Vrije Universiteit Brussel

The Fourier Transform

Adrian Munteanu
Electronics and Informatics department

Overview

Review of the Fourier Theory

The Fourier Transform

2. Fourier Theory

- The signal to be expanded is either continuous or discrete in time.
- The expansion involves an integral (a transform) or a summation (a series).

$\Rightarrow$ Four possible combinations of continuous/discrete time and integral/series expansions

General Case

$\{\psi_\omega\}$, $\{\psi_k\}$ a continuous and a discrete set of basis functions respectively.

(a) *Continuous-time Integral Expansion*

$$x(t) = \int X_\omega \psi_\omega(t)\, d\omega, \quad X_\omega = \langle \tilde{\psi}_\omega(t), x(t) \rangle$$

(b) *Continuous-time Series Expansion*

$$x(t) = \sum_k X_k \psi_k(t), \quad X_k = \langle \tilde{\psi}_k(t), x(t) \rangle$$

(c) *Discrete-time Integral Expansion*

$$x[n] = \int X_\omega \psi_\omega[n]\, d\omega, \quad X_\omega = \langle \tilde{\psi}_\omega[n], x[n] \rangle$$

(d) *Discrete-time Series Expansion*

$$x[n] = \sum_k X_k \psi_k[n], \quad X_k = \langle \tilde{\psi}_k[n], x[n] \rangle$$

2. Fourier Theory

Fourier Case

(a) *Continuous-time Fourier Transform* **(CTFT)** - *Fourier Transform*

(b) *Continuous-time Fourier Series* **(CTFS)** - *Fourier Series*

(c) *Discrete-time Fourier Transform* **(DTFT)**

(d) *Discrete-time Fourier Series* **(DTFS)**

All cases: $\{\psi\} = \{\tilde{\psi}\}$

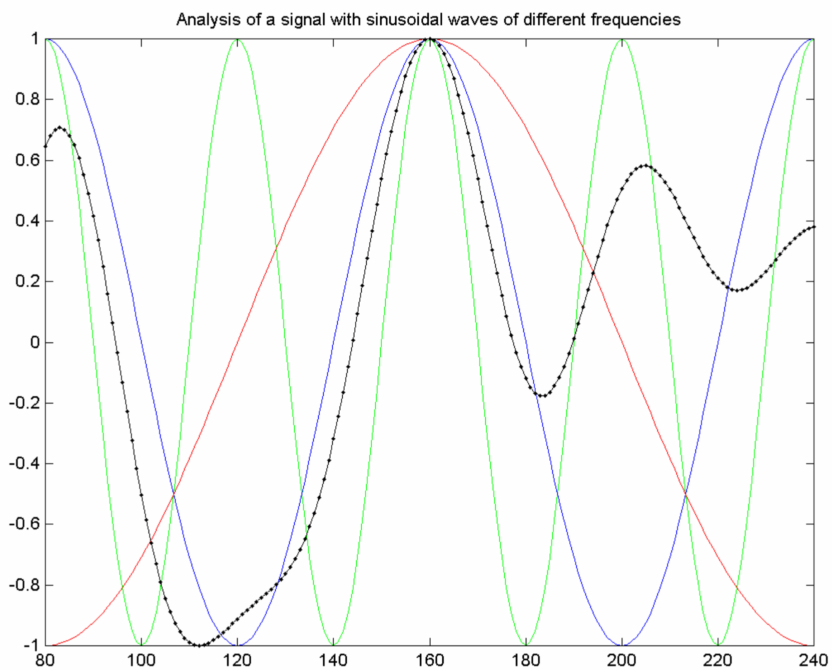
2. The Fourier Transform

The Fourier Transform in $L^1(\Re)$

$$F(\omega)=\int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt, \qquad (1)$$

$$\psi_{\omega}(t)=e^{j\omega t} \qquad F(\omega)=\left\langle e^{j\omega t}, f(t)\right\rangle$$

✍ It measures "the intensity" of the oscillations at the frequency ω in $f(t)$



2. The Fourier Transform

The Inverse Fourier Transform in $L^1(\Re)$

- If $f \in L^1(\Re)$ and $F \in L^1(\Re)$ then:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t} d\omega \qquad (2)$$

✍ Synthesize $f(t)$ as a sum of sinusoidal waves $e^{j\omega t}$ of amplitude $F(\omega)$

- Note 1: the inversion formula is exact if f is continuous.
- Note 2: the inversion formula is exact if $f(t)$ is defined as $\left(f(t^+) + f(t^-)\right)/2$ at a point of discontinuity

Fourier Transform in $L^2(\Re)$

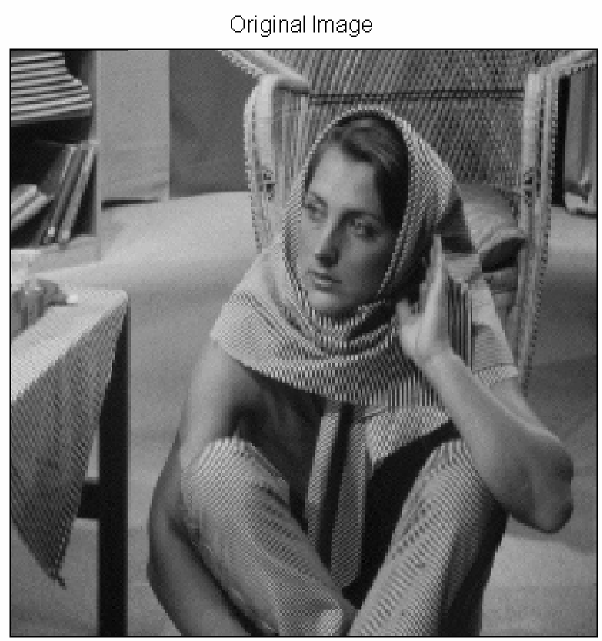
- The formulas above hold in the L^2 sense: if $\hat{f}(t)$ is the result of (1) followed by (2), then:

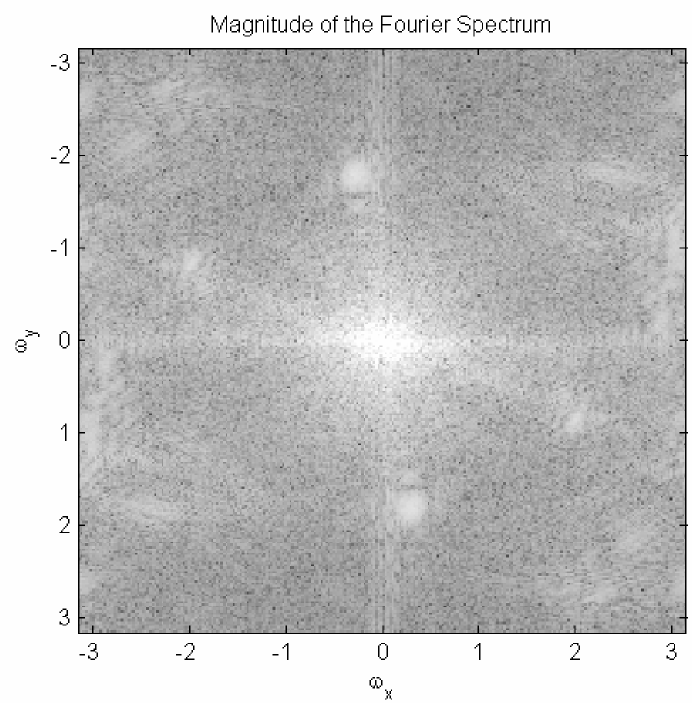
$$\left\| f(t) - \hat{f}(t) \right\|_2 = 0$$

2. The Fourier Transform

How does it look??

