Summary of the things that I learned

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CHAPTER 1

Week 08

1. Univalent Categories

A univalent category is a category in which the univalence axiom holds. I.e., a category $\mathcal{C}$ in which, for all $A, B \in \mathcal{C}_0$, the canonical map $(A =_{\mathcal{C}} B) \to (A \cong B)$ is an equivalence.

2. Categories

An **n-category** is a category with 0-cells (objects), 1-cells (morphisms), 2-cells (morphisms between morphisms), up to n-cells and various compositions: $A \to B \to C$. $A \xrightarrow{f,g,h} B$, $f \Rightarrow g \Rightarrow h$. $A \xrightarrow{f,g} B \xrightarrow{f',g'} C$, $\alpha : f \Rightarrow g$, and identities $\alpha' : f' \Rightarrow g'$ gives $\alpha' * \alpha : f' \circ f \Rightarrow g' \circ g$. These all need to work together 'nicely'. An ω-category is the same, but all the way up.

A topological space gives a (weak) ω-category. 0-cells are points, 1-cells are paths, 2-cells are homotopies etc. Composition is by glueing. It is a 'groupoid', in the sense that all homotopies of dimension ≥ 1 are invertible. However, glueing is not associative, so it is a 'weak' ω-category.

A category with only one object $\star$ is equivalent to a monoid (with elements being the set $\mathcal{C}(\star, \star)$). A 2-category with only one 0-cell is the same thing as a monoidal category (objects: the 1-cells. Morphisms: the 2-cells). A monoidal category with just one object gives 2 monoid structures on its set of morphisms. These are the same, and commutative.

A **monoid** is a set with a multiplication and a unit. A **monad** on a category $\mathcal{C}$ is a functor $T : \mathcal{C} \to \mathcal{C}$, together with natural transformations $\mu : T \circ T \to T$ (satisfying associativity) and $\eta : 1_{\mathcal{A}} \to T$ (acting as a two-sided unit).

A **presheaf** on a category $\mathcal{A}$ is a functor $\mathcal{A}^{opp} \to \mathbf{Set}$.

Given a category $\mathcal{E}$ and an object $E \in \mathcal{E}_0$, the **slice category** $\mathcal{E}/E$ with objects being the maps $D \xrightarrow{p} E$ and morphisms being commutative triangles.

A **multicategory**, not necessarily the same as an n-category, is a category in which arrows go from multiple objects to one, instead of from one object to one. I.e. it is a category with a class C_0 of objects, for all n, and all $a, a_1, \dots, a_n \in C_0$, a class $C(a_1, \dots, a_n; a)$ of 'morphisms', and a composition

$$C(a_1, \dots, a_n; a) \times C(a_{1,1}, \dots, a_{1,k_1}; a_1) \times \dots \times C(a_{n,1}, \dots, a_{n,k_n}; a_n) \to C(a_{1,1}, \dots, a_{n,k_n}; a),$$

written $(\theta, \theta_1, \dots, \theta_n) \mapsto \theta(\theta_1, \dots, \theta_n)$ and for each $a \in C_0$ an identity $1_a \in C(a; a)$. It must satisfy associativity

$$\theta \circ (\theta_1 \circ (\theta_{1,1}, \dots, \theta_{1,k_1}), \dots, \theta_n \circ (\theta_{n,1}, \dots, \theta_{n,k_n})) = (\theta \circ (\theta_1, \dots, \theta_n)) \circ (\theta_{1,1}, \dots, \theta_{n,k_n})$$

and identity

$$\theta \circ (1_{a_1}, \dots, 1_{a_n}) = \theta = 1_a \circ \theta.$$

A **map of multicategories** is a function $f_0 : C_0 \to C'_0$ and maps $C(a_1, \ldots, a_n; a) \to C(f_0(a_1), \ldots, f_0(a_n); f_0(a))$, preserving composition and identities.

For C a multicategory, a C**-algebra** is a map from C into the multicategory **Set** (with objects $\mathbf{Set}_0$ and maps $\mathbf{Set}(a_1, \ldots, a_n; a) = \mathbf{Set}(a_1 \times \cdots \times a_n; a)$). I.e., for each $a \in C_0$, a set $X(a)$, and for each map $\theta : a_1, \ldots, a_n \to a$, a function $X(\theta) : X(a_1) \times X(a_n) \to X(a)$. An example is, for a multicategory C, to take $X(a) = C(; a)$ (maps from the empty sequence into a).

