Ordinary Differential Equations

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Abstract

This lab takes a step back from the other lab's concepts of iterating simple functions for the purpose of generating chaos, and introduces continuous systems. It also introduces the concept of initial conditions and how important they are in order to accurately predict the final outcome with precision and accuracy.

1 Introduction to Ordinary Differential Equations (ODE)

The simplest form of an ODE is the following:

$$(dx/dt) = ax \tag{1}$$

Here in this equation ”t” is a real variable and ”d” is the derivative of the function x=x(t). This can be expanded to the following:

$$x'(t) = ax(t) \tag{2}$$

Calculus allows us to derive the following solution to the equation:

$$f'(t) = aK^{at} = af(t) \tag{3}$$

Therefore the solution can be identified as f(t)=K^{at}.
In this equation k is any constant derived via a certain initial condition.
If a doesn't equal 0, than the equation is stable (to a point) and the initial condition has some degree of freedom. Zero is therefore, sometimes referred to as the bifurcation point, otherwise known as the point where chaos can emerge from the system given the slightest change in the initial condition.

2 ODE- Expanding beyond the basics

For example, let's examine a system of two differential equations in two unknown functions: x$'_1(t) = a_1 x_1, x'_2(t) = a_2 x_2$
Immediately these functions can be solved because there is no specific relationship identified between them.

$$x_1(t) = K_1^{a_1 t} \tag{4}$$

$$x_2(t) = K_2^{a_2 t} \tag{5}$$

Although this is a simple system, more complicated systems can be reduced to this form. Using vector notation, one can visualize equations 4 and 5 as a dynamical system, where t is time and the resultant cure "x(t)" is the movement of a particle through a plane.
Let's create another variable "u" that represents the particle's position in the plane at time = 0. To indicate the dependence of the position on t and u we denote it by the following equation:

$$\phi_t(u) = (u_1^{a_1 t}, u_2^{a_2 t}) \tag{6}$$

For each point in the plane there is a certain transformation assigning to each point u, represented by $\phi_1(u)$.
The transformation is denoted by the following linear equation:

$$\phi_1 = R^1 -> R^2 \tag{7}$$

As time continues, Each point in the plane moves simultaneously along it's own trajectory and this forms a collection of maps. This family of transformations is called the flow or dynamical system.

3 Newton's Second Law

The concept between the physical force field and the mathematical concept of differential equations is Newton's second law: "F = m a".
We say that x(t) denotes the position vector of the particle at time t, where x: R -¿ R' is a sufficiently differentiable curve. Since the acceleration vector is the second derivative of x(t) with respect to time, one can say that:

$$a(t) = \ddot{x}(t) \tag{8}$$

Newton's second law states:

$$F(x(t)) = m\ddot{x}(t) \tag{9}$$

Thus we obtain a second order differential equation:

$$\ddot{x} = \frac{1}{m} F(x) \tag{10}$$

4 Procedures

```
[ ] #imports python libraries
    import numpy as np
    import scipy.integrate as spi
    import matplotlib.pyplot as plt
    %matplotlib inline
```

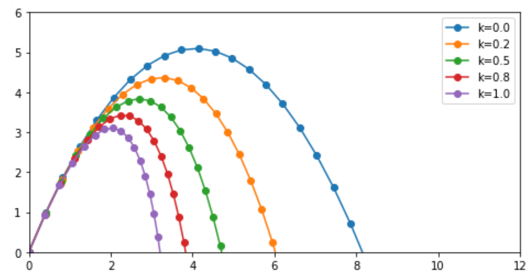
```
[ ] m = 1.  # particle's mass
    k = 0.  # drag coefficient
    g = 9.81  # gravity acceleration
```

```
[ ] # v has four components: v=[u, u']
    v0 = np.zeros(4)
    v0[2] = 4.
    v0[3] = 10.
```

```
[ ] #defines the function
    def f(v, t0, k):
        u, udot = v[:2], v[2:]

        udotdot = -k / m * udot
        udotdot[1] -= g

        return np.r_[udot, udotdot]
```

```
[ ] fig, ax = plt.subplots(1, 1, figsize=(8, 4))

    t = np.linspace(0., 3., 30)

    #the drag coefficient, k, between 0 and 1, divided evenly by 5.
    for k in np.linspace(0., 1., 5):

    #defines v, velocity,
        v = spi.odeint(f, v0, t, args=(k,))

        ax.plot(v[:, 0], v[:, 1], 'o-', label=f'k={k:.1f}')

    ax.legend()
    ax.set_xlim(0, 12)
    ax.set_ylim(0, 6)
```

```
(0.0, 6.0)
```



5 Conclusions

In this lab we analyzed the importance of initial conditions in the context of dynamical systems. We looked at an example with a particle moving through a two dimensional plane. The real applications of this are seen in the documentary featuring the women behind the ENIAC computer.