Linear Algebra and its Applications 491 (2016) 56–72

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Hankel tensors, Vandermonde tensors and their positivities ☆

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A R T I C L E I N F O

Article history:
Received 20 August 2014
Accepted 1 February 2015
Available online 4 March 2015
Submitted by F. Zhang

MSC:
53A45
15A69

Keywords:
Tensor
Decomposition
Hankel tensor
V-tensor
Copositive tensor

A B S T R A C T

An mth order n-dimensional Hankel tensor is defined as a tensor $\mathcal{A}$ satisfying $A_{i_1 \dots i_m} \equiv A_{i_1+\dots+i_m-m}$ for some numbers $A_0, A_1, \dots, A_{m(n-1)}$. A Hankel tensor possesses a Vandermonde decomposition (VD) $\mathcal{A} = \sum_{k=1}^{r} \lambda_k \mathbf{u}_k^m$ where $\mathbf{u}_k = (1, w_k, w_k^2, \dots, w_k^{n-1})^T$ is called a Vandermonde vector (V-vector). $\mathcal{A}$ is called a Vandermonde tensor (V-tensor) if $\mathcal{A}$ has a VD with each $\lambda_k = 1$. V-tensors are the natural extension of Vandermonde matrices. It is easy to see that all even order V-tensors are positive semidefinite (psd) and thus copositive. An odd order real symmetric tensor is psd only if it is zero. The problem when an odd order Hankel tensor is copositive is open. We present a necessary and sufficient condition for a rank-2 odd-order symmetric real tensor to be copositive. Some necessary conditions for a general V-tensor to be copositive are also presented. The singularity of V-tensors is also investigated, and we show that a V-tensor $\mathcal{A}$ is singular if its V-rank is less than its dimension. This condition becomes necessary if $\mathcal{A}$ is of odd order.

☆ This work is partially supported by the Hong Kong Research Grant Council (Grant Nos. PolyU 501909, 502510, 502111, 501212) and was initialized when the author was visiting the Hong Kong Polytechnic University.

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http://dx.doi.org/10.1016/j.laa.2015.02.012

1. Introduction

We use $[n]$ to denote the set $\{1, 2, \dots, n\}$ for any given positive integer n, $N(m, n)$ for the number $(n-1)m$ where m, n are two positive integers. For simplicity, we denote $N := N(m, n)$ when m, n are clear from the context. In this paper we mainly deal with problems on the field of real numbers.

Let $A = (A_{i_1 i_2 \dots i_m}) \in \mathcal{T}_{m,n}$ where $\mathcal{T}_{m,n}$ denotes the set of all mth order n-dimensional real tensors. For any given positive numbers $m, n \geq 1$, we denote

$$S(m, n) := \{\tau = (i_1, \dots, i_m)\colon\ i_1, \dots, i_m \in [n]\}$$

and for each $k = 0, 1, \dots, N \equiv (n-1)m$, we denote

$$S(k; m, n) = \{\tau = (i_1, \dots, i_m) \in S(m, n)\colon\ i_1 + \dots + i_m = m + k\}.$$

For simplicity, we sometimes use notation $S(k)$ instead of $S(k; m, n)$ when no confusion arises, and use s_k to denote the cardinality of $S(k)$. For convenience, we sometimes use A_σ to denote $A_{i_1 i_2 \dots i_m}$ ($\mathcal{A}$'s entry) for $\sigma = (i_1, i_2, \dots, i_m) \in S(m, n)$. It is obvious that set $\{S(k; m, n)\colon\ k = 0, 1, \dots, N\}$ gives a partition of $S(m, n)$. A tensor $\mathcal{A}$ of m-order n-dimension is called a *Hankel tensor* if there exists an $N+1$-dimensional vector $\mathbf{v} = (v_0, v_1, \dots, v_N)^T$ such that

$$A_{i_1 i_2 \dots i_m} \equiv v_{i_1 + i_2 + \dots + i_m - m}, \quad \forall \sigma \equiv (i_1, \dots, i_m) \in S(m, n). \tag{1.1}$$

If $\sigma = (i_1, i_2, \dots, i_m) \in S(k; m, n)$ ($k \in [N] \cup \{0\}$), then (1.1) is equivalent to

$$A_\sigma = v_k, \quad \forall \tau \in S(k),\ k = 0, 1, 2, \dots, N.$$

In this case $\mathcal{A}$ is called a Hankel tensor determined by $\mathbf{v}$. We use $\mathcal{H}_{m,n}$ to denote the set of all mth order n-dimensional Hankel tensors. It is shown in [12] that

Lemma 1.1. *An mth order n-dimensional Hankel tensor $\mathcal{A}$ can always be decomposed as*

$$\mathcal{A} = \sum_{j=1}^{r} \lambda_j \mathbf{u}_j^m \tag{1.2}$$

where each λ_j is a scalar and $\mathbf{u}_j := (1, \omega_j, \dots, \omega_j^{n-1})^T$ for some complex number ω_j.

The decomposition (1.2) in Lemma 1.1 is called a *Vandermonde decomposition* (briefly *VD*, or a V-decomposition) of $\mathcal{A}$ in [12]. We call the minimal number r of the summands in (1.2) a *Hankel-rank*, denoted by *Hrank*$(\mathcal{A})$. Note that the Hankel-rank is called Vandermonde rank (V-rank) in [12] and is denoted by *Vrank*$(\mathcal{A})$. But here the V-rank will be defined differently in the following.

We recall that a real *mth order n-dimensional symmetric tensor* $\mathcal{A}$ is a tensor $\mathcal{A} = (A_{i_1 i_2 \ldots i_m})$ that satisfies

$$A_{i_{\tau(1)} i_{\tau(2)} \ldots i_{\tau(m)}} = A_{i_1 i_2 \ldots i_m}, \quad \forall \tau \in Sym_m, \tag{1.3}$$

where Sym_m denotes the set of all permutations on set $[m]$. We denote $\mathcal{S}_{m,n}$ for the set of all mth order n-dimensional symmetric tensors. A Hankel tensor is obviously symmetric. A Hankel tensor can be constructed from any given symmetric or a general tensor, as to be illustrated.

