Interpolation using Hankel tensor completion

Stewart Trickett, Lynn Burroughs, and Andrew Milton*
CGG

Summary

We present a novel multidimensional seismic trace interpolator that works on constant-frequency slices. It performs completion on Hankel tensors whose order is twice the number of spatial dimensions. Completion is done by fitting a PARAFAC model using an Alternating Least Squares algorithm. The new interpolator runs quickly and can better handle large gaps and high sparsity than existing completion methods.

Introduction

Interpolating in four spatial dimensions simultaneously, known as 5D interpolation, has become widespread as it can overcome acquisition constraints for 3D seismic surveys (Trad, 2009). Prestack traces, however, are often both noisy and sparse when placed on a regular four-dimensional grid, and so we require interpolators that perform well under these conditions.

A *tensor* is a multi-way array (Kolda and Bader, 2009). For example, a vector is a first-order tensor, a matrix is a second-order tensor, and a cube of values is a third-order tensor.

An *outer product* (designated "∘") is the multiplication of *n* vectors to form a tensor of order *n*. For example, the outer product of two vectors **a** and **b** forms a matrix **M**:

$$\mathbf{M} = \mathbf{a} \circ \mathbf{b} = \mathbf{a}\,\mathbf{b}^{\mathrm{T}} \quad \text{where} \quad M(i,j) = a(i)\, b(j).$$

The outer product of three vectors **a**, **b**, and **c** forms a third-order tensor **T** (Figure 1):

$$\mathbf{T} = \mathbf{a} \circ \mathbf{b} \circ \mathbf{c} \quad \text{where} \quad T(i,j,k) = a(i)\, b(j)\, c(k).$$

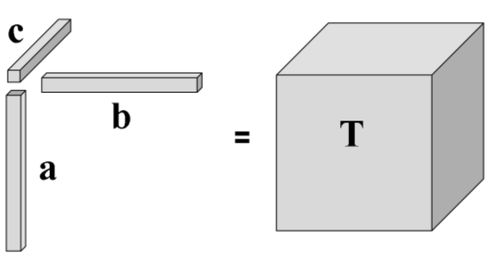

Figure 1: The outer product of three vectors forms a third-order tensor.

There are many ways to define tensor rank. Here we say a tensor has rank *k* if it can be written as the sum of *k* (but no fewer) outer products. Thus the tensor in Figure 1 has rank one.

Seismic trace interpolators have been developed recently based on tensor completion. Beginning with a multidimensional grid of traces with some traces missing, the general method is as follows:

Take the DFT of every trace in the grid.
For every frequency...
1. Form a complex-valued tensor **T** from the frequency slice.
2. Perform tensor completion on **T**.
3. Recover the interpolated frequency slice from the completed tensor.

Take the inverse DFT of each trace.

Two methods for forming the tensor in step 1 have been proposed. The first forms block Hankel matrices (Trickett et al., 2010; Oropeza and Sacchi, 2011). We will call this *Hankel matrix completion*. The second method takes the grid of complex values as a tensor without rearranging the values (Kreimer and Sacchi, 2012), so that the number of spatial dimensions equals the tensor order. We will call this *direct tensor completion*. Some of the tensor elements will be unknown (and thus in need of interpolating) due to the missing traces.

Tensor completion in step 2 finds a low-rank tensor **R** which fits as closely as possible the known elements of **T**. That is, **R** minimizes

$$\| Z\,(\mathbf{T} - \mathbf{R}) \|_{F} \qquad (1)$$

where $\|\cdot\|_F$ is the Frobenius norm (root-mean-square of the tensor elements) and $Z(\cdot)$ is an operator that zeroes out all elements that are unknown in **T**. Tensor **R** provides an approximation to the unknown tensor elements, and thus to the missing traces.

Step 3 is typically done by averaging over every tensor element in which each frequency slice value was originally placed.

