THE CATEGORIES OF RETRACTS

Locally cartesian closed

Let C be any category, and take $f : C(B, A)$.

$$\begin{tikzcd} C \downarrow A \arrow[r, "f^*"] & C \downarrow B \arrow[l, bend right, "\sum_f"'] \arrow[l, bend left, "\prod_f"] \end{tikzcd}$$

Relatively cartesian closed

Some categories do not have f^* or $\prod_f$ for all f. But sometimes we can choose full subcategories of 'display maps' (or 'fibrations') $C \downarrow_H B \subseteq C \downarrow B$ such that

- Composition of display maps is a display map;
- Terminal projections are display maps (so $C \downarrow_H T = C \downarrow T \simeq C$);
- Pullbacks of display maps along any map exist.

$$\begin{tikzcd} C \downarrow_H A \arrow[r, dashed, "f^*"] & C \downarrow_H B \arrow[l, bend right, "\sum_f"'] \arrow[l, bend left, "\prod_f"] \end{tikzcd}$$

Cartesian closed relative to H if f^* has a right adjoint.

Let A be a category. Let $\text{よ} : A \hookrightarrow \operatorname{Psh} A$ be the Yoneda embedding. Let B, C, D be the categories given by:

Scott's version

$$B_0 = \sum_{X:A} \sum_{f:A(X,X)} f \cdot f = f$$

$$B((X_1, f_1), (X_2, f_2)) = \{g : A(X_1, X_2) \mid f_1 \cdot g \cdot f_2 = g\} :$$

$$\begin{tikzcd} X_1 \arrow[loop left, "f_1"] \arrow[r, "g"] & X_2 \arrow[loop right, "f_2"] \end{tikzcd}$$

First interpretation of the equivalent notion

$$C_0 = \sum_{P:\operatorname{Psh} A} \sum_{X:A} \sum_{s:A(P,\text{よ}(X))} \sum_{r:A(\text{よ}(X),P)} s \cdot r = \mathrm{id}$$

$$C((P_1, X_1, r_1, s_1), (P_2, X_2, r_2, s_2)) = \operatorname{Psh} A(P_1, P_2) :$$

$$\begin{tikzcd} \text{よ}(X_1) \arrow[r, bend left, "r_1"] & P_1 \arrow[l, bend left, "s_1"] \arrow[r, "g"] & P_2 \arrow[r, bend left, "s_2"] & \text{よ}(X_2) \arrow[l, bend left, "r_2"] \end{tikzcd}$$

Second interpretation of the equivalent notion

$$D_0 = \sum_{P:\operatorname{Psh} A} \left\Vert \sum_{X:A} \sum_{s:A(P,\text{よ}(X))} \sum_{r:A(\text{よ}(X),P)} s \cdot r = \mathrm{id} \right\Vert$$

$$D((P_1, X_1, r_1, s_1), (P_2, X_2, r_2, s_2)) = \operatorname{Psh} A(P_1, P_2) :$$

$$\begin{tikzcd} \text{よ}(X_1) \arrow[r, bend left, dashed, "r_1"] & P_1 \arrow[l, bend left, dashed, "s_1"] \arrow[r, "g"] & P_2 \arrow[r, bend left, dashed, "s_2"] & \text{よ}(X_2) \arrow[l, bend left, dashed, "r_2"] \end{tikzcd}$$

B

$$\begin{tikzcd} X_1 \arrow[loop left, "f_1"] \arrow[r, "g"] & X_2 \arrow[loop right, "f_2"] \end{tikzcd}$$

C

$$\begin{tikzcd} \text{よ}(X_1) \arrow[r, bend left, "r_1"] & P_1 \arrow[l, bend left, "s_1"] \arrow[r, "g"] & P_2 \arrow[r, bend left, "s_2"] & \text{よ}(X_2) \arrow[l, bend left, "r_2"] \end{tikzcd}$$

D

$$\begin{tikzcd} \text{よ}(X_1) \arrow[r, dashed, bend left, "r_1"] & P_1 \arrow[l, dashed, bend left, "s_1"] \arrow[r, "g"] & P_2 \arrow[r, dashed, bend left, "s_2"] & \text{よ}(X_2) \arrow[l, dashed, bend left, "r_2"] \end{tikzcd}$$

PROPERTIES

	B	C	D
Embeds fully faithfully into Psh A	✓	✓	✓
Subcategory of Psh A	✗	✗	✓
Univalent	✗	✗	✓
Scott's construction	✓	✗	✗
Taylor's construction	✓	✗	✗

$$\begin{tikzcd} B \arrow[r, "\sim"] & C \arrow[r, hook, two heads] & D & \subseteq & \operatorname{Psh} A \end{tikzcd}$$

