Decorating "Classical Lambda Calculus in Modern Dress"

April 12, 2023

Definitions - Algebraic Theory

Definition

An algebraic theory consists of

- a functor $\mathcal{T} : F \to \mathrm{SET}$,
- elements (projections) $\pi_{n,i} \in \mathcal{T}(n)$ for all $1 \leq i \leq n$,
- a composition morphism $\bullet : \mathcal{T}(m) \times \mathcal{T}(n)^m \to \mathcal{T}(n)$ for all m, n.

The composition must be

- associative,
- unital,
- compatible with projections,
- dinatural in m.

Definitions - Abstract Clone

Definition

An abstract clone is

- a function $C : \mathbb{N} \to \mathrm{SET}$,
- a composition morphism $\bullet : C(m) \times C(n)^m \to C(n)$ for all m, n,
- elements (projections) $\pi_{n,i} \in C(n)$ for all $1 \leq i \leq n$.

The composition must satisfy

- $\pi_{n,i} \bullet (f_1, \ldots, f_n) = f_i$,
- $f \bullet (\pi_{1,n}, \ldots, \pi_{n,n}) = f$,
- $(f \bullet (g_1, \ldots, g_m)) \bullet (h_1, \ldots, h_n) = f \bullet (g_1 \bullet (h_1, \ldots, h_n), \ldots, g_m \bullet (h_1, \ldots, h_n))$.

Definitions - Algebra for an Algebraic Theory

Definition

An algebra for an algebraic theory $\mathcal{T}$ consists of

1. a set A,
2. an action $\alpha_n : \mathcal{T}(n) \times A^n \to A$.

Such that α_n satisfies:

1. naturality in n,
2. associativity,
3. unitality.

Definitions - λ-theory

Definition

A λ-theory consists of

- an algebraic theory $\mathcal{L}$,
- a retraction $\rho : \mathcal{L}(n) \to \mathcal{L}(n+1)$,
- a section $\lambda : \mathcal{L}(n+1) \to \mathcal{L}(n)$.

Such that

- ρ and λ are natural in n;
- ρ and λ are compatible (?) with the actions $\mathcal{L}(m) \times \mathcal{L}(n)^m \to \mathcal{L}(n)$ and $\mathcal{L}(m+1) \times \mathcal{L}(n)^m \to \mathcal{L}(n+1)$ (which "ignores the last variable").

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An algebra for a λ-theory $\mathcal{L}$ is an algebra A for the underlying algebraic theory.

For each term $t(\mathbf{x}) \in \mathcal{L}(n)$ and each tuple $\mathbf{a} \in A^n$, we get an interpretation $t(\mathbf{a}) \in A$.

Given a λ-theory $\mathcal{L}$, we can interpret a term t of the lambda calculus (that has a context Γ of length n) as an element $[\![t]\!] \in \mathcal{L}(n)$.

Examples

Definition

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Example

Take $C(n) = \{\star\}$, $\pi_{n,i} = \star$ and $\star \bullet \{\star, \dots, \star\} = \star$.
This theory is the terminal λ-theory.

Examples

The λ-calculus Λ, in which $\Lambda(n)$ consists of the terms with n free variables, $\pi_{n,i} = \mathtt{Var}(i)$ (with De Bruijn indices) and $\bullet$ is substitution.

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This is a λ-theory. According to the paper, it is the initial λ-theory.

Examples

More generally, we can create abstract clones using algebraic signatures: Given sets of n-ary constructors Σ_n and a sequence of variables $x_1, \ldots, x_n$, we can build the $C(n)$ inductively:

- $x_i : C(n)$ for $1 \leq i \leq n$;
- $\sigma(c_1, \ldots, c_n) : C(n)$ for $c_1, \ldots, c_n : C(n)$ and $\sigma : \Sigma_n$.

Then the $\pi_{n,i}$ are the x_i, and $f \bullet (g_1, \ldots, g_m)$ substitutes the g_i for the x_i in f.

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For example, if we talk about rings, we have $\Sigma_0 = \{0, 1\}$, $\Sigma_1 = \{-\}$ (negation) and $\Sigma_2 = \{+, \cdot\}$. Then $C(n)$ is almost the polynomial ring over $\mathbb{Z}$ in n variables (but not quite, because we distinguish, for example, between 0, $0 + 0$ and $x_1 - x_1$).

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If we have a type A and we have for all $\sigma \in \Sigma_n$, a map $[\![\sigma]\!] : A^n \to A$. Then we can make A into an algebra for this theory.

