The Z-Transform

7. The Z-Transform

Definition: Let $\{x[n]\}_{n\in Z}$ be a discrete signal. The **forward Z-transform** is defined as:

$$X(z)=\sum_{n=-\infty}^{\infty}x[n]z^{-n}$$

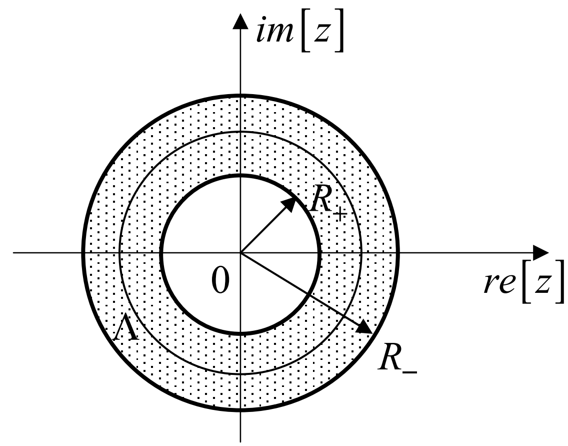
which is denoted as: $X(z)=\mathcal{Z}\{x[n]\}$

-The Z-transform is a Laurent series which is converging for some values of the variable z situated in domain of convergence defined by the bounds (see figure):

$$R_+\leq|z|\leq R_-$$

The Z-transform can be written as:

$$X(z)=\sum_{n=-\infty}^{0}x[n]z^{-n}+\sum_{n=0}^{\infty}x[n]z^{-n}=X_-(z)+X_+(z)$$



- It can be shown that:
    - $X_-(z)$ converges for $|z|\leq R_-$
    - $X_+(z)$ converges for $|z|\geq R_+$

7. The Z-Transform

Terminology:

- If the **signal is causal**, that is, if $x[n]=0, \forall n<0$, then the transform contains only negative powers of z. In this case the transform is called the **unilateral Z-transform**
- If the opposite case, the transform is called the **bilateral Z-transform**

- For causal signals which are in $l^2(\mathbb{Z})$, the Z-transform exists for $|z|> R_+$, and $R_+ \leq 1$

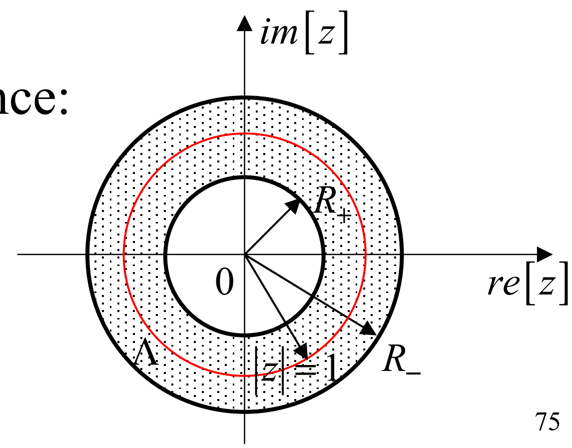
Inverse Z-transform:

$$x_n = \frac{1}{2\pi j} \oint_{\Lambda} X(z) z^{n-1} dz$$

- For usual signals, which are causal and in $l^2(\mathbb{Z})$, one can take Λ as being the unit-circle in the Z-domain, corresponding to: $|z|=1$

- In this case, in polar coordinates: $z=e^{j\theta}, dz = je^{j\theta} d\theta$. Hence:

$$x_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} X\left(e^{j\theta}\right) e^{jn\theta} d\theta$$



7. The Z-Transform

Properties:

- The properties below refer to the unilateral Z-transform, but they can be adapted to the bilateral Z-transform as well.

