

Semantics for the λ -calculus

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CHAPTER 1

Algebraic Structures

1. Algebraic Theories

DEFINITION 1 (algebraic theory). We define an algebraic theory T to be a sequence of sets T_n indexed over \mathbb{N} with for all $1 \leq i \leq n$ elements ("variables" or "projections") $x_{n,i} : T_n$ (we usually leave n implicit), together with a substitution operation

$$- \bullet - : T_m \times T_n^m \rightarrow T_n$$

for all m, n , such that

$$\begin{aligned} x_j \bullet g &= g_j \\ f \bullet (x_{l,i})_i &= f \\ (f \bullet g) \bullet h &= f \bullet (g_i \bullet h)_i \end{aligned}$$

for all $1 \leq j \leq l$, $f : T_l$, $g : T_m^l$ and $h : T_n^m$.

DEFINITION 2 (algebraic theory morphism). A morphism F between algebraic theories T and T' is a sequence of functions $F_n : T_n \rightarrow T'_n$ (we usually leave the n implicit) such that

$$\begin{aligned} F_n(x_j) &= x_j \\ F_n(f \bullet g) &= F_m(f) \bullet (F_n(g_i))_i \end{aligned}$$

for all $1 \leq j \leq n$, $f : T_m$ and $g : T_n^m$.

REMARK 1. We can construct binary products of algebraic theories, with sets $(T \times T')_n = T_n \times T'_n$, variables (x_i, x'_i) and substitution

$$(f, f') \bullet (g, g') = (f \bullet g, f' \bullet g').$$

In the same way, the category of algebraic theories has all limits.

2. Algebras

DEFINITION 3 (algebra). An algebra A for an algebraic theory T is a set A , together with an action

$$\bullet : T_n \times A^n \rightarrow A$$

for all n , such that

$$\begin{aligned} x_j \bullet a &= a_j \\ (f \bullet g) \bullet a &= f \bullet (g_i \bullet a)_i \end{aligned}$$

for all j , $f : T_m$, $g : T_n^m$ and $a : A^n$.

DEFINITION 4 (algebra morphism). For an algebraic theory T , a morphism F between T -algebras A and A' is a function $F : A \rightarrow A'$ such that

$$F(f \bullet a) = f \bullet (F(a_i))_i$$

for all $f : T_n$ and $a : A^n$.

REMARK 2. The category of algebras has all limits. The set of a limit of algebras is the limit of the underlying sets.

REMARK 3. Note that for an algebraic theory T , the T_n are all algebras for T , with the action given by \bullet .

3. Presheaves

DEFINITION 5 (presheaf). A presheaf P for an algebraic theory T is a sequence of sets P_n indexed over \mathbb{N} , together with an action

$$\bullet : P_m \times T_n^m \rightarrow P_n$$

for all m, n , such that

$$\begin{aligned} t \bullet (x_{l,i})_i &= t \\ (t \bullet f) \bullet g &= t \bullet (f_i \bullet g)_i \end{aligned}$$

for all $t : P_l$, $f : T_m^l$ and $g : T_n^m$.

DEFINITION 6 (presheaf morphism). For an algebraic theory T , a morphism F between T -presheaves P and P' is a sequence of functions $F_n : P_n \rightarrow P'_n$ such that

$$F_n(t \bullet f) = F_n(t) \bullet f$$

for all $t : P_m$ and $f : T_n^m$.

We will write PT for the category of T -presheaves and their morphisms.

REMARK 4. The category of presheaves has all limits. The n th set \bar{P}_n of a limit \bar{P} of presheaves P_i is the limit of the n th sets $P_{i,n}$ of the presheaves in the limit diagram.

4. λ -theories

DEFINITION 7 (λ -theory). A λ -theory is an algebraic theory L , together with sequences of functions $\lambda_n : L_{n+1} \rightarrow L_n$ and $\rho_n : L_n \rightarrow L_{n+1}$, such that

$$\begin{aligned} \lambda_m(f) \bullet h &= \lambda_n(f \bullet (h_1 \bullet (x_{n+1,1}, \dots, x_{n+1,n}), \dots, h_m \bullet (x_{n+1,1}, \dots, x_{n+1,n}), x_{n+1,n+1})) \\ \rho_n(g \bullet h) &= \rho_m(g) \bullet (h_1 \bullet (x_{n+1,1}, \dots, x_{n+1,n}), \dots, h_m \bullet (x_{n+1,1}, \dots, x_{n+1,n}), x_{n+1,n+1}) \end{aligned}$$

for all $f : L_{m+1}$, $g : L_m$ and $h : L_n^m$.

DEFINITION 8 (β - and η -equality). We say that a λ -theory L satisfies β -equality (or that it is a λ -theory with β) if $\rho_n \circ \lambda_n = \text{id}_{L_n}$ for all n . We say that it satisfies η -equality if $\lambda_n \circ \rho_n = \text{id}_{L_{n+1}}$ for all n .

DEFINITION 9 (λ -theory morphism). A morphism F between λ -theories L and L' is an algebraic theory morphism F such that

$$\begin{aligned} F_n(\lambda_n(f)) &= \lambda_n(F_{n+1}(f)) \\ \rho_n(F_n(g)) &= F_{n+1}(\rho_n(g)) \end{aligned}$$

for all $f : L_{n+1}$ and $g : L_n$.

REMARK 5. The category of lambda theories has all limits, with the underlying algebraic theory of a limit being the limit of the underlying algebraic theories.

A λ -theory algebra or presheaf is a presheaf for the underlying algebraic theory.

5. Examples

5.1. The free algebraic theory on a set.

5.2. The free object algebraic theory.

5.3. The initial algebraic theory.

5.4. The terminal theory.

5.5. The Endomorphism Theory.

5.6. The free λ -theory.

5.7. The λ -calculus.

5.8. The theory presheaf.

5.9. The theory algebra.

5.10. The “+l” presheaf. Let $\iota_{m,n} : T_m \rightarrow T_{m+n}$ denote the function that sends f to $f \bullet (x_{m+n,1}, \dots, x_{m+n,m})$. Note that

$$\iota_{m,n}(f) \bullet g = f \bullet (g_i)_{i \leq m}$$

and

$$\iota_{m,n}(f \bullet g) = f \bullet g \bullet (x_i)_i = f \bullet (g_i \bullet (x_j)_j)_i = f \bullet (\iota_{m,n}(g_i))_i.$$

For tuples $x : X^m$ and $y : X^n$, let $x+y$ denote the tuple $(x_1, \dots, x_m, y_1, \dots, y_n) : X^{m+n}$.

DEFINITION 10 (The ‘+l’ presheaf). Given a T -presheaf Q , we can construct a presheaf $A(Q, l)$, given by $A(Q, l)_n = Q_{n+l}$. Then, for $q : A(Q, l)_m$ and $f : T_n^m$, the substitution is given by

$$q \bullet_{A(Q, l)} f = q \bullet_Q ((\iota_{n, l}(f_i))_i + (x_{n+i})_i)$$

LEMMA 1. The +l presheaf is a presheaf

PROOF. We have, for $q : A(Q, l)_n$,

$$\begin{aligned} q \bullet_{A(Q, l)} (x_i)_i &= q \bullet_Q ((\iota_{n, l}(x_i))_i + (x_{n+i})_i) \\ &= q \bullet_Q ((x_i)_i + (x_{n+i})_i) \\ &= q \bullet_Q (x_i)_i \\ &= q. \end{aligned}$$

We have, for $q : A(Q, k)_l$, $f : T_m^l$ and $g : T_n^m$,

$$\begin{aligned} q \bullet_{A(Q, k)} f \bullet_{A(Q, k)} g &= q \bullet_Q ((\iota_{m, l}(f_i))_i + (x_{m+i})_i) \bullet_Q ((\iota_{n, l}(g_j))_j + (x_{n+j})_j) \\ &= q \bullet_Q (((\iota_{m, l}(f_i) \bullet_T ((\iota_{n, l}(g_j))_j + (x_{n+j})_j))_i + (x_{m+i} \bullet_T ((\iota_{n, l}(g_j))_j + (x_{n+j})_j))_i)) \\ &= q \bullet_Q ((f_i \bullet_T (\iota_{n, l}(g_j))_j)_i + (x_{n+i})_i) \\ &= q \bullet_Q ((\iota_{n, l}(f_i \bullet_T g))_i + (x_{n+i})_i) \\ &= q \bullet_{A(Q, k)} (f_i \bullet_T g). \end{aligned}$$

□

CHAPTER 2

Category Theoretic Preliminaries

I will assume a familiarity with the category-theoretical concepts presented in [AW23]. These include categories, functors, isomorphisms, natural transformations, adjunctions, equivalences and limits.

1. Notation

For an object c in a category C , I will write $c : C$.

For a morphism f between objects c and c' in a category C , I will write $f : C(c, c')$ or $f : c \rightarrow c'$.

For composition of morphisms $f : C(c, d)$ and $g : C(d, e)$, I will write $f \cdot g$.

For composition of functors $F : A \rightarrow B$ and $G : B \rightarrow C$, I will write $F \bullet G$.

2. Quick preliminaries

2.1. Adjunctions. An adjoint equivalence of categories has multiple definitions. The one we will use here is the following:

DEFINITION 11. An adjoint equivalence between categories C and D is a pair of adjoint functors

$$\begin{array}{ccc} & L & \\ D & \xleftarrow{\quad} & C \\ & R & \end{array}$$

such that the unit $\eta : \text{id}_C \Rightarrow L \bullet R$ and counit $\epsilon : R \bullet L \Rightarrow \text{id}_D$ are isomorphisms of functors.