Hankel tensors are a class of structured tensors being investigated very recently [12]. They were originated from Hankel matrices, which found their important applications in many fields such as linear dynamic systems [9]. The research work already done on Hankel tensors involves their spectral characterizations [12,17], tensor decompositions (see e.g. [4]) and their positivities (including positive definiteness, copositivity, and complete positivity [11,13,15], etc.) etc. Hankel tensors can be applied to data fitting [5]. In [16], the authors study the Hilbert tensor which is a special kind of Hankel tensor determined by $\mathbf{v} = (1, 1/2, 1/3, \ldots, 1/(N+1))$. Recently, some progress has been made on the study of other kinds of structured tensors such as Cauchy tensor, P-tensor and B-tensor, and Centro-Symmetric tensor [12,2,17,18,3].

A Hankel tensor demonstrates a more symmetric form than a general symmetric tensor and thus has better properties. For example, the degree of freedom (*dof*) of a Hankel tensor is much less than the *dof* of a symmetric tensor. In fact, it is easy to see that $\mathcal{H}_{m,n}$ is a linear subspace of the linear space $\mathcal{S}_{m,n}$, which is a linear subspace of $\mathcal{T}_{m,n}$, the set of all mth order n-dimensional real tensors (under the operations of the usual tensor addition and scalar multiplication). Furthermore, we have $dim(\mathcal{H}_{m,n}) = N + 1$ and $dim(\mathcal{S}_{m,n}) = \sum_{j=0}^{N} q_j$ where q_j denotes the partition number of j into m groups, with each group no larger than n (the empty subset allowed). Note that $q_j = \binom{m+j-1}{j}$ when $0 \le j \le n$. For example, let $m = 3, n = 4$. Then $N = 9$, $q_0 = 1$, $q_1 = 1$, $q_2 = 2$, $q_3 = 3$, and $4 = q_4 = q_5 = q_6 = \cdots$. We present here without proof a property of $\mathcal{S}_{m,n}$.

Proposition 1.2. $dim(\mathcal{S}_{m,n}) = \binom{m+n-1}{m}$.

In this paper, we investigate properties of Hankel tensors and V-tensors. From Lemma 1.1 we can see that all the even order real symmetric tensors with each coefficient $\lambda_j \ge 0$ in (1.2) are psd (positive semidefinite), and so a V-tensor of an even order is also psd. It follows that a V-tensor of an even order is copositive. The problem when a Hankel tensor of an odd order is copositive is put forward by Qi in [12] and still remains open. We show in this paper that a rank-2 odd-order real symmetric tensor $\mathcal{A}$ (i.e., $\mathcal{A} = \mathbf{u}^{2k+1} + \mathbf{v}^{2k+1}$) is copositive if and only if its induced 1-tensor (to be defined in the following) is copositive, i.e., $\mathbf{u} + \mathbf{v} \ge 0$. Some necessary conditions for a general V-tensor to be copositive are also presented.

A tensor $\mathcal{A} \in \mathcal{T}_{m,n}$ is called a *singular tensor* if it has a zero eigenvalue. We show that a Vandermonde tensor is singular if its Vandermonde rank is less than its dimension. This condition is also necessary if m is odd.

2. Hankel tensors and rank-one symmetric tensors

For any given nonzero (real) vector $\mathbf{x} = (x_1, \dots, x_n)^T \in \mathbb{R}^n$, we can form a *rank-one mth order n-dimensional symmetric tensor* $\mathbf{x}^m = (x_{i_1 i_2 \dots i_m})$ where

$$x_{i_1 i_2 \dots i_m} = x_{i_1} x_{i_2} \dots x_{i_m}.$$

This rank-one symmetric real tensor is crucial in our following discussion (note that notation $\mathbf{x}^{[k]}$ denotes a vector $(x_1^k, x_2^k, \dots, x_n^k)^T$). An important known fact about a symmetric tensor is that every symmetric tensor $\mathcal{A} \in \mathcal{S}_{m,n}$ can be written as the sum of rank-one symmetric tensors. This is called the symmetric rank-one decomposition of tensors. The *symmetric tensor rank* of $\mathcal{A}$, denoted $rank_s(\mathcal{A})$, is defined as the minimal number of the rank-one tensors in the summation.

In order to consider when a given symmetric tensor becomes Hankel, we first consider a rank-one tensor. The following lemma can be deduced from the VD of a Hankel tensor [12]. For the completeness and the following discussion, we present an alternative proof without using the VD of $\mathcal{A}$.

Lemma 2.1. *A rank-one mth order n-dimensional real tensor $\mathcal{A}$ with $m > 1$ is a Hankel tensor if and only if $\mathcal{A} = \lambda \mathbf{x}^m$ where*

$$\mathbf{x} = (1, \omega, \omega^2, \dots, \omega^{n-1})^T \tag{2.1}$$

is a real column vector of dimension n, and λ and ω are both nonzero scalars.

Proof. Let $\mathcal{A} = \lambda \mathbf{x}^m$ with $x = (1, \omega, \omega^2, \dots, \omega^{n-1})^T$. Then by definition $\mathcal{A} = (A_{i_1 \dots i_m}) \in \mathcal{T}_{m,n}$ satisfies

$$A_{i_1 \dots i_m} = \lambda x_{i_1} x_{i_2} \dots x_{i_m} = \lambda \omega^{i_1 + \dots + i_m - m}. \tag{2.2}$$

Thus $\mathcal{A}$ is a Hankel tensor determined by $\mathbf{v} = (v_0, v_1, \dots, v_N)$ where $v_k = \lambda \omega^k$ for each $k \in [N] \cup \{0\}$.

Conversely, if $\mathcal{A}$ is an mth order n-dimensional Hankel tensor with $rank(\mathcal{A}) = 1$, then $\mathcal{A}$ is symmetric, and thus can be written in form $\mathcal{A} = \lambda \mathbf{x}^m$ by [4]. Denote $\mathbf{x} = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$. Note that $\mathbf{x} \neq 0$ since otherwise $rank(\mathcal{A}) = 0$. We may assume that $x_1 \neq 0$ (through reordering the coordinates of $\mathbf{x}$). Now for our convenience we write $\mathbf{x} = \mu(1, x_1, x_2, \dots, x_{n-1})^T$. So $\mathcal{A} = \lambda \mu^m \mathbf{y}$ where $\mathbf{y} := (1, y_1, y_2, \dots, y_{n-1})^T \in \mathbb{R}^n$. Thus it suffices to consider the case $\mathcal{A} = \mathbf{x}^m$ with $\mathbf{x} = (1, x_1, x_2, \dots, x_{n-1})^T$ and to show that

$x_k = x_1^k$ for all $k = 1, 2, \ldots, n$. We use the induction to the dimension of $\mathbf{x}$ to prove this in the following.