- a two-dimensional example…





2. The Fourier Transform

Properties

- **Linearity**

$$\mathcal{F}\left\{\sum_k c_k x_k(t)\right\} = \sum_k c_k \mathcal{F}\{x_k(t)\} \stackrel{\Delta}{=} \sum_k c_k X_k(\omega)$$

- **Shifting in time**

$$\mathcal{F}\{x(t-t_0)\} = e^{-j\omega t_0}\mathcal{F}\{x(t)\} = e^{-j\omega t_0}X(\omega)$$

$$\mathcal{F}\{x(t-t_0)\} = \int_{-\infty}^{\infty} x(t-t_0)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(u)e^{-j\omega(u+t_0)}du = e^{-j\omega t_0}\int_{-\infty}^{\infty} x(u)e^{-j\omega u}du = e^{-j\omega t_0}X(\omega)$$

- **Shifting in frequency** (modulation)

$$\mathcal{F}^{-1}\{X(\omega-\omega_0)\} = e^{j\omega_0 t}x(t)$$

$$\mathcal{F}^{-1}\{X(\omega-\omega_0)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega-\omega_0)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(u)e^{j(u+\omega_0)t}d\omega =$$

$$= \frac{e^{j\omega_0 t}}{2\pi}\int_{-\infty}^{\infty} X(u)e^{j\omega u}du = e^{j\omega_0 t}\mathcal{F}^{-1}\{X(\omega)\} = e^{j\omega_0 t}x(t)$$

2. The Fourier Transform

Properties

- **Time-reversal**

$$\mathcal{F}\{x(-t)\} = X(-\omega)$$

$$\mathcal{F}\{x(-t)\} = \int_{-\infty}^{\infty} x(-t)e^{-j\omega t}dt = -\int_{\infty}^{-\infty} x(u)e^{j\omega u}du = \int_{-\infty}^{\infty} x(u)e^{j\omega u}du = X(-\omega)$$

- **Symmetric signals**

$$\textit{If } x(t) = x(-t), \textit{ then } X(\omega) = X(-\omega)$$

$$\text{Proof:} \quad X(\omega) = \mathcal{F}\{x(t)\} = \mathcal{F}\{x(-t)\} = X(-\omega)$$

- **Anti-symmetric signals**

$$\textit{If } x(t) = -x(-t), \textit{ then } X(\omega) = -X(-\omega)$$

$$X(\omega) = \mathcal{F}\{x(t)\} = \mathcal{F}\{-x(-t)\} = -\mathcal{F}\{x(-t)\} = -X(-\omega)$$

- **Complex Conjugation**

$$\textit{If } \mathcal{F}\{x(t)\} = X(\omega) \textit{ then } \mathcal{F}\{x^*(t)\} = X^*(-\omega)$$

$$\mathcal{F}\{x^*(t)\} = \int_{-\infty}^{\infty} x^*(t)e^{-j\omega t}dt = \left(\int_{-\infty}^{\infty} x(t)e^{j\omega t}dt\right)^* = X^*(-\omega)$$

2. The Fourier Transform

Properties

- **Duality**

$$\mathcal{F}\{x(t)\} = X(\omega) \Leftrightarrow \mathcal{F}\{X(t)\} = 2\pi x(-\omega) \Leftrightarrow \mathcal{F}^{-1}\{x(\omega)\} = \frac{1}{2\pi} X(-t)$$

Start from the definition of the inverse
Fourier transform:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} dt$$

Replacing ω with u and t with ω :

$$x(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(u) e^{ju\omega} du \Leftrightarrow 2\pi x(\omega) = \int_{-\infty}^{\infty} X(u) e^{ju\omega} du$$

Replacing ω with $-\omega$ and u with t :

$$2\pi x(-\omega) = \int_{-\infty}^{\infty} X(u) e^{-ju\omega} du \Leftrightarrow 2\pi x(-\omega) = \int_{-\infty}^{\infty} X(t) e^{-jt\omega} dt \Leftrightarrow 2\pi x(-\omega) = \mathcal{F}\{X(t)\}$$

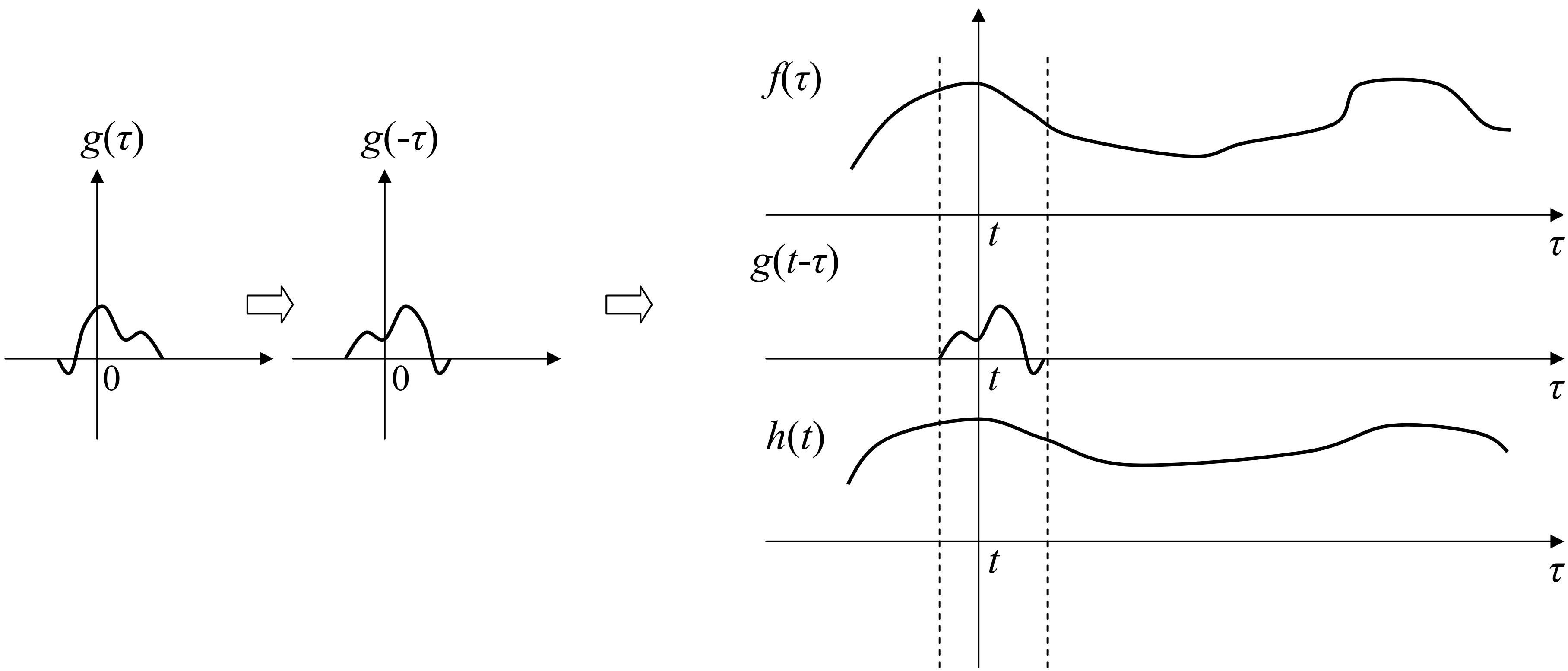
$$\mathcal{F}\{x(t)\} = X(\omega) \Leftrightarrow \mathcal{F}\{X(t)\} = 2\pi x(-\omega) \Leftrightarrow \mathcal{F}^{-1}\{x(-\omega)\} = \frac{1}{2\pi} X(t) \Leftrightarrow \mathcal{F}^{-1}\{x(\omega)\} = \frac{1}{2\pi} X(-t)$$