Of course, there is a concept of **free multicategory**: Given a set X, and for all $n \in N$, and $x, x_1, \ldots, x_n \in X$, a set $X(x_1, \ldots, x_n; x)$, we get a multicategory X' with $X'_0 = X_0$, and $X'(x_1, \ldots, x_n; x)$ given by formal compositions of elements of the $X(y_1, \ldots, y_m; y)$.

A **bicategory** consists of a class $\mathcal{B}_0$ of 0-cells, or objects; For each $A, B \in \mathcal{B}_0$, a category $\mathcal{B}(A, B)$ of 1-cells (objects) and 2-cells (morphisms); for each $A, B, C \in \mathcal{B}_0$, a functor $\mathcal{B}(B, C) \times \mathcal{B}(A, B) \to \mathcal{B}(A, C)$ written $(g, f) \mapsto g \circ f$ on 1-cells and $(\delta, \gamma) \mapsto \delta * \gamma$ on 2-cells; For each $A \in \mathcal{B}_0$ an object $1_A \in \mathcal{B}(A, A)$; isomorphisms representing associativity and identity axioms (e.g. $f \circ 1_A \cong f \in \mathcal{B}(A, B)$), natural in their arguments, satisfying pentagon and triangle axioms.

The collection of categories Cat forms a bicategory. In analogy, we define a monad in a bicategory to be an object A, together with a 1-cell $t : A \to A$ and 2-cells $\mu : t \circ t \to t$ and $\eta : 1_A \to t$ satisfying a couple of commutativity axioms (those of 1.1.3 in [**Lei03**]).

3. Operads

3.1. Definitions. An **operad** is a multicategory with only one object. More explicitly, an operad has a set $P(k)$ for every $k \in \mathbb{N}$, whose elements can be thought of as k-ary operations. It also has, for all $n, k_1, \ldots, k_n \in \mathbb{N}$, a *composition* function

$$P(n) \times P(k_1) \times \cdots \times P(k_n) \to P(k_1 + \cdots + k_n)$$

and an element $1 = 1_P \in P(1)$ called the **identity**, satisfying

$$\theta \circ (1, 1, \ldots, 1) = \theta = \theta \circ 1$$

for all θ, and

$$\theta \circ (\theta_1 \circ (\theta_{1,1}, \ldots, \theta_{1,k_1}), \ldots, \theta_n \circ (\theta_{n,1}, \ldots, \theta_{n,k_n})) = (\theta \circ (\theta_1, \ldots, \theta_n)) \circ (\theta_{1,1}, \ldots, \theta_{n,k_n})$$

for all $\theta \in P(n)$, $\theta_1 \in P(k_1)$, $\ldots$, $\theta_n \in P(k_n)$ and all $\theta_{1,1} \ldots \theta_{n,k_n}$.

A **morphism of operads** is a family

$$f_n : (P(n) \to Q(n))_{n \in \mathbb{N}}$$

of functions, preserving composition and identities.

A P**-algebra** for P an operad, is a set X and, for each n, and $\theta \in P(n)$, a function $\overline{\theta} : X^n \to X$, satisfying the evident axioms (identity is the identity function, the function of a composition is the composition of the functions?).

3.2. Examples. For any vector space V, there is an operad with $P(k) = V^{\otimes k} \to V$.

The terminal operad 1 has $P(n) = \{\star_1\}$ for all n. An algebra for 1 is a set X together with a function $X^n \to X$, denoted as $(x_1, \ldots, x_n) \mapsto (x_1 \cdot \cdots \cdot x_n)$, satisfying

$$((x_{1,1} \cdot \ldots x_{1,k_1}) \cdot \cdots \cdot (x_{n,1} \cdot \cdots \cdot x_{n,k_n})) = (x_{1,1} \cdot \cdots \cdot x_{n,k_n})$$

and

$$x = (x).$$

The category of 1-algebras is the category of monoids.

There exist various sub-operads of 1. For example, the smallest one has $P(1) = \{\star\}$ and $P(n) = \emptyset$ for $n \neq 1$.

Or the operad with $P(0) = \emptyset$ and $P(n) = \{\star_n\}$ for $n > 0$, which has semigroups as its algebras (sets with associative binary operations).

