Semantics for the λ-calculus

Contents

CHAPTER 1

Definitions

1. Algebraic Theories

DEFINITION 1 (algebraic theory). We define an algebraic theory T to be a sequence of sets T_n indexed over $\mathbb{N}$ with for all $1 \leq i \leq n$ elements ("variables" or "projections") $x_{n,i} : T_n$ (we usually leave n implicit), together with a substitution operation

$$\_ \bullet \_ : T_m \times T_n^m \to T_n$$

for all m, n, such that

$$x_j \bullet g = g_j$$
$$f \bullet (x_{l,i})_i = f$$
$$(f \bullet g) \bullet h = f \bullet (g_i \bullet h)_i$$

for all $1 \leq j \leq l$, $f : T_l$, $g : T_m^l$ and $h : T_n^m$.

DEFINITION 2 (algebraic theory morphism). A morphism F between algebraic theories T and T' is a sequence of functions $F_n : T_n \to T'_n$ (we usually leave the n implicit) such that

$$F_n(x_j) = x_j$$
$$F_n(f \bullet g) = F_m(f) \bullet (F_n(g_i))_i$$

for all $1 \leq j \leq n$, $f : T_m$ and $g : T_n^m$.

REMARK 1. We can construct binary products of algebraic theories, with sets $(T \times T')_n = T_n \times T'_n$, variables (x_i, x_i) and substitution

$$(f, f') \bullet (g, g') = (f \bullet g, f' \bullet g').$$

In the same way, the category of algebraic theories has all limits.

2. Algebras

DEFINITION 3 (algebra). An algebra A for an algebraic theory T is a set A, together with an action

$$\bullet : T_n \times A^n \to A$$

for all n, such that

$$x_j \bullet a = a_j$$
$$(f \bullet g) \bullet a = f \bullet (g_i \bullet a)_i$$

for all j, $f : T_m$, $g : T_n^m$ and $a : A^n$.

DEFINITION 4 (algebra morphism). For an algebraic theory T, a morphism F between T-algebras A and A' is a function $F : A \to A$ such that

$$F(f \bullet a) = f \bullet (F(a_i))_i$$

for all $f : T_n$ and $a : A^n$.

Remark 2. The category of algebras has all limits. The set of a limit of algebras is the limit of the underlying sets.

Remark 3. Note that for an algebraic theory T, the T_n are all algebras for T, with the action given by $\bullet$.

3. Presheaves

Definition 5 (presheaf). A presheaf P for an algebraic theory T is a sequence of sets P_n indexed over $\mathbb{N}$, together with an action

$$\bullet : P_m \times T_n^m \to P_n$$

for all m, n, such that

$$t \bullet (x_{l,i})_i = t$$
$$(t \bullet f) \bullet g = t \bullet (f_i \bullet g)_i$$

for all $t : P_l$, $f : T_m^l$ and $g : T_n^m$.

Definition 6 (presheaf morphism). For an algebraic theory T, a morphism F between T-presheaves P and P' is a sequence of functions $F_n : P_n \to P'_n$ such that

$$F_n(t \bullet f) = F_m(t) \bullet f$$

for all $t : P_m$ and $f : T_n^m$.

We will write PT for the category of T-presheaves and their morphisms.

Remark 4. The category of presheaves has all limits. The nth set $\overline{P}_n$ of a limit $\overline{P}$ of presheaves P_i is the limit of the nth sets $P_{i,n}$ of the presheaves in the limit diagram.

4. λ-theories

Definition 7 (λ-theory). A λ-theory is an algebraic theory L, together with sequences of functions $\lambda_n : L_{n+1} \to L_n$ and $\rho_n : L_n \to L_{n+1}$, such that

$$\lambda_m(f) \bullet h = \lambda_n(f \bullet (h_1, \dots, h_m, x_{n+1}))$$
$$\rho_n(g \bullet h) = \rho_m(g) \bullet (h_1, \dots, h_m, x_{n+1})$$

for all $f : L_{m+1}$, $g : L_m$ and $h : L_n^m$.

Definition 8 (β- and η-equality). We say that a λ-theory L satisfies β-equality (or that it is a λ-theory with β) if $\rho_n \circ \lambda_n = \mathrm{id}_{L_n}$ for all n. We say that is satisfies η-equality if $\lambda_n \circ \rho_n = \mathrm{id}_{L_{n+1}}$ for all n.

Definition 9 (λ-theory morphism). A morphism F between λ-theories L and L' is an algebraic theory morphism F such that

$$F_n(\lambda_n(f)) = \lambda_n(F_{n+1}(f))$$
$$\rho_n(F_n(g)) = F_{n+1}(\rho_n(g))$$

for all $f : L_{n+1}$ and $g : L_n$.

Remark 5. The category of lambda theories has all limits, with the underlying algebraic theory of a limit being the limit of the underlying algebraic theories.

A λ-theory algebra or presheaf is a presheaf for the underlying algebraic theory.

5. Alternate definitions

Definition 10. Lawvere theory: **(TODO)**

Definition 11. Relative monad: **(TODO)**

Definition 12. Abstract clone: **(TODO)**

Definition 13. Cartesian Operad: **(TODO)**

(https://ncatlab.org/nlab/show/lambda+theory)

CHAPTER 2

Category Theoretic Preliminaries

I will assume a familiarity with the category-theoretical concepts presented in [**AW23**]. These include categories, functors, isomorphisms, natural transformations, adjunctions, equivalences and limits.

1. Notation

For an object c in a category C, I will write $c : C$.

For a morphism f between objects c and c' in a category C, I will write $f : C(c, c')$ or $f : c \to c'$.

For composition of morphisms $f : C(c, d)$ and $g : C(d, e)$, I will write $f \cdot g$.

For composition of functors $F : A \to B$ and $G : B \to C$, I will write $F \bullet G$.

2. Adjunctions

An adjoint equivalence of categories has multiple definitions. The one we will use here is the following:

Definition 14. An adjoint equivalence between categories C and D is a pair of adjoint functors

$$\begin{tikzcd} D \arrow[r, bend right, "R"'] \arrow[r, phantom, "\perp"] & C \arrow[l, bend right, "L"'] \end{tikzcd}$$

such that the unit $\eta : \mathrm{id}_C \Rightarrow L \bullet R$ and counit $\epsilon : R \bullet L \Rightarrow \mathrm{id}_D$ are isomorphisms of functors.

3. Kan Extensions

One of the most general and abstract concepts in category theory is the concept of *Kan extensions.* In [**ML98**], Section X.7, MacLane notes that

> The notion of Kan extensions subsumes all the other fundamental concepts of category theory.

In this thesis, we will use left Kan extension a handful of times. It comes in handy when we want to extend a functor along another functor in the following way:

Let A, B and C be categories and let $F : A \to B$ be a functor.

Definition 15. Precomposition gives a functor between functor categories $F_* : [B, C] \to [A, C]$. If F_* has a left adjoint, we will denote call this adjoint functor the *left Kan extension* along F and denote it $\mathrm{Lan}_F : [A, C] \to [B, C]$.

$$\begin{tikzcd} A \arrow[rr, "F"] \arrow[dr, dashed, "F_* G"'] & & B \arrow[dl, "G"] \\ & C & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rr, "F"] \arrow[dr, "G"'] & & B \arrow[dl, dashed, "\mathrm{Lan}_F G"] \\ & C & \end{tikzcd}$$

Analogously, when F_* has a right adjoint, one calls this the *right Kan extension* along F and denote it $\mathrm{Ran}_F : [A, C] \to [B, C]$.

If a category has limits (resp. colimits), we can construct the right (resp. left) Kan extension in a ‘pointwise’ fashion (see Theorem X.3.1 in [**ML98**] or Theorem 2.3.3 in [**KS06**]). Below, I will outline the parts of the construction that we will need explicitly in this thesis.