The result is trivial if $n = 1$. For $n = 2$, the result is also obvious. Now we assume $n > 2$. We first want to prove that $\mathbf{x}$ has no zero entry. Suppose that x has zero entries, then we may assume without loss of generality that

$$supp(\mathbf{x}) = \{1, 2, \ldots, r\}$$

where $r \in [n-1]$, i.e., $x_1, \ldots, x_r \neq 0$, and $x_{r+1} = x_{r+2} = \cdots = x_{n-1} = 0$. Now we take $s := m - r + 1$ and $\tau = (i_1, \ldots, i_m) = (\overbrace{1, \ldots, 1}^{s}, 2, 3, \ldots, r) \in S(k; m, n)$ where

$$k = i_1 + \cdots + i_m - m = \frac{1}{2} r(r-1).$$

Denote $\sigma = (\overbrace{1, \ldots, 1}^{s-1}, 2, 3, \ldots, r-1, r+1)$. Then obviously $\sigma \in S(k; m, n)$. Thus $A_\tau = A_\sigma$ since $\mathcal{A}$ is a Hankel tensor. But this is impossible since $A_\tau \neq 0$ and $A_\sigma = 0$ by the definition of $supp(\mathbf{x})$. This proves that $\mathbf{x}$ has no zero entry.

If $m = 2$, A is a Hankel matrix with $A_{ij} = x_{i+1} x_{j+1}$ where we denote $x_0 := 1$. Thus $A_{i+1,j} = A_{i,j+1}$, so $x_{i+1} x_j = x_i x_{j+1}$ for all possible i, j. It follows that

$$\frac{x_i}{x_j} = \frac{x_{i+1}}{x_{j+1}}. \tag{2.3}$$

So we have

$$\frac{x_0}{x_1} = \frac{x_1}{x_2} = \cdots = \frac{x_{n-2}}{x_{n-1}}.$$

Consequently we get the result.

Now we let $m > 2$ and $1 \leq i < j \leq n$. We write

$$\tau = (i+1, \overbrace{j, \ldots, j}^{m-1}) \in S(k; m, n)$$

where $k = i + 1 + (m-1)j - m$, and

$$\tau' = (i, j+1, \overbrace{j, \ldots, j}^{m-2}).$$

Then $\tau' \in S(k; m, n)$. Thus we have $A_\tau = A_{\tau'}$, i.e.,

$$x_{i+1} x_j \ldots x_j = x_i x_{j+1} x_j \ldots x_j.$$

So we also get (2.3). The proof is completed by using the same argument as above. $\square$

We can see from the proof of Lemma 2.1 that a rank-one Hankel tensor has no zero entry. In fact, we have the following:

Corollary 2.2. *An mth order n-dimensional real tensor $\mathcal{A} = (A_{i_1 i_2 \ldots i_m})$ is a rank-one Hankel tensor if and only if there exist two nonzero numbers $a, b \in \mathbb{R}$ such that*

$$A_{i_1 i_2 \ldots i_m} = a^{1-k} b^k, \quad \forall (i_1, \ldots, i_m) \in S(k, m, n), \tag{2.4}$$

holds for each $k \in \{0, 1, 2, \ldots, N\}$.

Proof. If $\mathcal{A}$ satisfies condition (2.4), then it is obvious that $\mathcal{A}$ is a Hankel tensor determined by vector

$$\mathbf{v} = (a, b, \frac{b^2}{a}, \frac{b^3}{a^2}, \ldots, \frac{b^n}{a^{n-1}}).$$

Write $\omega = \frac{b}{a}$, then we have $\mathcal{A} = a\mathbf{x}^m$ where $\mathbf{x} = (1, \omega, \omega^2, \ldots, \omega^{n-1})^T$. Thus $\mathcal{A}$ is a rank-one Hankel tensor.

Conversely, if $\mathcal{A}$ is a rank-one Hankel tensor, then by Lemma 2.1, we have $\mathcal{A} = a\mathbf{x}^m$ where $\mathbf{x} = (1, \omega, \omega^2, \ldots, \omega^{n-1})^T$ for some nonzero number a and nonzero $\omega \in \mathbb{R}$. Denote $b = a\omega$. Then we may verify that $\mathcal{A}$ satisfies (2.4). $\square$

A vector in form (2.1) is called a *Vandermonde vector*, or simply a *V-vector*, which plays an important role in the study of Hankel tensors. By Lemma 1.1, a Vandermonde matrix of order n can always be written in form $\sum_{j=1}^{r} \mathbf{x}_j^2$ with $\mathbf{x}_j$ in form (2.1). An mth order n-dimensional symmetric tensor $\mathcal{A}$ is said to have a *Vandermonde decomposition*, or briefly *VD*, if it can be expressed as a linear combination of some rank-one symmetric tensor $\mathbf{u}^m$, where $\mathbf{u}$ is a V-vector. A *Vandermonde tensor*, or briefly a *V-tensor*, is defined as a sum of some rank-one V-tensors, i.e.,

$$\mathcal{A} = \sum_{j=1}^{r} \mathbf{u}_j^m \tag{2.5}$$

where $\mathbf{u}_j = (1, \omega_j, \omega_j^2, \ldots, \omega_j^{n-1})^T \in \mathbb{R}^n$ is a V-vector, and all $\omega_j \in \mathbb{R}$ are distinct nonzero numbers. The *Vandermonde rank* of a V-tensor $\mathcal{A}$, denoted *Vrank*($\mathcal{A}$), is the smallest number r such that (2.5) holds [12].

A symmetric tensor $\mathcal{A}$ is uniquely associated with a multivariate polynomial $F_{\mathcal{A}}(x)$ by

$$F_{\mathcal{A}}(x) := \sum_{i_1, \ldots, i_m = 1}^{n} A_{i_1 i_2 \ldots i_m} x_{i_1} x_{i_2} \ldots x_{i_m}. \tag{2.6}$$

Recall that an mth order n-dimensional symmetric tensor $\mathcal{A}$ is called *positive semi-definite* (*psd*) if $F_{\mathcal{A}}(x) \geq 0$ for all $x \in \mathbb{R}^n$. $\mathcal{A}$ is called a *positive definite* (*pd*) tensor if $F_{\mathcal{A}}(x) > 0$ for all nonzero vector $x \in \mathbb{R}^n$. An mth order n-dimensional symmetric tensor $\mathcal{A}$ is called *copositive* if $F_{\mathcal{A}}(x) \geq 0$ for all nonnegative vector $x \in R^n_+$. It is called *strictly copositive* if $F_{\mathcal{A}}(x) > 0$ for all nonzero nonnegative vector $x \in R^n_+$. The problem of determining the copositivity of a given mth order n-dimensional Hankel tensor is NP-hard [12].

