MATH 476 Portfolio

Andres Rocha

March 30, 2025

1 Problem Set 1

1.1 Notation

1.1.1 Definition 1: Delivery Price or Strike Price

Description of the term: Purchase price agreed by long position.

Symbol(s): K

Formal Definition: The delivery price of the forward contract.

1.1.2 Definition 2: Spot Price

Description of the term: Purchase price agreed by long position at time T.

Symbol(s): S_T

Formal Definition: Spot price of the asset at maturity of the contract.

Exercise 1

Question: Find the payoff from a long position in a forward contract and the payoff from a short position in a forward contract on one unit of an asset.

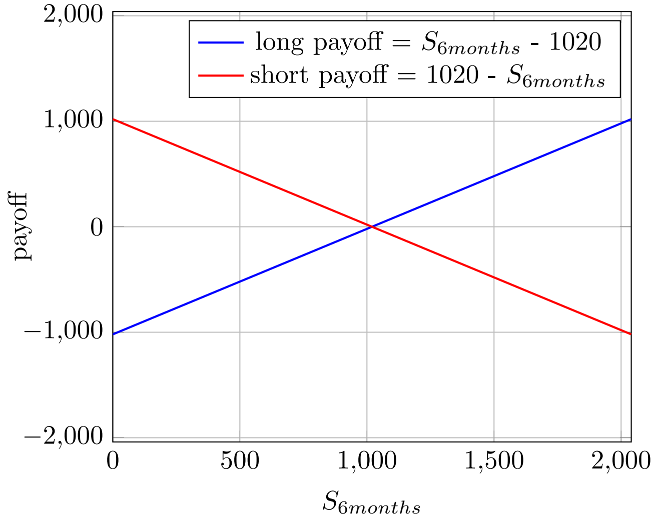
Solution:

- The payoff from a long position in a forward contract on one unit of an asset is $S_T - K$.
- The payoff from a short position in a forward contract on one unit of an asset is $K - S_T$.

Exercise 2

Question: Suppose that the S&R 500 index has a current price of $1000, and the 6-month forward price is $1020. What happens if the index price is $950 in 6 months? $1200 in 6 months? Construct payoff diagrams for the long and short position on this contract. What would be an advantage of using the forward contract to buy the index in 6 months, as opposed to buying it outright at time t = 0?

Solution:



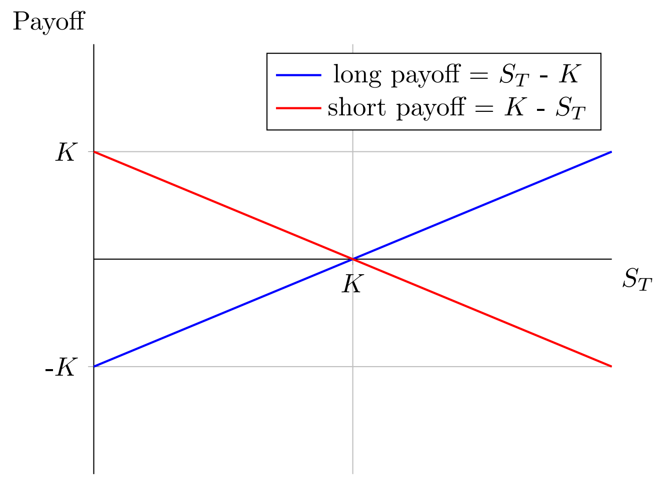
- If S_t = \$950, the payoff for the long position is \$950 - \$1020 = -\$70
- If S_t = \$950, the payoff for the short position is \$1020 - \$950 = \$70
- If S_t = \$1200, the payoff for the long position is \$1200 - \$1020 = \$180
- If S_t = \$1200, the payoff for the short position is \$1020 - \$1200 = -\$180

The advantage of the forward contract is that you are guaranteed a future price to buy the asset at.

Exercise 3

Question: Construct payoff diagrams for the long and short positions in a forward contract with delivery price K and spot price S_T

Solution:



Exercise 4

Question: Forward contracts on foreign exchange are very popular. Most large banks employ both spot and forward foreign-exchange traders. The table below shows quotes for the exchange rate between the British pound (GBP) and the U.S. dollar (USD) in May 2020. The quote is for the number of USD per GBP.

	Bid	Ask
Spot	1.2217	1.2220
1-month forward	1.2218	1.2222
3-month forward	1.2220	1.2225
6-month forward	1.2224	1.2230

Suppose that the treasurer of a US corporation knows it will pay 1 million in GBP in 6 months and wants to hedge against exchange rate changes. Suppose that the bank agrees to a 6-month forward contract to purchase 1 million GBP in 6 months. (i) What happens if the spot exchange rate is 1.3000 in 6 months? (ii) What is the sport exchange rate is 1.2000 in 6 months?

Solution:

(i) If the spot exchange rate is 1.3000 in 6 months, given the asking price of 1.2230, the Treasurer's payoff is:

(Spot Price) - (Delivery Price), the long position

$= (1000000 \text{ GBP} \times 1.30 \text{ USD/GBP}) - (1000000 \text{ GBP} \times 1.2230 \text{ USD/GBP})$

$= 77000$ USD

(ii) If the spot exchange rate is 1.2000 in 6 months, given the asking price of 1.2230, the Treasurer's payoff is:

(Spot Price) - (Delivery Price), the long position

= (1000000 GBP $\times$ 1.20 USD/GBP) - (1000000 GBP - 1.2230 USD/GBP)

= -23000 USD

Exercise 5

Question: An investor enters into a short forward contract to sell 100,000 British Pounds for U.S dollars at an exchange rate of 1.3000 USD per pound. How much does the investor gain or lose if the exchange rate at the end of the contract is (i) 1.2900 and (ii) 1.3200?

Solution:

(i) If the exchange rate at the end of the contract is 1.2900 then the investor's payoff would be:

(Delivery Price) - (Spot Price), the short position

= (100,000 GBP x 1.3 USD/GBP) - (100,000 GBP x 1.29 USD/GBP)

= 1000 USD

(ii) If the exchange rate at the end of the contract is 1.2900 then the investor's payoff would be:

(Delivery Price) - (Spot Price), the short position

= (100,000 GBP x 1.3 USD/GBP) - (100,000 GBP x 1.32 USD/GBP)

= -2000 USD

Exercise 6

Question: A trader enters into a short forward contract on 100 million yen. The forward exchange rate is \$0.0090 per yen. How much does the trader gain or lose if the exchange rate at then end of the contract is (i) \$0.0084 per yen and (ii) \$0.0101.

Solution:

(i) If the exchange rate at the end of the contract is \$0.0084 per yen the investors payoff is:

(Delivery Price) - (Spot Price), the short position

= (100 Million Yen x 0.0090 USD/YEN) - (100 Million Yen x 0.0089 USD/YEN)

= 60,000 USD

(ii) If the exchange rate at the end of the contract is \$0.0101 per yen
the investors payoff is:

(Delivery Price) - (Spot Price), the short position

= (100 Million Yen x 0.0090 USD/YEN) - (100 Million Yen x 0.0101 USD/YEN)

= -110,000 USD

Exercise 7

Question: Consider the following European call option. Let T denote the date exctly 10 days from now. At time T, the holder of the option may purchase one share of XYZ stock for \$250. To gain an understanding of how call options work and what might be reasonable for the price of this option, we will consider two possible situations that might occur on the expiry time T. Let S_T denote the price of one share of XYZ stock at time T. (i) What happens if S_{10days} = \$270 ? (ii) What happens if S_{10days} = \$230 ?

Solution:

(i) If S_{10days} = \$270 then the payoff is,

max[\$0, S_{10days} - K] ; the holder is in the long position

= max[\$0, \$270 - \$250]

= max[\$0, \$20]

= \$20

$\therefore$ the holder uses his right to buy the stock

(ii) If S_{10days} = \$230 then the payoff is,

max[0, S_{10days} - K] ; the holder is in the long position

= max[0, \$230 - \$250]

= max[0, -\$20]

= \$0

$\therefore$ the holder does not use his right to buy the stock

Exercise 8

Question: Suppose that the XYZ share in example 7 only takes the values \$230 or \$270 with equal probability. Find the expected payoff at time T on the call option. This expected value is a useful approximation for what a reasonable amount to pay for the call option would be.

Solution:

$\mathscr{E} = \frac{1}{2} \cdot \$20 + \frac{1}{2} \cdot \$0 = \$10$

Exercise 9

Question: Let c denote the expected payoff that you obtained in Exercise 8. Although option pricing is, in general, more complicated, suppose that the holder of the option did pay c for this option:

(i) What is his net profit or loss if $S_{10days} = \$270$? Express the net profit or loss in this case as a percentage of the initial cost of the option.

(ii) What is his net profit or loss if $S_{10days} = \$230$? Express the net profit or loss in this case as a percentage of the initial cost of the option.

Solution:

(i) If $S_{10days} = \$270$ then the profit is,

max[\$0, S_{10days} - K] - c

= max[\$0, \$270 - \$250] - \$10 ; holder is in the long position

= max[\$0, \$20] - \$10

= \$20 - \$10

= \$10

$\therefore$ since \$10/\$10 = 100%, the gain is 100% of the initial cost

(ii) If $S_{10days} = \$230$ then the profit is,

max[\$0, S_{10days} - K] - c ; holder is in the long position

= max[\$0, \$230 - \$250] - \$10

= max[\$0, -\$20] - \$10

= \$0 - \$10

= -\$10

$\therefore$ since $-\$10/\$10$ = -100%, the loss is 100% of the initial cost

Exercise 10

Question: Suppose that, instead, the investor purchased the share for \$250 instead of purchasing the option. Express his net profit or loss in each case (i.e.

(i) S_{10days} = \$230 or (ii) S_{10days} = \$270) as a percentage of the initial cost of purchasing the share. Compare with the results of the previous exercise:

Solution:
(i) If S_{10days} = \$230 the profit is,

S_{10days} - K ; the investor is taking the long position

= \$230 - \$250

= -\$20
∴ since -\$20/\$250 = -8% the investor loses 8% of the initial cost

(ii) If S_{10days} = \$270 the profit is,

S_{10days} - K ; the investor is taking the long position

= \$270 - \$250

= \$20
∴ since \$20/\$250 = 8% the investor gains 8% of the initial cost

Exercise 11

Question: Consider an investor who but a European put option to sell 100 shares of stock XYZ with a strike price of \$70. Suppose that the current stock price is (i)\$65. (ii) What happens if S_T = \$55?

Solution:
The holder of the EPO has the right to sell the asset at the strike price.

(i) If S_T = \$65 the holder of the EPO will use his right to sell the option for \$70 per share. The payoff is,

(K - S_T) · 100 ; the holder has the long position

= (\$70 - \$65) · 100

= \$500

(ii) If S_T = \$55 the holder of the EPO will use his right to sell the option for \$70 per share. The payoff is,

(K - S_T) · 100 ; the holder has the long position

= (\$70 - \$55) · 100

= \$1500

Exercise 12

Question: It is important to observe that sometimes an investor chooses to exercise an option even though he may make a loss overall. For example, suppose that an investor buys a European call option with a strike price of \$100 per share to buy 100 shares of XYS stock, and that the current stock price is \$98 per share. The price of the option to purchase these 100 shares is \$500. Suppose that the price of the stock is \$102 per share at expiry. Explain why it is preferable for the investor to exercise the option in this case, even though he makes a loss overall.

Solution:
There are two cases:
(i) The investor buys XYS stock at expiry
(ii) The investor does not buy XYS stock at expiry

(i)
$Profit_1 = S_{expiry}$ - K - (initial cost); investor has the long position
= 100 · (\$102 - \$100) - \$500
= -\$300
(ii)
$Profit_2 = S_{expiry}$ - K - (initial cost); investor has the long position
= 100 · (0 - \$100)
= -\$10000
∴ It is preferable to exercise the option to minimize the loss.

Exercise 13

Question: Find each of the following:

(i) The payoff to the holder of a long position in a European call option.

(ii) The payoff to the holder of a short position in a European call option.

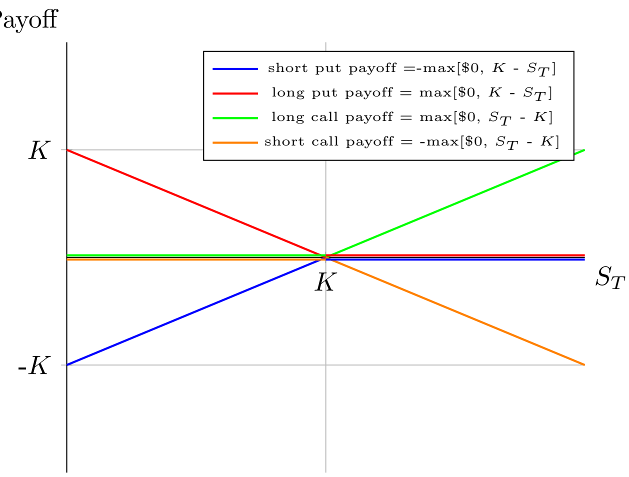
(iii) The payoff to the holder of a long position in a European put option.

(iv) The payoff to the holder of a short position in a European put option
Solution:
(i) Payoff = max[\$0, S_T - K]
(ii) Payoff = -max[\$0, S_T - K]
(iii) Payoff = max[\$0, K - S_T]
(iv) Payoff = -max[\$0, K - S_T]

Exercise 14

Question: Construct payoff diagrams for each of the four positions above (long call, short call, long pout, short put).

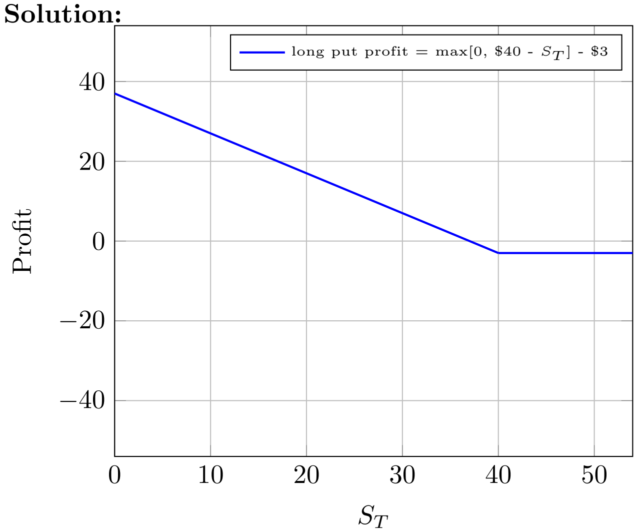
Solution:



Exercise 15

Question: An investor buys a European put on a share for \$3. The stock price is \$42, and the strike price is \$40. Under what circumstances does the investor make a profit? Under what circumstances will the option be exercised? Draw a profit diagram illustrating the variation of the investor's profit (not the payoff) with the stock price at the maturity of the option.

Solution:



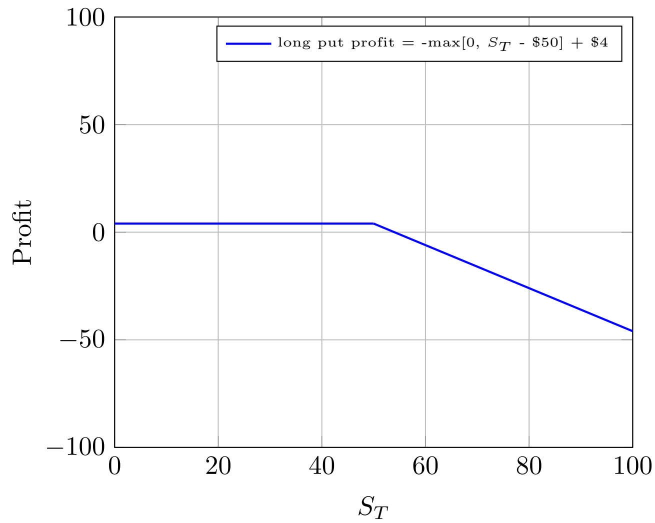
The investor will exercise the option if $S_T < 37$ (x intercept), else the investor will choose

not to exercise the option.

Exercise 16

Question: An investor sells a European call on a share for \$4. The stock price is \$47, and the strike price is \$50. Under what circumstances does the investor make a profit? Under what circumstances will the option be exercised? Draw a diagram showing the variation of the investor's profit with the stock price at the maturity of the option.