LEMMA 1. *If C has colimits, Lan_F exists.*

PROOF. First of all, for objects $b : B$, we take

$$\mathrm{Lan}_F G(b) := \mathrm{colim}\left((F \downarrow b) \to A \xrightarrow{G} C\right).$$

Here, $(F \downarrow b)$ denotes the comma category with as objects the morphisms $B(F(a), b)$ for all $a : A$, and as morphisms from $f : B(F(a), b)$ to $f' : B(F(a'), b)$ the morphisms $g : A(a, a')$ that make the diagram commute:

$$\begin{tikzcd} F(a) \arrow[rr, "F(g)"] \arrow[dr, "f'"'] & & F(a') \arrow[dl, "f'"] \\ & b & \end{tikzcd}$$

and $(F \downarrow b) \to A$ denotes the projection functor that sends $f : B(F(a_1), b)$ to a_1.

Now, a morphism $h : B(b, b')$ gives a morphism of diagrams, sending the $F(a)$ corresponding to $f : B(G(a), b)$ to the $F(a)$ corresponding to $f \cdot h : B(G(a), b')$. From this, we get a morphism $\mathrm{Lan}_F G(h) : C(\mathrm{Lan}_F G(b), \mathrm{Lan}_F G(b'))$.

The unit of the adjunction is a natural transformation $\eta : \mathrm{id}_{[A,C]} \Rightarrow \mathrm{Lan}_F \bullet F_*$. We will define this pointwise, for $G : [A, C]$ and $a : A$. Our diagram contains the $G(a)$ corresponding to $\mathrm{id}_{F(a)} : (F \downarrow F(a))$ and the colimit cocone gives a morphism

$$\eta_G(a) : C(G(a), \mathrm{Lan}_F G(F(a))),$$

the latter being equal to $(\mathrm{Lan}_F \bullet F_*)(G)(a)$.

The counit of the adjunction is a natural transformation $\epsilon : F_* \bullet \mathrm{Lan}_F \Rightarrow \mathrm{id}_{[B,C]}$. We will also define this pointwise, for $G : [B, C]$ and $b : B$. The diagram for $\mathrm{Lan}_F(F_* G)(b)$ consists of $G(F(a))$ for all $f : B(F(a), b)$. Then, by the universal property of the colimit, the morphisms $G(f) : C(G(F(a)), G(b))$ induce a morphism

$$\epsilon_G(b) : C(\mathrm{Lan}_F(F_* G)(b), G(b)).$$

□

LEMMA 2. *If $F : A \to B$ is a fully faithful functor, and C is a category with colimits, η is a natural isomorphism.*

PROOF. To show that η is a natural isomorphism, we have to show that $\eta_G(a') : G(a') \Rightarrow \mathrm{Lan}_F G(F(a'))$ is an isomorphism for all $G : [A, C]$ and $a' : A$. Since a left adjoint is unique up to natural isomorphism, we can assume that $\mathrm{Lan}_F G(F(a'))$ is given by

$$\mathrm{colim}((F \downarrow F(a')) \to A \xrightarrow{G} C).$$

Now, the diagram for this colimit consists of $G(a)$ for each arrow $f : B(F(a), F(a'))$. Since F is fully faithful, we have $f = F(\overline{f})$ for some $\overline{f} : A(a, a')$. If we now take the arrows $G(\overline{f}) : C(G(a), G(a'))$, the universal property of the colimit gives an arrow

$$\varphi : C(\mathrm{Lan}_F G(F(a')), G(a'))$$

which constitutes an inverse to $\eta_G(a')$. □

REMARK 6. In the same way, if C has limits, ϵ is a natural isomorphism.

COROLLARY 1. *If C has limits or colimits, precomposition of functors $[B, C]$ along a fully faithful functor is (split) essentially surjective.*

PROOF. For each $G : [A, C]$ we take $\mathrm{Lan}_F G : [B, C]$, and we have $F_*(\mathrm{Lan}_F G) \cong G$. $\square$

COROLLARY 2. *If C has colimits (resp. limits), left (resp. right) Kan extension of functors $[A, C]$ along a fully faithful functor is fully faithful.*

PROOF. Since left Kan extension along F is the left adjoint to precomposition, we have

$$[A, C](\mathrm{Lan}_F G, \mathrm{Lan}_F G') \cong [B, C](G, F_*(\mathrm{Lan}_F G')) \cong [B, C](G, G').$$

$\square$

4. The Karoubi envelope

Let C be a category. If we have a retraction-section pair $c \underset{s}{\overset{r}{\rightleftarrows}} d$ we have (by definition) $s{\cdot}r = \mathrm{id}_d$. On the other hand, $s{\cdot}r : c \to c$ is an idempotent morphism. Conversely, we can wonder whether for any idempotent morphism $a : c \to c$, we can find a retraction-section pair $r : c \to d$ and $s : d \to c$ such that $a = r \cdot s$. If this is the case, we say that the idempotent a *splits*. If a does not split, we can wonder whether we can find an embedding $\iota_C : C \hookrightarrow \overline{C}$ such that the idempotent $\iota_C(a) : \iota_C(c) \to \iota_C(c)$ does split.

DEFINITION 16. We define the category $\overline{C}$. The objects of $\overline{C}$ are tuples (c, a) with $c : C$, $a : C(c, c)$ such that $a \cdot a = a$. The morphisms between (c, a) and (d, b) are morphisms $f : C(a, b)$ such that $a \cdot f \cdot b = f$. The identity morphism on (c, a) is given by a and $\overline{C}$ inherits morphism composition from C.

This category is called the *Karoubi Envelope*, the *idempotent completion*, the *category of retracts*, or the *Cauchy completion* of C.

REMARK 7. Note that for a morphism $f : \overline{C}((c, a), (d, b))$,

$$a \cdot f = a \cdot a \cdot f \cdot b = a \cdot f \cdot b = f$$

and in the same way, $f \cdot b = f$.

DEFINITION 17. We have an embedding $\iota_C : C \to \overline{C}$, sending $c : C$ to (c, id_c) and $f : C(c, d)$ to f.

LEMMA 3. *Every object $c : \overline{C}$ is a retract of $\iota_C(c_0)$ for some $c_0 : C$.*

PROOF. Note that $c = (c_0, a)$ for some $c_0 : C$ and an idempotent $a : c \to c$. We have morphisms $\iota_C(c) \underset{a_\leftarrow}{\overset{a_\rightarrow}{\rightleftarrows}} (c, a)$, both given by a. We have $a_\leftarrow \cdot a_\rightarrow = a = \mathrm{id}_{(c,a)}$, so (c, a) is a retract of $\iota_C(c)$. $\square$

LEMMA 4. *Every idempotent splits in $\overline{C}$.*

PROOF. Take an idempotent $e : \overline{C}(c, c)$. Note that c is given by an object $c_0 : C$ and an idempotent $a : C(c_0, c_0)$. Also, e is given by some idempotent $e : C(c_0, c_0)$ with $a \cdot e \cdot a = e$.

