

## Semantics for the $\lambda$ -calculus



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## CHAPTER 1

# Definitions

### 1. Algebraic Theories

DEFINITION 1 (algebraic theory). We define an algebraic theory  $T$  to be a sequence of sets  $T_n$  indexed over  $\mathbb{N}$  with for all  $1 \leq i \leq n$  elements ("variables" or "projections")  $x_{n,i} : T_n$  (we usually leave  $n$  implicit), together with a substitution operation

$$- \bullet - : T_m \times T_n^m \rightarrow T_n$$

for all  $m, n$ , such that

$$\begin{aligned} x_j \bullet g &= g_j \\ f \bullet (x_{l,i})_i &= f \\ (f \bullet g) \bullet h &= f \bullet (g_i \bullet h)_i \end{aligned}$$

for all  $1 \leq j \leq l$ ,  $f : T_l$ ,  $g : T_m^l$  and  $h : T_n^m$ .

DEFINITION 2 (algebraic theory morphism). A morphism  $F$  between algebraic theories  $T$  and  $T'$  is a sequence of functions  $F_n : T_n \rightarrow T'_n$  (we usually leave the  $n$  implicit) such that

$$\begin{aligned} F_n(x_j) &= x_j \\ F_n(f \bullet g) &= F_m(f) \bullet (F_n(g_i))_i \end{aligned}$$

for all  $1 \leq j \leq n$ ,  $f : T_m$  and  $g : T_n^m$ .

REMARK 1. We can construct binary products of algebraic theories, with sets  $(T \times T')_n = T_n \times T'_n$ , variables  $(x_i, x'_i)$  and substitution

$$(f, f') \bullet (g, g') = (f \bullet g, f' \bullet g').$$

In the same way, the category of algebraic theories has all limits.

### 2. Algebras

DEFINITION 3 (algebra). An algebra  $A$  for an algebraic theory  $T$  is a set  $A$ , together with an action

$$\bullet : T_n \times A^n \rightarrow A$$

for all  $n$ , such that

$$\begin{aligned} x_j \bullet a &= a_j \\ (f \bullet g) \bullet a &= f \bullet (g_i \bullet a)_i \end{aligned}$$

for all  $j$ ,  $f : T_m$ ,  $g : T_n^m$  and  $a : A^n$ .

DEFINITION 4 (algebra morphism). For an algebraic theory  $T$ , a morphism  $F$  between  $T$ -algebras  $A$  and  $A'$  is a function  $F : A \rightarrow A'$  such that

$$F(f \bullet a) = f \bullet (F(a_i))_i$$

for all  $f : T_n$  and  $a : A^n$ .

REMARK 2. The category of algebras has all limits. The set of a limit of algebras is the limit of the underlying sets.

REMARK 3. Note that for an algebraic theory  $T$ , the  $T_n$  are all algebras for  $T$ , with the action given by  $\bullet$ .

### 3. Presheaves

DEFINITION 5 (presheaf). A presheaf  $P$  for an algebraic theory  $T$  is a sequence of sets  $P_n$  indexed over  $\mathbb{N}$ , together with an action

$$\bullet : P_m \times T_n^m \rightarrow P_n$$

for all  $m, n$ , such that

$$\begin{aligned} t \bullet (x_{l,i})_i &= t \\ (t \bullet f) \bullet g &= t \bullet (f_i \bullet g)_i \end{aligned}$$

for all  $t : P_l$ ,  $f : T_m^l$  and  $g : T_n^m$ .

DEFINITION 6 (presheaf morphism). For an algebraic theory  $T$ , a morphism  $F$  between  $T$ -presheaves  $P$  and  $P'$  is a sequence of functions  $F_n : P_n \rightarrow P'_n$  such that

$$F_n(t \bullet f) = F_n(t) \bullet f$$

for all  $t : P_m$  and  $f : T_n^m$ .

We will write  $PT$  for the category of  $T$ -presheaves and their morphisms.

REMARK 4. The category of presheaves has all limits. The  $n$ th set  $\bar{P}_n$  of a limit  $\bar{P}$  of presheaves  $P_i$  is the limit of the  $n$ th sets  $P_{i,n}$  of the presheaves in the limit diagram.

### 4. $\lambda$ -theories

DEFINITION 7 ( $\lambda$ -theory). A  $\lambda$ -theory is an algebraic theory  $L$ , together with sequences of functions  $\lambda_n : L_{n+1} \rightarrow L_n$  and  $\rho_n : L_n \rightarrow L_{n+1}$ , such that

$$\begin{aligned} \lambda_m(f) \bullet h &= \lambda_n(f \bullet (h_1, \dots, h_m, x_{n+1})) \\ \rho_n(g \bullet h) &= \rho_m(g) \bullet (h_1, \dots, h_m, x_{n+1}) \end{aligned}$$

for all  $f : L_{m+1}$ ,  $g : L_m$  and  $h : L_n^m$ .

DEFINITION 8 ( $\beta$ - and  $\eta$ -equality). We say that a  $\lambda$ -theory  $L$  satisfies  $\beta$ -equality (or that it is a  $\lambda$ -theory with  $\beta$ ) if  $\rho_n \circ \lambda_n = \text{id}_{L_n}$  for all  $n$ . We say that  $L$  satisfies  $\eta$ -equality if  $\lambda_n \circ \rho_n = \text{id}_{L_{n+1}}$  for all  $n$ .

DEFINITION 9 ( $\lambda$ -theory morphism). A morphism  $F$  between  $\lambda$ -theories  $L$  and  $L'$  is an algebraic theory morphism  $F$  such that

$$\begin{aligned} F_n(\lambda_n(f)) &= \lambda_n(F_{n+1}(f)) \\ \rho_n(F_n(g)) &= F_{n+1}(\rho_n(g)) \end{aligned}$$

for all  $f : L_{n+1}$  and  $g : L_n$ .

REMARK 5. The category of lambda theories has all limits, with the underlying algebraic theory of a limit being the limit of the underlying algebraic theories.

A  $\lambda$ -theory algebra or presheaf is a presheaf for the underlying algebraic theory.

**5. Alternate definitions**

DEFINITION 10. Lawvere theory: **(TODO)**

DEFINITION 11. Relative monad: **(TODO)**

DEFINITION 12. Abstract clone: **(TODO)**

DEFINITION 13. Cartesian Operad: **(TODO)**

(<https://ncatlab.org/nlab/show/lambda+theory>)





## CHAPTER 2

# Category Theoretic Preliminaries

I will assume a familiarity with the category-theoretical concepts presented in [AW23]. These include categories, functors, isomorphisms, natural transformations, adjunctions, equivalences and limits.

### 1. Notation

For an object  $c$  in a category  $C$ , I will write  $c : C$ .

For a morphism  $f$  between objects  $c$  and  $c'$  in a category  $C$ , I will write  $f : C(c, c')$  or  $f : c \rightarrow c'$ .

For composition of morphisms  $f : C(c, d)$  and  $g : C(d, e)$ , I will write  $f \cdot g$ .

For composition of functors  $F : A \rightarrow B$  and  $G : B \rightarrow C$ , I will write  $F \bullet G$ .

### 2. Adjunctions

An adjoint equivalence of categories has multiple definitions. The one we will use here is the following:

DEFINITION 14. An adjoint equivalence between categories  $C$  and  $D$  is a pair of adjoint functors

$$\begin{array}{ccc} & L & \\ D & \xleftarrow{\quad} & C \\ & R & \end{array}$$

such that the unit  $\eta : \text{id}_C \Rightarrow L \bullet R$  and counit  $\epsilon : R \bullet L \Rightarrow \text{id}_D$  are isomorphisms of functors.