Solution:

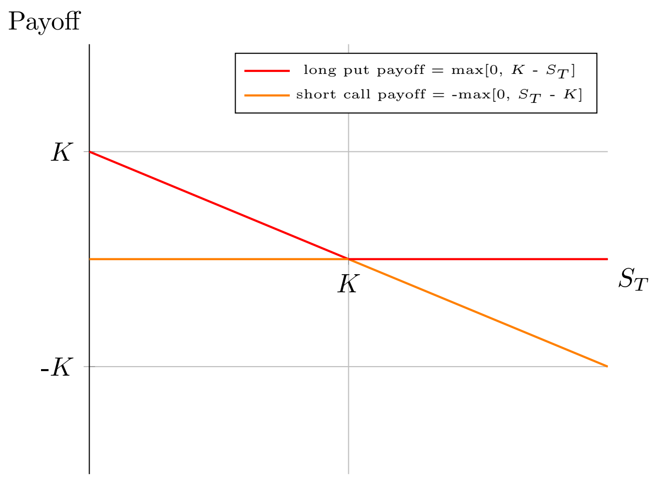


The investor will profit if the $S_T <$ \$54. (x-intercept)

Exercise 17

Question: An investor sells a European call option with strike price K and maturity T , and buys a put with the same strike price and maturity. Describe the investor's position–describe all of the possible situations at maturity, explain which (if any) of the options the investor should exercise at maturity, and find the investor's payoff in each case.

Solution:

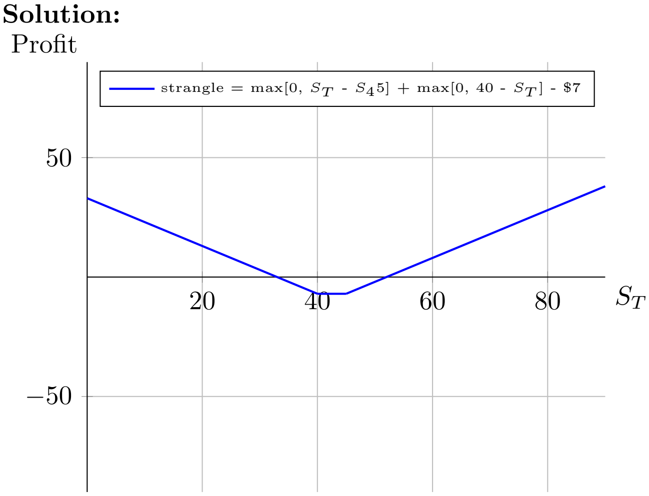


This is the same as if the investor took the short position in a forward contract. In other words K - S_T, the short position in a forward contract.

Exercise 18

Question: A trader buys a call option with a strike price of \$45 and a put option with a strike price of \$40. Both options have the same maturity. The call costs \$3 and the put costs \$4. Draw a diagram showing the variation of the trader's profit with the asset price. Note: this type of trading strategy is known as a strangle.

Solution:



A strangle is a good strategy for volatile assets

Exercise 19

Question: Explain why an American option is always worth at least as much as a European option on the same asset with the same strike price and exercise date.

Solution: Let's suppose that the American option was worth less then the European option in some instance where the strike price and expiry date were the same. If we can show that this makes no sense then it will explain why the American option is always worth at least as much a European option.

- The American option gives the holder the right to buy or sell an asset before and including at the expiry date
- The European option gives the holder the right to buy or sell an asset at the expiry date.

If the American option was worth less then the European option this would not make sense, since your are paying less for more rights on when to buy or sell the asset on top of already being able to buy or sell at the expiry date.

Exercise 20

Question: Complete the following table to summarize the effect on the price of a stock option of increasing one variable while keeping all others fixed. Write a "+" to indicate that an increase in the variable causes the option price to increase, and write a "-" to indicate that an increase in the variable causes the option price to decrease. Write a ? if the relationship is uncertain.

Variable	European Call	European Put	American Call	American Put
Current Price	+	-	+	-
Stock Price	-	+	-	+
Time to expiration	+	+	+	+
Volatility	+	+	+	+
Risk-free interest rate	+	?	+	?

Exercise 21

Question: A trader writes a December put option with a strike price of \$30. The price of the option is \$4. Under what circumstances does the trader make a profit?

Solution:

The profit is,

max[\$0, K - S_T] - \$4 ; the writer has the short position

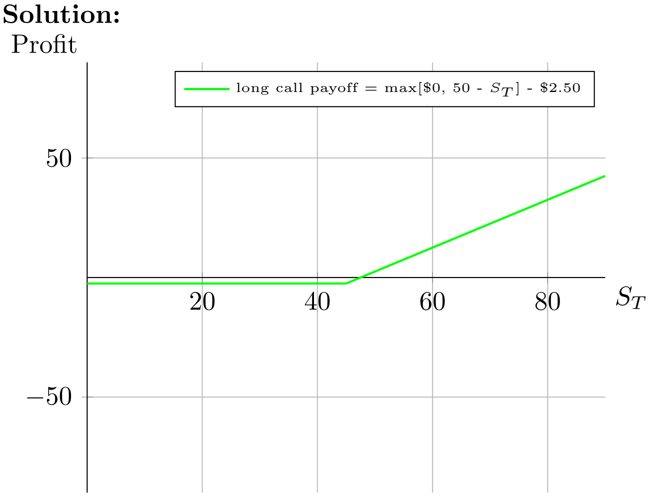
= -max[0, \$30 - S_T] + \$4

$\therefore$ the trader makes profit if $S_T > \$26$

Exercise 22

Question: Suppose that a March call option to buy a share for \$50 costs \$2.50 and is held until March. Under what circumstances will the holder of the option make a profit? Under what circumstances will the option be exercised? Draw a diagram illustrating how the profit from a long position in the option depends on the stock price at maturity of the option.

Solution:



The option will not be exercised if $S_T < \$47.50$. If $S_T > \$47.50$ the option will be exercised. Note that \$47.50 is the x-intercept.

Exercise 23

Question: It is May and a trader writes a September call option with a strike price of \$20. The stock price is \$18 and the option price is \$2. Describe the trader's cash flows if the option is held until September and the stock price is \$25 at that time.

Solution:

If $S_{September}$ = \$25,

The profit is,

-max[\$0, S_T - K] - (initial cost) ; the writer has the short position

= -max[\$0, \$25 - \$20] + \$2

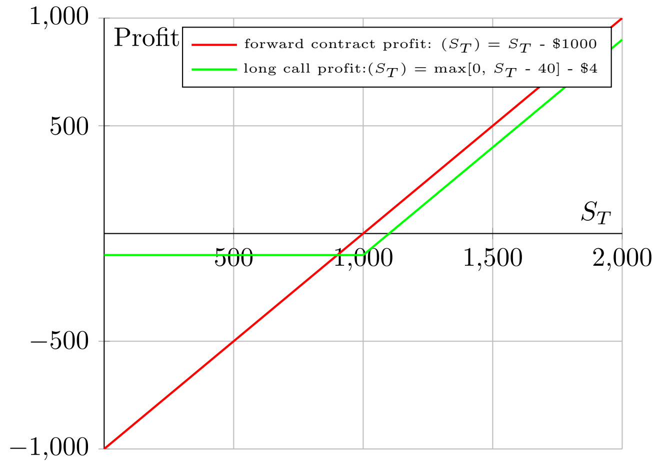
= -\$3

and the payoff is \$5.

Exercise 24

Question: Trader A enters into a forward contract to buy an asset for \$1,000 in one year. Trader B buys a call option to buy the asset for \$1,000 in one year. The cost of the option is \$100. What is the difference between the positions of the traders? Show the profit as a function of the price of the asset in one year for the two traders.

Solution:



The main difference between the positions is that Trader A is taking on more risk to losses compared to Trader B in exchange for the possibility of higher profits.

Exercise 25

Question: A trader is considering two alternatives: buy 100 shares of the stock and buy 100 September call options with a strike price of $320. For each alternative, find each of the following (i) the upfront cost, (ii) the total profit if the stock price in September is $400, and (iii) the total loss if the stock price in September is $300. With the cost of a single option being $21.70.

Solution:

(i)

- If the trader buys 100 shares of stock the upfront cost is;

(100 Share) · 316.50 (USD/Share) = $31,650

- If the trader buys 100 call options the upfront cost is;

(100 Option) · 21.70 (USD/Option) = $2,170

(ii)

- If the trader buys 100 shares and S_T = $400 the profit is;

$100 \cdot (S_{September}$ - $K)$

= 100 · ($400 - $320)

= $8000

- If the trader buys 100 options and S_T = $400 the profit is;

100 · (max[$0, $S_{September}$ - K] - (initial cost))

= 100 · (max[$0, $400 - $320] - $21.70)

= $5830

(iii)

- If the trader buys 100 shares and S_T = $400 the profit is;

$100 \cdot (S_{September}$ - $K)$

= 100 · ($300 - $320)

= -$2000

- If the trader buys 100 options and S_T = $300 the profit is;

100 · (max[0, $S_{September}$ - K] - (initial cost))

$= 100 \cdot$ (max[0, \$300 - \$320] - \$21.70)

= -\$2170

Exercise 26

Question: On May 21, 2020, an investor owns 100 Apple shares. The investor is comparing two alternatives to limit risk. The first involves buying one December put option contract at \$3.30 with a strike price of \$290. The second involves instructing a broker to sell the 100 shares as soon as Apple's price reaches \$290. Discuss the advantages and disadvantages of the two strategies.

Solution:

By choosing the put option contract the most investor can lose is \$3300, since that is what is payed to be in the long position of the option.

By choosing the second alternative the investor is making a the assumption that the asset will reach \$290. This could be more of risk compared to the first alternative, depending on what this assumption was based on.

Therefore, if the goal of the investor is to limit risk then choosing a put option contract would be better compared to the other alternative.

Exercise 27

Question: Suppose that a deposit of \$150 attracts simple annual interest at a rate of 8%. Find the value of the deposit after 20 days. Assume that there are 365 days in one year.

Solution:

The future value of simple annual interest is given by,

P(1 + tr)

$= \$150 \cdot (1 + 20/365 * 8/100)$

= \$150.66

Exercise 28

Question: Find the principal to be deposited initially in an account attracting simple annual interest at a rate of 8% if \$1,000 is needed after three months. Assume that there are 12 months in one year.

Solution:

The principal (P) is given by,

P(1 + tr) = V(t) ; where V(t) is the future value

$= \text{P} \cdot (1 + (31 \cdot 3)/365 \cdot 8/100) = \$1000$; assuming 31 days per month

$\therefore \text{P} = \$980.02$

Exercise 29

Question: Find and compare the future value after two years of a deposit of \$100 attracting interest at an annual interest rate of 10% is compounded (a) annually and (b) monthly.

Solution:

(a) If the deposit compounds annually, the future value is given by,

$\text{V(t)} = \text{P} \cdot (1 + r/m)^{tm}$; with m = 1

$= \$100 \cdot (1 + .10/1)^{2 \cdot 1}$

$= \$121$

(b) If the deposit compounds monthly, the future value is given by,

$\text{V(t)} = \text{P} \cdot (1 + r/m)^{tm}$; with m = 12

$= \$100 \cdot (1 + .10/12)^{2 \cdot 12}$

$= \$122.03$

Exercise 30

Question: Show that if m < k, then $(1 + r/m)^m < (1 + r/k)^k$. Interpret this in the context of future value (i.e. write a sentence using financial terms that describes what this inequality tells us in terms of future value of money).

Solution:

Define $f(\text{x}) = (1 + 1/x)^x$. If we can show $f'(\text{x}) > 0$ then the proof is complete, since this would show that f is increasing and therefore by definition of an increasing function $f(\text{m}) < f(\text{k})$ since m < k.

Assuming x is positive since m and k represent the number of interest payments,

$f'(\text{x}) = \ln(1 + 1/x)(1 + 1/\text{x})^x > 0$, since 1 + 1/x > 1.

The inequality $(1 + r/m)^m < (1 + r/k)^k$ shows that future values increase along with the number of interest payments.

Exercise 31

Question: In the case of continuously compounded interest, interest is added continuously to the principal. If V (t) is the amount in the bank at time t and if r is the constant interest rate, then we obtain the following differential equation for V as a function of t: $dV/dt = r \cdot v$

Let P denote the initial principal invested (i.e. P = V (0)), and solve the differential equation above to find a formula for V as a function of t

Solution: This is a seperable differential equation. Therefore,
$dV/dt =$ rv
$\implies dv/v =$ r dt
$\implies ln|v(t)| =$ rt + c ; by taking the integral of both sides
$\implies v(t) = e^{rt+c} =$ ce^{rt}

Apply initial condition,
$\implies$ v(0) = c = P
$\therefore$ v(t) = Pe^{rt}

Exercise 32

Question: Continuously compounded interest can also be viewed as periodically compounded interest in which we take the limit as m (the number of interest payments made per year) goes to infinity, i.e.

$$V(t) = \lim_{x\to\infty}(1 + r/m)^{tm} \cdot P$$

(a) Show that

$$e = \lim_{x\to\infty}(1 + 1/x)^x$$

(b) Use the result above and V(t) = $\lim_{x\to\infty}(1 + r/m)^{tm} \cdot P$ to obtain a closed form (i.e. without a limit) expression for V (t). You should, of course, obtain the same expression that you obtained by solving the differential equation in Continuous Compounding I.

Solution:
(a) $\lim_{x\to\infty}(1 + 1/x)^x$, let y = $1/x$

$\implies \lim_{y\to0}(1 + y)^{1/y}$, rewriting the original limit

$\implies \lim_{y\to0}(1 + y)^{1/y} = \lim_{y\to0} e^{ln(1+y)^{1/y}} = \lim_{y\to0} e^{1/y\cdot ln(1+y)}$

Now,

$\lim_{y\to0} ln(1 + y)/y =_{lh} \lim_{y\to0}(1 + y)^{-1} = 1$

$\therefore e = \lim_{x\to\infty}(1 + 1/x)^x$

(b)
$\lim_{m\to\infty}(1+r/m)^{tm}$, let y = $1/m$

$\implies \lim_{y\to 0}(1+ry)^{t/y}$, rewriting the original limit

$\implies \lim_{y\to 0}(1+ry)^{t/y} = \lim_{y\to 0} e^{ln(1+ry)^{t/y}} = \lim_{y\to 0} e^{t/y\cdot ln(1+ry)}$

Now taking the limit of the power,

$\lim_{y\to 0} tln(1+ry)/y =_{lh} \lim_{y\to 0} tr(1+ry)^{-1} = rt$

$\therefore Pe^{rt} = \lim_{m\to\infty} P(1+r/m)^{tm}$

Exercise 33

Question: Show that the stock price is an upper bound on the option price: c $\leq S_0$. In other words, the option can never be worth more than the stock. Hint: Argue that if $S_0 <$ c, an arbitrage opportunity exists.

Solution:
Assume c $> S_0$ this would imply the option price is worth more than the stock price. This leads to a contradiction with the no arbitrage principle since the stock could be bought at t=0 and call option could be sold for a guaranteed profit.

Specifically,
If $S_T > K$ (the owner of call option exercises the option),
$\implies$ profit = c + K - S_0 ; which is clearly positive based on assumption

If $S_T \leq$ K (the owner of call option does not exercise the option),
$\implies$ profit = c - S_0 ; which is clearly positive based on assumption
$\therefore$ An arbitrage opportunity exists.

Exercise 34

Question: Show that the put option cannot be worth more than the present value of K today: $p \leq Ke^{-rT}$.

Solution:
Let's suppose for contradiction that $p > Ke^{-rT}$. This leads to a contradiction with the no arbitrage principle, since selling the put option is a guaranteed profit.

Specifically,
If $S_T > K$ (the owner of the put option does not exercises the option),
$\implies$ profit = p - K , which is clearly positive based on assumption

If $S_T \leq K$ (the owner of the put option does not exercises the option),

$\implies$ profit = p , which is clearly positive
$\therefore$ An arbitrage opportunity exists.

Exercise 35

Question: Show that c $\geq S_0 - Ke^{-rt}$
Solution:
Consider two portfolios:

Portfolio A: One European call option and an amount of cash equal to Ke^{-rT}.
Portfolio B: One share of the stock.

Assume that at time T, the stock price is S_T. The payoff of Portfolio B at time T is simply S_T.

If $S_T \geq K$,
- The payoff of portfolio A at time T is, $S_T - K + K$
- The payoff of portfolios B is, S_T

If $S_T < K$,
- The payoff of portfolio A at time T is, K
- The payoff of portfolio B is, S_T

$\therefore$ Since Portfolio A has a higher or equal to payoff in both scenarios, it must have a
higher value at time T than Portfolio B under no-arbitrage conditions, directly implying that c $\geq S_0 - Ke^{-rt}$.

Exercise 36

Question: Show that p $\geq Ke^{-rt} - S_0$
Solution: Consider two portfolios:

Portfolio A: One European put option and one share of stock.
Portfolio B: K dollars in savings account with a risk free interest rate

Assume that at time T, the stock price is S_T. The payoff of Portfolio B at time T is simply S_T.

If $S_T \geq K$,
- The payoff of portfolio A at time T is, $Ke^{-rT} - S_T + S_T$
- The payoff of portfolios B at time T is, Ke^{-rt}

If $S_T < K$,
- The payoff of portfolio A at time T is, Ke^{-rT}
- The payoff of portfolio B at time T is , S_T

$\therefore$ Since Portfolio A has a higher or equal to payoff in both scenarios, it must have a higher value at time T than Portfolio B under no-arbitrage conditions, directly implying that p $\geq Ke^{-rt}$ - S_0.

