SUMMARY OF THE THINGS THAT I LEARNED

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1. Week 08

1.1. **Univalent Categories.** A univalent category is a category in which the univalence axiom holds. I.e., a category $\mathcal{C}$ in which, for all $A, B \in \mathcal{C}_0$, the canonical map $(A =_{\mathcal{C}} B) \to (A \cong B)$ is an equivalence.

1.2. **Categories.** An n**-category** is a category with 0-cells (objects), 1-cells (morphisms), 2-cells (morphisms between morphisms), up to n-cells and various compositions: $A \to B \to C$. $A \xrightarrow{f,g,h} B$, $f \Rightarrow g \Rightarrow h$. $A \xrightarrow{f,g} B \xrightarrow{f',g'} C$, $\alpha : f \Rightarrow g$, and identities $\alpha' : f' \Rightarrow g'$ gives $\alpha' * \alpha : f' \circ f \Rightarrow g' \circ g$. These all need to work together 'nicely'. An ω-category is the same, but all the way up.

A topological space gives a (weak) ω-category. 0-cells are points, 1-cells are paths, 2-cells are homotopies etc. Composition is by glueing. It is a 'groupoid', in the sense that all homotopies of dimension ≥ 1 are invertible. However, glueing is not associative, so it is a 'weak' ω-category.

A category with only one object $\star$ is equivalent to a monoid (with elements being the set $\mathcal{C}(\star, \star)$). A 2-category with only one 0-cell is the same thing as a monoidal category (objects: the 1-cells. Morphisms: the 2-cells). A monoidal category with just one object gives 2 monoid structures on its set of morphisms. These are the same, and commutative.

A **monoid** is a set with a multiplication and a unit. A **monad** on a category $\mathcal{C}$ is a functor $T : \mathcal{C} \to \mathcal{C}$, together with natural transformations $\mu : T \circ T \to T$ (satisfying associativity) and $\eta : 1_{\mathcal{A}} \to T$ (acting as a two-sided unit).

A **presheaf** on a category $\mathcal{A}$ is a functor $\mathcal{A}^{opp} \to \mathbf{Set}$.

Given a category $\mathcal{E}$ and an object $E \in \mathcal{E}_0$, the **slice category** $\mathcal{E}/E$ with objects being the maps $D \xrightarrow{p} E$ and morphisms being commutative triangles.

A **multicategory**, not necessarily the same as an n-category, is a category in which arrows go from multiple objects to one, instead of from one object to one. I.e. it is a category with a class C_0 of objects, for all n, and all $a, a_1, \dots, a_n \in C_0$, a class $C(a_1, \dots, a_n; a)$ of 'morphisms', and a composition

$$C(a_1, \dots, a_n; a) \times C(a_{1,1}, \dots, a_{1,k_1}; a_1) \times \cdots \times C(a_{n,1}, \dots, a_{n,k_n}; a_n) \to C(a_{1,1}, \dots, a_{n,k_n}; a),$$

written $(\theta, \theta_1, \dots, \theta_n) \mapsto \theta(\theta_1, \dots, \theta_n)$ and for each $a \in C_0$ an identity $1_a \in C(a; a)$. It must satisfy associativity

$$\theta \circ (\theta_1 \circ (\theta_{1,1}, \dots, \theta_{1,k_1}), \dots, \theta_n \circ (\theta_{n,1}, \dots, \theta_{n,k_n})) = (\theta \circ (\theta_1, \dots, \theta_n)) \circ (\theta_{1,1}, \dots, \theta_{n,k_n})$$

and identity

$$\theta \circ (1_{a_1}, \dots, 1_{a_n}) = \theta = 1_a \circ \theta.$$

A **map of multicategories** is a function $f_0 : C_0 \to C_0'$ and maps $C(a_1, \dots, a_n; a) \to C(f_0(a_1), \dots, f_0(a_n); f_0(a))$, preserving composition and identities.

For C a multicategory, a C**-algebra** is a map from C into the multicategory **Set** (with objects $\mathbf{Set}_0$ and maps $\mathbf{Set}(a_1, \ldots, a_n; a) = \mathbf{Set}(a_1 \times \cdots \times a_n; a)$). I.e., for each $a \in C_0$, a set $X(a)$, and for each map $\theta : a_1, \ldots, a_n \to a$, a function $X(\theta) : X(a_1) \times X(a_n) \to X(a)$. An example is, for a multicategory C, to take $X(a) = C(;a)$ (maps from the empty sequence into a).

