Extreme Integration

Start with Taylor Series.

If f(z) is regular in D, than all derivitives exist, therfore we can expand f(z) in a taylor series about some point z_0.

$$a = f(z_0)$$
$$a_n = \frac{1}{n!} f^n(z_0) \tag{1}$$

The question is if the series converges? If z_0 is in the center of a disk iwth radius r, such that r is equal to the distance between z_0 and teh closest singularity.

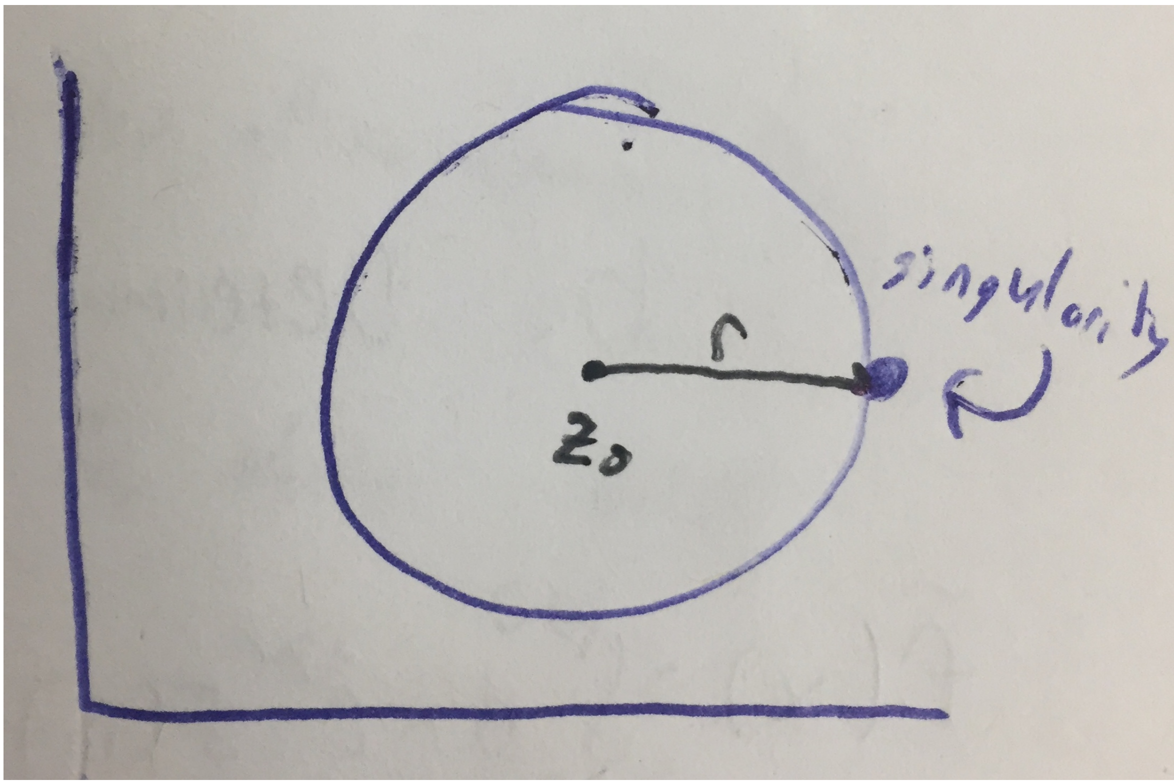

Figure 1: Make a caption. Radius of Convergence defined by r. R is length to closest singulaiity.

As an example consider

$$f(z) = \frac{1}{z} \tag{2}$$

The function exists everywhere, except for the singulairty at z=0. To evaluate we can expand the funciton

$$\frac{1}{z} = \frac{1}{1 - \left(\frac{z_0 - z}{z_0}\right)^n} \tag{3}$$

We can now use a geometrix series The Taylor series is

$$\frac{1}{z_0} \sum_{n=0}^{\infty} \left(\frac{z_0 - z}{z_o}\right)^n \tag{4}$$

To convege then

$$\frac{z - z_0}{z_0} < 1 \tag{5}$$

We know z-z_0 < z_0.