The impetus for the above method is an *Exactness Theorem*, which holds for every method of forming **T** described here:

Suppose a multidimensional trace grid has no more than k dips. Then for every frequency there exists a rank-k tensor which fits the known elements of tensor **T** *exactly.*

SEG Houston 2013 Annual Meeting
DOI http://dx.doi.org/10.1190/segam2013-0416.1

Method

Here we combine ideas from the direct tensor and Hankel matrix completion to produce what we will call *Hankel tensor completion*. There are two questions to answer: How do we form tensor **T** in step 1 and how do we derive a low-rank approximation **R** in step 2?

To form the tensor, suppose we are given a raw frequency slice **S** having two spatial dimensions with lengths s_1 and s_2. Form a fourth–order Hankel tensor **T** by generating two tensor orders for every spatial dimension:

$$T(i,j,m,n) = S(i+j-1,\ m+n-1)$$

where the lengths of the four tensor directions are (in order) $s_1/2+1$, $(s_1+1)/2$, $s_2/2+1$, and $(s_2+1)/2$.

Figure 2 depicts the conversion of a 5x5 frequency slice into 5x5 direct tensor, a 9x9 block-Hankel matrix, and a 3x3x3x3 fourth-order Hankel tensor.

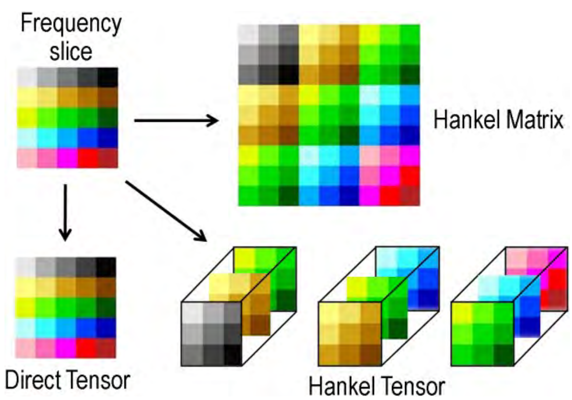

Figure 2: Three strategies for converting a frequency slice for two spatial dimensions into a tensor.

In four spatial dimensions we build an eighth-order tensor:

$$T(i,j,m,n.p,q,r,s) = S(i+j-1,\ m+n-1,\ p+q-1,\ r+s-1).$$

There are many other ways to create a tensor from a frequency slice. The above scheme has the same elements as the Hankel matrix method, but arranged in a different pattern.

Given tensor **T**, we must find a low-rank approximation **R**. There are many strategies for this, including Tucker decomposition or HOSVD (Kreimer and Sacchi, 2011) and nuclear norm minimization (Kreimer and Sacchi, 2012). Here we use PARAFAC decomposition (Kolda and Bader, 2009). For tensor order p and rank k, we model:

$$\mathbf{R} = \sum_{i=1}^{k} \mathbf{u}_1^i \circ \mathbf{u}_2^i \circ \ldots \circ \mathbf{u}_p^i.$$

There is no algorithm to determine vectors $\mathbf{u}_j^i$, $i = 1,\ldots,k$, $j = 1,\ldots,p$ to minimize equation (1) in every case. Nevertheless, there are many that give reasonable solutions, the simplest being Alternating Least Squares, or ALS:

> Make an initial estimate of $\mathbf{u}_j^i$.
> Iterate until equation (1) stops decreasing…
>   Iterate for $j = 1,\ldots,p$
>   Update $\mathbf{u}_j^i$, $i = 1,\ldots,k$ to minimize equation (1).

Each minimization step is a series of linear least-squares problems, one for every element of $\mathbf{u}_j^i$. Missing tensor elements, representing missing traces in the grid, are handled by ignoring these elements (that is, by omitting their rows in the linear system) during each least-squares solution, so that they have no effect on the minimization.

Why might Hankel tensors make for a better interpolator than Hankel matrices? The tensor outer-product vectors are much shorter than the matrix outer-product vectors. For example, suppose we are filtering a multi-dimensional frequency slice that is 15 traces on each side. Here is the number of parameters needed to model a single rank (and thus a single dip) using the two methods:

Spatial Dimensions	Hankel Matrix	Hankel Tensor	Ratio
1	16	16	1
2	128	32	4
3	1024	48	21
4	8192	64	128

Table 1: The number of outer-product parameters estimated for each rank for Hankel matrix and Hankel tensor interpolation as a function of spatial dimensions. The ratio of these two numbers is in the final column. The data grid is 15 traces in each direction.