The suboperad with $P(n) = \{\star_n\}$ exactly when $n \leq 1$ has as its algebras the pointed sets.

The **operad of curves** has $P(n) = \{\text{smooth maps } \mathbb{R} \to \mathbb{R}^n\}$.

Given a monad on **Set**, we get a natural operad structure $T(n)_{n\in\mathbb{N}}$, with $T(n)$ the set of words in n variables and composition given by 'substitution'.

Given a monoid M (a category with one object), there is a operad given by $P(n) = M^n$ and composition

$$(\alpha_1, \ldots, \alpha_n) \circ ((\alpha_{1,1}, \ldots, \alpha_{1,k_1}), \ldots, (\alpha_{n,1}, \ldots, \alpha_{n,k_n})).$$

The **Little 2-disks** operad D has

$$D(n) = \{\text{set of non-overlapping disks contained within the unit disk}\},$$

with composition being geometric "substitution". I.e.: scale and move a unit disk and its contained disks to match one of the smaller disks, and replace the smaller disk with the transformed contents of our original unit disk. See also: this image that explains a lot

Given sets $X(n)$ for all $n \in \mathbb{N}$, the **free operad** X' on these is defined exactly by $X(n) \subseteq X'(n)$, $1 \in X'(1)$ and for all $m, n_1, \ldots, n_m \in \mathbb{N}$ and $f \in X(m)$ and $f_i \in X'(n_i)$, we have $f \circ (f_1, \ldots, f_m) \in X'(n_1 + \cdots + n_m)$.

4. T-operads

4.1. Definitions. A category is **cartesian** if it has all pullbacks. A functor is cartesian if it preserves pullbacks. A natural transformation $\alpha : S \to T$ is cartesian if for all $f : A \to B$, the naturality diagram

$$\begin{array}{ccc} SA & \xrightarrow{Sf} & SB \\ \downarrow{\scriptstyle \alpha_A} & & \downarrow{\scriptstyle \alpha_B} \\ TA & \xrightarrow{Tf} & TB \end{array}$$

is a pullback. A monad (T, μ, η) on a category $\mathcal{E}$ is cartesian if the category $\mathcal{E}$, the functor T and the natural transformations μ and η are cartesian.

We can represent (the morphism structure of) an ordinary category using diagrams $C_0 \xleftarrow{\text{domain}} C_1 \xrightarrow{\text{codomain}} C_0$, $C_1 \times_{C_0} C_1 \xrightarrow{\text{composition}} C_1$ and $C_0 \xrightarrow{\text{id}} C_1$ together with some axioms. For a multicategory, we need to slightly modify this, using a functor $T : \mathbf{Set} \to \mathbf{Set}$, $A \mapsto \coprod A^n$, to $TC_0 \xleftarrow{d} C_1 \xrightarrow{c} C_0$ and $C_1 \times_{TC_0} TC_1 \xrightarrow{\circ} C_1$.

Given a cartesian monad (T, μ, η) on a category $\mathcal{E}$, we can define a bicategory $\mathcal{E}_{(T)}$, with the class of 0-cells being $\mathcal{E}_0$, the 1-cells $E \to E'$ being diagrams $TE \xleftarrow{d} M \xrightarrow{c} E'$, 2-cells $(M, d, c) \to (N, q, p)$ are maps $M \to N$ such that the diagram with E, E', M, N commutes. The composite of 1-cells $TE \xleftarrow{d} M \xrightarrow{c} E'$ and $TE \xleftarrow{d'}$

$M' \xrightarrow{c'} E''$ is given by

$$TE \xleftarrow{\mu_E} T^2E \xleftarrow{Td} TM \leftarrow TM \times_{TE'} M' \to M' \xrightarrow{c'} E''$$

in which the coproduct in the middle is formed using Tc and d. We can define a T-multicategory to be a monad on $\mathcal{E}_{(T)}$. Equivalently, we can define it as an object $C_0 \in \mathcal{E}$, together with a diagram $t : TC_0 \xleftarrow{d} C_1 \xrightarrow{c} C_0$ and maps $C_1 \circ C_1 := TC_1 \times_{TC_0} C_1 \xrightarrow{\circ} C_1$ and $C_0 \xrightarrow{id} C_1$ satisfying associativity and identity axioms.