1. Linearity

Let $x[n] = Ax_1[n] + bx_2[n]$. The Z-transform is linear, that is:

$$X(z) = \mathcal{Z}\{x[n]\} = \mathcal{Z}\{Ax_1[n] + Bx_2[n]\} = AX_1(z) + BX_2(z)$$

- The convergence radius R for $X(z)$ is: $R = \max\left(R_{1,+}, R_{2,+}\right)$

2. Scaling

$$\mathcal{Z}\left\{a^n x[n]\right\} = \sum_{n=0}^{\infty} a^n x[n] z^{-n} = \sum_{n=0}^{\infty} x[n]\left(\frac{z}{a}\right)^{-n} = X\left(\frac{z}{a}\right)$$

3. Shifting in time

- positive shifts in time

$$\mathcal{Z}\{x[n-k]\}_{k>0} = \sum_{n=0}^{\infty} x[n-k] z^{-n} = z^{-k} \sum_{m=-k}^{\infty} x[m] z^{-m} = z^{-k} \sum_{m=0}^{\infty} x[m] z^{-m} = z^{-k} X(z)$$

7. The Z-Transform

Properties:

3. Delay

- negative shifts in time

$$\mathcal{Z}\{x[n-k]\}_{k<0} = \sum_{n=0}^{\infty} x[n-k]z^{-n} = z^{-k}\sum_{m=-k}^{\infty} x[m]z^{-m} = z^{-k}\left(\sum_{m=0}^{\infty} x[m]z^{-m} - \sum_{m=0}^{|k|} x[m]z^{-m}\right) =$$

$$= z^{-k}X(z) - z^{-k}\sum_{m=0}^{|k|} x[m]z^{-m}$$

4. Multiplication in time

$$\mathcal{Z}\{nx[n]\} = -z\frac{dX}{dz}$$

Proof:

$$\frac{dX}{dz} = -\sum_{n=0}^{\infty} nx[n]z^{-n-1} = -\frac{1}{z}\sum_{n=0}^{\infty} nx[n]z^{-n} = -\frac{1}{z}\mathcal{Z}\{nx[n]\}$$

5. Limits

$$x[0] = \lim_{z\to\infty} X(z)$$

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n} = x[0] + \sum_{n=1}^{\infty} x[n]z^{-n} \Leftrightarrow \lim_{z\to\infty} X(z) = x[0]$$

Properties:

6. Convolution of discrete signals

$$\mathcal{Z}\{x[n]*y[n]\}=X(z)\cdot Y(z)$$

Proof:

$$X(z)\cdot Y(z)=\left(\sum_{n=0}^{\infty}x[n]z^{-n}\right)\left(\sum_{m=0}^{\infty}y[m]z^{-m}\right)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}x[n]y[m]z^{-n-m}$$

$$n+m=p\Rightarrow n=p-m$$

$$X(z)\cdot Y(z)=\sum_{p=0}^{\infty}\sum_{m=0}^{\infty}x[p-m]y[m]z^{-p}=\sum_{p=0}^{\infty}\left(\sum_{m=0}^{\infty}x[p-m]y[m]\right)z^{-p}\qquad[*]$$

$$Let:\quad s[p]=\sum_{m=0}^{\infty}x[p-m]y[m]\overset{\Delta}{=}x[p]*y[p]$$

From [*] we write: $X(z)\cdot Y(z)=S(z)$.

In other words:

$$\mathcal{Z}\{x[n]*y[n]\}=X(z)\cdot Y(z)$$

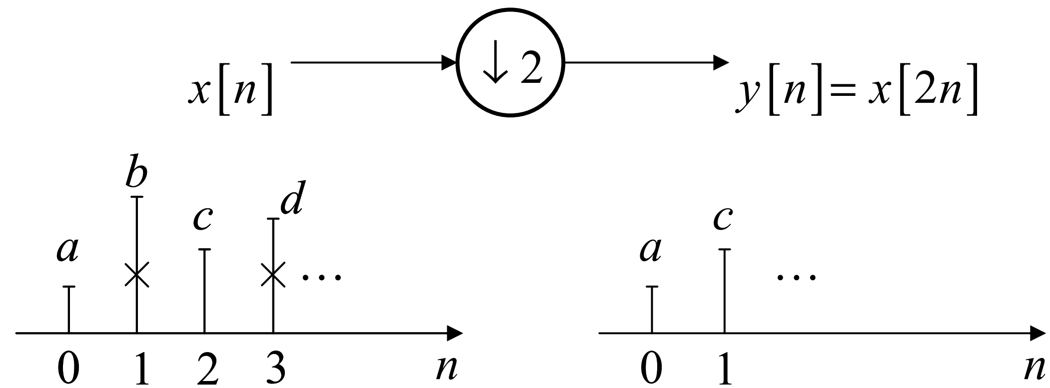
7. The Z-Transform

Properties:

