

Semantics for the λ -calculus

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CHAPTER 1

Definitions

1. Algebraic Theories

DEFINITION 1 (algebraic theory). We define an algebraic theory T to be a sequence of sets T_n indexed over \mathbb{N} with for all $1 \leq i \leq n$ elements ("variables" or "projections") $x_{n,i} : T_n$ (we usually leave n implicit), together with a substitution operation

$$- \bullet - : T_m \times T_n^m \rightarrow T_n$$

for all m, n , such that

$$\begin{aligned} x_j \bullet g &= g_j \\ f \bullet (x_{l,i})_i &= f \\ (f \bullet g) \bullet h &= f \bullet (g_i \bullet h)_i \end{aligned}$$

for all $1 \leq j \leq l$, $f : T_l$, $g : T_m^l$ and $h : T_n^m$.

DEFINITION 2 (algebraic theory morphism). A morphism F between algebraic theories T and T' is a sequence of functions $F_n : T_n \rightarrow T'_n$ (we usually leave the n implicit) such that

$$\begin{aligned} F_n(x_j) &= x_j \\ F_n(f \bullet g) &= F_m(f) \bullet (F_n(g_i))_i \end{aligned}$$

for all $1 \leq j \leq n$, $f : T_m$ and $g : T_n^m$.

REMARK 1. We can construct binary products of algebraic theories, with sets $(T \times T')_n = T_n \times T'_n$, variables (x_i, x'_i) and substitution

$$(f, f') \bullet (g, g') = (f \bullet g, f' \bullet g').$$

In the same way, the category of algebraic theories has all limits.

2. Algebras

DEFINITION 3 (algebra). An algebra A for an algebraic theory T is a set A , together with an action

$$\bullet : T_n \times A^n \rightarrow A$$

for all n , such that

$$\begin{aligned} x_j \bullet a &= a_j \\ (f \bullet g) \bullet a &= f \bullet (g_i \bullet a)_i \end{aligned}$$

for all j , $f : T_m$, $g : T_n^m$ and $a : A^n$.

DEFINITION 4 (algebra morphism). For an algebraic theory T , a morphism F between T -algebras A and A' is a function $F : A \rightarrow A'$ such that

$$F(f \bullet a) = f \bullet (F(a_i))_i$$

for all $f : T_n$ and $a : A^n$.

REMARK 2. The category of algebras has all limits. The set of a limit of algebras is the limit of the underlying algebras.

REMARK 3. Note that for an algebraic theory T , the T_n are all algebras for T , with the action given by \bullet .

3. Presheaves

DEFINITION 5 (presheaf). A presheaf P for an algebraic theory T is a sequence of sets P_n indexed over \mathbb{N} , together with an action

$$\bullet : P_m \times T_n^m \rightarrow P_n$$

for all m, n , such that

$$\begin{aligned} t \bullet (x_{l,i})_i &= t \\ (t \bullet f) \bullet g &= t \bullet (f_i \bullet g)_i \end{aligned}$$

for all $t : P_l$, $f : T_m^l$ and $g : T_n^m$.

DEFINITION 6 (presheaf morphism). For an algebraic theory T , a morphism F between T -presheaves P and P' is a sequence of functions $F_n : P_n \rightarrow P'_n$ such that

$$F_n(t \bullet f) = F_m(t) \bullet f$$

for all $t : P_m$ and $f : T_n^m$.

We will write PT for the category of T -presheaves and their morphisms.

REMARK 4. The category of presheaves has all limits. The n th set \bar{P}_n of a limit \bar{P} of presheaves P_i is the limit of the n th sets $P_{i,n}$ of the presheaves in the limit diagram.

4. λ -theories

DEFINITION 7 (λ -theory). A λ -theory is an algebraic theory L , together with sequences of functions $\lambda_n : L_{n+1} \rightarrow L_n$ and $\rho_n : L_n \rightarrow L_{n+1}$, such that

$$\begin{aligned} \lambda(f) \bullet g &= \lambda(f \bullet (g_1, \dots, g_m, x_{n+1})) \\ \rho(f \bullet g) &= \rho(f) \bullet (g_1, \dots, g_m, x_{n+1}) \end{aligned}$$

for all $f : L_m$ and $g : L_n^m$. ((**TODO**) : Fix)

DEFINITION 8 (β - and η -equality). We say that a λ -theory L satisfies β -equality (or that it is a λ -theory with β) if $\rho_n \circ \lambda_n = \text{id}_{L_n}$ for all n . We say that it satisfies η -equality if $\lambda_n \circ \rho_n = \text{id}_{L_{n+1}}$ for all n .