### 3. Kan Extensions

One of the most general and abstract concepts in category theory is the concept of *Kan extensions*. In [ML98], Section X.7, MacLane notes that

The notion of Kan extensions subsumes all the other fundamental concepts of category theory.

In this thesis, we will use left Kan extension a handful of times. It comes in handy when we want to extend a functor along another functor in the following way:

Let  $A$ ,  $B$  and  $C$  be categories and let  $F : A \rightarrow B$  be a functor.

DEFINITION 15. Precomposition gives a functor between functor categories  $F_* : [B, C] \rightarrow [A, C]$ . If  $F_*$  has a left adjoint, we will denote call this adjoint functor the *left Kan extension* along  $F$  and denote it  $\text{Lan}_F : [A, C] \rightarrow [B, C]$ .

$$\begin{array}{ccc} A & \xrightarrow{F} & B \\ \swarrow F_* G & & \searrow G \\ & C & \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{F} & B \\ \swarrow G & & \searrow \text{Lan}_F G \\ & C & \end{array}$$

Analogously, when  $F_*$  has a right adjoint, one calls this the *right Kan extension* along  $F$  and denote it  $\text{Ran}_F : [A, C] \rightarrow [B, C]$ .

If a category has limits (resp. colimits), we can construct the right (resp. left) Kan extension in a ‘pointwise’ fashion (see Theorem X.3.1 in [ML98] or Theorem 2.3.3 in [KS06]). Below, I will outline the parts of the construction that we will need explicitly in this thesis.

LEMMA 1. *If  $C$  has colimits,  $\text{Lan}_F$  exists.*

PROOF. First of all, for objects  $b : B$ , we take

$$\text{Lan}_F G(b) := \text{colim} \left( (F \downarrow b) \rightarrow A \xrightarrow{G} C \right).$$

Here,  $(F \downarrow b)$  denotes the comma category with as objects the morphisms  $B(F(a), b)$  for all  $a : A$ , and as morphisms from  $f : B(F(a), b)$  to  $f' : B(F(a'), b)$  the morphisms  $g : A(a, a')$  that make the diagram commute:

$$\begin{array}{ccc} F(a) & \xrightarrow{F(g)} & F(a') \\ & \searrow f' & \swarrow f' \\ & b & \end{array}$$

and  $(F \downarrow b) \rightarrow A$  denotes the projection functor that sends  $f : B(F(a_1), b)$  to  $a_1$ .

Now, a morphism  $h : B(b, b')$  gives a morphism of diagrams, sending the  $F(a)$  corresponding to  $f : B(G(a), b)$  to the  $F(a)$  corresponding to  $f \cdot h : B(G(a), b')$ . From this, we get a morphism  $\text{Lan}_F G(h) : C(\text{Lan}_F G(b), \text{Lan}_F G(b'))$ .

The unit of the adjunction is a natural transformation  $\eta : \text{id}_{[A, C]} \Rightarrow \text{Lan}_F \bullet F_*$ . We will define this pointwise, for  $G : [A, C]$  and  $a : A$ . Our diagram contains the  $G(a)$  corresponding to  $\text{id}_{F(a)} : (F \downarrow F(a))$  and the colimit cocone gives a morphism

$$\eta_G(a) : C(G(a), \text{Lan}_F G(F(a))),$$

the latter being equal to  $(\text{Lan}_F \bullet F_*)(G)(a)$ .

The counit of the adjunction is a natural transformation  $\epsilon : F_* \bullet \text{Lan}_F \Rightarrow \text{id}_{[B, C]}$ . We will also define this pointwise, for  $G : [B, C]$  and  $b : B$ . The diagram for  $\text{Lan}_F (F_* G)(b)$  consists of  $G(F(a))$  for all  $f : B(F(a), b)$ . Then, by the universal property of the colimit, the morphisms  $G(f) : C(G(F(a)), G(b))$  induce a morphism

$$\epsilon_G(b) : C(\text{Lan}_F (F_* G)(b), G(b)).$$

□

LEMMA 2. *If  $F : A \rightarrow B$  is a fully faithful functor, and  $C$  is a category with colimits,  $\eta$  is a natural isomorphism.*

PROOF. To show that  $\eta$  is a natural isomorphism, we have to show that  $\eta_G(a') : G(a') \Rightarrow \text{Lan}_F G(F(a'))$  is an isomorphism for all  $G : [A, C]$  and  $a' : A$ . Since a left adjoint is unique up to natural isomorphism, we can assume that  $\text{Lan}_F G(F(a'))$  is given by

$$\text{colim}((F \downarrow F(a')) \rightarrow A \xrightarrow{G} C).$$

Now, the diagram for this colimit consists of  $G(a)$  for each arrow  $f : B(F(a), F(a'))$ . Since  $F$  is fully faithful, we have  $f = F(\bar{f})$  for some  $\bar{f} : A(a, a')$ . If we now take the arrows  $G(\bar{f}) : C(G(a), G(a'))$ , the universal property of the colimit gives an arrow

$$\varphi : C(\text{Lan}_F G(F(a')), G(a'))$$

which constitutes an inverse to  $\eta_G(a')$ . □

REMARK 6. In the same way, if  $C$  has limits,  $\epsilon$  is a natural isomorphism.

COROLLARY 1. *If  $C$  has limits or colimits, precomposition of functors  $[B, C]$  along a fully faithful functor is (split) essentially surjective.*

PROOF. For each  $G : [A, C]$  we take  $\text{Lan}_F G : [B, C]$ , and we have  $F_*(\text{Lan}_F G) \cong G$ .  $\square$

COROLLARY 2. *If  $C$  has colimits (resp. limits), left (resp. right) Kan extension of functors  $[A, C]$  along a fully faithful functor is fully faithful.*

PROOF. Since left Kan extension along  $F$  is the left adjoint to precomposition, we have

$$[A, C](\text{Lan}_F G, \text{Lan}_F G') \cong [B, C](G, F_*(\text{Lan}_F G')) \cong [B, C](G, G').$$

$\square$

#### 4. The Karoubi envelope

Let  $C$  be a category. If we have a retraction-section pair  $c \xrightleftharpoons[s]{r} d$  we have (by definition)  $s \cdot r = \text{id}_d$ . On the other hand,  $s \cdot r : c \rightarrow c$  is an idempotent morphism. Conversely, we can wonder whether for any idempotent morphism  $a : c \rightarrow c$ , we can find a retraction-section pair  $r : c \rightarrow d$  and  $s : d \rightarrow c$  such that  $a = r \cdot s$ . If this is the case, we say that the idempotent  $a$  *splits*. If  $a$  does not split, we can wonder whether we can find an embedding  $\iota_C : C \hookrightarrow \overline{C}$  such that the idempotent  $\iota_C(a) : \iota_C(c) \rightarrow \iota_C(c)$  does split.

DEFINITION 16. We define the category  $\overline{C}$ . The objects of  $\overline{C}$  are tuples  $(c, a)$  with  $c : C$ ,  $a : C(c, c)$  such that  $a \cdot a = a$ . The morphisms between  $(c, a)$  and  $(d, b)$  are morphisms  $f : C(a, b)$  such that  $a \cdot f \cdot b = f$ . The identity morphism on  $(c, a)$  is given by  $a$  and  $\overline{C}$  inherits morphism composition from  $C$ .

This category is called the *Karoubi Envelope*, the *idempotent completion*, the *category of retracts*, or the *Cauchy completion* of  $C$ .

REMARK 7. Note that for a morphism  $f : \overline{C}((c, a), (d, b))$ ,

$$a \cdot f = a \cdot a \cdot f \cdot b = a \cdot f \cdot b = f$$

and in the same way,  $f \cdot b = f$ .