Laurent Series

Expand the series about point z$_0$

$$
\begin{aligned}
f(z) &= \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \\
a_n &= \frac{1}{2\pi i}\oint_\gamma dz \frac{f(z)}{(z-z_0)^{n+1}}
\end{aligned}
\tag{6}
$$

Where teh second line comes from cauchy.

As an example we can again use f(z) = $\frac{1}{z}$ whih is naturally the form for a laurentz series.

$$
\frac{1}{z} = \frac{a_1}{z-0} \tag{7}
$$

Where we define a$_1$ = 1 and z$_0$ = 0

The contour would need to go around z$_0$, but we may not need to include z$_0$ in our domain (i.e. we can make the contour as small as we want around z$_0$ infinately small about the point.

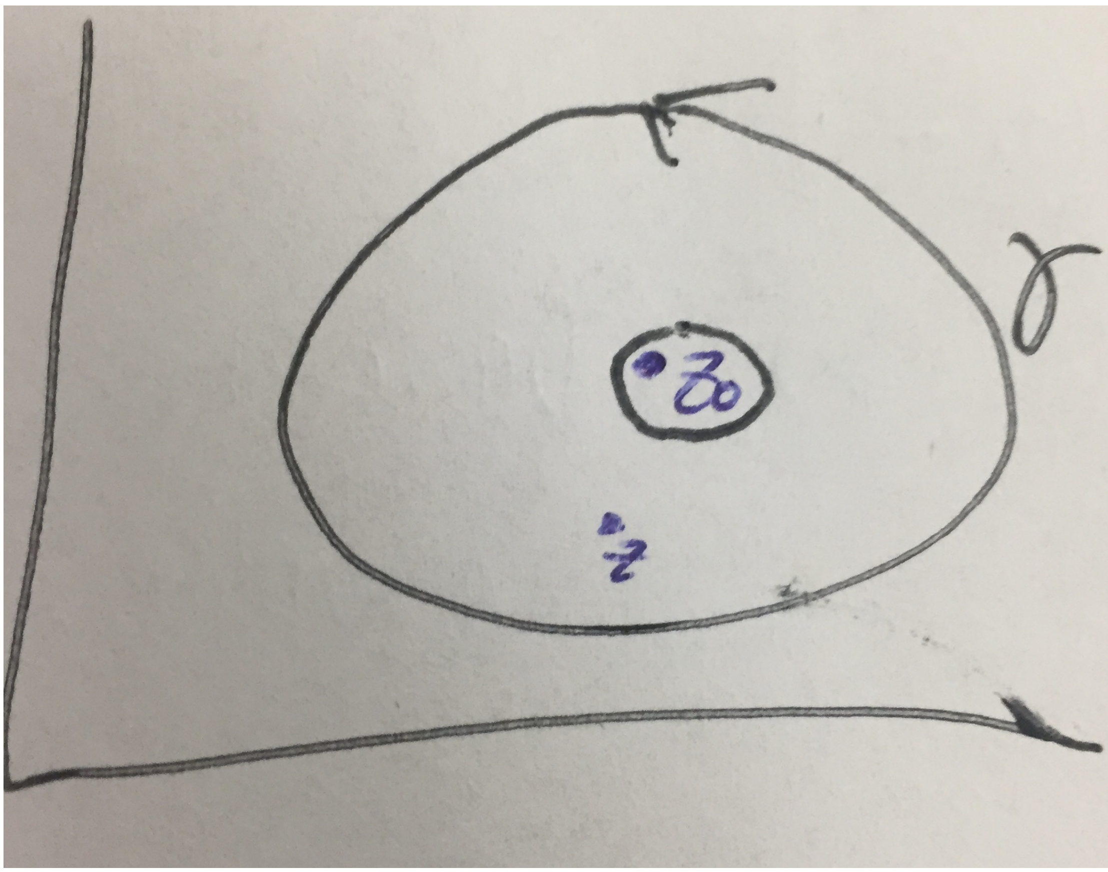

Figure 2: Make a caption. Radius of Convergence defined by r. R is length to closest singulaiity.