DEFINITION 9 (λ -theory morphism). A morphism F between λ -theories L and L' is an algebraic theory morphism F such that

$$\begin{aligned} F_n(\lambda_n(f)) &= \lambda_n(F_{n+1}(f)) \\ \rho_n(F_n(g)) &= F_{n+1}(\rho_n(g)) \end{aligned}$$

for all $f : L_{n+1}$ and $g : L_n$.

REMARK 5. The category of lambda theories has all limits, with the underlying algebraic theory of a limit being the limit of the underlying algebraic theories.

A λ -theory algebra or presheaf is a presheaf for the underlying algebraic theory.

CHAPTER 2

Lemmas

1. The endomorphism theory

DEFINITION 10 (Endomorphism theory). Suppose that we have a category C and an object $X : C$, such that all powers X^n of X are also in C . The endomorphism theory $E(X)$ of X is the algebraic theory given by $E(X)_n = C(X^n, X)$ with projections as variables $x_{n,i} : X^n \rightarrow X$ and a substitution that sends $f : X^m \rightarrow X$ and $g_1, \dots, g_m : X^n \rightarrow X$ to $f \circ \langle g_i \rangle_i : X^n \rightarrow X^m \rightarrow X$.

LEMMA 1. $E(X)$ is indeed an algebraic theory.

PROOF. For $1 \leq j \leq l$, $f : E(X)_l$, $g : E(X)_m^l$ and $h : E(X)_n^m$, we have

$$\begin{aligned} x_j \bullet g &= x_j \circ \langle g_i \rangle_i = g_j, \\ f \bullet (x_{l,i})_i &= f \circ \langle x_{l,i} \rangle_i = f \circ \text{id}_{X^l} = f \end{aligned}$$

and

$$(f \bullet g) \bullet h = f \circ \langle g_i \rangle_i \circ \langle h_i \rangle_i = f \circ \langle g_i \circ \langle h_{i'} \rangle_{i'} \rangle_i = f \bullet (g_i \bullet h)_i.$$

□

DEFINITION 11 (Endomorphism λ -theory). Now, suppose that the exponential object X^X exists, and that we have morphisms back and forth $\text{abs} : X^X \rightarrow X$ and $\text{app} : X \rightarrow X^X$. Let, for $Y : C$, φ_Y be the isomorphism $C(X \times Y, X) \xrightarrow{\sim} C(Y, X^X)$. We can give $E(X)$ a λ -theory structure by setting, for $f : E(X)_{n+1}$ and $g : E(X)_n$,

$$\lambda(f) = \text{abs} \circ \varphi_{X^n}(f) \quad \rho(g) = \varphi_{X^n}^{-1}(\text{app} \circ g).$$

LEMMA 2. $E(X)$ is indeed a λ -theory.

PROOF. Note that $\varphi : C(- \times X, X) \xrightarrow{\sim} C(-, X^X)$ is a natural isomorphism, so for $g : E(X)_n^m$, the following diagram commutes

$$\begin{array}{ccc} C(X^m \times X, X) & \xrightarrow{- \circ (\langle g_i \rangle_i \times \text{id}_X)} & C(X^n \times X, X^X) \\ \varphi_{X^m}^{-1} \left(\begin{array}{c} \uparrow \\ \downarrow \end{array} \right) \varphi_{X^m} & & \varphi_{X^n}^{-1} \left(\begin{array}{c} \uparrow \\ \downarrow \end{array} \right) \varphi_{X^n} \\ C(X^m, X^X) & \xrightarrow{- \circ \langle g_i \rangle_i} & C(X^n, X^X) \end{array}$$

and note that $\langle g_i \rangle_i \times \text{id}_X = \langle g_1, \dots, g_m, x_{n+1} \rangle$. Then we have, for all $f : E(X)_m$

$$\begin{aligned} \lambda_m(f) \bullet g &= \text{abs} \circ \varphi_{X^m}(f) \circ \langle g_i \rangle_i \\ &= \text{abs} \circ \varphi_{X^n}(f \circ \langle g_1, \dots, g_m, x_{n+1} \rangle) \\ &= \lambda_n(f \bullet (g_1, \dots, g_m, x_{n+1})) \end{aligned}$$

and

$$\begin{aligned} \rho_n(f \bullet g) &= \varphi_{X^n}^{-1}(\text{app} \circ f \circ \langle g_i \rangle_i) \\ &= \varphi_{X^m}^{-1}(\text{app} \circ f) \circ \langle g_1, \dots, g_m, x_{n+1} \rangle \\ &= \rho_m(f) \bullet (g_1, \dots, g_m, x_{n+1}). \end{aligned}$$

