



Machine Learning

Linear Algebra  
review (optional)

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Matrices and  
vectors

**Matrix:** Rectangular array of numbers:



$$\rightarrow \boxed{\mathbb{R}^{4 \times 2}}$$



$$\boxed{\mathbb{R}^{2 \times 3}}$$

Dimension of matrix: number of rows x number of columns

## Matrix Elements (entries of matrix)

$$A = \begin{bmatrix} 1402 & 191 \\ 1371 & 821 \\ 949 & 1437 \\ 147 & 1448 \end{bmatrix}$$

$A_{ij}$  = " $i, j$  entry" in the  $i^{th}$  row,  $j^{th}$  column.

$$A_{11} = 1402$$

$$A_{12} = 191$$

$$A_{32} = 1437$$

$$A_{41} = 147$$

~~$A_{43}$~~  = Undefined (error)

Vector: An  $n \times 1$  matrix.

$$\textcircled{y} = \begin{bmatrix} \textcircled{460} \\ \textcircled{232} \\ \textcircled{315} \\ 178 \end{bmatrix}$$

$n = 4$

← 4-dimensional vector.

~~$\mathbb{R}^{3 \times 2}$~~

$\mathbb{R}^4$

$y_i = i^{th}$  element

$$y_1 = 460$$

$$y_2 = 232$$

$$y_3 = 315$$

→ A, B, C, X

a, b, x, y

1-indexed vs 0-indexed:

$y[1]$

$$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

1-indexed

$y[0]$

$$y = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

0-indexed



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## Addition and scalar multiplication

# Matrix Addition

$$\begin{array}{c} \downarrow \quad \downarrow \\ \rightarrow \begin{bmatrix} \textcircled{1} & 0 \\ \textcircled{2} & 5 \\ \textcircled{3} & 1 \end{bmatrix} + \begin{bmatrix} \textcircled{4} & 0.5 \\ \textcircled{2} & 5 \\ \textcircled{0} & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0.5 \\ 4 & 10 \\ 3 & 2 \end{bmatrix} \\ \text{3} \times \text{2} \quad \text{3} \times \text{2} \quad \text{3} \times \text{2} \end{array}$$

$$\begin{array}{c} \rightarrow \begin{bmatrix} 1 & 0 \\ 2 & 5 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 0.5 \\ 2 & 5 \end{bmatrix} = \text{error} \\ \text{3} \times \text{2} \quad \text{2} \times \text{2} \end{array}$$

# Scalar Multiplication

← real number

$$3 \times \begin{bmatrix} 1 & 0 \\ 2 & 5 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 6 & 15 \\ 9 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 5 \\ 3 & 1 \end{bmatrix} \times 3$$

3x2                      3x2

$$\begin{bmatrix} 4 & 0 \\ 6 & 3 \end{bmatrix} / 4 = \frac{1}{4} \begin{bmatrix} 4 & 0 \\ 6 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{3}{2} & \frac{3}{4} \end{bmatrix}$$

# Combination of Operands

$$\begin{aligned}
 & \text{Scalar multiplication} \rightarrow 3 \times \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} / 3 \quad \text{Scalar division} \\
 & = \begin{bmatrix} 3 \\ 12 \\ 6 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ \frac{2}{3} \end{bmatrix} \quad \begin{array}{l} \text{matrix subtraction /} \\ \text{vector subtraction} \end{array} \\
 & = \begin{bmatrix} 2 \\ 12 \\ 10 \frac{1}{3} \end{bmatrix} \quad \begin{array}{l} \text{matrix addition /} \\ \text{vector addition} \end{array}
 \end{aligned}$$

$3 \times 1$  matrix  
 $3$ -dimensional vector





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## Matrix-vector multiplication

# Example

$$\begin{bmatrix} 1 & 3 \\ 4 & 0 \\ 2 & 1 \end{bmatrix}_{3 \times 2} \begin{bmatrix} 1 \\ 5 \end{bmatrix}_{2 \times 1} = \begin{bmatrix} 16 \\ 4 \\ 7 \end{bmatrix}_{3 \times 1} \text{ matrix}$$

$$1 \times 1 + 3 \times 5 = 16$$

$$4 \times 1 + 0 \times 5 = 4$$

$$2 \times 1 + 1 \times 5 = 7$$

## Details:

$$\underline{A} \times \underline{x} = \underline{y}$$

$\underline{A}$  is an  $m \times n$  matrix (m rows, n columns).  
 $\underline{x}$  is an  $n \times 1$  matrix (n-dimensional vector).  
 $\underline{y}$  is an  $m$ -dimensional vector.