We can now represent f(z) at any point z because the function is regular the integral is not a function of path.

Pole

z$_0$ is a simple pole of order M if a$_m \neq 0$, but for any m'>m a$_{m'}$=0.

If m is infinite, than z$_0$ is known as a essential singularity at this point.

There are other types of singularities, ofr example abranching point occurs for f(z) = $\sqrt{z}$. In polar coordinates we could write z = Re$^{i\theta}$.

$$
f(z) = \sqrt{z} = \sqrt{R}e^{i\theta} \tag{8}
$$

We can choose a new contour

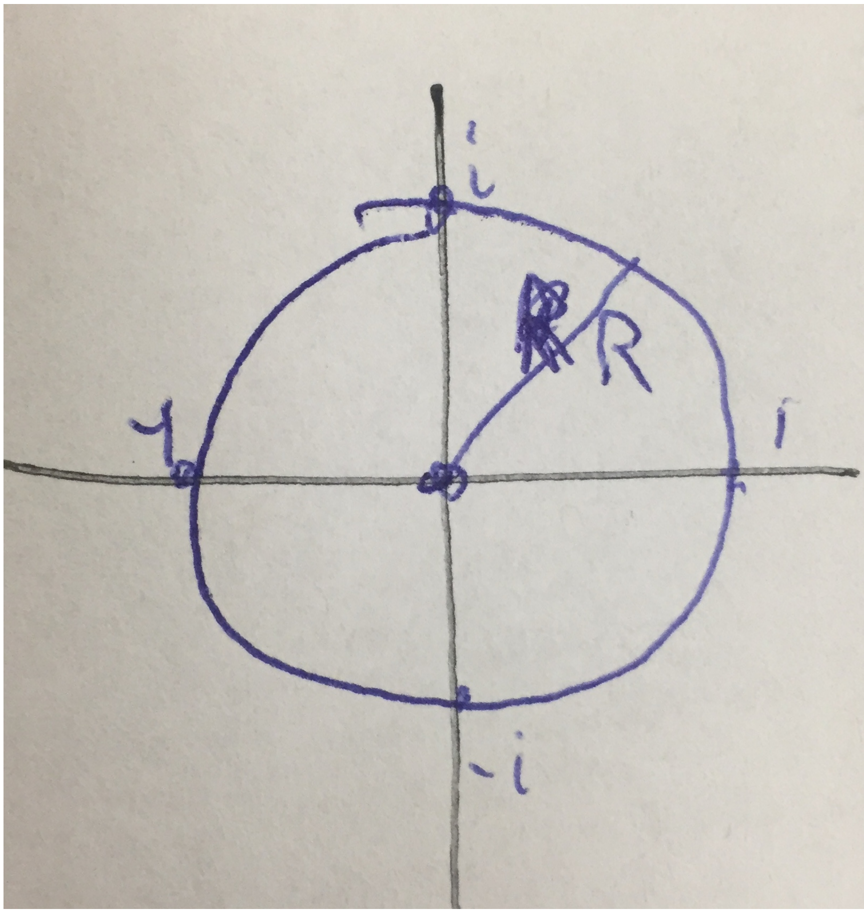

Figure 3: Make a caption. Radius of Convergence defined by r. R is length to closest singulaiity.

$$\begin{aligned} f(1) &= e^{i\theta/2} = 1 \\ f(i) &= e^{i\pi/4} = 1 \\ f(-1) &= e^{i\pi/2} = 1 \\ f(1) &= e^{i2\pi/2} = -1 \end{aligned} \tag{9}$$

The function continuously changes about the branching point, start at 1, end at -1, the function therefore cannot be continuous. If you go around another time, you will get a value of 1, and so on. This means $\sqrt{z}$ is a double-value function.