□

2. The theory presheaf

DEFINITION 12 (The theory presheaf). Let T be an algebraic theory. We can turn T into an T -presheaf \tilde{T} by setting $\tilde{T}_n = T_n$ and using the substitution from T :

$$\bullet : \tilde{T}_m \times T_n^m \rightarrow \tilde{T}_n.$$

LEMMA 3. \tilde{T} is indeed a presheaf.

PROOF. For all $t : \tilde{T}_l$, $f : T_m^l$ and $g : T_n^m$,

$$t \bullet (x_{l,i})_i = t$$

and

$$(t \bullet f) \bullet g = t \bullet (f_i \bullet g)_i$$

because T is an algebraic theory. \square

LEMMA 4. Given an algebraic theory T and a T -presheaf Q , we have for all n a bijection of sets

$$\varphi : PT(\tilde{T}^n, Q) \cong Q_n.$$

PROOF. Take $\varphi(f) = f_n(x_1, \dots, x_n)$.

Conversely, take $\varphi^{-1}(q)$ to be the presheaf morphism that sends $t : T_m^n$ to $q \bullet t : Q_m$. This is indeed a presheaf morphism, since for all $t : T_l^n$ and $f : T_m^l$,

$$\varphi^{-1}(q)(t \bullet f) = q \bullet t \bullet f = \varphi^{-1}(q)(t) \bullet f.$$

Now, for a presheaf morphism $f : T^n \rightarrow Q$ and $t : T_m^n$, we have

$$\varphi^{-1}(\varphi(f))(t) = f_n(x_1, \dots, x_n) \bullet t = f_n((x_1, \dots, x_n) \bullet t) = f_n(t_1, \dots, t_n) = f_n(t).$$

Conversely, given $q : Q_n$, we have

$$\varphi(\varphi^{-1}(q)) = q \bullet (x_1, \dots, x_n) = q.$$

which concludes the proof. \square

3. The ‘+l’ presheaf

Let $\iota_{m,n} : T_m \rightarrow T_{m+n}$ denote the function that sends f to $f \bullet (x_{m+n,1}, \dots, x_{m+n,m})$. Note that

$$\iota_{m,n}(f) \bullet g = f \bullet (g_i)_{i \leq m}$$

and

$$\iota_{m,n}(f \bullet g) = f \bullet g \bullet (x_i)_i = f \bullet (g_i \bullet (x_j)_j)_i = f \bullet (\iota_{m,n}(g_i))_i.$$

For tuples $x : X^m$ and $y : X^n$, let $x+y$ denote the tuple $(x_1, \dots, x_m, y_1, \dots, y_n) : X^{m+n}$.

DEFINITION 13 (The ‘+l’ presheaf). Given a T -presheaf Q , we can construct a presheaf $A(Q, l)$, given by $A(Q, l)_n = Q_{n+l}$. Then, for $q : A(Q, l)_m$ and $f : T_n^m$, the substitution is given by

$$q \bullet_{A(Q,l)} f = q \bullet_Q ((\iota_{n,l}(f_i))_i + (x_{n+i})_i)$$

LEMMA 5. The $+l$ presheaf is a presheaf

PROOF. We have, for $q : A(Q, l)_n$,

$$\begin{aligned} q \bullet_{A(Q,l)} (x_i)_i &= q \bullet_Q ((\iota_{n,l}(x_i))_i + (x_{n+i})_i) \\ &= q \bullet_Q ((x_i)_i + (x_{n+i})_i) \\ &= q \bullet_Q (x_i)_i \\ &= q. \end{aligned}$$

