-\lambda u} g(\mathbf{x}, u|\mathbf{x}_0, \mathbf{v}(), \mathbf{a}(), \Psi)\, \mathrm{d}u && (20)\\
&= \frac{N_0}{2\pi\sqrt{a_x a_y}} \int_0^{T_m} \frac{e^{-\lambda u}}{u + t_\varepsilon} \exp\left(-\frac{(x - x_0 - v_x u)^2}{2a_x(u + t_\varepsilon)} - \frac{(y - y_0 - v_y u)^2}{2a_y(u + t_\varepsilon)}\right) && (21)\\
&= N_0 h(\mathbf{x}|\mathbf{x}_0, \mathbf{v}(), \mathbf{a}(), \Psi) && (22)
\end{aligned}
$$

where

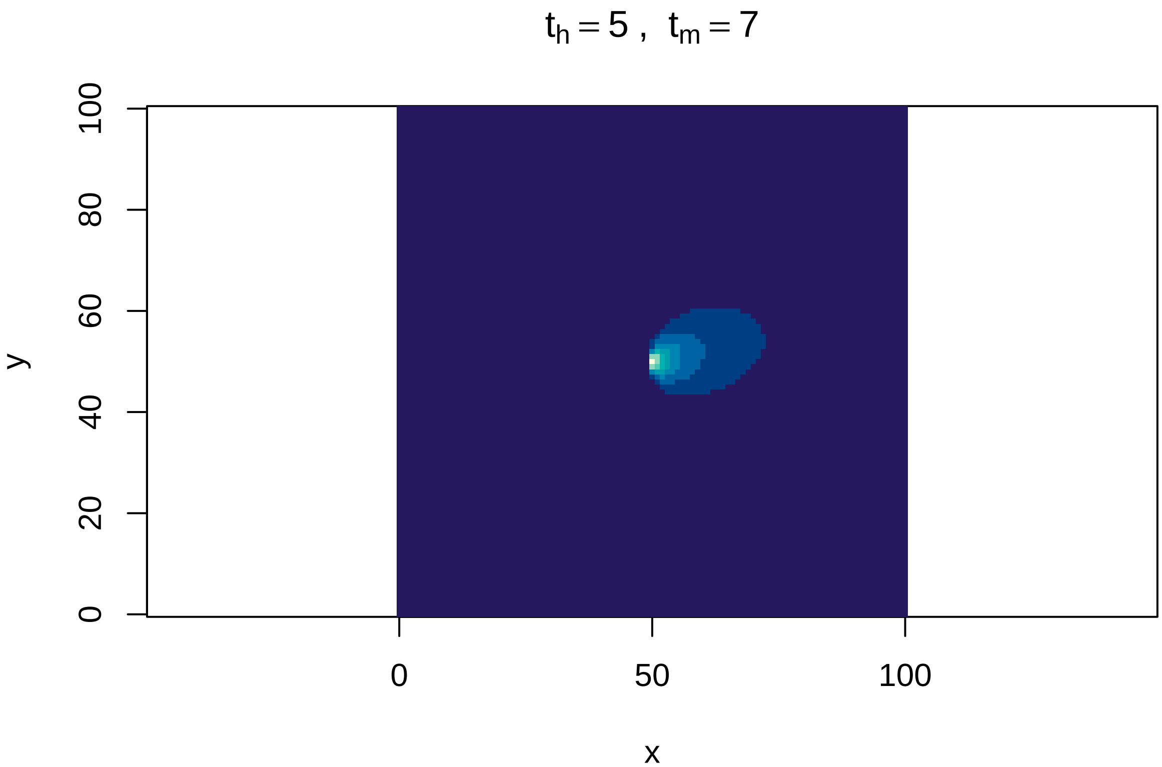
$$
h(\mathbf{x}|\mathbf{x}_0, \mathbf{v}(), \mathbf{a}(), \Psi) = \frac{1}{2\pi\sqrt{a_x a_y}} \int_0^{T_m} \frac{e^{-\lambda u}}{u + t_\varepsilon} \exp\left(-\frac{(x - x_0 - v_x u)^2}{2a_x(u + t_\varepsilon)} - \frac{(y - y_0 - v_y u)^2}{2a_y(u+, t_\varepsilon)}\right)
$$

```
th <- 5
tm <- 7
ax <- 2^2; ay <- 5^2
amat <- array(0,dim=c(nx,ny))
nc <- 100
```

```r
for(i in 1:nc) {
   t <- i/nc * 10
   amat <- amat + outer(xvec,yvec, denratefunc, t=t,
                        x0=x0,y0=y0,t0=0, vx=vx,vy=vy, ax=ax,ay=ay, teps=teps, n0=n0,th=th,tm=tm)
}
image(xvec, yvec, amat,asp=1,
      main=bquote(t[h]==.(th) ~ ", " ~ t[m]==.(tm)),
      xlab="x", ylab="y",col=colvec)
```




```r
th <- 5
tm <- 7
ax <- 2^2; ay <- 5^2
eps <- 0.001
amat <- n0*outer(xvec,yvec, hfunc,
                 x0=x0-eps,y0=y0-eps, vx=vx,vy=vy, ax=ax,ay=ay, teps=teps, th=th,tm=tm)
image(xvec, yvec, amat, asp=1,
      main=bquote(t[h]==.(th) ~ ", " ~ t[m]==.(tm)), xlab="x", ylab="y",col=colvec)
```




```
hfunc(x=60,y=60, x0=50,y0=50, vx=vx,vy=vy, ax=ax,ay=ay, teps=teps, th=5,tm=7)
```

