Semantics for the λ-calculus

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Classical lambda calculus in modern dress

- Paper by Martin Hyland.
- About models for the λ-calculus.
- Three 'big' theorems.
- My job: 'annotate'.

The **untyped** λ-calculus

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$$\lambda x_1, x_1$$

$$\lambda x_3x_2x_1, x_1(x_2x_3).$$

Can have β- and η-equality:

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The *(pure) λ-calculus*: Described exactly by the above.

Algebraic theories: objects with variables and substitution

Example

λ-calculus: $\Lambda_n = \{(\lambda x_1, x_1), x_5, (\lambda x_3, x_7)x_{42}\}$.

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Definition

An *algebraic theory* T is a sequence of sets T_n with variables $x_{i,n} \in T_n$ (for $0 \leq i < n$) and a substitution operation $\bullet : T_m \times T_n^m \to T_n$.

λ-theory: structure with app and abs

Definition

A λ-*theory* L is an algebraic theory, together with abstraction functions $\lambda : L_{n+1} \to L_n$ and application functions $\rho : L_n \to L_{n+1}$ (both compatible with the substitution).

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The pure λ-calculus Λ is the initial λ-theory.
β- and η-equality:

$$\rho_n \circ \lambda_n = \mathrm{Id}_{L_{n+1}} \qquad \lambda_n \circ \rho_n = \mathrm{Id}_{L_n}.$$

Algebras: Interpretations (or denotations)

We want to interpret terms with free variables as functions from a context to a set

Example

In $T(n) = \mathbb{Z}[x_1, \ldots, x_n]$, we can take a set $A = \mathbb{Q}$ and get

$$2x_1 + 3x_1^2 x_2 : A^2 \to A, \quad (a_1, a_2) \mapsto 2 \cdot a_1 + 3 \cdot a_1^2 \cdot a_2.$$

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Definition

For an algebraic theory T, a *T-algebra* A is a set A, together with interpretation functions $T_n \times A^n \to A$ for all n (respecting the variables and substitution).

Scott's representation theorem (1980)

For every λ-theory L, we can find a category C and an object $X : C_0$, such that L is isomorphic to the endomorphism theory of X: the λ-theory $E(X)$ given by $E(X)_n = X^n \to X$.

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$E(X)_n = X^n \to X$.
The variables of $E(X)_n$ are the projections $\pi_i : X^n \to X$. Also, substituting $g_1, \ldots, g_m : X^n \to X$ into $f : X^m \to X$ composes f with $\langle g_1, \ldots, g_m \rangle : X^n \to X^m$.

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For $\lambda : E(X)_{n+1} \to E(X)_n$, postcomposing with some morphism $\overline{abs} : (X \to X) \to X$ gives

$$\lambda : E(X)_{n+1} = (X^{n+1} \to X) \simeq (X^n \to (X \to X)) \xrightarrow{\overline{abs} \circ -} (X^n \to X) = E(X)_n.$$

In the same way, we get $\rho : E(X)_n \to E(X)_{n+1}$ from a morphism $\overline{app} : X \to (X \to X)$.

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C is the category of sequences of sets $(P_i)_i$ with a composition $P_m \times L_n^m \to P_n$ and X is the sequence $(L_i)_i$.
With Hyland's definitions and some lemmas, the representation theorem arises before you know it *(on paper)*.

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This functor again uses the endomorphism theory $E(X)$ for some object X to construct the λ-theory.
Hyland shows that these functors constitute an adjoint equivalence.

The category of retracts

Given a λ-theory L, we can view elements $f : L_1$ as one-argument functions, and we can compose them like $f \circ g := f \bullet g$.
Now we construct a category R

$$R_0 = \{a : L_1 \mid a \circ a = a\}, \qquad a \to b = \{f : L_1 \mid b \circ f \circ a = f\}.$$

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If we want to do *dependent type theory*, we need dependent products and sums.

For a morphism $f : A \to B$, we have a pullback functor between slice categories $f^* : R/B \to R/A$. We have a dependent product and sum along f if f^* has a right and a left adjoint.

If the category is *locally cartesian closed*, we have all dependent products, and so all pullback functors have both adjoints.

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But this is too strong a requirement. Not all pullback functors have both adjoints, but some do. R is *relatively cartesian closed*.
I am still working on understanding the proof.

Annotating the paper

An algebraic theory T is first a functor $T : \mathbf{F} \to Sets$: so we have sets $T(n)$ of n-ary multimaps with variable renamings. In addition, T is equipped with projections $pr_1, \ldots, pr_n : T(n)$ including as special case the identity $id \in T(1)$. Finally there are compositions $T(n) \times T(m)^n \to T(m)$ which are **associative, unital, compatible with projections and natural in n and m**. *A map $F : S \to T$ of algebraic theories is a natural transformation with components $F_n : S(n) \to T(n)$ preserving projections and composition.*

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- Learn the background.
- Decode the definitions and theorems.
- Find examples.
- Formalize.
- Mechanize.

Mechanization

- Displayed categories:
  - Univalence;
  - Limits (twice);
- Higher inductive types;
- $X^{n+1} = X \times X^n$;
- $X_{n+1} = X_{1+n}$;

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3 'big' theorems:

- Every model of the λ-calculus arises as the endomorphism theory of some category.
- There is an equivalence between models of the λ-calculus, and interpretations of the λ-calculus as functions on a set.
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Mechanization is hard.

Do you have questions?

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Because I have one: I am still a bit unsure about the exact 'meaning' of relative cartesian closedness. Can someone explain that better to me?