Now, we have $(c_0, e) : \overline{C}$ and morphisms $(c_0, a) \underset{e_\leftarrow}{\overset{e_\rightarrow}{\rightleftarrows}} (c_0, e)$, both given by e. We have $e_\leftarrow \cdot e_\rightarrow = e = \mathrm{id}_{(c_0,e)}$, so (c_0, e) is a retract of (c_0, a). Also, $e = e_\rightarrow \cdot e_\leftarrow$, so e is split. $\square$

REMARK 8. Note that the embedding is fully faithful, since

$$\overline{C}((c, \mathrm{id}_c), (d, \mathrm{id}_d)) = \{f : C(c, d) \mid \mathrm{id}_c \cdot f \cdot \mathrm{id}_d = f\} = C(c, d).$$

REMARK 9. Let D be a category. Suppose that we have a retraction-section pair in D, given by $\begin{tikzcd} d \arrow[r, shift left, "r"] & d' \arrow[l, shift left, "s"] \end{tikzcd}$. Now, suppose that we have an object $c : D$ and a morphism f with $(r \cdot s) \cdot f = f$. Then we get a morphism $s \cdot f : d' \to c$ such that f factors as $r \cdot (s \cdot f)$. Also, for any g with $r \cdot g = f$, we have

$$g = s \cdot r \cdot g = s \cdot f.$$

$$\begin{tikzcd} d \arrow[r, "r"] \arrow[dr, "f"'] & d' \arrow[r, "s"] \arrow[d, dashed, "s \cdot f"] & d \arrow[dl, "f"] \\ & c & \end{tikzcd}.$$

Therefore, d' is the equalizer of $\begin{tikzcd} d \arrow[r, shift left, "\mathrm{id}_d"] \arrow[r, shift right, "r \cdot s"'] & d \end{tikzcd}$. In the same way, it is also the coequalizer of this diagram.

Now, note that if we have a coequalizer c' of id_c and a, and an equalizer d' of id_d and b, the universal properties of these give an equivalence

$$D(c', d') \cong \{f : D(c, d') \mid a \cdot f = f\} \cong \{f : D(c, d) \mid a \cdot f = f = f \cdot b\}.$$

$$\begin{tikzcd} c \arrow[r, shift left, "\mathrm{id}_c"] \arrow[r, shift right, "a"'] & c \arrow[r] \arrow[d] \arrow[dr] & c' \arrow[d] \\ d & d \arrow[l, shift right, "\mathrm{id}_d"'] \arrow[l, shift left, "b"] & d' \arrow[l] \end{tikzcd}$$

Since a functor preserves retracts, and since every object of $\overline{C}$ is a retract of an object in C, one can lift a functor from C (to a category with (co)equalizers) to a functor on $\overline{C}$.

For convenience, the lemma below works with pointwise left Kan extension using colimits, but one could also prove this using just (co)equalizers (or right Kan extension using limits).

LEMMA 5. *Let D be a category with colimits. We have an adjoint equivalence between $[C, D]$ and $[\overline{C}, D]$.*

PROOF. We already have an adjunction $\mathrm{Lan}_{\iota_C} \dashv \iota_{C*}$. Also, since ι_C is fully faithful, we know that η is a natural isomorphism. Therefore, we only have to show that ϵ is a natural isomorphism. That is, we need to show that $\epsilon_G(c, a) : D(\mathrm{Lan}_{\iota_C}(\iota_{C*}G)(c, a), G(c, a))$ is an isomorphism for all $G : [\overline{C}, D]$ and $(c, a) : \overline{C}$.

One of the components in the diagram of $\mathrm{Lan}_{\iota_C}(\iota_{C*}G)(c, a)$ is the $\iota_{C*}G(c) = G(c, \mathrm{id}_c)$ corresponding to $a : \iota_C(c) \to (c, a)$. This component has a morphism into our colimit

$$\varphi : C(G(\iota_C(c)), \mathrm{Lan}_{\iota_C}(\iota_{C*}G)(c, a)).$$

Note that we can view a as a morphism $a : \overline{C}((c, a), \iota_C(c))$. This gives us our inverse morphism

$$G(a) \cdot \varphi : C(G(c, a), \mathrm{Lan}_{\iota_C}(\iota_{C*}G)(c, a)).$$

$\square$

LEMMA 6. *The formation of the opposite category commutes with the formation of the Karoubi envelope.*

PROOF. An object in $\overline{C^{\mathrm{op}}}$ is an object $c : C^{\mathrm{op}}$ (which is just an object $c : C$), together with an idempotent morphism $a : C^{\mathrm{op}}(c, c) = C(c, c)$. This is the same as an object in $\overline{C}^{\mathrm{op}}$.

A morphism in $\overline{C^{\mathrm{op}}}((c, a), (d, b))$ is a morphism $f : C^{\mathrm{op}}(c, d) = C(d, c)$ such that

$$b \cdot_C f \cdot_C a = a \cdot_{C^{\mathrm{op}}} f \cdot_{C^{\mathrm{op}}} b = f.$$

A morphism in $\overline{C}^{\mathrm{op}}((c,a),(d,b)) = \overline{C}((d,b),(c,a))$ is a morphism $f : C(d,c)$ such that $b \cdot f \cdot a = f$.

Now, in both categories, the identity morphism on (c,a) is given by a.

Lastly, $\overline{C^{\mathrm{op}}}$ inherits morphism composition from C^{op}, which is the opposite of composition in C. On the other hand, composition in $\overline{C}^{\mathrm{op}}$ is the opposite of composition in $\overline{C}$, which inherits composition from C. □

Corollary 3. *As the category* **SET** *is cocomplete, we have an equivalence between the category of presheaves on C and the category of presheaves on $\overline{C}$.*

5. Monoids as categories

Take a monoid M.

Definition 18. We can construct a category C_M with $C_{M0} = \{\star\}$, $C_M(\star,\star) = M$. The identity morphism on $\star$ is the identity $1 : M$. The composition is given by multiplication $g \cdot_{C_M} f = f \cdot_M g$.

Remark 10. An isomorphism of monoids gives an (adjoint) equivalence of categories.

Definition 19. A *right monoid action* of M on a set X is a function $X \times M \to X$ such that for all $x : X$, $m, m' : M$,

$$x1 = x \qquad \text{and} \qquad (xm)m' = x(m \cdot m').$$

Definition 20. A *morphism* between sets X and Y with a right M- action is an M-equivariant function $f : X \to Y$: a function such that $f(xm) = f(x)m$ for all $x : X$ and $m : M$.

Lemma 7. *Presheaves on C_M are equivalent to sets with a right M-action.*

Proof. This correspondence sends a presheaf F to the set $F(\star)$, and conversely, the set X to the presheaf F given by $F(\star) := X$. The M-action corresponds to the presheaf acting on morphisms as $xm = F(m)(x)$. A morphism (natural transformation) between presheaves $F \Rightarrow G$ corresponds to a function $F(\star) \to G(\star)$ that is M-equivariant, which is exactly a monoid action morphism. □

Definition 21. We can view M as a set U_M with right M-action $mn = m \cdot_M n$ for $m : U_M$ and $n : M$.

Remark 11. Since the category of sets with an M-action is equivalent to a presheaf category, it has all limits. However, we can make this concrete. The set of the product $\prod_i X_i$ is the product of the underlying sets. The action is given pointwise by $(x_i)_i m = (x_i m)_i$.

Definition 22. Given an object c in a category C with terminal object t. The global elements of c are the morphisms $C(t,c)$.

Note that the initial set with M-action is $\{\star\}$, with action $\star m = \star$.

Lemma 8. *The global elements of a set with right M-action correspond to the elements that are invariant under the M-action.*

Proof. A global element of X is a morphism $\varphi : \{\star\} \to X$ such that for all $m : M$, $\varphi(\star)m = \varphi(\star m) = \varphi(\star)$. Therefore, it is given precisely by the element $\varphi(\star) : X$, which must be invariant under the M-action. □

Lemma 9. *The category C of sets with an M-action has exponentials.*

Proof. Given sets with M-action X and Y. Consider the set $C(M \times X, Y)$ with an M-action given by $\phi m'(m,x) = \phi(m'm, x)$. This is the exponential object X^Y, with the evaluation morphism $X \times X^Y \to Y$ given by $(x,\phi) \mapsto \phi(1,x)$. □

5.1. Extension and restriction of scalars. Let $\varphi : M \to M'$ be a morphism of monoids.

LEMMA 10. *We get a restriction of scalars functor φ_* from sets with a right M'-action to sets with a right M-action.*

PROOF. Given a set X with right M'-action, take the set X again, and give it a right M-action, sending (x, m) to $x\varphi(m)$.

