Midterm review

July 12, 2023

Classical lambda calculus in modern dress

As far as I understand it:

- Paper by Martin Hyland.
- Talks about models for the λ-calculus.
- Scott's representation theorem (1980)

  Any model is isomorphic to the "endomorphism theory of a reflexive object in a cartesian closed category".

- Fundamental theorem of the λ-calculus.

  There is an adjoint equivalence between the category of models and the category of semantics for the λ-calculus with β-equality.

Algebraic theories: objects with variables and substitutions

Example

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Definition

An *algebraic theory* T is a sequence of sets $T(n)$ with variables $x_{i,n} \in T(n)$ (for $0 \leq i < n$) and a substitution operation $\bullet : T(m) \times T(n)^m \to T(n)$.

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Example

In $T(n) = \mathbb{Z}[x_1, \ldots, x_n]$ for $(m, n) = (2, 1)$:

$$(x_1 x_2) \bullet ((\lambda x_2, x_1 x_2), x_1 x_1) = (\lambda x_2, x_1 x_2)(x_1 x_1).$$

Algebras: semantics

We want to interpret terms with free variables as functions from a context to a set

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$$2x_1 + 3x_1^2 x_2 : A^2 \to A, \quad (a_1, a_2) \mapsto 2 \cdot a_1 + 3 \cdot a_1^2 \cdot a_2.$$

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Definition

For an algebraic theory T, a T*-algebra* A is a set A, together with an interpretation function $T(n) \times A^n \to A$ for all n (respecting the variables and substitution).

λ-theory: structure with app and abs

To obtain a model for the λ-calculus, we want to talk about structures that also have λ-abstraction and application.

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A *λ-theory* L is an algebraic theory, together with abstraction functions $\lambda : L(n+1) \to L(n)$ and application functions $\rho : L(n) \to L(n+1)$ (respecting the substitution).

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The λ-calculus Λ is the initial λ-theory.
β- and η-equality boil down to the requirement that respectively $\rho \circ \lambda$ and $\lambda \circ \rho$ are the identity.

Scott's representation theorem (1980)

Given an object U in a cartesian closed category C such that we have a retraction $U \xrightarrow{r} U^U$, we get a λ-theory with β-equality $\mathcal{U}$ (the endomorphism theory of U) by setting $\mathcal{U}(n) = C(U^n, U)$.

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The paper shows that λ-theory is the endomorphism theory for some object U.
It does this by explicitly constructing U.

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The functor the other way is a bit more complicated. Given a Λ-algebra, it

- constructs a monoid M_A;
- takes the category of presheaves PM_A;
- takes the object $U = M_A \in P(M_A)$;
- takes its endomorphism theory $\mathcal{U}(n)$;
- notes that Λ_A, the algebraic theory of extensions of A, is isomorphic to $\mathcal{U}(n)$.

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The fundamental theorem then states that $A \mapsto \Lambda_A$ and $L \mapsto L(0)$ constitute an adjoint equivalence of these categories.

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- Defined their categories as displayed categories;
- Showed univalence for all of these categories;
- Showed univalence of two standard constructions;
- Constructed the Λ-calculus with beta and eta equality as a (hypothetical) higher inductive type;

Displayed categories

Many categories are based on other categories. They just have 'more structure'. We formalized our categories as (towers of) displayed categories, because this matches their mathematical structure more closely, and helps when proving things like univalence. We have the following towers of displayed categories:

$$algebraic\_theory \to algebraic\_theory\_data \to pointed\_functor \to [F, HSET].$$

$$algebra \to algebra\_data \to algebraic\_theory \times HSET \to algebraic\_theory;$$

$$lambda\_theory \to lambda\_theory\_data \to algebraic\_theory;$$

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Given displayed categories D over C and E over $\int_C D$, we have the category $\sum_D E$, which is displayed over C. If D and E are displayed univalent, then $\sum_D E$ is displayed univalent as well.

The λ-calculus as a (hypothetical) higher inductive type

Need:

- Sets $\Lambda(n)$ for all n.
- Constructors.
- Substitution.
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Solution: Hypothetical higher inductive type.

Future work

- Remainder of the paper;
- Verify Hyland's claims about the inefficiency of alternative approaches;
- Compare Hyland's approach with the approaches of his predecessors;
- Explorations:
  - How about the λ-calculus with β-conversion (as a special relation) instead of β-equality?
  - What if we also have η-equality?
  - What if we don't have β-reduction?