A T-operad is a T-multicategory such that C_0 is the terminal object of $\mathcal{E}$. Equivalently, it is an object over $T1$, (so we have a morphism $P \to T1$), together with maps $P \times_{T1} TP \to P$ and $1 \xrightarrow{id} P$, both over $T1$, satisfying associativity and identity axioms.

4.2. Examples. For T the identity monad on **Set**, a T-operad is exactly a monoid (or an operad with only unary functions) (since there is always a unique map to $\{1\}$).

If $\mathcal{E}$ is **Set**, the terminal object 1 will always be $\{1\}$.

For the free monoid monad $TA = \bigsqcup A^n$, the T-operads are precisely the operads that we defined before.

For the monad $TA = 1 + A$, we can view TA as a subset of the free monoid on A, and this gives an operad with 0-ary and 1-ary functions. The 1-ary arrows form a monoid, and the 0-ary arrows are a set, with an action of the monoid.

5. Cartesian Operads

5.1. Theory. Using Towards a doctrine of operads.

NLab uses notation: **Fin** for what we would call a standard skeleton of finite sets (i.e. the category of finite sets $\{0, \dots, n-1\}$ and maps between them). A^B denotes all morphisms/functors $B \to A$. I.e., the class of functors $\mathbf{Fin} \to \mathbf{Set}$ is denoted $\mathbf{Set}^{\mathbf{Fin}}$.

Take $I = \mathbf{Fin}(1, -) : \mathbf{Set}^{\mathbf{Fin}} = \mathbf{Fin} \to \mathbf{Set}$.

Let $[\mathbf{Set}^{\mathbf{Fin}}, \mathbf{Set}^{\mathbf{Fin}}]$ be the category of finite-product-preserving, cocontinuous endofunctors on $\mathbf{Set}^{\mathbf{Fin}}$. The map $Ev_I : [\mathbf{Set}^{\mathbf{Fin}}, \mathbf{Set}^{\mathbf{Fin}}] \to \mathbf{Set}^{\mathbf{Fin}}$, given by $F \mapsto F(I)$ is an equivalence. $[\mathbf{Set}^{\mathbf{Fin}}, \mathbf{Set}^{\mathbf{Fin}}]$ has a monoidal product $\odot$ given by endofunctor composition, and we can transfer this to $\mathbf{Set}^{\mathbf{Fin}}$.

Concretely, we have $F \odot G = \int^{n \in \mathbf{Fin}} F(n) G^n$.

5.2. Cartesian operads. A **cartesian operad** is a monoid in this monoidal category $(\mathbf{Set}^{\mathbf{Fin}}, \odot, I)$. I.e., it is a triple (M, μ, η), with $M \in \mathbf{Set}^{\mathbf{Fin}}$, $\mu : M \odot M \to M$ and $\eta : I \to M$.

More concretely, this is a functor $M : \mathbf{Fin} \to \mathbf{Set}$, together with maps

$$m_{n,k} : M(n) \times M(k)^n \to M(k),$$

natural in k and dinatural in n, and an element $e \in M(1)$.

Dinaturality in n means the following. Fix $k \in \mathbf{Fin}$. We have the functors $\mathbf{Fin}^{\mathrm{op}} \times \mathbf{Fin} \to \mathbf{Set}$ given by

$$F : (n, n') \mapsto M(n') \times M(k)^n \quad \text{and} \quad G : (n, n') \mapsto M(k).$$

For all $n \in \mathbf{Fin}_0$, we have a morphism

$$\bullet : F(n, n) = M(n) \times M(k)^n \to M(k) = G(n, n).$$

Naturality means that for all $a : n \to n'$,

$$G(n,a) \circ \bullet \circ F(a,n) = F(a,n') \circ \bullet \circ G(n',a).$$

i.e., for all $f \in M(n)$, $g_1, \ldots, g_{n'} \in M(k)$,

$$f \bullet (g_{a(1)}, \ldots, g_{a(n)}) = M(a)(f) \bullet (g_1, \ldots, g_{n'}).$$

Now, if we have, in **Fin**, a decomposition $k = k_1 + \cdots + k_n$, and we have inclusion maps $i_j : k_j \hookrightarrow k$, then we have

$$M(n) \times M(k_1) \times \cdots \times M(k_n) \xrightarrow{1 \times M(i_1) \times \cdots \times M(i_n)} M(n) \times M(k)^n \xrightarrow{m_{n,k}} M(k),$$

which gives an operad structure.