Exercise 37

Question: What is a lower bound for the price of a 2-month European put option on a non- dividend-paying stock when the stock price is \$58, the strike price is \$65, and the risk- free interest rate is 5% per annum?

Solution:
Using the equation we derived in Exercise 36, p $\geq Ke^{-rt}$ - S_0.
$\implies p \geq 65 \cdot e^{(-0.05/12)\cdot 2}$ - 58
$\therefore p \geq$ \$6.46

Exercise 38

Question: Show that c + Ke^{-rt} = p + S_0
Solution: Consider two portfolios:

Portfolio A: One European call option and an amount of cash equal to Ke^{-rt}
Portfolio B: One European put option plus one share of the stock

If $S_T \geq$ K,
- The payoff of portfolio A at time T is, S_T - K + K
- The payoff of portfolio B at time T is, S_T

If $S_T <$ K,
- The payoff of portfolio A at time T is, K
- The payoff of portfolio B at time T is, K - S_T + S_T

$\therefore$ Since both portfolios are equal are time T, by the no-arbitrage conditions we conclude that c + Ke^{-rt} = p + S_0. We call this *put-call parity.*

Exercise 39

Question: The current price of a stock is S_0 = \$19 and the price of a 3-month European call option on the stock with a strike price of \$20 is \$1. The risk-free annual interest rate is 4%. What is the price of a 3-month European put option on the stock with strike price \$20?

Solution: Using the *put-call parity*, c + Ke^{-rt} = p + S_0,
$\implies$ p = c + Ke^{-rt} - S_0
$\implies$ p = 1 + $20e^{-(.04/12)\cdot 3}$ - \$19
$\therefore$ p = \$1.80

Exercise 40

Question: The prices of European call and put options on a stock with an expiration date in 12 months and a strike price of \$120 are \$20 and \$5, respectively. The current stock price is \$130. What is the implied risk-free interest rate?

Solution: Using the *put-call parity*, c + Ke^{-rt} = p + S_0,
$\implies$ r = -1/t · ln((p + S_0 - c) / k)
$\implies$ r = -ln((5 + 130 - 20)/120)
$\therefore$ p = 0.0425

Exercise 41

Question: Suppose that the price of a stock is \$31, and that the price of a a European call option on the stock with strike price \$30 is \$3, and that the price of a European put option on the stock with the same strike price and expiry is \$2.25. The expiry time is 3 months. The risk-free interest rate is 10% per year. Show that put-call parity does not hold, and construct an arbitrage opportunity. In particular, show that an arbitrager makes a risk-free profit by buying the call option and short-selling both the put and the stock.

Solution: Let's substitute the values into the put-call parity equation:
c + Ke^{-rt} = p + S_0
$\implies 3 + 30 \cdot e^{-(.10/12)\cdot 3)} = 2.25 + 31$
$\implies 32.26 \neq 33.25$
$\therefore$ put-call parity does not hold in this case.

An arbitrager can do the following:

1. Short sell the put option for \$2.25.
2. Short sell the stock for \$31.
3. Buy the call option for \$3.

At the expiration of the options:

- If the stock price is above \$30, the call option will be exercised, and the short put will expire worthless. With the stock bought through the call option the arbitrager will close the short sell of the stock.

  profit = [S_T - S_T] + [-\$30 - \$3 + \$2.25 + \$31] ; which is positive

- If the stock price is below \$30, the put option will be exercised, and the short call will expire worthless. With the stock sold to the arbitrager being in short position of the put option the arbitrager will close the short sell of the stock.

  profit = [S_T - S_T] + [-\$30 - \$3 + \$2.25 + \$31] ; which is positive

$\therefore$ The arbitrager will profit and therefore makes a risk free profit.

Exercise 42

Question: A 1-month European put option on a non-dividend-paying stock is currently selling for \$2.50. The stock price is \$47, the strike price is \$50, and the risk-free interest rate is 6% per annum. What opportunities are there for an arbitrageur?

Solution: Here's how an arbitrageur could exploit this situation:

1. **Borrow \$49.50 at 6%**: The arbitrageur could borrow the stock at \$49.50 at the risk-free interest rate for one month

2. **Buy the Stock**: The arbitrageur could buy the stock at the current market price of \$47

3. **Buy the Put Option**: The arbitrageur can buy the put option for \$2.50.

Now, regardless of the value S_T if the arbitrager always exercises their right to sell then they always profit since,

profit $= \$50 - \$49.50e^{.06/12}$
$\therefore$ profit $= \$0.2525$
$\therefore$ The arbitrageur can earn risk-free profit.

Exercise 43

Question: Explain why the arguments leading to put–call parity for European options cannot be used to give a similar result for American options.

Solution. When early exercise is not possible, we can argue that two portfolios that are worth the same at time T must be worth the same at earlier times. When early exercise is possible, the argument is no longer valid.

Exercise 44

Question: Show that it is never optimal to exercise an American call option prior to expiry.

Solution. There are two main reasons that an American call should never be exercised prior to expiry. One relates to the insurance that it provides. A call option, when held instead of the stock itself, in effect insures the holder against the stock price falling below the strike price. Once the option has been exercised and the strike price has been exchanged for the stock price, this insurance vanishes. The other reason concerns the time value of money. From the perspective of the option holder, the later the strike price is paid out the better. We will show that if C is the value of an American call option, then $C = c$. To prove this formally, note that we have already shown that $C \geq c$. To show that $C \leq c$, from put-call parity, we have:

$$c = p + S_0 - Ke^{-rT} > S_0 - Ke^{-rT} > S_0 - K.$$

Similarly, if c_t is the value (price) of a European call option at time t, then $c_t \leq S_t - K$, by the same reasoning as above. But, note that if an American call is exercised at time t, the payoff is $C_t = S_t - K$. Thus, $c_t \geq C_t$ for all t. We conclude that $c = C$.

Exercise 45

Question: Explain (with an example) why it can be optimal to exercise an American put option prior to expiry.

Solution. It can be optimal to exercise an American put option on a non-dividend-paying stock early. Indeed, at any given time during its life, the put option should always be exercised early if it is sufficiently deep in the money. To illustrate, consider an extreme situation. Suppose that the strike price is \$10 and the stock price is virtually zero. By exercising immediately, an investor makes an immediate gain of \$10. If the investor waits, the gain from exercise might be less than \$10, but it cannot be more than \$10, because negative stock prices are impossible. Furthermore, assuming the interest rate is positive, receiving \$10 now is preferable to receiving \$10 in the future. It follows that the option should be exercised immediately. Like a call option, a put option can be viewed as providing insurance. A put option, when held in conjunction with the stock, insures the holder against the stock price falling below a certain level. However, a put option is different from a call option in that it may be optimal for an investor to forgo this insurance and exercise early in order to realize the strike price immediately. In general, the early exercise of a put option becomes more attractive as S_0 decreases, as r increases, and as the volatility decreases.

Exercise 46

Question: Give an intuitive explanation for why the early exercise of an American put becomes more attractive as the risk-free rate increases and volatility decreases.

Solution. The early exercise of an American put is attractive when the interest earned on the strike price is greater than the insurance element lost. When interest rates increase, the value of the interest earned on the strike price increases making early exercise more attractive. When volatility decreases, the insurance element is less valuable. Again this makes early exercise more attractive.

Exercise 47

Question: Let C denote the value of an American call option to buy one share of a stock, and let P denote the value of an American put option to sell one share of a stock. Show that

$$S_0 - K \leq C - P \leq S_0 - Ke^{-rT}.$$

Solution. First, since $P \leq p$, from put-call parity, we have that

$P \le p = c + Ke^{-rT} - S_0$. Since $c = C$, we have $P \le C + Ke^{-rT} - S_0$. Thus, $C - P \le S_0 - Ke^{-rT}$.

Next, consider the following portfolios:

- Portfolio I: One European call option plus an amount of cash equal to K.
- Portfolio II: One American put option plus one share of the stock.

Both options have the same exercise price and expiration date. At time T, portfolio I is worth $\max(S_T - K, 0) + Ke^{-rT} = \max(S_T, K) - K + Ke^{-rT} = \max(S_T, K) + K(e^{-rT} - 1)$.

For portfolio II, there are two cases to consider: either the put option is exercised early, or the put option is not exercised early. If the put option is not exercised early, portfolio II is worth $\max(K - S_T, 0) + S_T = \max(K, S_T)$ at time T. Thus, in this case, since $\max(K, S_T) < \max(S_T, K) + K(e^{-rT} - 1)$, portfolio I is worth more than portfolio II at time T. For the second case, suppose that the put option in portfolio II is exercised early, say, at time $\tau < T$. This means that portfolio II is worth K at time τ. However, even if the call option were worthless, portfolio I would be worth $Ke^{-r\tau}$ at time $t = \tau$. It follows that portfolio I is worth at least as much as portfolio II in all circumstances. Thus, portfolio I must be worth at least as much as portfolio II at time $t = 0$, so we conclude that $c + K \ge P + S_0$, or $S_0 - K \le c - P = C - P$.

Exercise 48

Question: Different Strike Prices Suppose that $c(K_1)$, $c(K_2)$, and $c(K_3)$ are the prices of European call options with strike prices K_1, K_2, K_3, respectively, where $K_1 < K_2 < K_3$, and that $p(K_1)$, $p(K_2)$, and $p(K_3)$ are the prices of European put options with these strike prices. All options have the same maturity. Prove each of the following inequalities.

(Part i) $c(K_1) \le c(K_2)$

Solution. Proof: by contradiction. Suppose that $c(K_1) < c(K_2)$. An arbitrageur can make a riskless profit by purchasing the call option with strike price K_1 and selling the call option with strike price K_2. This is called a bull spread. At time $t = 0$, the investor has a positive cash flow of $c(K_2) - c(K_1)$. There are three cases to consider.

- If $S_T < K_1$, then neither option is exercised, so the cash flow at time T is 0. In this case, the investor has made a riskless profit of $c(K_2) - c(K_1)$.
- If $K_1 \le S_T \le K_2$, then the call option with strike price K_1 is exercised, so at time T the cash flow is $S_T - K_1$. In this case, the overall profit is $c(K_2) - c(K_1) + S_T - K_1$.
- If $S_T > K_2$, then both call options are exercised, so the cash flow at time T is $S_T - K_1 + K_2 - S_T = K_2 - K_1$. In this case, the overall profit is $c(K_2) - c(K_1) + K_2 - K_1$.

In all possible cases, the investor makes a profit, so we have reached a contradiction.

(Part ii) $p(K_2) \le p(K_1)$

Solution: by contradiction. Suppose that $p(K_2) < p(K_1)$. An arbitrageur can make a riskless profit by purchasing the put with strike price K_2 and selling the call option with strike price K_1. At time $t = 0$, the investor has a positive cash flow of $p(K_1) - p(K_1)$. There are three cases to consider.

In all possible cases, the investor makes a profit, so we have reached a contradiction. (Alternatively, use put-call parity.)

(Part iii) $c(K_1) - c(K_2) \le K_2 - K_1$

Solution. Proof: by contradiction. Suppose that $c(K_1)-c(K_2) > K_2-K_1$. An arbitrageur can make a riskless profit by selling the call option with strike price K_1, buying the call option with strike price K_2, and investing K_2-K_1 in the risk-free interest rate. At time $t = 0$, the initial cash flow is $c(K_1)-c(K_2)-(K_2-K_1)$, which is positive. There are three cases to consider.

- If $S_T < K_1$, then neither option is exercised, so the payoff at time T is $(K_2-K_1)e^{rT}$, and the overall profit is $c(K_1)-c(K_2)-(K_2-K_1)+(K_2-K_1)e^{rT}$.

- If $K_1 \le S_T \le K_2$, then the call option with strike price K_1 is exercised, so at time T the payoff is $K_1 - S_T + (K_2 - K_1)e^{rT}$. In this case, the overall profit is $c(K_1) - c(K_2) - (K_2 - K_1) + (K_2 - K_1)e^{rT} - (S_T - K_1)$. Since $S_T \le K_2$, profit $= c(K_1) - c(K_2) - (K_2 - K_1) + (K_2 - K_1)e^{rT} - (S_T - K_1) \ge c(K_1) - c(K_2) - (K_2 - K_1) + (K_2 - K_1)e^{rT} - (K_2 - K_1) = c(K_1) - c(K_2) + (K_2 - K_1)(e^{rT} - 1)$, so the profit is positive.

- If $S_T > K_2$, then both options are exercised, and the payoff at time T is $e^{rT}(K_2 - K_1) - (K_2 - K_1) = (K_2 - K_1)(e^{rT} - 1)$. Thus, in this case, the overall profit is $c(K_1) - c(K_2) + (K_2 - K_1)(e^{rT} - 1)$, which is positive.

In all possible cases, the investor makes a profit, so we have reached a contradiction.

(Part iv) $p(K_2) - p(K_1) \le K_2 - K_1$

Solution. The proof is similar to those given above. (Alternatively, use put-call parity.)

Exercise 49

Question:

Suppose that $c(K_1)$, $c(K_2)$, and $c(K_3)$ are the prices of European call options with strike prices $K_1,\ K_2,\ K_3$, respectively, where $K_1 < K_2 < K_3$.
All options have the same maturity.

(i) Show that K_2 is a convex combination of K_1 and K_3, i.e. there exists a real number λ such that $0 < \lambda < 1$ and

$$K_2 = \lambda K_1 + (1 - \lambda)K_3.$$

(ii) Show that, with the value of λ found above,

$$c(K_2) \leq \lambda c(K_1) + (1 - \lambda)c(K_2).$$

Solution. When early exercise is not possible, we can argue that two portfolios that are worth the same at time T must be worth the same at earlier times. When early exercise is possible, the argument is no longer valid.
(i) To find λ we do some simple algebra,

$K_2 = \lambda K_1 + K_3 \text{ - } \lambda K_3$

$\implies K_2 \text{ - } K_3 = \lambda(K_1 \text{ - } \lambda K_3)$

$\therefore \lambda = \frac{K_2 - K_3}{K_1 - K_3}$, which is positive

(ii) Using the λ defined above we will show this inequality is true.

We will start with a statement that we know is true,

$$c(K_2) \leq c(K_1)$$

Since the right to buy a stock at a lower price should not be worth more than the right to buy at a higher price ($K_1 < K_2$).

It follows,

$\frac{K_2 - K_3}{K_1 - K_3} \cdot c(K_2) \leq \frac{K_2 - K_3}{K_1 - K_3} \cdot c(K_1)$

$\implies 0 \leq \frac{K_2 - K_3}{K_1 - K_3} c(K_1) - \frac{K_2 - K_3}{K_1 - K_3} c(K_2)$

$\implies c(K_2) \leq \frac{K_2 - K_3}{K_1 - K_3} c(K_1) - \frac{K_2 - K_3}{K_1 - K_3} c(K_2) + c(K_2)$

$\therefore c(K_2) \leq \lambda c(K_1) + (1 - \lambda)c(K_2)$

Exercise 50

Question:

Suppose that $c(K_1)$, $c(K_2)$, and $c(K_3)$ are the prices of European call options with strike prices $K_1,\ K_2,\ K_3$, respectively, where $K_1 < K_2 < K_3$ and $K_3 - K_2 = K_2 - K_1$. All options have the same maturity. Show that

$$c(K_2) \leq \frac{c(K_1) + c(K_3)}{2}.$$

Hint: consider a portfolio that is long on one option with strike price K_1, long one option with strike price K_3, and short two options with strike price K_2.

Solution.

To show that $c(K_2) \leq \frac{c(K_1)+c(K_3)}{2}$, consider a portfolio consisting of:

- Long one option with strike price K_1, denoted as $C(K_1)$.
- Long one option with strike price K_3, denoted as $C(K_3)$.
- Short two options with strike price K_2, denoted as $2C(K_2)$.

We are going to consider four different cases,

Case 1: $S_T < K_1$
In this case the portfolio is worth 0 since none of the short nor long options will be exercised

Case 2: $K_1 \leq S_T \leq K_2$
In this case the portfolio is worth $S_T - K_1$ since the long option with strike price K_1 will be exercised.
Note: $S_T - K_1 \geq 0$

Case 3: $K_2 \leq S_T \leq K_3$
In this case the portfolio is worth $S_T - K_1$ - $2(S_T$ - $K_2)$ since the long option with strike price K_1 will be exercised and the two short options will be exercised.
Note: $S_T - K_1$ - $2(S_T$ - $K_2) = K_3 - S_T \geq 0$

Case 4: $K_3 \leq S_T$
In this case the portfolio is worth $S_T - K_1$ - $2(S_T$ - $K_2)$ + $(S_T$ - $K_3)$, since the long option with strike price K_1 will be exercised and the two short options will be exercised.
Note: $S_T - K_1$ - $2(S_T$ - $K_2)$ + $(S_T$ - $K_3) = 0$

$\implies$ The initial value of this portfolio is $C(K_1) + C(K_3) - 2C(K_2) \geq 0$.