Given a positive number $s \in [n]$ and an mth order n-dimensional real tensor $\mathcal{A}$. An mth order s-dimensional tensor B is called a *principal s-subtensor* of $\mathcal{A}$, if all elements of B are indexed within a subset $S \subseteq [n]$. Let $\mathcal{A}$ be an mth order n-dimensional Hankel tensor written in form (1.2). The *s-induced tensor* (where $s \in [m]$) $\mathcal{B} \equiv A_s := (B_{i_1 i_2 \ldots i_s})$ is defined as an s-order n-dimensional Hankel (and thus symmetric) tensor

$$\mathcal{B} = \sum_{k=0}^{r} \lambda_k \mathbf{u}_k^s. \tag{2.7}$$

The following lemma shows that the induced tensor is independent of any specific VD (1.2), and thus is well-defined.

Lemma 2.3. *Let $\mathcal{A}$ be an mth order n-dimensional Hankel tensor with VDs*

$$\mathcal{A} = \sum_{j=1}^{r_1} \lambda_j \mathbf{u}_j^m \quad \text{and} \quad \mathcal{A} = \sum_{j=1}^{r_2} \mu_j \mathbf{v}_j^m \tag{2.8}$$

where r_1, r_2 are two positive integers, and $\mathbf{u}_j, \mathbf{v}_j \in \mathbb{R}^n$ are described by

$$\mathbf{u}_j = (1, w_j, w_j^2, \ldots, w_j^{n-1})^T, \qquad \mathbf{v}_j = (1, z_j, z_j^2, \ldots, z_j^{n-1})^T,$$

where w_j, z_j are nonzero real numbers for all j. Then for any $s \in [m]$ we have

$$\sum_{j=1}^{r_1} \lambda_j \mathbf{u}_j^s = \sum_{j=1}^{r_2} \mu_j \mathbf{v}_j^s. \tag{2.9}$$

Proof. We denote the tensor on the left hand side of (2.9) by $\mathcal{B} = (B_{i_1 \ldots i_s})$ and the tensor on the right hand side by $\mathcal{C} = (C_{i_1 \ldots i_s})$. For any $\sigma = (i_1, i_2, \ldots, i_s) \in S(k; s, n)$, we denote

$$\tau = (i_1, i_2, \ldots, i_s, \overbrace{1, \ldots, 1}^{m-s}) \in S(m, n).$$

It follows that $\tau \in S(k; m, n)$, and

$$A_\tau = \sum_{j=1}^{r_1} \lambda_j w_j^m = B_\sigma \tag{2.10}$$

by the first equality of (2.8). This fact can be deduced as follows,

$$A_\tau = (\sum_{j=1}^{r_1} \lambda_j \mathbf{u}_j^s)_{i_1 \dots i_s 1 \dots 1} \tag{2.11}$$

$$= \sum_{j=1}^{r_1} \lambda_j u_j^{(i_1)} \dots u_j^{(i_s)} u_j^{(1)} \dots u_j^{(1)} \tag{2.12}$$

$$= \sum_{j=1}^{r_1} \lambda_j w_j^{i_1+\dots+i_s} \tag{2.13}$$

$$= \sum_{j=1}^{r_1} \lambda_j w_j^k = B_\sigma \tag{2.14}$$

where the kth entry of $\mathbf{u}_j$ is $u_j^{(k)} = w_j^{k-1}$ by the assumption (note that $u_j^{(1)} = w_j^0 = 1$). Thus we have (2.10). Similarly, by the second equality of (2.8), we get

$$A_\tau = \sum_{j=1}^{r_2} \mu_j z_j^m = C_\sigma. \tag{2.15}$$

Thus $\mathcal{B}_\sigma = C_\sigma$ for all $\sigma \in S(k; s, n)$. Consequently $\mathcal{B} = \mathcal{C}$. □

A symmetric tensor can always be written in form (1.2) where each nonzero vector $\mathbf{u}_j \in \mathbb{R}^n$ has no other restriction. It seems that the definition of the induced tensor can be generalized analogously to a general symmetric tensor. Unfortunately the independence of rank-one decompositions for a general symmetric tensor cannot be verified so easily as in the case of a Hankel tensor.

3. Copositivity of Vandermonde tensors

Given an mth order n-dimensional Hankel tensor $\mathcal{A}$ associated with vector $\mathbf{v} = (v_0, v_1, \dots, v_N)$. $\mathcal{A}$ can also be associated with the *plane tensor* $\mathcal{P} = (P_{i_1 i_2 \dots i_N})$ (we call it briefly the *AP-tensor* of $\mathcal{A}$) which is an Nth order 2-dimensional Hankel tensor defined [12] as

$$P_{i_1 i_2 \dots i_N} = \frac{s(k; m, n) v_k}{\binom{N}{k}}, \qquad \sigma := (i_1, i_2, \dots, i_N) \in S(k; N, 2) \tag{3.1}$$

where $s(k; m, n) = |S(k; m, n)|$. The following two lemmas are shown in [12].

Lemma 3.1. *Let the mth order n-dimensional Hankel tensor $\mathcal{A}$ associated with a real vector $\mathbf{v} = (v_0, v_1, \dots, v_N)$ be copositive. Then $v_k \geq 0$ if k is a multiple of m.*

Lemma 3.2. *Let $\mathcal{A}$ be an mth order n-dimensional Hankel tensor and copositive. Then the AP-tensor $\mathcal{P}$ of $\mathcal{A}$ is also copositive.*

Unfortunately this condition ($\mathcal{P}$ is copositive) is not sufficient for $\mathcal{A}$ to be copositive. Actually the determining of the copositivity of an even matrix is a difficult problem.