Thus Hankel tensor completion estimates fewer parameters, resulting in greater accuracy and robustness in the presence of noise or extreme sparsity, especially in higher dimensions.

A second advantage is that the method runs much faster than Hankel matrix completion, even with the speed-ups of Gao, Sacchi, and Chen (2013). The recursive nature of the model allows computations for each spatial dimension to separate, and we need not explicitly form Hankel tensors at any stage.

DOI http://dx.doi.org/10.1190/segam2013-0416.1

Examples

We first compare Hankel matrix to Hankel tensor interpolation on synthetic data in two spatial dimensions. Direct tensor interpolation is not compared, since it does a poor job in two spatial dimensions due its lack of constraints. Figure 3 shows that Hankel tensor is better able to handle large gaps. Figure 4 shows a synthetic in four spatial dimensions, demonstrating Hankel tensor interpolation's superior ability to handle extreme sparsity.

A real example is given in Figure 5, showing a single 3D CMP gather before and after 5D interpolation. The gather now has a complete set of offsets and azimuths. Its CMP stack in Figure 6 shows that a subtle event has been revealed by the interpolation.

Conclusions

Hankel tensor completion is a novel means of interpolation that is computationally fast and demonstrates a greater ability to handle large gaps or high sparsity than existing completion methods.

Much remains to be done. It's not clear what the best decomposition is for rank reduction, nor what the best algorithm is to calculate it, given that ALS can sometimes be slow to converge. More comparisons between Hankel tensor interpolation and other interpolators are also needed.

Acknowledgements

Real data is provided by the Consortium for Research in Elastic Wave Exploration Seismology ("CREWES").

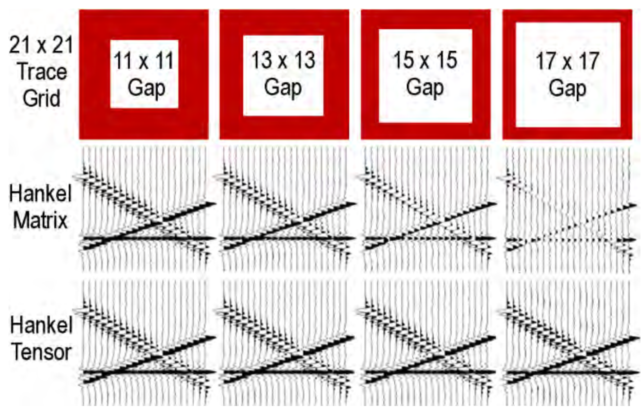

Figure 3: Comparing interpolators on a synthetic 21 by 21 trace grid with a gap in the center. Only a slice near the middle of the grid is shown. Hankel matrix interpolation does poorly when the gap is 15 by 15 traces or larger, while Hankel tensor interpolation does well even for a 17 by 17 trace gap.