$\implies -2C(K_2) \geq -C(K_1) - C(K_3)$

$\therefore C(K_2) \leq \frac{C(K_1)+C(K_3)}{2}$

Exercise 51

Question:

State and prove the result corresponding to the exercise above for European put options.

Solution.

To show that $c(K_2) \leq \frac{c(K_1)+c(K_3)}{2}$, consider a portfolio consisting of:

- Short one option with strike price K_1, denoted as $C(K_1)$.
- Short one option with strike price K_3, denoted as $C(K_3)$.
- Long two options with strike price K_2, denoted as $2C(K_2)$.

We are going to consider four different cases,

Case 1: $S_T < K_1$
In this case the portfolio is worth $K_1 - S_T$ - $2(K_2$ - $S_T)$ + $(K_3$ - $S_T)$, since all options will be exercised
Note: $K_1 - S_T$ - $2(K_2$ - $S_T)$ + $(K_3$ - $S_T)$ = K_1 - $2K_2$ = 0

Case 2: $K_1 \leq S_T \leq K_2$
In this case the portfolio is worth -$2(K_2$ - $S_T)$ + $(K_3$ - $S_T)$, since all options besides the short with strike price K_1 will be exercised.
Note: -$2(K_2$ - $S_T)$ + $(K_3$ - $S_T)$ = -K_1 + $S_T \geq 0$

Case 3: $K_2 \leq S_T \leq K_3$
In this case the portfolio is worth $(K_3$ - $S_T)$, since the long option with strike price K_3 will be the only option exercised,
Note: $K_3 - S_T \geq 0$

Case 4: $K_3 \leq S_T$
In this case the portfolio is worth 0 since none of the options will be exercised
Note: $S_T - K_1$ - $2(S_T$ - $K_2)$ + $(S_T$ - $K_3)$ = 0

$\implies$ The initial value of this portfolio is $C(K_1) + C(K_3) - 2C(K_2) \geq 0$.

$\implies -2C(K_2) \geq -C(K_1) - C(K_3)$

$\therefore C(K_2) \leq \frac{C(K_1)+C(K_3)}{2}$

Exercise 52

Question: Consider a bull spread. As usual, let S_T denote the price of the stock at expiry of the options. Find the payoff from a bull spread. Is an investor who enters into a bull spread hoping that the stock price will increase or decrease?

Solution. There are three cases to consider,

Case 1: $S_T < K_1 < K_2$
In this case the payoff is 0 since none of the options wil be exercised.

Case 2: $K_1 < S_T < K_2$
In this case the payoff is $(S_T - K_1)$, since the option with strike price K_1 will be exercised.

Case 3: $K_1 < K_2 < S_T$
In this case the payoff is $(S_T - K_1) - (S_T - K_2)$, since both of the options will be exercised.

Overall the investor is hoping for the price of the stock to increase

Exercise 53

Question: An investor buys for \$3 a 3-month European call option with a strike price of \$30 and sells for \$1 a 3-month European put with a strike price of \$35. Find the profit from this bull spread in each of the following cases: (i) S_T = \$25 (ii) S_T = \$34 (iii) S_T = \$40

Solution. There are three cases to consider,
Note: The premium for these options is -\$2

Case (i): We will leverage case 1 of of Exercise 52,
payoff = \$0
$\implies$ profit = \$0 - \$2 = -\$2

Case (ii): We will leverage case 2 of of Exercise 52,
payoff = $(S_T - K_1)$ = (\$34 - \$30)
$\implies$ profit = (\$34 - \$30) - \$2 = \$2

Case (iii): We will leverage case 3 of of Exercise 52,
payoff = $(S_T - K_1) - (S_T - K_2)$ = (\$40 - \$30) - (\$40 - \$35)
$\implies$ profit = (\$40 - \$30) - (\$40 - \$35) - \$2 = \$3

Exercise 54

Question: Consider a bear spread. As usual, let S_T denote the price of the stock at expiry of the options. Find the payoff from a bear spread. Is an investor who enters into a bear spread hoping that the stock price will increase or decrease?

Case 1: $S_T < K_2 < K_1$
In this case the payoff is $(K_1 - S_T)$ - $(K_2 - S_T)$, since both of the options will be exercised.

Case 2: $K_2 < S_T < K_1$
In this case the payoff is $(K_1 - S_T)$, since the option with strike price K_2 will be exercised.

Case 3: $K_2 < K_1 < S_T$
In this case the payoff is \$0, since none of the options will be exercised.

In this case the investor is hoping for the price of the stock to decrease

Exercise 55

Question: An investor buys for \$3 a 3-month European put option with a strike price of \$35 and sells for \$1 a 3-month European put with a strike price of \$30. Find the profit from this bear spread in each of the following cases: (i) $S_T =$ \$25 (ii) $S_T =$ \$34 (iii) $S_T =$ \$40

Solution. There are three cases to consider,
Note: The premium for these options is -\$2

Case (i): We will leverage case 1 of of Exercise 54,
payoff = $(K_1 - S_T)$ - $(K_2 - S_T)$ = (\$35 -\$25) - (\$30 -\$25)
$\implies$ profit = (\$35 -\$25) - (\$30 - \$25)- \$2 = \$3

Case (ii): We will leverage case 2 of of Exercise 54,
payoff = $(K_1 - S_T)$ = (\$35 - \$34)
$\implies$ profit = (\$34 - \$35) - \$2 = -\$1

Case (iii): We will leverage case 3 of of Exercise 54,
payoff = 0
$\implies$ profit = -\$2

Exercise 56

Question: Find the payoff from a straddle. As usual, let K denote the strike price and let S_T denote the price of the stock at expiry.

Solution. There are three cases to consider,
Case 1: $S_T <$ K
$\Longrightarrow$ payoff $= K - S_T$, here we exercise the call option
Case 2: $S_T =$ K
$\Longrightarrow$ payoff $= 0$, here we exercise neither option
Case 2: $S_T >$ K
$\Longrightarrow$ payoff $= S_T - K$, here we exercise the put option

Exercise 57

Question: A call with a strike price of \$60 costs \$6. A put with the same strike price and expiration date costs \$4. Construct a table that shows the profit from a straddle. For what range of stock prices would the straddle lead to a loss.

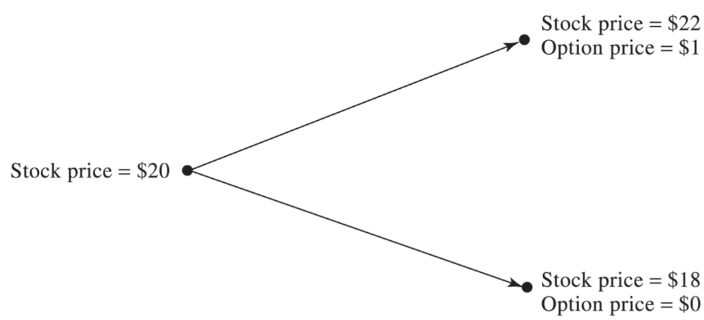
Solution:

	Call Payoff	Put Payoff	Profit
$S_T <$ \$60	0	\$60 - S_T	\$50 - S_T
$S_T =$ \$60	0	0	-\$10
$S_T >$ \$60	S_T - \$60	0	S_T - \$70

From the table we can conclude that in order to make a profit we need S_T either to be less than \$50 or greater than \$70 anything in between these to values would be a loss.

Exercise 58

Question: Show that at time T, the value of the option is either \$1 or \$0, as illustrated in the *one-step binomial tree* below.



Solution:
To determine the value of the European call option at time T, when the stock price can be either \$22 or \$18, we compare the option's strike price (\$21) with the potential stock prices.

1. If the stock price at time T is \$22, then exercising the call option allows the holder to buy the stock for \$21, which is less than the market price. Therefore, the option is worth \$22 - \$21 = \$1.

2. If the stock price at time T is \$18, then exercising the call option would result in buying the stock for \$21, which is more than the market price. In this case, it's not economically beneficial to exercise the option, so its value is \$0.

So, at time T, the value of the option is either \$1 or \$0, depending on whether the stock price is \$22 or \$18 respectively.

Exercise 59

Question: Show that if $S_T = 22$, then the total value of the portfolio at time T is 22Δ - 1

Solution:
In this case the owner of the portfolio will exercise the right to buy the stock since the strike price is lower than the present value. Exercising this at time T the portfolio will now be Δ shares valued at \$22 and minus \$1 as a result of being in one short position of the call option.

Exercise 60

Question: Show that if $S_T = 18$, then the total value of the portfolio at time T is 18Δ

Solution:

In this case none of the options will be exercised so at time T the portfolio will now be Δ shares valued at $18 each.

Exercise 61

Question: Find the value of Δ that makes this portfolio riskless, and conclude that, regardless of whether the stock price moves up or down, the value of the portfolio is always $4.5 at time T

Solution:

Leveraging exercises 60 and 59 we get,

22Δ - 1 = 4.5 and 18Δ = 4.5

$\implies$ 40Δ = 10

$\therefore$ Δ = .25

Exercise 62

Question: Suppose that the risk-free annual interest rate (compounded continuously) is 12% and that T = 3 months. Show that the value of the portfolio today is $4.367.

Solution: Assuming that Δ = .25, the future value is guaranteed to be 4.5.

$\implies$ present value = $4.5e^{-(.12/12)\cdot 3}$

$\therefore$ present value = $4.367.

Exercise 63

Question: Show that, in the absence of arbitrage, the price of the call option today is $0.633.

Solution:

Leveraging exercise 62,

$4.367 = $.25 * 20 - (Price of Call Option)

$\implies$ $4.367 - $.25 * 20 = -(Price of Call Option)

$\implies$ -0.367 = -(Price of Call Option)

$\therefore$ Price of Call Option = $0.633

Exercise 64

Question: Recall that the portfolio is riskless if the portfolio is worth the same amount at time T in both cases (i.e. $S_T = S_0 u$ or $S_T = S_0 d$). Find the value of Δ that makes the portfolio riskless.

Solution:

$\Delta S_0 u - f_u = \Delta S_0 d - f_d$

$\therefore \Delta = \frac{f_u - f_d}{S_0(u-d)}$

Exercise 65

Question: Use the present value of the portfolio and the expression for Δ that you obtained in the previous exercise to show that,

$$f = e^{-rT}\left[pf_u + (1-p)f_d\right],$$

where

$$p = \frac{e^{rT} - d}{u - d}.$$

Solution:
Generalizing Exercise 63 we get the present value of the option (f) to be,

f = Δ S_0 - (present value of portfolio)

$\implies$ f = $(\frac{f_u - f_d}{u-d})$ - $(e^{-rt}\ [\frac{f_u - f_d}{u-d}d - f_d])$

$\implies$ f = $e^{-rt}[\frac{f_u e^{rt}}{u-d}$ - $\frac{f_d e^{rt}}{u-d}$ - $\frac{f_u d}{u-d}$ + $\frac{f_d u}{u-d}$ + $\frac{f_d d}{u-d}$ - $\frac{f_d d}{u-d}]$

$\implies$ f = $e^{-rt}[\frac{e^{rt} - d}{u-d}f_u$ - $\frac{u - e^{rt}}{u-d}f_d]$

Now after setting $p = \frac{e^{rT} - d}{u-d}$ we conclude that,

$\therefore$ f = $e^{-rT}\ [pf_u + (1-p)f_d]$

Exercise 66

Question: Use this expression for f to show that f = \$0.633 for the numerical example considered previously in this section.

Solution:

$$\$0.633 = e^{-\left(\frac{0.12}{12}\cdot 3\right)} \cdot \left(\frac{e^{\left(\frac{0.12}{12}\cdot 3\right)} - \frac{18}{20}}{\frac{22}{20} - \frac{18}{20}} \cdot 1 + \left(1 - \frac{e^{\left(\frac{0.12}{12}\cdot 3\right)} - \frac{18}{20}}{\frac{22}{20} - \frac{18}{20}}\right) \cdot 0 \right)$$

Exercise 67

Question: A stock price is currently \$100. Over each of the next two 6-month periods it is expected to go up by 10% or down by 10%. The risk-free interest rate is 8% per annum with continuous compounding. What is the value of a 1-year European call option with a strike price of \$100

Solution:

$$\$9.61 = e^{-2(.5\cdot .08)} \cdot \left(\left(\frac{e^{(.5\cdot .08)} - \frac{90}{100}}{\frac{110}{100} - \frac{90}{100}} \right)^2 \cdot 21\ +\ \left(1 - \left(\frac{e^{(.5\cdot .08)} - \frac{90}{100}}{\frac{110}{100} - \frac{90}{100}} \right)^2 \right) \cdot 0 \right))$$

Exercise 68

Question: For the situation considered in the previous exercise, what is the value of a 1-year European put option with a strike price of \$100?

Solution:

Using put call parity c + Ke^{-rt} = p + S_0,

$\implies$ 9.61 + $100e^{-.08\cdot 1}$ = p + \$100

$\therefore$ p = 9.61 + $100e^{-.08\cdot 1}$ - \$100 = \$1.92

Exercise 69

Question: A stock price is currently \$25. It is known that at the end of 2 months it will be either \$23 or \$27. The risk-free interest rate is 10% per annum with continuous compounding. Suppose S_T is the stock price at the end of 2 months. What is the value of a derivative that pays S_T^2 at this time?

Solution:

If after two months, the price could either be \$23 (making the value of the option $23^2 = 529$) or \$27 (making the value of the option $27^2 = 729$), consider the portfolio to have Δ shares and one derivative. Then the value of the portfolio in two months is

$$27\Delta - 729 \text{ or } 23\Delta - 529.$$

$$\implies 27\Delta - 729 = 23\Delta - 529 \implies 4\Delta = 200 \implies \Delta = 50.$$

That means the value of portfolio is $27(50) - 729 = 621$.

This implies that the current value of the portfolio with the value of the option (derivative) to be y and assuming $\Delta = \$50$

$$\implies 50 \cdot 25 - y.$$

As it should at least earn the risk-free rate, that means the current value of the portfolio, earning risk-free rate should be the same as the future value of the portfolio = 621.

$$\implies (1250 - y)e^{0.10\cdot\frac{2}{12}} = 621.$$

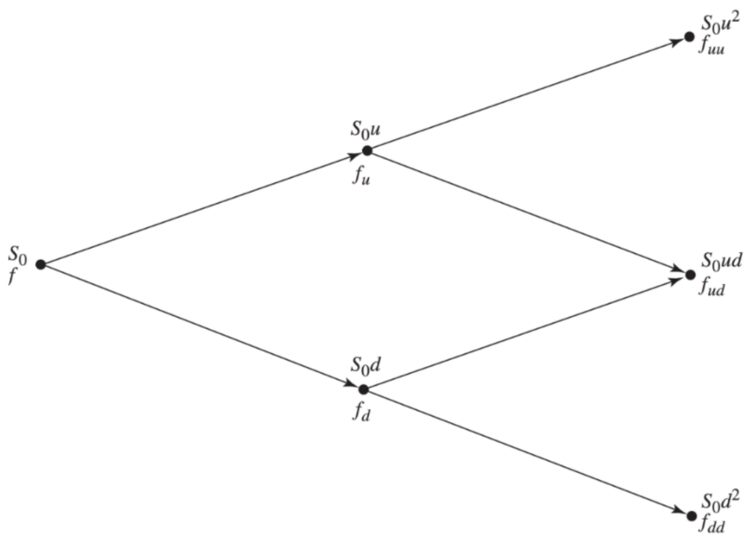
$$\implies 1250 \cdot e^{0.10\cdot\frac{2}{12}} - y \cdot e^{0.10\cdot\frac{2}{12}} = 621.$$

$$\implies y \cdot e^{0.10\cdot\frac{2}{12}} = 1250 \cdot e^{0.10\cdot\frac{2}{12}} - 621.$$

$$\therefore y = 639.3.$$

Exercise 70

Question: Consider a stock whose current price is S_0 and a European call option on the stock whose current price is f. Suppose that there are two time steps, each of length Δt, before expiry, and that during each time step, the stock price either moves up to u times its initial value ($u > 1$) or down to d times its initial value ($d < 1$). This is illustrated in the following figure, using the same notation as in the previous figure for the payoff from the option at each stage. (For example, f_{uu} is the value of the option at time $2\Delta t = T$ if $S_T = u^2 S_0$, i.e. if the stock increases in value at each time step.



Show that f $= e^{-2r\Delta t}[p^2 f_{uu} + 2p(1-p)f_{ud} + (1-p)^2 f_{dd}]$

Solution: Given the option value at time $t = 0$, f, is determined by:

$$f = e^{-r\Delta t}[pf_u + (1-p)f_d]$$

where:

- f_u is the value of the option at time Δt if the stock price increases to uS_0.
- f_d is the value of the option at time Δt if the stock price decreases to dS_0.
- p is the risk-neutral probability of an up movement, and $1 - p$ is the probability of a down movement.
- r is the risk-free interest rate.