6. Downsampling

Let: $y[n] = x[2n]$

In $\mathcal{Z}$: $Y(z) = \frac{1}{2}\left[X\left(z^{1/2}\right) + X\left(-z^{1/2}\right)\right]$



Proof:

Let: $X_e(z) = \sum_{n=0}^{\infty} x[2n] z^{-n}$ and $X_o(z) = \sum_{n=0}^{\infty} x[2n+1] z^{-n}$

We notice that: $Y(z) = \sum_{n=0}^{\infty} y[n] z^{-n} = X_e(z)$

Also: $X(z) = \sum_{n=0}^{\infty} x[n] z^{-n} = \sum_{m=0}^{\infty} x[2m] z^{-2m} + \sum_{m=0}^{\infty} x[2m+1] z^{-2m-1} \Leftrightarrow$

$$X(z) = X_e\left(z^2\right) + z^{-1} X_o\left(z^2\right) \qquad [*]$$

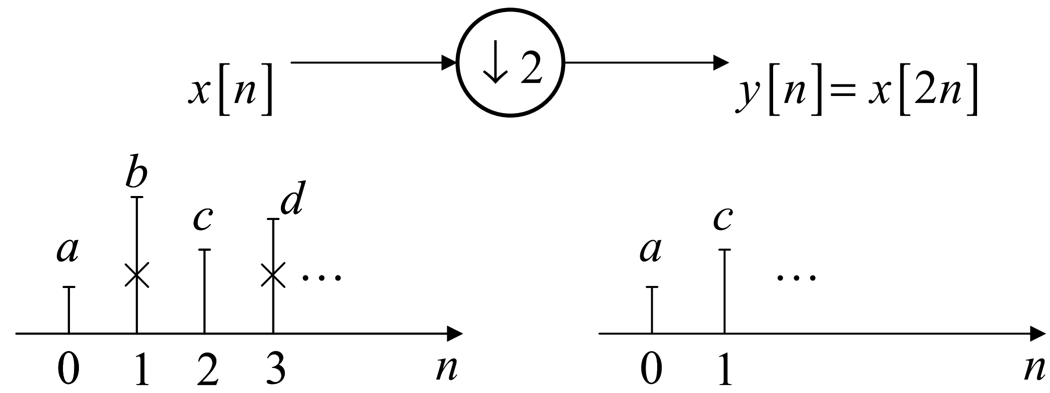
Replace $z \rightarrow -z$:

$$X(z) = X_e\left(z^2\right) - z^{-1} X_o\left(z^2\right) \qquad [**]$$

7. The Z-Transform

Properties:

6. Downsampling (contd.)



From [*] and [**] it results:

$$X_e\left(z^2\right)=\frac{1}{2}\left[X(z)+X(-z)\right]\Rightarrow X_e(z)=Y(z)=\frac{1}{2}\left[X\left(z^{1/2}\right)+X\left(-z^{1/2}\right)\right]$$

We also deduce that:

$$X_o\left(z^2\right)=\frac{z}{2}\left[X(z)-X(-z)\right]\Rightarrow X_o(z)=\frac{z^{1/2}}{2}\left[X\left(z^{1/2}\right)-X\left(-z^{1/2}\right)\right]$$

