Optimality of Myopic Ordering Policies for Inventory Model with Stochastic Supply

Article *in* Operations Research · February 2000

DOI: 10.1287/opre.48.1.181.12442

Khang Do Ba
Asian Institute of Technology

OPTIMALITY OF MYOPIC ORDERING POLICIES FOR INVENTORY MODEL WITH STOCHASTIC SUPPLY

DO BA KHANG and OKITSUGU FUJIWARA
School of Management, Asian Institute of Technology, Bangkok, Thailand
khang@ait.ac.th

(Received May 1991; revision received November 1996, January 1998, October 1998; accepted January 1999)

This paper addresses a discrete time inventory model where the maximum amount of supply from which instantaneous replenishment orders can be made is a random variable. Optimal ordering policies are characterized as critical number policies with monotone increasing optimal critical values. A sufficient condition is then presented for myopic ordering policies to be optimal.

Myopic or greedy optimal ordering policies have been thoroughly investigated for the classical dynamic inventory models with stochastic demands (see, for example, Heyman and Sobel 1984). As for periodic review models with supply uncertainty, Gerchak et al. (1988) consider a stationary model with the assumption that the distribution of the yield fraction is invariant over time. They show that myopic policies and order-up-to policies are generally not optimal in such a case. Bassok and Yano (1992) investigate problems with stochastically proportional yield rate, where demand is deterministic and must be satisfied with a specified probability in each period. They show that, under certain conditions, the optimal lot size is a real-valued multiple of net demand for an integer number of periods into the future. Sufficient conditions are then given for myopic policies to be optimal. Ciarallo et al. (1994) consider models where random yield is a consequence of random production capacity with stationary cost and demand distribution. They show that an order-up-to policy minimizes the total expected cost in both finite- and infinite-horizon models. More recently, Wang and Gerchak (1996) investigate a multiperiod production model with periodic review that simultanously incorporates variable capacity, random yield, and uncertain demand. They show that the objective function of total discounted expected costs is quasi-convex, and the optimal policy is of a "re-order point" type, which reduces to an "order-up-to" type if only variable capacity is considered.

This work investigates the effect of supply uncertainty on an optimal ordering policy in a multiperiod production inventory setting. Instead of a stochastically proportional yield or a variable production capacity, we consider a different assumption where the maximum amount of supply from which orders can be made to replenish the inventory is a random variable. Our model is motivated by a production planning problem observed in the frozen seafood industry where the daily supply of raw material varies randomly and the operator must decide how much of the available supply to process. Similar problems can be noted in various food processing companies that rely for raw materials on a rather random harvest, or in some manufacturing plants where the raw materials for one production stage are the random yields from the preceding stages. Other possible applications of the model could be found in economic use of reservoir water in generation of electric power.

We make some restrictive assumptions for our model, notably that the demand is deterministic and constant over time. This is presumably an acceptable approximation in the frozen seafood industry. In the companies where we made our observations, rather stable orders from large distributors are made on a long-term basis, usually from three to six months to one year in advance. The production volume will depend on a more volatile and random supply of raw material that comes on a daily or weekly basis. Another aspect of our model, also motivated by our frozen seafood plant observations, is that the raw material received should be processed immediately or it will perish and be discarded after a relatively short period of time. However, the finished product can be kept in inventory to meet demand. Set-up cost is assumed to be negligible. The optimal solutions for such a problem can then be characterized by a series of critical values similar to those obtained by Wang and Gerchak for their respective model. Aside from providing an explicit and convenient way for computing the optimal ordering policies, some structural results can be derived, notably the monotonicity of the critical ordering values and a sufficient optimality condition for the myopic ordering policy.

1. PROBLEM FORMULATION

Consider a production system for a single product over a finite horizon with independently and identically distributed random supplies of raw materials. For a planning horizon of T periods, at the beginning of each period $t\ (0 \leqslant t \leqslant T)$, the amount available s_t of the raw material supply for the period is observed. The actual demand of the finished product for

Subject classifications: Dynamic programming, optimal control. Inventory production, operating characteristics: supply uncertainty.
Area of review: MANUFACTURING OPERATIONS.

