Definition 1. An algebraic theory is a functor $\mathcal{T} : F \to SET$ together with a composition morphism $\bullet : \mathcal{T}(m) \times \mathcal{T}(n)^m \to \mathcal{T}(n)$ for all m, n and elements (projections) $\pi_{n,i} \in \mathcal{T}(n)$ for all $1 \leq i \leq n$. The composition must be associative, unital, compatible with projections and dinatural in m.

This is equivalent to:

Definition 2. An abstract clone is a function $C : \mathbb{N} \to SET$, together with a composition morphism $\bullet : C(m) \times C(n)^m \to C(n)$ for all m, n and elements (projections) $\pi_{n,i} \in C(n)$ for all $1 \leq i \leq n$, such that

$$\pi_{n,i} \bullet (f_1, \ldots, f_n) = f_i;$$
$$f \bullet (\pi_{1,n}, \ldots, \pi_{n,n}) = f;$$
$$(f \bullet (g_1, \ldots, g_m)) \bullet (h_1, \ldots, h_n) = f \bullet (g_1 \bullet (h_1, \ldots, h_n), \ldots, g_m \bullet (h_1, \ldots, h_n)).$$

Definition 3. An algebra for an algebraic theory $\mathcal{T}$ is a set A with an associative unital action $\mathcal{T}(n) \times A^n \to A$, natural in n.

Definition 4. A λ-theory is an algebraic theory $\mathcal{L}$ together with retractions $\mathcal{L}(n+1) \lhd \mathcal{L}(n)$ with retraction $\rho : \mathcal{L}(n) \to \mathcal{L}(n+1)$ and section $\lambda : \mathcal{L}(n+1) \to \mathcal{L}(n)$ natural in n and compatible (?) with the actions $\mathcal{L}(m) \times \mathcal{L}(n)^m \to \mathcal{L}(n)$ and $\mathcal{L}(m+1) \times \mathcal{L}(n)^m \to \mathcal{L}(n+1)$ (which "ignores the last variable").

Definition 5. An algebra for a λ-theory $\mathcal{L}$ is an algebra A for the underlying algebraic theory.

For each term $t(\mathbf{x}) \in \mathcal{L}(n)$ and each tuple $\mathbf{a} \in A^n$, we get an interpretation $t(\mathbf{a}) \in A$.

Given a λ-theory $\mathcal{L}$, we can interpret a term t of the lambda calculus (that has a context Γ of length n) as an element $[\![t]\!] \in \mathcal{L}(n)$.

Example 1. Take $T(n) = \{\star\}$, $\pi_{n,i} = \star$ and $\star \bullet \{\star, \ldots, \star\} = \star$.
This theory is the terminal λ-theory.

Example 2. The λ-calculus Λ, in which $\Lambda(n)$ consists of the terms with n free variables, $\pi_{n,i} = \mathtt{Var}(i)$ (with De Bruijn indices) and $\bullet$ is substitution.
For every theory $\mathcal{T}$ the $\mathcal{T}(n)$ are algebras, so the $\Lambda(n)$ are algebras.
This is a λ-theory. According to the paper, it is the initial λ-theory.

Example 3. We can create abstract clones using algebraic signatures. If we have, for all n, a set of constructors Σ_n, and we have a sequence of variables $x_1, x_2, \ldots$. Then we can build the elements iteratively with the rules $x_i \in T(n)$ if $i \leq n$, and for $x_1, \ldots, x_m \in T(n)$ and $\sigma \in \Sigma_m$, $\sigma(x_1, \ldots, x_m) \in T(n)$. Then the $\pi_{n,i}$ are the x_i, and $f \bullet (g_1, \ldots, g_m)$ substitutes the g_i for the x_i in f.

For example, if we talk about rings, we have $\Sigma_0 = \{0, 1\}$, $\Sigma_1 = \{-\}$ (negation) and $\Sigma_2 = \{+, \cdot\}$. Then $T(n)$ is almost the polynomial ring over $\mathbb{Z}$ in n variables (but not quite, because we distinguish, for example, between 0, $0+0$ and $x_1 - x_1$).

If we have a type A with for all $\sigma \in \Sigma_n$, a map $[\![\sigma]\!] : A^n \to A$. Then A is an algebra for this theory.

Example 4. Let R be a ring. Take $T(n) = R[x_1, \ldots, x_n]$ the polynomial ring in n variables. Take $\pi_{n,i} = x_i$ and let $f \bullet (g_1, \ldots, g_n)$ substitute the g_i for the x_i in f.