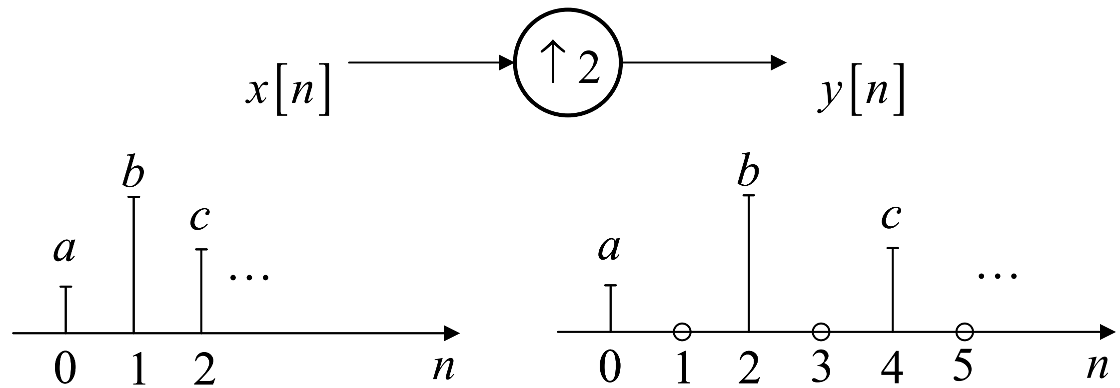
- $X_e(z), X_o(z)$ represent the Z-transforms of **the even and odd phases** of the input signal $x[n]$

Properties:

7. Upsampling

$$y[2n] = x[n]; \quad y[2n+1] = 0$$

In $\mathcal{Z}$: $Y(z) = X\left(z^2\right)$



Proof:

$$Y(z) = \sum_{n=0}^{\infty} y[n] z^{-n} = \sum_{m=0}^{\infty} y[2m] z^{-2m} + \sum_{m=0}^{\infty} y[2m+1] z^{-2m-1} =$$

$$= \sum_{m=0}^{\infty} x[m] z^{-2m} + 0 = X\left(z^2\right)$$

8. Z-transforms of a few classical functions

Transform:	Convergence Radius:
$\mathcal{Z}\{\delta[n]\}=1$	1
$\mathcal{Z}\{u[n]\}=\frac{z}{z-1}$	1
$\mathcal{Z}\{\delta_T[n]\}=\frac{z}{z-1}$	1
$\mathcal{Z}\{n\}=\frac{z}{(z-1)^2}$	1
$\mathcal{Z}\{a^n\}=\frac{z}{z-a}$	$\lvert a\rvert$
$\mathcal{Z}\{e^{-an}\}=\frac{z}{z-e^{-a}}$	e^{-a}
$\mathcal{Z}\{\sin\omega n\}=\frac{z\sin\omega}{z^2-2z\cos\omega+1}$	1
$\mathcal{Z}\{\cos\omega n\}=\frac{z(z-\cos\omega)}{z^2-2z\cos\omega+1}$	1

7. The Z-Transform

Poles and zeros

- Let $X(z)$ be a fraction of the form:

$$X\left(z\right)=\frac{A\left(z\right)}{B\left(z\right)}=\frac{\sum_{k=0}^{p}\alpha_k z^{-k}}{\sum_{k=0}^{q}\beta_k z^{-k}} \qquad [*]$$

where $A(z)$ and $B(z)$ are polynomials in z of orders p and q respectively.