2.2. Exponential objects. (TODO)

3. Kan Extensions

One of the most general and abstract concepts in category theory is the concept of *Kan extensions*. In [ML98], Section X.7, MacLane notes that

The notion of Kan extensions subsumes all the other fundamental concepts of category theory.

In this thesis, we will use left Kan extension a handful of times. It comes in handy when we want to extend a functor along another functor in the following way:

Let A , B and C be categories and let $F : A \rightarrow B$ be a functor.

DEFINITION 12. Precomposition gives a functor between functor categories $F_* : [B, C] \rightarrow [A, C]$. If F_* has a left adjoint, we will denote call this adjoint functor the *left Kan extension* along F and denote it $\text{Lan}_F : [A, C] \rightarrow [B, C]$.

$$\begin{array}{ccc} A & \xrightarrow{F} & B \\ \swarrow F_* G & & \searrow G \\ & C & \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{F} & B \\ \swarrow G & & \searrow \text{Lan}_F G \\ & C & \end{array}$$

Analogously, when F_* has a right adjoint, one calls this the *right Kan extension* along F and denote it $\text{Ran}_F : [A, C] \rightarrow [B, C]$.

If a category has limits (resp. colimits), we can construct the right (resp. left) Kan extension in a ‘pointwise’ fashion (see Theorem X.3.1 in [ML98] or Theorem 2.3.3 in [KS06]). Below, I will outline the parts of the construction that we will need explicitly in this thesis.

LEMMA 2. *If C has colimits, Lan_F exists.*

PROOF. First of all, for objects $b : B$, we take

$$\text{Lan}_F G(b) := \text{colim} \left((F \downarrow b) \rightarrow A \xrightarrow{G} C \right).$$

Here, $(F \downarrow b)$ denotes the comma category with as objects the morphisms $B(F(a), b)$ for all $a : A$, and as morphisms from $f : B(F(a), b)$ to $f' : B(F(a'), b)$ the morphisms $g : A(a, a')$ that make the diagram commute:

$$\begin{array}{ccc} F(a) & \xrightarrow{F(g)} & F(a') \\ & \searrow f' & \swarrow f' \\ & b & \end{array}$$

and $(F \downarrow b) \rightarrow A$ denotes the projection functor that sends $f : B(F(a_1), b)$ to a_1 .

Now, a morphism $h : B(b, b')$ gives a morphism of diagrams, sending the $F(a)$ corresponding to $f : B(G(a), b)$ to the $F(a)$ corresponding to $f \cdot h : B(G(a), b')$. From this, we get a morphism $\text{Lan}_F G(h) : C(\text{Lan}_F G(b), \text{Lan}_F G(b'))$.

The unit of the adjunction is a natural transformation $\eta : \text{id}_{[A, C]} \Rightarrow \text{Lan}_F \bullet F_*$. We will define this pointwise, for $G : [A, C]$ and $a : A$. Our diagram contains the $G(a)$ corresponding to $\text{id}_{F(a)} : (F \downarrow F(a))$ and the colimit cocone gives a morphism

$$\eta_G(a) : C(G(a), \text{Lan}_F G(F(a))),$$

the latter being equal to $(\text{Lan}_F \bullet F_*)(G)(a)$.

The counit of the adjunction is a natural transformation $\epsilon : F_* \bullet \text{Lan}_F \Rightarrow \text{id}_{[B, C]}$. We will also define this pointwise, for $G : [B, C]$ and $b : B$. The diagram for $\text{Lan}_F (F_* G)(b)$ consists of $G(F(a))$ for all $f : B(F(a), b)$. Then, by the universal property of the colimit, the morphisms $G(f) : C(G(F(a)), G(b))$ induce a morphism

$$\epsilon_G(b) : C(\text{Lan}_F (F_* G)(b), G(b)).$$

□

LEMMA 3. *If $F : A \rightarrow B$ is a fully faithful functor, and C is a category with colimits, η is a natural isomorphism.*

PROOF. To show that η is a natural isomorphism, we have to show that $\eta_G(a') : G(a') \Rightarrow \text{Lan}_F G(F(a'))$ is an isomorphism for all $G : [A, C]$ and $a' : A$. Since a left adjoint is unique up to natural isomorphism, we can assume that $\text{Lan}_F G(F(a'))$ is given by

$$\text{colim}((F \downarrow F(a')) \rightarrow A \xrightarrow{G} C).$$

Now, the diagram for this colimit consists of $G(a)$ for each arrow $f : B(F(a), F(a'))$. Since F is fully faithful, we have $f = F(\bar{f})$ for some $\bar{f} : A(a, a')$. If we now take the arrows $G(\bar{f}) : C(G(a), G(a'))$, the universal property of the colimit gives an arrow

$$\varphi : C(\text{Lan}_F G(F(a')), G(a'))$$

which constitutes an inverse to $\eta_G(a')$. □

REMARK 6. In the same way, if C has limits, ϵ is a natural isomorphism.

COROLLARY 1. *If C has limits or colimits, precomposition of functors $[B, C]$ along a fully faithful functor is (split) essentially surjective.*

PROOF. For each $G : [A, C]$ we take $\text{Lan}_F G : [B, C]$, and we have $F_*(\text{Lan}_F G) \cong G$. \square

COROLLARY 2. *If C has colimits (resp. limits), left (resp. right) Kan extension of functors $[A, C]$ along a fully faithful functor is fully faithful.*

PROOF. Since left Kan extension along F is the left adjoint to precomposition, we have

$$[A, C](\text{Lan}_F G, \text{Lan}_F G') \cong [B, C](G, F_*(\text{Lan}_F G')) \cong [B, C](G, G').$$

\square

4. The Karoubi envelope

Let C be a category. If we have a retraction-section pair $c \xrightleftharpoons[s]{r} d$ we have (by definition) $s \cdot r = \text{id}_d$. On the other hand, $s \cdot r : c \rightarrow c$ is an idempotent morphism. Conversely, we can wonder whether for any idempotent morphism $a : c \rightarrow c$, we can find a retraction-section pair $r : c \rightarrow d$ and $s : d \rightarrow c$ such that $a = r \cdot s$. If this is the case, we say that the idempotent a *splits*. If a does not split, we can wonder whether we can find an embedding $\iota_C : C \hookrightarrow \overline{C}$ such that the idempotent $\iota_C(a) : \iota_C(c) \rightarrow \iota_C(c)$ does split.

DEFINITION 13. We define the category \overline{C} . The objects of \overline{C} are tuples (c, a) with $c : C$, $a : C(c, c)$ such that $a \cdot a = a$. The morphisms between (c, a) and (d, b) are morphisms $f : C(a, b)$ such that $a \cdot f \cdot b = f$. The identity morphism on (c, a) is given by a and \overline{C} inherits morphism composition from C .

This category is called the *Karoubi Envelope*, the *idempotent completion*, the *category of retracts*, or the *Cauchy completion* of C .

REMARK 7. Note that for a morphism $f : \overline{C}((c, a), (d, b))$,

$$a \cdot f = a \cdot a \cdot f \cdot b = a \cdot f \cdot b = f$$

and in the same way, $f \cdot b = f$.

DEFINITION 14. We have an embedding $\iota_C : C \rightarrow \overline{C}$, sending $c : C$ to (c, id_c) and $f : C(c, d)$ to f .

LEMMA 4. *Every object $c : \overline{C}$ is a retract of $\iota_C(c_0)$ for some $c_0 : C$.*

PROOF. Note that $c = (c_0, a)$ for some $c_0 : C$ and an idempotent $a : c \rightarrow c$. We have morphisms $\iota_C(c) \xrightleftharpoons[a_{\leftarrow}]{a_{\rightarrow}} (c, a)$, both given by a . We have $a_{\leftarrow} \cdot a_{\rightarrow} = a = \text{id}_{(c, a)}$, so (c, a) is a retract of $\iota_C(c)$. \square

LEMMA 5. *Every idempotent splits in \overline{C} .*

PROOF. Take an idempotent $e : \overline{C}(c, c)$. Note that c is given by an object $c_0 : C$ and an idempotent $a : C(c_0, c_0)$. Also, e is given by some idempotent $e : C(c_0, c_0)$ with $a \cdot e \cdot a = e$.

Now, we have $(c_0, e) : \overline{C}$ and morphisms $(c_0, a) \xrightleftharpoons[e_{\leftarrow}]{e_{\rightarrow}} (c_0, e)$, both given by e . We have $e_{\leftarrow} \cdot e_{\rightarrow} = e = \text{id}_{(c_0, e)}$, so (c_0, e) is a retract of (c_0, a) . Also, $e = e_{\rightarrow} \cdot e_{\leftarrow}$, so e is split. \square

REMARK 8. Note that the embedding is fully faithful, since

$$\overline{C}((c, \text{id}_c), (d, \text{id}_d)) = \{f : C(c, d) \mid \text{id}_c \cdot f \cdot \text{id}_d = f\} = C(c, d).$$

REMARK 9. Let D be a category. Suppose that we have a retraction-section pair in D , given by $d \xrightleftharpoons[r]{r} d'$. Now, suppose that we have an object $c : D$ and a morphism f with $(r \cdot s) \cdot f = f$. Then we get a morphism $s \cdot f : d' \rightarrow c$ such that f factors as $r \cdot (s \cdot f)$. Also, for any g with $r \cdot g = f$, we have

$$g = s \cdot r \cdot g = s \cdot f.$$

$$\begin{array}{ccccc} d & \xrightarrow{r} & d' & \xrightarrow{s} & d \\ & \searrow f & \downarrow s \cdot f & \swarrow f & \\ & & c & & \end{array}.$$

Therefore, d' is the equalizer of $d \xrightleftharpoons[r \cdot s]{\text{id}_d} d$. In the same way, it is also the coequalizer of this diagram.