→ To get  $y_i$ , multiply  $\underline{A}$ 's  $i^{th}$  row with elements of vector  $\underline{x}$ , and add them up.

# Example

$$\begin{bmatrix} 1 & 2 & 1 & 5 \\ 0 & 3 & 0 & 4 \\ -1 & -2 & 0 & 0 \end{bmatrix}_{3 \times 4} \begin{matrix} \downarrow \\ \begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix}_{4 \times 1} \end{matrix} = \begin{bmatrix} 14 \\ 13 \\ -7 \end{bmatrix}_{3 \times 1} = \begin{bmatrix} 14 \\ 13 \\ -7 \end{bmatrix}$$

$$\left. \begin{array}{l} 1 \times 1 + 2 \times 3 + 1 \times 2 + 5 \times 1 = 14 \\ 0 \times 1 + 3 \times 3 + 0 \times 2 + 4 \times 1 = 13 \\ -1 \times 1 + (-2) \times 3 + 0 \times 2 + 0 \times 1 = -7 \end{array} \right\}$$

House sizes:

→ 2104

→ 1416

→ 1534

→ 852

Matrix

4x2

1	2104
1	1416
1	1534
1	852

$$h_{\theta}(x) = -40 + 0.25x$$

$h_{\theta}(x)$

2x1

Vector

X

$$\begin{bmatrix} -40 \\ 0.25 \end{bmatrix}$$

=

4x1 matrix

$-40 \times 1 + 0.25 \times 2104$
$-40 \times 1 + 0.25 \times 1416$

$h_{\theta}(1416)$

Prediction = Data Matrix \* Parameters

4x1

for  $i = 1:1000$ ,  
prediction(i) = ...



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## Matrix-matrix multiplication

# Example

$$\begin{array}{l} \begin{bmatrix} 1 & 3 & 2 \\ 4 & 0 & 1 \end{bmatrix} \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 0 & 1 \\ \hline 5 & 2 \\ \hline \end{array} = \begin{bmatrix} 11 & 10 \\ 9 & 14 \end{bmatrix} \\ \textcircled{2 \times 3} \quad \textcircled{3 \times 2} \\ \begin{bmatrix} 1 & 3 & 2 \\ 4 & 0 & 1 \end{bmatrix} \times \begin{array}{|c|} \hline 1 \\ \hline 0 \\ \hline 5 \\ \hline \end{array} = \begin{bmatrix} 11 \\ 9 \end{bmatrix} \\ \begin{bmatrix} 1 & 3 & 2 \\ 4 & 0 & 1 \end{bmatrix} \times \begin{array}{|c|} \hline 3 \\ \hline 1 \\ \hline 2 \\ \hline \end{array} = \begin{bmatrix} 10 \\ 14 \end{bmatrix} \end{array}$$

Handwritten green annotations: The first row of the first matrix is underlined. In the second equation, the first matrix is underlined. In the third equation, the first matrix is underlined. Arrows in the third equation point from the first and second rows of the first matrix to the first and second elements of the resulting column vector, respectively.

## Details:

$$\underline{A} \times \underline{B} = \underline{C}$$

$m \times n$  matrix  
( $m$  rows,  
 $n$  columns)

$n \times o$  matrix  
( $n$  rows,  
 $o$  columns)

$m \times o$   
matrix

The  $i^{th}$  column of the matrix  $C$  is obtained by multiplying  $A$  with the  $i^{th}$  column of  $B$ . (for  $i = 1, 2, \dots, o$ )



# Example

$$\overset{2 \times 2}{\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}} \overset{2 \times 2}{\begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}} =$$

$$\overset{2 \times 2}{\begin{bmatrix} 9 & 7 \\ 15 & 12 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \end{bmatrix} =$$

$$\begin{bmatrix} 1 \times 0 + 3 \times 3 \\ 2 \times 0 + 5 \times 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 15 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} =$$

$$\begin{bmatrix} 1 \times 1 + 3 \times 2 \\ 2 \times 1 + 5 \times 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 12 \end{bmatrix}$$

House sizes:

$$\begin{Bmatrix} \frac{2104}{1416} \\ \frac{1534}{852} \end{Bmatrix}$$

Matrix

$$\begin{bmatrix} 1 & \frac{2104}{1416} \\ 1 & \frac{1534}{852} \\ 1 & \frac{1534}{852} \\ 1 & \frac{1534}{852} \end{bmatrix} \times$$