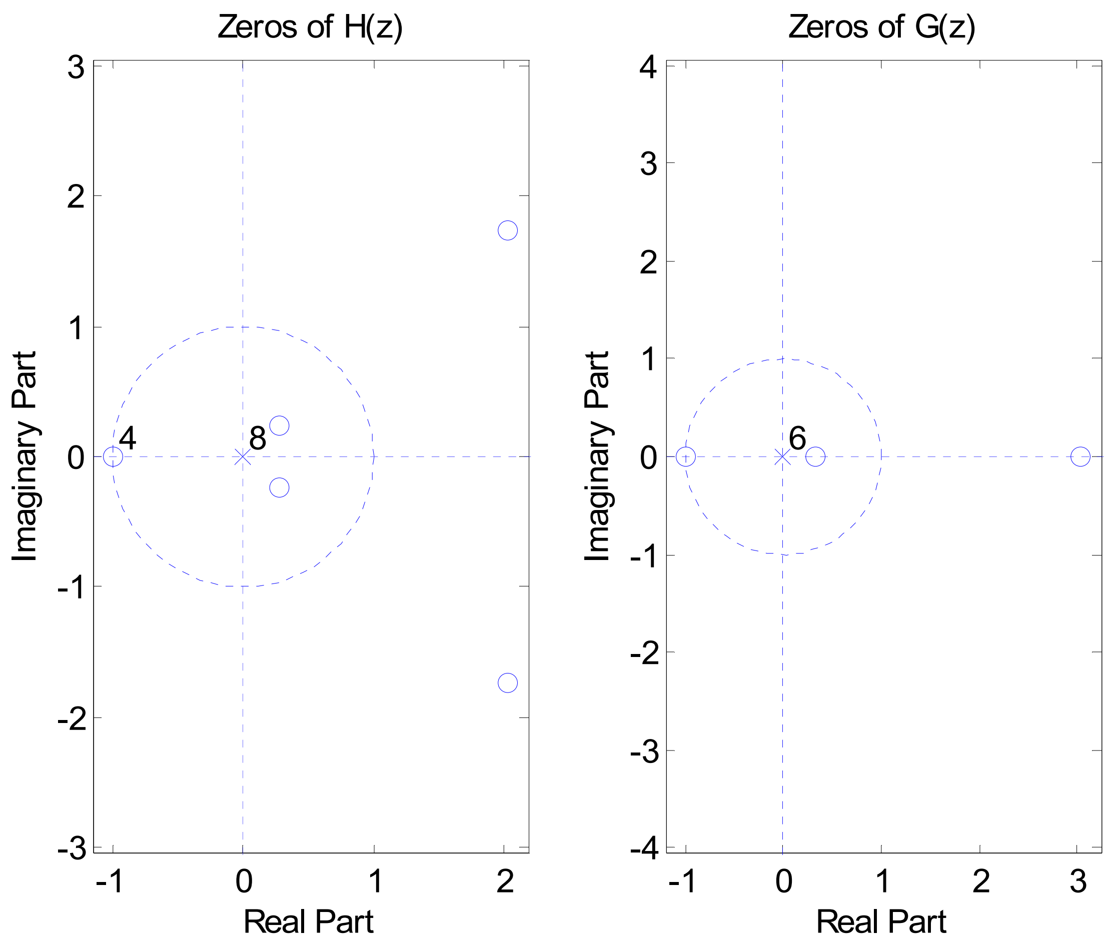
- The values z_p for which $A\left(z_p\right)=0$ are called **the zeros** of $X(z)$
- The values z_p for which $B\left(z_p\right)=0$ are called **the poles** of $X(z)$

- It can be shown that [*] corresponds to the Z-transform of a *causal signal* if $p \le q$
- Deriving the inverse Z-transform can be made using various methods. The most common is the factorization in simple fractions that correspond to known inverse transforms. Example:

$$X\left(z\right)=\frac{Az}{\left(z-a\right)}+\frac{Bz}{\left(z-a\right)^2} \Leftrightarrow x\left[n\right]=Aa^n+Bna^{n-1}$$

- in this example a is *a pole of order two*

Poles and zeros



Zeros of H(z)

2.0311 + 1.7390i

2.0311 - 1.7390i

-1.0001 + 0.0001i

-1.0001 - 0.0001i

-0.9999 + 0.0001i

-0.9999 - 0.0001i

0.2841 + 0.2432i

0.2841 - 0.2432i

Zeros of G(z)