We can apply this formula at time Δt again to find f_u and f_d:

$$f_u = e^{-r\Delta t}[pf_{uu} + (1-p)f_{ud}]$$
$$f_d = e^{-r\Delta t}[pf_{ud} + (1-p)f_{dd}]$$

Substituting these expressions back into the formula for f, we get:

$$f = e^{-r\Delta t}[p(e^{-r\Delta t}[pf_{uu} + (1-p)f_{ud}]) + (1-p)(e^{-r\Delta t}[pf_{ud} + (1-p)f_{dd}])]$$

Simplifying this expression, we obtain:

$$f = e^{-2r\Delta t}[p^2 f_{uu} + 2p(1-p)f_{ud} + (1-p)^2 f_{dd}]$$

Which is the desired result.

Exercise 71

Question: A stock price is currently \$50. Over each of the next two 3-month periods it is expected to go up by 6% or down by 5%. The risk-free interest rate is 5% per annum with continuous compounding. What is the value of a 6-month European call option with a strike price of \$51?

Solution:

$\$1.635 = e^{-\left(2\cdot\frac{.05}{12}\cdot 3\right)} \cdot \left(\left(\frac{e^{\left(\frac{.05}{12}\cdot 3\right)} - .95}{1.06 - .95}\right)^2 \cdot 5.18\right)$

Since f_{ud} and f_{dd} both equal to 0.

Exercise 72

Let S_n be the number of successes in n Bernoulli trials with probability p of success on each trial. We want to show that

$$E[S_n] = np.$$

Solution

Consider the random variable S_n, which represents the number of successes in n independent Bernoulli trials. We can write S_n as the sum of n independent Bernoulli random variables X_i, where $X_i = 1$ if the i-th trial is a success, and $X_i = 0$ otherwise. Therefore,

$$S_n = \sum_{i=1}^{n} X_i.$$

To find $E[S_n]$, we use the linearity of expectation:

$$E[S_n] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i].$$

Since each X_i is a Bernoulli random variable with parameter p, the expected value of X_i is:

$$E[X_i] = 1 \cdot p + 0 \cdot (1 - p) = p.$$

Substituting $E[X_i] = p$ into the sum, we get:

$$E[S_n] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} p = np.$$

Therefore, we have shown that

$$E[S_n] = np.$$

Exercise 73

Let X be a numerically-valued discrete random variable with expected value $\mu = E[X]$. We want to show that

$$V(X) = E[X^2] - \mu^2.$$

Solution

The variance of X, denoted by $V(X)$, is defined as:

$$V(X) = E[(X - \mu)^2].$$

We can expand the expression inside the expectation:

$$(X - \mu)^2 = X^2 - 2X\mu + \mu^2.$$

Using the linearity of expectation, we take the expectation of both sides:

$$E[(X - \mu)^2] = E[X^2 - 2X\mu + \mu^2].$$

Since expectation is a linear operator, we can separate the terms:

$$E[X^2 - 2X\mu + \mu^2] = E[X^2] - E[2X\mu] + E[\mu^2].$$

Next, we use the fact that μ is a constant and can be taken out of the expectation:

$$E[2X\mu] = 2\mu E[X].$$

Since $E[X] = \mu$, we have:

$$E[2X\mu] = 2\mu \cdot \mu = 2\mu^2.$$

Also, $E[\mu^2]$ is simply μ^2 because μ^2 is a constant:

$$E[\mu^2] = \mu^2.$$

Combining these results, we get:

$$E[X^2 - 2X\mu + \mu^2] = E[X^2] - 2\mu^2 + \mu^2.$$

Simplifying the right-hand side:

$$E[X^2] - 2\mu^2 + \mu^2 = E[X^2] - \mu^2.$$

Therefore, we have shown that the variance $V(X)$ is:

$$V(X) = E[X^2] - \mu^2.$$

Exercise 74

Let S_n be the number of successes in n Bernoulli trials with probability p of success on each trial, and let $q = 1 - p$. We want to show that

$$V[S_n] = \sigma^2 = npq.$$

Solution

Consider the random variable S_n, which represents the number of successes in n

independent Bernoulli trials. We can write S_n as the sum of n independent Bernoulli random variables X_i, where $X_i = 1$ if the i-th trial is a success, and $X_i = 0$ otherwise. Therefore,

$$S_n = \sum_{i=1}^{n} X_i.$$

To find $V[S_n]$, we use the properties of variance. The variance of the sum of independent random variables is the sum of their variances:

$$V[S_n] = V\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} V(X_i).$$

Since each X_i is a Bernoulli random variable with parameter p, the variance of X_i is given by:

$$V(X_i) = p(1-p) = pq.$$

Substituting $V(X_i) = pq$ into the sum, we get:

$$V[S_n] = \sum_{i=1}^{n} V(X_i) = \sum_{i=1}^{n} pq = npq.$$

Therefore, we have shown that

$$V[S_n] = npq.$$

Exercise 75

Show that for any x,

$$1 - N(x) = N(-x),$$

i.e., the probability that a random draw from the standard normal distribution is above x, $1 - N(x)$, equals the probability that the draw is below $-x$, $N(-x)$.

Solution

Let X be a random variable that follows the standard normal distribution, i.e., $X \sim N(0, 1)$.

The cumulative distribution function (CDF) of the standard normal distribution, $N(x)$, is given by:

$$N(x) = P(X \leq x).$$

We need to show that:

$$1 - N(x) = N(-x).$$

First, recall that the standard normal distribution is symmetric about zero. This implies that for any x:

$$P(X \leq -x) = P(X \geq x).$$

Next, using the definition of the CDF:

$$N(-x) = P(X \leq -x).$$

Because of the symmetry of the standard normal distribution, we have:

$$P(X \leq -x) = P(X \geq x).$$

Also, we know that:

$$P(X \geq x) = 1 - P(X \leq x).$$

Therefore:

$$P(X \leq -x) = 1 - P(X \leq x).$$

Substituting back into the CDF:

$$N(-x) = 1 - P(X \leq x) = 1 - N(x).$$

Thus, we have shown that:

$$1 - N(x) = N(-x).$$

Exercise 76

Show that the price of a European put option on a non-dividend paying stock with strike price K and expiry T is

$$p = Ke^{-rT}N(-d_2) - S_0N(-d_1),$$

where d_1 and d_2 are defined as follows:

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}},$$

$$d_2 = d_1 - \sigma\sqrt{T}.$$

Solution:

The Black-Scholes formula for the price of a European call option is given by:

$$c = S_0 N(d_1) - Ke^{-rT} N(d_2),$$

where $N(x)$ is the cumulative distribution function for a standard normal distribution.

To find the price of the corresponding European put option, we can use the put-call parity relationship for European options, which states:

$$c - p = S_0 - Ke^{-rT}.$$

Solving for p, we get:

$$p = c - S_0 + Ke^{-rT}.$$

Substitute the expression for c:

$$p = \left(S_0 N(d_1) - Ke^{-rT} N(d_2)\right) - S_0 + Ke^{-rT}.$$

Rearranging terms, we obtain:

$$p = S_0 N(d_1) - S_0 - Ke^{-rT} N(d_2) + Ke^{-rT}.$$

Factor out common terms:

$$p = S_0(N(d_1) - 1) + Ke^{-rT}(1 - N(d_2)).$$

Using the properties of the cumulative distribution function,
$N(x) + N(-x) = 1$, we have:

$$N(-d_1) = 1 - N(d_1),$$

$$N(-d_2) = 1 - N(d_2).$$

Thus, the put option price simplifies to:

$$p = S_0 N(-d_1) - Ke^{-rT} N(-d_2).$$

Hence, we have shown that the price of a European put option is:

$$p = Ke^{-rT} N(-d_2) - S_0 N(-d_1).$$

Exercise 77

A non-dividend paying stock has current value $S_0 = \$41$. The volatility is $\sigma = 0.3$ and the risk-free interest rate is r = 8%. Find the price of a European call option on the stock with strike price K = \$40 and expiry T = 3 months.

Solution:

c $= 41 \cdot N\left(d_1\right) - 40e^{-(.08 \cdot 3)} \cdot N\left(d_2\right)$

with,

$d_1 = \frac{\ln\left(\frac{41}{40}\right)+\left(.08+\ \frac{.3^2}{2}\right)3}{.3\cdot 3}$

and,

$d_2 =\ d_1\ -\ .3\cdot\sqrt{3}$

$\therefore c = \$3.399$

Exercise 78

Solution:

p $= 40e^{-(.08\cdot 3)}\cdot N\left(-d_2\right) - 41\cdot N\left(-d_1\right)$

with,

$d_1 = \frac{\ln\left(\frac{41}{40}\right)+\left(.08+\ \frac{.3^2}{2}\right)3}{.3\cdot 3}$

and,

$d_2 =\ d_1\ -\ .3\cdot\sqrt{3}$

$\therefore p = \$1.607$

Exercise 79

Find the binomial approximation for the call option in Exercise 77 with $n = 1, n = 2, n = 10, n = 12$, and $n = 100$. What happens to the approximation as n increases? Where the call option $c = e^{-rT}\sum_{j=0}^{n}\binom{n}{j}p^j(1-p)^{n-j}\max(S_0u^jd^{n-j} - K, 0)$.

Solution:

To find the binomial approximation for the call option, we use the binomial pricing model. We first need to determine the up and down factors, u and d, and the risk-neutral probability p.

Given:

$$S_0 = \$41, \quad \sigma = 0.3, \quad r = 0.08, \quad K = \$40, \quad T = \frac{3}{12} \text{ years}$$

The up and down factors are given by:

$$u = e^{\sigma\sqrt{\frac{T}{n}}}, \quad d = e^{-\sigma\sqrt{\frac{T}{n}}}$$

The risk-neutral probability p is given by:

$$p = \frac{e^{r\frac{T}{n}} - d}{u - d}$$

The value of the call option is then given by:

$$c = e^{-rT} \sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} \max(S_0 u^j d^{n-j} - K, 0)$$

We now compute the binomial approximation for different values of n:
For $n = 1$:

$$u = e^{0.3\sqrt{\frac{3}{12}}}, \quad d = e^{-0.3\sqrt{\frac{3}{12}}}$$

$$p = \frac{e^{0.08\frac{3}{12}} - d}{u - d}$$

$$3.96396 = e^{-0.08\cdot\frac{3}{12}} \left[p \max(S_0 u - K, 0) + (1-p) \max(S_0 d - K, 0)\right]$$

For $n = 2$:

$$3.33095 = e^{0.3\sqrt{\frac{3}{24}}}, \quad d = e^{-0.3\sqrt{\frac{3}{24}}}$$

$$p = \frac{e^{0.08\frac{3}{24}} - d}{u - d}$$

$$3.33095 = e^{-0.08\cdot\frac{3}{12}} \sum_{j=0}^{2} \binom{2}{j} p^j (1-p)^{2-j} \max(S_0 u^j d^{2-j} - K, 0)$$

For $n = 10$:

$$u = e^{0.3\sqrt{\frac{3}{120}}}, \quad d = e^{-0.3\sqrt{\frac{3}{120}}}$$

$$p = \frac{e^{0.08\frac{3}{120}} - d}{u - d}$$

$$3.42679 = e^{-0.08\cdot\frac{3}{12}} \sum_{j=0}^{10} \binom{10}{j} p^j (1-p)^{10-j} \max(S_0 u^j d^{10-j} - K, 0)$$

For $n = 12$:

$$u = e^{0.3\sqrt{\frac{3}{144}}}, \quad d = e^{-0.3\sqrt{\frac{3}{144}}}$$

$$p = \frac{e^{0.08\frac{3}{144}} - d}{u - d}$$

$$3.42677 = e^{-0.08\cdot\frac{3}{12}} \sum_{j=0}^{12} \binom{12}{j} p^j (1-p)^{12-j} \max(S_0 u^j d^{12-j} - K, 0)$$

For $n = 100$:

$$u = e^{0.3\sqrt{\frac{3}{1200}}}, \quad d = e^{-0.3\sqrt{\frac{3}{1200}}}$$

$$p = \frac{e^{0.08\frac{3}{1200}} - d}{u - d}$$

$$3.4003 = e^{-0.08\cdot\frac{3}{12}}\sum_{j=0}^{100}\binom{100}{j}p^j(1-p)^{100-j}\max(S_0u^jd^{100-j}-K,0)$$

As n increases, the binomial approximation converges to the Black-Scholes value.

Exercise 80

A non-dividend paying stock has current value $S_0 = \$120$. The volatility is $\sigma = 0.3$ and the risk-free interest rate is $r = 8\%$. (i) Find the price of a European call option on the stock with strike price $K = \$100$ and expiry $T = 1$ year. (ii) Compute the price of the European call option for several values of large T. What happens to the price of the European call option as $T \to \infty$?

Solution:

(i)
The price of a European call option is given by the Black-Scholes formula:

$$c = S_0N(d_1) - Ke^{-rT}N(d_2)$$

where,

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Given the parameters:

$$S_0 = 120, \quad K = 100, \quad \sigma = 0.3, \quad r = 0.08, \quad T = 1$$

First, compute d_1:

$$d_1 = \frac{\ln\left(\frac{120}{100}\right) + \left(0.08 + \frac{0.3^2}{2}\right)\cdot 1}{0.3\sqrt{1}} = \frac{\ln(1.2) + 0.08 + 0.045}{0.3} = \frac{0.1823 + 0.125}{0.3} = \frac{0.3073}{0.3} \approx 1.0243$$

Then compute d_2:

$$d_2 = d_1 - 0.3\sqrt{1} = 1.0243 - 0.3 = 0.7243$$

Using the cumulative distribution function of the standard normal distribution, $N(d_1)$ and $N(d_2)$:

$$N(d_1) \approx 0.8475, \quad N(d_2) \approx 0.7656$$

Now, compute the call price c:

$$c = 120 \cdot 0.8475 - 100e^{-0.08 \cdot 1} \cdot 0.7656 = 120 \cdot 0.8475 - 100 \cdot 0.9231 \cdot 0.7656 \approx 101.70 - 70.65 \approx 31.05$$

Therefore, the price of the European call option is $\boxed{\$31.05}$.

(ii)
As T increases, the terms in the Black-Scholes formula change.
For very large T, the behavior of d_1 and d_2 is dominated by T.

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T} = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} - \sigma\sqrt{T}$$

For large T:

$$d_1 \approx \frac{(r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} = \frac{(r + \frac{\sigma^2}{2})\sqrt{T}}{\sigma}$$

$$d_2 \approx \frac{(r + \frac{\sigma^2}{2})\sqrt{T}}{\sigma} - \sigma\sqrt{T} = \left(\frac{r + \frac{\sigma^2}{2}}{\sigma} - \sigma\right)\sqrt{T}$$

Since $\frac{r+\frac{\sigma^2}{2}}{\sigma} - \sigma$ is a constant, as $T \to \infty$, d_1 and d_2 increase indefinitely.

Consequently, $N(d_1)$ and $N(d_2)$ approach 1.

Therefore, the price of the call option approximates:

$$c \approx S_0 - Ke^{-rT}$$

As $T \to \infty$, $e^{-rT} \to 0$, so:

$$c \approx S_0$$

Hence, the price of the European call option approaches $S_0 = \$120$ as $T \to \infty$.

Exercise 81

Consider a bull spread in which you buy a 40-strike call and sell a 45-strike call. Suppose $S_0 = \$40$, $\sigma = 0.30$, $r = 0.08$, and $T = 0.5$.

Draw a graph with stock prices ranging from \$20 to \$60 depicting the profit on the bull spread at expiry.

Solution:
To draw the profit graph of a bull spread at expiry, we need to consider the payoff of the long call (strike price $K_1 = 40$) and the short call (strike price $K_2 = 45$).

The payoff of a long call option is:

$$\max(S_T - K_1, 0)$$

The payoff of a short call option is:

$$-\max(S_T - K_2, 0)$$

The payoff of the bull spread is the difference between these two payoffs:

$$\text{Payoff} = \max(S_T - 40, 0) - \max(S_T - 45, 0)$$

To calculate the profit, we subtract the net premium paid for the spread.