We have, for $q : A(Q, k)_l$, $f : T_m^l$ and $g : T_n^m$,

$$\begin{aligned}
q \bullet_{A(Q, k)} f \bullet_{A(Q, k)} g &= q \bullet_Q ((\iota_{m, l}(f_i))_i + (x_{m+i})_i) \bullet_Q ((\iota_{n, l}(g_i))_i + (x_{n+i})_i) \\
&= q \bullet_Q (((\iota_{m, l}(f_i) \bullet_T ((\iota_{n, l}(g_j))_j + (x_{n+j})_j))_i + (x_{m+i} \bullet_T ((\iota_{n, l}(g_j))_j + (x_{n+j})_j))_i)) \\
&= q \bullet_Q ((f_i \bullet_T (\iota_{n, l}(g_j))_j)_i + (x_{n+i})_i) \\
&= q \bullet_Q ((\iota_{n, l}(f_i \bullet_T g))_i + (x_{n+i})_i) \\
&= q \bullet_{A(Q, k)} (f_i \bullet_T g).
\end{aligned}$$

□

4. Exponentiability of the theory presheaf

LEMMA 6. *For all l , the presheaf \tilde{T}^l is exponentiable.*

PROOF. We will show that $A(-, l)$ constitutes a right adjoint to the functor $- \times \tilde{T}^l$. We will do this using universal arrows ([ML98], Chapter IV.1, Theorem 2 (iv)). To that end, we will need for all $Q : PT$ a universal arrow $\varphi : A(Q, l) \times \tilde{T}^l \rightarrow Q$.

For $q : A(Q, l)_n = Q_{n+l}$ and $t : \tilde{T}_n^l$, we take $\varphi(q, t) = q \bullet_Q ((x_{n,i})_i + t)$.

This is a presheaf morphism, since for all $q : A(Q, l)_m^l$, $t : \tilde{T}_m^l$ and $f : T_n^m$,

$$\begin{aligned}
\varphi((q, t) \bullet_{A(Q, l) \times \tilde{T}^l} f) &= \varphi(q \bullet_{A(Q, l)} f, t \bullet_{\tilde{T}^l} f) \\
&= q \bullet_{A(Q, l)} f \bullet_Q ((x_i)_i + (t \bullet_{\tilde{T}^l} f)) \\
&= q \bullet_Q ((\iota_{n, l}(f_i))_i + (x_{n+i})_i) \bullet_Q ((x_i)_i + (t \bullet_{\tilde{T}^l} f)) \\
&= q \bullet_Q ((\iota_{n, l}(f_i) \bullet_T ((x_j)_j + (t \bullet_{\tilde{T}^l} f)))_i + (x_{n+i} \bullet_T ((x_j)_j + (t \bullet_{\tilde{T}^l} f)))_i) \\
&= q \bullet_Q ((f_i \bullet_T (x_j)_j)_i + ((t \bullet_{\tilde{T}^l} f)_i)_i) \\
&= q \bullet_Q ((f_i)_i + (t_i \bullet_{\tilde{T}^l} f)_i) \\
&= q \bullet_Q ((x_i \bullet_T f)_i + (t_i \bullet_T f)_i) \\
&= q \bullet_Q ((x_i)_i + t) \bullet_Q f \\
&= \varphi(q, t) \bullet_Q f.
\end{aligned}$$

Now, given any presheaf $Q' : PT$ we need to show that any morphism $\psi : Q' \times \tilde{T}^l \rightarrow Q$ factors uniquely as $\varphi \circ (\tilde{\psi} \times \text{id}_{\tilde{T}^l})$ for some $\tilde{\psi} : Q' \rightarrow A(Q, l)$.

So, given such a ψ , and given $q : Q'_n$, we take $\tilde{\psi}(q) = \psi(\iota_{n, l}(q), (x_{n+i})_i)$

This is a presheaf morphism, since for all $q : Q'_m$ and $f : T_n^m$,

$$\begin{aligned}
\tilde{\psi}(q \bullet f) &= \psi(\iota_{n, l}(q \bullet f), (x_{n+i})_i) \\
&= \psi(q \bullet (\iota_{n, l}(f_i))_i, (x_{n+i})_i) \\
&= \psi((\iota_{m, l}(q), (x_{m+i})_i) \bullet_{Q' \times \tilde{T}^l} ((\iota_{n, l}(f_i))_i + (x_{n+i})_i)) \\
&= \psi(\iota_{m, l}(q), (x_{m+i})_i) \bullet_Q ((\iota_{n, l}(f_i))_i + (x_{n+i})_i) \\
&= \tilde{\psi}(q) \bullet_{A(Q, l)} f.
\end{aligned}$$