DEFINITION 17. We have an embedding  $\iota_C : C \rightarrow \overline{C}$ , sending  $c : C$  to  $(c, \text{id}_c)$  and  $f : C(c, d)$  to  $f$ .

LEMMA 3. *Every object  $c : \overline{C}$  is a retract of  $\iota_C(c_0)$  for some  $c_0 : C$ .*

PROOF. Note that  $c = (c_0, a)$  for some  $c_0 : C$  and an idempotent  $a : c \rightarrow c$ . We have morphisms  $\iota_C(c) \xrightleftharpoons[a_{\leftarrow}]{a_{\rightarrow}} (c, a)$ , both given by  $a$ . We have  $a_{\leftarrow} \cdot a_{\rightarrow} = a = \text{id}_{(c, a)}$ , so  $(c, a)$  is a retract of  $\iota_C(c)$ .  $\square$

LEMMA 4. *Every idempotent splits in  $\overline{C}$ .*

PROOF. Take an idempotent  $a : C(c, c)$ . We have an object  $(c, a) : \overline{C}$ . By the previous lemma,  $(c, a)$  is a retract of  $\iota_C(c)$ . We have  $\iota_C(a) = a = a_{\rightarrow} \cdot a_{\leftarrow}$ , so  $a$  splits in  $\overline{C}$ .  $\square$

REMARK 8. Note that the embedding is fully faithful, since

$$\overline{C}((c, \text{id}_c), (d, \text{id}_d)) = \{f : C(c, d) \mid \text{id}_c \cdot f \cdot \text{id}_d = f\} = C(c, d).$$

REMARK 9. Let  $D$  be a category. Suppose that we have a retraction-section pair in  $D$ , given by  $d \xrightleftharpoons[s]{r} d'$ . Now, suppose that we have an object  $c : D$  and a morphism  $f$  with  $(r \cdot s) \cdot f = f$ . Then we get a morphism  $s \cdot f : d' \rightarrow c$  such that  $f$  factors as  $r \cdot (s \cdot f)$ . Also, for any  $g$  with  $r \cdot g = f$ , we have

$$g = s \cdot r \cdot g = s \cdot f.$$

$$\begin{array}{ccccc}
d & \xrightarrow{r} & d' & \xrightarrow{s} & d \\
& \searrow f & \downarrow s \cdot f & \swarrow f & \\
& & c & & 
\end{array}$$

Therefore,  $d'$  is the equalizer of  $d \xrightarrow[r \cdot s]{\text{id}_d} d$ . In the same way, it is also the co-equalizer of this diagram.

Since a functor preserves retracts, and since every object of  $\overline{C}$  is a retract of an object in  $C$ , one can lift a functor from  $C$  (to a category with (co)equalizers) to a functor on  $\overline{C}$ .

For convenience, the lemma below works with pointwise left Kan extension using colimits, but one could also prove this using just (co)equalizers (or right Kan extension using limits).

LEMMA 5. *Let  $D$  be a category with colimits. We have an adjoint equivalence between  $[C, D]$  and  $[\overline{C}, D]$ .*

PROOF. We already have an adjunction  $\text{Lan}_{\iota_C} \dashv \iota_{C*}$ . Also, since  $\iota_C$  is fully faithful, we know that  $\eta$  is a natural isomorphism. Therefore, we only have to show that  $\epsilon$  is a natural isomorphism. That is, we need to show that  $\epsilon_G(c, a) : D(\text{Lan}_{\iota_C}(\iota_{C*}G)(c, a), G(c, a))$  is an isomorphism for all  $G : [\overline{C}, D]$  and  $(c, a) : \overline{C}$ .

One of the components in the diagram of  $\text{Lan}_{\iota_C}(\iota_{C*}G)(c, a)$  is the  $\iota_{C*}G(c) = G(c, \text{id}_c)$  corresponding to  $a : \iota_C(c) \rightarrow (c, a)$ . This component has a morphism into our colimit

$$\varphi : C(G(\iota_C(c)), \text{Lan}_{\iota_C}(\iota_{C*}G)(c, a)).$$

Note that we can view  $a$  as a morphism  $a : \overline{C}((c, a), \iota_C(c))$ . This gives us our inverse morphism

$$G(a) \cdot \varphi : C(G(c, a), \text{Lan}_{\iota_C}(\iota_{C*}G)(c, a)).$$

□

LEMMA 6. *The formation of the opposite category commutes with the formation of the Karoubi envelope.*

PROOF. An object in  $\overline{C}^{\text{op}}$  is an object  $c : C^{\text{op}}$  (which is just an object  $c : C$ ), together with an idempotent morphism  $a : C^{\text{op}}(c, c) = C(c, c)$ . This is the same as an object in  $\overline{C}^{\text{op}}$ .

A morphism in  $\overline{C}^{\text{op}}((c, a), (d, b))$  is a morphism  $f : C^{\text{op}}(c, d) = C(d, c)$  such that

$$b \cdot_C f \cdot_C a = a \cdot_{C^{\text{op}}} f \cdot_{C^{\text{op}}} b = f.$$

A morphism in  $\overline{C}^{\text{op}}((c, a), (d, b)) = \overline{C}((d, b), (c, a))$  is a morphism  $f : C(d, c)$  such that  $b \cdot f \cdot a = f$ .

Now, in both categories, the identity morphism on  $(c, a)$  is given by  $a$ .

Lastly,  $\overline{C}^{\text{op}}$  inherits morphism composition from  $C^{\text{op}}$ , which is the opposite of composition in  $C$ . On the other hand, composition in  $\overline{C}^{\text{op}}$  is the opposite of composition in  $\overline{C}$ , which inherits composition from  $C$ . □

COROLLARY 3. *As the category **SET** is cocomplete, we have an equivalence between the category of presheaves on  $C$  and the category of presheaves on  $\overline{C}$ .*

## 5. Monoids as categories

Take a monoid  $M$ .

DEFINITION 18. We can construct a category  $C_M$  with  $C_{M0} = \{\star\}$ ,  $C_M(\star, \star) = M$ . The identity morphism on  $\star$  is the identity  $1 : M$ . The composition is given by multiplication  $g \cdot_{C_M} f = f \cdot_M g$ .

DEFINITION 19. A *right monoid action* of  $M$  on a set  $X$  is a function  $X \times M \rightarrow X$  such that for all  $x : X$ ,  $m, m' : M$ ,

$$x1 = x \quad \text{and} \quad (xm)m' = x(m \cdot m').$$

DEFINITION 20. A *morphism* between sets  $X$  and  $Y$  with a right  $M$ -action is an  $M$ -equivariant function  $f : X \rightarrow Y$ : a function such that  $f(xm) = f(x)m$  for all  $x : X$  and  $m : M$ .

LEMMA 7. *Presheaves on  $C_M$  are equivalent to sets with a right  $M$ -action.*

PROOF. This correspondence sends a presheaf  $F$  to the set  $F(\star)$ , and conversely, the set  $X$  to the presheaf  $F$  given by  $F(\star) := X$ . The  $M$ -action corresponds to the presheaf acting on morphisms as  $xm = F(m)(x)$ . A morphism (natural transformation) between presheaves  $F \Rightarrow G$  corresponds to a function  $F(\star) \rightarrow G(\star)$  that is  $M$ -equivariant, which is exactly a monoid action morphism.  $\square$

DEFINITION 21. We can view  $M$  as an object  $U_M$  in  $C_M$ , with action  $mn = m \cdot_M n$  for  $m : U_M$  and  $n : M$ .

