Semantics for the λ-calculus

Contents

CHAPTER 1

Category Theoretic Preliminaries

I will assume a familiarity with the category-theoretical concepts presented in [**AW23**]. These include categories, functors, isomorphisms, natural transformations, adjunctions, equivalences and limits.

1. Notation

For an object c in a category C, I will write $c : C$.

For a morphism f between objects c and c' in a category C, I will write $f : C(c, c')$ or $f : c \to c'$.

For composition of morphisms $f : C(c, d)$ and $g : C(d, e)$, I will write $f \cdot g$.

For composition of functors $F : A \to B$ and $G : B \to C$, I will write $F \bullet G$.

2. Universal Arrows

([**ML98**], Chapter IV.1, Theorem 2 (iv)) **(TODO)**

3. Adjunctions

Recall that an *adjunction* $L \dashv R$ is a pair of functors

$$\begin{tikzcd} D \arrow[r, bend right, "R"'] & C \arrow[l, bend right, "L"'] \end{tikzcd}$$

with natural transformations (the unit and counit)

$$\eta : \mathrm{id}_C \Rightarrow L \bullet R \quad \text{and} \quad \epsilon : R \bullet L \Rightarrow \mathrm{id}_D$$

such that the diagrams

$$\begin{tikzcd} L \arrow[rr, "\mathrm{id}_L"] \arrow[dr, "\eta \bullet L"'] & & L \\ & L \bullet R \bullet L \arrow[ur, "L \bullet \epsilon"'] & \end{tikzcd} \qquad \begin{tikzcd} R \arrow[rr, "\mathrm{id}_R"] \arrow[dr, "R \bullet \eta"'] & & R \\ & R \bullet L \bullet R \arrow[ur, "\epsilon \bullet R"'] & \end{tikzcd}$$

commute. Here the natural transformation $\eta \bullet L : L \bullet R \bullet L$ is the natural transformation η whiskered on the right by L, and the other whiskered transformations are similar.

An alternative characterization of an adjunction $L \dashv R$ is as a natural bijection

$$\varphi : D(L(c), d) \xrightarrow{\sim} C(c, R(d)).$$

Naturality means that for all $f : C(c', c)$, $g : D(d, d')$ and $h : D(L(c), d)$,

$$\varphi(L(f) \cdot h \cdot g) = f \cdot \varphi(h) \cdot R(g).$$

3.1. Adjoint equivalences. An adjoint equivalence of categories has multiple definitions. The one we will use here is the following:

Definition 1. An *adjoint equivalence* between categories C and D is a pair of adjoint functors $L \dashv R$ like above such that the unit $\eta : \mathrm{id}_C \Rightarrow L \bullet R$ and counit $\epsilon : R \bullet L \Rightarrow \mathrm{id}_D$ are isomorphisms of functors.

3.2. Exponential objects. Note that in the category of sets, for all X, Y : **SET**, we have a set $(X \to Y)$. Also, for all X, Y, Z, there is a (natural) bijection

$$(X \times Y \to Z) \cong (X \to (Y \to Z))$$

which we can also write as

$$\mathbf{SET}(X \times Y, Z) \cong \mathbf{SET}(X, (Y \to Z)).$$

In other words, we have functors $X \mapsto X \times Y$ and $Z \mapsto (Y \to Z)$, and these two form an adjunction. The following generalizes this

Definition 2. A category C has *exponential objects* (or *exponentials*) if for all $c : C$, the functor $c' \mapsto c' \times c$ has a right adjoint, which we denote $d \mapsto d^c$.

Lemma 1. *For $c, d, d^c : C$, we have a natural bijection*

$$C(d' \times c, d) \cong C(d', d^c)$$

iff we have a universal arrow ...

Proof. **(TODO)** □

Remark 1. It is actually very well possible that a category does not have all exponentials, but it has some objects $c, d, d^c : C$ with a natural bijection

$$C(d' \times c, d) \cong C(d', d^c).$$

Then d^c is still called an exponential object.

4. (Co)slice categories

Given an object in a category $c : C$, the morphisms to and from c constitute the slice and coslice categories

Definition 3. The *slice category* $C \downarrow c$ is the category with as objects the morphisms to c:

$$(C \downarrow c)_0 = \sum_{c' : C} C(c', c).$$

The morphisms from (c', f) to (c'', f') are the morphisms $g : c' \to c''$ making the following diagram commute.

$$\begin{tikzcd} c' \arrow[rr, "g"] \arrow[dr, "f"'] & & c'' \arrow[dl, "f'"] \\ & c & \end{tikzcd}$$

The *coslice category* $c \downarrow C$ is similar, but with the morphisms from c:

$$(c \downarrow C)_0 = \sum_{c' : C} C(c, c').$$

5. (Weakly) terminal objects

Definition 4. If a category has an object t, such that there is a (not necessarily unique) morphism to it from every other object in the category, t is said to be a *weakly terminal object*.

Definition 5. Given an object c in a category C with terminal object t. A *global element* of c is a morphism $C(t, c)$.

6. Kan Extensions

One of the most general and abstract concepts in category theory is the concept of *Kan extensions*. In [**ML98**], Section X.7, MacLane notes that

> The notion of Kan extensions subsumes all the other fundamental concepts of category theory.

In this thesis, we will use left Kan extension a handful of times. It comes in handy when we want to extend a functor along another functor in the following way:

Let A, B and C be categories and let $F : A \to B$ be a functor.

Definition 6. Precomposition gives a functor between functor categories $F_* : [B, C] \to [A, C]$. If F_* has a left adjoint, we will denote call this adjoint functor the *left Kan extension* along F and denote it $\mathrm{Lan}_F : [A, C] \to [B, C]$.

$$\begin{tikzcd} A \arrow[rr, "F"] \arrow[dr, dashed, "F_* G"'] & & B \arrow[dl, "G"] \\ & C & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rr, "F"] \arrow[dr, "G"'] & & B \arrow[dl, dashed, "\mathrm{Lan}_F G"] \\ & C & \end{tikzcd}$$

Analogously, when F_* has a right adjoint, one calls this the *right Kan extension* along F and denote it $\mathrm{Ran}_F : [A, C] \to [B, C]$.

If a category has limits (resp. colimits), we can construct the right (resp. left) Kan extension in a 'pointwise' fashion (see Theorem X.3.1 in [**ML98**] or Theorem 2.3.3 in [**KS06**]). Below, I will outline the parts of the construction that we will need explicitly in this thesis.

Lemma 2. *If C has colimits, Lan_F exists.*

Proof. First of all, for objects $b : B$, we take

$$\mathrm{Lan}_F G(b) := \mathrm{colim}\left((F \downarrow b) \to A \xrightarrow{G} C\right).$$

Here, $(F \downarrow b)$ denotes the comma category with as objects the morphisms $B(F(a), b)$ for all $a : A$, and as morphisms from $f : B(F(a), b)$ to $f' : B(F(a'), b)$ the morphisms $g : A(a, a')$ that make the diagram commute:

$$\begin{tikzcd} F(a) \arrow[rr, "F(g)"] \arrow[dr, "f"'] & & F(a') \arrow[dl, "f'"] \\ & b & \end{tikzcd}$$

and $(F \downarrow b) \to A$ denotes the projection functor that sends $f : B(F(a_1), b)$ to a_1.

Now, a morphism $h : B(b, b')$ gives a morphism of diagrams, sending the $F(a)$ corresponding to $f : B(G(a), b)$ to the $F(a)$ corresponding to $f \cdot h : B(G(a), b')$. From this, we get a morphism $\mathrm{Lan}_F G(h) : C(\mathrm{Lan}_F G(b), \mathrm{Lan}_F G(b'))$.

The unit of the adjunction is a natural transformation $\eta : \mathrm{id}_{[A,C]} \Rightarrow \mathrm{Lan}_F \bullet F_*$. We will define this pointwise, for $G : [A, C]$ and $a : A$. Our diagram contains the $G(a)$ corresponding to $\mathrm{id}_{F(a)} : (F \downarrow F(a))$ and the colimit cocone gives a morphism

$$\eta_G(a) : C(G(a), \mathrm{Lan}_F G(F(a))),$$

the latter being equal to $(\mathrm{Lan}_F \bullet F_*)(G)(a)$.

The counit of the adjunction is a natural transformation $\epsilon : F_* \bullet \mathrm{Lan}_F \Rightarrow \mathrm{id}_{[B,C]}$. We will also define this pointwise, for $G : [B, C]$ and $b : B$. The diagram for $\mathrm{Lan}_F(F_* G)(b)$ consists of $G(F(a))$ for all $f : B(F(a), b)$. Then, by the universal property of the colimit, the morphisms $G(f) : C(G(F(a)), G(b))$ induce a morphism

$$\epsilon_G(b) : C(\mathrm{Lan}_F(F_* G)(b), G(b)).$$

$\square$

LEMMA 3. *If $F : A \to B$ is a fully faithful functor, and C is a category with colimits, $\eta : id_{[A,C]} \Rightarrow Lan_F \bullet F_*$ is a natural isomorphism.*

PROOF. To show that η is a natural isomorphism, we have to show that $\eta_G(a') : G(a') \Rightarrow \mathrm{Lan}_F G(F(a'))$ is an isomorphism for all $G : [A, C]$ and $a' : A$. Since a left adjoint is unique up to natural isomorphism ([**AW23**], Exercise 153), we can assume that $\mathrm{Lan}_F G(F(a'))$ is given by

$$\mathrm{colim}((F \downarrow F(a')) \to A \xrightarrow{G} C).$$

Now, the diagram for this colimit consists of $G(a)$ for each arrow $f : B(F(a), F(a'))$. Since F is fully faithful, we have $f = F(\overline{f})$ for some $\overline{f} : A(a, a')$. If we now take the arrows $G(\overline{f}) : C(G(a), G(a'))$, the universal property of the colimit gives an arrow

$$\varphi : C(\mathrm{Lan}_F G(F(a')), G(a'))$$

which constitutes an inverse to $\eta_G(a')$. The proof of this revolves around properties of the colimit and its (induced) morphisms. □

REMARK 2. In the same way, if C has limits, ϵ is a natural isomorphism.