It is natural to construct a symmetric tensor $\mathcal{S} = (S_{i_1 i_2 \ldots i_m})$ from any given mth order n-dimensional real tensor $\mathcal{A}$ by the following formula:

$$S_{i_1 i_2 \ldots i_m} = \frac{1}{m!} \sum_{\tau \in Sym_m} A_{i_{\tau(1)} i_{\tau(2)} \ldots i_{\tau(m)}}$$

where τ runs through the symmetric group Sym_m on set $[m]$. Similarly, we can also generate a Hankel tensor $\mathcal{H}_{\mathcal{A}}$ from an mth order n-dimensional real tensor $\mathcal{A}$ by

$$H_{i_1 i_2 \ldots i_m} = \frac{1}{s_k} \sum_{\sigma \in S(k;m,n)} A_{j_1 j_2 \ldots j_m}, \qquad \sigma = (j_1, j_2, \ldots, j_m).$$

We call $\mathcal{H}_{\mathcal{A}}$ the Hankel tensor generated from $\mathcal{A}$. The AP tensor of $\mathcal{H}_{\mathcal{A}}$ is called the Associated Plane tensor of $\mathcal{A}$, or the AP tensor of $\mathcal{A}$, and is denoted by $\mathcal{P}_{\mathcal{A}}$. We ask a more general question than Lemma 3.2: Let $\mathcal{A}$ be a copositive mth order n-dimensional symmetric tensor. Is the AP-tensor $\mathcal{P}$ of $\mathcal{A}$ also copositive?

For any given vector $\mathbf{x} \in \mathbb{R}^n$, we define $\mathcal{H}_{\mathbf{x}}$ to be the mth order n-dimensional Hankel tensor $\mathcal{H}_{\mathbf{x}^m}$ generated by $\mathbf{x}^m$. Denote

$$\mathcal{G}_{m,n} := \{\mathcal{H}_{\mathbf{x}} \colon \mathbf{x} \in \mathbb{R}^n\}.$$

It is obvious that each element $\mathcal{H}_{\mathbf{x}}$ in $\mathcal{G}_{m,n}$ is a copositive Hankel tensor since it is nonnegative. Note also that a rank-one tensor $\mathbf{x}^m$ can generate a Hankel tensor whose H-rank might be very large, but no larger than $N = (n-1)m$.

Denote by $\mathbf{x}_\sigma$ the entry $x_{i_1} x_{i_2} \ldots x_{i_m}$ of the rank-one tensor $\mathbf{x}^m$ where $\sigma = (i_1, i_2, \ldots, i_m)$. It is easy to see that

Theorem 3.3. *Let $\mathbf{x} \in \mathbb{R}^n$. Then $\mathcal{H}_{\mathbf{x}} = \mathbf{x}^m$ if and only if $\mathbf{x}^m$ is a Hankel tensor, if and only if $\mathbf{x}$ is a multiple of V-vector.*

Proof. The second part of the theorem is directly from Lemma 2.1. Now we prove the first part. If $\mathcal{H}_{\mathbf{x}} = \mathbf{x}^m$, then $\mathbf{x}^m$ is Hankel since $\mathcal{H}_{\mathbf{x}}$ is a Hankel tensor. By Lemma 2.1, $\mathbf{x}$ is a multiple of V-vector. Conversely, if $\mathbf{x}^m$ is Hankel, then $\mathbf{x}$ is a V-tensor, thus we may write $\mathbf{x} = \lambda(1, x, x^2, \ldots, x^{n-1})^T$. Let $\mathcal{H}_{\mathbf{x}} = (H_{i_1 i_2 \ldots i_m})$. Then we have

$$H_\sigma = \frac{1}{s_k} \sum_{\sigma \in S(k)} \mathbf{x}_\sigma = \frac{1}{s_k} \sum_{\sigma \in S(k)} x^k = x^k.$$

It follows that $\mathcal{H}_{\mathbf{x}} = \mathbf{x}^m$. $\square$

For simplicity, we use s_k to replace $s(k;m,n)$ and $S(k)$ to replace $S(k;m,n)$ when m,n are clear from the context. The inner product of two mth order n-dimensional real tensors $\mathcal{A},\mathcal{B}$, denoted $\langle\mathcal{A},\mathcal{B}\rangle$, is defined as

$$\langle\mathcal{A},\mathcal{B}\rangle = \sum_{\sigma\in S(m,n)} A_\sigma B_\sigma. \tag{3.2}$$

Define the Kronecker product of two tensors $\mathcal{A},\mathcal{B}\in\mathcal{T}_{m,n}$ as a tensor $\mathcal{C} := \mathcal{A}\circ\mathcal{B} = (C_\sigma)$ where $C_\sigma = A_\sigma B_\sigma$ for each $\sigma\in S(m,n)$. Then we have

$$\langle\mathcal{A},\mathcal{B}\rangle = (\mathcal{A}\circ\mathcal{B})\mathbf{e}^m, \quad \mathbf{e} = (\overbrace{1,1,\dots,1}^{n})^T\in\mathbb{R}^n.$$

Theorem 3.4. *Let $\mathcal{A}$ be an mth order n dimensional copositive tensor. Then $\mathcal{H}_\mathcal{A}$ is copositive if and only if $\langle\mathcal{A},\mathcal{H}_\mathbf{x}\rangle\geq 0$ for all $\mathbf{x}\in R^n_+$.*

Proof. We denote $\mathcal{H}_\mathcal{A} = (H_\sigma)$. For any given $\mathbf{x} = (x_1,x_2,\dots,x_n)^T\in R^n_+$, we denote $\mathcal{X} = (X_\sigma) = \mathcal{H}_\mathbf{x}$, i.e., the Hankel tensor generated from $\mathbf{x}$, that is, by the definition,

$$X_\sigma = \frac{1}{s_k}\sum_{\sigma\in S(k)} x_\sigma, \quad \forall\sigma\in S(k).$$

Then we have

$$\begin{aligned}
\mathcal{H}_\mathcal{A}\mathbf{x}^m &= \sum_{\sigma\in S(m,n)} H_\sigma x_\sigma\\
&= \sum_{k=0}^{N}\sum_{\tau\in S(k)} H_\tau x_\tau\\
&= \sum_{k=0}^{N}\Big(\frac{1}{s_k}\sum_{\sigma\in S(k)} A_\sigma\Big)\Big(\sum_{\sigma\in S(k)} x_\sigma\Big)\\
&= \sum_{k=0}^{N}\sum_{\tau\in S(k)} A_\tau\Big(\frac{1}{s_k}\sum_{\sigma\in S(k)} x_\sigma\Big)\\
&= \sum_{k=0}^{N}\sum_{\tau\in S(k)} A_\tau X_\tau\\
&= \langle\mathcal{A},\mathcal{X}\rangle.
\end{aligned}$$

The result follows by the definition of copositive tensors. $\square$

Let $\mathcal{A}$ be a rank-r Vandermonde tensor. Then $\mathcal{A}$ can be written in form (1.2). We denote $U = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r]$. Then U is an $n \times r$ Vandermonde matrix. For m to be an even number, we can see that $\mathcal{A}$ is positive semidefinite since

$$F_{\mathcal{A}}(\mathbf{x}) = \mathcal{A}\mathbf{x}^m = \langle \mathcal{A}, \mathbf{x}^m \rangle = \sum_{j=1}^{r} (\mathbf{u}^T \mathbf{x})^m \geq 0, \quad \forall \mathbf{x} \in \mathbb{R}^n.$$

Thus $\mathcal{A}$ is copositive when m is even. But the situation becomes much more complicated when m is an odd number.