3.0407

-1.0010 + 0.0010i

-1.0010 - 0.0010i

-0.9990 + 0.0010i

-0.9990 - 0.0010i

0.3289

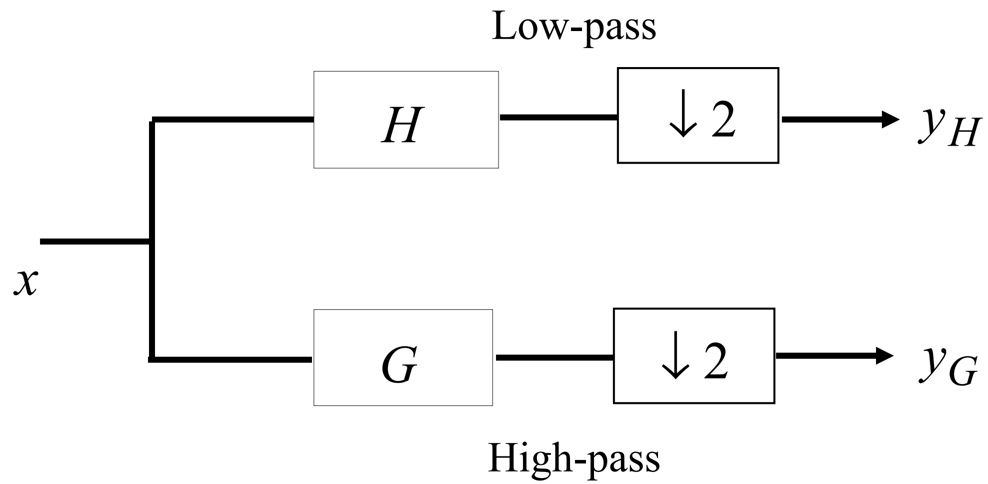
Two-Channel Filter Banks

- Downsampling a signal by 2:

$$X'(z) = \frac{1}{2}\left[X\left(z^{1/2}\right) + X\left(-z^{1/2}\right)\right]$$

- Upsampling by 2:

$$X''(z) = X'\left(z^2\right)$$



- Analysis

$$Y_H(z) = \frac{1}{2}\left[H\left(z^{1/2}\right)\cdot X\left(z^{1/2}\right) + H\left(-z^{1/2}\right)\cdot X\left(-z^{1/2}\right)\right]$$

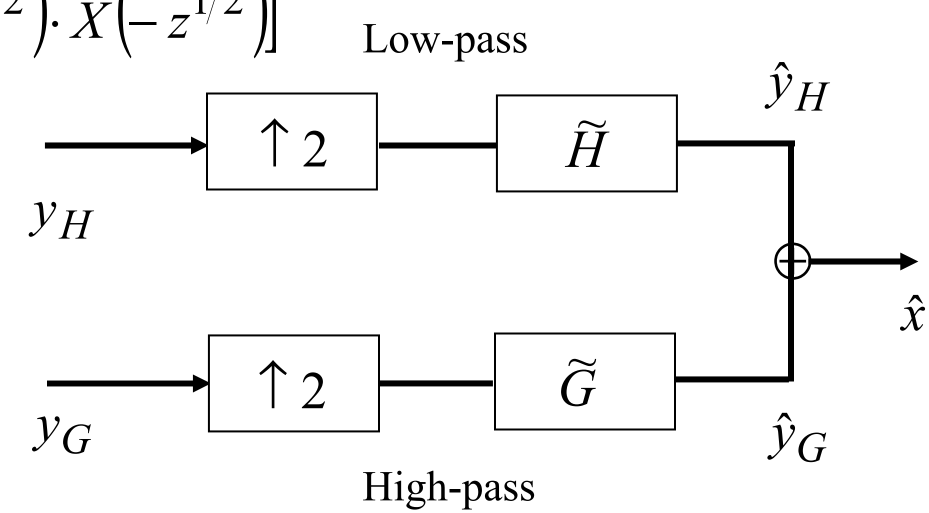
$$Y_G(z) = \frac{1}{2}\left[G\left(z^{1/2}\right)\cdot X\left(z^{1/2}\right) + G\left(-z^{1/2}\right)\cdot X\left(-z^{1/2}\right)\right]$$