Of course, there is a concept of **free multicategory**: Given a set X, and for all $n \in N$, and $x, x_1, \ldots, x_n \in X$, a set $X(x_1, \ldots, x_n; x)$, we get a multicategory X' with $X'_0 = X_0$, and $X'(x_1, \ldots, x_n; x)$ given by formal compositions of elements of the $X(y_1, \ldots, y_m; y)$.

A **bicategory** consists of a class $\mathcal{B}_0$ of 0-cells, or objects; For each $A, B \in \mathcal{B}_0$, a category $\mathcal{B}(A, B)$ of 1-cells (objects) and 2-cells (morphisms); for each $A, B, C \in \mathcal{B}_0$, a functor $\mathcal{B}(B, C) \times \mathcal{B}(A, B) \to \mathcal{B}(A, C)$ written $(g, f) \mapsto g \circ f$ on 1-cells and $(\delta, \gamma) \mapsto \delta * \gamma$ on 2-cells; For each $A \in \mathcal{B}_0$ an object $1_A \in \mathcal{B}(A, A)$; isomorphisms representing associativity and identity axioms (e.g. $f \circ 1_A \cong f \in \mathcal{B}(A, B)$), natural in their arguments, satisfying pentagon and triangle axioms.

The collection of categories Cat forms a bicategory. In analogy, we define a monad in a bicategory to be an object A, together with a 1-cell $t : A \to A$ and 2-cells $\mu : t \circ t \to t$ and $\eta : 1_A \to t$ satisfying a couple of commutativity axioms (those of 1.1.3 in [Lei03]).

1.3. **Operads.**

1.3.1. *Definitions.* An **operad** is a multicategory with only one object. More explicitly, an operad has a set $P(k)$ for every $k \in \mathbb{N}$, whose elements can be thought of as k-ary operations. It also has, for all $n, k_1, \ldots, k_n \in \mathbb{N}$, a *composition* function

$$P(n) \times P(k_1) \times \cdots \times P(k_n) \to P(k_1 + \cdots + k_n)$$

and an element $1 = 1_P \in P(1)$ called the **identity**, satisfying

$$\theta \circ (1, 1, \ldots, 1) = \theta = \theta \circ 1$$

for all θ, and

$$\theta \circ (\theta_1 \circ (\theta_{1,1}, \ldots, \theta_{1,k_1}), \ldots, \theta_n \circ (\theta_{n,1}, \ldots, \theta_{n,k_n})) = (\theta \circ (\theta_1, \ldots, \theta_n)) \circ (\theta_{1,1}, \ldots, \theta_{n,k_n})$$

for all $\theta \in P(n)$, $\theta_1 \in P(k_1)$, $\ldots$, $\theta_n \in P(k_n)$ and all $\theta_{1,1} \ldots \theta_{n,k_n}$.

A **morphism of operads** is a family

$$f_n : (P(n) \to Q(n))_{n \in \mathbb{N}}$$

of functions, preserving composition and identities.

A P**-algebra** for P an operad, is a set X and, for each n, and $\theta \in P(n)$, a function $\overline{\theta} : X^n \to X$, satisfying the evident axioms (identity is the identity function, the function of a composition is the composition of the functions?).

1.3.2. *Examples.* For any vector space V, there is an operad with $P(k) = V^{\otimes k} \to V$.

The terminal operad 1 has $P(n) = \{\star_1\}$ for all n. An algebra for 1 is a set X together with a function $X^n \to X$, denoted as $(x_1, \ldots, x_n) \mapsto (x_1 \cdot \cdots \cdot x_n)$, satisfying

$$((x_{1,1} \cdot \ldots x_{1,k_1}) \cdot \cdots \cdot (x_{n,1} \cdot \cdots \cdot x_{n,k_n})) = (x_{1,1} \cdot \cdots \cdot x_{n,k_n})$$

and

$$x = (x).$$

The category of 1-algebras is the category of monoids.