To find the premium for both options, we will compute their Black-Scholes values:

$$c_{40} = S_0 N(d_{1,40}) - K_1 e^{-rT} N(d_{2,40})$$

with,

$$d_{1,40} = \frac{\ln\left(\frac{S_0}{K_1}\right) + \left(0.08 + \frac{0.3^2}{2}\right) \cdot 0.5}{0.3\sqrt{0.5}}$$

$$d_{2,40} = d_{1,40} - 0.3\sqrt{0.5}$$

Similarly,

$$c_{45} = S_0 N(d_{1,45}) - K_2 e^{-rT} N(d_{2,45})$$

with,

$$d_{1,45} = \frac{\ln\left(\frac{S_0}{K_2}\right) + \left(0.08 + \frac{0.3^2}{2}\right) \cdot 0.5}{0.3\sqrt{0.5}}$$

$$d_{2,45} = d_{1,45} - 0.3\sqrt{0.5}$$

Calculating $d_{1,40}$ and $d_{2,40}$:

$$d_{1,40} = \frac{\ln\left(\frac{40}{40}\right) + \left(0.08 + \frac{0.3^2}{2}\right) \cdot 0.5}{0.3\sqrt{0.5}} = \frac{0 + (0.08 + 0.045) \cdot 0.5}{0.3\sqrt{0.5}} = \frac{0.0625}{0.2121} \approx 0.2946$$

$$d_{2,40} = d_{1,40} - 0.3\sqrt{0.5} = 0.2946 - 0.2121 \approx 0.0825$$

Calculating $d_{1,45}$ and $d_{2,45}$:

$$d_{1,45} = \frac{\ln\left(\frac{40}{45}\right) + \left(0.08 + \frac{0.3^2}{2}\right) \cdot 0.5}{0.3\sqrt{0.5}} = \frac{\ln\left(\frac{40}{45}\right) + 0.0625}{0.2121} = \frac{-0.1178 + 0.0625}{0.2121} \approx -0.2604$$

$$d_{2,45} = d_{1,45} - 0.3\sqrt{0.5} = -0.2604 - 0.2121 \approx -0.4725$$

Using the cumulative distribution function of the standard normal distribution, $N(d_{1,40}) \approx 0.6162$ and $N(d_{2,40}) \approx 0.5329$, and $N(d_{1,45}) \approx 0.3978$ and $N(d_{2,45}) \approx 0.3181$:

$$c_{40} = 40 \cdot 0.6162 - 40 \cdot e^{-0.04} \cdot 0.5329 \approx 24.65 - 40 \cdot 0.9612 \cdot 0.5329 \approx 24.65 - 20.47 \approx 4.18$$

$$c_{45} = 40 \cdot 0.3978 - 45 \cdot e^{-0.04} \cdot 0.3181 \approx 15.91 - 45 \cdot 0.9612 \cdot 0.3181 \approx 15.91 - 13.73 \approx 2.18$$

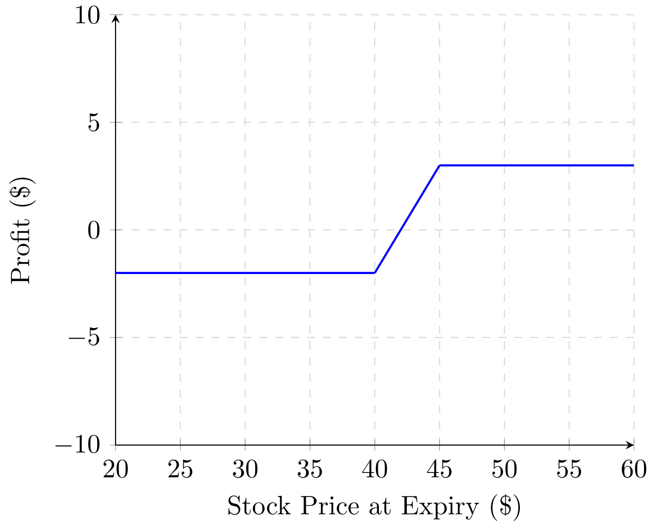
The net premium paid for the spread is:

$$\text{Net Premium} = c_{40} - c_{45} \approx 4.18 - 2.18 = 2.00$$

The profit function is then given by:

$$\text{Profit} = \begin{cases} -2.00 & \text{if } S_T \leq 40 \\ S_T - 40 - 2.00 & \text{if } 40 < S_T \leq 45 \\ 3.00 & \text{if } S_T > 45 \end{cases}$$

We now plot this using a graph.



The graph shows the profit of the bull spread at expiry for stock prices ranging from \$20 to \$60.

Exercise 98

Graph the feasible region for Example 10.1.

Solution:

The constraints are:

$$\begin{aligned} x_1 + 2x_2 &\le 4, \\ 4x_1 + 2x_2 &\le 12, \\ -x_1 + x_2 &\le 1, \\ x_1 &\ge 0, \\ x_2 &\ge 0. \end{aligned}$$

Exercise 99

Graph the lines $z = x_1 + x_2$ along with the feasible region for $z = 0, 1, 2, 5, 15$. Which of these represent possible values of the objective function?

Solution:

The lines for $z = 0, 1, 2$ are within the feasible region. The line for $z = 5, 15$ is outside the feasible region, indicating they are not possible value for the objective function.

Exercise 100

Find all 5 of the corner points for the feasible region of Example 10.1, and substitute each into the objective function. Verify that $\frac{10}{3}$ is the maximum value of the objective function and that it occurs at $(x_1, x_2) = \left(\frac{8}{3}, \frac{2}{3}\right)$.

Corner points:

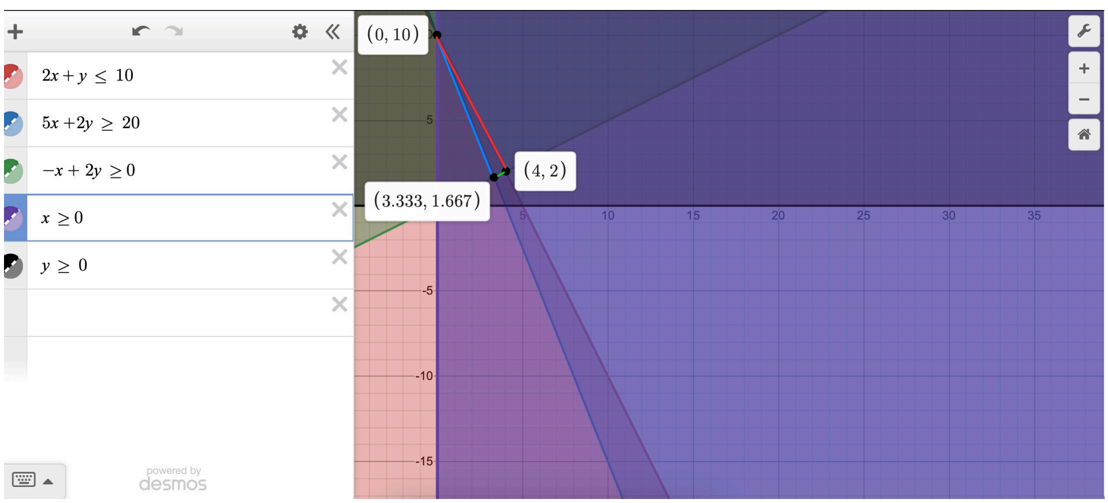
$$\begin{aligned}(0,0) &\Rightarrow z=0,\\ (0,2) &\Rightarrow z=2,\\ (2,1) &\Rightarrow z=3,\\ \left(\frac{8}{3},\frac{2}{3}\right) &\Rightarrow z=\frac{10}{3},\\ (3,0) &\Rightarrow z=3.\end{aligned}$$

The maximum value is $\frac{10}{3}$, occurring at $\left(\frac{8}{3},\frac{2}{3}\right)$.

Exercise 101

Consider the following linear programming problem: Find x_1 and x_2 to minimize x_1+3x_2 subject to the constraints:

$$\begin{aligned}2x_1+x_2 &\leq 10,\\ 5x_1+2x_2 &\geq 20,\\ -x_1+2x_2 &\geq 0,\\ x_1,x_2 &\geq 0.\end{aligned}$$



Solution The optimal solution is $x_1=\frac{10}{3}$, $x_2=\frac{5}{3}$ with minimum value $x_1+3x_2=\frac{10}{3}+\frac{15}{3}=\frac{25}{3}$.

Exercise 102

Determine the range of $\frac{c_1}{c_2}$ for which $(x_1, x_2) = (4, 3)$ is an optimal solution of the problem:

$$\begin{aligned} \text{maximize} \quad & c_1x_1 + c_2x_2 \\ \text{subject to} \quad & 2x_1 + x_2 \leq 11, \\ & -x_1 + 2x_2 \leq 2, \\ & x_1, x_2 \geq 0. \end{aligned}$$

Solution
First we calculate the slopes of the constraints:

$$\text{Slope of } 2x_1 + x_2 = 11 \implies 2,$$
$$\text{Slope of } -x_1 + 2x_2 = 2 \implies -\frac{1}{2}.$$

For $(4, 3)$ to be an optimal solution, the objective function $c_1x_1 + c_2x_2$ must be tangent to one of these constraints at the optimal point. This means the slope of the objective function $\frac{c_1}{c_2}$ must lie between the slopes of the two binding constraints.

Therefore, the range of $\frac{c_1}{c_2}$ for which $(4, 3)$ is an optimal solution is:

$$-\frac{1}{2} \leq \frac{c_1}{c_2} \leq 2.$$

Exercise 103

Consider the following linear programming problem, where the value of c_1 has not yet been determined. Find x_1 and x_2 to:

$$\text{maximize} \quad c_1x_1 + 2x_2$$

subject to the constraints:

$$\begin{cases} 4x_1 + x_2 \leq 12 \\ x_1 - x_2 \geq 2 \\ x_1, x_2 \geq 0 \end{cases}$$

Determine the optimal solutions for the various possible values of c_1 (both positive and negative).

Solution
To solve this linear programming problem, we will consider the constraints and find the feasible region. We will then determine the optimal solution for different values of c_1.

First, we plot the constraints:

1. $4x_1 + x_2 = 12$ 2. $x_1 - x_2 = 2$
The intersection points of these lines and the feasible region will help us determine the optimal solution.

Solving for the intersection of $4x_1 + x_2 = 12$ and $x_1 - x_2 = 2$:

1. From $x_1 - x_2 = 2$, we get $x_2 = x_1 - 2$.
2. Substitute $x_2 = x_1 - 2$ into $4x_1 + x_2 = 12$:

$$4x_1 + (x_1 - 2) = 12 \implies 5x_1 - 2 = 12 \implies 5x_1 = 14 \implies x_1 = \frac{14}{5}$$

$$x_2 = \frac{14}{5} - 2 = \frac{14}{5} - \frac{10}{5} = \frac{4}{5}$$

So, the intersection point is $\left(\frac{14}{5}, \frac{4}{5}\right)$.

The vertices of the feasible region are determined by the intersection of the constraints with the axes and each other:

1. Intersection of $4x_1 + x_2 = 12$ with $x_2 = 0$:

$$4x_1 = 12 \implies x_1 = 3 \quad \text{(Point: } (3, 0))$$

2. Intersection of $x_1 - x_2 = 2$ with $x_1 = 2$:

$$2 - x_2 = 2 \implies x_2 = 0 \quad \text{(Point: } (2, 0))$$

The feasible region vertices are:

$$(2, 0), \left(\frac{14}{5}, \frac{4}{5}\right), (3, 0)$$

To find the optimal solution, evaluate the objective function at these vertices for various values of c_1.

For $c_1 > 0$:

$$\begin{aligned}
&\text{At } (2, 0): && 2c_1 \\
&\text{At } \left(\frac{14}{5}, \frac{4}{5}\right): && c_1 \cdot \frac{14}{5} + 2 \cdot \frac{4}{5} = \frac{14c_1 + 8}{5} = 2.8c_1 + 1.6 \\
&\text{At } (3, 0): && 3c_1
\end{aligned}$$

For $c_1 < 0$:

$$\begin{aligned}
&\text{At } (2, 0): && 2c_1 \\
&\text{At } \left(\frac{14}{5}, \frac{4}{5}\right): && \frac{14c_1 + 8}{5} = 2.8c_1 + 1.6 \\
&\text{At } (3, 0): && 3c_1
\end{aligned}$$

Thus, the optimal solutions are:

- For $c_1 > 0$, the optimal point is $(3, 0)$ if $c_1 > 8$, else the optimal point is $(\frac{14}{5}, \frac{4}{5})$

- For $c_1 < 0$, the optimal point is $(2, 0)$ since $2c_1$ is greater than $3c_1$ and $2.8c_1 + 1.6$.

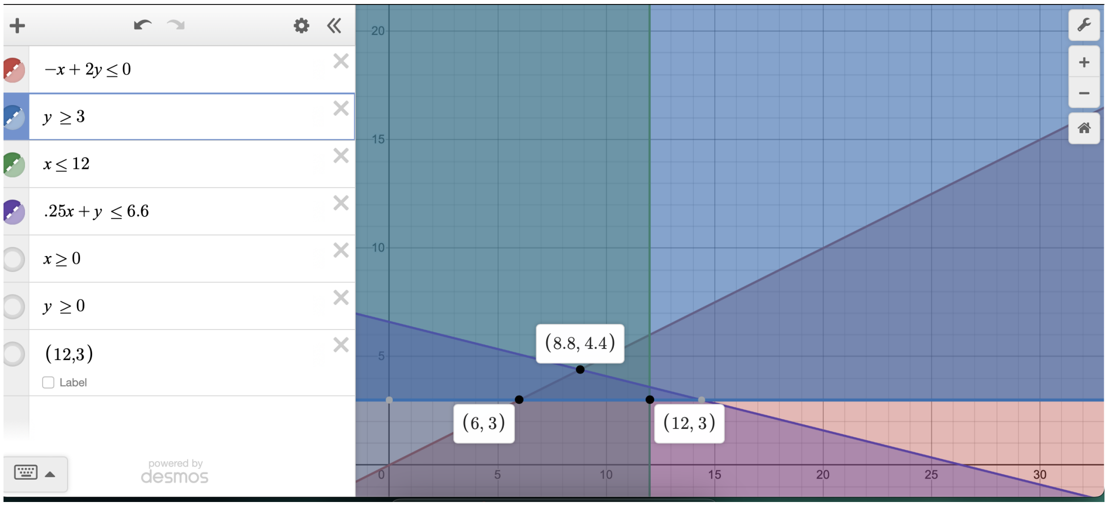
Exercise 104

A manufacturing process requires oil refineries to produce at least 2 gallons of gasoline for every gallon of fuel oil. To meet winter demand, at least 3 million gallons of fuel oil must be produced. The demand for gasoline is no more than 12 million gallons per day. It takes 0.25 hours to ship each million gallons of gasoline and 1 hour to ship each million gallons of fuel oil. No more than 6.6 hours are available for shipping. If the refinery sells gasoline for \$1.25 per gallon and fuel oil for \$1 per gallon, how much of each should be produced to maximize revenue?

Solution
Let x_1 be the million gallons of gasoline and x_2 be the million gallons of fuel oil.
Maximize $1.25x_1 + x_2$

$$\begin{aligned} 2x_2 &\leq x_1, \\ x_2 &\geq 3, \\ x_1 &\leq 12, \\ 0.25x_1 + x_2 &\leq 6.6, \\ x_1, x_2 &\geq 0. \end{aligned}$$

Solution:



$$x_1 = 12, \quad x_2 = 3 \implies \text{Maximum revenue} = 1.25(12) + 3 = 18$$

Exercise 105

Transform the following LP to a standard maximum problem:

$$\begin{aligned}
\text{minimize} \quad & x_1 - 12x_2 - 2x_3 \\
\text{subject to:} \quad & \\
5x_1 - x_2 - 2x_3 &= 10, \\
2x_1 + x_2 - 20x_3 &\geq -30, \\
x_2 &\leq 0, \\
1 \leq x_3 &\leq 4.
\end{aligned}$$

Solution
To convert this to a standard maximum problem:

1. Change the objective function to maximize by multiplying by -1:

$$\text{maximize} \; -x_1 + 12x_2 + 2x_3$$

2. Convert the inequality $2x_1 + x_2 - 20x_3 \geq -30$ to standard form:

$$-2x_1 - x_2 + 20x_3 \leq 30$$

3. Convert the upper and lower bounds to standard form using new variables:

$$w_2 = -x_2, \quad x_2 = -w_2 \Rightarrow w_2 \geq 0$$

$$x_3 = 1 + w_3, \quad w_3 \geq 0$$

$$x_3 = 4 - w_4, \quad w_4 \geq 0$$

The transformed problem is:

$$\text{maximize } -x_1 - 12w_2 + 2 + 2w_3$$

subject to:

$$\begin{aligned} 5x_1 + w_2 - 2(1 + w_3) &= 10, \\ -2x_1 - w_2 + 20(1 + w_3) &\leq 30, \\ w_2 &\geq 0, \\ w_3 &\geq 0, \\ w_4 &\geq 0. \end{aligned}$$

In matrix form:

$$\text{maximize } -x_1 - 12w_2 + 2 + 2w_3$$

subject to:

$$\begin{bmatrix} 5 & 1 & -2 & 0 & 0 \\ -2 & -1 & 20 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ w_2 \\ w_3 \end{bmatrix} \leq \begin{bmatrix} 12 \\ 10 \end{bmatrix}$$

Exercises 106-110

Given the following linear system and objective function, find the optimal solution.

$$\max\{2x_1 + 3x_2 + x_3\}$$

$$\begin{cases} x_1 + x_2 + 4x_3 \leq 100 \\ x_1 + 2x_2 + x_3 \leq 150 \\ 3x_1 + 2x_2 + x_3 \leq 320 \end{cases}$$

Solution

Add slack variables to turn all inequalities to equalities (Exercise 106).

$$\begin{cases} x_1 + x_2 + 4x_3 + s_1 = 100 \\ x_1 + 2x_2 + x_3 + s_2 = 150 \\ 3x_1 + 2x_2 + x_3 + s_3 = 320 \end{cases}$$

Create the initial tableau of the new linear system.

$$\begin{array}{c} \left[\begin{array}{cccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & b \\ \hline 1 & 1 & 4 & 1 & 0 & 0 & 100 \\ 1 & 2 & 1 & 0 & 1 & 0 & 150 \\ 3 & 2 & 1 & 0 & 0 & 1 & 320 \\ \hline -2 & -3 & -1 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \\ \\ \end{array} \end{array}$$

There are negative elements in the bottom row, so the current solution is not optimal. Thus, pivot to improve the current solution. The entering variable is x_2 and the departing variable is s_2.