COROLLARY 1. *If C has limits or colimits, precomposition of functors $[B, C]$ along a fully faithful functor is (split) essentially surjective.*

PROOF. For each $G : [A, C]$ we take $\mathrm{Lan}_F G : [B, C]$, and we have $F_*(\mathrm{Lan}_F G) \cong G$. □

COROLLARY 2. *If C has colimits (resp. limits), left (resp. right) Kan extension of functors $[A, C]$ along a fully faithful functor is fully faithful.*

PROOF. Since left Kan extension along F is the left adjoint to precomposition, we have

$$[A, C](\mathrm{Lan}_F G, \mathrm{Lan}_F G') \cong [B, C](G, F_*(\mathrm{Lan}_F G')) \cong [B, C](G, G').$$

□

7. The Karoubi envelope

Let C be a category. If we have a retraction-section pair $c \underset{s}{\overset{r}{\rightleftarrows}} d$ we have (by definition) $s \cdot r = \mathrm{id}_d$. On the other hand, $r \cdot s : c \to c$ is an idempotent morphism, since $r \cdot s \cdot r \cdot s = r \cdot s$. Conversely, we can wonder whether for some idempotent morphism $a : c \to c$, we can find a retraction-section pair (r, s) such that $a = r \cdot s$. If this is the case, we say that the idempotent a *splits*. If a does not split, we can wonder whether we can find an embedding $\iota_C : C \hookrightarrow \overline{C}$ such that the idempotent $\iota_C(a) : \iota_C(c) \to \iota_C(c)$ does split. This is one way to arrive at the *Karoubi envelope*.

DEFINITION 7. We define the category $\overline{C}$. The objects of $\overline{C}$ are tuples (c, a) with $c : C$, $a : C(c, c)$ such that $a \cdot a = a$. The morphisms between (c, a) and (d, b) are morphisms $f : C(a, b)$ such that $a \cdot f \cdot b = f$. The identity morphism on (c, a) is given by a and $\overline{C}$ inherits morphism composition from C.

This category is called the *Karoubi envelope*, the *idempotent completion*, the *category of retracts*, or the *Cauchy completion* of C.

REMARK 3. Note that for a morphism $f : \overline{C}((c, a), (d, b))$,

$$a \cdot f = a \cdot a \cdot f \cdot b = a \cdot f \cdot b = f$$

and in the same way, $f \cdot b = f$.

DEFINITION 8. We have an embedding $\iota_C : C \to \overline{C}$, sending $c : C$ to (c, id_c) and $f : C(c,d)$ to f.

LEMMA 4. *Every object $c : \overline{C}$ is a retract of $\iota_C(c_0)$ for some $c_0 : C$.*

PROOF. Note that $c = (c_0, a)$ for some $c_0 : C$ and an idempotent $a : c \to c$. We have morphisms $\iota_C(c) \underset{a_\leftarrow}{\overset{a_\to}{\rightleftarrows}} (c,a)$, both given by a. We have $a_\leftarrow \cdot a_\to = a = \mathrm{id}_{(c,a)}$, so (c,a) is a retract of $\iota_C(c)$. □

LEMMA 5. *In $\overline{C}$, every idempotent splits.*

PROOF. Take an idempotent $e : \overline{C}(c,c)$. Note that c is given by an object $c_0 : C$ and an idempotent $a : C(c_0, c_0)$. Also, e is given by some idempotent $e : C(c_0, c_0)$ with $a \cdot e \cdot a = e$.

Now, we have $(c_0, e) : \overline{C}$ and morphisms $(c_0, a) \underset{e_\leftarrow}{\overset{e_\to}{\rightleftarrows}} (c_0, e)$, both given by e. We have $e_\leftarrow \cdot e_\to = e = \mathrm{id}_{(c_0,e)}$, so (c_0, e) is a retract of (c_0, a). Also, $e = e_\to \cdot e_\leftarrow$, so e is split. □

REMARK 4. Note that the embedding is fully faithful, since

$$\overline{C}((c, \mathrm{id}_c), (d, \mathrm{id}_d)) = \{f : C(c,d) \mid \mathrm{id}_c \cdot f \cdot \mathrm{id}_d = f\} = C(c,d).$$

REMARK 5. Let D be a category. Suppose that we have a retraction-section pair in D, given by $d \underset{s}{\overset{r}{\rightleftarrows}} d'$. Now, suppose that we have an object $c : D$ and a morphism f with $(r \cdot s) \cdot f = f$. Then we get a morphism $s \cdot f : d' \to c$ such that f factors as $r \cdot (s \cdot f)$. Also, for any g with $r \cdot g = f$, we have

$$g = s \cdot r \cdot g = s \cdot f.$$

$$\begin{tikzcd} d \arrow[r, "r"] \arrow[dr, "f"'] & d' \arrow[r, "s"] \arrow[d, dashed, "s \cdot f"] & d \arrow[dl, "f"] \\ & c & \end{tikzcd}.$$

Therefore, d' is the equalizer of $d \underset{r \cdot s}{\overset{\mathrm{id}_d}{\rightrightarrows}} d$. In the same way, it is also the coequalizer of this diagram.

Now, note that if we have a coequalizer c' of id_c and a, and an equalizer d' of id_d and b, the universal properties of these give an equivalence

$$D(c', d') \cong \{f : D(c, d') \mid a \cdot f = f\} \cong \{f : D(c,d) \mid a \cdot f = f = f \cdot b\}.$$

$$\begin{tikzcd} c \arrow[r, shift left, "\mathrm{id}_c"] \arrow[r, shift right, "a"'] & c \arrow[r] \arrow[d] \arrow[dr] & c' \arrow[d] \\ d & d \arrow[l, shift right, "\mathrm{id}_d"'] \arrow[l, shift left, "b"] & d' \arrow[l] \end{tikzcd}$$

Since a functor preserves retracts, and since every object of $\overline{C}$ is a retract of an object in C, one can lift a functor from C (to a category with (co)equalizers) to a functor on $\overline{C}$.

For convenience, the lemma below works with pointwise left Kan extension using colimits, but one could also prove this using just (co)equalizers (or right Kan extension using limits).

LEMMA 6. *Let D be a category with colimits. We have an adjoint equivalence between $[C, D]$ and $[\overline{C}, D]$.*

PROOF. We already have an adjunction $\mathrm{Lan}_{\iota_C} \dashv \iota_{C*}$. Also, since ι_C is fully faithful, we know that η is a natural isomorphism. Therefore, we only have to show that ϵ is a natural isomorphism. That is, we need to show that $\epsilon_G(c,a) : D(\mathrm{Lan}_{\iota_C}(\iota_{C*}G)(c,a), G(c,a))$ is an isomorphism for all $G : [\overline{C}, D]$ and $(c,a) : \overline{C}$.

One of the components in the diagram of $\mathrm{Lan}_{\iota_C}(\iota_{C*}G)(c,a)$ is the $\iota_{C*}G(c) = G(c, \mathrm{id}_c)$ corresponding to $a : \iota_C(c) \to (c,a)$. This component has a morphism into our colimit

$$\varphi : C(G(\iota_C(c)), \mathrm{Lan}_{\iota_C}(\iota_{C*}G)(c,a)).$$

Note that we can view a as a morphism $a : \overline{C}((c,a), \iota_C(c))$. This gives us our inverse morphism

$$G(a) \cdot \varphi : C(G(c,a), \mathrm{Lan}_{\iota_C}(\iota_{C*}G)(c,a)).$$

□

LEMMA 7. *The formation of the opposite category commutes with the formation of the Karoubi envelope.*

PROOF. An object in $\overline{C^{\mathrm{op}}}$ is an object $c : C^{\mathrm{op}}$ (which is just an object $c : C$), together with an idempotent morphism $a : C^{\mathrm{op}}(c,c) = C(c,c)$. This is the same as an object in $\overline{C}^{\mathrm{op}}$.

A morphism in $\overline{C^{\mathrm{op}}}((c,a),(d,b))$ is a morphism $f : C^{\mathrm{op}}(c,d) = C(d,c)$ such that

$$b \cdot_C f \cdot_C a = a \cdot_{C^{\mathrm{op}}} f \cdot_{C^{\mathrm{op}}} b = f.$$

A morphism in $\overline{C}^{\mathrm{op}}((c,a),(d,b)) = \overline{C}((d,b),(c,a))$ is a morphism $f : C(d,c)$ such that $b \cdot f \cdot a = f$.

Now, in both categories, the identity morphism on (c,a) is given by a.

Lastly, $\overline{C^{\mathrm{op}}}$ inherits morphism composition from C^{op}, which is the opposite of composition in C. On the other hand, composition in $\overline{C}^{\mathrm{op}}$ is the opposite of composition in $\overline{C}$, which inherits composition from C. □

COROLLARY 3. *As the category* **SET** *is cocomplete, we have an equivalence between the category of presheaves on C and the category of presheaves on $\overline{C}$:*

$$[C^{op}, \mathbf{SET}] \cong [\overline{C^{op}}, \mathbf{SET}] \cong [\overline{C}^{op}, \mathbf{SET}].$$

8. Monoids as categories

Take a monoid M.

DEFINITION 9. We can construct a category C_M with $C_{M0} = \{\star\}$, $C_M(\star,\star) = M$. The identity morphism on $\star$ is the identity $1 : M$. The composition is given by multiplication $g \cdot_{C_M} f = f \cdot_M g$.

REMARK 6. Actually, we have a functor from the category of monoids to the category of setcategories (categories whose object type is a set).

A monoid morphism $f : M \to M'$ is equivalent to a functor $F_f : C_M \to C_{M'}$. Any functor between C_M and $C_{M'}$ sends $\star_M$ to $\star_{M'}$. The monoid morphism manifests as $F_f(m) = f(m)$ for $m : C_M(\star,\star) = M$.