2. The Fourier Transform

Properties

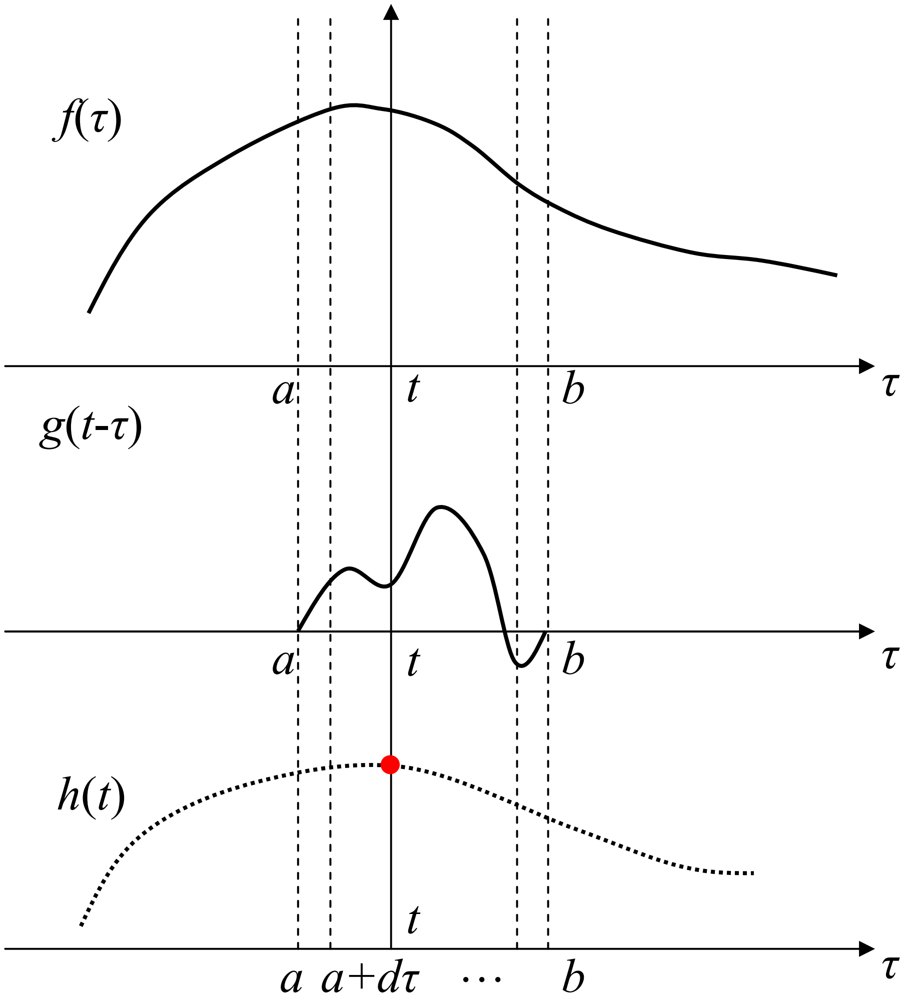
- **Convolution between two signals: definition**

$$h(t) = f(t) * g(t) = \int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, d\tau = \int_{-\infty}^{\infty} f(t-\tau)\, g(\tau)\, d\tau$$



Properties

- **Convolution between two signals: definition**



$$h(t) = f(t) * g(t) = \int_{-\infty}^{\infty} f(\tau) g(t-\tau) d\tau$$

The integral can be approximated by an infinite sum:

- Divide the interval [a,b] into K intervals of size $d\tau$
- We notice that $d\tau \to 0$ as $K \to \infty$
- Thus:

$$h(t) = f(t) * g(t) = \int_{-\infty}^{\infty} f(\tau) g(t-\tau) d\tau$$

$$h(t) = \lim_{K \to \infty} \sum_{k=0}^{K-1} f(a + kd\tau) g\big(t - (a + kd\tau)\big)$$

2. The Fourier Transform

Properties

- **Convolution between two signals: definition**

$$h(t) = f(t) * g(t) = \int_{-\infty}^{\infty} f(\tau) g(t-\tau) d\tau = \int_{-\infty}^{\infty} f(t-\tau) g(\tau) d\tau$$

- **Convolution theorem 1:**

$$\mathcal{F}\{f(t) * g(t)\} = \mathcal{F}\{f(t)\} \cdot \mathcal{F}\{g(t)\} = F(\omega) \cdot G(\omega)$$

Proof:

$$\mathcal{F}\{f(t) * g(t)\} = \int_{-\infty}^{\infty} \left(f(t) * g(t)\right) e^{-j\omega t} dt = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\tau) g(t-\tau) d\tau \right] e^{-j\omega t} dt =$$

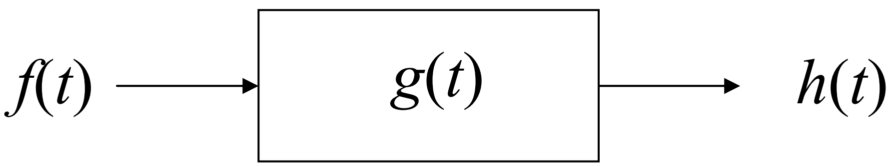
$$= \int_{-\infty}^{\infty} f(\tau) \left[\int_{-\infty}^{\infty} g(t-\tau) e^{-j\omega t} dt \right] d\tau = \int_{-\infty}^{\infty} f(\tau) e^{-j\omega \tau} \left[\int_{-\infty}^{\infty} g(u) e^{-j\omega u} du \right] d\tau =$$

$$= \left[\int_{-\infty}^{\infty} f(\tau) e^{-j\omega \tau} d\tau \right] \left[\int_{-\infty}^{\infty} g(u) e^{-j\omega u} du \right] = F(\omega) \cdot G(\omega)$$

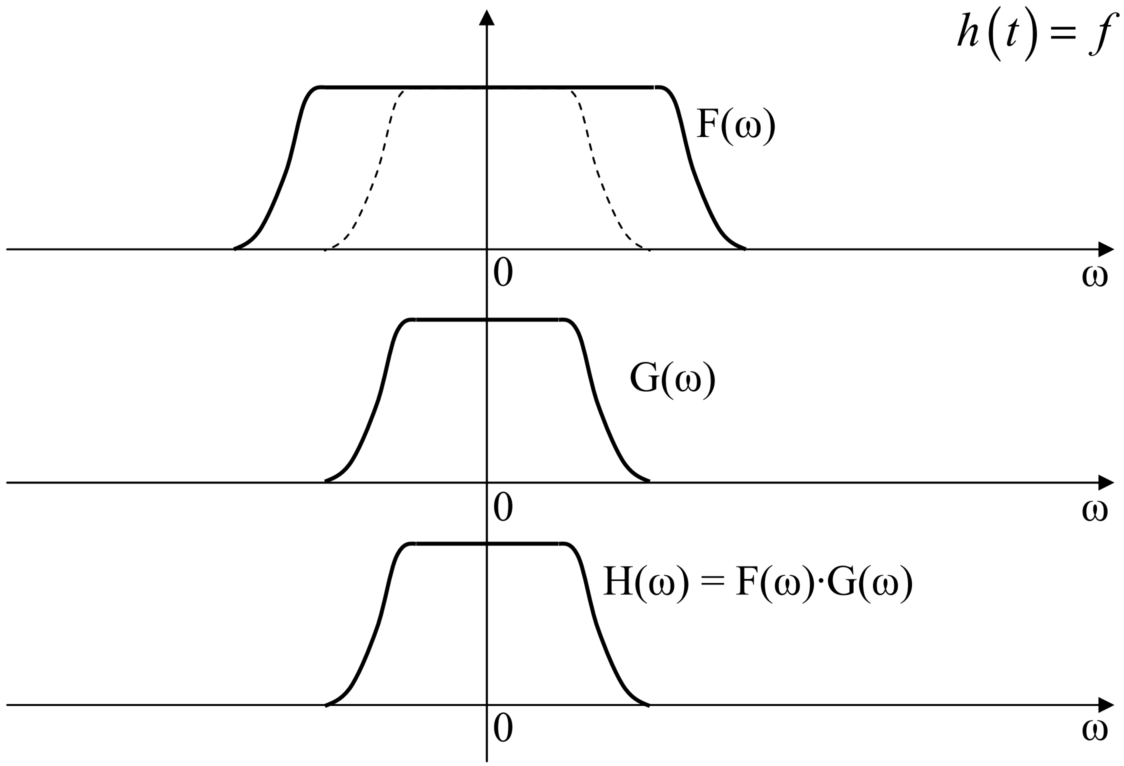
2. The Fourier Transform

Properties

- **Convolution between two signals: interpretation**



A filtering operation corresponds to a convolution in the time domain between the input signal, $f(t)$ and the filter $g(t)$, producing the output: $h(t) = f(t) * g(t)$



2. The Fourier Transform

Properties

- **Convolution theorem 2:**

$$\mathcal{F}\{f(t)\cdot g(t)\} = \frac{1}{2\pi}F(\omega)*G(\omega)$$

Proof.

$$\mathcal{F}\{f(t)\cdot g(t)\} = \int_{-\infty}^{\infty}\left(f(t)\cdot g(t)\right)e^{-j\omega t}dt = \int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}F(u)e^{jut}du\right]g(t)e^{-j\omega t}dt =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}F(u)\left[\int_{-\infty}^{\infty}g(t)e^{-j\omega t}e^{jut}dt\right]du = \frac{1}{2\pi}\int_{-\infty}^{\infty}F(u)\left[\int_{-\infty}^{\infty}g(t)e^{-j(\omega-u)t}dt\right]du =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}F(u)G(\omega-u)du = \frac{1}{2\pi}F(\omega)*G(\omega)$$

2. The Fourier Transform

Properties

- **Multiplication theorem**

$$\int_{-\infty}^{\infty} f^*(t)\,g(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F^*(\omega)\,G(\omega)\,d\omega$$

Proof.

$$\int_{-\infty}^{\infty} f^*(t)\,g(t)\,dt = \int_{-\infty}^{\infty} f^*(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\,e^{j\omega t}\,d\omega\right]dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\left[\int_{-\infty}^{\infty} f^*(t)\,e^{j\omega t}\,dt\right]d\omega =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F^*(\omega)\,G(\omega)\,d\omega$$

- **Parseval's theorem (or energy conservation property)**

$$\int_{-\infty}^{\infty} \left|f(t)\right|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|F(\omega)\right|^2 d\omega$$

Proof: consider $f(t) = g(t)$ in the multiplication theorem.