0030-364X/00/4801-0181 $05.00
1526-5463 electronic ISSN

the period is also known and is equal to $a_t = d - I_t$, where d is the constant demand in each period for the finished product and I_t is the inventory of the finished product left over from the previous period $t+1$. The decision variable x_t is the amount of raw material to be purchased for processing in that period. For simplicity of presentation and without any loss of generality, it is assumed that one unit of raw material is needed to produce one unit of the finished product. Then x_t is also the production volume of the period, which of course cannot exceed s_t. Because $I_{t-1} = I_t - d + x_t$, it can be observed that

$$a_{t-1} = a_t + d - x_t. \tag{1}$$

The shortage of period t is $(x_t - a_t)^-$, and the inventory kept at the end of the period is $(x_t - a_t)^+$, where $z^+ = \max(0, z)$ and $z^- = \max(0, -z)$. The total cost incurred in period t is thus

$$G(x_t, a_t) = cx_t + h(x_t - a_t)^+ + p(x_t - a_t)^-, \tag{2}$$

where c, h, and p (with $c < p$) are the unit production, inventory holding, and back-logging costs, respectively.

Let $f_t(a, s)$ denote the minimal expected total system cost from period t through the end of the planning horizon, given the actual demand a of the final product and the observed raw material supply s at the beginning of period t. The optimality principle of dynamic programming implies the following recursive functional equation for $f_t(a, s)$:

$$f_t(a,s) = \min_{0 \leqslant x \leqslant s} \left\{ G(x,a) + \alpha \int_0^\infty f_{t-1}(a + d - x, \xi)\phi(\xi)d\xi \right\} \quad \text{for } t = 1, 2, \ldots, \tag{3}$$

where

$$f_0(a,s) = 0 \quad \forall a, s. \tag{4}$$

In this equation, α is the discounting factor $(0 \leqslant \alpha \leqslant 1)$ and $\varphi(\cdot)$ is the density function for the random and stationary raw material supply in each period.

Equation (3) is similar to the functional equation for the classical single-item inventory model with uncertain demands (see, for example, Denardo 1982, Bensoussan et al. 1983). The difference is the presence of upper-bound constraint s for minimization. Note that this upper bound is given for the current period only and remains a random variable in the subsequent periods.

2. CHARACTERIZATION OF OPTIMAL ORDERING POLICIES

For simplicity of notation we denote

$$L_t(x,a) = G(x,a) + \alpha \int_0^\infty f_{t-1}(a + d - x, \xi)\phi(\xi)d\xi, \tag{5}$$

so (3) becomes

$$f_t(a,s) = \min_{0 \leqslant x \leqslant s} L_t(x,a). \tag{6}$$

Let $x_t^*(a,s)$ be the optimal production volume in period t, given the observed raw material supply s and actual demand a at the beginning of the period; that is,

$$0 \leqslant x_t^*(a,s) \leqslant s, \tag{7}$$

and

$$f_t(a,s) = L_t[x_t^*(a,s), a] = \min_{0 \leqslant x \leqslant s} L_t(x,a). \tag{8}$$

It is clear that $x_1^*(a,s) = \min\{a, s\}$ for all nonnegative values of a.

Theorem 1 below characterizes the optimal ordering policies for the problem just formulated.

THEOREM 1. *For any fixed positive integer t:*

(i) *$L_t(x,a)$ is continuous in (x,a) and piecewise differentiable in each variable;*

(ii) *the first order partial derivative $\partial_1 L_t(x,a)$ is monotone increasing in x for any fixed a;*

(iii) *there exists a unique real number a_t^* between 0 and $(t-1)d$ such that $\partial_1 L_t(x,a) < 0$ whenever $x < a + a_t^*$, and $\partial_1 L_t(x,a) \geqslant 0$ whenever $x > a + a_t^*$;*

(iv) *a solution for (7)–(8) can be obtained as follows:*

$$x_t^*(a,s) = \begin{cases} 0 & \text{if } a + a_t^* < 0, \\ a + a_t^* & \text{if } 0 \leqslant a + a_t^* < s, \\ s & \text{if } s \leqslant a + a_t^*; \end{cases} \tag{9}$$

(v) *moreover,*

$$\partial_1 L_{t+1}(x,a) = \begin{cases} c(1-\alpha) - p + \alpha \displaystyle\int_0^{a-d-x+a_t^*} \partial_1 L_t(\xi, a + d - x)\phi(\xi)d\xi & \text{if } x < a, \\ c(1-\alpha) + h + \alpha \displaystyle\int_0^{a-d-x+a_t^*} \partial_1 L_t(\xi, a + d - x)\phi(\xi)d\xi & \text{if } a < x < a + d + a_t^*, \\ c(1-\alpha) + h + \alpha\partial_1 L_1(0, a + d - x+) & \text{if } a + d + a_t^* < x. \end{cases}$$