LEMMA 8. *An isomorphism of monoids gives an (adjoint) equivalence of categories.*

PROOF. Given an isomorphism $f : M \to M'$. Then we have functors $F_f : C_M \to C_{M'}$ and $F_{f^{-1}} : C_{M'} \to C_M$. Take the identity natural transformations $\eta : \mathrm{id}_{C_M} \Rightarrow F_f \bullet F_{f^{-1}}$ and $\epsilon : F_{f^{-1}} \bullet F_f \Rightarrow \mathrm{id}_{C_{M'}}$. Of course these are natural isomorphisms. □

DEFINITION 10. A *right monoid action* of M on a set X is a function $X \times M \to X$ such that for all $x : X$, $m, m' : M$,

$$x1 = x \qquad \text{and} \qquad (xm)m' = x(m \cdot m').$$

DEFINITION 11. A *morphism* between sets X and Y with a right M-action is an M-equivariant function $f : X \to Y$: a function such that $f(xm) = f(x)m$ for all $x : X$ and $m : M$.

These, together with the identity and composition from **SET**, constitute a category $\mathbf{RAct}_M$ of right M-actions.

LEMMA 9. *Presheaves on C_M are equivalent to sets with a right M-action.*

PROOF. This correspondence sends a presheaf F to the set $F(\star)$, and conversely, the set X to the presheaf F given by $F(\star) := X$. The M-action corresponds to the presheaf acting on morphisms as $xm = F(m)(x)$. A morphism (natural transformation) between presheaves $F \Rightarrow G$ corresponds to a function $F(\star) \to G(\star)$ that is M-equivariant, which is exactly a monoid action morphism. $\square$

REMARK 7. Since the category of sets with an M-action is equivalent to a presheaf category, it has all limits. However, we can make this concrete. The set of the product $\prod_i X_i$ is the product of the underlying sets. The action is given pointwise by $(x_i)_i m = (x_i m)_i$.

Note that the initial set with M-action is $\{\star\}$, with action $\star m = \star$.

LEMMA 10. *The global elements of a set with right M-action correspond to the elements that are invariant under the M-action.*

PROOF. A global element of X is a morphism $\varphi : \{\star\} \to X$ such that for all $m : M$, $\varphi(\star)m = \varphi(\star m) = \varphi(\star)$. Therefore, it is given precisely by the element $\varphi(\star) : X$, which must be invariant under the M-action. $\square$

LEMMA 11. *The category C of sets with an M-action has exponentials.*

PROOF. Given sets with M-action X and Y. Consider the set $C(M \times X, Y)$ with an M-action given by $\phi m'(m, x) = \phi(m'm, x)$. This is the exponential object X^Y, with the (universal) evaluation morphism $X \times X^Y \to Y$ given by $(x, \phi) \mapsto \phi(1, x)$. $\square$

DEFINITION 12. We can view M as a set U_M with right M-action $mn = m \cdot_M n$ for $m : U_M$ and $n : M$.

8.1. Extension and restriction of scalars. Let $\varphi : M \to M'$ be a morphism of monoids.

Remember that sets with a right monoid action are equivalent to presheaves on the monoid category. Also, φ is equivalent to a functor between the monoid categories. The following is a specific case of the concepts in the section about Kan extension:

LEMMA 12. *We get a restriction of scalars functor φ_* from sets with a right M'-action to sets with a right M-action.*

PROOF. Given a set X with right M'-action, take the set X again, and give it a right M-action, sending (x, m) to $x\varphi(m)$.

On morphisms, send an M'-equivariant morphism $f : X \to X'$ to the M-equivariant morphism $f : X \to X'$. $\square$

Since **SET** has colimits, and restriction of scalars corresponds to precomposition of presheaves (on $C_{M'}$), we can give it a left adjoint. This is the (pointwise) left Kan extension, which boils down to:

LEMMA 13. *We get an extension of scalars functor φ^* from sets with a right M-action to sets with a right M'-action.*

PROOF. Given a set X with right M-action. Take $Y = X \times M'/\sim$ with the relation $(xm, m') \sim (x, f(m) \cdot m')$ for $m : M$. This has a right M'-action given by $(x, m')n' = (x, m'n')$.

On morphisms, it sends the m-equivariant $f : X \to X'$ to the morphism $(x, m') \mapsto (f(x), m')$. □

LEMMA 14. *For U_M the set M with right M-action, we have $\varphi^*(U_M) \cong U_{M'}$.*

PROOF. The proof relies on the fact that for all $m : U_M$ and $m' : M'$, we have

$$(m, m') \sim (1, \varphi(m)m').$$

□

Consider the category D with $D_0 = M'$ and

$$D(m', \overline{m}') = \{m : M \mid \varphi(m) \cdot m' = \overline{m}'\}.$$

LEMMA 15. *Suppose that D has a weakly terminal element. Then for I_M the terminal object in the category of sets with a right M-action, we have $\varphi^*(I_M) \cong I_{M'}$.*

PROOF. If D has a weakly terminal object, there exists $m_0 : M'$ such that for all $m' : M'$, there exists $m : M$ such that $\varphi(m) \cdot m' = m_0$.

The proof relies on the fact that every element of $\varphi^*(I_M)$ is given by some $(\star, m')$, but then

$$(\star, m') = (\star \cdot m, m') \sim (\star, \varphi(m) \cdot m') = (\star, \overline{m'}),$$

so $\varphi^*(I_M)$ has exactly 1 element. □

REMARK 8. For φ^* to preserve terminal objects, we actually only need D to be connected. The fact that $\varphi^*(I_M)$ is a quotient by a symmetric and transitive relation then allows us to 'walk' from any $(\star, m_1')$ to any other $(\star, m_2')$ in small steps.

For any $m_1', m_2' : M'$, consider the category $D_{m_1', m_2'}$, given by

$$D_{m_1', m_2', 0} = \{(m', m_1, m_2) : M' \times M \times M \mid m_i' = \varphi(m_i) \cdot m'\}$$

and

$$D_{m_1', m_2'}((m', m_1, m_2), (\overline{m}', \overline{m}_1, \overline{m}_2)) = \{m : M \mid \varphi(m) \cdot m' = \overline{m}', m_i = \overline{m}_i \cdot m\}.$$

LEMMA 16. *Suppose that $D_{m_1', m_2'}$ has a weakly terminal object for all $m_1', m_2' : M'$. Then for sets A and B with right M-action, we have $\varphi^*(A \times B) \cong \varphi^*(A) \times \varphi^*(B)$.*

PROOF. Now, any element in $\varphi^*(A) \times \varphi^*(B) = (A \times M'/\sim) \times (B \times M'/\sim)$ is given by some (a, m_1', b, m_2').

The fact that $D_{m_1', m_2'}$ has a weakly terminal object means that we have some $\overline{m}' : M'$ and $\overline{m}_1, \overline{m}_2 : M$ with $m_i' = \varphi(\overline{m}_i) \cdot \overline{m}'$. Therefore,

$$(a, m_1', b, m_2') = (a, \varphi(\overline{m}_1) \cdot \overline{m}', b, \varphi(\overline{m}_2) \cdot \overline{m}') \sim (a\overline{m}_1, \overline{m}', b\overline{m}_2, \overline{m}'),$$

so this is equivalent to some element in $\varphi^*(A \times B) = (A \times B \times M'/\sim)$. Note that this trivially respects the right M'-action.

The fact that $(\overline{m}', \overline{m}_1, \overline{m}_2)$ is weakly terminal also means that for all $m' : M'$ and $m_1, m_2 : M$ with $m_i' = \varphi(m_i) \cdot m'$, there exists $m : M$ such that $\varphi(m) \cdot m' = \overline{m}'$ and $m_i = \overline{m}_i \cdot m$. This means that the equivalence that we established is actually well-defined: equivalent elements in $\varphi^*(A) \times \varphi^*(B)$ are sent to equivalent elements in $\varphi^*(A \times B)$.

Therefore, we have an isomorphism $\psi : \varphi^*(A) \times \varphi^*(B) \xrightarrow{\sim} \varphi^*(A \times B)$. Now we only need to show that the projections are preserved by this isomorphism. To that end, take $x = (a, m_1', b, m_2') \sim (a\overline{m}_1, \overline{m}', b\overline{m}_2, \overline{m}') : \varphi^*(A) \times \varphi^*(B)$. We have

$$\varphi^*(\pi_1)(\psi(x)) = (a\overline{m}_1, \overline{m}') = \pi_1'(x).$$

In the same way, $\varphi^*(\pi_2) \circ \psi = \pi_2'$ and this concludes the proof. $\square$

CHAPTER 2

Univalent Foundations

CHAPTER 3

Algebraic Structures

1. Algebraic Theories

DEFINITION 13. We define an *algebraic theory* T to be a sequence of sets T_n indexed over $\mathbb{N}$ with for all $1 \leq i \leq n$ elements ("variables" or "projections") $x_{n,i} : T_n$ (we usually leave n implicit), together with a substitution operation

$$\_ \bullet \_ : T_m \times T_n^m \to T_n$$

for all m, n, such that

$$\begin{aligned} x_j \bullet g &= g_j \\ f \bullet (x_{l,i})_i &= f \\ (f \bullet g) \bullet h &= f \bullet (g_i \bullet h)_i \end{aligned}$$

for all $1 \leq j \leq l$, $f : T_l$, $g : T_m^l$ and $h : T_n^m$.

DEFINITION 14. A *morphism* F between algebraic theories T and T' is a sequence of functions $F_n : T_n \to T'_n$ (we usually leave the n implicit) such that

$$\begin{aligned} F_n(x_j) &= x_j \\ F_n(f \bullet g) &= F_m(f) \bullet (F_n(g_i))_i \end{aligned}$$

for all $1 \leq j \leq n$, $f : T_m$ and $g : T_n^m$.

REMARK 9. We can construct binary products of algebraic theories, with sets $(T \times T')_n = T_n \times T'_n$, variables (x_i, x_i) and substitution

$$(f, f') \bullet (g, g') = (f \bullet g, f' \bullet g').$$

In the same way, the category of algebraic theories has all limits.