2. The Fourier Transform

Properties

- The **energy spectral density** is defined as:

$$S_{xx}(\omega) \stackrel{\Delta}{=} |X(\omega)|^2$$

Interpretation: the energy spectral density informs us how the signal's energy is distributed along the frequency axes and about the frequency ranges where the energy of the signal is concentrated.

With this definition, Parsevals' theorem is written as:

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega) d\omega$$

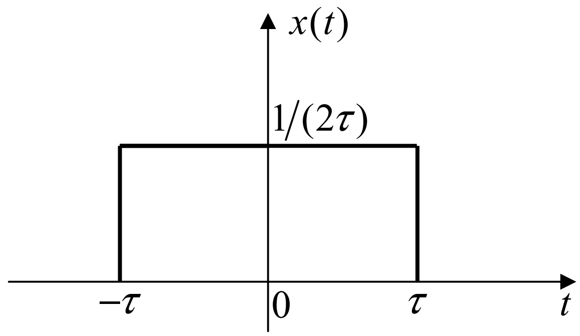
Interpretation: *Energy in the spatial domain = Energy in the Fourier domain*

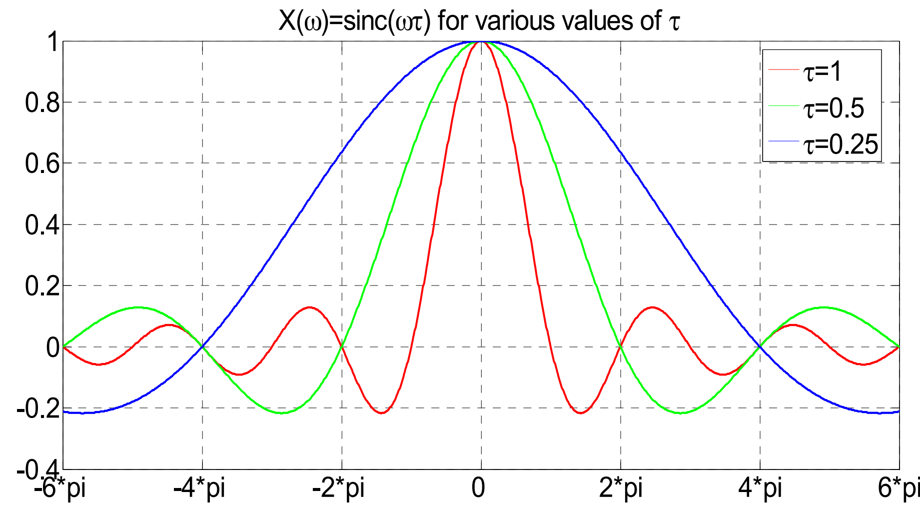
The total energy contained in the signal is obtained by integrating $S_{xx}(\omega)$ over the entire frequency axes

2. The Fourier Transform

- **Energy spectral density – Example**

$$x(t)=\begin{cases}\dfrac{1}{2\tau} & |t|\le\tau\\ 0 & otherwise\end{cases}$$





The Sinc function (for different values of τ)

$$X(\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt=\int_{-\tau}^{\tau}\frac{1}{2\tau}e^{-j\omega t}dt;\quad \left(e^{-j\omega t}\right)'=-j\omega e^{-j\omega t}\Rightarrow e^{-j\omega t}=\frac{1}{-j\omega}\left(e^{-j\omega t}\right)'$$

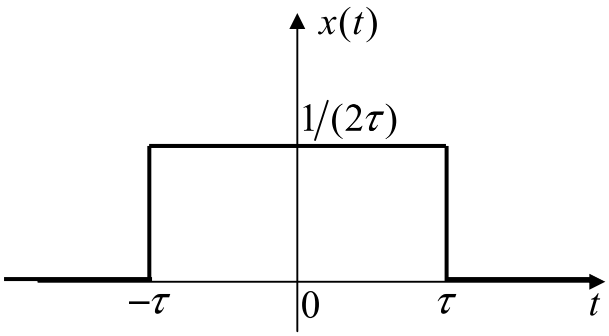
$$X(\omega)=\frac{1}{2\tau}\int_{-\tau}^{\tau}e^{-j\omega t}dt=\frac{1}{2\tau}\frac{1}{-j\omega}\int_{-\tau}^{\tau}\left(e^{-j\omega t}\right)'dt=-\frac{1}{2j\omega\tau}\quad e^{-j\omega t}\Big|_{-\tau}^{\tau}=-\frac{e^{-j\omega\tau}-e^{j\omega\tau}}{2j\omega\tau}=$$

$$=\frac{e^{j\omega\tau}-e^{-j\omega\tau}}{2j\omega\tau}=\frac{(\cos(\omega\tau)+j\sin(\omega\tau))-(\cos(-\omega\tau)+j\sin(-\omega\tau))}{2j\omega\tau}=\frac{\sin(\omega\tau)}{\omega\tau}\overset{\Delta}{=}\mathbf{sinc}(\boldsymbol{\omega\tau})$$

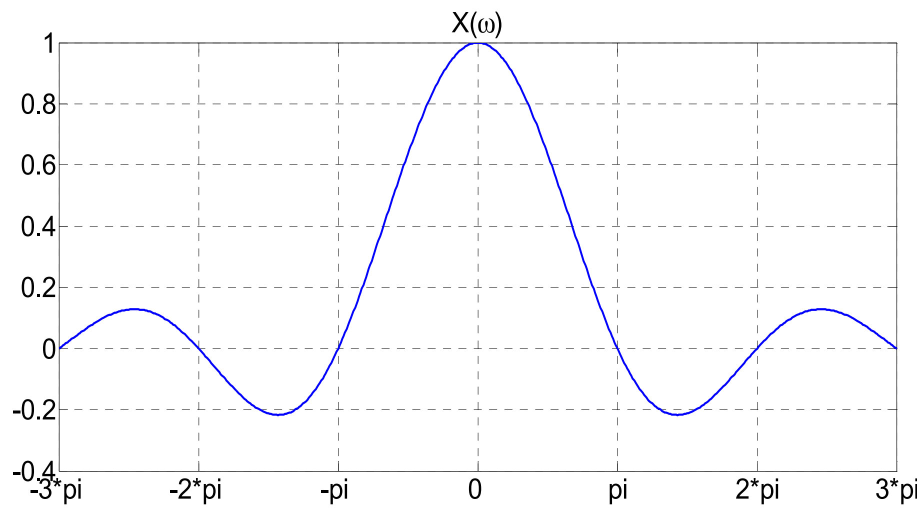
2. The Fourier Transform

- **Energy spectral density – Example**

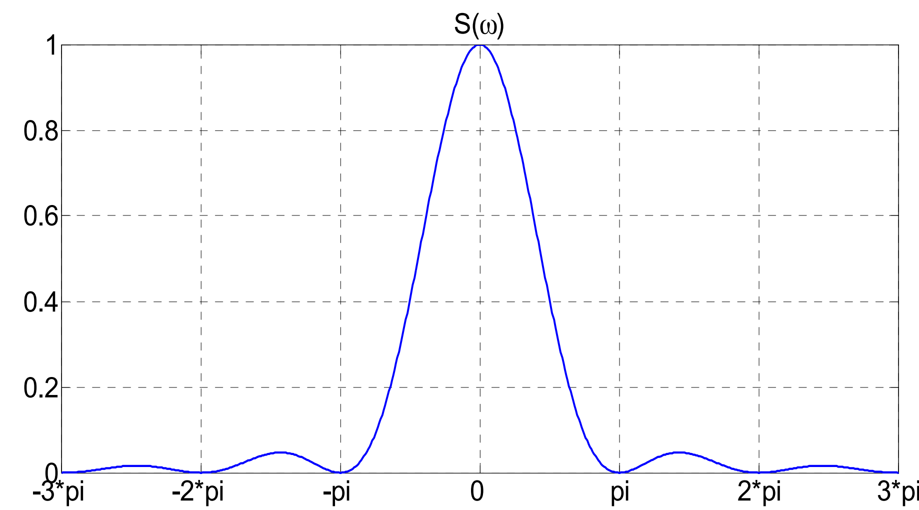
$$x(t)=\begin{cases}\dfrac{1}{2\tau} & |t|\le\tau \\ 0 & otherwise\end{cases}$$