Note that indeed $\varphi \circ (\tilde{\psi} \times \text{id}_{\tilde{T}^l}) = \psi$:

$$\begin{aligned}
\varphi(\tilde{\psi}(q), t) &= \varphi(\psi(\iota_{n, l}(q), (x_{n+i})_i), t) \\
&= \psi(\iota_{n, l}(q), (x_{n+i})_i) \bullet_Q ((x_i)_i + t) \\
&= \psi(\iota_{n, l}(q) \bullet_Q ((x_i)_i + t), (x_{n+i})_i \bullet_Q ((x_i)_i + t)) \\
&= \psi(q \bullet (x_i)_i, (t_i)_i) \\
&= \psi(q, t).
\end{aligned}$$

Now, suppose that we have another $\tilde{\psi}' : Q' \rightarrow A(Q, l)$ such that $\varphi \circ (\tilde{\psi}' \times \text{id}_{\tilde{T}^l}) = \psi$. Then we have

$$\begin{aligned}
 \tilde{\psi}(q) &= \psi(\iota_{n,l}(q), (x_{n+i})_i) \\
 &= (\varphi \circ (\tilde{\psi}' \times \text{id}_{\tilde{T}^l}))(\iota_{n,l}(q), (x_{n+i})_i) \\
 &= \varphi(\tilde{\psi}'(\iota_{n,l}(q)), (x_{n+i})_i) \\
 &= \tilde{\psi}'(\iota_{n,l}(q)) \bullet ((x_i)_i + (x_{n+i})_i) \\
 &= \iota_{n,l}(\tilde{\psi}'(q)) \bullet ((x_i)_i + (x_{n+i})_i) \\
 &= \tilde{\psi}'(q) \bullet (x_i)_i \\
 &= \tilde{\psi}'(q),
 \end{aligned}$$

so $\tilde{\psi}$ is unique, which completes the proof. \square

Now, this adjunction $- \times \tilde{T}^l \dashv A(-, l)$ induces a natural isomorphism

$$\varphi : PT(- \times \tilde{T}^l, \tilde{T}) \xrightarrow{\sim} PT(-, A(\tilde{T}, l))$$

LEMMA 7. For all $f : PT(\tilde{T}^n \times \tilde{T}^l, \tilde{T})$,

$$\varphi_{\tilde{T}^n}(f)(q) = f(\iota_{m,l}(q), (x_{m+i})_i)$$

PROOF. **(TODO)** \square

LEMMA 8. For all $f : PT(\tilde{T}^n, A(\tilde{T}, l))$,

$$\varphi_{\tilde{T}^n}^{-1}(f)(q, t) = f(q) \bullet ((x_i)_i + t).$$

PROOF. **(TODO)** \square

CHAPTER 3

Theorems

1. Scott's Representation Theorem

THEOREM 1. *Any λ -theory L is isomorphic to the endomorphism λ -theory $E(\tilde{L})$ of \tilde{L} in the presheaf category of L .*

PROOF. First of all, remember that \tilde{L} is indeed exponentiable and that $\tilde{L}^{\tilde{L}} = A(\tilde{L}, 1)$. Now, since L is a λ -theory, we have functions back and forth $\lambda : A(\tilde{L}, 1) \rightarrow \tilde{L}$ and $\rho : \tilde{L} \rightarrow A(\tilde{L}, 1)$. These are presheaf morphisms because for all $f : A(\tilde{L}, 1)_m$ and $g : \tilde{L}_m$ and $t : T_n^m$,

$$\lambda(f \bullet_{A(\tilde{L}, 1)} t) = \lambda(f \bullet_{\tilde{L}} ((\iota_{m,1}(t_i))_i + (x_{n+1}))) = \lambda(f) \bullet_{\tilde{L}} t$$

and

$$\rho(g \bullet_{\tilde{L}} t) = \rho(g) \bullet_{\tilde{L}} ((\iota_{m,1}(t_i))_i + (x_{n+1})) = \rho(g) \bullet_{A(\tilde{L}, 1)} t.$$

Therefore, $E(\tilde{L})$ is indeed a λ -theory.

