SUMMARY OF THE THINGS THAT I LEARNED

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1. Week 08

1.1. **Univalent Categories.** A univalent category is a category in which the univalence axiom holds. I.e., a category $\mathcal{C}$ in which, for all $A, B \in \mathcal{C}_0$, the canonical map $(A =_{\mathcal{C}} B) \to (A \cong B)$ is an equivalence.

1.2. **Categories.** An n**-category** is a category with 0-cells (objects), 1-cells (morphisms), 2-cells (morphisms between morphisms), up to n-cells and various compositions: $A \to B \to C$. $A \xrightarrow{f,g,h} B$, $f \Rightarrow g \Rightarrow h$. $A \xrightarrow{f,g} B \xrightarrow{f',g'} C$, $\alpha : f \Rightarrow g$, and identities $\alpha' : f' \Rightarrow g'$ gives $\alpha' * \alpha : f' \circ f \Rightarrow g' \circ g$. These all need to work together 'nicely'. An ω-category is the same, but all the way up.

A topological space gives a (weak) ω-category. 0-cells are points, 1-cells are paths, 2-cells are homotopies etc. Composition is by glueing. It is a 'groupoid', in the sense that all homotopies of dimension ≥ 1 are invertible. However, glueing is not associative, so it is a 'weak' ω-category.

A category with only one object $\star$ is equivalent to a monoid (with elements being the set $\mathcal{C}(\star, \star)$). A 2-category with only one 0-cell is the same thing as a monoidal category (objects: the 1-cells. Morphisms: the 2-cells). A monoidal category with just one object gives 2 monoid structures on its set of morphisms. These are the same, and commutative.

A **monoid** is a set with a multiplication and a unit. A **monad** on a category $\mathcal{A}$ is a functor $\mathcal{A} \to \mathcal{A}$, together with natural transformations $\mu : T \circ T \to T$ (satisfying associativity) and $\eta : 1_{\mathcal{A}} \to T$ (acting as a two-sided unit).

A **presheaf** on a category $\mathcal{A}$ is a functor $\mathcal{A}^{opp} \to \mathbf{Set}$.

Given a category $\mathcal{E}$ and an object $E \in \mathcal{E}_0$, the **slice category** $\mathcal{E}/E$ with objects being the maps $D \xrightarrow{p} E$ and morphisms being commutative triangles.

A **multicategory**, not necessarily the same as an n-category, is a category in which arrows go from multiple objects to one, instead of from one object to one. I.e. it is a category with a class C_0 of objects, for all n, and all $a, a_1, \ldots, a_n \in C_0$, a class $C(a_1, \ldots, a_n; a)$ of 'morphisms', and a composition

$$C(a_1, \ldots, a_n; a) \times C(a_{1,1}, \ldots, a_{1,k_1}; a_1) \times \cdots \times C(a_{n,1}, \ldots, a_{n,k_n}; a_n) \to C(a_{1,1}, \ldots, a_{n,k_n}; a),$$

written $(\theta, \theta_1, \ldots, \theta_n) \mapsto \theta(\theta_1, \ldots, \theta_n)$ and for each $a \in C_0$ an identity $1_a \in C(a; a)$. It must satisfy associativity

$$\theta \circ (\theta_1 \circ (\theta_{1,1}, \ldots, \theta_{1,k_1}), \ldots, \theta_n \circ (\theta_{n,1}, \ldots, \theta_{n,k_n})) = (\theta \circ (\theta_1, \ldots, \theta_n)) \circ (\theta_{1,1}, \ldots, \theta_{n,k_n})$$

and identity

$$\theta \circ (1_{a_1}, \ldots, 1_{a_n}) = \theta = 1_a \circ \theta.$$

A **map of multicategories** is a function $f_0 : C_0 \to C'_0$ and maps $C(a_1, \ldots, a_n; a) \to C(f_0(a_1), \ldots, f_0(a_n); f_0(a))$, preserving composition and identities.

For C a multicategory, a C**-algebra** is a map from C into the multicategory **Set** (with objects Set_0 and maps $\mathrm{Set}(a_1, \ldots, a_n; a) = \mathrm{Set}(a_1 \times \cdots \times a_n; a))$. I.e., for each $a \in C_0$, a set $X(a)$, and for each map $\theta : a_1, \ldots, a_n \to a$, a function $X(\theta) : X(a_1) \times X(a_n) \to X(a)$. An example is, for a multicategory C, to take $X(a) = C(; a)$ (maps from the empty sequence into a).

1.3. **Operads.**

1.3.1. *Definitions.* An **operad** is a multicategory with only one object. More explicitly, an operad has a set $P(k)$ for every $k \in \mathbb{N}$, whose elements can be thought of as k-ary operations. It also has, for all $n, k_1, \ldots, k_n \in \mathbb{N}$, a *composition* function

$$P(n) \times P(k_1) \times \cdots \times P(k_n) \to P(k_1 + \cdots + k_n)$$

and an element $1 = 1_P \in P(1)$ called the **identity**, satisfying

$$\theta \circ (1, 1, \ldots, 1) = \theta = \theta \circ 1$$

for all θ, and

$$\theta \circ (\theta_1 \circ (\theta_{1,1}, \ldots, \theta_{1,k_1}), \ldots, \theta_n \circ (\theta_{n,1}, \ldots, \theta_{n,k_n})) = (\theta \circ (\theta_1, \ldots, \theta_n)) \circ (\theta_{1,1}, \ldots, \theta_{n,k_n})$$

for all $\theta \in P(n)$, $\theta_1 \in P(k_1)$, $\ldots$, $\theta_n \in P(k_n)$ and all $\theta_{1,1} \ldots \theta_{n,k_n}$.

A **morphism of operads** is a family

$$f_n : (P(n) \to Q(n))_{n \in \mathbb{N}}$$

of functions, preserving composition and identities.

A P**-algebra** for P an operad, is a set X and, for each n, and $\theta \in P(n)$, a function $\overline{\theta} : X^n \to X$, satisfying the evident axioms (identity is the identity function, the function of a composition is the composition of the functions?).

1.3.2. *Examples.* For any vector space V, there is an operad with $P(k) = V^{\otimes k} \to V$.

The terminal operad 1 has $P(n) = \{\star_1\}$ for all n. An algebra for 1 is a set X together with a function $X^n \to X$, denoted as $(x_1, \ldots, x_n) \mapsto (x_1 \cdot \cdots \cdot x_n)$, satisfying

$$((x_{1,1} \cdot \ldots x_{1,k_1}) \cdot \cdots \cdot (x_{n,1} \cdot \cdots \cdot x_{n,k_n})) = (x_{1,1} \cdot \cdots \cdot x_{n,k_n})$$

and

$$x = (x).$$

The category of 1-algebras is the category of monoids.

There exist various sub-operads of 1. For example, the smallest one has $P(1) = \{\star\}$ and $P(n) = \emptyset$ for $n \neq 1$.

Or the operad with $P(0) = \emptyset$ and $P(n) = \{\star_n\}$ for $n > 0$, which has semigroups as its algebras (sets with associative binary operations).

The suboperad with $P(n) = \{\star_n\}$ exactly when $n \leq 1$ has as its algebras the pointed sets.