On morphisms, send an M'-equivariant morphism $f : X \to X'$ to the M-equivariant morphism $f : X \to X'$. □

Since **SET** has colimits, and restriction of scalars corresponds to precomposition of presheaves (on $C_{M'}$), we can give it a left adjoint. This is the (pointwise) left Kan extension, which boils down to:

LEMMA 11. *We get an extension of scalars functor φ^* from sets with a right M-action to sets with a right M'-action.*

PROOF. Given a set X with right M-action. Take $Y = X \times M' / \sim$ with the relation $(xm, m') \sim (x, f(m) \cdot m')$ for $m : M$. This has a right M'-action given by $(x, m')n' = (x, m'n')$.

On morphisms, it sends $f : X \to X'$ to the morphism $(x, m') \mapsto (f(x), m')$. □

LEMMA 12. *For U_M the set M with right M-action, we have $\varphi^*(U_M) \cong U_{M'}$.*

PROOF. The proof relies on the fact that for all $m : U_M$ and $m' : M'$, we have

$$(m, m') \sim (1, \varphi(m)m').$$

□

Consider the category D with $D_0 = M'$ and

$$D(m', \overline{m}') = \{m : M \mid \varphi(m) \cdot m' = \overline{m}'\}.$$

If a category has an object t, such that there is a morphism to it from every other object in the category, t is said to be *weakly terminal.*

LEMMA 13. *Suppose that D has a weakly terminal element. Then for I_M the terminal set with right M-action, we have $\varphi^*(I_M) \cong I_{M'}$.*

PROOF. If D has a weakly terminal object, there exists $\overline{m}' : M'$ such that for all $m' : M'$, there exists $m : M$ such that $\varphi(m) \cdot m' = \overline{m}'$.

The proof relies on the fact that every element of $\varphi^*(I_M)$ is given by some $(\star, m')$, but then

$$(\star, m') = (\star \cdot m, m') \sim (\star, \varphi(m) \cdot m') = (\star, \overline{m'}),$$

so $\varphi^*(I_M)$ has exactly 1 element. □

REMARK 12. For φ^* to preserve terminal objects, we actually only need D to be connected. The fact that $\varphi^*(I_M)$ is a quotient by a symmetric and transitive relation then allows us to 'walk' from any $(\star, m_1')$ to any other $(\star, m_2')$ in small steps.

For any $m_1', m_2' : M'$, consider the category $D_{m_1', m_2'}$, given by

$$D_{m_1', m_2', 0} = \{(m', m_1, m_2) : M' \times M \times M \mid m_i' = \varphi(m_i) \cdot m'\}$$

and

$$D_{m_1', m_2'}((m', m_1, m_2), (\overline{m}', \overline{m}_1, \overline{m}_2)) = \{m : M \mid \varphi(m) \cdot m' = \overline{m}', m_i = \overline{m}_i \cdot m\}.$$

LEMMA 14. *Suppose that $D_{m'_1,m'_2}$ has a weakly terminal object for all $m'_1, m'_2 : M'$. Then for sets A and B with right M-action, we have $\varphi^*(A \times B) \cong \varphi^*(A) \times \varphi^*(B)$.*

PROOF. Now, any element in $\varphi^*(A) \times \varphi^*(B) = (A \times M'/\sim) \times (B \times M'/\sim)$ is given by some (a, m'_1, b, m'_2).

The fact that $D_{m'_1,m'_2}$ has a weakly terminal object means that we have some $\overline{m}' : M'$ and $\overline{m}_1, \overline{m}_2 : M$ with $m'_i = \varphi(\overline{m}_i) \cdot \overline{m}'$, such that for all $m' : M'$ and $m_1, m_2 : M$ with $m'_i = \varphi(m_i) \cdot m'$, there exists $m : M$ such that $\varphi(m) \cdot m' = \overline{m}'$ and $m_i = \overline{m}_i \cdot m$.

Therefore,

$$(a, m'_1, b, m'_2) = (a, \varphi(\overline{m}_1) \cdot \overline{m}', b, \varphi(\overline{m}_2) \cdot \overline{m}') \sim (a\overline{m}_1, \overline{m}', b\overline{m}_2, \overline{m}'),$$

so this is equivalent to some element in $\varphi^*(A \times B) = (A \times B \times M'/\sim)$.

The second part of weak terminality means that this equivalence is actually well-defined: equivalent elements in $\varphi^*(A) \times \varphi^*(B)$ are sent to equivalent elements in $\varphi^*(A \times B)$.

(**(TODO)** Work this out more?) □

CHAPTER 3

Lemmas

1. The endomorphism theory

Definition 23 (Endomorphism theory). Suppose that we have a category C and an object $X : C$, such that all powers X^n of X are also in C. The endomorphism theory $E(X)$ of X is the algebraic theory given by $E(X)_n = C(X^n, X)$ with projections as variables $x_{n,i} : X^n \to X$ and a substitution that sends $f : X^m \to X$ and $g_1, \dots, g_m : X^n \to X$ to $f \circ \langle g_i \rangle_i : X^n \to X^m \to X$.

Lemma 15. *$E(X)$ is indeed an algebraic theory.*

Proof. For $1 \leq j \leq l$, $f : E(X)_l$, $g : {E(X)_m}^l$ and $h : {E(X)_n}^m$, we have

$$x_j \bullet g = x_j \circ \langle g_i \rangle_i = g_j,$$
$$f \bullet (x_{l,i})_i = f \circ \langle x_{l,i} \rangle_i = f \circ \mathrm{id}_{X^l} = f$$

and

$$(f \bullet g) \bullet h = f \circ \langle g_i \rangle_i \circ \langle h_i \rangle_i = f \circ \langle g_i \circ \langle h_{i'} \rangle_{i'} \rangle_i = f \bullet (g_i \bullet h)_i.$$

□

Definition 24 (Endomorphism λ-theory). Now, suppose that the exponential object X^X exists, and that we have morphisms back and forth $abs : X^X \to X$ and $app : X \to X^X$. Let, for $Y : C$, φ_Y be the isomorphism $C(X \times Y, X) \xrightarrow{\sim} C(Y, X^X)$. We can give $E(X)$ a λ-theory structure by setting, for $f : E(X)_{n+1}$ and $g : E(X)_n$,

$$\lambda(f) = abs \circ \varphi_{X^n}(f) \qquad \rho(g) = \varphi_{X^n}^{-1}(app \circ g).$$

Lemma 16. *$E(X)$ is indeed a λ-theory.*

Proof. Note that $\varphi : C(- \times X, X) \xrightarrow{\sim} C(-, X^X)$ is a natural isomorphism, so for $g : {E(X)_n}^m$, the following diagram commutes

$$\begin{tikzcd}
C(X^m \times X, X) \arrow[r, "- \circ (\langle g_i \rangle_i \times \mathrm{id}_X)"] \arrow[d, shift left, "\varphi_{X^m}"] & C(X^n \times X, X^X) \arrow[d, shift left, "\varphi_{X^n}"] \\
C(X^m, X^X) \arrow[r, "- \circ \langle g_i \rangle_i"] \arrow[u, shift left, "\varphi_{X^m}^{-1}"] & C(X^n, X^X) \arrow[u, shift left, "\varphi_{X^n}^{-1}"]
\end{tikzcd}$$

and note that $\langle g_i \rangle_i \times \mathrm{id}_X = \langle g_1, \dots, g_m, x_{n+1} \rangle$. Then we have, for all $f : E(X)_m$

$$\begin{aligned}
\lambda_m(f) \bullet g &= abs \circ \varphi_{X^m}(f) \circ \langle g_i \rangle_i \\
&= abs \circ \varphi_{X^n}(f \circ \langle g_1, \dots, g_m, x_{n+1} \rangle) \\
&= \lambda_n(f \bullet (g_1, \dots, g_m, x_{n+1}))
\end{aligned}$$

and

$$\begin{aligned}
\rho_n(f \bullet g) &= \varphi_{X^n}^{-1}(app \circ f \circ \langle g_i \rangle_i) \\
&= \varphi_{X^m}^{-1}(app \circ f) \circ \langle g_1, \dots, g_m, x_{n+1} \rangle \\
&= \rho_m(f) \bullet (g_1, \dots, g_m, x_{n+1}).
\end{aligned}$$