Note that because $L_t(x,a)$ is only piecewise differentiable, (ii), (iii), and (v) should be understood to be valid wherever the partial derivatives exist. The optimal ordering policy given in (iv) is closely related to the classical critical number policy. In fact, $d + a_t^*$ can be considered as the critical order-up-to value whenever the supply of raw material is sufficient: By ordering $a + a_t^* = d - I_t + a_t^*$, the initial inventory I_t is raised to the fixed level $d + a_t^*$, which is independent of I_t. The difference here is that the raw material supply may not be sufficient, in which case it is optimal to purchase all the available supply for production. Note that the nonnegative a_t^*

determined in (iii) can be interpreted as buffer stock needed to compensate for the stochastic supplies of raw material in the subsequent periods

As a consequence of Theorem 1, solving problem (7)–(8) is reduced to finding the critical values a_t^* for $t=1,2,\ldots$. These values can be computed as follows. Denote $\Lambda_t(x)=\partial_1 L_t(x,0)$, which is piecewise defined monotone increasing function. Thus a_t^* is determined by solving numerically the equation $\Lambda_t(x)=0$ for $0\leqslant x\leqslant(t-1)d$, where the function $\Lambda_t(x)$ can be defined recursively as follows:

$$\Lambda_t(x)=\begin{cases} c(1-\alpha)-p \\ \quad +\alpha\int_0^{d-x+a_{t-1}^*}\Lambda_{t-1}(\xi+x-d)\phi(\xi)d\xi \\ \quad \text{if } x<0, \\ c(1-\alpha)+h \\ \quad +\alpha\int_0^{d-x+a_{t-1}^*}\Lambda_{t-1}(\xi+x-d)\phi(\xi)d\xi \\ \quad \text{if } 0<x<d+a_{t-1}^*, \\ c(1-\alpha)+h+\alpha\Lambda_{t-1}(x-d) \\ \quad \text{if } d+a_{t-1}^*<x, \end{cases}$$

for $t=2,3,\ldots$ and $\Lambda_1(x)=c-p$ if $x<0$, and $=c+p$ otherwise.

The proof of Theorem 1 is rather straightforward using induction on t. The theorem can be extended to the case where a fixed production capacity K is added at each period t, and the stationary assumption on cost parameters and demand distribution is dropped. Interested readers are referred to Khang and Fujiwara (1998) for the detailed proofs as well as the computational applications of these results.

3. OPTIMALITY OF MYOPIC ORDERING POLICY

We first show that the sequence of the optimal critical values is monotone increasing when the length of the horizon increases.

THEOREM 2. *For any positive integer t,*
(i) $\partial_1 L_{t+1}(x,a)\leqslant\partial_1 L_t(x,a)$ *for all* $x<a+a_t^*$;
(ii) $a_t^*\leqslant a_{t-1}^*$.

PROOF. Case $t=1$ can be easily verified. Suppose that Theorem 2 has been proved for $t-1$. For $x<a+a_t^*$, note that

$$\int_0^{a-d-x+a_t^*}\partial_1 L_t(\xi,a+d-x)\phi(\xi)d\xi$$

$$\leqslant\int_0^{a-d-x+a_{t-1}^*}\partial_1 L_t(\xi,a+d-x)\phi(\xi)d\xi$$

$$\leqslant\int_0^{a+d-x+a_{t-1}^*}\partial_1 L_{t-1}(\xi,a+d-x)\phi(\xi)d\xi.$$

The first inequality holds because $a_{t-1}^*\leqslant a_t^*$ and $\partial_1 L_t(\xi,a+d-x)\leqslant 0$ for $\xi\leqslant a+d-x+a_t^*$, while the second inequality is a result of the induction hypothesis on (i). It follows then from Theorem 1(v) that (i) is true for t. As $\partial_1 L_{t+1}(x,a)\leqslant\partial_1 L_t(x,a)$ for $x\leqslant a+a_t^*$, (ii) also holds for t. □

Because the critical value a_t^* can be considered as buffer stock for subsequent periods, the monotonicity property presented in Theorem 2(ii) has a natural interpretation: The further the stock is from the end of the planning horizon, the more buffer stock is needed to compensate for the uncertainty of raw material supplies in the future.