Now, note that if we have a coequalizer c' of id_c and a , and an equalizer d' of id_d and b , the universal properties of these give an equivalence

$$D(c', d') \cong \{f : D(c, d') \mid a \cdot f = f\} \cong \{f : D(c, d) \mid a \cdot f = f = f \cdot b\}.$$

$$\begin{array}{ccccc} c & \xrightleftharpoons[a]{\text{id}_c} & c & \longrightarrow & c' \\ & & \downarrow & \searrow & \downarrow \\ d & \xleftarrow[b]{\text{id}_d} & d & \longleftarrow & d' \end{array}$$

Since a functor preserves retracts, and since every object of \overline{C} is a retract of an object in C , one can lift a functor from C (to a category with (co)equalizers) to a functor on \overline{C} .

For convenience, the lemma below works with pointwise left Kan extension using colimits, but one could also prove this using just (co)equalizers (or right Kan extension using limits).

LEMMA 6. *Let D be a category with colimits. We have an adjoint equivalence between $[C, D]$ and $[\overline{C}, D]$.*

PROOF. We already have an adjunction $\text{Lan}_{\iota_C} \dashv \iota_{C*}$. Also, since ι_C is fully faithful, we know that η is a natural isomorphism. Therefore, we only have to show that ϵ is a natural isomorphism. That is, we need to show that $\epsilon_G(c, a) : D(\text{Lan}_{\iota_C}(\iota_{C*}G)(c, a), G(c, a))$ is an isomorphism for all $G : [\overline{C}, D]$ and $(c, a) : \overline{C}$.

One of the components in the diagram of $\text{Lan}_{\iota_C}(\iota_{C*}G)(c, a)$ is the $\iota_{C*}G(c) = G(c, \text{id}_c)$ corresponding to $a : \iota_C(c) \rightarrow (c, a)$. This component has a morphism into our colimit

$$\varphi : C(G(\iota_C(c)), \text{Lan}_{\iota_C}(\iota_{C*}G)(c, a)).$$

Note that we can view a as a morphism $a : \overline{C}((c, a), \iota_C(c))$. This gives us our inverse morphism

$$G(a) \cdot \varphi : C(G(c, a), \text{Lan}_{\iota_C}(\iota_{C*}G)(c, a)).$$

□

LEMMA 7. *The formation of the opposite category commutes with the formation of the Karoubi envelope.*

PROOF. An object in $\overline{C^{\text{op}}}$ is an object $c : C^{\text{op}}$ (which is just an object $c : C$), together with an idempotent morphism $a : C^{\text{op}}(c, c) = C(c, c)$. This is the same as an object in \overline{C}^{op} .

A morphism in $\overline{C^{\text{op}}}((c, a), (d, b))$ is a morphism $f : C^{\text{op}}(c, d) = C(d, c)$ such that

$$b \cdot_C f \cdot_C a = a \cdot_{C^{\text{op}}} f \cdot_{C^{\text{op}}} b = f.$$

A morphism in $\overline{C}^{\text{op}}((c, a), (d, b)) = \overline{C}((d, b), (c, a))$ is a morphism $f : C(d, c)$ such that $b \cdot f \cdot a = f$.

Now, in both categories, the identity morphism on (c, a) is given by a .

Lastly, \overline{C}^{op} inherits morphism composition from C^{op} , which is the opposite of composition in C . On the other hand, composition in \overline{C}^{op} is the opposite of composition in \overline{C} , which inherits composition from C . \square

COROLLARY 3. *As the category **SET** is cocomplete, we have an equivalence between the category of presheaves on C and the category of presheaves on \overline{C} .*

5. Monoids as categories

Take a monoid M .

DEFINITION 15. We can construct a category C_M with $C_{M0} = \{\star\}$, $C_M(\star, \star) = M$. The identity morphism on \star is the identity $1 : M$. The composition is given by multiplication $g \cdot_{C_M} f = f \cdot_M g$.

REMARK 10. Actually, we have a functor from the category of monoids to the category of setcategories (categories whose object type is a set).

A monoid morphism $f : M \rightarrow M'$ is equivalent to a functor $F_f : C_M \rightarrow C_{M'}$. Any functor between C_M and $C_{M'}$ sends \star_M to $\star_{M'}$. The monoid morphism manifests as $F_f(m) = f(m)$ for $m : C_M(\star, \star) = M$.

LEMMA 8. *An isomorphism of monoids gives an (adjoint) equivalence of categories.*

PROOF. Given an isomorphism $f : M \rightarrow M'$. Then we have functors $F_f : C_M \rightarrow C_{M'}$ and $F_{f^{-1}} : C_{M'} \rightarrow C_M$. Take the identity natural transformations $\eta : \text{id}_{C_M} \Rightarrow F_f \bullet F_{f^{-1}}$ and $\epsilon : F_{f^{-1}} \bullet F_f \Rightarrow \text{id}_{C_{M'}}$. Of course these are natural isomorphisms. \square

DEFINITION 16. A *right monoid action* of M on a set X is a function $X \times M \rightarrow X$ such that for all $x : X$, $m, m' : M$,

$$x1 = x \quad \text{and} \quad (xm)m' = x(m \cdot m').$$

DEFINITION 17. A *morphism* between sets X and Y with a right M -action is an M -equivariant function $f : X \rightarrow Y$: a function such that $f(xm) = f(x)m$ for all $x : X$ and $m : M$.

LEMMA 9. *Presheaves on C_M are equivalent to sets with a right M -action.*

PROOF. This correspondence sends a presheaf F to the set $F(\star)$, and conversely, the set X to the presheaf F given by $F(\star) := X$. The M -action corresponds to the presheaf acting on morphisms as $xm = F(m)(x)$. A morphism (natural transformation) between presheaves $F \Rightarrow G$ corresponds to a function $F(\star) \rightarrow G(\star)$ that is M -equivariant, which is exactly a monoid action morphism. \square

DEFINITION 18. We can view M as a set U_M with right M -action $mn = m \cdot_M n$ for $m : U_M$ and $n : M$.

REMARK 11. Since the category of sets with an M -action is equivalent to a presheaf category, it has all limits. However, we can make this concrete. The set of the product $\prod_i X_i$ is the product of the underlying sets. The action is given pointwise by $(x_i)_i m = (x_i m)_i$.

DEFINITION 19. Given an object c in a category C with terminal object t . The global elements of c are the morphisms $C(t, c)$.

Note that the initial set with M -action is $\{\star\}$, with action $\star m = \star$.

LEMMA 10. *The global elements of a set with right M -action correspond to the elements that are invariant under the M -action.*

PROOF. A global element of X is a morphism $\varphi : \{\star\} \rightarrow X$ such that for all $m : M$, $\varphi(\star)m = \varphi(\star m) = \varphi(\star)$. Therefore, it is given precisely by the element $\varphi(\star) : X$, which must be invariant under the M -action. \square

LEMMA 11. *The category C of sets with an M -action has exponentials.*

PROOF. Given sets with M -action X and Y . Consider the set $C(M \times X, Y)$ with an M -action given by $\phi m'(m, x) = \phi(m'm, x)$. This is the exponential object X^Y , with the evaluation morphism $X \times X^Y \rightarrow Y$ given by $(x, \phi) \mapsto \phi(1, x)$. \square

5.1. Extension and restriction of scalars. Let $\varphi : M \rightarrow M'$ be a morphism of monoids.

LEMMA 12. *We get a restriction of scalars functor φ_* from sets with a right M' -action to sets with a right M -action.*

PROOF. Given a set X with right M' -action, take the set X again, and give it a right M -action, sending (x, m) to $x\varphi(m)$.

On morphisms, send an M' -equivariant morphism $f : X \rightarrow X'$ to the M -equivariant morphism $f : X \rightarrow X'$. \square

Since **SET** has colimits, and restriction of scalars corresponds to precomposition of presheaves (on $C_{M'}$), we can give it a left adjoint. This is the (pointwise) left Kan extension, which boils down to:

LEMMA 13. *We get an extension of scalars functor φ^* from sets with a right M -action to sets with a right M' -action.*

PROOF. Given a set X with right M -action. Take $Y = X \times M' / \sim$ with the relation $(xm, m') \sim (x, f(m) \cdot m')$ for $m : M$. This has a right M' -action given by $(x, m')n' = (x, m'n')$.

On morphisms, it sends $f : X \rightarrow X'$ to the morphism $(x, m') \mapsto (f(x), m')$. \square

LEMMA 14. *For U_M the set M with right M -action, we have $\varphi^*(U_M) \cong U_{M'}$.*

PROOF. The proof relies on the fact that for all $m : U_M$ and $m' : M'$, we have

$$(m, m') \sim (1, \varphi(m)m').$$

\square

Consider the category D with $D_0 = M'$ and

$$D(m', \overline{m}') = \{m : M \mid \varphi(m) \cdot m' = \overline{m}'\}.$$

If a category has an object t , such that there is a morphism to it from every other object in the category, t is said to be *weakly terminal*.