2. Algebras

DEFINITION 15. An *algebra* A for an algebraic theory T is a set A, together with an action

$$\bullet : T_n \times A^n \to A$$

for all n, such that

$$\begin{aligned} x_j \bullet a &= a_j \\ (f \bullet g) \bullet a &= f \bullet (g_i \bullet a)_i \end{aligned}$$

for all j, $f : T_m$, $g : T_n^m$ and $a : A^n$.

DEFINITION 16. For an algebraic theory T, a *morphism* F between T-algebras A and A' is a function $F : A \to A$ such that

$$F(f \bullet a) = f \bullet (F(a_i))_i$$

for all $f : T_n$ and $a : A^n$.

REMARK 10. The category of algebras has all limits. The set of a limit of algebras is the limit of the underlying sets.

REMARK 11. Note that for an algebraic theory T, the T_n are all algebras for T, with the action given by $\bullet$.

DEFINITION 17 (Pullback of algebras). If we have a morphism of algebraic theories $f : T' \to T$, we have a functor $\mathbf{Alg}_T \to \mathbf{Alg}'_T$. It endows T'-algebras with an action from T given by $g \bullet_{T'} a = f(g) \bullet_T a$. Then T'-algebra morphisms commute with this T-action, so we indeed have a functor.

LEMMA 17. ***(TODO)*** *Fibration*

3. Presheaves

DEFINITION 18. A *presheaf* P for an algebraic theory T is a sequence of sets P_n indexed over $\mathbb{N}$, together with an action

$$\bullet : P_m \times T_n^m \to P_n$$

for all m, n, such that

$$t \bullet (x_{l,i})_i = t$$
$$(t \bullet f) \bullet g = t \bullet (f_i \bullet g)_i$$

for all $t : P_l$, $f : T_m^l$ and $g : T_n^m$.

DEFINITION 19. For an algebraic theory T, a *morphism* F between T-presheaves P and P' is a sequence of functions $F_n : P_n \to P'_n$ such that

$$F_n(t \bullet f) = F_m(t) \bullet f$$

for all $t : P_m$ and $f : T_n^m$.

We will write $\mathbf{Pshf}_T$ for the category of T-presheaves and their morphisms.

REMARK 12. The category of presheaves has all limits. The nth set $\overline{P}_n$ of a limit $\overline{P}$ of presheaves P_i is the limit of the nth sets $P_{i,n}$ of the presheaves in the limit diagram.

LEMMA 18. ***(TODO)*** *Fibration*

4. λ-theories

Let $\iota_{m,n} : T_m \to T_{m+n}$ be the function that sends f to $f \bullet (x_{m+n,1}, \dots, x_{m+n,m})$. Note that

$$\iota_{m,n}(f) \bullet g = f \bullet (g_i)_{i \le m} \quad \text{and} \quad \iota_{m,n}(f \bullet g) = f \bullet (\iota_{m,n}(g_i))_i.$$

For tuples $x : X^m$ and $y : X^n$, let $x + y$ denote the tuple $(x_1, \dots, x_m, y_1, \dots, y_n) : X^{m+n}$.

DEFINITION 20 (λ-theory). A *λ-theory* is an algebraic theory L, together with sequences of functions $\lambda_n : L_{n+1} \to L_n$ and $\rho_n : L_n \to L_{n+1}$, such that

$$\lambda_m(f) \bullet h = \lambda_n(f \bullet ((\iota_{n,1}(h_i))_i + (x_{n+1})))$$
$$\rho_n(g \bullet h) = \rho_m(g) \bullet ((\iota_{n,1}(h_i))_i + (x_{n+1}))$$

for all $f : L_{m+1}$, $g : L_m$ and $h : L_n^m$.

DEFINITION 21 (β- and η-equality). We say that a λ-theory L satisfies β-equality (or that it is a λ-theory with β) if $\rho_n \circ \lambda_n = \mathrm{id}_{L_n}$ for all n. We say that is satisfies η-equality if $\lambda_n \circ \rho_n = \mathrm{id}_{L_{n+1}}$ for all n.

Definition 22 (λ-theory morphism). A *morphism* F between λ-theories L and L' is an algebraic theory morphism F such that

$$F_n(\lambda_n(f)) = \lambda_n(F_{n+1}(f))$$
$$\rho_n(F_n(g)) = F_{n+1}(\rho_n(g))$$

for all $f : L_{n+1}$ and $g : L_n$.

Remark 13. The category of lambda theories has all limits, with the underlying algebraic theory of a limit being the limit of the underlying algebraic theories.

A λ-theory algebra or presheaf is a presheaf for the underlying algebraic theory.

5. Examples

There is a lot of different examples of algebraic theories and their algebras. Some of these even turn out to be λ-theories. In this section, we will discuss a couple of these.

5.1. The free algebraic theory on a set.

Example 1. Let S be a set. We can construct an algebraic theory $F(S)$ by taking $F(S)_n = S \sqcup \{1, \dots, n\}$ with projections $x_i = i$ and substitution

$$i \bullet g = g_i \qquad\qquad s \bullet g = s$$

for $i : \{1, \dots, n\}$ and $s : S$.

If we have a function $f : S \to S'$, we get a morphism $F(f) : F(S) \to F(S')$ given by

$$F(f)_n(i) = i \qquad\qquad F(f)_n(s) = f(s)$$

for $i : \{1, \dots, n\}$ and $s : S$.

Also, F obviously respects the identity and substitution morphisms, so it is a functor.

Note that we have a forgetful functor $(\cdot)_0$ that sends a morphism of algebraic theories $g : T \to T'$ to the function $f_0 : T_0 \to T_0'$.

Lemma 19. *The algebraic theory $F(S)$ defined above, is the free algebraic theory on the set S.*

Proof. Let T be an algebraic theory. We have an equivalence

$$\mathbf{AlgTh}(F(S), T) \cong \mathbf{SET}(S, T_0),$$

sending $f : \mathbf{AlgTh}(F(S), T)$ to $f_0 : S = S \sqcup \emptyset \to T_0$ (this is trivially natural in S and T) and $f : \mathbf{SET}(S, T_0)$ to the functions $g_n : F(S)_n \to T_n$ given by

$$g_n(i) = x_i \qquad\qquad g_n(s) = f(s) \bullet ().$$

□

The proofs that $F(S)$ is an algebraic theory and that $F(f)$ and g are algebraic theory morphisms is an easy exercise in case distinction.

Corollary 4. *$F(\emptyset)$ is the initial algebraic theory.*

Proof. For $S = \emptyset$, the equivalence of hom-sets becomes

$$\mathbf{AlgTh}(F(\emptyset), T) \cong \mathbf{SET}(\emptyset, T_0)$$

and the latter has exactly one element. □

Lemma 20. *There is an adjoint equivalence between the category $\mathbf{Alg}_{F(S)}$ and the coslice category **((TODO)**) $S \downarrow \mathbf{SET}$.*

PROOF. For the equivalence, we send a $F(S)$-algebra A to the set A with morphism $s \mapsto s \bullet ()$. An algebra morphism $f : A \to B$ is sent to the coslice morphism $f : (S \to A) \to (S \to B)$. This constitutes a functor.

Note that the category of $F(S)$-algebras is univalent.

Also, the functor is fully faithful, since one can show that for $F(S)$-algebras, the coslice morphism $\varphi : (f : S \to A) \to (f' : S \to B)$ also has the structure of an algebra morphism $\varphi : A \to B$.

Lastly, the functor is essentially surjective, since we can lift a coslice $f : S \to X$ to a $F(S)$-algebra X, with action

$$i \bullet x = x_i \quad \text{and} \quad s \bullet x = f(s).$$

Therefore, the functor $\mathbf{Alg}_{F(S)} \to S \downarrow \mathbf{SET}$ is an adjoint equivalence.

The proofs of these facts work by simple case distinction, and by using the properties of the coslice and algebra morphisms. □

$F(\emptyset)$ is, in some sense, the smallest nontrivial algebraic theory. Then $F(S)$ is the smallest nontrivial algebraic theory that has the elements of S as constants.

5.2. The free λ-theory on a set. Like with the free algebraic theory, we will construct the free λ-theory as the smallest nontrivial λ-theory (which is the λ-calculus) with some additional constants.

Let S be a set. Consider the sequence of inductive types $(\Lambda(S)_n)_n$ with the following constructors:

$$\begin{aligned}
&\mathtt{Var}_n : \{1, \dots, n\} \to \Lambda(S)_n;\\
&\mathtt{App}_n : \Lambda(S)_n \to \Lambda(S)_n \to \Lambda(S)_n;\\
&\mathtt{Abs}_n : \Lambda(S)_{n+1} \to \Lambda(S)_n;\\
&\mathtt{Con}_n : S \to \Lambda(S)_n.
\end{aligned}$$

Define a substitution operator $\bullet : \Lambda(S)_m \times \Lambda(S)_n^m \to \Lambda(S)_n$ by induction on the first argument:

$$\begin{aligned}
\mathtt{Var}_m(i) \bullet g &= g_i;\\
\mathtt{App}_m(a, b) \bullet g &= \mathtt{App}_n(a \bullet g, b \bullet g);\\
\mathtt{Abs}_m(a) \bullet g &= \mathtt{Abs}_n(a \bullet ((g_i \bullet (x_{n+1,j})_j)_i + (x_{n+1})));\\
\mathtt{Con}_m(s) \bullet g &= \mathtt{Con}_n(s).
\end{aligned}$$

And then quotient $\Lambda(S)$ by the relation generated by

$$\mathtt{App}_m(\mathtt{Abs}_m(f), g) \sim f \bullet ((x_{n,i})_i + (g))$$

for all $f : \Lambda(S)_{n+1}$ and $g : \Lambda(S)_n$.

EXAMPLE 2. We can give the sequence of sets $\Lambda(S)$ an algebraic theory structure with variables $x_{m,i} = \mathtt{Var}_m(i)$ and the substitution operator $\bullet$ defined above. We can give $\Lambda(S)$ a λ-theory structure with β-equality by taking

$$\lambda_n(f) = \mathtt{Abs}_n(f) \quad \text{and} \quad \rho_n(f) = \mathtt{App}_{n+1}(f \bullet (\mathtt{Var}_{n+1}(i))_i, \mathtt{Var}_{n+1}(n+1)).$$

Now, given a function $S \to S'$, we define a morphism $\mathbf{LamTh}(\Lambda(S), \Lambda(S'))$ by induction, sending $\mathtt{Var}(i)$, $\mathtt{App}(a, b)$ and $\mathtt{Abs}(a)$ in $\Lambda(S)$ to their corresponding elements in $\Lambda(S')$ and sending $\mathtt{Con}(s)$ to $\mathtt{Con}(f(s))$.