The finite-pulse function in time



The Sinc function (for τ=1)



Energy spectral density (for τ=1)

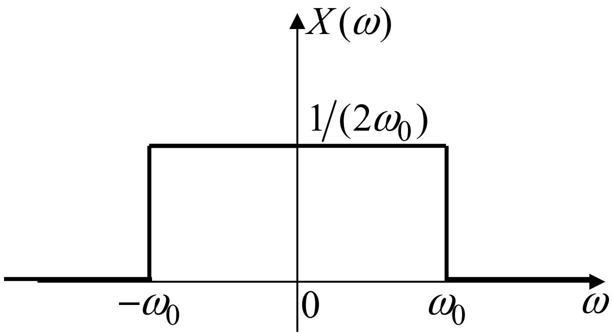
The energy spectral density:

$$S_{xx}(\omega)=\left|X(\omega)\right|^{2}=sinc^{2}(\omega t)$$

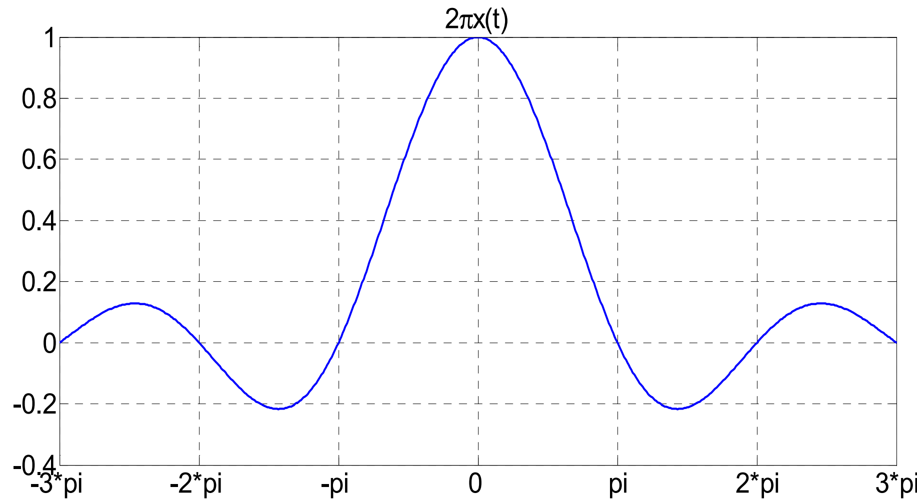
2. The Fourier Transform

- **The ideal low-pass filter (rectangle function)**

$$X(\omega)=\begin{cases}\dfrac{1}{2\omega_0} & |\omega|\le\omega_0\\ 0 & otherwise\end{cases}$$



The finite-pulse function in frequency



The Sinc function in time domain (for ω_0=1)

We apply the duality property, given by:

Let : $y(\omega)=X(\omega)$ as illustrated above. If we replace ω by t and ω_0 by τ, we know that:

$$\mathcal{F}\{y(t)\}=sinc(\omega\omega_0)\overset{\Delta}{=}Y(\omega)\Leftrightarrow\ \mathcal{F}\{y(t)\}=Y(\omega)\Leftrightarrow\mathcal{F}^{-1}\{y(\omega)\}=\frac{1}{2\pi}Y(-t)$$

$$x(t)=\mathcal{F}^{-1}\{X(\omega)\}=\mathcal{F}^{-1}\{y(\omega)\}=\frac{1}{2\pi}Y(-t)=\frac{1}{2\pi}sinc(-\omega_0 t)=\frac{1}{2\pi}sinc(\omega_0 t)$$

2. The Fourier Transform

- The **cross-correlation** between two signals $x(t), y(t)$ is defined as:

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t)\, y(t-\tau)\, dt = x(t) * y(-t)$$

Fourier transform relations:

$$\mathcal{F}\{R_{xy}(\tau)\} = \mathcal{F}\{x(t) * y(-t)\} = X(\omega) \cdot Y(-\omega)$$

- When $y(t) = x(t)$, the above becomes the **auto-correlation**:

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)\, x(t-\tau)\, dt$$

Measures the similarity of a signal with a delayed version of itself.

- **Spectral-density theorem**

$$S_{xx}(\omega) = \mathcal{F}\{R_{xx}(\tau)\}$$

Proof. $$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)\, x(t-\tau)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \left(X(\omega) e^{-j\omega t} \right)^* d\omega \Leftrightarrow$$

$$\Leftrightarrow R_{xx}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, X^*(\omega)\, e^{j\omega t} d\omega \Leftrightarrow R_{xx}(\tau) = \mathcal{F}^{-1}\{S_{xx}(\omega)\}$$

2. The Fourier Transform

Properties

- **Differentiation in time** (becomes multiplication in frequency)

$$\mathcal{F}\left\{x'(t)\right\} = j\omega X(\omega); \quad \mathcal{F}\left\{x^{(n)}(t)\right\} = (j\omega)^n X(\omega)$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \Leftrightarrow x'(t) = \frac{1}{2\pi}\frac{d}{dt}\left[\int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega\right] =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(j\omega X(\omega)\right) e^{j\omega t} d\omega = \mathcal{F}^{-1}\left\{j\omega X(\omega)\right\}$$

- **Integration in time** (becomes division in frequency)

$$\mathcal{F}\left\{\int_{-\infty}^{t} x(\tau) d\tau\right\} = \frac{X(\omega)}{j\omega} \quad \textit{for signals with} \quad X(0) = 0$$

Proof: will be shown latter on using distributions

2. The Fourier Transform

Properties

- **Differentiation in frequency** (becomes multiplication in time)

$$\mathcal{F}^{-1}\left\{X'(\omega)\right\} = (-jt)\,x(t)$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \Leftrightarrow X'(\omega) = \frac{d}{d\omega}\left[\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt\right] = \int_{-\infty}^{\infty}(-jt)\,x(t)e^{-j\omega t}dt = \mathcal{F}\left\{(-jt)\,x(t)\right\}$$

- **Integration in frequency** (becomes division in time)

$$\mathcal{F}^{-1}\left\{\int_{-\infty}^{\omega} X(u)\,du\right\} = \frac{x(t)}{-jt}$$

Proof: Let $Y(\omega) = \int_{-\infty}^{\omega} X(u)\,du \Rightarrow Y'(\omega) = X(\omega)$

From the Differentiation in Frequency property (see above):

$$\left.\begin{aligned}\mathcal{F}^{-1}\left\{Y'(\omega)\right\} &= (-jt)\,y(t)\\ \mathcal{F}^{-1}\left\{Y'(\omega)\right\} = \mathcal{F}^{-1}\left\{X(\omega)\right\} &= x(t)\end{aligned}\right\} \Rightarrow y(t) = \frac{x(t)}{(-jt)} \Rightarrow Y(\omega) = \mathcal{F}\left\{\frac{x(t)}{(-jt)}\right\}$$