LEMMA 15. *Suppose that D has a weakly terminal element. Then for I_M the terminal set with right M -action, we have $\varphi^*(I_M) \cong I_{M'}$.*

PROOF. If D has a weakly terminal object, there exists $\overline{m}' : M'$ such that for all $m' : M'$, there exists $m : M$ such that $\varphi(m) \cdot m' = \overline{m}'$.

The proof relies on the fact that every element of $\varphi^*(I_M)$ is given by some (\star, m') , but then

$$(\star, m') = (\star \cdot m, m') \sim (\star, \varphi(m) \cdot m') = (\star, \overline{m}'),$$

so $\varphi^*(I_M)$ has exactly 1 element. \square

REMARK 12. For φ^* to preserve terminal objects, we actually only need D to be connected. The fact that $\varphi^*(I_M)$ is a quotient by a symmetric and transitive relation then allows us to ‘walk’ from any (\star, m'_1) to any other (\star, m'_2) in small steps.

For any $m'_1, m'_2 : M'$, consider the category $D_{m'_1, m'_2}$, given by

$$D_{m'_1, m'_2, 0} = \{(m', m_1, m_2) : M' \times M \times M \mid m'_i = \varphi(m_i) \cdot m'\}$$

and

$$D_{m'_1, m'_2}((m', m_1, m_2), (\bar{m}', \bar{m}_1, \bar{m}_2)) = \{m : M \mid \varphi(m) \cdot m' = \bar{m}', m_i = \bar{m}_i \cdot m\}.$$

LEMMA 16. *Suppose that $D_{m'_1, m'_2}$ has a weakly terminal object for all $m'_1, m'_2 : M'$. Then for sets A and B with right M -action, we have $\varphi^*(A \times B) \cong \varphi^*(A) \times \varphi^*(B)$.*

PROOF. Now, any element in $\varphi^*(A) \times \varphi^*(B) = (A \times M' / \sim) \times (B \times M' / \sim)$ is given by some (a, m'_1, b, m'_2) .

The fact that $D_{m'_1, m'_2}$ has a weakly terminal object means that we have some $\bar{m}' : M'$ and $\bar{m}_1, \bar{m}_2 : M$ with $m'_i = \varphi(\bar{m}_i) \cdot \bar{m}'$, such that for all $m' : M'$ and $m_1, m_2 : M$ with $m'_i = \varphi(m_i) \cdot m'$, there exists $m : M$ such that $\varphi(m) \cdot m' = \bar{m}'$ and $m_i = \bar{m}_i \cdot m$.

Therefore,

$$(a, m'_1, b, m'_2) = (a, \varphi(\bar{m}_1) \cdot \bar{m}', b, \varphi(\bar{m}_2) \cdot \bar{m}') \sim (a\bar{m}_1, \bar{m}', b\bar{m}_2, \bar{m}'),$$

so this is equivalent to some element in $\varphi^*(A \times B) = (A \times B \times M' / \sim)$.

The second part of weak terminality means that this equivalence is actually well-defined: equivalent elements in $\varphi^*(A) \times \varphi^*(B)$ are sent to equivalent elements in $\varphi^*(A \times B)$.

((TODO) Work this out more?)

□

CHAPTER 3

Univalent Foundations

1. Equality and homotopy
2. hProps and hSets
3. Transports and transport hell
4. The univalence axiom
5. Univalent categories

CHAPTER 4

Previous work in categorical semantics

1. The correspondence between categories and λ -calculi

In [SH80], Scott and Lambek argue that there is a correspondence between simply typed λ -calculi and cartesian closed categories (categories with products and ‘function objects’). (See page 413).

Types in the λ -calculus correspond to objects in the category.

Types $A \rightarrow B$ in the λ -calculus correspond to exponential objects B^A in the category.

Terms in the λ -calculus of type B , with free variables $x_1 : A_1, \dots, x_n : A_n$, correspond to morphisms $A_1 \times \dots \times A_n \rightarrow B$.

A free variable $x_i : A_i$ in a context with free variables $x_1 : A_1, \dots, x_n : A_n$ corresponds to the projection morphism $\pi_i : A_1 \times \dots \times A_n \rightarrow A_i$.

Given a term $s : B_1 \rightarrow B_2$ and a term $t : B_1$, both with free variables $x_1 : A_1, \dots, x_n : A_n$, corresponding to morphisms $\bar{s} : A_1 \times \dots \times A_n \rightarrow B_2$ and $\bar{t} : A_1 \times \dots \times A_n \rightarrow B_1$, the application $st : B_2$ corresponds to the composite of the product morphism with the evaluation morphism $A_1 \times \dots \times A_n \rightarrow B_2^{B_1} \times B_1 \rightarrow B_2$.

$$\begin{array}{ccccc}
 & A_1 \times \dots \times A_n & & & \\
 & \swarrow \bar{s} & \downarrow \langle \bar{s}, \bar{t} \rangle & \searrow \bar{t} & \\
 B_2^{B_1} & \xleftarrow{\pi_1} & B_2^{B_1} \times B_1 & \xrightarrow{\pi_2} & B_1 \\
 & & \downarrow ev & & \\
 & & B_2 & &
 \end{array}$$

Given a term $t : B$ with free variables $x_1 : A_1, \dots, x_n : A_n$, the abstraction $(\lambda x_n, t) : A_n \rightarrow B$ corresponds to using the adjunction $- \times A_n \dashv (-)^{A_n}$:

$$C(A_1 \times \dots \times A_{n-1} \times A_n, B) \simeq C(A_1 \times \dots \times A_{n-1}, B^{A_n}).$$

(TODO) How about the untyped λ -calculus?

2. Scott’s Representation Theorem

3. The Taylor Fibration

CHAPTER 5

The paper

1. The endomorphism theory

DEFINITION 20 (Endomorphism theory). Suppose that we have a category C and an object $X : C$, such that all powers X^n of X are also in C . The endomorphism theory $E(X)$ of X is the algebraic theory given by $E(X)_n = C(X^n, X)$ with projections as variables $x_{n,i} : X^n \rightarrow X$ and a substitution that sends $f : X^m \rightarrow X$ and $g_1, \dots, g_m : X^n \rightarrow X$ to $f \circ \langle g_i \rangle_i : X^n \rightarrow X^m \rightarrow X$.

LEMMA 17. $E(X)$ is indeed an algebraic theory.

PROOF. For $1 \leq j \leq l$, $f : E(X)_l$, $g : E(X)_m^l$ and $h : E(X)_n^m$, we have

$$\begin{aligned} x_j \bullet g &= x_j \circ \langle g_i \rangle_i = g_j, \\ f \bullet (x_{l,i})_i &= f \circ \langle x_{l,i} \rangle_i = f \circ \text{id}_{X^l} = f \end{aligned}$$

and

$$(f \bullet g) \bullet h = f \circ \langle g_i \rangle_i \circ \langle h_i \rangle_i = f \circ \langle g_i \circ \langle h_{i'} \rangle_{i'} \rangle_i = f \bullet (g_i \bullet h)_i.$$

□

DEFINITION 21 (Endomorphism λ -theory). Now, suppose that the exponential object X^X exists, and that we have morphisms back and forth $\text{abs} : X^X \rightarrow X$ and $\text{app} : X \rightarrow X^X$. Let, for $Y : C$, φ_Y be the isomorphism $C(X \times Y, X) \xrightarrow{\sim} C(Y, X^X)$. We can give $E(X)$ a λ -theory structure by setting, for $f : E(X)_{n+1}$ and $g : E(X)_n$,

$$\lambda(f) = \text{abs} \circ \varphi_{X^n}(f) \quad \rho(g) = \varphi_{X^n}^{-1}(\text{app} \circ g).$$

LEMMA 18. $E(X)$ is indeed a λ -theory.

PROOF. Note that $\varphi : C(- \times X, X) \xrightarrow{\sim} C(-, X^X)$ is a natural isomorphism, so for $g : E(X)_n^m$, the following diagram commutes

$$\begin{array}{ccc} C(X^m \times X, X) & \xrightarrow{- \circ (\langle g_i \rangle_i \times \text{id}_X)} & C(X^n \times X, X^X) \\ \varphi_{X^m}^{-1} \left(\begin{array}{c} \uparrow \\ \downarrow \end{array} \right) \varphi_{X^m} & & \varphi_{X^n}^{-1} \left(\begin{array}{c} \uparrow \\ \downarrow \end{array} \right) \varphi_{X^n} \\ C(X^m, X^X) & \xrightarrow{- \circ \langle g_i \rangle_i} & C(X^n, X^X) \end{array}$$

and note that $\langle g_i \rangle_i \times \text{id}_X = \langle g_1, \dots, g_m, x_{n+1} \rangle$. Then we have, for all $f : E(X)_m$

$$\begin{aligned} \lambda_m(f) \bullet g &= \text{abs} \circ \varphi_{X^m}(f) \circ \langle g_i \rangle_i \\ &= \text{abs} \circ \varphi_{X^n}(f \circ \langle g_1, \dots, g_m, x_{n+1} \rangle) \\ &= \lambda_n(f \bullet (g_1, \dots, g_m, x_{n+1})) \end{aligned}$$

and

$$\begin{aligned} \rho_n(f \bullet g) &= \varphi_{X^n}^{-1}(\text{app} \circ f \circ \langle g_i \rangle_i) \\ &= \varphi_{X^m}^{-1}(\text{app} \circ f) \circ \langle g_1, \dots, g_m, x_{n+1} \rangle \\ &= \rho_m(f) \bullet (g_1, \dots, g_m, x_{n+1}). \end{aligned}$$

□

2. The theory presheaf

DEFINITION 22 (The theory presheaf). Let T be an algebraic theory. We can turn T into an T -presheaf \tilde{T} by setting $\tilde{T}_n = T_n$ and using the substitution from T :

$$\bullet : \tilde{T}_m \times T_n^m \rightarrow \tilde{T}_n.$$

LEMMA 19. \tilde{T} is indeed a presheaf.