Matrix

$$\begin{bmatrix} -40 \\ 0.25 \end{bmatrix} \begin{bmatrix} 200 \\ 0.1 \end{bmatrix} \begin{bmatrix} -150 \\ 0.4 \end{bmatrix} =$$

$$\begin{bmatrix} 486 \\ 314 \\ 344 \\ 173 \end{bmatrix} \begin{bmatrix} 410 \\ 342 \\ 353 \\ 285 \end{bmatrix} \begin{bmatrix} 692 \\ 416 \\ 464 \\ 191 \end{bmatrix}$$

Prediction  
of first  
 $h_\theta$

Predictions  
of 2nd  
 $h_\theta$

Have 3 competing hypotheses:

1.  $h_\theta(x) = -40 + 0.25x$

2.  $h_\theta(x) = 200 + 0.1x$

3.  $h_\theta(x) = -150 + 0.4x$



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## Matrix multiplication properties

$$3 \times 5 = 5 \times 3$$


"Commutative"

Let  $A$  and  $B$  be matrices. Then in general,  
 $A \times B \neq B \times A$ . (not commutative.)

E.g.

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$


$$\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix}$$


$A \times B$   
 $m \times n \quad n \times m$

$A \times B$  is  $m \times m$

$B \times A$  is  $n \times n$

↑

$$\underline{3 \times 5 \times 2}$$

$$3 \times 10 = 30 = 15 \times 2$$

$$3 \times (5 \times 2) = (3 \times 5) \times 2$$

"Associative"

$$\begin{array}{l} A \times (B \times C) \leftarrow \\ \underline{(A \times B)} \times C \leftarrow \end{array}$$



$$A \times B \times C.$$

Let  $D = B \times C$ . Compute  $A \times D$ .

Let  $E = A \times B$ . Compute  $E \times C$ .

$$\begin{array}{l} A \times (B \times C) \\ (A \times B) \times C \\ \searrow \\ \text{Some} \\ \text{answe.} \end{array}$$

1 is identity

$$1 \times z = z \times 1 = z$$

for any  $z$

# Identity Matrix

Denoted  $I$  (or  $I_{n \times n}$ ).

Examples of identity matrices:

$[1]$   
 $1 \times 1$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$2 \times 2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$3 \times 3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$4 \times 4$

Informally:

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

For any matrix  $A$ ,

$$A \cdot I = I \cdot A = A$$

$m \times n$     $n \times n$     $m \times m$     $m \times n$     $m \times n$

$I_{n \times n}$

Note:

$AB \neq BA$  in general

$$AI = IA \checkmark$$



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Inverse and  
transpose

$$\underline{1 = \text{"identity"}}$$

$$3 \underbrace{(3^{-1})}_{\frac{1}{3}} = 1$$

$$12 \times \underbrace{(12^{-1})}_{\frac{1}{12}} = 1$$

$$0 \underbrace{(0^{-1})}_{\text{undefined}}$$

Not all numbers have an inverse.

**Matrix inverse:**  $\swarrow$  square matrix  
(#rows = #columns)  $A^{-1}$

If  $A$  is an  $m \times m$  matrix, and if it has an inverse,

$$\rightarrow \underline{A(A^{-1})} = \underline{A^{-1}A} = \underline{I}.$$

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \swarrow$$

$$\text{e.g. } \underbrace{\begin{bmatrix} 3 & 4 \\ 2 & 16 \end{bmatrix}}_A \underbrace{\begin{bmatrix} 0.4 & -0.1 \\ -0.05 & 0.075 \end{bmatrix}}_{A^{-1}} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{A^{-1}A} = I_{2 \times 2}$$

Matrices that don't have an inverse are "singular" or "degenerate"



# Matrix Transpose

Example:

$$\underline{A} = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 5 & 9 \end{bmatrix}_{2 \times 3}$$
$$\underline{B} = \underline{A}^T = \begin{bmatrix} 1 & 3 \\ 2 & 5 \\ 0 & 9 \end{bmatrix}_{3 \times 2}$$

Let  $A$  be an  $m \times n$  matrix, and let  $B = A^T$ .

Then  $B$  is an  $n \times m$  matrix, and

$$\underline{B}_{ij} = \underline{A}_{ji}.$$

$$B_{12} = A_{21} = 2$$

$$B_{32} = 9 \quad A_{23} = 9.$$