□

2. The theory presheaf

DEFINITION 25 (The theory presheaf). Let T be an algebraic theory. We can turn T into an T-presheaf $\tilde{T}$ by setting $\tilde{T}_n = T_n$ and using the substitution from T:

$$\bullet : \tilde{T}_m \times T_n^m \to \tilde{T}_n.$$

LEMMA 17. *$\tilde{T}$ is indeed a presheaf.*

PROOF. For all $t : \tilde{T}_l$, $f : T_m^l$ and $g : T_n^m$,

$$t \bullet (x_{l,i})_i = t$$

and

$$(t \bullet f) \bullet g = t \bullet (f_i \bullet g)_i$$

because T is an algebraic theory. □

LEMMA 18. *Given an algebraic theory T and a T-presheaf Q, we have for all n a bijection of sets*

$$\varphi : PT(\tilde{T}^n, Q) \cong Q_n.$$

PROOF. Take $\varphi(f) = f_n(x_1, \dots, x_n)$.

Conversely, take $\varphi^{-1}(q)$ to be the presheaf morphism that sends $t : T_m^n$ to $q \bullet t : Q_m$. This is indeed a presheaf morphism, since for all $t : T_l^n$ and $f : T_m^l$,

$$\varphi^{-1}(q)(t \bullet f) = q \bullet t \bullet f = \varphi^{-1}(q)(t) \bullet f.$$

Now, for a presheaf morphism $f : T^n \to Q$ and $t : T_m^n$, we have

$$\varphi^{-1}(\varphi(f))(t) = f_n(x_1, \dots, x_n) \bullet t = f_n((x_1, \dots, x_n) \bullet t) = f_n(t_1, \dots, t_n) = f_n(t).$$

Conversely, given $q : Q_n$, we have

$$\varphi(\varphi^{-1}(q)) = q \bullet (x_1, \dots, x_n) = q.$$

which concludes the proof. □

3. The '+l' presheaf

Let $\iota_{m,n} : T_m \to T_{m+n}$ denote the function that sends f to $f \bullet (x_{m+n,1}, \dots, x_{m+n,m})$. Note that

$$\iota_{m,n}(f) \bullet g = f \bullet (g_i)_{i \leq m}$$

and

$$\iota_{m,n}(f \bullet g) = f \bullet g \bullet (x_i)_i = f \bullet (g_i \bullet (x_j)_j)_i = f \bullet (\iota_{m,n}(g_i))_i.$$

For tuples $x : X^m$ and $y : X^n$, let $x + y$ denote the tuple $(x_1, \dots, x_m, y_1, \dots, y_n) : X^{m+n}$.

DEFINITION 26 (The '+l' presheaf). Given a T-presheaf Q, we can construct a presheaf $A(Q, l)$, given by $A(Q, l)_n = Q_{n+l}$. Then, for $q : A(Q, l)_m$ and $f : T_n^m$, the substitution is given by

$$q \bullet_{A(Q,l)} f = q \bullet_Q ((\iota_{n,l}(f_i))_i + (x_{n+i})_i)$$

LEMMA 19. *The $+l$ presheaf is a presheaf*

PROOF. We have, for $q : A(Q, l)_n$,

$$\begin{aligned} q \bullet_{A(Q,l)} (x_i)_i &= q \bullet_Q ((\iota_{n,l}(x_i))_i + (x_{n+i})_i) \\ &= q \bullet_Q ((x_i)_i + (x_{n+i})_i) \\ &= q \bullet_Q (x_i)_i \\ &= q. \end{aligned}$$

We have, for $q : A(Q,k)_l$, $f : T^l_m$ and $g : T^m_n$,

$$\begin{aligned} q \bullet_{A(Q,k)} f \bullet_{A(Q,k)} g &= q \bullet_Q ((\iota_{m,l}(f_i))_i + (x_{m+i})_i) \bullet_Q ((\iota_{n,l}(g_i))_i + (x_{n+i})_i) \\ &= q \bullet_Q (((\iota_{m,l}(f_i) \bullet_T ((\iota_{n,l}(g_j))_j + (x_{n+j})_j))_i + (x_{m+i} \bullet_T ((\iota_{n,l}(g_j))_j + (x_{n+j})_j))_i)) \\ &= q \bullet_Q ((f_i \bullet_T (\iota_{n,l}(g_j))_j)_i + (x_{n+i})_i) \\ &= q \bullet_Q ((\iota_{n,l}(f_i \bullet_T g))_i + (x_{n+i})_i) \\ &= q \bullet_{A(Q,k)} (f_i \bullet_T g). \end{aligned}$$

□

4. Exponentiability of the theory presheaf

LEMMA 20. *For all l, the presheaf $\tilde{T}^l$ is exponentiable.*

PROOF. We will show that $A(-,l)$ constitutes a right adjoint to the functor $- \times \tilde{T}^l$. We will do this using universal arrows ([**ML98**], Chapter IV.1, Theorem 2 (iv)). To that end, we will need for all $Q : PT$ a universal arrow $\varphi : A(Q,l) \times \tilde{T}^l \to Q$.

For $q : A(Q,l)_n = Q_{n+l}$ and $t : \tilde{T}^l_n$, we take $\varphi(q,t) = q \bullet_Q ((x_{n,i})_i + t)$.

This is a presheaf morphism, since for all $q : A(Q,l)^l_m$, $t : \tilde{T}^l_m$ and $f : T^m_n$,

$$\begin{aligned} \varphi((q,t) \bullet_{A(Q,l)\times\tilde{T}^l} f) &= \varphi(q \bullet_{A(Q,l)} f, t \bullet_{\tilde{T}^l} f) \\ &= q \bullet_{A(Q,l)} f \bullet_Q ((x_i)_i + (t \bullet_{\tilde{T}^l} f)) \\ &= q \bullet_Q ((\iota_{n,l}(f_i))_i + (x_{n+i})_i) \bullet_Q ((x_i)_i + (t \bullet_{\tilde{T}^l} f)) \\ &= q \bullet_Q ((\iota_{n,l}(f_i) \bullet_T ((x_j)_j + (t \bullet_{\tilde{T}^l} f)))_i + (x_{n+i} \bullet_T ((x_j)_j + (t \bullet_{\tilde{T}^l} f)))_i) \\ &= q \bullet_Q ((f_i \bullet_T (x_j)_j)_i + ((t \bullet_{\tilde{T}^l} f)_i)_i) \\ &= q \bullet_Q ((f_i)_i + (t_i \bullet_{\tilde{T}} f)_i) \\ &= q \bullet_Q ((x_i \bullet_T f)_i + (t_i \bullet_T f)_i) \\ &= q \bullet_Q ((x_i)_i + t) \bullet_Q f \\ &= \varphi(q,t) \bullet_Q f. \end{aligned}$$

Now, given any presheaf $Q' : PT$ we need to show that any morphism $\psi : Q' \times \tilde{T}^l \to Q$ factors uniquely as $\varphi \circ (\tilde{\psi} \times \mathrm{id}_{\tilde{T}^l})$ for some $\tilde{\psi} : Q' \to A(Q,l)$.