REMARK 10. Since the category of sets with an  $M$ -action is equivalent to a presheaf category, it has all limits. However, we can make this concrete. The set of the product  $\prod_i X_i$  is the product of the underlying sets. The action is given pointwise by  $(x_i)_i m = (x_i m)_i$ .

LEMMA 8. *The category  $C$  of sets with an  $M$ -action has exponentials.*

PROOF. Given sets with  $M$ -action  $X$  and  $Y$ . Consider the set  $C(M \times X, Y)$  with an  $M$ -action given by  $\phi m'(m, x) = \phi(m'm, x)$ .  $\square$

Exponentials: **(TODO)**  
 Global elements: **(TODO)**  
 Restriction of scalars: **(TODO)**  
 Extension of scalars: **(TODO)**  
 Extension of scalars preserves the monoid action: **(TODO)**  
 Extension of scalars preserves limits: **(TODO)**



## CHAPTER 3

### Lemmas

#### 1. The endomorphism theory

DEFINITION 22 (Endomorphism theory). Suppose that we have a category  $C$  and an object  $X : C$ , such that all powers  $X^n$  of  $X$  are also in  $C$ . The endomorphism theory  $E(X)$  of  $X$  is the algebraic theory given by  $E(X)_n = C(X^n, X)$  with projections as variables  $x_{n,i} : X^n \rightarrow X$  and a substitution that sends  $f : X^m \rightarrow X$  and  $g_1, \dots, g_m : X^n \rightarrow X$  to  $f \circ \langle g_i \rangle_i : X^n \rightarrow X^m \rightarrow X$ .

LEMMA 9.  $E(X)$  is indeed an algebraic theory.

PROOF. For  $1 \leq j \leq l$ ,  $f : E(X)_l$ ,  $g : E(X)_m^l$  and  $h : E(X)_n^m$ , we have

$$\begin{aligned} x_j \bullet g &= x_j \circ \langle g_i \rangle_i = g_j, \\ f \bullet (x_{l,i})_i &= f \circ \langle x_{l,i} \rangle_i = f \circ \text{id}_{X^l} = f \end{aligned}$$

and

$$(f \bullet g) \bullet h = f \circ \langle g_i \rangle_i \circ \langle h_i \rangle_i = f \circ \langle g_i \circ \langle h_{i'} \rangle_{i'} \rangle_i = f \bullet (g_i \bullet h)_i.$$

□

DEFINITION 23 (Endomorphism  $\lambda$ -theory). Now, suppose that the exponential object  $X^X$  exists, and that we have morphisms back and forth  $\text{abs} : X^X \rightarrow X$  and  $\text{app} : X \rightarrow X^X$ . Let, for  $Y : C$ ,  $\varphi_Y$  be the isomorphism  $C(X \times Y, X) \xrightarrow{\sim} C(Y, X^X)$ . We can give  $E(X)$  a  $\lambda$ -theory structure by setting, for  $f : E(X)_{n+1}$  and  $g : E(X)_n$ ,

$$\lambda(f) = \text{abs} \circ \varphi_{X^n}(f) \quad \rho(g) = \varphi_{X^n}^{-1}(\text{app} \circ g).$$

LEMMA 10.  $E(X)$  is indeed a  $\lambda$ -theory.

PROOF. Note that  $\varphi : C(- \times X, X) \xrightarrow{\sim} C(-, X^X)$  is a natural isomorphism, so for  $g : E(X)_n^m$ , the following diagram commutes

$$\begin{array}{ccc} C(X^m \times X, X) & \xrightarrow{- \circ (\langle g_i \rangle_i \times \text{id}_X)} & C(X^n \times X, X^X) \\ \varphi_{X^m}^{-1} \left( \begin{array}{c} \uparrow \\ \downarrow \end{array} \right) \varphi_{X^m} & & \varphi_{X^n}^{-1} \left( \begin{array}{c} \uparrow \\ \downarrow \end{array} \right) \varphi_{X^n} \\ C(X^m, X^X) & \xrightarrow{- \circ \langle g_i \rangle_i} & C(X^n, X^X) \end{array}$$

and note that  $\langle g_i \rangle_i \times \text{id}_X = \langle g_1, \dots, g_m, x_{n+1} \rangle$ . Then we have, for all  $f : E(X)_m$

$$\begin{aligned} \lambda_m(f) \bullet g &= \text{abs} \circ \varphi_{X^m}(f) \circ \langle g_i \rangle_i \\ &= \text{abs} \circ \varphi_{X^n}(f \circ \langle g_1, \dots, g_m, x_{n+1} \rangle) \\ &= \lambda_n(f \bullet (g_1, \dots, g_m, x_{n+1})) \end{aligned}$$

and

$$\begin{aligned} \rho_n(f \bullet g) &= \varphi_{X^n}^{-1}(\text{app} \circ f \circ \langle g_i \rangle_i) \\ &= \varphi_{X^m}^{-1}(\text{app} \circ f) \circ \langle g_1, \dots, g_m, x_{n+1} \rangle \\ &= \rho_m(f) \bullet (g_1, \dots, g_m, x_{n+1}). \end{aligned}$$

□

## 2. The theory presheaf

DEFINITION 24 (The theory presheaf). Let  $T$  be an algebraic theory. We can turn  $T$  into an  $T$ -presheaf  $\tilde{T}$  by setting  $\tilde{T}_n = T_n$  and using the substitution from  $T$ :

$$\bullet : \tilde{T}_m \times T_n^m \rightarrow \tilde{T}_n.$$

LEMMA 11.  $\tilde{T}$  is indeed a presheaf.

PROOF. For all  $t : \tilde{T}_l$ ,  $f : T_m^l$  and  $g : T_n^m$ ,

$$t \bullet (x_{l,i})_i = t$$

and

$$(t \bullet f) \bullet g = t \bullet (f_i \bullet g)_i$$

because  $T$  is an algebraic theory.  $\square$

LEMMA 12. Given an algebraic theory  $T$  and a  $T$ -presheaf  $Q$ , we have for all  $n$  a bijection of sets

$$\varphi : PT(\tilde{T}^n, Q) \cong Q_n.$$

PROOF. Take  $\varphi(f) = f_n(x_1, \dots, x_n)$ .

Conversely, take  $\varphi^{-1}(q)$  to be the presheaf morphism that sends  $t : T_m^n$  to  $q \bullet t : Q_m$ . This is indeed a presheaf morphism, since for all  $t : T_l^n$  and  $f : T_m^l$ ,

$$\varphi^{-1}(q)(t \bullet f) = q \bullet t \bullet f = \varphi^{-1}(q)(t) \bullet f.$$

Now, for a presheaf morphism  $f : T^n \rightarrow Q$  and  $t : T_m^n$ , we have

$$\varphi^{-1}(\varphi(f))(t) = f_n(x_1, \dots, x_n) \bullet t = f_n((x_1, \dots, x_n) \bullet t) = f_n(t_1, \dots, t_n) = f_n(t).$$

Conversely, given  $q : Q_n$ , we have

$$\varphi(\varphi^{-1}(q)) = q \bullet (x_1, \dots, x_n) = q.$$

which concludes the proof.  $\square$

## 3. The ‘+l’ presheaf

Let  $\iota_{m,n} : T_m \rightarrow T_{m+n}$  denote the function that sends  $f$  to  $f \bullet (x_{m+n,1}, \dots, x_{m+n,m})$ . Note that

$$\iota_{m,n}(f) \bullet g = f \bullet (g_i)_{i \leq m}$$

and

$$\iota_{m,n}(f \bullet g) = f \bullet g \bullet (x_i)_i = f \bullet (g_i \bullet (x_j)_j)_i = f \bullet (\iota_{m,n}(g_i))_i.$$

For tuples  $x : X^m$  and  $y : X^n$ , let  $x+y$  denote the tuple  $(x_1, \dots, x_m, y_1, \dots, y_n) : X^{m+n}$ .