- Synthesis

$$\hat{Y}_H(z) = \tilde{H}(z)\cdot Y_H\left(z^2\right)$$

$$\hat{Y}_G(z) = \tilde{G}(z)\cdot Y_G\left(z^2\right)$$

$$\hat{X}(z) = \hat{Y}_H(z) + \hat{Y}_G(z)$$



7.1. Two-Channel Filter Banks

Two-Channel Filter Banks

$$\hat{Y}_H(z) = \frac{1}{2}\widetilde{H}(z)\cdot[H(z)\cdot X(z) + H(-z)\cdot X(-z)]$$

$$\hat{Y}_G(z) = \frac{1}{2}\widetilde{G}(z)\cdot[G(z)\cdot X(z) + G(-z)\cdot X(-z)]$$

$$\hat{X}(z) = \frac{1}{2}\begin{pmatrix}\widetilde{H}(z) & \widetilde{G}(z)\end{pmatrix}\cdot\underbrace{\begin{pmatrix}H(z) & H(-z)\\ G(z) & G(-z)\end{pmatrix}}_{H_m(z)}\cdot\underbrace{\begin{pmatrix}X(z)\\ X(-z)\end{pmatrix}}_{X_m(z)}$$

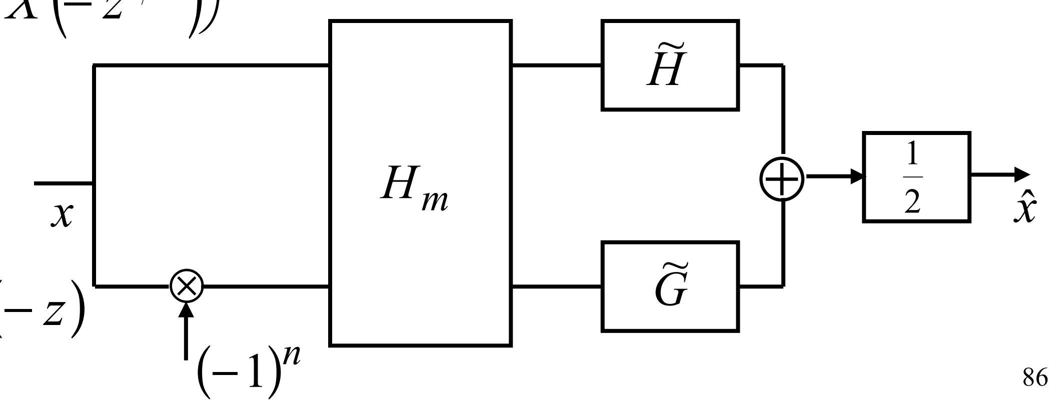
$H_m(z)$ is the analysis *modulation matrix*

$$\begin{pmatrix}Y_H(z)\\ Y_G(z)\end{pmatrix} = \frac{1}{2}\begin{pmatrix}H\left(z^{1/2}\right) & H\left(-z^{1/2}\right)\\ G\left(z^{1/2}\right) & G\left(-z^{1/2}\right)\end{pmatrix}\cdot\begin{pmatrix}X\left(z^{1/2}\right)\\ X\left(-z^{1/2}\right)\end{pmatrix}$$

$$Y(z) = \frac{1}{2}H_m\left(z^{1/2}\right)\cdot X_m\left(z^{1/2}\right)$$

Modulation-domain analysis of the two-channel filter bank

$$(-1)^n x[n] \rightarrow X(-z)$$



7.1. Two-Channel Filter Banks

Perfect Reconstruction Condition

$$\hat{X}(z) = X(z) \Rightarrow \begin{cases} \tilde{H}(z)\cdot H(z) + \tilde{G}(z)\cdot G(z) = 2 \\ \tilde{H}(z)\cdot H(-z) + \tilde{G}(z)\cdot G(-z) = 0 \end{cases}$$

Matrix notation:

$$\begin{pmatrix} \tilde{H}(z) & \tilde{G}(z) \end{pmatrix} \cdot H_m(z) = \begin{pmatrix} 2 & 0 \end{pmatrix}$$

Transpose, and multiply by $\left(H_m{}^T(z)\right)^{-1}$:

$$\begin{pmatrix} \tilde{H}(z) \\ \tilde{G}(z) \end{pmatrix} = \frac{2}{\det(H_m(z))} \begin{pmatrix} G(-z) \\ -H(-z) \end{pmatrix}$$