5.3. Clones. In other parts of mathematics, a cartesian operad is called a **clone**. An abstract clone consists of sets $M(n)$ for all $n \in \mathbb{N}$, for all $n, k \in \mathbb{N}$ a function $\bullet : M(n) \times M(k)^n \to M(k)$ and for each $1 \leq i \leq n \in \mathbb{N}$, an element $\pi_{i,n} \in M(n)$ such that for $f \in M(i)$, $g_1, \ldots, g_i \in M(j)$ and $h_1, \ldots, h_j \in M(k)$,

$$f \bullet (g_1 \bullet (h_1, \ldots, h_j), \ldots, g_i \bullet (h_1 \ldots, h_j)) = (f \bullet (g_1, \ldots, g_i)) \bullet (h_1, \ldots, h_j),$$

for $f_1, \ldots, f_n \in M(k)$,

$$\pi_{i,n} \bullet (f_1, \ldots, f_n) = f_i,$$

and for $f \in M(n)$,

$$f \bullet (\pi_{1,n}, \ldots, \pi_{n,n}) = f.$$

(It is claimed that this automatically gives naturality)

Naturality means for all $a \in \mathbf{Fin}(n, n')$, for all $f \in M(n)$, $g_1, \ldots, g_{n'} \in M(k)$,

$$f \bullet (g_{a(1)}, \ldots, g_{a(n)}) = (f \bullet (\pi_{a(1),n'}, \ldots, \pi_{a(n),n'})) \bullet (g_1, \ldots, g_{n'}).$$

However, by associativity and since $\pi_{i,n} \bullet (f_1, \ldots, f_n)$, we have

$$\begin{aligned}
&(f \bullet (\pi_{a(1),n'}, \ldots, \pi_{a(n),n'})) \bullet (g_1, \ldots, g_{n'}) \\
&= (f \bullet (\pi_{a(1),n'} \bullet (g_1, \ldots, g_{n'}), \ldots, \pi_{a(n),n'} \bullet (g_1, \ldots, g_{n'}))) \\
&= f \bullet (g_{a(1)}, \ldots, g_{a(n)}).
\end{aligned}$$

Naturality in k is as follows. Fix $n \in \mathbf{Fin}_0$. We have functors $F, G : \mathbf{Fin} \to \mathbf{Set}$, given by

$$F : k \mapsto M(n) \times M(k)^n \quad \text{and} \quad G : k \mapsto M(k).$$

For $a : k \to k'$, we must have

$$G(a) \circ \bullet = \bullet \circ F(a).$$

That is, for all $f \in M(n)$, $g_1, \ldots, g_n \in M(k)$,

$$M(a)(f \bullet (g_1, \ldots, g_n)) = f \bullet (M(a)(g_1), \ldots, M(a)(g_n)).$$

Now, $M(a)$ is given by

$$M(a)(f) = f \bullet (\pi_{a(1),k'}, \ldots, \pi_{a(k),k'}).$$

Therefore, we have

$$\begin{aligned}
&M(a)(f \bullet (g_1, \ldots, g_n)) \\
&= (f \bullet (g_1, \ldots, g_n)) \bullet (\pi_{a(1),k'}, \ldots, \pi_{a(k),k'}) \\
&= f \bullet (g_1 \bullet (\pi_{a(1),k'}, \ldots, \pi_{a(k),k'}), \ldots, g_n \bullet (\pi_{a(1),k'}, \ldots, \pi_{a(k),k'})) \\
&= f \bullet (M(a)(g_1), \ldots, M(a)(g_n)).
\end{aligned}$$

So the associativity and projection axioms ensure naturality.

CHAPTER 2

Week 09

1. Adjunctions

For three sets X, Y, Z, we have a bijection

$$\mathbf{Set}(X \times Y, Z) \cong \mathbf{Set}(X, Y \to Z)$$

that is natural in X and Z. This is an example of a general notion:

For functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$, F is a **left adjoint** for G if there is an isomorphism $\mathcal{C}(FX, Y) \cong \mathcal{D}(X, GY)$ that is dinatural in X and Y.