DEFINITION 25 (The ‘+l’ presheaf). Given a  $T$ -presheaf  $Q$ , we can construct a presheaf  $A(Q, l)$ , given by  $A(Q, l)_n = Q_{n+l}$ . Then, for  $q : A(Q, l)_m$  and  $f : T_n^m$ , the substitution is given by

$$q \bullet_{A(Q,l)} f = q \bullet_Q ((\iota_{n,l}(f_i))_i + (x_{n+i})_i)$$

LEMMA 13. The  $+l$  presheaf is a presheaf

PROOF. We have, for  $q : A(Q, l)_n$ ,

$$\begin{aligned} q \bullet_{A(Q,l)} (x_i)_i &= q \bullet_Q ((\iota_{n,l}(x_i))_i + (x_{n+i})_i) \\ &= q \bullet_Q ((x_i)_i + (x_{n+i})_i) \\ &= q \bullet_Q (x_i)_i \\ &= q. \end{aligned}$$



We have, for  $q : A(Q, k)_l$ ,  $f : T_m^l$  and  $g : T_n^m$ ,

$$\begin{aligned}
q \bullet_{A(Q, k)} f \bullet_{A(Q, k)} g &= q \bullet_Q ((\iota_{m, l}(f_i))_i + (x_{m+i})_i) \bullet_Q ((\iota_{n, l}(g_i))_i + (x_{n+i})_i) \\
&= q \bullet_Q (((\iota_{m, l}(f_i) \bullet_T ((\iota_{n, l}(g_j))_j + (x_{n+j})_j))_i + (x_{m+i} \bullet_T ((\iota_{n, l}(g_j))_j + (x_{n+j})_j))_i)) \\
&= q \bullet_Q ((f_i \bullet_T (\iota_{n, l}(g_j))_j)_i + (x_{n+i})_i) \\
&= q \bullet_Q ((\iota_{n, l}(f_i \bullet_T g))_i + (x_{n+i})_i) \\
&= q \bullet_{A(Q, k)} (f_i \bullet_T g).
\end{aligned}$$

□

#### 4. Exponentiability of the theory presheaf

LEMMA 14. *For all  $l$ , the presheaf  $\tilde{T}^l$  is exponentiable.*

PROOF. We will show that  $A(-, l)$  constitutes a right adjoint to the functor  $- \times \tilde{T}^l$ . We will do this using universal arrows ([ML98], Chapter IV.1, Theorem 2 (iv)). To that end, we will need for all  $Q : PT$  a universal arrow  $\varphi : A(Q, l) \times \tilde{T}^l \rightarrow Q$ .

For  $q : A(Q, l)_n = Q_{n+l}$  and  $t : \tilde{T}_n^l$ , we take  $\varphi(q, t) = q \bullet_Q ((x_{n,i})_i + t)$ .

This is a presheaf morphism, since for all  $q : A(Q, l)_m^l$ ,  $t : \tilde{T}_m^l$  and  $f : T_n^m$ ,

$$\begin{aligned}
\varphi((q, t) \bullet_{A(Q, l) \times \tilde{T}^l} f) &= \varphi(q \bullet_{A(Q, l)} f, t \bullet_{\tilde{T}^l} f) \\
&= q \bullet_{A(Q, l)} f \bullet_Q ((x_i)_i + (t \bullet_{\tilde{T}^l} f)) \\
&= q \bullet_Q ((\iota_{n, l}(f_i))_i + (x_{n+i})_i) \bullet_Q ((x_i)_i + (t \bullet_{\tilde{T}^l} f)) \\
&= q \bullet_Q ((\iota_{n, l}(f_i) \bullet_T ((x_j)_j + (t \bullet_{\tilde{T}^l} f)))_i + (x_{n+i} \bullet_T ((x_j)_j + (t \bullet_{\tilde{T}^l} f))))_i \\
&= q \bullet_Q ((f_i \bullet_T (x_j)_j)_i + ((t \bullet_{\tilde{T}^l} f)_i)_i) \\
&= q \bullet_Q ((f_i)_i + (t_i \bullet_{\tilde{T}^l} f)_i) \\
&= q \bullet_Q ((x_i \bullet_T f)_i + (t_i \bullet_T f)_i) \\
&= q \bullet_Q ((x_i)_i + t) \bullet_Q f \\
&= \varphi(q, t) \bullet_Q f.
\end{aligned}$$

Now, given any presheaf  $Q' : PT$  we need to show that any morphism  $\psi : Q' \times \tilde{T}^l \rightarrow Q$  factors uniquely as  $\varphi \circ (\tilde{\psi} \times \text{id}_{\tilde{T}^l})$  for some  $\tilde{\psi} : Q' \rightarrow A(Q, l)$ .

So, given such a  $\psi$ , and given  $q : Q'_n$ , we take  $\tilde{\psi}(q) = \psi(\iota_{n, l}(q), (x_{n+i})_i)$

This is a presheaf morphism, since for all  $q : Q'_m$  and  $f : T_n^m$ ,

$$\begin{aligned}
\tilde{\psi}(q \bullet f) &= \psi(\iota_{n, l}(q \bullet f), (x_{n+i})_i) \\
&= \psi(q \bullet (\iota_{n, l}(f_i))_i, (x_{n+i})_i) \\
&= \psi((\iota_{m, l}(q), (x_{m+i})_i) \bullet_{Q' \times \tilde{T}^l} ((\iota_{n, l}(f_i))_i + (x_{n+i})_i)) \\
&= \psi(\iota_{m, l}(q), (x_{m+i})_i) \bullet_Q ((\iota_{n, l}(f_i))_i + (x_{n+i})_i) \\
&= \tilde{\psi}(q) \bullet_{A(Q, l)} f.
\end{aligned}$$

Note that indeed  $\varphi \circ (\tilde{\psi} \times \text{id}_{\tilde{T}^l}) = \psi$ :

$$\begin{aligned}
\varphi(\tilde{\psi}(q), t) &= \varphi(\psi(\iota_{n, l}(q), (x_{n+i})_i), t) \\
&= \psi(\iota_{n, l}(q), (x_{n+i})_i) \bullet_Q ((x_i)_i + t) \\
&= \psi(\iota_{n, l}(q) \bullet_Q ((x_i)_i + t), (x_{n+i})_i \bullet_Q ((x_i)_i + t)) \\
&= \psi(q \bullet (x_i)_i, (t_i)_i) \\
&= \psi(q, t).
\end{aligned}$$

Now, suppose that we have another  $\tilde{\psi}' : Q' \rightarrow A(Q, l)$  such that  $\varphi \circ (\tilde{\psi}' \times \text{id}_{\tilde{T}^l}) = \psi$ . Then we have

$$\begin{aligned}
 \tilde{\psi}(q) &= \psi(\iota_{n,l}(q), (x_{n+i})_i) \\
 &= (\varphi \circ (\tilde{\psi}' \times \text{id}_{\tilde{T}^l}))(\iota_{n,l}(q), (x_{n+i})_i) \\
 &= \varphi(\tilde{\psi}'(\iota_{n,l}(q)), (x_{n+i})_i) \\
 &= \tilde{\psi}'(\iota_{n,l}(q)) \bullet ((x_i)_i + (x_{n+i})_i) \\
 &= \iota_{n,l}(\tilde{\psi}'(q)) \bullet ((x_i)_i + (x_{n+i})_i) \\
 &= \tilde{\psi}'(q) \bullet (x_i)_i \\
 &= \tilde{\psi}'(q),
 \end{aligned}$$

so  $\tilde{\psi}$  is unique, which completes the proof.  $\square$

Now, this adjunction  $- \times \tilde{T}^l \dashv A(-, l)$  induces a natural isomorphism

$$\varphi : PT(- \times \tilde{T}^l, \tilde{T}) \xrightarrow{\sim} PT(-, A(\tilde{T}, l))$$