Note that, like with the previous example, we have a forgetful functor $\dot{()}_0$: $\mathbf{LamTh} \to \mathbf{SET}$.

LEMMA 21. *$\Lambda(S)$ is the free λ-theory on S.*

Proof. Let L be a λ-theory. We have an equivalence

$$\mathbf{LamTh}(\Lambda(S), L) \cong \mathbf{SET}(S, L_0),$$

sending $f : \mathbf{LamTh}(\Lambda(S), L)$ to $f_0|_S : S \to L_0$ (again, trivially natural in S and L) and conversely, $g : \mathbf{SET}(S, L_0)$ to the inductively defined $f : \mathbf{LamTh}(\Lambda(S), L)$ given by

$$\begin{aligned} f(\mathtt{Var}(i)) &= x_i;\\ f(\mathtt{App}(a, b)) &= \rho(f(a)) \bullet ((x_{n,i})_i + (b));\\ f(\mathtt{Abs}(a)) &= \lambda(f(a));\\ f(\mathtt{Con}(s)) &= g(s) \bullet (). \end{aligned}$$

□

The proofs that $\Lambda(S)$ is indeed a λ-theory and that $\Lambda(f)$ and g are λ-theory morphisms, mainly work by definition of $\bullet$, λ and ρ, by induction on the terms of $\Lambda(S)$ and by invoking the properties of the λ-theory L.

Corollary 5. *The 'pure' lambda calculus is the initial λ-theory.*

Proof. If we take $S = \emptyset$, $\Lambda(\emptyset)$ is the lambda calculus, which we will call Λ. We have, like with the free algebraic theory, that $\Lambda(\emptyset)$ is the initial λ-theory. □

5.3. The free object algebraic theory.

Example 3. Take a category C, with a forgetful functor $G : C \to \mathbf{SET}$ and a free functor $F : \mathbf{SET} \to C$. Let $\eta : \mathrm{id}_{\mathbf{SET}} \Rightarrow F \bullet G$ be the unit of the adjunction and let $\varphi : C(F(c), d) \cong \mathbf{SET}(c, G(d))$ be the natural equivalence of homsets.

We define an algebraic theory T with $T_n = G(F(\{1, \dots, n\}))$, projections $x_{n,i} = \eta_{\{1,\dots,n\}}(i)$. For the substitution, note that we take $t_1, \dots, t_m : T_n$, so we have $t : \{1, \dots, m\} \to G(F(\{1, \dots, n\}))$. We then take

$$s \bullet t = G(\varphi^{-1}(t))(s).$$

Now, given an object $c : C$, we can create a T-algebra $\alpha(c)$, with set $G(c)$ and action

$$s \bullet t = G(\varphi^{-1}(t))(s).$$

Also, given a morphism $f : C(c, d)$. This gives a morphism $G(f) : \alpha(c) \to \alpha(d)$. Therefore, $\alpha : C \to \mathbf{Alg}_T$ is a functor.

The proofs that T is an algebraic theory, that $G(c)$ is an algebra and that $G(f)$ is an algebra morphism mainly rely on the fact that φ is natural.

So we have a functor from C to the category of T-algebras. One can wonder whether there also is a functor the other way, or whether α is even an equivalence. This is hard to characterize precisely, but in algebra, there is a broad class of examples where the functor is an equivalence, so where C is equivalent to $\mathbf{Alg}_T$. That is probably why T is called an *algebraic theory*.

The idea is that if an object of C is a set, together with some operations between its elements, one can carefully choose some elements of T_0, T_1, T_2 etc., which act on an algebra like the specified operations.

Example 4. For C the category of monoids, $\alpha : C \to \mathbf{Alg}_T$ is an adjoint equivalence.

Note that T_n is the free monoid on n elements. Its elements can be viewed as strings $(x_1 x_5 x_3 x_{18} \dots x_7)$ with the characters $x_1, \dots, x_n$, with the x_i the generators of the monoid, acting as the projections of the algebraic theory.

Let A be a T-algebra. We can give A a monoid structure by taking, for $a, b : A$,

$$ab = (x_1 x_2) \bullet (a, b)$$

and unit element

$$1 = () \bullet ().$$

Then the laws like associativity follow from those laws on the monoid and from the fact that the action on the algebra commutes with the substitution:

$$a(bc) = (x_1(x_2x_3)) \bullet (a, b, c) = ((x_1x_2)x_3) \bullet (a, b, c) = (ab)c.$$

Note that if we take a monoid, turn it into a T-algebra and then into a monoid again, we still have the same underlying set, and it turns out that the monoid operation and unit element are equal to the original monoid operation and unit element. Therefore, α is essentially surjective. It is also fully faithful, since any T-algebra morphism respects the action of T, which makes it into a monoid morphism. Therefore, α is an adjoint equivalence.

Remark 14. In the same way, one can characterize groups, rings and R-algebras (for R a ring) as algebras of some algebraic theory. On the other hand, one can not use this method to describe fields as algebras for some theory T, because one would need to describe the inverse $z \mapsto z^{-1}$ operation as $t \bullet (z)$ for some $t : T_1$, with $zz^{-1} = 1$, but since the elements of the algebraic theory act on all (combinations of) elements of the algebra, one would be able to take the inverse $0^{-1} = t \bullet (0)$ with $00^{-1} = 1$, which would make no sense.

Remark 15. Another counterexample is the category **Top** of topological spaces. We have a forgetful functor $G : \mathbf{Top} \to \mathbf{SET}$ that just forgets the topology. On the other hand, we have a free functor $F : \mathbf{SET} \to \mathbf{Top}$ which endows a set with the discrete topology. The construction above yields the inital algebraic theory $T_n = \{1, \dots, n\}$, with an algebra action on every topological space $i \bullet (a_1, \dots, a_n) = a_i$. Now, note that we can endow the set $\{\top, \bot\}$ with four different, nonisomorphic topologies, which all yield the same T-algebra. In other words: the T-algebra structure does not preserve the topological information. Therefore, the functor $\alpha : \mathbf{Top} \to \mathbf{Alg}_T$ is not an equivalence.

5.4. The terminal theory.

Example 5. We can create a (somewhat trivial) algebraic theory T by taking $T_n = \{\star\}$, with projections $x_i = \star$ and substitution $\star \bullet \star = \star$. Taking $\lambda(\star) = \star$ and $\rho(\star) = \star$, we give it a λ-theory structure (with β and η-equality). Checking that this is indeed an algebraic theory and even a λ-theory is trivial.

Now, given any other algebraic theory T', there exists a unique function $T'_n \to T_n$ for every n, sending everything to $\star$. These functions actually constitute an algebraic theory morphism $T' \to T$. If T' is a λ-theory, the algebraic theory morphism is actually a λ-theory morphism. Again, checking this is trivial.

Therefore, T is the terminal algebraic theory and λ-theory.

Lemma 22. $\{\star\}$ *is the only algebra of the terminal theory.*

Proof. Let A be a T-algebra. First of all, we have an element $\star_A = \star_T \bullet_0 ()$. Secondly, for all elements $\star, \star' : A$, we have

$$\star = x_1 \bullet (\star, \star') = \star \bullet (\star, \star') = x_2 \bullet (\star, \star') = \star'.$$

Therefore, $A = \{\star\}$, which allows exactly one possible T-action:

$$\star \bullet (\star, \dots, \star) = \star.$$

$\square$

5.5. The endomorphism theory.

Definition 23. Suppose that we have a category C and an object $X : C$, such that all powers X^n of X are also in C. The *endomorphism theory* $E(X)$ of X is the algebraic theory given by $E(X)_n = C(X^n, X)$ with projections as variables $x_{n,i} : X^n \to X$ and a substitution that sends $f : X^m \to X$ and $g_1, \dots, g_m : X^n \to X$ to $f \circ \langle g_i \rangle_i : X^n \to X^m \to X$.

Definition 24. Now, suppose that the exponential object X^X exists, and that we have morphisms back and forth $abs : X^X \to X$ and $app : X \to X^X$. Let, for $Y : C$, φ_Y be the isomorphism $C(X \times Y, X) \xrightarrow{\sim} C(Y, X^X)$. We can give $E(X)$ a λ-theory structure by setting, for $f : E(X)_{n+1}$ and $g : E(X)_n$,

$$\lambda(f) = abs \circ \varphi_{X^n}(f) \qquad \rho(g) = \varphi_{X^n}^{-1}(app \circ g).$$

The proofs that $E(X)$ is an algebraic theory and a λ-theory, use properties of the product, and naturality of the isomorphism φ_Y.

5.6. The theory algebra.

Example 6. Let T be an algebraic theory and n a natural number. We can endow the T_n with a T-algebra structure, by taking the substitution operator of T as the T-action. Since this commutes with the substitution operator and the projections, T_n is a T-algebra.

5.7. The theory presheaf.

Example 7. Let T be an algebraic theory. We can endow T with a T-presheaf structure, by taking the substitution operator of T as the action on T. Since this commutes with the substitution operator and the projections, T is a T-presheaf.