Next we consider the "myopic" ordering policy, which is often observed in management practice (see also Heyman and Sobel 1984). Given limited and random supplies of raw materials, this policy means ordering exactly the amount of raw material needed to satisfy the actual demand of the current period if the supply is sufficient; otherwise, all the available supply is purchased for production. Thus, the myopic policy corresponds to choosing $a_t^*=0$ for all t in Equation (9). In general, such a policy is not optimal because one can choose, for example, $t=2$ and the parameters such that $h<\alpha(p-c)\Phi(d)$, in which case $a_2^*=d-\Phi^{-1}(h/\alpha(p-c))>0$, where $\Phi(\cdot)$ is the cumulative function of the raw material supply. The following theorem, however, establishes a sufficient condition for the myopic ordering policy to be optimal.

THEOREM 3. *Suppose that for some integer $T>1$,*

$$h+c(1-\alpha)\geqslant\alpha\left[-c+p\sum_{\tau=0}^{T-2}\alpha^\tau\right]\Phi(d).$$

Then for all positive integers t not exceeding T, $a_t^*=0$.

PROOF. The proof is based on the following observation.

LEMMA. *For every positive integer t and for every (x,a), the following inequality holds:*

$$\partial_1 L_t(x,a)\geqslant c-p\sum_{\tau=0}^{t-1}\alpha^\tau.$$

For $t=1$ the lemma can be verified by a simple computation. Assume that it has been proved for some positive integer t and consider case $t+1$. Because $\partial_1 L_{t+1}(x,a)$ is nondecreasing in x for any fixed a, it suffices to prove the lemma for the case where $x<a$. It follows from Theorem 1(v) and the inductive assumption that

$$\partial_l L_{t+1}(x,a)=c(1-\alpha)-p+\alpha\int_0^{a+d-x+a_t^*}\partial_1 L_t(\xi,a+d-x)\phi(\xi)d\xi$$

$$\geqslant c(1-\alpha)-p-\alpha\left(c-p\sum_{\tau=0}^{t-1}\alpha^\tau\right)$$

$$\begin{aligned}&\Phi(a+d-x+a_t^*)\\ &\geqslant c(1-\alpha)-p+\alpha\left(c-p\sum_{\tau=0}^{t-1}\alpha^{\tau}\right)\\ &=c-p\sum_{\tau=0}^{t}\alpha^{\tau},\end{aligned}$$

and the lemma is proved.

Now given any positive integer t not exceeding T, it follows from Theorem 1(v) and the lemma that, for $a<x<a+d$,

$$\begin{aligned}\hat{c}_1L_t(x,a)&=c(1-\alpha)+h\\ &\quad+\alpha\int_0^{a+d-x}\hat{c}_1L_{t-1}(\xi,a+d-x)\phi(\xi)d\xi\\ &\geqslant h+c(1-\alpha)+\alpha\left[c-p\sum_{\tau=0}^{t-2}\alpha^{\tau}\right]\Phi(d)\\ &\geqslant h+c(1-\alpha)+\alpha\left[c-p\sum_{\tau=0}^{T-2}\alpha^{\tau}\right]\Phi(d)\\ &\geqslant 0.\end{aligned}$$

Thus Theorem 1 (iii) implies that $a_t^*=0$. □

REFERENCES

Bassok, Y., C. A. Yano. 1992. Optimal finite and infinite horizon policies for a single stage production system with variable yields. Working Paper, Department of Industrial and Operations Engineering, University of Michigan, Ann Arbor, MI.

Bensoussan, A., M. Crouchy, J. Proth. 1983. *Mathematical Theory of Production Planning*. North-Holland, Amsterdam.

Ciarallo, F., R. Akella, T. Morton. 1994. A periodic review production planning model with uncertain capacity and uncertain demand—optimality of extended myopic policies. *Management Sci.* **40** 320–332.

Denardo, E. 1982. *Dynamic Programming*. Prentice-Hall, Englewood Cliffs, NJ.

Gerchak, Y., R. J. Vickson, M. Parlar. 1988. Periodic review production models with variable yield and uncertain demand. *IIE Trans.* **20**(2) 144–150.

Henig, M., Y. Gerchak. 1990. The structure of periodic review policies in the presence of random yield. *Oper. Res.* **38**(4) 634–643.

Heyman, D. P., M. J. Sobel. 1984. *Stochastic Models in Operations Research, II. Stochastic Optimization*. McGraw-Hill, New York.

Khang, D. B., O. Fujiwara. 1998. Optimal ordering policies for an inventory model with stochastic supplies. Working Paper, Asian Institute of Technology, Bangkok, Thailand.

Wang, Y., Y. Gerchak. 1996. Periodic review production models with variable capacity, random yield, and uncertain demand. *Management Sci.* **42**(1) 130–137.