1.1. Examples. A special case is for $G : \mathcal{C} \to \mathbf{Set}$ the forgetful functor: if we have a left adjoint $F : \mathbf{Set} \to \mathcal{C}$ to G, we call $F(X)$ the **free** $\mathcal{C}$ of X (for example: free group, free monoid, free category, etc.). That means that $\mathcal{C}(F(X), Y) \cong \mathbf{Set}(X, G(Y))$, or in other words:

Adjunctions can be composed: If F and G both forget part of the structure of an object, then we can construct the free object 'piecewise' using the composition of the left adjoints.

Let $G : \mathbf{Group} \to \mathbf{Set}$ be the forgetful functor and $F : \mathbf{Set} \to \mathbf{Group}$ be the free group functor. For any set S, we have an inclusion $i : S \hookrightarrow FS$. Now, given any group Y and any morphism of sets from S to Y (formally: $f \in \mathbf{Set}(S, GY)$). Then the fact that $\mathbf{Set}(FS, Y) \cong \mathbf{Set}(S, GY)$ is expressed in the fact that f factors uniquely as $g \circ i$.

$$\begin{tikzcd} S \arrow[r, "i"] \arrow[dr, "f"'] & F(S) \arrow[d, "\exists! g"] \\ & Y \end{tikzcd}$$

In topology, the forgetful functor from topological spaces to sets has a left adjoint that turns a set S into a space $F(S)$ with the discrete topology, since it has the smallest topology on S that still can factor all morphisms $S \to [0, 1]$ for example.

In algebra, the inclusion functor from the category of monoids (sets with an associative operation and a unit) to the category of semigroups (sets with an associative operation) has a left adjoint that sends the semigroup S to the monoid $S \sqcup \{\star\}$ and gives it operations such that $\star$ is the unit.

In algebra, the inclusion functor from the category of rings to the category of rngs (rings but without an identity) has a left adjoint that sends the rng R to $R \times \mathbb{Z}$ and defines $(r, x)(s, y) = (rs + xs + ry, xy)$.

For $\mathbf{Dom_m}$ the category of integral domains with injective morphisms. Then the forgetful functor $\mathbf{Fields} \to \mathbf{Dom_m}$ has a left adjoint that sends a domain to its field of fractions.

For $\rho : R \to S$ a morphism of rings, we can view every S-module as an R-module: we have a functor $S-\mathbf{Mod} \to R-\mathbf{Mod}$. This has a left adjoint, $M \mapsto M \otimes_R S$.

2. Operads and cartesian operads

2.1. Recap operads and cartesian operads. In this section, we will represent a cartesian operad C by a clone:

- For all $n \in \mathbb{N}$, a set $C(n)$;
- For all $1 \leq i \leq n$ projections $\pi_{n,i} \in P(n)$;
- For all m, n, a function $\rho_{m,n} : C(m) \times C(n)^m \to C(n)$.

such that

- For $f \in C(l)$, $g_1, \ldots, g_l \in C(m)$ and $h_1, \ldots, h_m \in C(n)$,

$$\rho(f, (\rho(g_1, (h_1, \ldots, h_m)), \ldots, \rho(g_l, (h_1, \ldots, h_m)))) = \rho(\rho(f, (g_1, \ldots, g_l)), (h_1, \ldots, h_l)).$$

- for $f_1, \ldots, f_n \in M(n)$ and $1 \leq i \leq n$, $\rho(\pi_{i,n}, (f_1, \ldots, f_n)) = f_i$;
- For $f \in M(n)$, $\rho(f, (\pi_{1,n}, \ldots, \pi_{n,n})) = f$.

and a morphism of cartesian operads $\varphi : C \to C'$ by componentwise functions $\varphi_n : C(n) \to C'(n)$ such that

- For all $1 \leq i \leq n$, $\varphi_n(\pi_{n,i}) = \pi'_{n,i}$;
- For all m, n, $f \in C(m)$, $g_1, \ldots, g_m \in C(n)$

$$\rho(\varphi(f), (\varphi(g_1), \ldots, \varphi(g_m))) = \varphi(\rho(f, (g_1, \ldots, g_m))).$$

We will represent an operad P by

- For all $n \in \mathbb{N}$, a set $P(n)$;
- An element $e \in P(1)$;
- For all $m, n_1, \ldots, n_m \in \mathbb{N}$, a function

  $$\gamma_{m,n_1,\ldots,n_m} : C(m) \times C(n_1) \times \cdots \times C(n_m) \to C(n_1 + \cdots + n_m)$$

