

Math 210 Distributing Balls into Boxes

The same combinatorial problem frequently can be phrased in many different ways, and one of the most common ways to phrase combinatorial problems is in terms of distributing balls into boxes. For this reason, it is important to devote some time to becoming familiar with this terminology.

In this section, we want to consider the problem of how to count the number of ways of distributing k balls into n boxes, under various conditions. The conditions that are generally imposed are the following:

- 1) The balls can be either distinguishable or indistinguishable.
- 2) The boxes can be either distinguishable or indistinguishable.
- 3) The distribution can take place either with exclusion or without exclusion.

Let us discuss these terms briefly. The term "distinguishable" refers to the fact that the balls, or-boxes, are marked in some way or have some feature about them that makes each one distinguishable from the others. For example, they may be numbered, each with a different number, they may each be a different color, or they may each be a different size or shape. For the purposes of our discussion, when we speak of k distinguishable balls, we will assume that they are numbered with the consecutive integers 1 through k , and when we speak of n distinguishable boxes, we will assume that they are numbered with the consecutive integers 1 through n .

The term "indistinguishable" refers to the fact that the balls, or boxes, are so identical that there is no way to tell them apart (not even by their location!). In particular then, when placing indistinguishable balls into distinguishable boxes, it makes no difference which balls go into which boxes. In fact, it is not even possible to tell which balls go into which boxes! All that we are able to tell is the number of balls that end up in each box.

In our discussion, we will consider only the case of distinguishable boxes. As you might imagine, the case of indistinguishable boxes turns out to be much more complicated than that of distinguishable boxes and is usually studied in a course in combinatorics.

As to the third condition, the phrase "with exclusion" means that no box can contain more than one ball, and the phrase "without exclusion" means that a box may contain more than one ball. Fortunately, we can use our knowledge of permutations and combinations to help us with the problems of distributing balls into boxes. Therefore, in each case, we will first try to rephrase the problem in terms of permutations and combinations.

Before considering the various cases, we should clear up one possible point of confusion. Namely, the order in which the balls are placed into the boxes is not important. To help keep this in mind, it is a good idea to think of the balls as being placed into the boxes at exactly the same time.

Let us now turn to the various cases. (Incidentally, how many cases are there?)

Case 1

How many ways are there to distribute k distinguishable balls into n distinguishable boxes, with exclusion?

In this case, we consider putting k balls, numbered 1 through k , into n boxes, numbered 1 through n , in such a way that no box receives more than one ball.

Now, we can translate this into the language of permutations and combinations as follows. Putting k distinguishable balls into n boxes, with exclusion, amounts to the same thing as making an ordered selection of k of the n boxes, where the balls do the selecting for us. The ball labeled 1 selects the first box, the ball labeled 2 selects the second box, and so on. In other words, distributing k distinguishable balls into n distinguishable boxes, with exclusion, is the same as forming a permutation of size k , taken from the set of n boxes. This gives us the following theorem.

Theorem 1

Distributing k distinguishable balls into n distinguishable boxes, with exclusion, corresponds to forming a permutation of size k , taken from a set of size n . Therefore, there are $P(n, k) = (n)_k = n(n-1)(n-2)\cdots(n-k+1)$ different ways to distribute k distinguishable balls into n distinguishable boxes, with exclusion.

Case 2

How many ways are there to distribute k distinguishable balls into n distinguishable boxes, without exclusion?

In this case, we consider putting k balls, numbered 1 through k , into n boxes, numbered 1 through n , but this time with no restriction on the number of balls that can go into each box.

Again, instead of thinking in terms of putting k balls into n boxes, we can think in terms of selecting k of the n boxes. As before, the balls do the selecting for us

but this time more than one ball may go into the same box, which means that the same box may be chosen more than once.

Therefore, we are still dealing with ordered selections, or permutations, of the boxes, but now with unrestricted repetitions. In particular, we have the following theorem.

Theorem 2

Distributing k distinguishable balls into n distinguishable boxes, without exclusion, corresponds to forming a permutation of size k , with unrestricted repetitions, taken from a set of size n . Therefore, there are n^k different ways to distribute k distinguishable balls into n distinguishable boxes, without exclusion.

Case 3

How many ways are there to distribute k indistinguishable balls into n distinguishable boxes, with exclusion?

In this case, we have k identical balls, and we wish to place them into n distinguishable boxes in such a way that no box receives more than one ball.

Once such a placement of balls has been made, then, since the balls are identical, all we can say is which boxes have received a ball and which have not. In other words, placing the balls has the same effect as simply choosing k of the n boxes.

Those boxes that receive a ball are the ones that are chosen, and those that do not receive a ball are not chosen.

Hence, in this case we are making unordered selections, that is, forming combinations of size k , taken from the set of n boxes. This gives the following theorem.

Theorem 3

Distributing k indistinguishable balls into n distinguishable boxes, with exclusion corresponds to forming a combination of size k , taken from a set of size n . Therefore,

there are $C(n, k) = \binom{n}{k}$ different ways to distribute k indistinguishable balls into n distinguishable boxes, with exclusion.

Case 4

How many ways are there to distribute indistinguishable balls into n distinguishable boxes, without exclusion?

In this case, we have k identical balls, to be distributed into n distinguishable boxes, but with no restriction on the number of balls that can occupy a given box.