Dirac and Heaviside distributions

3. Dirac and Heaviside distributions

Definition

- Processing signals that are discontinuous is often a difficult task, because one cannot use classical mathematical tools, applicable in case of continuous signals
- In this context, distributions and distribution theory play an important role
- A **distribution** is the process by which a real-valued function $x(t)$ is mapped by a functional $f(t)$ to a a set of values $N_f(x)$ by the relation:

$$\langle x(t), f(t)\rangle = N_f(x) \Leftrightarrow \int_{-\infty}^{\infty} x(t) f(t) dt = N_f(x)$$

- The Dirac and Heaviside distributions are the most known distributions
- The **Dirac distribution** $\delta(t)$ is a functional that maps a function $x(t)$, which is continuous in $t = 0$, to $x(0)$:

$$\int_{-\infty}^{\infty} x(t)\delta(t) dt = x(0)$$

- The **Heaviside distribution** $u(t)$ is the functional that maps a function $x(t)$, which is integrable in $\mathbb{R}_+$, to:

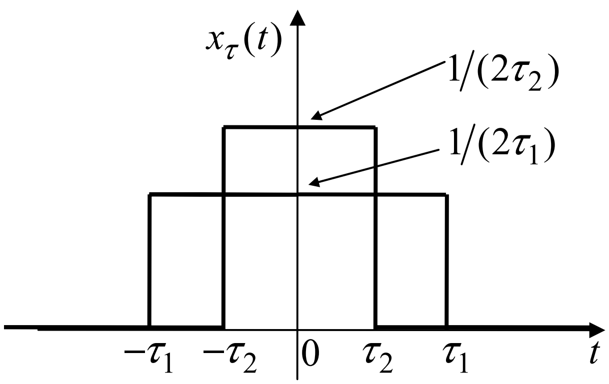
$$\int_{-\infty}^{\infty} x(t)u(t) dt = \int_{0}^{\infty} x(t) dt$$

3. Dirac and Heaviside distributions

- **Dirac distribution - definition as a limit of a series of rectangle functions**

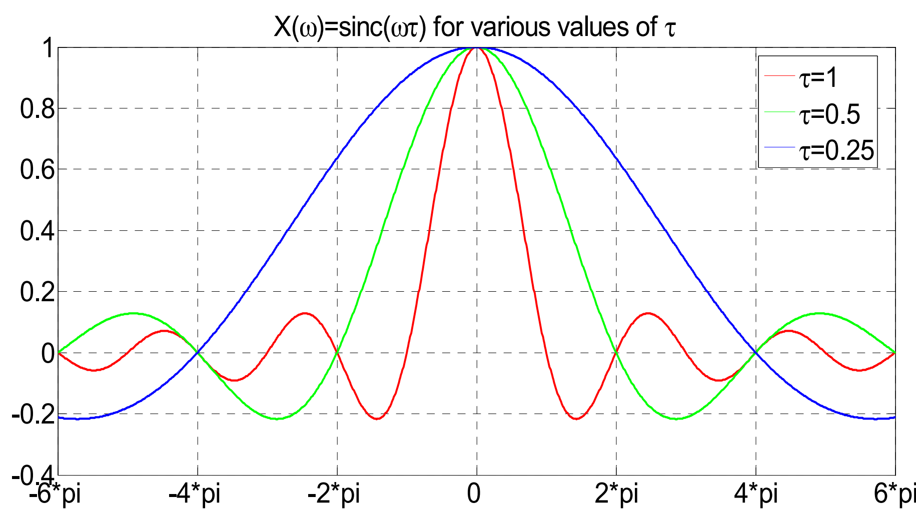
$$\delta(t) = \lim_{\tau \to 0} x_\tau(t)$$

$$x_\tau(t) = \begin{cases} \dfrac{1}{2\tau} & |t| \le \tau \\ 0 & otherwise \end{cases}$$



The finite-pulse function in time

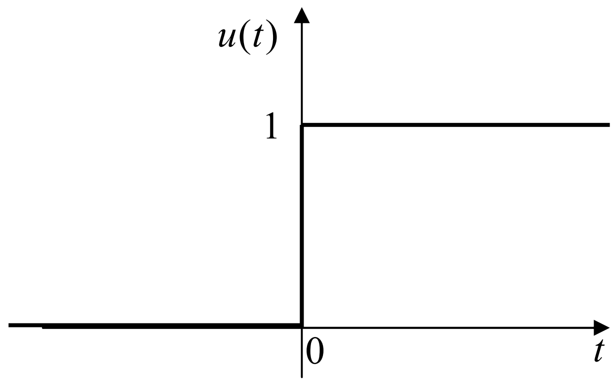
$$X_\tau(\omega) = sinc(\omega\tau)$$



The Sinc function (for different values of τ)

3. Dirac and Heaviside distributions

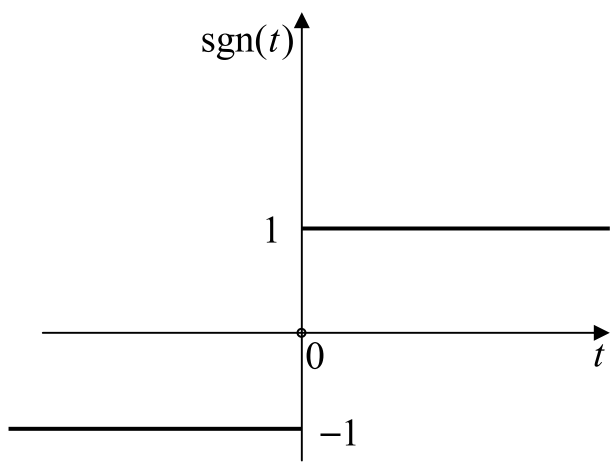
- **The Heaviside distribution (or Heaviside step function) - definition**



The Heaviside step function in time

$$\int_{-\infty}^{\infty} x(t)u(t)dt = \int_{0}^{\infty} x(t)dt$$

$$u(t) = \frac{1}{2} + \frac{1}{2}\operatorname{sgn}(t)$$



The sgn(t) function

$$\operatorname{sgn}(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}$$

$Note:$ $u(0)$ is often defined as $u(0) = 0$; $u(0) = 0.5$, or $u(0) = 1$

$\rightarrow$ Most of the time, $u(0) = 0.5$, making the link between $u(t)$ and $\operatorname{sgn}(t)$

3. Dirac and Heaviside distributions

- **Property 1**

$$\delta(at) = \frac{1}{|a|}\delta(t)$$

Proof:

$$\int_{-\infty}^{\infty} x(t)\delta(t)\,dt = x(0); \qquad (*)$$

$$\int_{-\infty}^{\infty} x(t)\delta(at)\,dt = ? \quad t_1 = at \Rightarrow \begin{cases} t_1 \,|_{-\infty}^{\infty}, & if\ a > 0 \\ t_1 \,|_{\infty}^{-\infty}, & if\ a < 0 \end{cases} \Rightarrow$$

$$\int_{-\infty}^{\infty} x(t)\delta(at)\,dt = \frac{1}{|a|}\int_{-\infty}^{\infty} x\left(\frac{t_1}{a}\right)\delta(t_1)\,dt_1 = \frac{x(0)}{|a|} \Rightarrow$$

$$\int_{-\infty}^{\infty} x(t)\frac{\delta(at)}{|a|}\,dt = x(0) \qquad (**);$$ From (*) and (**) it results property 1

Note: $a = -1 \Leftrightarrow \delta(-t) = \delta(t)$

3. Dirac and Heaviside distributions

- **Property 2 – Time- and Frequency-sampling property of the Dirac function**

$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)\,dt = \int_{-\infty}^{\infty} x(t)\delta(t_0-t)\,dt = x(t_0)$$

Proof:

The first two equalities are due to the symmetry of the Dirac function

Change the variables:

$$t_1 = t - t_0 \Rightarrow dt_1 = dt_0 \Rightarrow \int_{-\infty}^{\infty} x(t)\delta(t-t_0)\,dt = \int_{-\infty}^{\infty} x(t_1+t_0)\delta(t_1)\,dt_1 =$$

$$= \int_{-\infty}^{\infty} y(t_1)\delta(t_1)\,dt_1 = y(0) = x(t_0)$$

The same property can be applied in the frecquency domain:

$$\int_{-\infty}^{\infty} X(\omega)\delta(\omega-\omega_0)\,d\omega = X(\omega_0)$$