For any presheaf Q and for any n , we have a bijection $PL(L^n, Q) \cong Q_n$. Then we have $\varphi : E(\tilde{L})_n \cong L_n$. This bijection is an isomorphism of λ -theories, since it preserves the x_i , \bullet , ρ and λ : for all $1 \leq j \leq n$, $f : E(\tilde{L})_m$, $g : E(\tilde{L})_{m+1}$ and $h : E(\tilde{L})_n^m$.

$$\begin{aligned} \varphi(x_j) &= x_j(x_1, \dots, x_n) \\ &= x_j; \\ \varphi(f \bullet h) &= f \circ \langle h_i \rangle_i((x_i)_i) \\ &= f((h_i((x_j)_j))_i) \\ &= f((x_i)_i \bullet (h_i((x_j)_j))_i) \\ &= f((x_i)_i) \bullet (h_i((x_j)_j))_i \\ &= \varphi(f) \bullet (\varphi(h_i))_i; \\ \varphi(\rho(f)) &= \rho(f)((x_i)_i) \\ &= \rho(f((x_i)_i)) \bullet (x_i)_i \\ &= \rho(f((x_i)_i)) \\ &= \rho(\varphi(f)); \\ \varphi(\lambda(g)) &= \lambda(g)((x_i)_i) \\ &= \lambda(\varphi_{X^n}(g)((x_i)_i)) \\ &= \lambda(g(\iota_{m,l}((x_i)_i) + (x_{m+1}))) \\ &= \lambda(g((x_i)_i)) \\ &= \lambda(\varphi(g)). \end{aligned}$$

□

2. Locally cartesian closedness of the category of retracts

DEFINITION 14 (Category of retracts). The category of retracts for a λ -theory L is the category with objects $f : L_n$ such that $f \bullet f = f$ and it has as morphisms $g : f \rightarrow f'$ the terms $g : L_n$ such that $f' \bullet g \bullet f = g$. The object $f : L_n$ has identity element f , and we have composition $g \circ g' = g \bullet g'$. These are morphisms (**TODO**)

LEMMA 9. *The category of retracts is indeed a category.*

PROOF. (**TODO**) □

THEOREM 2. *The category of retracts is locally cartesian closed (**TODO**) .*

3. The Fundamental Theorem of the λ -calculus

DEFINITION 15 (Λ). There is a special λ -theory, given by the λ -calculus itself. Λ_n is the set of λ -terms with n free variables, the x_i are the free variables, and \bullet is given by substitution. λ sends $f : \Lambda_{n+1}$ to $\lambda x_{n+1}. f$ and ρ sends $f : \Lambda_n$ to $\iota_{n,1}(f)x_{n+1}$ in Λ_n .

LEMMA 10. *Λ is indeed a λ -theory.*

PROOF. (**TODO**) □

LEMMA 11. *Λ is the initial λ -theory.*

PROOF. Given a λ -theory L , we construct a morphism $f : \Lambda \rightarrow L$ by induction on the λ -terms. We set $f(x_i) = x_i$, $f(\lambda(t)) = \lambda(f(t))$ and $f(st) = \rho(f(s)) \bullet ((x_i)_i + (f(t)))$.

This is a λ -theory morphism because (**TODO**)

It is unique, since (**TODO**) □

DEFINITION 16 (Pullback of algebras). If we have a morphism of algebraic theories $f : T' \rightarrow T$, we have a functor $AT \rightarrow AT'$.

On objects, it sends a T -algebra A to a T' -algebra with set A and action $g \bullet_{T'} a = f(g) \bullet_T a$. This is a T' -algebra because (**TODO**) .

On morphisms, it sends $\varphi : A \rightarrow A$ to $\varphi : A \rightarrow A$. This is a T' -algebra morphism because for all $g : T'_n$ and $a : A^n$, we have

$$\varphi(g \bullet_{T'} a) = \varphi(f(g) \bullet_T a) = f(g) \bullet_T \varphi(a) = g \bullet_{T'} \varphi(a).$$

LEMMA 12. *This is indeed a functor.*

PROOF. (**TODO**) □

DEFINITION 17 (Term algebra). Given an algebraic theory T , for every n , T_n together with the action operator $\bullet : T_m \times T_n^m \rightarrow T_n$ gives a T -algebra.

LEMMA 13. *T_n is indeed a T -algebra.*

PROOF. (**TODO**) □

DEFINITION 18. For all n , we have a functor from lambda theories to Λ -algebras. It sends the λ -theory L to the L -algebra L_n and then turns this into a Λ -algebra via the morphism $\Lambda \rightarrow L$.