Define $P(z)$ as:

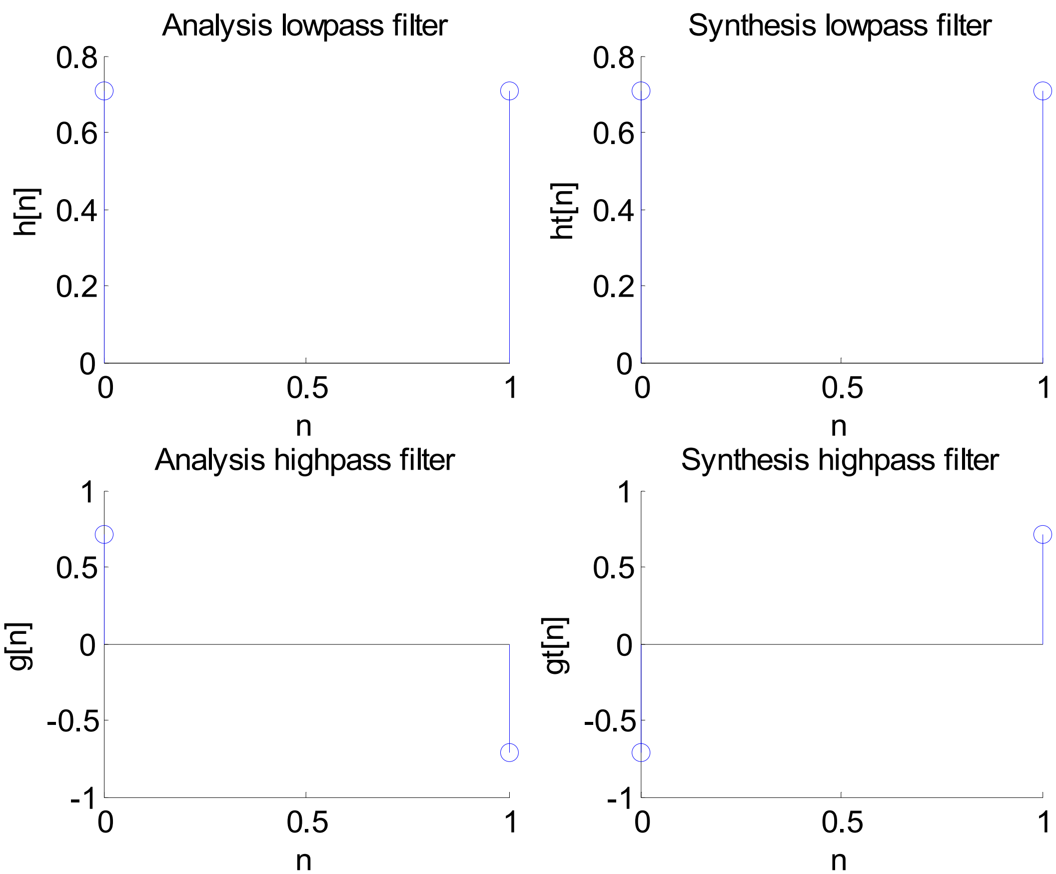
$$P(z) = \tilde{H}(z)\cdot H(z) = \frac{2}{\det(H_m(z))} H(z)\cdot G(-z)$$

$$\tilde{G}(z)\cdot G(z) = \frac{-2}{\det(H_m(z))} H(-z)\cdot G(z) = P(-z), \textit{ because } \det(H_m(z)) = -\det(H_m(-z))$$

Perfect reconstruction condition: $P(z) + P(-z) = 2$

7.1. Two-Channel Filter Banks

Example of a two-channel filter bank achieving perfect reconstruction



Haar filter bank

Analysis filters

$$h[n] = \left\{ \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \mid n = 0,1 \right\}$$

$$g[n] = \left\{ \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \mid n = 0,1 \right\}$$

Synthesis filters

$$ht[n] = \left\{ \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \mid n = 0,1 \right\}$$

$$gt[n] = \left\{ -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \mid n = 0,1 \right\}$$