LEMMA 15. *For all  $f : PT(\tilde{T}^n \times \tilde{T}^l, \tilde{T})$ ,*

$$\varphi_{\tilde{T}^n}(f)(q) = f(\iota_{m,l}(q), (x_{m+i})_i)$$

PROOF. **(TODO)**  $\square$

LEMMA 16. *For all  $f : PT(\tilde{T}^n, A(\tilde{T}, l))$ ,*

$$\varphi_{\tilde{T}^n}^{-1}(f)(q, t) = f(q) \bullet ((x_i)_i + t).$$

PROOF. **(TODO)**  $\square$

## CHAPTER 4

# Theorems

### 1. Scott's Representation Theorem

**THEOREM 1.** *Any  $\lambda$ -theory  $L$  is isomorphic to the endomorphism  $\lambda$ -theory  $E(\tilde{L})$  of  $\tilde{L}$  in the presheaf category of  $L$ .*

**PROOF.** First of all, remember that  $\tilde{L}$  is indeed exponentiable and that  $\tilde{L}^{\tilde{L}} = A(\tilde{L}, 1)$ . Now, since  $L$  is a  $\lambda$ -theory, we have functions back and forth  $\lambda : A(\tilde{L}, 1) \rightarrow \tilde{L}$  and  $\rho : \tilde{L} \rightarrow A(\tilde{L}, 1)$ . These are presheaf morphisms because for all  $f : A(\tilde{L}, 1)_m$  and  $g : \tilde{L}_m$  and  $t : T_n^m$ ,

$$\lambda(f \bullet_{A(\tilde{L}, 1)} t) = \lambda(f \bullet_{\tilde{L}} ((\iota_{m,1}(t_i))_i + (x_{n+1}))) = \lambda(f) \bullet_{\tilde{L}} t$$

and

$$\rho(g \bullet_{\tilde{L}} t) = \rho(g) \bullet_{\tilde{L}} ((\iota_{m,1}(t_i))_i + (x_{n+1})) = \rho(g) \bullet_{A(\tilde{L}, 1)} t.$$

Therefore,  $E(\tilde{L})$  is indeed a  $\lambda$ -theory.

For any presheaf  $Q$  and for any  $n$ , we have a bijection  $PL(L^n, Q) \cong Q_n$ . Then we have  $\varphi : E(\tilde{L})_n \cong L_n$ . This bijection is an isomorphism of  $\lambda$ -theories, since it preserves the  $x_i$ ,  $\bullet$ ,  $\rho$  and  $\lambda$ : for all  $1 \leq j \leq n$ ,  $f : E(\tilde{L})_m$ ,  $g : E(\tilde{L})_{m+1}$  and  $h : E(\tilde{L})_n^m$ .

$$\begin{aligned} \varphi(x_j) &= x_j(x_1, \dots, x_n) \\ &= x_j; \\ \varphi(f \bullet h) &= f \circ \langle h_i \rangle_i((x_i)_i) \\ &= f((h_i((x_j)_j))_i) \\ &= f((x_i)_i \bullet (h_i((x_j)_j))_i) \\ &= f((x_i)_i) \bullet (h_i((x_j)_j))_i \\ &= \varphi(f) \bullet (\varphi(h_i))_i; \\ \varphi(\rho(f)) &= \rho(f)((x_i)_i) \\ &= \rho(f((x_i)_i)) \bullet (x_i)_i \\ &= \rho(f((x_i)_i)) \\ &= \rho(\varphi(f)); \\ \varphi(\lambda(g)) &= \lambda(g)((x_i)_i) \\ &= \lambda(\varphi_{X^n}(g)((x_i)_i)) \\ &= \lambda(g(\iota_{m,l}((x_i)_i) + (x_{m+1}))) \\ &= \lambda(g((x_i)_i)) \\ &= \lambda(\varphi(g)). \end{aligned}$$

□

## 2. Locally cartesian closedness of the category of retracts

DEFINITION 26 (Category of retracts). The category of retracts for a  $\lambda$ -theory  $L$  is the category with objects  $f : L_n$  such that  $f \bullet f = f$  and it has as morphisms  $g : f \rightarrow f'$  the terms  $g : L_n$  such that  $f' \bullet g \bullet f = g$ . The object  $f : L_n$  has identity element  $f$ , and we have composition  $g \circ g' = g \bullet g'$ . These are morphisms (**TODO**)

LEMMA 17. *The category of retracts is indeed a category.*

PROOF. (**TODO**) □

THEOREM 2. *The category of retracts is locally cartesian closed (**TODO**) .*

## 3. The Fundamental Theorem of the $\lambda$ -calculus

DEFINITION 27 ( $\Lambda$ ). There is a special  $\lambda$ -theory, given by the  $\lambda$ -calculus itself.  $\Lambda_n$  is the set of  $\lambda$ -terms with  $n$  free variables, the  $x_i$  are the free variables, and  $\bullet$  is given by substitution.  $\lambda$  sends  $f : \Lambda_{n+1}$  to  $\lambda x_{n+1}. f$  and  $\rho$  sends  $f : \Lambda_n$  to  $\iota_{n,1}(f)x_{n+1}$  in  $\Lambda_n$ .

LEMMA 18.  *$\Lambda$  is indeed a  $\lambda$ -theory.*

PROOF. (**TODO**) □

LEMMA 19.  *$\Lambda$  is the initial  $\lambda$ -theory.*

PROOF. Given a  $\lambda$ -theory  $L$ , we construct a morphism  $f : \Lambda \rightarrow L$  by induction on the  $\lambda$ -terms. We set  $f(x_i) = x_i$ ,  $f(\lambda(t)) = \lambda(f(t))$  and  $f(st) = \rho(f(s)) \bullet ((x_i)_i + (f(t)))$ .

This is a  $\lambda$ -theory morphism because (**TODO**)

It is unique, since (**TODO**) □

DEFINITION 28 (Pullback of algebras). If we have a morphism of algebraic theories  $f : T' \rightarrow T$ , we have a functor  $AT \rightarrow AT'$ .

On objects, it sends a  $T$ -algebra  $A$  to a  $T'$ -algebra with set  $A$  and action  $g \bullet_{T'} a = f(g) \bullet_T a$ . This is a  $T'$ -algebra because (**TODO**) .

On morphisms, it sends  $\varphi : A \rightarrow A$  to  $\varphi : A \rightarrow A$ . This is a  $T'$ -algebra morphism because for all  $g : T'_n$  and  $a : A^n$ , we have

$$\varphi(g \bullet_{T'} a) = \varphi(f(g) \bullet_T a) = f(g) \bullet_T \varphi(a) = g \bullet_{T'} \varphi(a).$$

LEMMA 20. *This is indeed a functor.*

PROOF. (**TODO**) □

DEFINITION 29 (Term algebra). Given an algebraic theory  $T$ , for every  $n$ ,  $T_n$  together with the action operator  $\bullet : T_m \times T_n^m \rightarrow T_n$  gives a  $T$ -algebra.

LEMMA 21.  *$T_n$  is indeed a  $T$ -algebra.*

PROOF. (**TODO**) □

DEFINITION 30. For all  $n$ , we have a functor from lambda theories to  $\Lambda$ -algebras. It sends the  $\lambda$ -theory  $L$  to the  $L$ -algebra  $L_n$  and then turns this into a  $\Lambda$ -algebra via the morphism  $\Lambda \rightarrow L$ .