It sends morphisms $f : L \rightarrow L'$ to $f_n : L_n \rightarrow L'_n$. This is a Λ -algebra morphism because (**TODO**)

LEMMA 14. *This indeed constitutes a functor.*

PROOF. (**TODO**) □

REMARK 6. Note that for a monoid M , if we view M as a category, the category $[M^{\text{op}}, \mathbf{SET}]$ consists of sets with a right M -action.

DEFINITION 19 (The exponential object in the presheaf category). Given a monoid M , if we have two presheaves (sets with right M -actions) P and P' , we have a set of M -equivariant maps

$$F_{P,P'} = \left\{ f : M \times P \rightarrow P' \mid \prod_{p:P, m,m':M} f(m,p)m' = f(mm',pm') \right\}$$

with a right M -action, given by $f m'(m,p) = f(m'm,p)$. This is again M -equivariant because

$$f m'(m,p)m'' = f(m'm,p)m'' = f(m'mm'',pm'') = f m'(mm'',pm''),$$

so $F_{P,P'}$ is a presheaf.

Now, to show that $F_{P,P'}$ is the exponential object P'^P , we show that for any P , $F_{P,-}$ is the left adjoint of $- \times P$. So we need for all $P' : PT$, a universal arrow $\varphi : F_{P,P'} \times P \rightarrow P'$.

First of all, we have an evaluation map $\varphi : F_{P,P'} \times P \rightarrow P'$ given by $(f,p) \mapsto f(I,p)$ for I the unit of the monoid. This map is equivariant because for all m ,

$$(f,p)m = (fm,pm) \mapsto fm(I,pm) = f(m,pm) = f(I,p)m.$$

Now, given any presheaf Q and any morphism $\psi : Q \times P \rightarrow P'$, take $\tilde{\psi} : Q \rightarrow F_{P,P'}$ given by $\tilde{\psi}(q)(m,p) = \psi(qm,p)$. This is equivariant because

$$\tilde{\psi}(q)m(m',p) = \tilde{\psi}(q)(mm',p) = \psi(qmm',p) = \tilde{\psi}(qm)(m',p)$$

and we have

$$\varphi(\tilde{\psi}(q),p) = \tilde{\psi}(q)(I,p) = \psi(q,p).$$

Now, suppose that we have $\tilde{\psi}' : Q \rightarrow F_{P,P'}$ such that $\varphi \circ (\tilde{\psi}' \times \text{id}_P) = \psi$. Then for all $q : Q$, $m : M$ and $p : P$,

$$\tilde{\psi}(q)(m,p) = \psi(qm,p) = \varphi(\tilde{\psi}'(qm),p) = \tilde{\psi}'(qm)(I,p) = \psi'(q)m(I,p) = \psi'(q)(m,p),$$

so $\tilde{\psi}$ is unique and $F_{P,P'}$ is an exponential object.

DEFINITION 20 (n-functional terms). Let A be a Λ -algebra. We define

$$A(n) = \{a : A \mid (\lambda x_2 x_3 \dots x_{n+1}, x_1 x_2 x_3 \dots x_{n+1}) \bullet a = a\}.$$

DEFINITION 21. Take $\mathbf{1}_n = (\lambda x_1 \dots x_n, x_1 \dots x_n) \bullet () : A$.

DEFINITION 22. We define composition as $a \circ b = (\lambda x_3, x_1(x_2 x_3)) \circ (a, b)$ for $a, b : A$.

LEMMA 15. *This composition is associative.*

PROOF. (TODO) □

DEFINITION 23 (The monoid of a Λ -algebra). Now we make $A(1)$ into a monoid with unit $\lambda x_1, x_1$.

LEMMA 16. *This is indeed a monoid.*

PROOF. (TODO) □

From here on, we will assume that Λ (and therefore, any λ -theory) satisfies β -equality.