Lemma 23. *Given an algebraic theory T and a T-presheaf Q, we have for all n a bijection of sets*

$$\varphi : \mathbf{Pshf}_T(T^n, Q) \cong Q_n.$$

Proof. For $f : \mathbf{Pshf}_T(T^n, Q)$, take $\varphi(f) = f_n(x_1, \dots, x_n)$.
Conversely, for all $q : Q_n$ and all $t_1, \dots, t_n : T_m^n$ take

$$\varphi^{-1}(q)_m(t_1, \dots, t_n) = q \bullet t.$$

□

5.8. The "+l" presheaf.

Example 8 (The '+l' presheaf). Given a T-presheaf Q, we can construct a presheaf $A(Q, l)$ with $A(Q, l)_n = Q_{n+l}$ and, for $q : A(Q, l)_m$ and $f : T_n^m$, action

$$q \bullet_{A(Q,l)} f = q \bullet_Q ((\iota_{n,l}(f_i))_i + (x_{n+i})_i).$$

Lemma 24. *For all l and T-presheaves Q, $A(Q, l)$ is the exponential object Q^{T^l}.*

Proof. We will show that $A(-, l)$ constitutes a right adjoint to the functor $- \times T^l$. We will do this using universal arrows.

For Q a T-presheaf, take the arrow $\varphi : A(Q, l) \times T^l \to Q$ given by $\varphi(q, t) = q \bullet_Q ((x_{n,i})_i + t)$ for $q : A(Q, l)_n = Q_{n+l}$ and $t : T_n^l$.

Now, given a T-presheaf Q' and a morphism $\psi : Q' \times T^l \to Q$. Define $\tilde{\psi} : Q'_n \to A(Q, l)_n$ by $\tilde{\psi}(q) = \psi(\iota_{n,l}(q), (x_{n+i})_i)$.

Then ψ factors as $\varphi \circ (\tilde{\psi} \times \mathrm{id}_{T^l})$. Also, some equational reasoning shows that $\tilde{\psi}$ is unique, which proves that φ indeed is a universal arrow. □

CHAPTER 4

Previous work in categorical semantics

1. The correspondence between categories and λ-calculi

In [**SH80**], Scott and Lambek argue that there is a correspondence between simply typed λ-calculi and cartesian closed categories (categories with products and 'function objects'). (See page 413).

Types in the λ-calculus correspond to objects in the category.

Types $A \to B$ in the λ-calculus correspond to exponential objects B^A in the category.

Terms in the λ-calculus of type B, with free variables $x_1 : A_1, \ldots, x_n : A_n$, correspond to morphisms $A_1 \times \cdots \times A_n \to B$.

A free variable $x_i : A_i$ in a context with free variables $x_1 : A_1, \ldots, x_n : A_n$ corresponds to the projection morphism $\pi_i : A_1 \times \cdots \times A_n \to A_i$.

Given a term $s : B_1 \to B_2$ and a term $t : B_1$, both with free variables $x_1 : A_1, \ldots, x_n : A_n$, corresponding to morphisms $\overline{s} : A_1 \times \cdots \times A_n \to B_2$ and $\overline{t} : A_1 \times \cdots \times A_n \to B_1$, the application $st : B_2$ corresponds to the composite of the product morphism with the evaluation morphism $A_1 \times \cdots \times A_n \to B_2^{B_1} \times B_1 \to B_2$.

$$\begin{tikzcd}
& A_1 \times \cdots \times A_n \arrow[dl, "\overline{s}"'] \arrow[d, dashed, "\langle \overline{s}, \overline{t} \rangle"] \arrow[dr, "\overline{t}"] & \\
B_2^{B_1} & B_2^{B_1} \times B_1 \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] \arrow[d, "ev"] & B_1 \\
& B_2 &
\end{tikzcd}$$

Given a term $t : B$ with free variables $x_1 : A_1, \ldots, x_n : A_n$, the abstraction $(\lambda x_n, t) : A_n \to B$ corresponds to using the adjunction $- \times A_n \dashv (-)^{A_n}$:

$$C(A_1 \times \cdots \times A_{n-1} \times A_n, B) \simeq C(A_1 \times \cdots \times A_{n-1}, B^{A_n}).$$

(TODO) How about the untyped λ-calculus?

2. Scott's Representation Theorem

3. The Taylor Fibration

CHAPTER 5

The paper

1. Scott's Representation Theorem

THEOREM 1. *Any λ-theory L is isomorphic to the endomorphism λ-theory $E(L)$ of $L : \mathbf{Pshf}_L$, which is L, viewed as a presheaf in its own presheaf category.*

PROOF. First of all, remember that L is indeed exponentiable and that $L^L = A(L, 1)$. Now, since L is a λ-theory, we have sequences of functions back and forth $\lambda_n : A(L,1)_n \to L_n$ and $\rho_n : L_n \to A(L,1)_n$. These commute with the L-actions, so they constitute presheaf morphisms and $E(L)$ is indeed a λ-theory.

Lemma 23 gives a sequence of bijections $\varphi_n : \mathbf{Pshf}_L(L^n, L) \cong L_n$ for all n, sending $F : \mathbf{Pshf}_L(L^n, L)$ to $F(x_1, \dots, x_n)$, and conversely sending $s : L_n$ to $((t_1, \dots, t_n) \mapsto s \bullet (t_1, \dots, t_n))$. It considers λ-terms in n variables as n-ary functions on the λ-calculus. Therefore, it should come as no surprise that φ preserves the x_i, $\bullet$, ρ and λ, which makes it into an isomorphism of λ-theories and this concludes the proof. $\square$

2. Locally cartesian closedness of the category of retracts

DEFINITION 25 (Category of retracts). The category of retracts for a λ-theory L is the category with objects $f : L_n$ such that $f \bullet f = f$ and it has as morphisms $g : f \to f'$ the terms $g : L_n$ such that $f' \bullet g \bullet f = g$. The object $f : L_n$ has identity element f, and we have composition $g \circ g' = g \bullet g'$. These are morphisms **(TODO)**

LEMMA 25. *The category of retracts is indeed a category.*

PROOF. **(TODO)** $\square$

THEOREM 2. *The category of retracts is locally cartesian closed* ***(TODO)*** .

3. Equivalences

4. Terms of a Λ-algebra

Let A be an algebra for the initial λ-theory Λ. We will assume that Λ (and therefore, any λ-theory) satisfies β-equality.

The Λ-algebra structure gives the terms of A quite a lot of behaviour. For example, we can define 'function application' as

$$ab = (x_1 x_2) \bullet (a, b)$$

and composition as

$$a \circ b = (x_1 \circ x_2) \bullet (a, b)$$

for $a, b : A$, with $x_1 \circ x_2 = \lambda x_3, x_1(x_2 x_3) : \Lambda_2$.

REMARK 16. Recall that in Example 4, we constructed an algebraic theory T with a monoid structure. This allowed us to define a monoid operation on T-algebras as well. We then were able to transfer associativity of the operation on the T_n to associativity of the operation on the algebras. In exactly the same way, the function composition on A is associative because composition on Λ_n is associative.

DEFINITION 26. We can consider the sets of elements of A that behave like functions in n variables:

$$A_n = \{a : A \mid (\lambda x_2 x_3 \dots x_{n+1}, x_1 x_2 x_3 \dots x_{n+1}) \bullet a = a\}.$$

DEFINITION 27. Take $\mathbf{1}_n = (\lambda x_1 \dots x_n, x_1 \dots x_n) \bullet () : A$.

REMARK 17. Some straightforward rewriting, shows that for all $a : A$,

$$\mathbf{1}_n \circ a = (\lambda x_2 x_3 \dots x_{n+1}, x_1 x_2 \dots x_{n+1}) \bullet a.$$

In other words, $A_n = \{a : A \mid \mathbf{1}_n \circ a = a\}$.

REMARK 18. Also note that $\mathbf{1}_n \circ a \circ \mathbf{1}_n = \mathbf{1}_n \circ a$, so for $a : A_n$, $a \circ \mathbf{1}_n = a$.

LEMMA 26. *For $t : \Lambda_{m+n}$ and $a_1, \dots, a_m : A$, we have $(\lambda^n t) \bullet (a_1, \dots, a_m) : A$ and we have*

$$\mathbf{1}_n \circ ((\lambda^n t) \bullet (a_1, \dots, a_m)) = (\lambda^n t) \bullet (a_1, \dots, a_m),$$

so $(\lambda^n t) \bullet (a_1, \dots, a_m) : A_n$.

PROOF. This follows by straightforward rewriting. □

COROLLARY 6. *By the previous remark,*

$$((\lambda^n t) \bullet (a_1, \dots, a_m)) \circ \mathbf{1}_n = (\lambda^n t) \bullet (a_1, \dots, a_m).$$

COROLLARY 7. *In particular, $\mathbf{1}_m \circ \mathbf{1}_n = \mathbf{1}_{\max(m,n)}$. From this, it follows that $A_m \subseteq A_n$ for $m \leq n$. It also follows that $a \mapsto \mathbf{1}_n \circ a$ gives a function from A to A_n (and also from $A_m \subseteq A$ to A_n).*

5. The Fundamental Theorem of the λ-calculus

The fundamental theorem states that there is an equivalence of categories

$$\mathbf{LamTh} \cong \mathbf{Alg}_\Lambda.$$

We will prove this by showing that there is a fully faithful and essentially surjective functor.

5.1. The functor.

DEFINITION 28. For all n, we have a functor from lambda theories to Λ-algebras. It sends the λ-theory L to the L-algebra L_n and then turns this into a Λ-algebra via the morpism $\Lambda \to L$. It sends morphisms $f : L \to L'$ to the algebra morphism $f_n : L_n \to L'_n$.

5.2. Lifting Λ-algebras.

DEFINITION 29 (The monoid of a Λ-algebra). Now we make A_1 into a monoid under composition $\circ$ with unit $\mathbf{1}_1$. The fact that this is a monoid follows from the remarks in the last section.

Recall that we have an equivalence $[C_{A_1}^{\mathrm{op}}, \mathbf{SET}] \cong \mathbf{RAct}_{A_1}$.

DEFINITION 30. Now, composition $\circ$ gives a right A_1-action on the A_n, so we have $A_n : \mathbf{RAct}_{A_1}$.

LEMMA 27. *We have $A_1^{A_1} \cong A_2$ in $\mathbf{RAct}_{A_1}$.*

PROOF. Recall that $A_1^{A_1}$ consists of the set of A_1-equivariant morphisms $A_1 \times A_1 \to A_1$.