The following theorem presents a necessary and sufficient condition for an odd order symmetric tensor of rank-2 to be copositive (note that a third order tensor cannot be positive semidefinite unless it is zero).

Theorem 3.5. *Let m be an odd number and $\mathcal{A} = \mathbf{u}^m + \mathbf{v}^m$ be an mth order n dimensional real tensor. Then $\mathcal{A}$ is copositive if and only if*

$$\mathbf{u} + \mathbf{v} \geq 0. \tag{3.3}$$

Proof. We write $m = 2k + 1$. For any vector $x \in \mathbb{R}^n$, we denote

$$F_m(\mathbf{x}) := \mathcal{A}x^m = (\mathbf{u}^T \mathbf{x})^m + (\mathbf{v}^T \mathbf{x})^m$$

and $\alpha_x = \mathbf{u}^T \mathbf{x}, \beta_x = \mathbf{v}^T \mathbf{x}$. The condition (3.3) is equivalent to

$$\alpha_x + \beta_x \geq 0, \quad \forall \mathbf{x} \in R^n_+. \tag{3.4}$$

We want to prove that (3.4) holds if $\mathcal{A} = \mathbf{u}^{2k+1} + \mathbf{v}^{2k+1}$ is copositive. Suppose that $\mathcal{A} = \mathbf{u}^{2k+1} + \mathbf{v}^{2k+1}$ is copositive. Then we have

$$\mathcal{A}\mathbf{x}^{2k+1} = \alpha_x^{2k+1} + \beta_x^{2k+1} \geq 0, \quad \forall x \in R^n_+. \tag{3.5}$$

We consider three cases:

Case 1. $\alpha_x \beta_x > 0$. Then $\alpha_x > 0, \beta_x > 0$. Thus (3.4) holds.

Case 2. $\alpha_x \beta_x = 0$. Then $\alpha_x = 0$, which implies $\beta_x \geq 0$ when combining with (3.5); or $\beta_x = 0$, which implies $\alpha_x \geq 0$. Thus we have (3.4).

Case 3. $\alpha_x \beta_x < 0$. Then $\alpha_x > 0 > \beta_x$ or $\beta_x > 0 > \alpha_x$. Either of those, when combined with (3.5), yields (3.4).

Conversely we suppose $\alpha_x + \beta_x \geq 0$ for all $x \in R^n_+$. We use the induction to k to prove that $\tilde{\mathcal{A}}_k$ is copositive where $\tilde{\mathcal{A}}_k := \mathbf{u}^{2k+1} + \mathbf{v}^{2k+1}$.

For $k = 0$, $m = 1$ and $\tilde{\mathcal{A}}_0 = \mathbf{u}+\mathbf{v}$. The result is obvious since $\tilde{\mathcal{A}}_0\mathbf{x} = \alpha_x+\beta_x \geq 0$ for all nonnegative x. Thus $\tilde{\mathcal{A}}_0$ is copositive. Now suppose this assertion is true for $k-1$, that is, tensor $\tilde{\mathcal{A}}_{k-1} = \mathbf{u}^{2k-1} + \mathbf{v}^{2k-1}$ is copositive. We want to prove that $\tilde{\mathcal{A}}_k = \mathbf{u}^{2k+1} + \mathbf{v}^{2k+1}$ is copositive. Denote $f_k(x) = \alpha_x^{2k+1} + \beta_x^{2k+1}$ for each x. Then we need to prove that

$f_k(x) \geq 0$ for all $x \in R^n_+$. For any $x \in R^n_+$, we consider the sign of $\alpha_x\beta_x$. If $\alpha_x\beta_x \geq 0$, then $\lambda_x \geq 0, \beta_x \geq 0$, due to (3.4). Thus we have $f_k(x) = \alpha_x^{2k+1} + \beta_x^{2k+1} \geq 0$. Otherwise, $\alpha_x\beta_x < 0$. Since

$$f_k(x) = (\alpha_x + \beta_x)(\alpha_x^{2k} + \beta_x^{2k}) - \alpha_x\beta_x(\alpha_x^{2k-1} + \beta_x^{2k-1}) \tag{3.6}$$

we have

$$f_k(x) = f_0(x)F_{2k}(x) - \alpha_x\beta_x f_{k-1}(x).$$

Noting that $f_0(x) \geq 0$, $F_{2k}(x) \geq 0$ and $f_{k-1}(x) \geq 0$ due to the hypothesis, we obtain $f_k(x) \geq 0$ for all $x \in R^n_+$. The proof is completed. □

By Theorem 3.5 and its proof, we can easily obtain

Corollary 3.6. *Let m be an odd number and $\mathcal{A} = \mathbf{u}^m + \mathbf{v}^m \in \mathcal{S}_{m,n}$ be a rank-2 symmetric real tensor. Then $\mathcal{A}$ is copositive if and only if the induced s-tensor $\mathcal{A}_s$ of $\mathcal{A}$ is copositive for some odd number $s \leq m$, if and only if all its induced s-tensors are copositive.*

One may ask the question:

Problem 3.7. *Can we extend the result in Theorem 3.5 to a more general case?*

For a rank-3 Vandermonde tensor, it is natural to seek an analogous condition to (3.3) to guarantee the copositivity. Unfortunately, this is not true as displayed by the following example.