As in the last case, since the balls are indistinguishable, we can only tell how many balls each box has received. This translates into making a choice of k of the n boxes, but with the possibility that a box may be chosen more than once. Thus, placing k balls into n boxes in this case corresponds to forming an unordered selection, or combination, of size k , taken from the set of n boxes, but with unrestricted repetitions. This gives the following theorem.

Theorem 4

Distributing k indistinguishable balls into n distinguishable boxes, without exclusion, corresponds to forming a combination of size k with unrestricted repetitions, taken from a set of size n . Therefore, there are $\binom{n+k-1}{k}$ different ways to k distribute k indistinguishable balls into n distinguishable boxes, without exclusion.

We should discuss another condition that is commonly placed on the distribution of balls into boxes, namely, the condition that no box be empty. The next theorem summarizes the possibilities. We will prove part 2 of this theorem later and leave the other parts for you.

Theorem 5

1) The number of ways to distribute k distinguishable balls into n distinguishable boxes, with exclusion, in such a way that no box is empty, is $n!$ if $k = n$ and 0 if $k \neq n$

2) The number of ways to distribute k distinguishable balls into n distinguishable boxes, without exclusion, in such a way that no box is empty is:

$$\binom{n}{0}(n-0)^k - \binom{n}{1}(n-1)^k + \binom{n}{2}(n-2)^k - \cdots + (-1)^{n-1} \binom{n}{n-1}(1)^k$$

for $k \geq n$. If $k < n$, then, of course, there is no way. (We have written $(n-0)^k$ instead of n in order to preserve the pattern in the formula. Also, the factor $(-1)^{n-1}$ is in the last term in order to make it positive if n is odd and negative if n is even.

3) The number of ways to distribute k indistinguishable balls into n distinguishable boxes, with exclusion, in such a way that no box is empty, is 1 if $k = n$ and 0 if $k \neq n$.

4) The number of ways to distribute k indistinguishable balls into n distinguishable boxes, without exclusion, in such a way that no box is empty, is $\binom{k-1}{n-1}$

Here is a summary of our results:

Theorem #	Balls	Boxes	Excl	No box empty	# ways of putting k balls into n boxes
1	Dist	Dist	with		$(n)_k = n(n-1)(n-2)\cdots(n-k+1)$
5-1	Dist	Dist	with	yes	$n!$ if $k = n$ 0 if $k \neq n$
2	Dist	Dist	without		n^k
5-2	Dist	Dist	without	yes	see formula below (*)
3	Indist	Dist	with		$\binom{n}{k}$
5-3	Indist	Dist	with	yes	1 if $k = n$ 0 if $k \neq n$
5-4	Indist	Dist	without		$\binom{n+k-1}{k}$
5	Indist	Dist	without	yes	$\binom{k-1}{n-1}$

$$(*) \sum_{i=0}^{n-1} (-1)^i \binom{n}{i} (n-i)^k = \binom{n}{0}(n-0)^k - \binom{n}{1}(n-1)^k + \binom{n}{2}(n-2)^k - \cdots + (-1)^{n-1} \binom{n}{n-1}(1)^k$$

Exercises

Unless mentioned to the contrary, a distribution is assumed to be without exclusion.

1. a) How many ways are there to put 5 distinguishable balls into 7 distinguishable boxes, with exclusion?

2. How many ways are there to put 7 distinguishable balls into 5 distinguishable boxes
 - a. with exclusion?
 - b. without exclusion
3. How many ways are there to put 6 distinguishable balls into 9 distinguishable boxes in such a way that no box is empty and
 - a. without exclusion
 - b. with exclusion
4. How many ways are there to distribute n indistinguishable balls into m distinguishable boxes in such a way that each box receives at least one ball?
5. A certain computer room has 12 computers and 18 printers. Each printer must be connected to a computer and each computer must be connected to a printer. In how many ways can the connections be made?
6. In how many ways are there to assign 50 students to
 - a. 25 desks?
 - b. to 25 desks if each desk must be used?
 - c. 25 desks if each desk must be assigned to the same number of students?
7. In how many ways can 5 people divide 4 apples, 3 oranges, 6 bananas and 2 pears?

Ans: 1. $P(7,5) = (n)_k = (7)_5 = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 2520$

2. a. 0 b. $5^7 = 78125$

3. a. 0 b. 0

4. $\binom{n-1}{m-1}$

5. The printers are distinguishable balls and the computers are distinguishable boxes and the distribution is done so that no box is empty. According to Theorem 5 the answer is

$$n = 12$$

$$k = 18$$

$$\sum_{i=0}^{n-1} (-1)^i \binom{n}{i} (n-i)^k = 601783536940185600$$

6. a. $25^{50} = 7888609052210118054117285652827862296732064351090230047702789306640625$

b.

$$n = 25$$

$$k = 50$$

$$\sum_{i=0}^{n-1} (-1)^i \binom{n}{i} (n-i)^k = 11561749082085339049630894478582594023602447070429600727040000000000$$

$$\text{c. } \binom{50}{2, 2, \dots, 2} = \frac{50!}{2^{25}} = 906410610726874412405866627873920465838242816000000000000$$

$$7. \quad \binom{10}{6} \binom{8}{4} \binom{7}{3} \binom{6}{2} = 7717500$$