There exist various sub-operads of 1. For example, the smallest one has $P(1) = \{\star\}$ and $P(n) = \emptyset$ for $n \neq 1$.

Or the operad with $P(0) = \emptyset$ and $P(n) = \{\star_n\}$ for $n > 0$, which has semigroups as its algebras (sets with associative binary operations).

The suboperad with $P(n) = \{\star_n\}$ exactly when $n \leq 1$ has as its algebras the pointed sets.

The **operad of curves** has $P(n) = \{\text{smooth maps } \mathbb{R} \to \mathbb{R}^n\}$.

Given a monad on **Set**, we get a natural operad structure $T(n)_{n\in\mathbb{N}}$, with $T(n)$ the set of words in n variables and composition given by 'substitution'.

Given a monoid M (a category with one object), there is a operad given by $P(n) = M^n$ and composition

$$(\alpha_1, \dots, \alpha_n) \circ ((\alpha_{1,1}, \dots, \alpha_{1,k_1}), \dots, (\alpha_{n,1}, \dots, \alpha_{n,k_n})).$$

The **Little 2-disks** operad D has

$$D(n) = \{\text{set of non-overlapping disks contained within the unit disk}\},$$

with composition being geometric "substitution". I.e.: scale and move a unit disk and its contained disks to match one of the smaller disks, and replace the smaller disk with the transformed contents of our original unit disk. See also: this image that explains a lot

Given sets $X(n)$ for all $n \in \mathbb{N}$, the **free operad** X' on these is defined exactly by $X(n) \subseteq X'(n)$, $1 \in X'(1)$ and for all $m, n_1, \dots, n_m \in \mathbb{N}$ and $f \in X(m)$ and $f_i \in X'(n_i)$, we have $f \circ (f_1, \dots, f_m) \in X'(n_1 + \dots + n_m)$.

1.4. T-operads.

1.4.1. *Definitions.* A category is **cartesian** if it has all pullbacks. A functor is cartesian if it preserves pullbacks. A natural transformation $\alpha : S \to T$ is cartesian if for all $f : A \to B$, the naturality diagram

$$\begin{tikzcd} SA \arrow[r, "Sf"] \arrow[d, "\alpha_A"] & SB \arrow[d, "\alpha_B"] \\ TA \arrow[r, "Tf"] & TB \end{tikzcd}$$

is a pullback. A monad (T, μ, η) on a category $\mathcal{E}$ is cartesian if the category $\mathcal{E}$, the functor T and the natural transformations μ and η are cartesian.

We can represent (the morphism structure of) an ordinary category using diagrams $C_0 \xleftarrow{\text{domain}} C_1 \xrightarrow{\text{codomain}} C_0$, $C_1 \times_{C_0} C_1 \xrightarrow{\text{composition}} C_1$ and $C_0 \xrightarrow{\text{id}} C_1$ together with some axioms. For a multicategory, we need to slightly modify this, using a functor $T : \mathbf{Set} \to \mathbf{Set}$, $A \mapsto \coprod A^n$, to $TC_0 \xleftarrow{d} C_1 \xrightarrow{c} C_0$ and $C_1 \times_{TC_0} TC_1 \xrightarrow{\circ} C_1$.

Given a cartesian monad (T, μ, η) on a category $\mathcal{E}$, we can define a bicategory $\mathcal{E}_{(T)}$, with the class of 0-cells being $\mathcal{E}_0$, the 1-cells $E \to E'$ being diagrams $TE \xleftarrow{d} M \xrightarrow{c} E'$, 2-cells $(M, d, c) \to (N, q, p)$ are maps $M \to N$ such that the diagram with E, E', M, N commutes. The composite of 1-cells $TE \xleftarrow{d} M \xrightarrow{c} E'$ and $TE \xleftarrow{d'} M' \xrightarrow{c'} E''$ is given by

$$TE \xleftarrow{\mu_E} T^2E \xleftarrow{Td} TM \leftarrow TM \times_{TE'} M' \to M' \xrightarrow{c'} E''$$

in which the coproduct in the middle is formed using Tc and d. We can define a T-multicategory to be a monad on $\mathcal{E}_{(T)}$. Equivalently, we can define it as an object $C_0 \in \mathcal{E}$, together with a diagram $t : TC_0 \xleftarrow{d} C_1 \xrightarrow{c} C_0$ and maps $C_1 \circ C_1 := TC_1 \times_{TC_0} C_1 \xrightarrow{\circ} C_1$ and $C_0 \xrightarrow{id} C_1$ satisfying associativity and identity axioms.