PROOF. For all $t : \tilde{T}_l$, $f : T_m^l$ and $g : T_n^m$,

$$t \bullet (x_{l,i})_i = t$$

and

$$(t \bullet f) \bullet g = t \bullet (f_i \bullet g)_i$$

because T is an algebraic theory. \square

LEMMA 20. Given an algebraic theory T and a T -presheaf Q , we have for all n a bijection of sets

$$\varphi : PT(\tilde{T}^n, Q) \cong Q_n.$$

PROOF. Take $\varphi(f) = f_n(x_1, \dots, x_n)$.

Conversely, take $\varphi^{-1}(q)$ to be the presheaf morphism that sends $t : T_m^n$ to $q \bullet t : Q_m$. This is indeed a presheaf morphism, since for all $t : T_l^n$ and $f : T_m^l$,

$$\varphi^{-1}(q)(t \bullet f) = q \bullet t \bullet f = \varphi^{-1}(q)(t) \bullet f.$$

Now, for a presheaf morphism $f : T^n \rightarrow Q$ and $t : T_m^n$, we have

$$\varphi^{-1}(\varphi(f))(t) = f_n(x_1, \dots, x_n) \bullet t = f_n((x_1, \dots, x_n) \bullet t) = f_n(t_1, \dots, t_n) = f_n(t).$$

Conversely, given $q : Q_n$, we have

$$\varphi(\varphi^{-1}(q)) = q \bullet (x_1, \dots, x_n) = q.$$

which concludes the proof. \square

3. The ‘+l’ presheaf

LEMMA 21. For all l , the presheaf \tilde{T}^l is exponentiable.

PROOF. We will show that $A(-, l)$ constitutes a right adjoint to the functor $- \times \tilde{T}^l$. We will do this using universal arrows ([ML98], Chapter IV.1, Theorem 2 (iv)). To that end, we will need for all $Q : PT$ a universal arrow $\varphi : A(Q, l) \times \tilde{T}^l \rightarrow Q$.

For $q : A(Q, l)_n = Q_{n+l}$ and $t : \tilde{T}_n^l$, we take $\varphi(q, t) = q \bullet_Q ((x_{n,i})_i + t)$.

This is a presheaf morphism, since for all $q : A(Q, l)_m^l$, $t : \tilde{T}_m^l$ and $f : T_n^m$,

$$\begin{aligned} \varphi((q, t) \bullet_{A(Q, l) \times \tilde{T}^l} f) &= \varphi(q \bullet_{A(Q, l)} f, t \bullet_{\tilde{T}^l} f) \\ &= q \bullet_{A(Q, l)} f \bullet_Q ((x_i)_i + (t \bullet_{\tilde{T}^l} f)) \\ &= q \bullet_Q ((\iota_{n,l}(f_i))_i + (x_{n+i})_i) \bullet_Q ((x_i)_i + (t \bullet_{\tilde{T}^l} f)) \\ &= q \bullet_Q ((\iota_{n,l}(f_i) \bullet_T ((x_j)_j + (t \bullet_{\tilde{T}^l} f)))_i + (x_{n+i} \bullet_T ((x_j)_j + (t \bullet_{\tilde{T}^l} f))))_i \\ &= q \bullet_Q ((f_i \bullet_T (x_j)_j)_i + ((t \bullet_{\tilde{T}^l} f)_i)_i) \\ &= q \bullet_Q ((f_i)_i + (t_i \bullet_{\tilde{T}} f)_i) \\ &= q \bullet_Q ((x_i \bullet_T f)_i + (t_i \bullet_T f)_i) \\ &= q \bullet_Q ((x_i)_i + t) \bullet_Q f \\ &= \varphi(q, t) \bullet_Q f. \end{aligned}$$

Now, given any presheaf $Q' : PT$ we need to show that any morphism $\psi : Q' \times \tilde{T}^l \rightarrow Q$ factors uniquely as $\varphi \circ (\tilde{\psi} \times \text{id}_{\tilde{T}^l})$ for some $\tilde{\psi} : Q' \rightarrow A(Q, l)$.

So, given such a ψ , and given $q : Q'_n$, we take $\tilde{\psi}(q) = \psi(\iota_{n,l}(q), (x_{n+i})_i)$. This is a presheaf morphism, since for all $q : Q'_m$ and $f : T_n^m$,

$$\begin{aligned}\tilde{\psi}(q \bullet f) &= \psi(\iota_{n,l}(q \bullet f), (x_{n+i})_i) \\ &= \psi(q \bullet (\iota_{n,l}(f_i))_i, (x_{n+i})_i) \\ &= \psi((\iota_{m,l}(q), (x_{m+i})_i) \bullet_{Q' \times \tilde{T}^l} ((\iota_{n,l}(f_i))_i + (x_{n+i})_i)) \\ &= \psi(\iota_{m,l}(q), (x_{m+i})_i) \bullet_Q ((\iota_{n,l}(f_i))_i + (x_{n+i})_i) \\ &= \tilde{\psi}(q) \bullet_{A(Q,l)} f.\end{aligned}$$

Note that indeed $\varphi \circ (\tilde{\psi} \times \text{id}_{\tilde{T}^l}) = \psi$:

$$\begin{aligned}\varphi(\tilde{\psi}(q), t) &= \varphi(\psi(\iota_{n,l}(q), (x_{n+i})_i), t) \\ &= \psi(\iota_{n,l}(q), (x_{n+i})_i) \bullet ((x_i)_i + t) \\ &= \psi(\iota_{n,l}(q) \bullet ((x_i)_i + t), (x_{n+i})_i \bullet ((x_i)_i + t)) \\ &= \psi(q \bullet (x_i)_i, (t_i)_i) \\ &= \psi(q, t).\end{aligned}$$

Now, suppose that we have another $\tilde{\psi}' : Q' \rightarrow A(Q, l)$ such that $\varphi \circ (\tilde{\psi}' \times \text{id}_{\tilde{T}^l}) = \psi$. Then we have

$$\begin{aligned}\tilde{\psi}(q) &= \psi(\iota_{n,l}(q), (x_{n+i})_i) \\ &= (\varphi \circ (\tilde{\psi}' \times \text{id}_{\tilde{T}^l}))(\iota_{n,l}(q), (x_{n+i})_i) \\ &= \varphi(\tilde{\psi}'(\iota_{n,l}(q)), (x_{n+i})_i) \\ &= \tilde{\psi}'(\iota_{n,l}(q)) \bullet ((x_i)_i + (x_{n+i})_i) \\ &= \iota_{n,l}(\tilde{\psi}'(q)) \bullet ((x_i)_i + (x_{n+i})_i) \\ &= \tilde{\psi}'(q) \bullet (x_i)_i \\ &= \tilde{\psi}'(q),\end{aligned}$$

so $\tilde{\psi}$ is unique, which completes the proof. \square

Now, this adjunction $- \times \tilde{T}^l \dashv A(-, l)$ induces a natural isomorphism

$$\varphi : PT(- \times \tilde{T}^l, \tilde{T}) \xrightarrow{\sim} PT(-, A(\tilde{T}, l))$$

LEMMA 22. For all $f : PT(\tilde{T}^n \times \tilde{T}^l, \tilde{T})$,

$$\varphi_{\tilde{T}^n}(f)(q) = f(\iota_{m,l}(q), (x_{m+i})_i)$$

PROOF. (TODO) \square

LEMMA 23. For all $f : PT(\tilde{T}^n, A(\tilde{T}, l))$,

$$\varphi_{\tilde{T}^n}^{-1}(f)(q, t) = f(q) \bullet ((x_i)_i + t).$$

PROOF. (TODO) \square

4. Scott's Representation Theorem

THEOREM 1. Any λ -theory L is isomorphic to the endomorphism λ -theory $E(\tilde{L})$ of \tilde{L} in the presheaf category of L .

PROOF. First of all, remember that \tilde{L} is indeed exponentiable and that $\tilde{L}^{\tilde{L}} = A(\tilde{L}, 1)$. Now, since L is a λ -theory, we have functions back and forth $\lambda : A(\tilde{L}, 1) \rightarrow \tilde{L}$ and $\rho : \tilde{L} \rightarrow A(\tilde{L}, 1)$. These are presheaf morphisms because for all $f : A(\tilde{L}, 1)_m$ and $g : \tilde{L}_m$ and $t : T_n^m$,

$$\lambda(f \bullet_{A(\tilde{L}, 1)} t) = \lambda(f \bullet_{\tilde{L}} ((\iota_{m,1}(t_i))_i + (x_{n+1}))) = \lambda(f) \bullet_{\tilde{L}} t$$

and

$$\rho(g \bullet_{\tilde{L}} t) = \rho(g) \bullet_{\tilde{L}} ((\iota_{m,1}(t_i))_i + (x_{n+1})) = \rho(g) \bullet_{A(\tilde{L}, 1)} t.$$

Therefore, $E(\tilde{L})$ is indeed a λ -theory.