We can consider instead $\ln(z)$, and again let z=$Re^{i\theta}$, recall n=$\pm$ (1,2,3,..).

$$\begin{aligned} f(z) = \ln(z) &= \ln(Re^{i\theta}) = \ln(Re^{i\theta+2\pi ni}) = \ln(R) + \ln(e^{i\theta+2\pi in}) \\ &= \ln(R) + 2\pi ni + i\theta \end{aligned} \tag{10}$$

Therefore ln is an infinite valued function (because n runs from 0 to infinity), we collect a phase each iteration

$$\begin{aligned} \ln(1) &= 0 \\ \ln(2) &= 2\pi i \end{aligned} \tag{11}$$

Another tyoe of singularity is an esseintail singularity, consider a Taylor expansion.

$$f(z) = e^{1/z} = n\sum_{n=0}^{\infty} \frac{1}{n!}\frac{1}{z^n} \tag{12}$$

All the a_n terms are $\frac{1}{n!}$ so they all exist, z=0 is an essential singularty.

The Residue Theorem

$$\oint dz \quad f(z) = 2\pi i \sum_i \text{res} \tag{13}$$

Consider some function f(z) in a domain D, with isolated residues z_1, z_2, ...

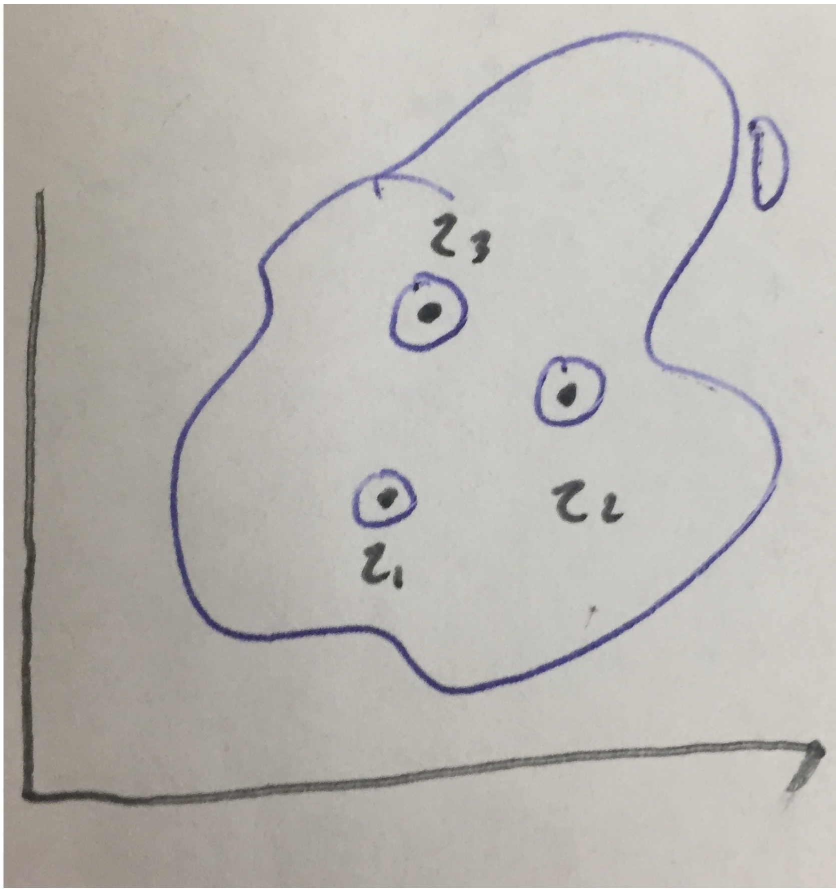

Figure 4: Make a caption.isolated singularities

If you define a contour over a set of isolated isngularities (poles) than the value of the integral is simply the values associated with the singularities through the residue theorem.

$$\text{residue} = \lim_{z \to z_0} (z - z_0) = a_1(z_0) \Rightarrow \lim_{z \to z_0} f(z) \frac{a_1}{(z - z_0} \Rightarrow a_1(z_0) = \text{residue} \tag{14}$$

So you need to cmpute a limit for each singularity to get the residue, and compute the integral through all of the residues.

Isolated Singularities

There own tyoe of singularity (finite power), these methods do not work forbranching points and esenitial singularities, keep that in mind.

As an example consider

$$f(z) = \frac{e^z}{z^5} \Rightarrow \oint dz \quad \frac{e^z}{z^5} \tag{15}$$