Taylor

Taylor works in Scott's category of retracts B of a category A that is based on a monoid:

$$\begin{tikzcd} * \arrow[loop left, "f_1"] \arrow[r, "g"] & * \arrow[loop right, "f_2"] \end{tikzcd}$$

For $X : B$, Taylor constructs a category of 'indexed types' B^X (which behaves like functions $X \to B$) with a fully faithful embedding

$$\sum_X : B^X \hookrightarrow B \downarrow X.$$

He chooses D such that i is essentially surjective onto $B \downarrow_H X$ and shows that we have an adjunction

$$\begin{tikzcd} B^X \arrow[r, "f^*"] \arrow[d, hook, two heads, "\sum_X"] & B^Y \arrow[l, bend left, "\prod_f"] \arrow[d, hook, two heads, "\sum_Y"] \\ B \downarrow_H X & B \downarrow_H Y \end{tikzcd}$$

In this case, B is a setcategory, so the functors $\sum$ become equivalences under the axiom of choice and we can lift $f^* \dashv \prod_f$ to $B \downarrow_H X$ and $B \downarrow_H Y$. In fact we have mere existence of pullbacks and to turn this into the mere existence of a pullback functor already requires the axiom of choice.

Note that if we try to do a similar thing in C, we would expect to have easy candidates for $\prod_f$ and $\sum_f$, but again we only have mere existence of pullbacks and now that is an even bigger problem because C is not a setcategory.

$$\begin{tikzcd} よ(*) \arrow[r, bend left, "r_1"] & P_1 \arrow[l, bend left, "s_1"] \arrow[r, "g"] & P_2 \arrow[r, bend left, "s_2"] & よ(*) \arrow[l, bend left, "r_2"] \end{tikzcd}$$

Hyland

For $X : \mathrm{Psh}\, A$, Hyland considers 'the category of retracts' $\mathbb{R}(X)$ of

$$\pi_1 : X \times \text{よ}(*) \to X$$

in $\mathrm{Psh}\, A \downarrow X$. He proceeds to construct retractions of $(Y \times \text{よ}(*), \pi_1)$ onto $\prod_f (X, g)$ and $\sum_f (X, g)$ for $f : Y \to X$ in $\mathbb{R}(X)$ and for a retraction of $(X \times \text{よ}(*), \pi_1)$ onto (X, g). That is,

$$\prod_f, \sum_f : \mathrm{Psh}\, A \downarrow X \to \mathrm{Psh}\, A \downarrow Y$$

send objects in $\mathbb{R}(X)$ to $\mathbb{R}(Y)$.

Furthermore, he claims that

- Scott's category of retracts B $(= \mathbb{R}(\text{よ}(*)))$ is a subcategory of $\mathrm{Psh}\, A$.
- For $X \in B$ and $Y \in \mathbb{R}(X)$, $Y \in B$.
- From this, Taylor's theorem follows.

This suggests that Hyland attempts to lift $\prod_f$ and $\sum_f$ along the weak equivalences

$$\begin{tikzcd}
(\mathbb{R} \downarrow_H X) \arrow[rr, dashed, bend left, "\sum_f"] \arrow[rr, dashed, bend right, "\prod_f"'] \arrow[d, hook, two heads, "\psi_X"] & & (\mathbb{R} \downarrow_H Y) \arrow[ll, dashed, "f^*" description] \arrow[d, hook, two heads, "\psi_Y"] \\
\mathbb{R}(\varphi(X)) \arrow[rr, bend left, "\sum_{\varphi(f)}"] \arrow[rr, bend right, "\prod_{\varphi(f)}"'] & & \mathbb{R}(\varphi(Y)) \arrow[ll, "\varphi(f)^*" description]
\end{tikzcd}$$

But since we are working with presheaf categories here, a weak equivalence does not give an adjoint equivalence.

The solution

If we abandon the need to replicate Taylor's proof, we can actually do a lot better: We will work in D, the Rezk completion of B and C:

$$\begin{tikzcd}
\text{よ}(X_1) \arrow[r, dashed, bend left, "r_1"] & P_1 \arrow[l, dashed, bend left, "s_1"] \arrow[r, "g"] & P_2 \arrow[r, dashed, bend left, "s_2"] & \text{よ}(X_2) \arrow[l, dashed, bend left, "r_2"]
\end{tikzcd}$$

We take $(Y, f) : D \downarrow_H X$ if there merely exists a retraction from $\pi_1 : X \times \text{よ}(*) \to X$ onto (Y, f). We can show that this forms a class of display maps, and using Hyland's proof, we can show that $\prod_f$ restricts to the relative slices, so D is cartesian closed relative to H.