For any presheaf Q and for any n , we have a bijection $PL(L^n, Q) \cong Q_n$. Then we have $\varphi : E(\tilde{L})_n \cong L_n$. This bijection is an isomorphism of λ -theories, since it preserves the x_i , \bullet , ρ and λ : for all $1 \leq j \leq n$, $f : E(\tilde{L})_m$, $g : E(\tilde{L})_{m+1}$ and $h : E(\tilde{L})_n^m$.

$$\begin{aligned} \varphi(x_j) &= x_j(x_1, \dots, x_n) \\ &= x_j; \\ \varphi(f \bullet h) &= f \circ \langle h_i \rangle_i((x_i)_i) \\ &= f((h_i((x_j)_j))_i) \\ &= f((x_i)_i \bullet (h_i((x_j)_j))_i) \\ &= f((x_i)_i) \bullet (h_i((x_j)_j))_i \\ &= \varphi(f) \bullet (\varphi(h_i))_i; \\ \varphi(\rho(f)) &= \rho(f)((x_i)_i) \\ &= \rho(f((x_i)_i)) \bullet (x_i)_i \\ &= \rho(f((x_i)_i)) \\ &= \rho(\varphi(f)); \\ \varphi(\lambda(g)) &= \lambda(g)((x_i)_i) \\ &= \lambda(\varphi_{X^n}(g))((x_i)_i) \\ &= \lambda(g(\iota_{m,t}((x_i)_i) + (x_{m+1}))) \\ &= \lambda(g((x_i)_i)) \\ &= \lambda(\varphi(g)). \end{aligned}$$

□

5. Locally cartesian closedness of the category of retracts

DEFINITION 23 (Category of retracts). The category of retracts for a λ -theory L is the category with objects $f : L_n$ such that $f \bullet f = f$ and it has as morphisms $g : f \rightarrow f'$ the terms $g : L_n$ such that $f' \bullet g \bullet f = g$. The object $f : L_n$ has identity element f , and we have composition $g \circ g' = g \bullet g'$. These are morphisms (**TODO**)

LEMMA 24. *The category of retracts is indeed a category.*

PROOF. (**TODO**)

□

THEOREM 2. *The category of retracts is locally cartesian closed (**TODO**) .*

6. Equivalences

7. The Fundamental Theorem of the λ -calculus

DEFINITION 24 (Λ). There is a special λ -theory, given by the λ -calculus itself. Λ_n is the set of λ -terms with n free variables, the x_i are the free variables, and

\bullet is given by substitution. λ sends $f : \Lambda_{n+1}$ to $\lambda x_{n+1}.f$ and ρ sends $f : \Lambda_n$ to $\iota_{n,1}(f)x_{n+1}$ in Λ_n .

LEMMA 25. Λ is indeed a λ -theory.

PROOF. (TODO) □

LEMMA 26. Λ is the initial λ -theory.

PROOF. Given a λ -theory L , we construct a morphism $f : \Lambda \rightarrow L$ by induction on the λ -terms. We set $f(x_i) = x_i$, $f(\lambda(t)) = \lambda(f(t))$ and $f(st) = \rho(f(s)) \bullet ((x_i)_i + (f(t)))$.

This is a λ -theory morphism because (TODO)

It is unique, since (TODO) □

DEFINITION 25 (Pullback of algebras). If we have a morphism of algebraic theories $f : T' \rightarrow T$, we have a functor $AT \rightarrow AT'$.

On objects, it sends a T -algebra A to a T' -algebra with set A and action $g \bullet_{T'} a = f(g) \bullet_T a$. This is a T' -algebra because (TODO).

On morphisms, it sends $\varphi : A \rightarrow A$ to $\varphi : A \rightarrow A$. This is a T' -algebra morphism because for all $g : T'_n$ and $a : A^n$, we have

$$\varphi(g \bullet_{T'} a) = \varphi(f(g) \bullet_T a) = f(g) \bullet_T \varphi(a) = g \bullet_{T'} \varphi(a).$$

LEMMA 27. This is indeed a functor.

PROOF. (TODO) □

DEFINITION 26 (Term algebra). Given an algebraic theory T , for every n , T_n together with the action operator $\bullet : T_m \times T_n^m \rightarrow T_n$ gives a T -algebra.

LEMMA 28. T_n is indeed a T -algebra.

PROOF. (TODO) □

DEFINITION 27. For all n , we have a functor from lambda theories to Λ -algebras. It sends the λ -theory L to the L -algebra L_n and then turns this into a Λ -algebra via the morphism $\Lambda \rightarrow L$.

It sends morphisms $f : L \rightarrow L'$ to $f_n : L_n \rightarrow L'_n$. This is a Λ -algebra morphism because (TODO)

LEMMA 29. This indeed constitutes a functor.

PROOF. (TODO) □

REMARK 13. Note that for a monoid M , if we view M as a category, the category $[M^{\text{op}}, \mathbf{SET}]$ consists of sets with a right M -action.

DEFINITION 28 (The exponential object in the presheaf category). Given a monoid M , if we have two presheaves (sets with right M -actions) P and P' , we have a set of M -equivariant maps

$$F_{P,P'} = \left\{ f : M \times P \rightarrow P' \mid \prod_{p:P, m,m':M} f(m,p)m' = f(mm',pm') \right\}$$

with a right M -action, given by $f m'(m,p) = f(m'm,p)$. This is again M -equivariant because

$$f m'(m,p)m'' = f(m'm,p)m'' = f(m'mm'',pm'') = f m'(mm'',pm''),$$

so $F_{P,P'}$ is a presheaf.

Now, to show that $F_{P,P'}$ is the exponential object P'^P , we show that for any P , $F_{P,-}$ is the left adjoint of $- \times P$. So we need for all $P' : PT$, a universal arrow $\varphi : F_{P,P'} \times P \rightarrow P'$.

First of all, we have an evaluation map $\varphi : F_{P,P'} \times P \rightarrow P'$ given by $(f, p) \mapsto f(I, p)$ for I the unit of the monoid. This map is equivariant because for all m ,

$$(f, p)m = (fm, pm) \mapsto fm(I, pm) = f(m, pm) = f(I, p)m.$$

Now, given any presheaf Q and any morphism $\psi : Q \times P \rightarrow P'$, take $\tilde{\psi} : Q \rightarrow F_{P,P'}$ given by $\tilde{\psi}(q)(m, p) = \psi(qm, p)$. This is equivariant because

$$\tilde{\psi}(q)m(m', p) = \tilde{\psi}(q)(mm', p) = \psi(qmm', p) = \tilde{\psi}(qm)(m', p)$$

and we have

$$\varphi(\tilde{\psi}(q), p) = \tilde{\psi}(q)(I, p) = \psi(q, p).$$

Now, suppose that we have $\tilde{\psi}' : Q \rightarrow F_{P,P'}$ such that $\varphi \circ (\tilde{\psi}' \times \text{id}_P) = \psi$. Then for all $q : Q$, $m : M$ and $p : P$,

$$\tilde{\psi}(q)(m, p) = \psi(qm, p) = \varphi(\tilde{\psi}'(qm), p) = \tilde{\psi}'(qm)(I, p) = \psi'(q)m(I, p) = \psi'(q)(m, p),$$

so $\tilde{\psi}$ is unique and $F_{P,P'}$ is an exponential object.

DEFINITION 29 (n-functional terms). Let A be a Λ -algebra. We define

$$A(n) = \{a : A \mid (\lambda x_2 x_3 \dots x_{n+1}, x_1 x_2 x_3 \dots x_{n+1}) \bullet a = a\}.$$

DEFINITION 30. Take $\mathbf{1}_n = (\lambda x_1 \dots x_n, x_1 \dots x_n) \bullet () : A$.

DEFINITION 31. We define composition as $a \circ b = (\lambda x_3, x_1(x_2 x_3)) \circ (a, b)$ for $a, b : A$.

LEMMA 30. *This composition is associative.*

PROOF. **(TODO)** □

DEFINITION 32 (The monoid of a Λ -algebra). Now we make $A(1)$ into a monoid with unit $\lambda x_1, x_1$.

LEMMA 31. *This is indeed a monoid.*

PROOF. **(TODO)** □

From here on, we will assume that Λ (and therefore, any λ -theory) satisfies β -equality.