It sends morphisms  $f : L \rightarrow L'$  to  $f_n : L_n \rightarrow L'_n$ . This is a  $\Lambda$ -algebra morphism because (**TODO**)

LEMMA 22. *This indeed constitutes a functor.*

PROOF. (**TODO**) □

REMARK 11. Note that for a monoid  $M$ , if we view  $M$  as a category, the category  $[M^{\text{op}}, \mathbf{SET}]$  consists of sets with a right  $M$ -action.

DEFINITION 31 (The exponential object in the presheaf category). Given a monoid  $M$ , if we have two presheaves (sets with right  $M$ -actions)  $P$  and  $P'$ , we have a set of  $M$ -equivariant maps

$$F_{P,P'} = \left\{ f : M \times P \rightarrow P' \mid \prod_{p:P, m,m':M} f(m,p)m' = f(mm',pm') \right\}$$

with a right  $M$ -action, given by  $f m'(m,p) = f(m'm,p)$ . This is again  $M$ -equivariant because

$$f m'(m,p)m'' = f(m'm,p)m'' = f(m'mm'',pm'') = f m'(mm'',pm''),$$

so  $F_{P,P'}$  is a presheaf.

Now, to show that  $F_{P,P'}$  is the exponential object  $P'^P$ , we show that for any  $P$ ,  $F_{P,-}$  is the left adjoint of  $- \times P$ . So we need for all  $P' : PT$ , a universal arrow  $\varphi : F_{P,P'} \times P \rightarrow P'$ .

First of all, we have an evaluation map  $\varphi : F_{P,P'} \times P \rightarrow P'$  given by  $(f,p) \mapsto f(I,p)$  for  $I$  the unit of the monoid. This map is equivariant because for all  $m$ ,

$$(f,p)m = (fm,pm) \mapsto fm(I,pm) = f(m,pm) = f(I,p)m.$$

Now, given any presheaf  $Q$  and any morphism  $\psi : Q \times P \rightarrow P'$ , take  $\tilde{\psi} : Q \rightarrow F_{P,P'}$  given by  $\tilde{\psi}(q)(m,p) = \psi(qm,p)$ . This is equivariant because

$$\tilde{\psi}(q)m(m',p) = \tilde{\psi}(q)(mm',p) = \psi(qmm',p) = \tilde{\psi}(qm)(m',p)$$

and we have

$$\varphi(\tilde{\psi}(q),p) = \tilde{\psi}(q)(I,p) = \psi(q,p).$$

Now, suppose that we have  $\tilde{\psi}' : Q \rightarrow F_{P,P'}$  such that  $\varphi \circ (\tilde{\psi}' \times \text{id}_P) = \psi$ . Then for all  $q : Q$ ,  $m : M$  and  $p : P$ ,

$$\tilde{\psi}(q)(m,p) = \psi(qm,p) = \varphi(\tilde{\psi}'(qm),p) = \tilde{\psi}'(qm)(I,p) = \psi'(q)m(I,p) = \psi'(q)(m,p),$$

so  $\tilde{\psi}$  is unique and  $F_{P,P'}$  is an exponential object.

DEFINITION 32 (n-functional terms). Let  $A$  be a  $\Lambda$ -algebra. We define

$$A(n) = \{a : A \mid (\lambda x_2 x_3 \dots x_{n+1}, x_1 x_2 x_3 \dots x_{n+1}) \bullet a = a\}.$$

DEFINITION 33. Take  $\mathbf{1}_n = (\lambda x_1 \dots x_n, x_1 \dots x_n) \bullet () : A$ .

DEFINITION 34. We define composition as  $a \circ b = (\lambda x_3, x_1(x_2 x_3)) \circ (a, b)$  for  $a, b : A$ .

LEMMA 23. *This composition is associative.*

PROOF. (TODO) □

DEFINITION 35 (The monoid of a  $\Lambda$ -algebra). Now we make  $A(1)$  into a monoid with unit  $\lambda x_1, x_1$ .

LEMMA 24. *This is indeed a monoid.*

PROOF. (TODO) □

From here on, we will assume that  $\Lambda$  (and therefore, any  $\lambda$ -theory) satisfies  $\beta$ -equality.

LEMMA 25. *For  $a : A$ ,  $a$  is in  $A(n)$  iff  $\mathbf{1}_n \circ a = a$ .*

PROOF.

$$\begin{aligned}
\mathbf{1}_n \circ a &= (\lambda x_3, x_1(x_2x_3)) \bullet (((\lambda x_1 \dots x_n, x_1 \dots x_n) \bullet ()), a) \\
&= (\lambda x_3, x_1(x_2x_3)) \bullet (((\lambda x_2 \dots x_{n+1}, x_2 \dots x_{n+1}) \bullet a), x_1 \bullet a) \\
&= ((\lambda x_3, x_1(x_2x_3)) \bullet ((\lambda x_2 \dots x_{n+1}, x_2 \dots x_{n+1}), x_1)) \bullet a \\
&= (\lambda x_2, (\lambda x_3 \dots x_{n+2}, x_3 \dots x_{n+2})(x_1x_2)) \bullet a \\
&= (\lambda x_2x_3 \dots x_{n+1}, x_1x_2 \dots x_{n+1}) \bullet a.
\end{aligned}$$

□

DEFINITION 36 (The presheaf category of a  $\Lambda$ -algebra). Let  $A$  be a  $\Lambda$ -algebra. If we view the monoid  $A(1)$  as a one-object category, we define the category  $PA$  to be the category of presheaves  $[A(1)^{\text{op}}, \mathbf{SET}]$ .

DEFINITION 37 (The objects  $A(n)$  in  $PA$ ). Given  $a : A(n)$  and  $b : A(1)$ , we have

$$\mathbf{1}_n \circ (a \circ b) = (\mathbf{1}_n \circ a) \circ b = a \circ b,$$

so  $a \circ b : A(n)$  and we have a right  $A(1)$ -action on  $A(n)$ , which makes  $A(n)$  into an object in  $PA$ .

LEMMA 26. We have  $A(1)^{A(1)} \cong A(2)$ .

PROOF. We have a bijection  $\varphi : A(2) \cong F_{A(1), A(1)}$ , given by

$$\varphi(a)(b, b') = (\lambda x_4, x_1(x_2x_4)(x_3x_4)) \bullet (a, b, b').$$

Note that  $\varphi(d)$  is equivariant because **(TODO)** Now,  $\varphi$  is a presheaf morphism because **(TODO)**

Take  $p = \lambda x_1, x_1(\lambda x_2x_3, x_2)$  and  $q = \lambda x_1, x_1(\lambda x_2x_3, x_3)$ . These are elements of  $A(1)$ . Note that for terms  $c_1, c_2$

$$\begin{aligned}
p(\lambda x_1, x_1c_1c_2) &= (\lambda x_1, x_1c_1c_2)(\lambda x_2x_3, x_2) \\
&= (\lambda x_1x_3, x_2)c_1c_2 \\
&= c_1.
\end{aligned}$$

In the same way,  $q \circ (\lambda x_1x_2, x_2c_1c_2) = c_2$ .