So, given such a ψ, and given $q : Q'_n$, we take $\tilde{\psi}(q) = \psi(\iota_{n,l}(q), (x_{n+i})_i)$

This is a presheaf morphism, since for all $q : Q'_m$ and $f : T^m_n$,

$$\begin{aligned} \tilde{\psi}(q \bullet f) &= \psi(\iota_{n,l}(q \bullet f), (x_{n+i})_i) \\ &= \psi(q \bullet (\iota_{n,l}(f_i))_i, (x_{n+i})_i) \\ &= \psi((\iota_{m,l}(q), (x_{m+i})_i) \bullet_{Q'\times\tilde{T}^l} ((\iota_{n,l}(f_i))_i + (x_{n+i})_i)) \\ &= \psi(\iota_{m,l}(q), (x_{m+i})_i) \bullet_Q ((\iota_{n,l}(f_i))_i + (x_{n+i})_i) \\ &= \tilde{\psi}(q) \bullet_{A(Q,l)} f. \end{aligned}$$

Note that indeed $\varphi \circ (\tilde{\psi} \times \mathrm{id}_{\tilde{T}^l}) = \psi$:

$$\begin{aligned} \varphi(\tilde{\psi}(q), t) &= \varphi(\psi(\iota_{n,l}(q), (x_{n+i})_i), t) \\ &= \psi(\iota_{n,l}(q), (x_{n+i})_i) \bullet ((x_i)_i + t) \\ &= \psi(\iota_{n,l}(q) \bullet ((x_i)_i + t), (x_{n+i})_i \bullet ((x_i)_i + t)) \\ &= \psi(q \bullet (x_i)_i, (t_i)_i) \\ &= \psi(q,t). \end{aligned}$$

Now, suppose that we have another $\tilde{\psi}' : Q' \to A(Q, l)$ such that $\varphi \circ (\tilde{\psi}' \times \mathrm{id}_{\tilde{T}^l}) = \psi$. Then we have

$$\begin{aligned}
\tilde{\psi}(q) &= \psi(\iota_{n,l}(q), (x_{n+i})_i) \\
&= (\varphi \circ (\tilde{\psi}' \times \mathrm{id}_{\tilde{T}^l}))(\iota_{n,l}(q), (x_{n+i})_i) \\
&= \varphi(\tilde{\psi}'(\iota_{n,l}(q)), (x_{n+i})_i) \\
&= \tilde{\psi}'(\iota_{n,l}(q)) \bullet ((x_i)_i + (x_{n+i})_i) \\
&= \iota_{n,l}(\tilde{\psi}'(q)) \bullet ((x_i)_i + (x_{n+i})_i) \\
&= \tilde{\psi}'(q) \bullet (x_i)_i \\
&= \tilde{\psi}'(q),
\end{aligned}$$

so $\tilde{\psi}$ is unique, which completes the proof. □

Now, this adjunction $- \times \tilde{T}^l \dashv A(-, l)$ induces a natural isomorphism

$$\varphi : PT(- \times \tilde{T}^l, \tilde{T}) \xrightarrow{\sim} PT(-, A(\tilde{T}, l))$$

LEMMA 21. *For all $f : PT(\tilde{T}^n \times \tilde{T}^l, \tilde{T})$,*

$$\varphi_{\tilde{T}^n}(f)(q) = f(\iota_{m,l}(q), (x_{m+i})_i)$$

PROOF. **(TODO)** □

LEMMA 22. *For all $f : PT(\tilde{T}^n, A(\tilde{T}, l))$,*

$$\varphi^{-1}_{\tilde{T}^n}(f)(q, t) = f(q) \bullet ((x_i)_i + t).$$

PROOF. **(TODO)** □

CHAPTER 4

Theorems

1. Scott's Representation Theorem

THEOREM 1. *Any λ-theory L is isomorphic to the endomorphism λ-theory $E(\tilde{L})$ of $\tilde{L}$ in the presheaf category of L.*

PROOF. First of all, remember that $\tilde{L}$ is indeed exponentiable and that $\tilde{L}^{\tilde{L}} = A(\tilde{L}, 1)$. Now, since L is a λ-theory, we have functions back and forth $\lambda : A(\tilde{L}, 1) \to \tilde{L}$ and $\rho : \tilde{L} \to A(\tilde{L}, 1)$. These are presheaf morphisms because for all $f : A(\tilde{L}, 1)_m$ and $g : \tilde{L}_m$ and $t : T_n^m$,

$$\lambda(f \bullet_{A(\tilde{L},1)} t) = \lambda(f \bullet_{\tilde{L}} ((\iota_{m,1}(t_i))_i + (x_{n+1}))) = \lambda(f) \bullet_{\tilde{L}} t$$

and

$$\rho(g \bullet_{\tilde{L}} t) = \rho(g) \bullet_{\tilde{L}} ((\iota_{m,1}(t_i))_i + (x_{n+1})) = \rho(g) \bullet_{A(\tilde{L},1)} t.$$

Therefore, $E(\tilde{L})$ is indeed a λ-theory.

For any presheaf Q and for any n, we have a bijection $PL(L^n, Q) \cong Q_n$. Then we have $\varphi : E(\tilde{L})_n \cong L_n$. This bijection is an isomorphism of λ-theories, since it preserves the x_i, $\bullet$, ρ and λ: for all $1 \leq j \leq n$, $f : E(\tilde{L})_m$, $g : E(\tilde{L})_{m+1}$ and $h : E(\tilde{L})_n^m$.

$$\begin{aligned}
\varphi(x_j) &= x_j(x_1, \dots, x_n) \\
&= x_j; \\
\varphi(f \bullet h) &= f \circ \langle h_i \rangle_i((x_i)_i) \\
&= f((h_i((x_j)_j))_i) \\
&= f((x_i)_i \bullet (h_i((x_j)_j))_i) \\
&= f((x_i)_i) \bullet (h_i((x_j)_j))_i \\
&= \varphi(f) \bullet (\varphi(h_i))_i; \\
\varphi(\rho(f)) &= \rho(f)((x_i)_i) \\
&= \rho(f((x_i)_i)) \bullet (x_i)_i \\
&= \rho(f((x_i)_i)) \\
&= \rho(\varphi(f)); \\
\varphi(\lambda(g)) &= \lambda(g)((x_i)_i) \\
&= \lambda(\varphi_{X^n}(g)((x_i)_i)) \\
&= \lambda(g(\iota_{m,l}((x_i)_i) + (x_{m+1}))) \\
&= \lambda(g((x_i)_i)) \\
&= \lambda(\varphi(g)).
\end{aligned}$$

□

2. Locally cartesian closedness of the category of retracts

DEFINITION 27 (Category of retracts). The category of retracts for a λ-theory L is the category with objects $f : L_n$ such that $f \bullet f = f$ and it has as morphisms $g : f \to f'$ the terms $g : L_n$ such that $f' \bullet g \bullet f = g$. The object $f : L_n$ has identity element f, and we have composition $g \circ g' = g \bullet g'$. These are morphisms **(TODO)**

LEMMA 23. *The category of retracts is indeed a category.*

PROOF. **(TODO)** □

THEOREM 2. *The category of retracts is locally cartesian closed* ***(TODO)*** *.*

3. The Fundamental Theorem of the λ-calculus

DEFINITION 28 (Λ). There is a special λ-theory, given by the λ-calculus itself. Λ_n is the set of λ-terms with n free variables, the x_i are the free variables, and $\bullet$ is given by substitution. λ sends $f : \Lambda_{n+1}$ to $\lambda x_{n+1}, f$ and ρ sends $f : \Lambda_n$ to $\iota_{n,1}(f)x_{n+1}$ in Λ_n.