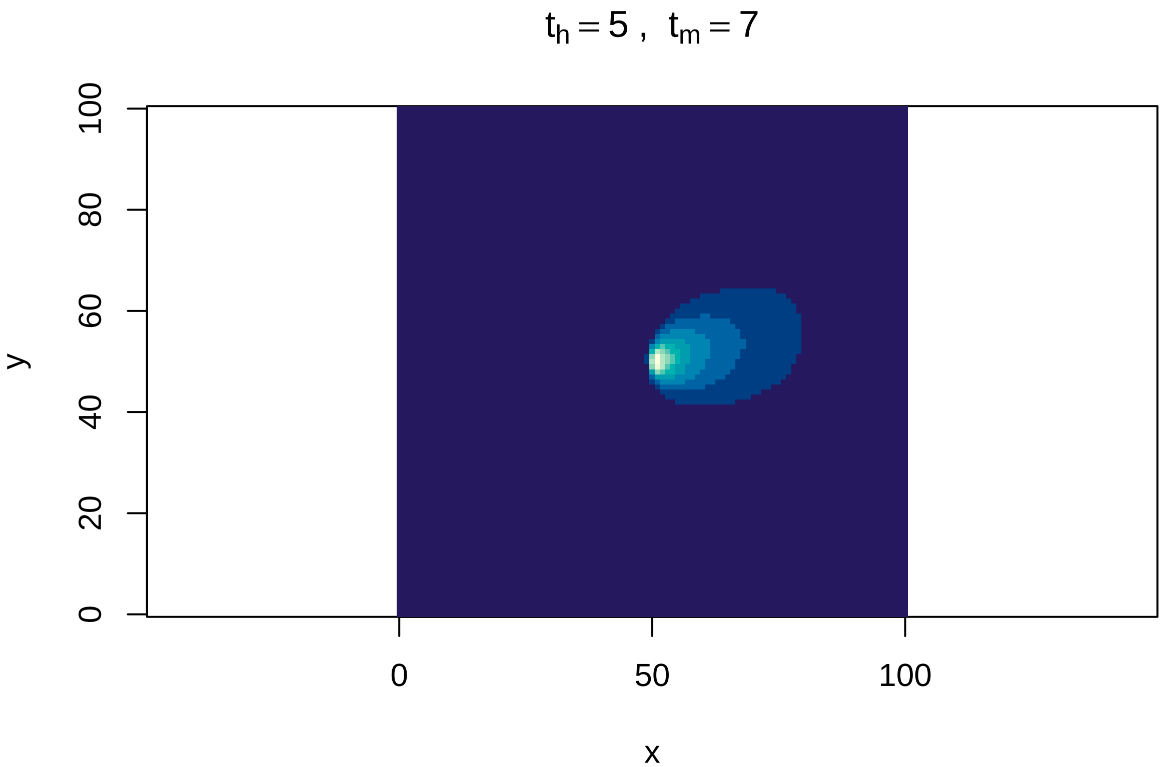
```
## [1] 0.004329505
```

```
hfunc(x=60,y=40, x0=50,y0=50, vx=vx,vy=vy, ax=ax,ay=ay, teps=teps, th=5,tm=7)
```

```
## [1] 0.001953023
```

```
th <- 5
tm <- 7
ax <- 2^2; ay <- 5^2
hmat <- array(0,dim=c(nx,ny))


nc <- 100
for(i in 1:nc) {
   t <- i/nc * 10
   amat <- amat + outer(xvec,yvec, denratefunc, t=t,
                        x0=x0,y0=y0,t0=0, vx=vx,vy=vy, ax=ax,ay=ay, teps=teps, n0=n0,th=th,tm=tm)
}
image(xvec, yvec, amat,asp=1,
      main=bquote(t[h]==.(th) ~ ", " ~ t[m]==.(tm)),
      xlab="x", ylab="y",col=colvec)
```



$t_h = 5$, $t_m = 7$
100
80
60
40
20
0
y
0
50
100
x