An inverse is given by

$$\psi : f \mapsto \lambda x_1x_2, f(p, q)(\lambda x_3, x_3x_1x_2).$$

This is a presheaf morphism because **(TODO)**

This is an inverse, because given  $f : F_{A(1), A(1)}$  and  $(a_1, a_2) : A(1) \times A(1)$ , we have

$$\begin{aligned}
\varphi(\psi(f))(a_1, a_2) &= u(\lambda x_1x_2, f(p, q)(\lambda x_3, x_3x_1x_2))(a_1, a_2) \\
&= \lambda x_1, (\lambda x_2x_3, f(p, q)(\lambda x_4, x_4x_2x_3))(a_1x_1)(a_2x_1) \\
&= \lambda x_1, f(p, q)(\lambda x_2, x_2(a_1x_1)(a_2x_1)) \\
&= f(p, q) \circ (\lambda x_1, (\lambda x_2, x_2(a_1x_1)(a_2x_1))) \\
&= f(p \circ (\lambda x_1, (\lambda x_2, x_2(a_1x_1)(a_2x_1))), q \circ (\lambda x_1, (\lambda x_2, x_2(a_1x_1)(a_2x_1)))) \\
&= f(\lambda x_1, p(\lambda x_2, x_2(a_1x_1)(a_2x_1)), \lambda x_1, q(\lambda x_2, x_2(a_1x_1)(a_2x_1))) \\
&= f(\lambda x_1, a_1x_1, \lambda x_1, a_2x_1) \\
&= f(a_1, a_2).
\end{aligned}$$

The last line is because  $a_i : A(1)$  and therefore  $\lambda x_1, a_ix_1 = a_i$ .

On the other hand, if we have  $a_1 : A(2)$ , we have

$$\begin{aligned}\psi(\varphi(a_1)) &= \psi((a_2, a_3) \mapsto \lambda x_1, a_1(a_2 x_1)(a_3 x_1)) \\ &= \lambda x_1 x_2, (\lambda x_3, a_1(p x_3)(q x_3))(\lambda x_3, x_3 x_1 x_2) \\ &= \lambda x_1 x_2, a_1(p(\lambda x_3, x_3 x_1 x_2))(q(\lambda x_3, x_3 x_1 x_2)) \\ &= \lambda x_1 x_2, a_1 x_1 x_2 \\ &= a_1.\end{aligned}$$

The last line is because  $a_1 : A(2)$  and therefore  $\lambda x_1 x_2, a_1 x_1 x_2 = a_1$ .

Therefore, this map is a bijection and an isomorphism.  $\square$

DEFINITION 38 (Endomorphism  $\lambda$ -theory of a  $\Lambda$ -algebra).  $PA$  borrows products from **SET**. Therefore, the algebraic theory  $E(A(1))$  exists. Now note that  $A(1)$  is exponentiable and  $A(1)^{A(1)} \cong A(2)$ . Note that  $A(2) \subseteq A(1)$  and that  $(\lambda x_2 x_3, x_1 x_2 x_3) \bullet -$  gives a function from  $A(1)$  to  $A(2)$ . This gives  $E(A(1))$  a  $\lambda$ -theory structure.

DEFINITION 39 (Pullback functor on presheaves for a  $\Lambda$ -algebra). Let  $f : A \rightarrow A'$  be a  $\Lambda$ -algebra morphism. Then for all  $a : A(n)$ ,

$$\mathbf{1}_n \circ f(a) = f(\mathbf{1}_n) \circ f(a) = f(\mathbf{1}_n \circ a),$$

so we have an induced morphism  $f : A(n) \rightarrow A'(n)$ .

Now, given a presheaf  $P : PA'$ . We can create a presheaf  $f^*P : PA$  by taking the set of  $P$ , and, for  $p : P$  and  $a : A$ , setting  $pa = p \circ f(a)$ . This is indeed a presheaf because **(TODO)**

Now, given a morphism  $g : P \rightarrow P'$ , we get a morphism by taking the function on the sets of  $P$  and  $P'$ . This is a morphism because **(TODO)**

LEMMA 27. *The above indeed constitutes a functor.*

PROOF. **(TODO)**  $\square$

Left Kan extension then gives a left adjoint  $f_* : PA \rightarrow PA'$ .

LEMMA 28. *We have  $f_*(A(1)) \cong A'(1)$ .*

PROOF. **(TODO)**  $\square$

LEMMA 29.  *$f_*$  preserves finite products.*

PROOF. **(TODO)**  $\square$

DEFINITION 40. Since  $f_*$  preserves finite products, given an element of  $g : E(A(1))(n) = PA(A(1)^n, A(1))$ , we get

$$\#f_*(g) : PA'(f(A(1)^n), f(A(1))) \cong PA'(A'(1)^n, A'(1)) = E(A'(1))(n).$$

LEMMA 30.  *$\#f_* : E(A(1)) \rightarrow E(A'(1))$  is a map of  $\lambda$ -theories.*

PROOF. **(TODO)**  $\square$

DEFINITION 41. We have an isomorphism  $E(A(1))(0) \cong A$  given by  $a \mapsto aI$ .

LEMMA 31. *This is indeed an isomorphism of  $\Lambda$ -algebras.*

PROOF. **(TODO)**  $\square$

LEMMA 32. *Given  $g : A \rightarrow A'$ ,*

THEOREM 3. *There exists an adjoint equivalence between the category of  $\lambda$ -theories, and the category of algebras of  $\Lambda$ .*

PROOF. We will show that the functor  $L \mapsto L_0$  is an equivalence of categories.

It is essentially surjective, because  $L$  is isomorphic **(TODO)** to  $E(A(1))$ .

Now, given morphisms  $f, f' : L \rightarrow L'$ . Suppose that  $f_0 = f'_0$ . Suppose that  $L$  and  $L'$  have  $\beta$ -equality. Then, given  $l : L_n$ , we have

$$f_n(l) = \rho^n(\lambda^n(f_n(l))) = \rho^n(f_0(\lambda^n(l))) = \rho^n(f'_0(\lambda^n(l))) = \rho^n(\lambda^n(f'_n(l))) = f'_n(l),$$

so the functor is faithful.

The functor is full because a  $\Lambda$ -algebra morphism  $f : A \rightarrow A'$  induces a functor  $f^* : PA' \rightarrow PA$ , and via left Kan extension we get a left adjoint  $f_* : PA \rightarrow PA'$  with  $f_*(A(1)) \cong A'(1)$ . Now,  $f_*$  preserves (finite) products, so we have maps  $PA(A(1)^n, A(1)) \rightarrow PA'(A'(1)^n, A'(1))$  and so a map  $E(A(1)) \rightarrow E(A'(1))$ . This map, when restricted to a map  $PA(1, A(1)) \rightarrow PA'(1, A(1))$ , and transported along the isomorphism  $a \mapsto aI$  **(TODO)**, is equal to  $f$  **(TODO)**.  $\square$

LEMMA 33. *The category of  $T$ -algebras has coproducts.*

PROOF. **(TODO)**  $\square$

DEFINITION 42 (Theory of extensions). Let  $T$  be an algebraic theory and  $A$  a  $T$ -algebra. We can define an algebraic theory  $T_A$  called ‘the theory of extensions of  $A$ ’ with  $(T_A)_n = T_n + A$ . The left injection of the variables  $x_i : T_n$  gives the variables. Now, take  $h : (T_n + A)^m$ . Sending  $g : T_m$  to  $\varphi(g) := g \bullet h$  gives a  $T$ -algebra morphism  $T_m \rightarrow T_n + A$  since

$$\varphi(f \bullet g) = f \bullet g \bullet h = f \bullet (g_i \bullet h) = f \bullet (\varphi(g_i))_i.$$

This, together with the injection morphism of  $A$  into  $T_n + A$ , gives us a  $T$ -algebra morphism from the coproduct:  $T_m + A \rightarrow T_n + A$ . We especially have a function on sets  $(T_m + A) \times (T_n + A)^m \rightarrow T_n + A$ , which we will define our substitution to be.

LEMMA 34.  *$T_A$  is indeed an algebraic theory.*

PROOF. **(TODO)**  $\square$



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