A T-operad is a T-multicategory such that C_0 is the terminal object of $\mathcal{E}$. Equivalently, it is an object over $T1$, (so we have a morphism $P \to T1$), together with maps $P \times_{T1} TP \to P$ and $1 \xrightarrow{id} P$, both over $T1$, satisfying associativity and identity axioms.

1.4.2. *Examples.* For T the identity monad on **Set**, a T-operad is exactly a monoid (or an operad with only unary functions) (since there is always a unique map to $\{1\}$).

If $\mathcal{E}$ is **Set**, the terminal object 1 will always be $\{1\}$.

For the free monoid monad $TA = \bigsqcup A^n$, the T-operads are precisely the operads that we defined before.

For the monad $TA = 1 + A$, we can view TA as a subset of the free monoid on A, and this gives an operad with 0-ary and 1-ary functions. The 1-ary arrows form a monoid, and the 0-ary arrows are a set, with an action of the monoid.

1.5. **Cartesian Operads.**

1.5.1. *Theory.* Using Towards a doctrine of operads.

NLab uses notation: **Fin** for what we would call a standard skeleton of finite sets (i.e. the category of finite sets $\{0, \dots, n-1\}$ and maps between them). A^B denotes all morphisms/functors $B \to A$. I.e., the class of functors $\mathbf{Fin} \to \mathbf{Set}$ is denoted $\mathbf{Set}^{\mathbf{Fin}}$.

Take $I = \mathbf{Fin}(1, -) : \mathbf{Set}^{\mathbf{Fin}} = \mathbf{Fin} \to \mathbf{Set}$.

Let $[\mathbf{Set}^{\mathbf{Fin}}, \mathbf{Set}^{\mathbf{Fin}}]$ be the category of finite-product-preserving, cocontinuous endofunctors on $\mathbf{Set}^{\mathbf{Fin}}$. The map $Ev_I : [\mathbf{Set}^{\mathbf{Fin}}, \mathbf{Set}^{\mathbf{Fin}}] \to \mathbf{Set}^{\mathbf{Fin}}$, given by $F \mapsto F(I)$ is an equivalence. $[\mathbf{Set}^{\mathbf{Fin}}, \mathbf{Set}^{\mathbf{Fin}}]$ has a monoidal product $\odot$ given by endofunctor composition, and we can transfer this to $\mathbf{Set}^{\mathbf{Fin}}$.

Concretely, we have $F \odot G = \int^{n \in \mathbf{Fin}} F(n) G^n$.

1.5.2. *Cartesian operads.* A **cartesian operad** is a monoid in this monoidal category $(\mathbf{Set}^{\mathbf{Fin}}, \odot, I)$. I.e., it is a triple (M, μ, η), with $M \in \mathbf{Set}^{\mathbf{Fin}}$, $\mu : M \odot M \to M$ and $\eta : I \to M$.

More concretely, this is a functor $M : \mathbf{Fin} \to \mathbf{Set}$, together with maps

$$m_{n,k} : M(n) \times M(k)^n \to M(k),$$

natural in k and dinatural in n, and an element $e \in M(1)$.

Dinaturality in n means the following. Fix $k \in \mathbf{Fin}$. We have the functors $\mathbf{Fin}^{\mathrm{op}} \times \mathbf{Fin} \to \mathbf{Set}$ given by

$$F : (n, n') \mapsto M(n') \times M(k)^n \quad \text{and} \quad G : (n, n') \mapsto M(k).$$

For all $n \in \mathbf{Fin}_0$, we have a morphism

$$\bullet : F(n, n) = M(n) \times M(k)^n \to M(k) = G(n, n).$$

Naturality means that for all $a : n \to n'$,

$$G(n, a) \circ \bullet \circ F(a, n) = F(a, n') \circ \bullet \circ G(n', a).$$

i.e., for all $f \in M(n)$, $g_1, \dots, g_{n'} \in M(k)$,

$$f \bullet (g_{a(1)}, \dots, g_{a(n)}) = M(a)(f) \bullet (g_1, \dots, g_{n'}).$$

Now, if we have, in **Fin**, a decomposition $k = k_1 + \cdots + k_n$, and we have inclusion maps $i_j : k_j \hookrightarrow k$, then we have

$$M(n) \times M(k_1) \times \cdots \times M(k_n) \xrightarrow{1 \times M(i_1) \times \cdots \times M(i_n)} M(n) \times M(k)^n \xrightarrow{m_{n,k}} M(k),$$

which gives an operad structure.