3. Dirac and Heaviside distributions

- **Property 3 – Definition of the n-th order derivative of the Dirac function**

$$\int_{-\infty}^{\infty} x(t)\delta^{(n)}(t)\,dt = (-1)^n x^{(n)}(0), \quad where\ x^{(n)}(t) = \frac{d^n}{dx^n}x(t)$$

Proof:

Start from the definition of the Dirac function:

$$\int_{-\infty}^{\infty} x(t)\delta(t-\tau)\,dt = x(\tau)$$

Derive once with respect to τ in both sides:

$$\frac{d}{d\tau}\int_{-\infty}^{\infty} x(t)\delta(t-\tau)\,dt = \frac{d}{d\tau}x(\tau) \Leftrightarrow \int_{-\infty}^{\infty} x(t)\left(\frac{d}{d\tau}\delta(t-\tau)\right)dt = x^{(1)}(\tau)$$

$$\Leftrightarrow \int_{-\infty}^{\infty} x(t)\delta^{(1)}(t-\tau)\,dt = -x^{(1)}(\tau);\ \text{take } \tau = 0 \Leftrightarrow \int_{-\infty}^{\infty} x(t)\delta^{(1)}(t)\,dt = -x^{(1)}(0)$$

Deriving again with respect to τ, gives:

$$\int_{-\infty}^{\infty} x(t)\delta^{(2)}(t-\tau)\,dt = x^{(2)}(\tau) \Leftrightarrow \int_{-\infty}^{\infty} x(t)\delta^{(2)}(t)\,dt = x^{(2)}(0), \text{ and so on.}$$

3. Dirac and Heaviside distributions

- **Property 4 – The link between the Dirac and Heaviside distributions**

$$\frac{du(t)}{dt} = \delta(t)$$

Proof:

Start from the definition of the Heaviside function:

$$\int_{-\infty}^{\infty} x(t)u(t)dt = \int_{0}^{\infty} x(t)dt$$

From the property of derivation by parts, and knowing that $x(\infty)=x(-\infty)=0$ it results:

$$\int_{-\infty}^{\infty} x(t)\frac{du(t)}{dt}dt = x(t)u(t)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} x'(t)u(t)dt = 0 - \int_{0}^{\infty} x'(t)dt = x(0)$$

On the other hand:

$$\int_{-\infty}^{\infty} x(t)\delta(t)dt = x(0)$$

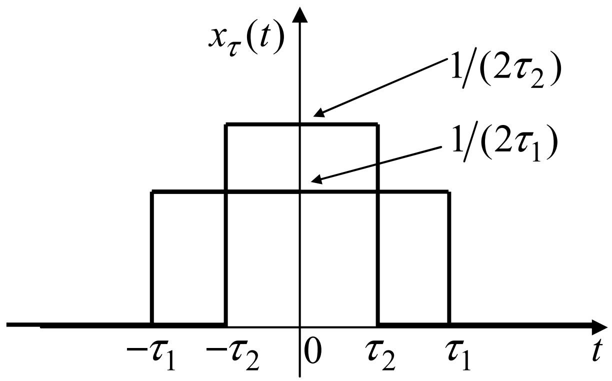
which implies that $\frac{du(t)}{dt} = \delta(t)$. Hence: $u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau$

3. Dirac and Heaviside distributions

- **Property 5 – Definition of the Dirac function as a limit of series of functions**

✍ A series of rectangle functions

$$x_\tau(t) = \begin{cases} \dfrac{1}{2\tau} & |t| \le \tau \\ 0 & otherwise \end{cases}$$



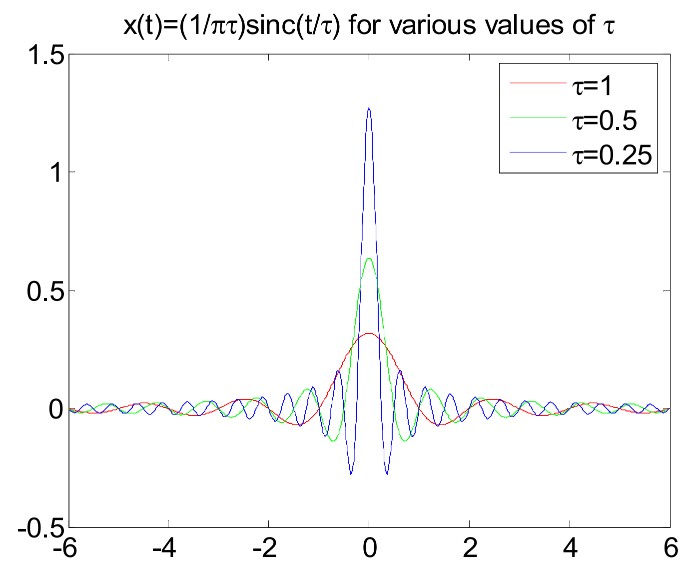
The finite-pulse function in time

$$\delta(t) = \lim_{\tau \to 0} x_\tau(t)$$

✍ A series of sinc functions

$$x_\tau(t) = \frac{1}{\pi\tau} \sin c\left(\frac{t}{\tau}\right)$$

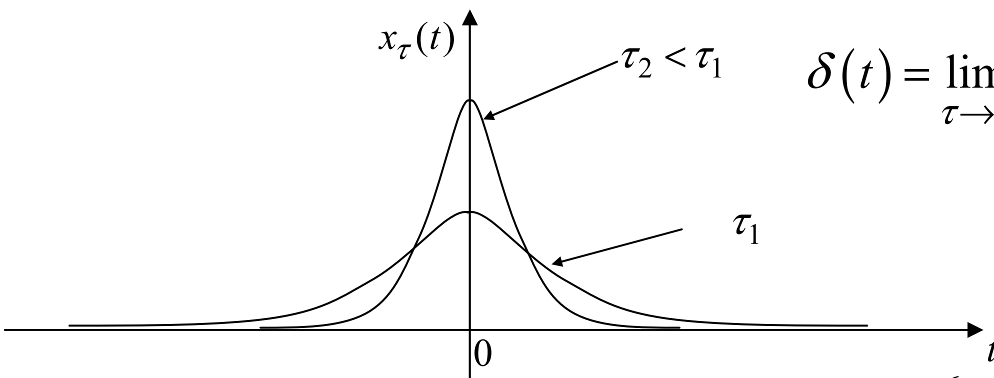
$$\delta(t) = \lim_{\tau \to 0} x_\tau(t)$$



3. Dirac and Heaviside distributions

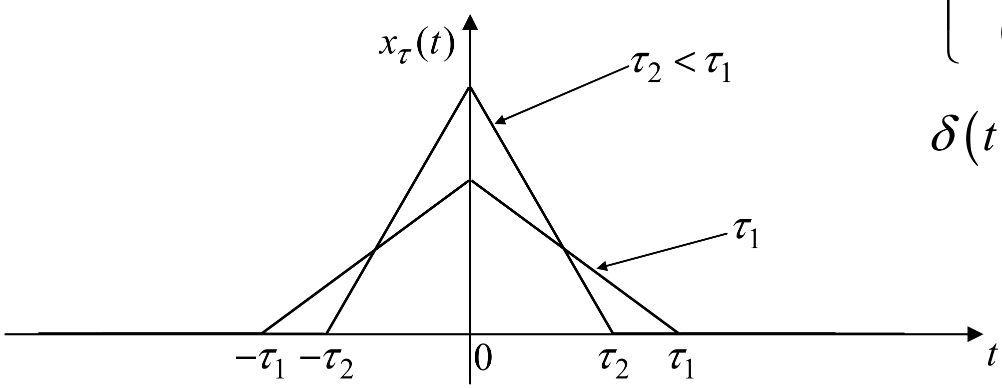
- **Property 5 – Definition of the Dirac function as a limit of series of functions**