Perform elementary row operations until the pivot element is 1 and all other elements in the entering column are 0 (Exercise 107/108).

$$\begin{array}{c} \left[\begin{array}{cccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & b \\ \hline 1/2 & 0 & 7/2 & 1 & -1/2 & 0 & 25 \\ 1/2 & 1 & 1/2 & 0 & 1/2 & 0 & 75 \\ 2 & 0 & 0 & 0 & -1 & 1 & 170 \\ \hline -1/2 & 0 & 1/2 & 0 & 3/2 & 0 & 225 \end{array}\right] \begin{array}{c} \\ s_1 \\ x_2 \\ s_3 \\ \\ \end{array} \end{array}$$

There are negative elements in the bottom row, so the current solution is not optimal. Thus, pivot to improve the current solution. The entering variable is x_1 and the departing variable is s_1.

Perform elementary row operations until the pivot element is 1 and all other elements in the entering column are 0 (Exercise 109).

$$\begin{array}{c}
\left[\begin{array}{cccccc|c}
x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & b \\
\hline
1 & 0 & 7 & 2 & -1 & 0 & 50 \\
0 & 1 & -3 & -1 & 1 & 0 & 50 \\
0 & 0 & -14 & -4 & 1 & 1 & 70 \\
\hline
0 & 0 & 4 & 1 & 1 & 0 & 250
\end{array}\right]
\end{array}
\begin{array}{c} \\ x_1 \\ x_2 \\ s_3 \\ \\ \end{array}$$

There are no negative elements in the bottom row, so we know the solution is optimal. Thus, the solution is (Exercise 110):

$$s_1 = 0,\ s_2 = 0,\ s_3 = 70,\ x_1 = 50,\ x_2 = 50,\ x_3 = 0,\ z = 250$$

Exercise 111

An office manager needs to purchase new filing cabinets. She knows that Ace cabinets cost \$40 each, require 6 square feet of floor space, and hold 8 cubic feet of files. On the other hand, each Excello cabinet costs \$80, requires 8 square feet of floor space, and holds 12 cubic feet. The budget permits her to spend no more than \$560, while the office has room for no more than 72 square feet of cabinets. The manager desires the greatest storage capacity within the limitations imposed by funds and space. How many of each type of cabinet should she buy?

Solution:

Let x_1 = number of Ace Cabinets and let x_2 = number of Excello Cabinets.

We can express our problem mathematically as,

$$\begin{aligned}
\text{maximize} \quad & z = 8x_1 + 12x_2 \\
\text{subject to:} \quad & \\
40x_1 + 80x_2 &\le 560, \\
6x_1 + 8x_2 &\le 72, \\
x_1, x_2 &\ge 0.
\end{aligned}$$

Now, we add slack variables to turn all inequalities to equalities.

$$\begin{cases} 40x_1 + 80x_2 + s_1 = 560 \\ 6x_1 + 8x_2 + s_2 = 72 \end{cases}$$

Create the initial tableau of the new linear system.

$$\left[\begin{array}{cccc|c}
x_1 & x_2 & s_1 & s_2 & b \\
\hline
40 & 80 & 1 & 0 & 560 \\
6 & 8 & 0 & 1 & 72 \\
\hline
-8 & -12 & 0 & 0 & 0
\end{array}\right]
\begin{array}{c} \\ s_1 \\ s_2 \\ \\ \end{array}$$

There are negative elements in the bottom row, so the current solution is not optimal. Thus, pivot to improve the current solution. The entering variable is x_2 and the departing variable is s_1.

Perform elementary row operations until the pivot element is 1 and all other elements in the entering column are 0.

$$\begin{array}{c} \left[\begin{array}{cccc|c} x_1 & x_2 & s_1 & s_2 & b \\ \hline 1/2 & 1 & 1/80 & 0 & 7 \\ 2 & 0 & -1/10 & 1 & 16 \\ \hline -2 & 0 & 3/20 & 0 & 84 \end{array}\right] \end{array} \begin{array}{c} \\ x_2 \\ s_2 \\ \\ \end{array}$$

There are negative elements in the bottom row, so the current solution is not optimal. Thus, pivot to improve the current solution. The entering variable is x_1 and the departing variable is s_2.

Perform elementary row operations until the pivot element is 1 and all other elements in the entering column are 0.

$$\left[\begin{array}{cccc|c} x_1 & x_2 & s_1 & s_2 & b \\ \hline 0 & 1 & 3/80 & -1/4 & 3 \\ 1 & 0 & -1/20 & 1/2 & 8 \\ \hline 0 & 0 & 1/20 & 1 & 100 \end{array}\right] \begin{array}{c} \\ x_2 \\ x_1 \\ \\ \end{array}$$

There are no negative elements in the bottom row, so we know the solution is optimal. Thus, the solution is:

$$s_1 = 0,\ s_2 = 0,\ x_1 = 8,\ x_2 = 3,\ z = 100$$

Exercise 112

Given the following linear system and objective function, find the optimal solution.

$$\max x_1 + 8x_2 + 2x_3$$

$$\begin{cases} x_1 + x_2 + x_3 \leq 90 \\ 2x_1 + 5x_2 + x_3 \leq 120 \\ x_1 + 3x_2 \leq 80 \end{cases}$$

Solution

Add slack variables to turn all inequalities to equalities.

$$\begin{cases} x_1 + x_2 + x_3 + s_1 = 90 \\ 2x_1 + 5x_2 + x_3 + s_2 = 120 \\ x_1 + 3x_2 + s_3 = 80 \end{cases}$$

Create the initial tableau of the new linear system.

$$
\begin{array}{cc}
\left[\begin{array}{cccccc|c}
x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & b \\
\hline
1 & 1 & 1 & 1 & 0 & 0 & 90 \\
2 & 5 & 1 & 0 & 1 & 0 & 120 \\
1 & 3 & 0 & 0 & 0 & 1 & 80 \\
\hline
-1 & -8 & -2 & 0 & 0 & 0 & 0
\end{array}\right]
&
\begin{array}{c}
\\
s_1 \\
s_2 \\
s_3 \\
\\
\end{array}
\end{array}
$$

There are negative elements in the bottom row, so the current solution is not optimal. Thus, pivot to improve the current solution. The entering variable is x_2 and the departing variable is s_2.

Perform elementary row operations until the pivot element is 1 and all other elements in the entering column are 0.

$$
\begin{array}{cc}
\left[\begin{array}{cccccc|c}
x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & b \\
\hline
\frac{3}{5} & 0 & \frac{4}{5} & 1 & -\frac{1}{5} & 0 & 66 \\
\frac{2}{5} & 1 & \frac{1}{5} & 0 & \frac{1}{5} & 0 & 24 \\
-\frac{1}{5} & 0 & -\frac{3}{5} & 0 & -\frac{3}{5} & 1 & 8 \\
\hline
\frac{11}{5} & 0 & -\frac{2}{5} & 0 & \frac{8}{5} & 0 & 192
\end{array}\right]
&
\begin{array}{c}
\\
s_1 \\
x_2 \\
s_3 \\
\\
\end{array}
\end{array}
$$

There are negative elements in the bottom row, so the current solution is not optimal. Thus, pivot to improve the current solution. The entering variable is x_3 and the departing variable is s_1.

Perform elementary row operations until the pivot element is 1 and all other elements in the entering column are 0.

$$
\begin{array}{cc}
\left[\begin{array}{cccccc|c}
x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & b \\
\hline
\frac{3}{4} & 0 & 1 & \frac{5}{4} & -\frac{1}{4} & 0 & \frac{165}{2} \\
\frac{1}{4} & 1 & 0 & -\frac{1}{4} & \frac{1}{4} & 0 & \frac{15}{2} \\
\frac{1}{4} & 0 & 0 & \frac{3}{4} & -\frac{3}{4} & 1 & \frac{115}{2} \\
\hline
\frac{5}{2} & 0 & 0 & \frac{1}{2} & \frac{3}{2} & 0 & 225
\end{array}\right]
&
\begin{array}{c}
\\
x_3 \\
x_2 \\
s_3 \\
\\
\end{array}
\end{array}
$$

There are no negative elements in the bottom row, so we know the solution is optimal. Thus, the solution is:

$$
s_1 = 0, \quad s_2 = 0, \quad s_3 = \frac{115}{2}, \quad x_1 = 0, \quad x_2 = \frac{15}{2}, \quad x_3 = \frac{165}{2}, \quad z = 225
$$

Exercise 113

Given the following linear system and objective function, find the optimal solution.

$$\max 3x_1 + 2x_2$$

$$\begin{cases} x_1 + 2x_2 \leq 6 \\ 3x_1 + 2x_2 \leq 12 \end{cases}$$

Solution Add slack variables to turn all inequalities to inequalities.

$$\begin{cases} x_1 + 2x_2 + s_1 = 6 \\ 3x_1 + 2x_2 + s_2 = 12 \end{cases}$$

Create the initial tableau of the new linear system.

$$\begin{array}{c} \left[\begin{array}{cccc|c} x_1 & x_2 & s_1 & s_2 & b \\ \hline 1 & 2 & 1 & 0 & 6 \\ 3 & 2 & 0 & 1 & 12 \\ \hline -3 & -2 & 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{c} \\ s_1 \\ s_2 \\ \\ \end{array}$$

There are negative elements in the bottom row, so the current solution is not optimal. Thus, pivot to improve the current solution. The entering variable is x_1 and the departing variable is s_2.

Perform elementary row operations until the pivot element is 1 and all other elements in the entering column are 0.

$$\left[\begin{array}{cccc|c} x_1 & x_2 & s_1 & s_2 & b \\ \hline 0 & \frac{4}{3} & 1 & -\frac{1}{3} & 2 \\ 1 & \frac{2}{3} & 0 & \frac{1}{3} & 4 \\ \hline 0 & 0 & 0 & 1 & 12 \end{array}\right] \begin{array}{c} \\ s_1 \\ x_1 \\ \\ \end{array}$$

There are no negative elements in the bottom row, so we know the solution is optimal. Thus, the solution is:

$$s_1 = 2, s_2 = 0, x_1 = 4, x_2 = 0, z = 12$$

Exercise 114

A farmer has 110 acres of available land he wishes to plant with a mixture of potatoes, corn, and cabbage. It costs him \$400 to produce an acre of potatoes, \$160 to produce an acre of corn, and \$280 to produce an acre of cabbage. He has a maximum of \$20,000 to spend. He makes a profit of \$120 per acre of potatoes, \$40 per acre of corn, and \$60 per acre of cabbage. (a) How many acres of each crop should he plant to maximize his profit? (b) If the farmer maximizes his profit, how much land will remain unplanted? What is the explanation for this

Solution

We are going to define x_1 = acres of potatoes, x_2 = acres of corn, x_3 = acres of cabbage.

We want to maximize profit which we define in terms of our variables as, profit = $120x_1 + 40x_2 + 60x_3$.

We can express our budget constraint as,
$400x_1 + 160x_2 + 60x_3 \leq \$20{,}000$

We can express our acreage constraint as,
$x_1 + x_2 + x_3 \leq 110$ acres

Now that we have defined a LP Problem we can solve using Simplex Method,

Add slack variables to turn all inequalities into equalities.

$$\begin{cases} 400x_1 + 160x_2 + 280x_3 + s_1 = 20000 \\ x_1 + x_2 + x_3 + s_2 = 110 \end{cases}$$

Create the initial tableau of the new linear system.

$$\left[\begin{array}{ccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & b \\ \hline 400 & 160 & 280 & 1 & 0 & 20000 \\ 1 & 1 & 1 & 0 & 1 & 110 \\ \hline -120 & -40 & -60 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} \\ s_1 \\ s_2 \\ \\ \end{array}$$

There are negative elements in the bottom row, so the current solution is not optimal. Thus, pivot to improve the current solution. The entering variable is x_1 and the departing variable is s_1.

Perform elementary row operations until the pivot element is 1 and all other elements in the entering column are 0.

$$\left[\begin{array}{ccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & b \\ \hline 1 & \frac{2}{5} & \frac{7}{10} & \frac{1}{400} & 0 & 50 \\ 0 & \frac{3}{5} & \frac{3}{10} & -\frac{1}{400} & 1 & 60 \\ \hline 0 & 8 & 24 & \frac{3}{10} & 0 & 6000 \end{array}\right] \begin{array}{c} \\ x_1 \\ s_2 \\ \\ \end{array}$$

There are no negative elements in the bottom row, so we know the solution is optimal. Thus, the solution is:

$$s_1 = 0, \ s_2 = 60, \ x_1 = 50, \ x_2 = 0, \ x_3 = 0, \ z = 6000$$

(a) Interpreting these results in context, this implies that the farmer should plant 50 acres of potatoes and plant 0 acres of corn and cabbage.

(b) Based on the slack variable s_2, 60 acres of land will remain unused. The explanation for not using the unplanted acres of land is that we are also constrained by budget, which has been exausted given these values.

Exercise 115

A product can be made in three sizes, large, medium, and small, which yield a net unit profit of \$12, \$10, and \$9, respectively. The company has three centers where this product can be manufactured and these centers have a capacity of turning out 550, 750, and 275 units of the product per day, respectively, regardless of the size or combination of sizes involved. Manufacturing this product requires cooling water and each unit of large, medium, and small sizes produced require 21, 17, and 9 gallons of water, respectively. The centers 1, 2, and 3 have 10,000, 7000, and 4200 gallons of cooling water available per day, respectively. Market studies indicate that there is a market for 700, 900, and 340 units of the large, medium, and small sizes, respectively, per day. The problem is to determine how many units of each of the sizes should be produced at the various centers in order to maximize the profit

(a) Formulate and solve the linear programming model for this problem. How many units of each of the sizes should be produced at the various centers to maximize the profit?

(b) Next, suppose that the following additional constraint is introduced: By company policy, the fraction (scheduled production)/(center's capacity) must be the same at all the centers. Using this additional information, formulate and solve a new linear programming model to maximize the profit. How has your result changed? Why do you think that this might be a policy that an actual manufacturing company might implement?

(c) Which, if any, of the water capacity constraints was binding? What happens to the solution and to the overall maximum profit if you increase the water capacity at each center by 1%? 5%? 10%? Vary each water capacity one at a time, while holding the others fixed, and write a report on your findings. Include graphs and tables to illustrate your results. Then do the same thing for the other constraints, and discuss your results in detail.

Solution:
Code I used for the Simplex Method: SimplexSolver Repository
(I used this for previous exercises as well)

Let $x_{i,j}$ be the number of products of type j made at Center i.
(1 for large, 2 for medium and 3 for small)

$$\text{Maximize } \sum_{i=1}^{3}(12x_{i,1} + 10x_{i,2} + 9x_{i,3})$$

We can express the center turn out constraints mathematically as:

$$\begin{aligned} x_{1,1} + x_{1,2} + x_{1,3} &\leq 550 \\ x_{2,1} + x_{2,2} + x_{2,3} &\leq 750 \\ x_{3,1} + x_{3,2} + x_{3,3} &\leq 275 \end{aligned}$$

We can express the water cooling constraints mathematically as:

$$\begin{aligned} 21x_{1,1} + 17x_{1,2} + 9x_{1,3} &\leq 10000 \\ 21x_{2,1} + 17x_{2,2} + 9x_{2,3} &\leq 7000 \\ 21x_{3,1} + 17x_{3,2} + 9x_{3,3} &\leq 420 \end{aligned}$$

We can express the market demand constraints as:

$$\begin{aligned} x_{1,1} + x_{2,1} + x_{3,1} &\leq 340 \\ x_{1,2} + x_{2,2} + x_{3,2} &\leq 900 \\ x_{1,3} + x_{2,3} + x_{3,3} &\leq 700 \end{aligned}$$

Now that we have a linear programming problem, we can set up our initial matrix for the simplex algorithm.

$$\left[\begin{array}{cccccccccccccccccc|c}
x_{1,1} & x_{1,2} & x_{1,3} & x_{2,1} & x_{2,2} & x_{2,3} & x_{3,1} & x_{3,2} & x_{3,3} & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & s_9 & b \\
\hline
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 550 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 750 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 275 \\
21 & 17 & 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 10000 \\
0 & 0 & 0 & 21 & 17 & 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 7000 \\
0 & 0 & 0 & 0 & 0 & 0 & 21 & 17 & 9 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 420 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 340 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 900 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 700 \\
\hline
-12 & -10 & -9 & -12 & -10 & -9 & -12 & -10 & -9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]$$

Part a:

After numerous steps of the simplex algorithm it outputs,

$$s_1 = 0, s_2 = \frac{1570}{51}, s_3 = \frac{685}{3}, s_4 = 0, s_5 = 0, s_6 = 0, s_7 = \frac{355}{2}, s_8 = \frac{15185}{34}, s_9 = 0, x_{1,1} = \frac{325}{2}, x_{1,2} = \frac{775}{2}$$

$$x_{1,3} = 0, x_{2,1} = 0, x_{2,2} = \frac{1120}{17}, x_{2,3} = \frac{1960}{3}, x_{3,1} = 0, x_{3,2} = 0, x_{3,3} = \frac{140}{3}, z = \frac{217325}{17}$$

Note: Since some of the values $x_{i,j}$ are fractions, some values will need to be rounded up/down so that these values make sense in context.