LEMMA 32. *For $a : A$, a is in $A(n)$ iff $\mathbf{1}_n \circ a = a$.*

PROOF.

$$\begin{aligned} \mathbf{1}_n \circ a &= (\lambda x_3, x_1(x_2 x_3)) \bullet (((\lambda x_1 \dots x_n, x_1 \dots x_n) \bullet ()), a) \\ &= (\lambda x_3, x_1(x_2 x_3)) \bullet (((\lambda x_2 \dots x_{n+1}, x_2 \dots x_{n+1}) \bullet a), x_1 \bullet a) \\ &= ((\lambda x_3, x_1(x_2 x_3)) \bullet ((\lambda x_2 \dots x_{n+1}, x_2 \dots x_{n+1}), x_1)) \bullet a \\ &= (\lambda x_2, (\lambda x_3 \dots x_{n+2}, x_3 \dots x_{n+2})(x_1 x_2)) \bullet a \\ &= (\lambda x_2 x_3 \dots x_{n+1}, x_1 x_2 \dots x_{n+1}) \bullet a. \end{aligned}$$

□

DEFINITION 33 (The presheaf category of a Λ -algebra). Let A be a Λ -algebra. If we view the monoid $A(1)$ as a one-object category, we define the category PA to be the category of presheaves $[A(1)^{\text{op}}, \mathbf{SET}]$.

DEFINITION 34 (The objects $A(n)$ in PA). Given $a : A(n)$ and $b : A(1)$, we have

$$\mathbf{1}_n \circ (a \circ b) = (\mathbf{1}_n \circ a) \circ b = a \circ b,$$

so $a \circ b : A(n)$ and we have a right $A(1)$ -action on $A(n)$, which makes $A(n)$ into an object in PA .

LEMMA 33. We have $A(1)^{A(1)} \cong A(2)$.

PROOF. We have a bijection $\varphi : A(2) \cong F_{A(1), A(1)}$, given by

$$\varphi(a)(b, b') = (\lambda x_4, x_1(x_2 x_4)(x_3 x_4)) \bullet (a, b, b').$$

Note that $\varphi(d)$ is equivariant because **(TODO)** Now, φ is a presheaf morphism because **(TODO)**

Take $p = \lambda x_1, x_1(\lambda x_2 x_3, x_2)$ and $q = \lambda x_1, x_1(\lambda x_2 x_3, x_3)$. These are elements of $A(1)$. Note that for terms c_1, c_2

$$\begin{aligned} p(\lambda x_1, x_1 c_1 c_2) &= (\lambda x_1, x_1 c_1 c_2)(\lambda x_2 x_3, x_2) \\ &= (\lambda x_1 x_3, x_2) c_1 c_2 \\ &= c_1. \end{aligned}$$

In the same way, $q \circ (\lambda x_1 x_2, x_2 c_1 c_2) = c_2$.

An inverse is given by

$$\psi : f \mapsto \lambda x_1 x_2, f(p, q)(\lambda x_3, x_3 x_1 x_2).$$

This is a presheaf morphism because **(TODO)**

This is an inverse, because given $f : F_{A(1), A(1)}$ and $(a_1, a_2) : A(1) \times A(1)$, we have

$$\begin{aligned} \varphi(\psi(f))(a_1, a_2) &= u(\lambda x_1 x_2, f(p, q)(\lambda x_3, x_3 x_1 x_2))(a_1, a_2) \\ &= \lambda x_1, (\lambda x_2 x_3, f(p, q)(\lambda x_4, x_4 x_2 x_3))(a_1 x_1)(a_2 x_1) \\ &= \lambda x_1, f(p, q)(\lambda x_2, x_2(a_1 x_1)(a_2 x_1)) \\ &= f(p, q) \circ (\lambda x_1, (\lambda x_2, x_2(a_1 x_1)(a_2 x_1))) \\ &= f(p \circ (\lambda x_1, (\lambda x_2, x_2(a_1 x_1)(a_2 x_1))), q \circ (\lambda x_1, (\lambda x_2, x_2(a_1 x_1)(a_2 x_1)))) \\ &= f(\lambda x_1, p(\lambda x_2, x_2(a_1 x_1)(a_2 x_1)), \lambda x_1, q(\lambda x_2, x_2(a_1 x_1)(a_2 x_1))) \\ &= f(\lambda x_1, a_1 x_1, \lambda x_1, a_2 x_1) \\ &= f(a_1, a_2). \end{aligned}$$

The last line is because $a_i : A(1)$ and therefore $\lambda x_1, a_i x_1 = a_i$.

On the other hand, if we have $a_1 : A(2)$, we have

$$\begin{aligned} \psi(\varphi(a_1)) &= \psi((a_2, a_3) \mapsto \lambda x_1, a_1(a_2 x_1)(a_3 x_1)) \\ &= \lambda x_1 x_2, (\lambda x_3, a_1(p x_3)(q x_3))(\lambda x_3, x_3 x_1 x_2) \\ &= \lambda x_1 x_2, a_1(p(\lambda x_3, x_3 x_1 x_2))(q(\lambda x_3, x_3 x_1 x_2)) \\ &= \lambda x_1 x_2, a_1 x_1 x_2 \\ &= a_1. \end{aligned}$$

The last line is because $a_1 : A(2)$ and therefore $\lambda x_1 x_2, a_1 x_1 x_2 = a_1$.

Therefore, this map is a bijection and an isomorphism. \square

DEFINITION 35 (Endomorphism λ -theory of a Λ -algebra). PA borrows products from **SET**. Therefore, the algebraic theory $E(A(1))$ exists. Now note that $A(1)$ is exponentiable and $A(1)^{A(1)} \cong A(2)$. Note that $A(2) \subseteq A(1)$ and that $(\lambda x_2 x_3, x_1 x_2 x_3) \bullet -$ gives a function from $A(1)$ to $A(2)$. This gives $E(A(1))$ a λ -theory structure.

DEFINITION 36 (Pullback functor on presheaves for a Λ -algebra). Let $f : A \rightarrow A'$ be a Λ -algebra morphism. Then for all $a : A(n)$,

$$\mathbf{1}_n \circ f(a) = f(\mathbf{1}_n) \circ f(a) = f(\mathbf{1}_n \circ a),$$

so we have an induced morphism $f : A(n) \rightarrow A'(n)$.

Now, given a presheaf $P : PA'$. We can create a presheaf $f^*P : PA$ by taking the set of P , and, for $p : P$ and $a : A$, setting $pa = p \circ f(a)$. This is indeed a presheaf because **(TODO)**

Now, given a morphism $g : P \rightarrow P'$, we get a morphism by taking the function on the sets of P and P' . This is a morphism because **(TODO)**

LEMMA 34. *The above indeed constitutes a functor.*

PROOF. **(TODO)** □

Left Kan extension then gives a left adjoint $f_* : PA \rightarrow PA'$.

LEMMA 35. *We have $f_*(A(1)) \cong A'(1)$.*

PROOF. **(TODO)** □

LEMMA 36. *f_* preserves finite products.*

PROOF. **(TODO)** □

DEFINITION 37. Since f_* preserves finite products, given an element of $g : E(A(1))(n) = PA(A(1)^n, A(1))$, we get

$$\#f_*(g) : PA'(f(A(1)^n), f(A(1))) \cong PA'(A'(1)^n, A'(1)) = E(A'(1))(n).$$

LEMMA 37. *$\#f_* : E(A(1)) \rightarrow E(A'(1))$ is a map of λ -theories.*

PROOF. **(TODO)** □

DEFINITION 38. We have an isomorphism $E(A(1))(0) \cong A$ given by $a \mapsto aI$.

LEMMA 38. *This is indeed an isomorphism of Λ -algebras.*

PROOF. **(TODO)** □

LEMMA 39. *Given $g : A \rightarrow A'$,*

THEOREM 3. *There exists an adjoint equivalence between the category of λ -theories, and the category of algebras of Λ .*

PROOF. We will show that the functor $L \mapsto L_0$ is an equivalence of categories. It is essentially surjective, because L is isomorphic **(TODO)** to $E(A(1))$.

Now, given morphisms $f, f' : L \rightarrow L'$. Suppose that $f_0 = f'_0$. Suppose that L and L' have β -equality. Then, given $l : L_n$, we have

$$f_n(l) = \rho^n(\lambda^n(f_n(l))) = \rho^n(f_0(\lambda^n(l))) = \rho^n(f'_0(\lambda^n(l))) = \rho^n(\lambda^n(f'_n(l))) = f'_n(l),$$

so the functor is faithful.

The functor is full because a Λ -algebra morphism $f : A \rightarrow A'$ induces a functor $f^* : PA' \rightarrow PA$, and via left Kan extension we get a left adjoint $f_* : PA \rightarrow PA'$ with $f_*(A(1)) \cong A'(1)$. Now, f_* preserves (finite) products, so we have maps $PA(A(1)^n, A(1)) \rightarrow PA'(A'(1)^n, A'(1))$ and so a map $E(A(1)) \rightarrow E(A'(1))$. This map, when restricted to a map $PA(1, A(1)) \rightarrow PA'(1, A'(1))$, and transported along the isomorphism $a \mapsto aI$ **(TODO)**, is equal to f **(TODO)**. □

8. An alternative proof for the fundamental theorem

9. Theory of extensions

LEMMA 40. *The category of T -algebras has coproducts.*

PROOF. **(TODO)** □

DEFINITION 39 (Theory of extensions). Let T be an algebraic theory and A a T -algebra. We can define an algebraic theory T_A called ‘the theory of extensions of A ’ with $(T_A)_n = T_n + A$. The left injection of the variables $x_i : T_n$ gives the variables. Now, take $h : (T_n + A)^m$. Sending $g : T_m$ to $\varphi(g) := g \bullet h$ gives a T -algebra morphism $T_m \rightarrow T_n + A$ since

$$\varphi(f \bullet g) = f \bullet g \bullet h = f \bullet (g_i \bullet h) = f \bullet (\varphi(g_i))_i.$$

This, together with the injection morphism of A into $T_n + A$, gives us a T -algebra morphism from the coproduct: $T_m + A \rightarrow T_n + A$. We especially have a function on sets $(T_m + A) \times (T_n + A)^m \rightarrow T_n + A$, which we will define our substitution to be.