1.5.3. *Clones.* In other parts of mathematics, a cartesian operad is called a **clone**. An abstract clone consists of sets $M(n)$ for all $n \in \mathbb{N}$, for all $n, k \in \mathbb{N}$ a function $\bullet : M(n) \times M(k)^n \to M(k)$ and for each $1 \leq i \leq n \in \mathbb{N}$, an element $\pi_{i,n} \in M(n)$ such that for $f \in M(i)$, $g_1, \ldots, g_i \in M(j)$ and $h_1, \ldots, h_j \in M(k)$,

$$f \bullet (g_1 \bullet (h_1, \ldots, h_j), \ldots, g_i \bullet (h_1 \ldots, h_j)) = (f \bullet (g_1, \ldots, g_i)) \bullet (h_1, \ldots, h_j),$$

for $f_1, \ldots, f_n \in M(k)$,

$$\pi_{i,n} \bullet (f_1, \ldots, f_n) = f_i,$$

and for $f \in M(n)$,

$$f \bullet (\pi_{1,n}, \ldots, \pi_{n,n}) = f.$$

(It is claimed that this automatically gives naturality)

Naturality means for all $a \in \mathbf{Fin}(n, n')$, for all $f \in M(n)$, $g_1, \ldots, g_{n'} \in M(k)$,

$$f \bullet (g_{a(1)}, \ldots, g_{a(n)}) = (f \bullet (\pi_{a(1),n'}, \ldots, \pi_{a(n),n'})) \bullet (g_1, \ldots, g_{n'}).$$

However, by associativity and since $\pi_{i,n} \bullet (f_1, \ldots, f_n)$, we have

$$\begin{aligned} &(f \bullet (\pi_{a(1),n'}, \ldots, \pi_{a(n),n'})) \bullet (g_1, \ldots, g_{n'}) \\ &= (f \bullet (\pi_{a(1),n'} \bullet (g_1, \ldots, g_{n'}), \ldots, \pi_{a(n),n'} \bullet (g_1, \ldots, g_{n'}))) \\ &= f \bullet (g_{a(1)}, \ldots, g_{a(n)}). \end{aligned}$$

Naturality in k is as follows. Fix $n \in \mathbf{Fin}_0$. We have functors $F, G : \mathbf{Fin} \to \mathbf{Set}$, given by

$$F : k \mapsto M(n) \times M(k)^n \quad \text{and} \quad G : k \mapsto M(k).$$

For $a : k \to k'$, we must have

$$G(a) \circ \bullet = \bullet \circ F(a).$$

That is, for all $f \in M(n)$, $g_1, \ldots, g_n \in M(k)$,

$$M(a)(f \bullet (g_1, \ldots, g_n)) = f \bullet (M(a)(g_1), \ldots, M(a)(g_n)).$$

Now, $M(a)$ is given by

$$M(a)(f) = f \bullet (\pi_{a(1),k'}, \ldots, \pi_{a(k),k'}).$$

Therefore, we have

$$\begin{aligned} &M(a)(f \bullet (g_1, \ldots, g_n)) \\ &= (f \bullet (g_1, \ldots, g_n)) \bullet (\pi_{a(1),k'}, \ldots, \pi_{a(k),k'}) \\ &= f \bullet (g_1 \bullet (\pi_{a(1),k'}, \ldots, \pi_{a(k),k'}), \ldots, g_n \bullet (\pi_{a(1),k'}, \ldots, \pi_{a(k),k'})) \\ &= f \bullet (M(a)(g_1), \ldots, M(a)(g_n)). \end{aligned}$$

So the associativity and projection axioms ensure naturality.

References

[Lei03] Tom Leinster. Higher operads, higher categories, 2003.