Example 3.8. Let $\mathcal{A} = \mathbf{u}^3 + \mathbf{v}^3 + \mathbf{w}^3$, where

$$\mathbf{u} = (2, 1, 1)^T, \qquad \mathbf{v} = (0, -2, -3)^T, \qquad \mathbf{w} = (-1, 1, 3)^T.$$

Then $\mathcal{A}$ is a 3-order 3-dimensional V-tensor. Now we choose vector $\mathbf{x} = (1, 1, 1)^T \in R^3_+$. Then

$$\begin{aligned}\mathcal{A}\mathbf{x}^3 &= (\mathbf{u}^T\mathbf{x})^3 + (\mathbf{v}^T\mathbf{x})^3 + (\mathbf{w}^T\mathbf{x})^3 \\ &= 64 - 125 + 27 = -34 < 0. \end{aligned} \tag{3.7}$$

Thus $\mathcal{A}$ is not a copositive tensor. But $\mathbf{u} + \mathbf{v} + \mathbf{w} = (1, 0, 1)^T \geq 0$.

In order to present a necessary condition for a Vandermonde tensor to be copositive, we denote

$$A_v = \begin{bmatrix} v_0 & v_0 & \cdots & \cdots & v_0 \\ v_1 & v_2 & \cdots & \cdots & v_{n-1} \\ v_2 & v_4 & \cdots & \cdots & v_{2(n-1)} \\ \vdots & \vdots & & & \vdots \\ v_{m-1} & v_{2(m-1)} & \cdots & \cdots & v_{(n-1)(m-1)} \\ v_m & v_{2m} & \cdots & \cdots & v_{(n-1)m} \end{bmatrix} \tag{3.8}$$

and

$$H = \begin{bmatrix} & & & & \binom{m}{0} \\ & & \cdots & \binom{m}{1} & \\ & & \cdot & & \\ & \cdot & & & \\ \cdot & & & & \\ \binom{m}{m} & & & & \end{bmatrix}. \tag{3.9}$$

Here A is an $(m+1) \times (n-1)$ matrix and H is an $(m+1) \times (m+1)$ matrix. We note that Lemma 3.1 is equivalent to saying that the last row of A is nonnegative. But this condition is far from being sufficient for $\mathcal{A}$ to be copositive.

The following result is another necessary condition for a Vandermonde tensor to be copositive.

Theorem 3.9. *Let an mth order n-dimensional Vandermonde tensor $\mathcal{A}$ be copositive which is associated with a vector $\mathbf{v} = (v_0, v_1, \ldots, v_N)$. Then $\mathcal{A}$ is copositive only if*

$$A_v^T H A_v \geq 0 \tag{3.10}$$

where A_v and H are defined respectively by (3.8) and (3.9).

Proof. Since $\mathcal{A}$ is copositive, we have $F_{\mathcal{A}}(\mathbf{x}) = \mathcal{A}\mathbf{x}^m \geq 0$ for all $\mathbf{x} \in R_+^n$. For any pair (i,j): $i,j \in [n]$, $i < j$, we take $\mathbf{x}$ to be $\mathbf{v}_{i,j} = e_i + e_j$ where e_i is the ith coordinate vector whose unique nonzero entry (equals 1) is its ith entry. Then we have $F_{\mathcal{A}}(\mathbf{v}_{i,j}) \geq 0$. Denote $f(i,j) \equiv F_{\mathcal{A}}(\mathbf{v}_{i,j})$. Then by (2.5) we have

$$\begin{aligned} f(i+1,j+1) &= \sum_{s=0}^{m} \binom{m}{s} \sum_{k=1}^{r} \omega_k^{si+(m-s)j} \\ &= \sum_{s=0}^{m} \binom{m}{s} \sum_{k=1}^{r} \omega_k^{si} \omega_k^{(m-s)j} \\ &= \sum_{s=0}^{m} \binom{m}{s} v_{si} v_{(m-s)j} \end{aligned}$$

where $i,j = 0,1,\ldots,n-1$, $i < j$. It follows that $f(i,j) = (A_v^T H A_v)_{i,j}$ for all $1 \leq i < j \leq n$, and for $i = j \in \{0,1,2,\ldots,n-1\}$, we have

$$\begin{aligned}(A_v^T H A_v)_{i,i} &= \sum_{s=0}^{m} \binom{m}{s} v_{si} v_{(m-s)i} \\ &= \sum_{s=0}^{m} \binom{m}{s} v_{mi} \\ &= 2^m v_{mi} \geq 0.\end{aligned}$$

The last inequality is due to Lemma 3.1. Thus we have $A_v^T H A_v \geq 0$. □

4. The singularity of Vandermonde tensors

The singularity of a cubic tensor is introduced through its eigenvalues. In matrix theory, we call an $n \times n$ matrix A to be singular if $rank(A) < n$, or equivalently, $det(A) = 0$, or A is not invertible. Unfortunately the rank of a high order tensor is too complicated to calculate, the determinant of a high order tensor is also very intriguing even to be defined (not least to say the computation), and the definition of the inverse of a tensor is still a disputable issue. The singularity of a square matrix can also be checked by its spectrum, that is, A is singular if and only if A has zero as one of its eigenvalues. This also applies to the high order tensors due to Hu [7] and the recent work by Shao [14] where it is shown that the determinant of a cubic tensor can be written as a product of some powers of eigenvalues of $\mathcal{A}$. There are many definitions of singularities of tensors. From Qi [10], to Cartwright and Sturmfels [1], to Hu and Qi [6] in which the singularity of a tensor is defined as the vanishing of the determinant.

For our purpose, let us review some terminology related to the spectrum of a tensor. Let $\mathcal{A}$ be an mth order n-dimensional symmetric tensor and $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ be a nonzero vector of dimension n (complex or real). A scalar $\lambda \in \mathcal{C}$ is called an *eigenvalue* of $\mathcal{A}$ corresponding to $\mathbf{x}$ if it satisfies

$$\mathcal{A}\mathbf{x}^{m-1} = \lambda \mathbf{x}^{[m-1]} \tag{4.1}$$

where $\mathcal{A}\mathbf{x}^{m-1}$ is defined as an n-dimensional vector $y = \mathcal{A}\mathbf{x}^{m-1} = (y_1, y_2, \ldots, y_n)^T$ with

$$y_i = \sum_{i_2,\ldots,i_m=1}^{n} A_{ii_2\ldots i_m} x_{i_2} x_{i_3} \ldots x_{i_m}, \quad i = 1, 2, \ldots, n,$$

and $\mathbf{x}^{[k]} := (x_1^k, x_2^k, \ldots, x_n^k)^T$. Such a vector $\mathbf{x}$ is called an eigenvector of $\mathcal{A}$ associated with λ, and $(\lambda, \mathbf{x})$ is called an eigen-pair of $\mathcal{A}$. They are respectively called an H-eigenvalue and an H-eigenvector of $\mathcal{A}$ if both λ and $\mathbf{x}$ are real ($(\lambda, \mathbf{x})$ is called an H-eigen-pair of $\mathcal{A}$). If (4.1) is replaced by $\mathcal{A}\mathbf{x}^{m-1} = \lambda\mathbf{x}$ for some $\lambda \in \mathcal{C}$ and $\mathbf{x} \in \mathcal{C}^n$ satisfying $\mathbf{x}^T\mathbf{x} = 1$, then λ is called an E-eigenvalue and $\mathbf{x}$ is called an E-eigenvector associated with λ, and $(\lambda, \mathbf{x})$ is called an E-eigen-pair of $\mathcal{A}$. If both λ and $\mathbf{x}$ are real, then they are called a Z-eigenvalue and a Z-eigenvector of A, respectively.