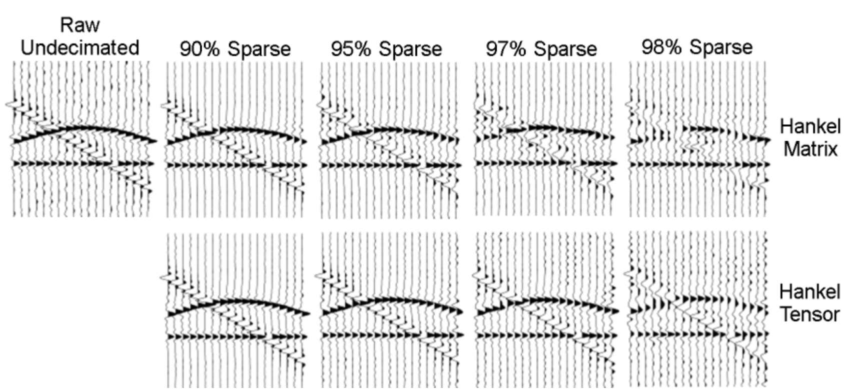

Figure 4: A synthetic in four spatial dimensions (a one-dimensional slice is shown) with an event curving in two dimensions and another planar event dipping in two dimensions. The raw undecimated synthetic is on the left. Most of the traces were removed at random, and then both Hankel matrix and Hankel tensor interpolators were applied to recreate the synthetic. "90% sparse" means that 90% of the traces were removed, leaving only 10% of the raw traces. Hankel tensor interpolation withstands greater sparsity, and in particular does a better job of preserving curvature.

DOI http://dx.doi.org/10.1190/segam2013-0416.1

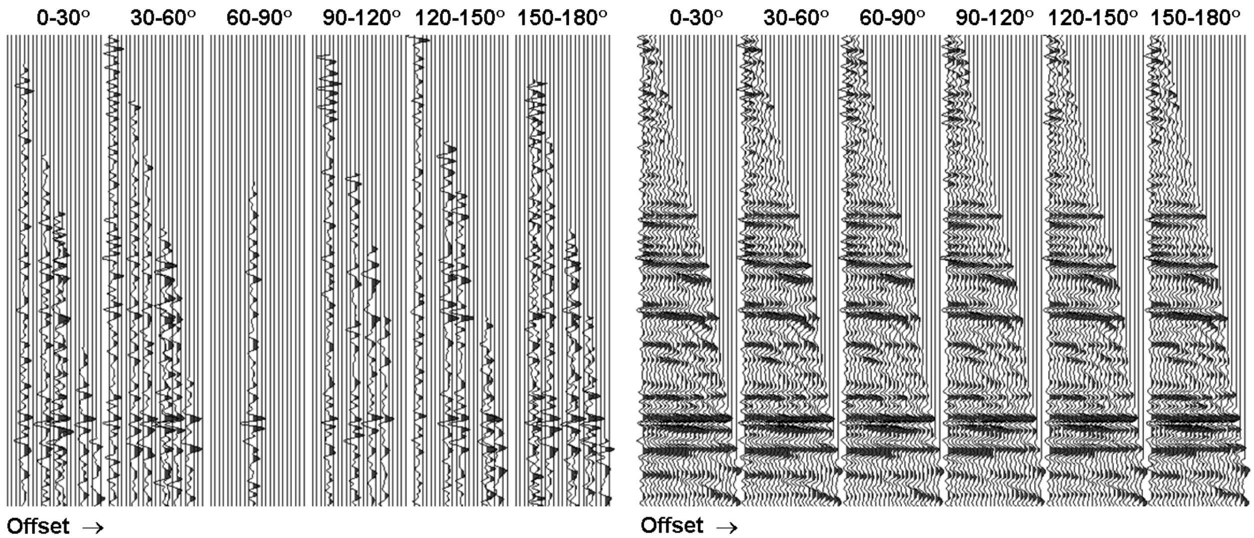

Figure 5: A real 3D CMP gather plotted by azimuth sector and offset (left) and the same gather after 5D Hankel tensor interpolation. Data is provided by the Consortium for Research in Elastic Wave Exploration Seismology ("CREWES").