We have a bijection $\varphi : A_2 \xrightarrow{\sim} A_1^{A_1}$, given by

$$\varphi(a)(b, b') = (\lambda x_4, x_1(x_2x_4)(x_3x_4)) \bullet (a, b, b'),$$

(TODO) : relate to product in category of retracts.

with an inverse given by

$$\varphi^{-1}(f) = \lambda x_1x_2, f(p_1, p_2)(\lambda x_3, x_3x_1x_2)$$

for $p_i = \lambda x_1, x_1(\lambda x_2x_3, x_{i+1})$. Note that for terms c_1, c_2, we have $p_i(\lambda x_1, x_1c_1c_2) = c_i$.

This is an inverse, because given $f : A_1^{A_1}$ and $(a_1, a_2) : A_1 \times A_1$, we have

$$\varphi(\psi(f))(a_1, a_2) = f(p_1, p_2) \circ q = f(p_1 \circ q, p_2 \circ q) = f(a_1, a_2)$$

for $q = \lambda x_1, (\lambda x_2, x_2(a_1x_1)(a_2x_1))$. In the last step of this proof, we use, among other things, the fact that the $a_i : A_1$ and therefore $\lambda x_1, a_ix_1 = a_i$.

Some straightforward rewriting shows that for $a : A_2$, we have $\psi(\varphi(a)) = a$. In the last step of this proof, we use the fact that $a : A_2$ and therefore $\lambda x_1x_2, ax_1x_2 = a$.

Therefore, φ is a bijection and, as it turns out, an isomorphism. □

DEFINITION 31 (Construction of the λ-theory). Since $\mathbf{RAct}_{A_1}$ has products, the algebraic theory $E(A_1)$ exists.

Recall that $A_2 \subseteq A_1$ and that $a \mapsto \mathbf{1}_2 \circ a$ gives a function from A_1 to A_2, which is, by definition, the identity on A_2. This gives $E(A_1)$ a λ-theory structure with β-equality.

5.3. Lifting algebra morphisms.

DEFINITION 32 (Pullback functor on presheaves for a Λ-algebra). A Λ-algebra morphism preserves $\mathbf{1}_n$ and $\circ$, so it sends elements of A_n to A'_n. In particular, it gives a monoid morphism $f : A_1 \to A'_1$.

Therefore, as described in Section 1.8, we get pullback and pushforward functors $f_* : \mathbf{RAct}_{A'_1} \to \mathbf{RAct}_{A_1}$ and $f^* : \mathbf{RAct}_{A_1} \to \mathbf{RAct}_{A'_1}$.

REMARK 19. By Lemma 14, we have $f^*(U_{A_1}) \cong U_{A'_1}$.

LEMMA 28. *f^* preserves finite products.*

PROOF. We will show that f^* preserves binary products and the terminal object.

We use Lemma 15 to show that f^* preserves the terminal object. We take

$$a_0 = (\lambda x_1x_2, x_2) \bullet () : A_1 \quad \text{and} \quad a'_0 = (\lambda x_1x_2, x_2) \bullet () : A'_1$$

and a'_0 is weakly terminal because for all $a : A_1$, we have $f(a_0) \circ a = a'_0$.

We use Lemma 16 to show that f^* also preserves the product. Therefore, given $a_1, a_2 : A_1$.

(**(TODO)** : relate to product) Take $a = \lambda x_1x_2, x_2(a_1x_1)(a_2x_1)$ and $\pi_i = \lambda x_1, x_1(\lambda x_2x_3, x_{n+1})$. We have $a_i = f(\pi_i) \circ a$.

Need to show that exists $(\overline{m}', \overline{m}_1, \overline{m}_2) : A'_1 \times A_1 \times A_1$ with $a_i = f(\overline{m}_i) \circ \overline{m}'$ such that for all (m', m_1, m_2), there exists $m : A_1$ such that $f(m) = m' = \overline{m}'$ and $m_i = \overline{m}_i \circ m$.

(TODO) □

DEFINITION 33. Since f^* preserves finite products, given an element of $g : E(U_{A_1})_n = \mathbf{RAct}_A(U_{A_1}^n, U_{A_1})$, we get

$$\#f^*(g) : \mathbf{RAct}_{A'}(f(A_1^n), f(A_1)) \cong \mathbf{RAct}_{A'}((A'_1)^n, A'_1) = E(A'_1)_n.$$

LEMMA 29. *$\#f^* : E(U_{A_1}) \to E(U_{A'_1})$ is a map of λ-theories.*

PROOF. **(TODO)** $\square$

DEFINITION 34. We have an isomorphism $E(U_{A_1})_0 \cong A$ given by $a \mapsto aI$.

LEMMA 30. *This is indeed an isomorphism of Λ-algebras.*

PROOF. **(TODO)** $\square$

LEMMA 31. *Given $g : A \to A'$,*

THEOREM 3. *There exists an adjoint equivalence between the category of λ-theories, and the category of algebras of Λ.*

PROOF. We will show that the functor $L \mapsto L_0$ is an equivalence of categories.

It is essentially surjective, because L is isomorphic **(TODO)** to $E(U_{A_1})$.

Now, given morphisms $f, f' : L \to L'$. Suppose that $f_0 = f'_0$. Suppose that L and L' have β-equality. Then, given $l : L_n$, we have

$$f_n(l) = \rho^n(\lambda^n(f_n(l))) = \rho^n(f_0(\lambda^n(l))) = \rho^n(f'_0(\lambda^n(l))) = \rho^n(\lambda^n(f'_n(l))) = f'_n(l),$$

so the functor is faithful.

The functor is full because a Λ-algebra morphism $f : A \to A'$ induces a functor $f^* : \mathbf{RAct}_{A'} \to \mathbf{RAct}_A$, and via left Kan extension we get a left adjoint $f^* : \mathbf{RAct}_A \to \mathbf{RAct}_{A'}$ with $f^*(A_1) \cong A'_1$. Now, f^* preserves (finite) products, so we have maps $\mathbf{RAct}_A(A_1^n, A_1) \to \mathbf{RAct}_{A'}((A'_1)^n, A'_1)$ and so a map $E(U_{A_1}) \to E(U_{A'_1})$. This map, when restricted to a map $\mathbf{RAct}_A(1, A_1) \to \mathbf{RAct}_{A'}(1, A_1)$, and transported along the isomorphism $a \mapsto aI$ **(TODO)** , is equal to f **(TODO)** . $\square$

6. An alternative proof for the fundamental theorem

7. Theory of extensions

LEMMA 32. *The category of T-algebras has coproducts.*

PROOF. **(TODO)** $\square$

DEFINITION 35 (Theory of extensions). Let T be an algebraic theory and A a T-algebra. We can define an algebraic theory T_A called 'the theory of extensions of A' with $(T_A)_n = T_n + A$. The left injection of the variables $x_i : T_n$ gives the variables. Now, take $h : (T_n + A)^m$. Sending $g : T_m$ to $\varphi(g) := g \bullet h$ gives a T-algebra morphism $T_m \to T_n + A$ since

$$\varphi(f \bullet g) = f \bullet g \bullet h = f \bullet (g_i \bullet h) = f \bullet (\varphi(g_i))_i.$$

This, together with the injection morphism of A into $T_n + A$, gives us a T-algebra morphism from the coproduct: $T_m + A \to T_n + A$. We especially have a function on sets $(T_m + A) \times (T_n + A)^m \to T_n + A$, which we will define our substitution to be.

LEMMA 33. *T_A is indeed an algebraic theory.*

PROOF. **(TODO)** $\square$

CHAPTER 6

The formalization

1. Statistics

2. Components

3. Displayed categories

3.1. Defininig objects by their category instead of the other way around.

3.2. _ax for categories and their objects.

3.3. Cartesian vs cartesian'.

4. Inductive types

5. The formalization of the λ-calculus

Defining Lambda Calculus in a different way (not as an axiomatized HIT) - As set quotient instead of HIT - With a signature

6. Tuples

$stnm \to A$ vs vec A

7. Products

$T \times (T \times \cdots \times T)$ vs $T \times T^n vsT^(Sn)$ Terminal as product over empty set over any set with a function to empty.

8. The $n + p$-presheaf

L (S n) (for lambda) vs L (n + 1) (stemming from the naive implementation of the L (n + p) presheaf)

9. Quotients

Quotients (by hrel or eqrel) vs coproducts (generalizing to arbitrary category with coproduct) vs a category with some structure

10. The Karoubi envelope

KanExtension instead of specific construction at KaroubiEnvelope

11. Univalence

Univalence bewijzen via isweqhomot vs direct

12. Equality, Iso's and Equivalence (of categories)

It is important to choose the right kind of equality to prove.

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APPENDIX A

Alternative definitions

The literature, there are many different but equivalent definitions, carrying many different names, for the objects that called 'algebraic theories' in Section 3.1. Also, many of these different names are attached to differently defined objects in different sources. This section will showcase some of the various definitions.

In this section, we will denote the finite set $[\![n]\!] = \{1, \dots, n\}$.

1. Abstract Clone

Definition 36. An algebraic theory as presented in Section 3.1, is usually called an *abstract clone*. In this thesis, outside of this specific section, we will call it algebraic theory to be consistent with the names that Hyland attaches to objects.

Remark 20. The definition of algebraic theory that Hyland gives is closest to that of an abstract clone. However, instead of a sequence of sets $(T_n)_n$, he requires a functor $T : F \to \mathbf{SET}$ (with $F \subseteq \mathbf{FinSET}$ the skeleton category of finite sets $F_0 = \{[\![0]\!], [\![1]\!], \dots\}$), and he requires $\bullet : T_m \times T_n^m \to T_n$ to be dinatural in n and natural in m.

Using naturality, one can show that with such a functor we have $x_{n,i} = f(a)(x_{1,1})$ for the function $a(1) = i : [\![n]\!]$.

Alternatively, using the same naturality, one can show that this functor sends a morphism $a : [\![m]\!] \to [\![n]\!]$ to the function $T_m \to T_n$ given by

$$f \mapsto f \bullet (x_{n,a(i)})_i.$$

If we take this to be the definition of our functor on morphisms, the (di)naturality in m and n can be shown using the associativity and the laws about the interaction between $\bullet$ and the x_i.