These spectrum of tensors are introduced by Qi [10] and Lim [8] almost simultaneously. We say that an mth order n-dimensional real tensor A is *singular* if it has zero as one of its eigenvalues. Thus the existence of a zero eigenvalue for A can be used to determine its singularity. We have

Theorem 4.1. *Let $m \geq 2$ and $n > 1$ and $\mathcal{A} \in \mathbf{V}_{m;n}$ be a V-tensor. Then:*

(1) *If $m > 0$ is an even integer, then $\mathcal{A}$ is singular if and only if Vrank$(\mathcal{A}) < n$.*
(2) *If $m > 0$ is an odd integer, then $\mathcal{A}$ is singular if Vrank$(\mathcal{A}) < n$.*

Proof. Let $Vrank(A) = r$. We recall that r is the minimal number of summands in the VD (2.5) with

$$\mathbf{u}_j = (1, \omega_j, \omega_j^2, \dots, \omega_j^{n-1})^T, \quad j = 1, 2, \dots, r,$$

where $\omega_1, \omega_2, \dots, \omega_r$ are required to be distinct.

(1) m is an even number. First we prove the necessity. Suppose that $\mathcal{A}$ is singular, that is, $\mathcal{A}$ has zero as one of its eigenvalues. We want to prove that $r < n$. Denote $U = [\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_r]$, then $U \in \mathbb{R}^{n\times r}$. Let $\mathbf{x} = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$ be an H-eigenvector vector corresponding to $\lambda = 0$, that is, it satisfies $\mathcal{A}\mathbf{x}^{m-1} = 0$. Then we have $F_{\mathcal{A}}(\mathbf{x}) = \mathcal{A}\mathbf{x}^m = \mathbf{x}^T\mathcal{A}\mathbf{x}^{m-1} = 0$. Denote $f_j(\mathbf{x}) := (\mathbf{u}_j^T\mathbf{x})^m$. Then we have $f_j(\mathbf{x}) \geq 0$ for each j since m is even and both $\mathbf{x}$ and $\mathbf{u}_j$ are real vectors. Thus $f_j(\mathbf{x}) = 0$ for $j = 1, 2, \dots, r$. It follows that $\mathbf{x}^T U = 0$. Thus we have $r = rank(U) < n$. For the sufficiency, we let $r < n$. Then there exists a nonzero vector $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{x}^T U = 0$ since $Vrank(\mathcal{A}) = r < n$, that is, $\mathbf{u}_j^T\mathbf{x} = 0$, $j = 1, 2, \dots, r$. It follows that

$$\begin{aligned}\mathbf{y} = \mathcal{A}\mathbf{x}^{m-1} &= (\sum_{j=1}^{r} \mathbf{u}_j^m)\mathbf{x}^{m-1} \\ &= \sum_{j=1}^{r} (\mathbf{u}_j^T\mathbf{x})^{m-1}\mathbf{u}_j \\ &= 0.\end{aligned}$$

Thus $(0, \mathbf{x})$ is an H-eigen-pair of $\mathcal{A}$. Consequently $\mathcal{A}$ is singular.

(2) m is an odd number. The singularity of $\mathcal{A}$ when m is odd can be verified by employing the similar technique we have used in the above argument. $\square$

Given an mth order n-dimensional real tensor $\mathcal{A}$. It is easy to see that the set of all the vectors satisfying $\mathcal{A}\mathbf{x}^{m-1} = 0$, denoted by $\mathcal{N}(\mathcal{A})$, forms a subspace of $\mathbb{R}^n$. We define the *degree of the singularity* of tensor $\mathcal{A}$ to be the dimension of $\mathcal{N}(\mathcal{A})$, and is denoted by $n_0(\mathcal{A})$. The following properties concerning $n_0(\mathcal{A})$ can be proved by using linear algebra theory.

Corollary 4.2. *Let $\mathcal{A}$ be an mth order n-dimensional Vandermonde tensor with $r = Vrank(\mathcal{A})$. Then*

(1) $n_0(\mathcal{A}) = n - r$ *if* $r < n$.
(2) $n_0(\mathcal{A}) = 0$ *if* $r \geq n$.

Recall that a Hankel tensor is called a *complete Hankel tensor* [12] if it has a positive VD, that is, all $\lambda_j \geq 0$ in (1.2). Thus a V-tensor is a complete Hankel tensor. An even order complete Hankel tensor is shown to be positive semi-definite (Theorem 3 in [12]). Similar to Proposition 3 of [12], we find that the Hadamard product of two V-tensors is also a V-tensor. For a V-tensor with $r = Vrank(\mathcal{A}) < n$, Theorem 4.1 answers Question 9 in [12] in the V-tensor case. Thus by Proposition 4 of [12], all the H-eigenvalues of $\mathcal{A}$ are nonnegative if $\mathcal{A}$ is a V-tensor with $r = Vrank(\mathcal{A}) < n$ and if m is odd.

Acknowledgements

The author wants to express his gratitude to Prof. Liqun Qi for his support and his helpful comments and suggestions on Hankel tensors. Thanks are also given to Prof. Bernd Sturmfels whose remarks during the 19th ILAS conference are helpful for the finalization of this paper. The author also wants to thank Prof. Jiayu Shao who constantly encourages the author to pay attention to the spectra of tensors. Thanks shall be also given to the referee(s) and the editor whose suggestions, remarks and critics are crucial for the improvement of the paper. The referee's remarks also lead the author to noticing the latest work on the singularities and the eigenvalues of tensors in [1] and [6].

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