Part b:

We can represent our new constraints mathematically as,

$$\frac{x_{1,1} + x_{1,2} + x_{1,3}}{550} = \frac{x_{2,1} + x_{2,2} + x_{2,3}}{750} = \frac{x_{3,1} + x_{3,2} + x_{3,3}}{275}$$

$$\equiv$$

$$\frac{x_{1,1} + x_{1,2} + x_{1,3}}{550} - \frac{x_{2,1} + x_{2,2} + x_{2,3}}{750} = 0$$

and

$$\frac{x_{3,1} + x_{3,2} + x_{3,3}}{275} - \frac{x_{2,1} + x_{2,2} + x_{2,3}}{750} = 0$$

$$\equiv$$

$$750(x_{1,1} + x_{1,2} + x_{1,3}) - 550(x_{2,1} + x_{2,2} + x_{2,3}) = 0$$

and

$$750(x_{3,1} + x_{3,2} + x_{3,3}) - 275(x_{2,1} + x_{2,2} + x_{2,3}) = 0$$

$$\equiv$$

$$750(x_{1,1} + x_{1,2} + x_{1,3}) - 550(x_{2,1} + x_{2,2} + x_{2,3}) \leq 0$$

$$-750(x_{1,1} - x_{1,2} - x_{1,3}) + 550(x_{2,1} + x_{2,2} + x_{2,3}) \leq 0$$

and

$$750(x_{3,1} + x_{3,2} + x_{3,3}) - 275(x_{2,1} + x_{2,2} + x_{2,3}) \leq 0$$

and

$$-750(x_{3,1} + x_{3,2} + x_{3,3}) + 275(x_{2,1} + x_{2,2} + x_{2,3}) \leq 0$$

We can now set up our initial matrix for the simplex algorithm,

$$\left[\begin{array}{cccccccccccccccccccccc|c}
x_{1,1} & x_{1,2} & x_{1,3} & x_{2,1} & x_{2,2} & x_{2,3} & x_{3,1} & x_{3,2} & x_{3,3} & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & s_9 & s_{10} & s_{11} & s_{12} & s_{13} & b \\
\hline
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 550 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 750 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 275 \\
21 & 17 & 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10000 \\
0 & 0 & 0 & 21 & 17 & 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7000 \\
0 & 0 & 0 & 0 & 0 & 0 & 21 & 17 & 9 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 420 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 340 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 900 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 700 \\
750 & 750 & 750 & -550 & -550 & -550 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-750 & -750 & -750 & 550 & 550 & 550 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -275 & -275 & -275 & 750 & 750 & 750 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 275 & 275 & 275 & -750 & -750 & -750 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\hline
-12 & -10 & -9 & -12 & -10 & -9 & -12 & -10 & -9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]$$

After various steps of the algorithm we get,

$s_1 = \frac{1370}{3}, s_{10} = 0, s_{11} = 0, s_{12} = 0, s_{13} = 0, s_2 = \frac{6850}{11}, s_3 = \frac{685}{3},$
$s_4 = 8040, s_5 = \frac{47600}{11}, s_6 = 0, s_7 = \frac{3940}{33}\ s_8 = 900, s_9 = \frac{1960}{3}$
$x_{1,1} = \frac{280}{3}, x_{1,2} = 0, x_{1,3} = 0, x_{2,1} = \frac{1400}{11}, x_{2,2} = 0, x_{2,3} = 0, x_{3,1} = 0$
$x_{3,2} = 0, x_{3,3} = \frac{140}{3}, z = \frac{33740}{11}$

Note: Since some of the values $x_{i,j}$ are fractions, some values will need to be rounded up/down so that these values make sense in context.

Based on the z values profit clearly went down. This new constraint could be interpreted as one imposed by a union to ensure that employees are not being overworked.

Part c:
In part (a) $s_1 = 0$ implies $x_{1,1} + x_{1,2} + x_{1,3}$ is a binding constraint. In part (b) since $s_1, s_2, s_3 > 0$ none of the water constraints are binding.

	Center 1 Turnout	Center 2 Turnout	Center 3 Turnout
1% increase	$3067	$3067	$3067
5% increase	$3067	$3067	$3067
10% increase	$3067	$3067	$3067

Table 1: Profit (z) given increase in turnout conditions

	Center 1 Cooling	Center 2 Cooling	Center 3 Cooling
1% increase	$3067	$3067	$3067
5% increase	$3067	$3067	$3067
10% increase	$3067	$3067	$3067

Table 2: Profit (z) given increase in cooling conditions

	Large Product Demand	Medium Product Demand	Small Product Demand
1% increase	$3067	$3067	$3067
5% increase	$3067	$3067	$3067
10% increase	$3067	$3067	$3067

Table 3: Profit (z) given increase in demand conditions

Note: That these profit values do not take into account rounding of x_i variables, however we can still make conclusions since the changes in the profit values would be negligible.

In summary we conclude that the increases *individually* imposed on each of our constraints are not enough to have a significant impact on the max profit.

2 Call BSM Exercises

Problem 1

Show that

$$c = e^{-rT} \sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} \max(S_0 u^j d^{n-j} - K, 0).$$

Solution
Consider the binomial model for a European call option. At each node, the stock price either goes up by a factor of $u = e^{\sigma\sqrt{T/n}}$ or goes down by a factor of $d = e^{-\sigma\sqrt{T/n}}$. The risk-neutral probability of an up move is $p = \frac{e^{rT/n} - d}{u - d}$.

Using this fact, we can now define all the possible steps in the binomial tree as,

$$c = e^{-rT} \sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} \max(S_0 u^j d^{n-j} - K, 0).$$

Problem 2

Show that the terms of the summation in the equation above are nonzero if and only if

$$j > \alpha = \frac{n}{2} - \frac{\ln(S_0/K)}{2\sigma\sqrt{T/n}}.$$

Thus,

$$c = e^{-rT} \sum_{j>\alpha} \binom{n}{j} p^j (1-p)^{n-j} (S_0 u^j d^{n-j} - K) = e^{-rT}(S_0 U_1 - K U_2).$$

Solution
The payoff $\max(S_0 u^j d^{n-j} - K, 0)$ is nonzero when $S_0 u^j d^{n-j} > K$.
Taking the logarithm and simplifying, we get

$$\ln(S_0 u^j d^{n-j}/K) > 0 \implies j \ln u + (n-j) \ln d > \ln K - \ln S_0.$$

Substituting $u = e^{\sigma\sqrt{T/n}}$ and $d = e^{-\sigma\sqrt{T/n}}$, we get

$$j\sigma\sqrt{T/n} - (n-j)\sigma\sqrt{T/n} > \ln K - \ln S_0,$$

$$(2j - n)\sigma\sqrt{T/n} > \ln K - \ln S_0,$$

$$j > \frac{n}{2} - \frac{\ln(S_0/K)}{2\sigma\sqrt{T/n}} = \alpha.$$

Problem 3(a)

We are asked to show that

$$\lim_{n\to\infty} p(1-p) = \frac{1}{4}.$$

Solution

To show this, we start by recalling the definition of p in the binomial model for option pricing:

$$p = \frac{e^{rT/n} - d}{u - d},$$

$$u \approx 1 + \sigma\sqrt{\frac{T}{n}},$$

$$d \approx 1 - \sigma\sqrt{\frac{T}{n}}.$$

Plugging these approximations into the formula for p, we get:

$$p = \frac{e^{rT/n} - (1 - \sigma\sqrt{T/n})}{(1 + \sigma\sqrt{T/n}) - (1 - \sigma\sqrt{T/n})}.$$

Since $e^{rT/n} \approx 1 + \frac{rT}{n}$ for large n, we can write:

$$p \approx \frac{1 + \frac{rT}{n} - 1 + \sigma\sqrt{\frac{T}{n}}}{2\sigma\sqrt{\frac{T}{n}}}$$

$$p \approx \frac{\frac{rT}{n} + \sigma\sqrt{\frac{T}{n}}}{2\sigma\sqrt{\frac{T}{n}}}$$

$$p \approx \frac{rT}{2\sigma\sqrt{Tn}} + \frac{1}{2}.$$

Rewriting it in a simpler form:

$$p \approx \frac{1}{2} + \frac{rT}{2\sigma\sqrt{Tn}}.$$

As $n \to \infty$, the second term $\frac{rT}{2\sigma\sqrt{Tn}}$ tends to zero, so p approaches $\frac{1}{2}$. Therefore, for large n, we approximate p as:

$$p \approx \frac{1}{2}.$$

Using this approximation, we can show:

$$p(1-p) \approx \left(\frac{1}{2}\right)\left(1 - \frac{1}{2}\right) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

Thus:

$$\lim_{n\to\infty} p(1-p) = \frac{1}{4}.$$

Problem 4

Use the Central Limit Theorem to show that $U_2 = N(d_2)$.

Solution

The Central Limit Theorem (CLT) states that if $X_1, X_2, \ldots, X_n$ are independent random variables with mean μ and variance σ^2, then the sum $\sum_{i=1}^{n} X_i$ is approximately normally distributed with mean $n\mu$ and variance $n\sigma^2$.

In the context of the binomial model, let $X_j = 1$ if the j-th trial is a success (price goes up), and $X_j = 0$ if the j-th trial is a failure (price goes down). For the binomial distribution $B(n, p)$, the mean μ is np and the variance σ^2 is $np(1-p)$.

The value of p in the binomial model for option pricing is given by:

$$p = \frac{e^{rT/n} - d}{u - d}$$

Approximating u and d for large n:

$$u \approx 1 + \sigma\sqrt{\frac{T}{n}}$$

$$d \approx 1 - \sigma\sqrt{\frac{T}{n}}$$

Thus,

$$p \approx \frac{e^{rT/n} - (1 - \sigma\sqrt{\frac{T}{n}})}{(1 + \sigma\sqrt{\frac{T}{n}}) - (1 - \sigma\sqrt{\frac{T}{n}})}$$

$$p \approx \frac{1 + \frac{rT}{n} - 1 + \sigma\sqrt{\frac{T}{n}}}{2\sigma\sqrt{\frac{T}{n}}}$$

$$p \approx \frac{\frac{rT}{n} + \sigma\sqrt{\frac{T}{n}}}{2\sigma\sqrt{\frac{T}{n}}}$$

$$p \approx \frac{rT}{2\sigma\sqrt{Tn}} + \frac{1}{2}$$

As $n \to \infty$,

$$p \to \frac{1}{2}$$

The binomial distribution $B(n, p)$ can be approximated by the normal distribution $N(np, np(1-p))$. As $n \to \infty$,

$$np \approx \frac{n}{2}$$

$$np(1-p) \approx \frac{n}{4}$$

Therefore, the binomial random variable X converges to the normal random variable:

$$X \sim N\left(\frac{n}{2}, \frac{n}{4}\right)$$

To show $U_2 \approx N(d_2)$, we consider the standard normal variable:

$$Z = \frac{X-\mu}{\sigma} = \frac{X-\frac{n}{2}}{\sqrt{\frac{n}{4}}}$$

Since the normal distribution approximates the binomial distribution as $n \to \infty$, we have:

$$U_2 = \sum_{j>\alpha} \binom{n}{j} p^j (1-p)^{n-j} \approx N(d_2)$$

Where:

$$d_2 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

Thus, using the Central Limit Theorem, we have shown that $U_2 \approx N(d_2)$.

3 Volatility Exercises

Code I wrote to complete these exercises: MATH476 Code

Exercise 1

Given observations on a stock price (in dollars) at the end of each of 15 consecutive weeks:

$$30.2, 32.0, 31.1, 30.1, 30.2, 30.3, 30.6, 33.0, 32.9, 33.0, 33.5, 33.5, 33.7, 33.5, 33.2$$

Assuming there are 52 trading weeks in one year.

Solution

To estimate the stock price volatility:

1. Calculate the continuously compounded returns u_i:

$$u_i = \ln\left(\frac{S_i}{S_{i-1}}\right)$$

2. Calculate the mean of the returns $\bar{u}$:

$$\bar{u} = \frac{1}{n} \sum_{i=1}^{n} u_i$$

3. Calculate the standard deviation s of the returns:

$$s = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (u_i - \bar{u})^2}$$

4. Annualize the standard deviation to get the volatility σ:

$$\sigma = s\sqrt{52}$$

Using the Google Colab code we get $\sigma = 0.20794001923088867$

Exercise 2:

Given:

- Market price of the call option: $C_{\text{market}} = 2.50$
- Stock price $S_0 = 15$
- Exercise price $K = 13$
- Time to maturity $T = 3$ months $= 0.25$ years
- Risk-free interest rate $r = 5\%$ per annum

To find the implied volatility σ using the Black-Scholes formula for a call option price:

$$C = S_0 N(d_1) - K e^{-rT} N(d_2)$$

where

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Solution

Using the Google Colab code we get $\sigma = 0.3964355285962891$

Exercise 3:

This is the data of DASH stock from May 6th to June 6th,

110.93, 112.00, 110.55, 110.25, 111.00 , 110.47 , 111.87, 113.35, 111.15, 112.19, 113.02, 114.71, 117.81, 116.28, 115.19, 116.22, 116.35, 115.37, 116.72, 113.00, 113.53, 114.48, 114.44, 116.48

Solution

Using the Google Colab code we get $\sigma = 0.09750432790261566$

Exercise 4

Given:

- Stock price $S_0 = 100$
- Strike price $K = 50$
- Risk-free interest rate $r = 6\%$ per annum
- Time to maturity $T = 0.01$ years

Solution
(a) Calculate the option price for values of σ from 0.05 to 1 in increments of .005

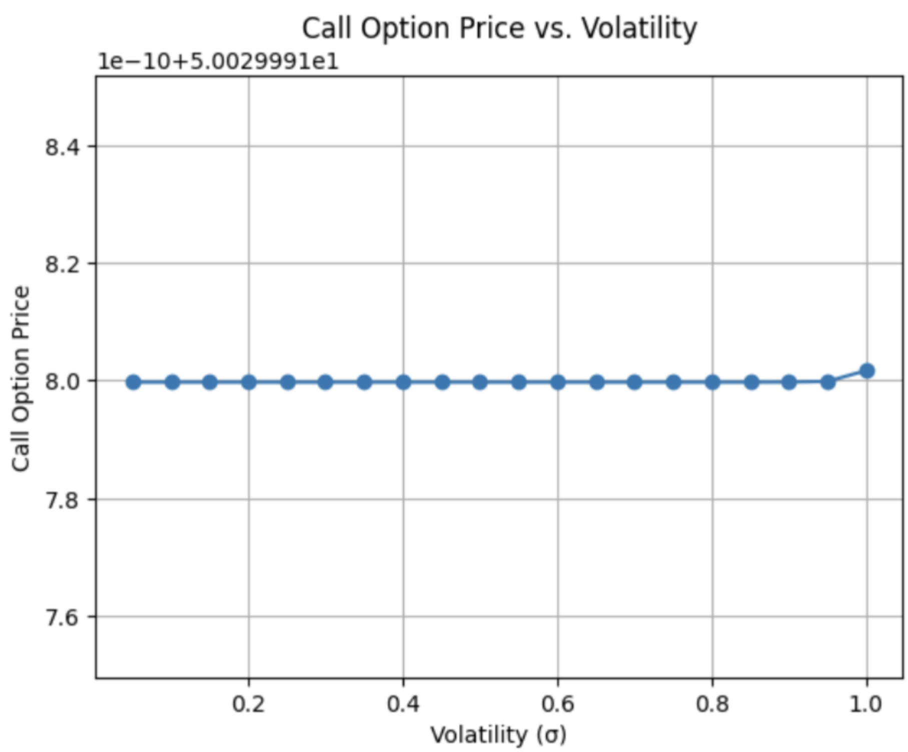


Figure 1: Call Option Price vs. Volatility

(b) Interpret the results.

As volatility increases from .005 to 1 we do not see much change in the call option price.

(c) Calculate the option price for $\sigma = 5$.

Using the Google Colab code we see a signifcant increase in the call option price with c = \$64.00670002074642.