Since any additional properties mean extra complexity when formalizing, and since the proofs rarely use the functor structure, we decided to reduce the functor $T : \mathbf{FinSET} \to \mathbf{SET}$ to a sequence of sets $(T_n)_n$.

2. Lawvere theory

Definition 37. An *algebraic theory* as presented in [**ARV10**] is a small category with finite products.

Definition 38. An *algebra* for an algebraic theory T is a finite-product-preserving functor $T \to \mathbf{SET}$.

This definition is more general than the definition of algebraic theory in Section 3.1. To make it equivalent, we have to be more specific about the objects of the category:

Definition 39. A *Lawvere theory*, or *one-sorted algebraic theory* is a category L, with $L_0 = \{0, 1, \dots\}$, such that $n = 1^n$, the n-fold product.

Lemma 34. *There is an equivalence between abstract clones and Lawvere theories.*

Proof. Let C be an abstract clone. We construct a Lawvere theory L as follows: We have objects $L_0 = \{0, 1, \dots\}$ and morphisms $L(m, n) = C_m^n$. The identity morphism is $\mathrm{id}_n = (x_i)_i : L(n, n)$ and for $f : L(l, m)$, $g : L(m, n)$, we have composition

$$f \cdot g = (g_i \bullet f)_i : L(l, n).$$

Lastly, we have product projections $\pi_{n,i} = x_{n,i} : L(n, 1)$ for all $1 \leq i \leq n$.

Conversely, if L is a Lawvere theory, we construct an abstract clone C as follows: We take $C_n = L(n, 1)$. We take the $x_{n,i}$ to be the product projections $\pi_i : L(n, 1)$ and we define the substitution $f \bullet g = \langle g_i \rangle_i \cdot f$ the composite of the product morphism $\langle g_i \rangle_i$ with f. □

2.1. Algebras for Lawvere Theories.

Lemma 35. *Let C be an abstract clone, and L be its associated Lawvere theory by the equivalence given above. A C-algebra is equivalent to an algebra for L.*

Proof. Let A be a C-algebra. We will construct a functor $F : L \to \mathbf{SET}$ as follows: We take $F(n) = A^n$. We define the action on morphisms as

$$F(f)(a) = (f_i \bullet a)_i$$

for $f : L(m, n) = L(m, 1)^n$ and $a : A^m$.

Conversely, let $F : L \to \mathbf{SET}$ be a functor. We take the C-algebra A, with $A = F(1)$ and for $f : C_n = L(n, 1)$ and $a : A^n$, $f \bullet a = F(f)(a)$. □

3. Cartesian Operad

Definition 40. A *cartesian operad* T is a functor $T : F \to \mathbf{SET}$, together with an 'identity' element $\mathrm{id} : T(1)$ and for all $m, n_1, \dots, n_m : \mathbb{N}$, a composition map

$$T(m) \times \prod_i T(n_i) \to T\left(\sum_i n_i\right),$$

written $(f, g_1, \dots, g_m) \mapsto f[g_1, \dots, g_m]$, satisfying some identity, associativity and naturality conditions (see [**Hyl14**], Definition 2.1, for more details).

Remark 21. One can arrive at this concept as a standalone definition, or one can view a cartesian operad as a *cartesian multicategory* with one element. A multicategory is a category in which morphisms have type $C((X_1, X_2, \dots, X_n), Y)$ instead of $C(X, Y)$. A cartesian multicategory is a multicategory in which one can permute the X_i of a morphism and has 'contraction' and 'weakening' operations:

$$C((X_1, \dots, X_i, X_i, \dots, X_n), Y) \to C((X_1, \dots, X_i, \dots, X_n), Y),$$
$$C((X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n), Y) \to C((X_1, \dots, X_{i-1}, X_i, X_{i+1}, \dots, X_n), Y).$$

Lemma 36. *There is an equivalence between abstract clones and cartesian operads.*

Proof. Let C be an abstract clone. We define a cartesian operad T with $T(n) = C_n$ and $T(f)(t) = t \bullet (x_{f(1)}, \dots, x_{f(m)})$ for $f : [\![m]\!] \to [\![n]\!]$ and $t : C_m$. The identity element is $x_{1,1}$ and the composition $f[g_1, \dots, g_n]$, for $f : C_m$ and $g_i : C_{n_i}$, is given by lifting all terms to $C_{\sum_i n_i}$ and then substituting:

$$f[g_1, \dots, g_n] = f \bullet (T(\iota_i)(g_i))$$

for $\iota_i : [\![n_i]\!] \hookrightarrow [\![\sum_i n_i]\!]$ the pairwise disjoint injections.

Conversely, given a cartesian operad T, we construct an abstract clone C with $C_n = T([\![n]\!])$. The variables are $x_{n,i} = \iota_{n,i}(\mathrm{id})$, for $\iota_{n,i} : [\![1]\!] \hookrightarrow [\![n]\!]$ the morphism that sends 1 to i. The substitution $f \bullet (g_i)_i$ for $g_1, \dots, g_m : T(n)$ is given by composing and then identifying some variables:

$$f \bullet (g_i)_i = T(\pi)(f[g_1, \dots, g_m])$$

for $\pi : [\![mn]\!] \to [\![n]\!]$ the function that sends $i+1$ to $(i \mod n) + 1$. □

4. Relative Monad

Definition 41. Let $S : C \to D$ be a functor. A *relative monad* on S is a functor $T : C \to D$, together with a natural transformation $\eta : S \Rightarrow T$ and a 'kleisli extension' $(-)^* : D(S(X), T(Y)) \to D(T(X), T(Y))$, natural in both S and T, such that for all $f : D(S(X), T(Y))$ and $g : D(S(Y), T(Z))$,

$$\eta_X^* = \mathrm{id}_{TX}, \qquad f = \eta_X \cdot f^* \qquad \text{and} \qquad (f \cdot g^*)^* = f^* \cdot g^*.$$

Remark 22. Note that for an adjunction $F \dashv G$, $F \bullet G$ gives a monad. In the same way, there exists a notion of *relative adjunction*, from which we can obtain a relative monad (see [**ACU14**], Theorem 2.10).

Remark 23. Now, there is a result that states: There is an equivalence between abstract clones and relative monads on the embedding $\iota : \mathbf{FinSET} \hookrightarrow \mathbf{SET}$.

Note that the objects of **FinSET** are defined to be sets X, together with a proof that there exists some $n : \mathbb{N}$ and some bijection $f : X \xrightarrow{\sim} [\![n]\!]$. This existence of n and f is given by the propositional truncation $\left\| \sum_{n:\mathbb{N}}, X \xrightarrow{\sim} [\![n]\!] \right\|$.

Classically, this construction starts with "fix, for all $X : \mathbf{FinSET}$, a bijection $f : X \xrightarrow{\sim} [\![n]\!]$". However, since X only provides *mere existence* of such a bijection without choosing one (by the propositional truncation), there is no way to obtain f without using the axiom of choice.

Now, we can partially circumvent this problem by noting that we have a fully faithful and essentially surjective embedding $F \to \mathbf{FinSET}$ (for F a skeleton of finite sets), which induces an adjoint equivalence $[F, \mathbf{SET}] \xrightarrow{\sim} [\mathbf{FinSET}, \mathbf{SET}]$, so we can lift the functor part of a relative monad on $F \to \mathbf{SET}$ to $\mathbf{FinSET} \to \mathbf{SET}$. We can also lift the natural transformation using this equivalence. However, the kleisli extension cannot be lifted using this and we are stuck.

Therefore, we will prove a modified statement:

Lemma 37. *There is an equivalence between abstract clones and relative monads on the embedding* $\iota : F \hookrightarrow \mathbf{SET}$.

Proof. Let C be an abstract clone. We define a relative monad T as follows: We take $T([\![n]\!]) = C_n$. For a morphism $a : F([\![m]\!], [\![n]\!])$. We take $T(a)(f) = f \bullet (x_{a(i)})_i$. We define $\eta_{[\![n]\!]}(i) = x_{n,i}$. Finally, for $g : \mathbf{SET}([\![m]\!], T([\![n]\!]))$, we define $g^*(f) = f \bullet g$.

Conversely, let $(T, \eta, (\cdot)^*)$ be a relative monad on the embedding $\iota : F \hookrightarrow \mathbf{SET}$. We define an abstract clone C with $C_n = T([\![n]\!])$. Substitution is defined as $f \bullet g = g^*(f)$ for $f : C_m$ and $g : C_n^m$ and we have variables $x_{n,i} = \eta_{[\![n]\!]}(i)$. □

5. Monoid in a skew-monoidal category

For details and a general treatment, see [**ACU14**], Section 3 and specifically Theorem 3.4.

Definition 42. Consider the category $[F, \mathbf{SET}]$. Note that we have a functor $\iota : F \hookrightarrow \mathbf{SET}$ and that **SET** has colimits. Therefore, we can define a 'tensor product' on functors $[F, \mathbf{SET}]$ as

$$F \otimes G = G \bullet \mathrm{Lan}_\iota F.$$

Together with the 'unit' ι, this gives $[F, \mathbf{SET}]$ a 'skew-monoidal category' structure.

Definition 43. Given a (skew-)monoidal category $(C, \otimes, I)$, a *monoid* in this category is an object $T : C$, together with a 'multiplication' $\mu : C(T \otimes T, T)$ and a 'unit' morphism $\eta : C(I, T)$ satisfying a couple of laws (see [**ML98**], Section III.6).

Lemma 38. *There exists an equivalence between relative monads on* $\iota : F \hookrightarrow$ $\mathbf{SET}$ *and monoids in the skew-monoidal category* $[F, \mathbf{SET}]$.

Proof. A monoid in $[F, \mathbf{SET}]$ consists of an object $T : [F, \mathbf{SET}]$, together with natural transformations $\mu : T \otimes T \Rightarrow T$ and $\eta : \iota \Longrightarrow T$. We can immediately see the functor T and natural transformation η of the relative monad pop up here. The kleisli extension corresponds to μ; they are related to each other via the properties of the left Kan extension of T. $\square$