LEMMA 24. *Λ is indeed a λ-theory.*

PROOF. **(TODO)** □

LEMMA 25. *Λ is the initial λ-theory.*

PROOF. Given a λ-theory L, we construct a morphism $f : \Lambda \to L$ by induction on the λ-terms. We set $f(x_i) = x_i$, $f(\lambda(t)) = \lambda(f(t))$ and $f(st) = \rho(f(s)) \bullet ((x_i)_i + (f(t)))$.

This is a λ-theory morphism because **(TODO)**

It is unique, since **(TODO)** □

DEFINITION 29 (Pullback of algebras). If we have a morphism of algebraic theories $f : T' \to T$, we have a functor $AT \to AT'$.

On objects, it sends a T-algebra A to a T'-algebra with set A and action $g \bullet_{T'} a = f(g) \bullet_T a$. This is a T'-algebra because **(TODO)** .

On morphisms, it sends $\varphi : A \to A$ to $\varphi : A \to A$. This is a T'-algebra morphism because for all $g : T'_n$ and $a : A^n$, we have

$$\varphi(g \bullet_{T'} a) = \varphi(f(g) \bullet_T a) = f(g) \bullet_T \varphi(a) = g \bullet_{T'} \varphi(a).$$

LEMMA 26. *This is indeed a functor.*

PROOF. **(TODO)** □

DEFINITION 30 (Term algebra). Given an algebraic theory T, for every n, T_n together with the action operator $\bullet : T_m \times T_n^m \to T_n$ gives a T-algebra.

LEMMA 27. *T_n is indeed a T-algebra.*

PROOF. **(TODO)** □

DEFINITION 31. For all n, we have a functor from lambda theories to Λ-algebras. It sends the λ-theory L to the L-algebra L_n and then turns this into a Λ-algebra via the morpism $\Lambda \to L$.

It sends morphisms $f : L \to L'$ to $f_n : L_n \to L'_n$. This is a Λ-algebra morphism because **(TODO)**

LEMMA 28. *This indeed constitutes a functor.*

PROOF. **(TODO)** □

REMARK 13. Note that for a monoid M, if we view M as a category, the category $[M^{\mathrm{op}}, \mathbf{SET}]$ consists of sets with a right M-action.

DEFINITION 32 (The exponential object in the presheaf category). Given a monoid M, if we have two presheaves (sets with right M-actions) P and P', we have a set of M-equivariant maps

$$F_{P,P'} = \left\{ f : M \times P \to P' \mid \prod_{p:P,m,m':M} f(m,p)m' = f(mm', pm') \right\}$$

with a right M-action, given by $fm'(m,p) = f(m'm,p)$. This is again M-equivariant because

$$fm'(m,p)m'' = f(m'm,p)m'' = f(m'mm'', pm'') = fm'(mm'', pm''),$$

so $F_{P,P'}$ is a presheaf.

Now, to show that $F_{P,P'}$ is the exponential object ${P'}^P$, we show that for any P, $F_{P,-}$ is the left adjoint of $- \times P$. So we need for all $P' : PT$, a universal arrow $\varphi : F_{P,P'} \times P \to P'$.

First of all, we have an evaluation map $\varphi : F_{P,P'} \times P \to P'$ given by $(f,p) \mapsto f(I,p)$ for I the unit of the monoid. This map is equivariant because for all m,

$$(f,p)m = (fm, pm) \mapsto fm(I,pm) = f(m,pm) = f(I,p)m.$$

Now, given any presheaf Q and any morphism $\psi : Q \times P \to P'$, take $\tilde{\psi} : Q \to F_{P,P'}$ given by $\psi(q)(m,p) = \psi(qm,p)$. This is equivariant because

$$\tilde{\psi}(q)m(m',p) = \tilde{\psi}(q)(mm',p) = \psi(qmm',p) = \tilde{\psi}(qm)(m',p)$$

and we have

$$\varphi(\tilde{\psi}(q),p) = \tilde{\psi}(q)(I,p) = \psi(q,p).$$

Now, suppose that we have $\tilde{\psi}' : Q \to F_{P,P'}$ such that $\varphi \circ (\tilde{\psi}' \times \mathrm{id}_P) = \psi$. Then for all $q : Q$, $m : M$ and $p : P$,

$$\tilde{\psi}(q)(m,p) = \psi(qm,p) = \varphi(\tilde{\psi}'(qm),p) = \tilde{\psi}'(qm)(I,p) = \psi'(q)m(I,p) = \psi'(q)(m,p),$$

so $\tilde{\psi}$ is unique and $F_{P,P'}$ is an exponential object.

DEFINITION 33 (n-functional terms). Let A be a Λ-algebra. We define

$$A(n) = \{a : A \mid (\lambda x_2 x_3 \dots x_{n+1}, x_1 x_2 x_3 \dots x_{n+1}) \bullet a = a\}.$$

DEFINITION 34. Take $\mathbf{1}_n = (\lambda x_1 \dots x_n, x_1 \dots x_n) \bullet () : A$.

DEFINITION 35. We define composition as $a \circ b = (\lambda x_3, x_1(x_2 x_3)) \circ (a,b)$ for $a, b : A$.

LEMMA 29. *This composition is associative.*

PROOF. **(TODO)** □

DEFINITION 36 (The monoid of a Λ-algebra). Now we make $A(1)$ into a monoid with unit $\lambda x_1, x_1$.

LEMMA 30. *This is indeed a monoid.*

PROOF. **(TODO)** □

From here on, we will assume that Λ (and therefore, any λ-theory) satisfies β-equality.

LEMMA 31. *For $a : A$, a is in $A(n)$ iff $\mathbf{1}_n \circ a = a$.*

PROOF.

$$\begin{aligned}
\mathbf{1}_n \circ a &= (\lambda x_3, x_1(x_2x_3)) \bullet (((\lambda x_1 \dots x_n, x_1 \dots x_n) \bullet ()), a) \\
&= (\lambda x_3, x_1(x_2x_3)) \bullet (((\lambda x_2 \dots x_{n+1}, x_2 \dots x_{n+1}) \bullet a), x_1 \bullet a) \\
&= ((\lambda x_3, x_1(x_2x_3)) \bullet ((\lambda x_2 \dots x_{n+1}, x_2 \dots x_{n+1}), x_1)) \bullet a \\
&= (\lambda x_2, (\lambda x_3 \dots x_{n+2}, x_3 \dots x_{n+2})(x_1x_2)) \bullet a \\
&= (\lambda x_2x_3 \dots x_{n+1}, x_1x_2 \dots x_{n+1}) \bullet a.
\end{aligned}$$

□

DEFINITION 37 (The presheaf category of a Λ-algebra). Let A be a Λ-algebra. If we view the monoid $A(1)$ as a one-object category, we define the category PA to be the category of presheaves $[A(1)^{\mathrm{op}}, \mathbf{SET}]$.

DEFINITION 38 (The objects $A(n)$ in PA). Given $a : A(n)$ and $b : A(1)$, we have

$$\mathbf{1}_n \circ (a \circ b) = (\mathbf{1}_n \circ a) \circ b = a \circ b,$$

so $a \circ b : A(n)$ and we have a right $A(1)$-action on $A(n)$, which makes $A(n)$ into an object in PA.

LEMMA 32. *We have $A(1)^{A(1)} \cong A(2)$.*

PROOF. We have a bijection $\varphi : A(2) \cong F_{A(1),A(1)}$, given by

$$\varphi(a)(b, b') = (\lambda x_4, x_1(x_2x_4)(x_3x_4)) \bullet (a, b, b').$$

Note that $\varphi(d)$ is equivariant because **(TODO)** Now, φ is a presheaf morphism because **(TODO)**

Take $p = \lambda x_1, x_1(\lambda x_2x_3, x_2)$ and $q = \lambda x_1, x_1(\lambda x_2x_3, x_3)$. These are elements of $A(1)$. Note that for terms c_1, c_2

$$\begin{aligned}
p(\lambda x_1, x_1c_1c_2) &= (\lambda x_1, x_1c_1c_2)(\lambda x_2x_3, x_2) \\
&= (\lambda x_1x_3, x_2)c_1c_2 \\
&= c_1.
\end{aligned}$$

In the same way, $q \circ (\lambda x_1x_2, x_2c_1c_2) = c_2$.