LEMMA 41. *T_A is indeed an algebraic theory.*

PROOF. **(TODO)** □

CHAPTER 6

The formalization

1. Statistics

2. Components

3. Displayed categories

3.1. Defininig objects by their category instead of the other way around.

3.2. `_ax` for categories and their objects.

3.3. Cartesian vs cartesian'.

4. Inductive types

5. The formalization of the λ -calculus

Defining Lambda Calculus in a different way (not as an axiomatized HIT) - As set quotient instead of HIT - With a signature

6. Tuples

$stnm \rightarrow A$ vs $\text{vec } A$

7. Products

$T \times (T \times \cdots \times T)$ vs $T \times T^{n \text{ vs } T(Sn)}$ Terminal as product over empty set over any set with a function to empty.

8. The $n + p$ -presheaf

$L(S\ n)$ (for lambda) vs $L(n + 1)$ (stemming from the naive implementation of the $L(n + p)$ presheaf)

9. Quotients

Quotients (by `hrel` or `eqrel`) vs coproducts (generalizing to arbitrary category with coproduct) vs a category with some structure

10. The Karoubi envelope

`KanExtension` instead of specific construction at `KaroubiEnvelope`

11. Univalence

Univalence bewijzen via `isweqhomot` vs direct

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APPENDIX A

Alternative definitions

The literature, there are many different but equivalent definitions, carrying many different names, for the objects that called ‘algebraic theories’ in Section 1. What makes matters even more confusing is the fact that the literature does not provide a consistent bijection between names and definitions. This section will showcase some of the various definitions.

1. Abstract Clone

DEFINITION 40 (abstract clone). An algebraic theory as presented in Section 1, is usually called an *abstract clone*. In this thesis, outside of this specific section, we will call it algebraic theory to be consistent with the names that Hyland attaches to objects.

2. Lawvere theory

DEFINITION 41 (algebraic theory). An *algebraic theory* as presented in [ARV10] is a small category with finite products.

DEFINITION 42 (algebraic theory algebra). An *algebra* for an algebraic theory T is a finite-product-preserving functor $T \rightarrow \mathbf{SET}$.

This definition is more general than the definition of algebraic theory in Section 1. To make it equivalent, we have to be more specific about the objects of the category:

DEFINITION 43 (Lawvere theory). A *Lawvere theory*, or *one-sorted algebraic theory* is a category L , with $L_0 = \{0, 1, \dots\}$, such that $n = 1^n$, the n -fold product.

2.1. Algebras for Lawvere Theories.

3. Relative Monad

DEFINITION 44 (relative monad). Let $S : C \rightarrow D$ be a functor. A *relative monad* on S is a functor $T : C \rightarrow D$, together with a natural transformation $\eta : S \Rightarrow T$ and a transformation $(-)^* : D(S(X), T(Y)) \rightarrow D(T(X), T(Y))$, natural in both S and T , such that for all $f : S(X) \rightarrow T(Y)$ and $g : S(Y) \rightarrow T(Z)$,

$$f = \eta_X \cdot f^*, \quad \eta_X^* = \text{id}_{TX} \quad \text{and} \quad (f \cdot g^*)^* = f^* \cdot g^*.$$

4. Cartesian Operad

DEFINITION 45 (multicategory). A *multicategory* C consists of

- A collection of objects C_0 ;
- For all $c_1, \dots, c_n, d : C_0$, a collection of multimorphisms $C((c_1, \dots, c_n), d)$;
- For all $c : C_0$, an identity morphism $\text{id}_c : C(c, c)$;
- For all multimorphisms $f : C((d_1, \dots, d_n), e)$ and $g_i : C((c_{i,1}, \dots, c_{i,m_i}), d_i)$ for $1 \leq n \leq i$, a composite multimorphism

$$(g_1, \dots, g_n) \cdot f : C((c_{1,1}, \dots, c_{n,m_n}), e)$$

such that these satisfy certain identity and associativity axioms (see for example [Lei04], page 35).

Let S_n be the symmetric group on n elements (the group consisting of all ways to shuffle the numbers $(1, \dots, n)$).

DEFINITION 46 (symmetric multicategory). A *symmetric* multicategory is a multicategory C , together with, for each $\sigma : S_n$ and all c_1, \dots, c_n, d , a bijection

$$\bar{\sigma} : C((c_1, \dots, c_n), d) \rightarrow C((c_{\sigma(1)}, \dots, c_{\sigma(n)}), d),$$

such that for all $\rho, \sigma : S_n$ and $\theta : C((c_1, \dots, c_n), d)$,

$$\bar{\rho}(\bar{\sigma}(\theta)) = \overline{(\sigma\rho)}(\theta) \quad \text{and} \quad \overline{1_{S_n}}(\theta) = \theta,$$

compatible with multimorphism composition (see [Lei04], page 52, equation (2:7)).

In some sense, for a symmetric multicategory, the order of the objects in the domain of a multimorphism does not matter.

For a collection S and a tuple $x = x_1, \dots, x_n : S$ and $1 \leq i \leq j \leq n$,

- let $u_i(x) : S^{n+i+1-j}$ denote the tuple x , but with the subtuple (x_i, \dots, x_j) repeated once:

$$(x_1, \dots, x_i, \dots, x_j, x_i, \dots, x_j, \dots, x_n).$$

- let $v_i(x) : S^{n-i-1+j}$ denote the tuple x , but with the subtuple (x_i, \dots, x_j) removed:

$$(x_1, \dots, x_{i-1}, x_{j+1}, \dots, x_n).$$

DEFINITION 47 (cartesian multicategory). A *cartesian multicategory* is a multicategory C , with, for all $c_1, \dots, c_n, d : C_0$ and $1 \leq i \leq n$

- a *contraction* operation $\bar{u}_i : C(u_{i,i}(c), d) \rightarrow C(c, d)$,
- a *deletion* operation $\bar{v}_i : C(v_{i,i}(c), d) \rightarrow C(c, d)$,

that satisfy some axioms (**TODO**)

DEFINITION 48 (cartesian operad). A *cartesian operad* is a cartesian multicategory with one object.

LEMMA 42. *The following are equivalent:*

- (1) *An abstract clone.*
- (2) *A Lawvere theory.*
- (3) *A relative monad on the embedding $\iota : \mathbf{FinSET} \hookrightarrow \mathbf{SET}$.*
- (4) *A cartesian operad.*

PROOF. (1) \Rightarrow (2): Given an abstract clone C , we construct a Lawvere theory L as follows: We have objects $L_0 = \{0, 1, \dots\}$ and morphisms $L(m, n) = C_m^n$. The identity morphism is $\text{id}_n = (x_i)_i : L(n, n)$ and for $f : L(l, m)$, $g : L(m, n)$, we have composition

$$f \cdot g = (g_i \bullet f)_i : L(l, n).$$

Lastly, we have product projections $\pi_{n,i} = x_{n,i} : L(n, 1)$ for all $1 \leq i \leq n$.

(2) \Rightarrow (3): Given a Lawvere theory L , we construct a relative monad (F, η) as follows: For $n : \mathbf{FinSET}$, take $F(n) = L(n, 1)$ and for $a : \mathbf{FinSET}(m, n)$, let $F(a) : \mathbf{SET}(F(m), F(n))$ be the morphism given by precomposition with the product morphism

$$\langle \pi_{n,a(1)}, \dots, \pi_{n,a(m)} \rangle : L(n, m).$$

The natural transformation is given by $\eta_n(i) = \pi_{n,i} : F(n)$ for all $i : \iota(n) = \{1, \dots, n\}$. Lastly, for $f : \mathbf{SET}(\iota(m), F(n))$, let $f : \mathbf{SET}(F(m), F(n))$ be given by precomposition with the product morphism

$$\langle f(1), \dots, f(m) \rangle : L(n, m)$$

(3) \Rightarrow (4): Given a relative monad (F, η) , we construct a cartesian operad A as follows: We take a multicategory with $A_0 : \{\star\}$. We take as the set of n -fold morphisms $A((\star)_i, \star) = F(n)$. We take an identity multimorphism $\text{id}_\star = \eta_1(1) : A(\star, \text{star})$. For some natural number m , natural numbers n_1, \dots, n_m and $i : 1 \leq i \leq m$ let $\phi_i : \mathbf{FinSET}(n_i, \sum n_i)$ be the standard injections into the coproduct. For $f : F(m)$ and $g_i : F(n_i)$, we have $F(\phi_i) : \mathbf{SET}(F(n_i), F(\sum n_i))$, so we have $(F(\phi_i)(g_i))_i : \mathbf{SET}(\iota(m), F(\sum n_i))$ and the composition is given by

$$f \cdot g = (F(\phi_i)(g_i))_i^*(f) : F\left(\sum n_i\right).$$

The permutation, contraction and deletion operations given by

$$\bar{\sigma} = F(\sigma) : A((\star_1, \dots, \star_n), \star) \rightarrow A((\star_{\sigma(1)}, \dots, \star_{\sigma(n)}), \star)$$

Given a cartesian operad A (**TODO**)

□