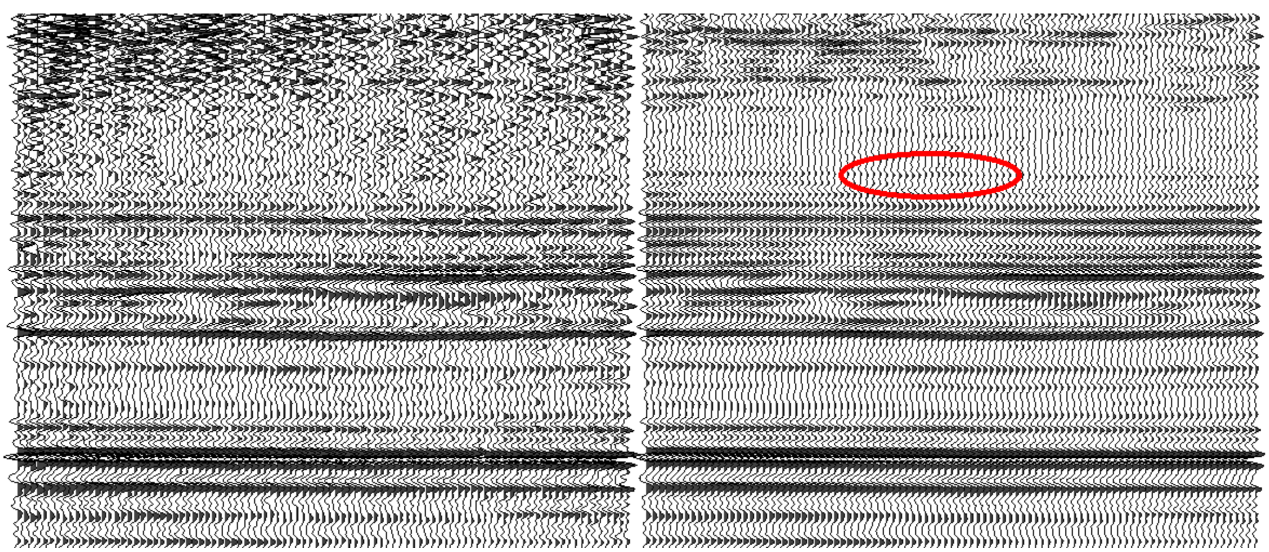

Figure 6: CMP stack of the gathers from figure 5 without (left) and with (right) 5D Hankel tensor interpolation. Note how a subtle event within the ellipse has been revealed. Data is provided by the Consortium for Research in Elastic Wave Exploration Seismology ("CREWES").

DOI http://dx.doi.org/10.1190/segam2013-0416.1

http://dx.doi.org/10.1190/segam2013-0416.1

EDITED REFERENCES

Note: This reference list is a copy-edited version of the reference list submitted by the author. Reference lists for the 2013 SEG Technical Program Expanded Abstracts have been copy edited so that references provided with the online metadata for each paper will achieve a high degree of linking to cited sources that appear on the Web.

REFERENCES

Gao, J., M. Sacchi, and X. Chen, 2013, A fast reduced-rank interpolation method for prestack seismic volumes that depend on four spatial dimensions: Geophysics, **78**, no. 1, V21–V30, http://dx.doi.org/10.1190/geo2012-0038.1.

Kolda, T. G., and B. W. Bader, 2009, Tensor decompositions and applications: SIAM (Society for Industrial and Applied Mathematics) Review, **51**, no. 3, 455–500, http://dx.doi.org/10.1137/07070111X.

Kreimer, N., and M. D. Sacchi, 2011, A tensor higher-order singular value decomposition (HOSVD) for pre-stack simultaneous noise-reduction and interpolation: 81st Annual International Meeting, SEG, Expanded Abstracts, 3069–3074.

Kreimer, N., and M. D. Sacchi, 2012, Tensor completion via nuclear norm minimization for 5D seismic data reconstruction: 82nd Annual International Meeting, SEG, Expanded Abstracts, doi: 0.1190/segam2012-0529.1.

Oropeza, V. E., and M. D. Sacchi, 2011, Simultaneous seismic data denoising and reconstruction via multichannel singular spectrum analysis: Geophysics, **76**, no. 3, V25–V32, http://dx.doi.org/10.1190/1.3552706.

Trad, D., 2009, Five-dimensional interpolation: Recovering from acquisition constraints: Geophysics, **74**, no. 6, V123–V132, http://dx.doi.org/10.1190/1.3245216.

Trickett, S. R., L. Burroughs, A. Milton, L. Walton, and R. Dack, 2010, Rank-reduction-based trace interpolation: 80th Annual International Meeting, SEG, Expanded Abstracts, 3829–3833.