An inverse is given by

$$\psi : f \mapsto \lambda x_1x_2, f(p, q)(\lambda x_3, x_3x_1x_2).$$

This is a presheaf morphism because **(TODO)**

This is an inverse, because given $f : F_{A(1),A(1)}$ and$(a_1, a_2) : A(1) \times A(1)$, we have

$$\begin{aligned}
\varphi(\psi(f))(a_1, a_2) &= u(\lambda x_1x_2, f(p, q)(\lambda x_3, x_3x_1x_2))(a_1, a_2) \\
&= \lambda x_1, (\lambda x_2x_3, f(p, q)(\lambda x_4, x_4x_2x_3))(a_1x_1)(a_2x_1) \\
&= \lambda x_1, f(p, q)(\lambda x_2, x_2(a_1x_1)(a_2x_1)) \\
&= f(p, q) \circ (\lambda x_1, (\lambda x_2, x_2(a_1x_1)(a_2x_1))) \\
&= f(p \circ (\lambda x_1, (\lambda x_2, x_2(a_1x_1)(a_2x_1))), q \circ (\lambda x_1, (\lambda x_2, x_2(a_1x_1)(a_2x_1)))) \\
&= f(\lambda x_1, p(\lambda x_2, x_2(a_1x_1)(a_2x_1)), \lambda x_1, q(\lambda x_2, x_2(a_1x_1)(a_2x_1))) \\
&= f(\lambda x_1, a_1x_1, \lambda x_1, a_2x_1) \\
&= f(a_1, a_2).
\end{aligned}$$

The last line is because $a_i : A(1)$ and therefore $\lambda x_1, a_ix_1 = a_i$.

On the other hand, if we have $a_1 : A(2)$, we have

$$\begin{aligned}\psi(\varphi(a_1)) &= \psi((a_2, a_3) \mapsto \lambda x_1, a_1(a_2 x_1)(a_3 x_1))\\ &= \lambda x_1 x_2, (\lambda x_3, a_1(p x_3)(q x_3))(\lambda x_3, x_3 x_1 x_2)\\ &= \lambda x_1 x_2, a_1(p(\lambda x_3, x_3 x_1 x_2))(q(\lambda x_3, x_3 x_1 x_2))\\ &= \lambda x_1 x_2, a_1 x_1 x_2\\ &= a_1.\end{aligned}$$

The last line is because $a_1 : A(2)$ and therefore $\lambda x_1 x_2, a_1 x_1 x_2 = a_1$.

Therefore, this map is a bijection and an isomorphism. □

DEFINITION 39 (Endomorphism λ-theory of a Λ-algebra). PA borrows products from **SET**. Therefore, the algebraic theory $E(A(1))$ exists. Now note that $A(1)$ is exponentiable and $A(1)^{A(1)} \cong A(2)$. Note that $A(2) \subseteq A(1)$ and that $(\lambda x_2 x_3, x_1 x_2 x_3) \bullet -$ gives a function from $A(1)$ to $A(2)$. This gives $E(A(1))$ a λ-theory structure.

DEFINITION 40 (Pullback functor on presheaves for a Λ-algebra). Let $f : A \to A'$ be a Λ-algebra morphism. Then for all $a : A(n)$,

$$\mathbf{1}_n \circ f(a) = f(\mathbf{1}_n) \circ f(a) = f(\mathbf{1}_n \circ a),$$

so we have an induced morphism $f : A(n) \to A'(n)$.

Now, given a presheaf $P : PA'$. We can create a presheaf $f^* P : PA$ by taking the set of P, and, for $p : P$ and $a : A$, setting $pa = p \circ f(a)$. This is indeed a presheaf because **(TODO)**

Now, given a morphism $g : P \to P'$, we get a morphism by taking the function on the sets of P and P'. This is a morphism because **(TODO)**

LEMMA 33. *The above indeed constitutes a functor.*

PROOF. **(TODO)** □

Left Kan extension then gives a left adjoint $f_* : PA \to PA'$.

LEMMA 34. *We have $f_*(A(1)) \cong A'(1)$.*

PROOF. **(TODO)** □

LEMMA 35. *f_* preserves finite products.*

PROOF. **(TODO)** □

DEFINITION 41. Since f_* preserves finite products, given an element of $g : E(A(1))(n) = PA(A(1)^n, A(1))$, we get

$$\#f_*(g) : PA'(f(A(1)^n), f(A(1))) \cong PA'(A'(1)^n, A'(1)) = E(A'(1))(n).$$

LEMMA 36. *$\#f_* : E(A(1)) \to E(A'(1))$ is a map of λ-theories.*

PROOF. **(TODO)** □

DEFINITION 42. We have an isomorphism $E(A(1))(0) \cong A$ given by $a \mapsto aI$.

LEMMA 37. *This is indeed an isomorphism of Λ-algebras.*

PROOF. **(TODO)** □

LEMMA 38. *Given $g : A \to A'$,*

THEOREM 3. *There exists an adjoint equivalence between the category of λ-theories, and the category of algebras of Λ.*

Proof. We will show that the functor $L \mapsto L_0$ is an equivalence of categories.

It is essentially surjective, because L is isomorphic **(TODO)** to $E(A(1))$.

Now, given morphisms $f, f' : L \to L'$. Suppose that $f_0 = f'_0$. Suppose that L and L' have β-equality. Then, given $l : L_n$, we have

$$f_n(l) = \rho^n(\lambda^n(f_n(l))) = \rho^n(f_0(\lambda^n(l))) = \rho^n(f'_0(\lambda^n(l))) = \rho^n(\lambda^n(f'_n(l))) = f'_n(l),$$

so the functor is faithful.

The functor is full because a Λ-algebra morphism $f : A \to A'$ induces a functor $f^* : PA' \to PA$, and via left Kan extension we get a left adjoint $f_* : PA \to PA'$ with $f_*(A(1)) \cong A'(1)$. Now, f_* preserves (finite) products, so we have maps $PA(A(1)^n, A(1)) \to PA'(A'(1)^n, A'(1))$ and so a map $E(A(1)) \to E(A'(1))$. This map, when restricted to a map $PA(1, A(1)) \to PA'(1, A(1))$, and transported along the isomorphism $a \mapsto aI$ **(TODO)** , is equal to f **(TODO)** . □

Lemma 39. *The category of T-algebras has coproducts.*

Proof. **(TODO)** □

Definition 43 (Theory of extensions). Let T be an algebraic theory and A a T-algebra. We can define an algebraic theory T_A called 'the theory of extensions of A' with $(T_A)_n = T_n + A$. The left injection of the variables $x_i : T_n$ gives the variables. Now, take $h : (T_n + A)^m$. Sending $g : T_m$ to $\varphi(g) := g \bullet h$ gives a T-algebra morphism $T_m \to T_n + A$ since

$$\varphi(f \bullet g) = f \bullet g \bullet h = f \bullet (g_i \bullet h) = f \bullet (\varphi(g_i))_i.$$

This, together with the injection morphism of A into $T_n + A$, gives us a T-algebra morphism from the coproduct: $T_m + A \to T_n + A$. We especially have a function on sets $(T_m + A) \times (T_n + A)^m \to T_n + A$, which we will define our substitution to be.

Lemma 40. *T_A is indeed an algebraic theory.*

Proof. **(TODO)** □

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