Find Chebyshev Center of a Polyhedron Using Linear Programming

Tianchen Liu

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Problem

Given a polyhedron. Find the largest Euclidean ball that lies inside a (convex) polyhedron.

Many equivalent ways of framing this problem. *e.g.* Find a point that maximizes the miminum distance (l_2 norm) to a set of (hyper)planes; Find Chebyshev center; etc.

ELI5 (3D Scenario)

Given any (weirdly shaped) cage with flat surfaces, what's the radius of the largest sphere that could fit inside the cage?

Examples in 2D and 3D

A polyhedron is a system of linear inequalities

Any polyhedron can be described in the following way.

$$\mathcal{P} = \{x \in \mathbb{R}^n \mid a_i^T x \leq b_i,\ i = 1, \ldots, m\}$$

In other words, a polyhedron is the set of points that satisfy a system of linear inequalities.

Euclidean ball

Any Euclidean ball can be described as

$$\mathcal{B} = \{x_c + u \mid \|u\|_2 \leq r\}$$

$x_c \in \mathbb{R}^n$ is the center. r is the radius.
Intuitively: the center $+$ a vector whose length shorter than r.

Rephrasing the problem

$$\mathcal{P} = \{x \in \mathbb{R}^n \mid a_i^T x \leq b_i,\ i = 1, \ldots, m\} \text{ (given)}$$
$$\mathcal{B} = \{x_c + u \mid \|u\|_2 \leq r\}$$

Adjust variables x_c and r to maximize r subject to the constraint $\mathcal{B} \subseteq \mathcal{P}$.

Getting closer to a LP! Now the question is how to formulate $\mathcal{B} \subseteq \mathcal{P}$ as *linear* constraints.

One inequality at a time

$\mathcal{B} \subseteq \mathcal{P} \iff$ every point in $\mathcal{B}$ satisfies every inequality required by $\mathcal{P}$.

Math

For the ith inequality: $a_i^T(x_c + u) \leq b_i$ for all $\|u\|_2 \leq r$.

$a_i^T u$ attains its maximum when $\|u\|_2 = r$ and u is in the same direction as a_i, specifically $u = \frac{a_i}{\|a_i\|_2} r$.
Intuitively, we're looking for the vector that points straight to the hyperplane (perpendicular) with length r. If that vector satisfies then all vectors in the ball satisfies.

So the ith inequality is equivalent to

$$a_i^T(x_c + \frac{a_i}{\|a_i\|_2} r) = a_i^T x_c + r\|a_i\|_2 \leq b_i$$

Formulation as a linear program

$$\begin{aligned} \underset{x_c \in \mathbb{R}^n, r \in \mathbb{R}}{\text{maximize}} \quad & r \\ \text{subject to} \quad & a_i^T x_c + r\|a_i\|_2 \leq b_i, \quad i = 1, \dots, m \end{aligned}$$