Negative binomial likelihood

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Assume the larva are distributed according to a negative binomial distribution with pdf

$$P(Z = z; m, k) = \binom{z+k-1}{z} \left(\frac{k}{k+m}\right)^{k+z} \left(\frac{m}{k}\right)^{z}$$

For $q = 5000$ individuals, we found a positive result, and for $n - q = 10^6 - 5000$ individuals a negative result. The probabilty of finding zero larva is

$$p_0 = P(Z = 0; m, k) = \left(\frac{k}{k+m}\right)^{k},$$

and the probability of finding one or more larva is

$$p_1 = 1 - p_0$$

The contribution to the likelihood is

$$p_0^{n-q} p_1^{q}$$

For $r = 35$ individuals, additionally the larvae were counted, after we knew they were positive. Hence, for the i-th individual, the contribution to the likelihood is

$$P(Z = z_i | Z > 0; m, k) = \frac{P(Z = z_i \cap Z > 0; m, k)}{P(Z > 0; m, k)} = \begin{cases} 0 & \text{for } z_i = 0, \\ \frac{P(Z=z_i;m,k)}{p_1} & \text{for } z_i > 0. \end{cases}$$

Since for those individuals zero is not possible, we can safely use the second case. The total likelihood becomes

$$l(m, k) = p_0^{n-q} p_1^{q} \prod_{i=1}^{r} \frac{P(Z = z_i; m, k)}{p_1}$$

But this is also equal to

$$l(m, k) = p_0^{n-q} p_1^{q-r} \prod_{i=1}^{r} P(Z = z_i; m, k)$$

This has the interpretation "$n-q$ were negative, $q-r$ were positive, and r were counted", but that is not what happened in our experiment. What goes wrong?

Or is it the case that the likelihoods are different, but I should include the binomial coefficients. So for experiment 1

$$l(m,k) = \binom{n}{q} p_0^{n-q} p_1^{q-r} \binom{q}{r} \prod_{i=1}^{r} P(Z = z_i; m, k)$$

but for experiment 2,

$$l(m,k) = \binom{n-r}{q-r} p_0^{n-q} p_1^{q-r} \binom{n}{r} \prod_{i=1}^{r} P(Z = z_i; m, k).$$

Since

$$\binom{n}{q}\binom{q}{r} = \frac{n!q!}{q!r!(n-q)!(q-r)!} = \frac{n!}{r!(n-q)!(q-r)!}$$

and

$$\binom{n-r}{q-r}\binom{n}{r} = \frac{(n-r)!n!}{(q-r)!(n-r-q+r)!r!(n-r)!} = \frac{n!}{(q-r)!(n-q)!r!}$$

No, again they are equal. I'm puzzled.