  (we will usually just call this γ).

such that

- For all $f \in P(n)$,

$$\gamma(f, (1, \ldots, 1)) = f = \gamma(1, f);$$

- For all $f \in P(l)$, $g_1 \in P(m_1)$, $\ldots$, $g_l \in P(m_l)$ and all $h_{1,1} \in P(n_{1,1})$, $\ldots$, $h_{l,m_l} \in P(n_{l,m_l})$,

$$f \circ (g_1 \circ (h_{1,1}, \ldots, h_{1,m_1}), \ldots, g_l \circ (h_{l,1}, \ldots, h_{l,m_l})) = (f \circ (g_1, \ldots, g_l)) \circ (h_{1,1}, \ldots, h_{l,m_l}).$$

and a morphism of operads $\psi : P \to P'$ by componentwise functions $\psi_n : P(n) \to P'(n)$ such that

- $\psi_1(e) = e'$;
- For all $f \in P(m)$, $g_1 \in P(n_1)$, $\ldots$, $g_m \in P(n_m)$,

$$\gamma(\psi(f), (\psi(g_1), \ldots, \psi(g_m))) \psi(\gamma(f, (g_1, \ldots, g_m))).$$

2.2. Free operad? Now, for all $1 \leq i \leq m$, we have an injection $j_i : C(n_i) \to C(n_1 + \cdots + n_m)$. Intuitively, it maps terms in a context with n_i variables to terms in a context with $n_1 + \cdots + n_m$ variables, by mapping variable $1 \leq k \leq n_1$ to variable $n_1 + \cdots + n_{i-1} + k$. This gives a morphism

$$\gamma : C(m) \times C(n_1) \times \cdots \times C(n_m) \xrightarrow{id \times j_1 \times \cdots \times j_m} C(m) \times C(n_1 + \cdots + n_m)^m \xrightarrow{\rho} C(n_1 + \cdots + n_m),$$

which gives an operad structure on the same sets. This gives a functor from cartesian operads to operads.

Then, the question arises whether this functor has a left adjoint.

Still open

3. Cartesian multicategory

Yet another way to think of a cartesian operad is as a one-object cartesian multicategory. A **cartesian multicategory** is a multicategory with an S_n-action on the hom-sets (in other words: we can permute the arguments of the morphsims) and duplication/diagonal/contraction operations

$$\mathrm{Hom}(c_1, \ldots, c_k, c_k, \ldots, c_n; c) \to \mathrm{Hom}(c_1, \ldots, c_k, \ldots, c_n; c)$$

and deletion/projection/weakening operations

$$\mathrm{Hom}(c_1, \ldots, c_k, \ldots, c_n; c) \to \mathrm{Hom}(c_1, \ldots, c_{k-1}, c_{k+1}, \ldots, c_n; c)$$

4. The paper

A summary of the results in the paper:

(1) A definition of the category of algebraic theories (clones).
(2) A definition of the category of T-algebras (for T an algebraic theory), the pullback of an algebra, the free algebra, and some properties of these.
(3) A definition of a presheaf (like an algebra, but with a right action instead of a left one) and some properties of this.
(4) A sidenote about the more classical approach to algebra.
(5) A definition for a λ-theory and some properties of its constituents.
(6) A definition for an interpretation of the λ-calculus in a λ-theory, and some properties.
(7) The notion that the algebraic theory Λ of all terms of the λ-calculus is the initial λ-theory.
(8) The notion that we can add constants to a theory to make sure that it 'has enough points'.
(9) A (new) proof that any λ-theory is isomorphic to the endomorphism λ-theory of some object.
(10) A section that concludes an equivalence between the presheaf categories of a Lawvere theory, a λ-theory and the category of retracts.
(11) The definition of Λ-algebra and a couple of its properties. In particular, a functor from λ-theories to Λ-algebras.
(12) An equivalene between Λ-algebras and their presheaves.
(13) A characterisation of the function space U^U for $U \in PA$ the universal.
(14) An equivalence of categories between the category of Λ-algebras and λ-theories.
(15) The Fundamental Theorem of the λ-calculus: there is an adjoint equivalence between λ-theories and Λ-algebras.

(16) A remark that this is much harder without the category theoretic framework.

Bibliography

[Lei03] Tom Leinster. Higher operads, higher categories, 2003.