# The Laplace Transform

# 6. The Laplace Transform

**Definition**: Let $x(t)$ be a continuous signal. The **forward bilateral Laplace transform** is defined as:

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt, \quad where \ s = \sigma + j\omega \in \mathbb{C}$$

which is denoted as: $X(s) = \mathcal{L}_{\mathcal{B}}\{x(t)\}$

- $X(s)$ exists if the following condition is satisfied: $\int_{-\infty}^{\infty} |x(t)| e^{-\sigma t} dt < \infty$

The **inverse bilateral Laplace transform** is defined as:

$$x(t) = \mathcal{L}_{\mathcal{B}}^{-1}\{X(s)\} = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds$$

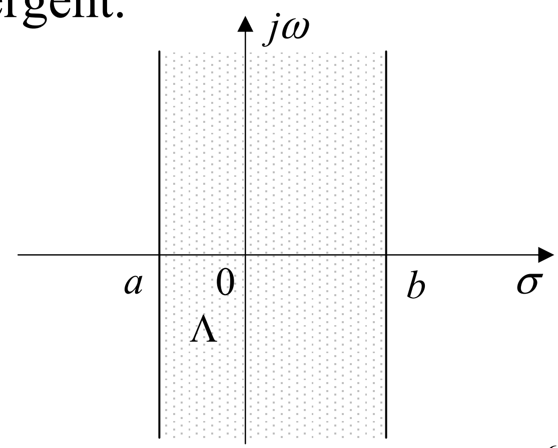
where the σ is chosen such that the integral above is convergent.

- It can be shown that the bilateral Laplace transform is convergent if there exist the constants $A$, $a$, $b$, such that:

$$|x(t)| \le \begin{cases} Ae^{at}, & for \quad t < 0 \\ Ae^{bt}, & for \quad t > 0 \end{cases}, \quad with \quad a < b$$

This is equivalent to the convergence condition:

$$a < \sigma < b$$



# 6. The Laplace Transform

**Definition**: Let $x(t)$ be a continuous signal. The **forward unilateral Laplace transform** is defined as:

$$X(s) = \int_0^{\infty} x(t) e^{-st} dt, \quad where \ s = \sigma + j\omega \in \mathbb{C}$$

which is denoted as: $X(s) = \mathcal{L}\{x(t)\}$

- $X(s)$ exists if the following condition is satisfied: $\int_0^{\infty} |x(t)| e^{-\sigma t} dt < \infty$

The **inverse unilateral Laplace transform** is defined as:

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds$$

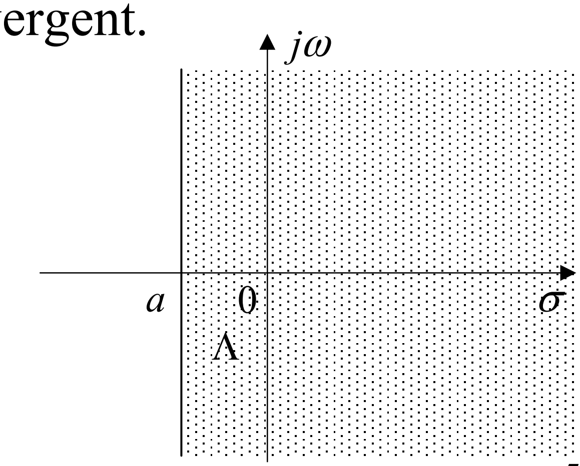
where the σ is chosen such that the integral above is convergent.

- It can be shown that the bilateral Laplace transform is convergent if there exist the constants $A$, α, such that:

$$|x(t)| \le A e^{\alpha t}$$

This is equivalent to the convergence condition:

$$\alpha < \sigma$$



# 6. The Laplace Transform

| Properties | Time | | (Unilateral) Laplace transform |
|---|---|---|---|
| - Linearity: | $\sum_k a_k x_k(t)$ | $\longleftrightarrow$ | $\sum_k a_k X_k(s)$ |
| - Derivation in time: | $\frac{dx(t)}{dt}$ | $\longleftrightarrow$ | $sX(s) - x(0)$ |
| - Integration in time: | $\int_0^t x(\tau)d\tau$ | $\longleftrightarrow$ | $\frac{X(s)}{s}$ |
| - Derivation in frequency: | $-tx(t)$ | $\longleftrightarrow$ | $\frac{dX(s)}{ds}$ |
| - Integration in frequency: | $\frac{x(t)}{t}$ | $\longleftrightarrow$ | $\int_0^\infty X(s)ds$ |
| - Scaling: | $x(at)$ | $\longleftrightarrow$ | $\frac{1}{a}X\left(\frac{s}{a}\right)$ |
| - Time shifting: | $x(t-t_o)$ | $\longleftrightarrow$ | $X(s)e^{-st_0}$ |
| - Frequency shifting: | $x(t)e^{s_0 t}$ | $\longleftrightarrow$ | $X(s-s_0)$ |

# 6. The Laplace Transform

| **Properties** | **Time** | | **(Unilateral) Laplace transform** |
|---|---|---|---|
| - Convolution in time: | $x(t) * y(t)$ | $\longleftrightarrow$ | $X(s) \cdot Y(s)$ |
| - Convolution in frequency | $x(t) \cdot y(t)$ | $\longleftrightarrow$ | $\frac{1}{2\pi j} X(s) * Y(s)$ |

**Laplace transforms for some usual signals** (see also Appendix given in the notes)

$$\delta(t) \longleftrightarrow 1$$

$$u(t) \longleftrightarrow \frac{1}{s}$$

$$\frac{t^n}{n!} \longleftrightarrow \frac{1}{s^{n+1}}$$

$$e^{-at} \longleftrightarrow \frac{1}{s+a}$$

$$\sin(\omega t) \longleftrightarrow \frac{\omega}{s^2 + \omega^2}$$

$$\cos(\omega t) \longleftrightarrow \frac{s}{s^2 + \omega^2}$$

**Inverting given Laplace transforms -> the residue method**