If we have a homomorphism of rings $R \to S$ and we take $\mathcal{T}(n) = R[X_1, \ldots, X_n]$, then S is a $\mathcal{T}$-algebra.

Example 5. Take $T(n) = \{1, 2, \ldots, n\}$, $\pi_{n,i} = i$ and $i \bullet (f_1, \ldots, f_n) = f_i$.

Any type A can be an algebra, with $i(a_1, \ldots, a_n) = a_i$.

Example 6. Take a semiring R with identities $0, 1$ and operations $+, \cdot$. Take $T(n) = R^n$, $\pi_{n,i} = (0, \ldots, 0, 1, 0, \ldots, 0)$ and take

$$f \bullet (g_1, \ldots, g_n) = \begin{pmatrix} g_{1,1} & \cdots & g_{n,1} \\ \vdots & \ddots & \vdots \\ g_{1,m} & \cdots & g_{n,m} \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} := \left(\sum_i f_i \cdot g_{i,1}, \ldots, \sum_i f_i \cdot g_{i,m} \right)$$

in a matrix multiplication like fashion.

For example, take S a set, $\mathcal{T}$ a topology on S. Then we can take $T(n) = \mathcal{T}^n$, with operations $\cup$, $\cap$, and units $\emptyset$, S. Then we have $\pi_{n,i} = (\emptyset, \ldots, \emptyset, S, \emptyset, \ldots, \emptyset)$. For $U = (U_1, \ldots, U_n)$, $V_i = (V_{i,1}, \ldots, V_{i,m})$ we have

$$U \bullet (V_1, \ldots, V_n) = (U_1 \cap V_{1,1} \cup \cdots \cup U_n \cap V_{n,1}, \ldots, U_1 \cap V_{1,m} \cup \cdots \cup U_n \cap V_{n,m}).$$

Or take $R = \mathbb{N}$, with operations $+, \cdot$ and units $0, 1$.

Example 5, and example 6 with R a field are special cases of the following:

Example 7. In any category with finite products, take an object X. Then we can set $T(n) = (X^n \to X)$. Then $\pi_{n,i}$ is the ith projection morphism. Also, by the universal property of the product, if we have m terms of $(X^n \to X)$, we get a term of $(X^n \to X^m)$ which we can compose with a term of $(X^m \to X)$ to get a term of $(X^n \to X)$.

If we have a retraction $r : X \to A$ with section s, then A is an algebra with $\varphi \cdot (a_1, \ldots, a_n) = r(\varphi(s(a_1), \ldots, s(a_n)))$. (?)

If we have a retraction $X \to X^X$, the theory that we get is a λ-theory.

1. Presheaves

Let A be a Λ-algebra.

Definition 6. We define the monoid M_A with underlying set $\{a \in A \mid \mathbf{1}a = a\}$ and composition $(a, b) \mapsto a \circ b = \lambda x, a(bx)$ (?).

Definition 7. We define PA to be the category of presheaves on the category M_A. (I.e. the set of contravariant functors into set). This has 'universal object' $U_A = M_A$ with the obvious right action of M_A.

Lemma 1. *For U_A, we have a retraction $U_A \to U_A^{U_A}$.*

Proof. We have $M_A = \{a \in A \mid \mathbf{1}a = a\}$. We can identify U^U with $\{a \in A \mid \mathbf{1}_2 a = a\}$.

Composition on the left with $\mathbf{1}$ gives the retraction. $\square$

Definition 8. We take $\mathcal{U}_A$ to be the theory of the reflexive universal $U_A \in P(A)$.

2. The main theorem

We have a map $\Lambda \to \mathcal{L}$, which makes $\mathcal{L}(0)$ into a Λ-algebra.

Given a Λ-algebra A. Take $\Lambda_A(n) = A + \Lambda(n)$ (which is a coproduct of $\mathcal{T}$-algebras, defined using a coend).

Lemma 2. *Λ_A is a λ-theory.*

Proof. We can identify Λ_A with $\mathcal{U}_A$.

We have a retraction $U_A \to U_A^{U_A}$. Composition with this gives a retraction $\mathcal{U}_A(n) \to \mathcal{U}_A(n+1)$. $\square$

Theorem 1. *There is an adjoint equivalence $\mathcal{L} \mapsto \mathcal{L}(0)$ and $A \mapsto \Lambda_A$ between λ-theories and Λ-algebras.*

In particular, each λ-theory $\mathcal{L}$ is isomorphic to the theory of extensions of its initial algebra $\mathcal{L}(0)$.