LEMMA 17. *For $a : A$, a is in $A(n)$ iff $\mathbf{1}_n \circ a = a$.*

PROOF.

$$\begin{aligned}
\mathbf{1}_n \circ a &= (\lambda x_3, x_1(x_2 x_3)) \bullet (((\lambda x_1 \dots x_n, x_1 \dots x_n) \bullet ()), a) \\
&= (\lambda x_3, x_1(x_2 x_3)) \bullet (((\lambda x_2 \dots x_{n+1}, x_2 \dots x_{n+1}) \bullet a), x_1 \bullet a) \\
&= ((\lambda x_3, x_1(x_2 x_3)) \bullet ((\lambda x_2 \dots x_{n+1}, x_2 \dots x_{n+1}), x_1)) \bullet a \\
&= (\lambda x_2, (\lambda x_3 \dots x_{n+2}, x_3 \dots x_{n+2})(x_1 x_2)) \bullet a \\
&= (\lambda x_2 x_3 \dots x_{n+1}, x_1 x_2 \dots x_{n+1}) \bullet a.
\end{aligned}$$

□

DEFINITION 24 (The presheaf category of a Λ -algebra). Let A be a Λ -algebra. If we view the monoid $A(1)$ as a one-object category, we define the category PA to be the category of presheaves $[A(1)^{\text{op}}, \mathbf{SET}]$.

DEFINITION 25 (The objects $A(n)$ in PA). Given $a : A(n)$ and $b : A(1)$, we have

$$\mathbf{1}_n \circ (a \circ b) = (\mathbf{1}_n \circ a) \circ b = a \circ b,$$

so $a \circ b : A(n)$ and we have a right $A(1)$ -action on $A(n)$, which makes $A(n)$ into an object in PA .

LEMMA 18. We have $A(1)^{A(1)} \cong A(2)$.

PROOF. (TODO)

□

DEFINITION 26 (Endomorphism λ -theory of a Λ -algebra). PA borrows products from \mathbf{SET} . Therefore, the algebraic theory $E(A(1))$ exists. Now note that $A(1)$ is exponentiable and $A(1)^{A(1)} \cong A(2)$. Note that $A(2) \subseteq A(1)$ and that $(\lambda x_2 x_3, x_1 x_2 x_3) \bullet -$ gives a function from $A(1)$ to $A(2)$. This gives $E(A(1))$ a λ -theory structure.

THEOREM 3. There exists an adjoint equivalence between the category of λ -theories, and the category of algebras of Λ .

PROOF. We will show that the functor $L \mapsto L_0$ is an equivalence of categories.

It is essentially surjective, because L is isomorphic (TODO) to $E(A(1))$.

Now, given morphisms $f, f' : L \rightarrow L'$. Suppose that $f_0 = f'_0$. Suppose that L and L' have β -equality. Then, given $l : L_n$, we have

$$f_n(l) = \rho^n(\lambda^n(f_n(l))) = \rho^n(f_0(\lambda^n(l))) = \rho^n(f'_0(\lambda^n(l))) = \rho^n(\lambda^n(f'_n(l))) = f'_n(l),$$

so the functor is faithful.

The functor is full because a Λ -algebra morphism $f : A \rightarrow A'$ induces a morphism $f^* : PA' \rightarrow PA$ (TODO), and via left Kan extension we get a left adjoint $f_* : PA \rightarrow PA'$ with $f_*(A(1)) \cong A'(1)$ (TODO). Now, f_* preserves (finite) products (TODO), so we have maps $PA(A(1)^n, A(1)) \rightarrow PA'(A'(1)^n, A'(1))$ and so a map $E(A(1)) \rightarrow E(A'(1))$ (TODO). This map, when restricted to a map $PA(1, A(1)) \rightarrow PA'(1, A(1))$, and transported along the isomorphism $a \mapsto aI$ (TODO), is equal to f (TODO). □

LEMMA 19. The category of T -algebras has coproducts.

PROOF. (TODO)

□

DEFINITION 27 (Theory of extensions). Let T be an algebraic theory and A a T -algebra. We can define an algebraic theory T_A called ‘the theory of extensions of A ’ with $(T_A)_n = T_n + A$. The left injection of the variables $x_i : T_n$ gives the variables. Now, take $h : (T_n + A)^m$. Sending $g : T_m$ to $\varphi(g) := g \bullet h$ gives a T -algebra morphism $T_m \rightarrow T_n + A$ since

$$\varphi(f \bullet g) = f \bullet g \bullet h = f \bullet (g_i \bullet h) = f \bullet (\varphi(g_i))_i.$$

This, together with the injection morphism of A into $T_n + A$, gives us a T -algebra morphism from the coproduct: $T_m + A \rightarrow T_n + A$. We especially have a function on sets $(T_m + A) \times (T_n + A)^m \rightarrow T_n + A$, which we will define our substitution to be.

LEMMA 20. T_A is indeed an algebraic theory.

PROOF. (TODO)

□

Bibliography

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