Examples

Let R be a ring. Take $T(n) = R[x_1, \dots, x_n]$ the polynomial ring in n variables. Take $\pi_{n,i} = x_i$ and let $f \bullet (g_1, \dots, g_n)$ substitute the g_i for the x_i in f.

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Take $T(n) = \{1, 2, \ldots, n\}$, $\pi_{n,i} = i$ and $i \bullet (f_1, \ldots, f_n) = f_i$.
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I am still pondering whether $T(n) = \mathbb{Z}$ could be a λ-algebra.

Examples

Take a semiring R with identities $0, 1$ and operations $+, \cdot$. Take $T(n) = R^n$, $\pi_{n,i} = (0, \dots, 0, 1, 0, \dots, 0)$ and take

$$f \bullet (g_1, \dots, g_n) = \begin{pmatrix} g_{1,1} & \cdots & g_{n,1} \\ \vdots & \ddots & \vdots \\ g_{1,m} & \cdots & g_{n,m} \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} := \left(\sum_i f_i \cdot g_{i,1}, \dots, \sum_i f_i \cdot g_{i,m} \right)$$

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For example, take S a set, $\mathcal{T}$ a topology on S. Then we can take $T(n) = \mathcal{T}^n$, with operations $\cup$, $\cap$, and units $\emptyset$, S. Then we have $\pi_{n,i} = (\emptyset, \ldots, \emptyset, S, \emptyset, \ldots, \emptyset)$. For $U = (U_1, \ldots, U_n)$, $V_i = (V_{i,1}, \ldots, V_{i,m})$ we have

$$U \bullet (V_1, \ldots, V_n) = (U_1 \cap V_{1,1} \cup \cdots \cup U_n \cap V_{n,1}, \ldots, U_1 \cap V_{1,m} \cup \cdots \cup U_n \cap V_{n,m}).$$

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Or take $R = \mathbb{N}$, with operations $+, \cdot$ and units $0, 1$.

Examples

In any category with finite products, take an object X. Then we can set $T(n) = (X^n \to X)$. Then $\pi_{n,i}$ is the ith projection morphism. Also, by the universal property of the product, if we have m terms of $(X^n \to X)$, we get a term of $(X^n \to X^m)$ which we can compose with a term of $(X^m \to X)$ to get a term of $(X^n \to X)$.

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I believe that if we have a retraction $r : X \to A$ with section s, then A is an algebra with $\varphi \cdot (a_1, \ldots, a_n) = r(\varphi(s(a_1), \ldots, s(a_n)))$.

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If we have a retraction $X \to X^X$, the theory that we get is a λ-theory.

About λ-theories

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Any λ-theory $\mathcal{T}$ has an element $e \in \mathcal{T}(1)$, retractions $\rho : \mathcal{T}(n) \to \mathcal{T}(n+1)$ and sections $\sigma : \mathcal{T}(n+1) \to \mathcal{T}(n)$. Define $app = \rho(e)$. For $\mathcal{T} = \Lambda$, we have $app = xy$.

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Take $\mathbf{1}_n = \sigma^{n+1}(\rho^n(e))$. Note that we can inject this as $\mathbf{1}_n \bullet ()$ into any $\mathcal{T}(n)$. For $\mathcal{T} = \Lambda$, this is

$$\lambda x, \lambda y_1, \ldots, \lambda y_n, x y_1 \ldots y_n.$$

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For $n = 1$, this is

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Preliminaries to the main theorem

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Lemma

Λ_A is a λ-theory.

Proof.

We can identify Λ_A with $\mathcal{U}_A$.
We have a retraction $U_A \to U_A^{U_A}$. Composition with this gives a retraction $\mathcal{U}_A(n) \to \mathcal{U}_A(n+1)$. $\square$

$\mathcal{U}_A$

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We define $P(A)$ to be the category of presheaves on the category M_A. This has 'universal object' $U_A = M_A$ with the obvious right action of M_A.

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We have $M_A = \{a \in A \mid \mathbf{1}a = a\}$. We can identify $U_A^{U_A}$ with $\{a \in A \mid \mathbf{1}_2 a = a\}$. Composition on the left with $\mathbf{1}$ gives the retraction. $\square$

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Definition

We take $\mathcal{U}_A$ to be the endomorphism theory of the reflexive universal $U_A \in P(A)$.

The Main Theorem of the Lambda Calculus

Theorem

There is an adjoint equivalence $\mathcal{L} \mapsto \mathcal{L}(0)$ and $A \mapsto \Lambda_A$ between λ-theories and Λ-algebras.
In particular, each λ-theory $\mathcal{L}$ is isomorphic to the theory of extensions of its initial algebra $\mathcal{L}(0)$.