✍ A series of Gaussian functions $x_\tau(t) = \frac{1}{\tau} e^{-\frac{\pi t^2}{\tau^2}}$



$$\delta(t) = \lim_{\tau \to 0} x_\tau(t)$$

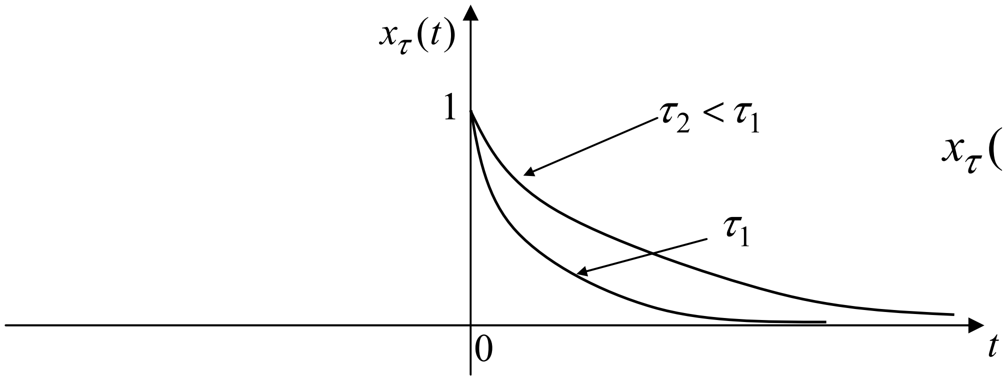
✍ A series of triangular functions $x_\tau(t) = \begin{cases} \frac{1}{\tau}\left(1 - \frac{|t|}{\tau}\right), & if\ |t| < \tau \\ 0, & otherwise \end{cases}$



$$\delta(t) = \lim_{\tau \to 0} x_\tau(t)$$

3. Dirac and Heaviside distributions

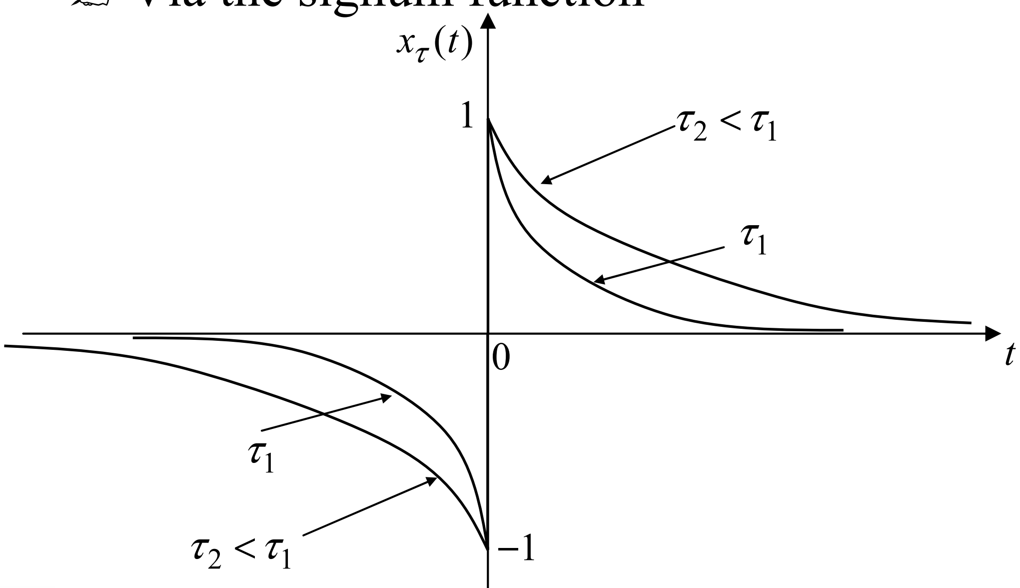
- **Property 5 – Definition of the Heaviside function as a limit of series of functions**
  - ✍ A series of exponentially-decaying functions



$$x_\tau(t) = \begin{cases} e^{-t\tau}, & \text{for } \; t > 0, \; \text{with } \tau > 0 \\ 0, & \text{for } \; t < 0 \end{cases}$$

$$u(t) = \lim_{\tau \to 0} x_\tau(t)$$

  - ✍ Via the signum function



$$u(t) = \frac{1}{2} + \frac{1}{2}\operatorname{sgn}(t)$$

$$\operatorname{sgn}(t) = \begin{cases} \lim_{\tau \to 0} x_\tau(t) & t \neq 0 \\ 0 & t = 0 \end{cases}$$

$$x_\tau(t) = \begin{cases} e^{-t\tau}, & \text{for } \; t > 0 \\ -e^{t\tau}, & \text{for } \; t < 0 \end{cases}, \quad \text{with } \tau > 0$$

3. Dirac and Heaviside distributions

- **Property 6 – The Fourier transforms of the Dirac and Heaviside distributions**
  - Dirac:

$$\mathcal{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = e^{-j\omega 0} = 1 \Leftrightarrow \mathcal{F}\{\delta(t)\} = 1$$

  - Duality property of the Fourier transform:

$$\mathcal{F}\{x(t)\} = X(\omega) \Leftrightarrow \mathcal{F}^{-1}\{x(\omega)\} = \frac{1}{2\pi} X(-t)$$

$$\mathcal{F}\{\delta(t)\} = 1 \Leftrightarrow \mathcal{F}^{-1}\{\delta(\omega)\} = \frac{1}{2\pi} \Leftrightarrow \mathcal{F}\{1\} = 2\pi\delta(\omega)$$

In general, we can calculate the FT of a constant:

$$\mathcal{F}\{A\delta(t)\} = A \Leftrightarrow \mathcal{F}^{-1}\{A\delta(\omega)\} = \frac{A}{2\pi} \Leftrightarrow \mathcal{F}\{A\} = 2\pi A\delta(\omega)$$

3. Dirac and Heaviside distributions

- **Property 6 – The Fourier transforms of the Dirac and Heaviside distributions**
  - ✍ Heaviside:

$$u(t)=\frac{1}{2}+\frac{1}{2}\operatorname{sgn}(t) \qquad \operatorname{sgn}(t)=\begin{cases}\lim\limits_{\tau\to 0} x_\tau(t) & t\neq 0\\ 0 & t=0\end{cases}$$

$$x_\tau(t)=\begin{cases}e^{-t\tau}, & for \quad t>0\\ -e^{t\tau}, & for \quad t<0\end{cases}, \quad with\ \tau>0$$

$$\mathcal{F}\{\operatorname{sgn}(t)\}=\lim_{\tau\to 0}\mathcal{F}\{x_\tau(t)\}=\lim_{\tau\to 0}\left[-\int_{-\infty}^{0} e^{t\tau}e^{-j\omega t}dt+\int_{0}^{\infty} e^{-t\tau}e^{-j\omega t}dt\right]$$

$$-\int_{-\infty}^{0} e^{(\tau-j\omega)t}dt=-\frac{1}{\tau-j\omega}e^{(\tau-j\omega)t}\Big|_{-\infty}^{0}=-\frac{1}{\tau-j\omega};\int_{0}^{\infty} e^{-(\tau+j\omega)t}dt=-\frac{1}{\tau+j\omega}e^{-(\tau+j\omega)t}\Big|_{0}^{\infty}=\frac{1}{\tau+j\omega}$$

$$\mathcal{F}\{\operatorname{sgn}(t)\}=\lim_{\tau\to 0}\left[\frac{1}{\tau+j\omega}-\frac{1}{\tau-j\omega}\right]=\lim_{\tau\to 0}\frac{-2j\omega}{\tau^2+\omega^2}=\frac{2}{j\omega}$$

$Hence:$

$$\mathcal{F}\{u(t)\}=\mathcal{F}\left\{\frac{1}{2}+\frac{1}{2}u(t)\right\}=\pi\delta(\omega)+\frac{1}{j\omega}$$

3. Dirac and Heaviside distributions

- **Property 6 –Fourier transforms of other functions**
  - ✍ Sine and Cosine:

$$\mathcal{F}\{1\} = 2\pi\delta(\omega) \Leftrightarrow \mathcal{F}\left\{1 \cdot e^{j\omega_0 t}\right\} = 2\pi\delta(\omega - \omega_0)$$

Also: $\mathcal{F}\left\{e^{-j\omega_0 t}\right\} = 2\pi\delta(\omega + \omega_0)$

Hence:

$$\mathcal{F}\{\cos\omega_0 t\} = \frac{1}{2}\mathcal{F}\left\{\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)\right\} = \pi\left(\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right)$$

$$\mathcal{F}\{\sin\omega_0 t\} = \frac{1}{2}\mathcal{F}\left\{\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right)\right\} = \frac{\pi}{j}\left(\delta(\omega - \omega_0) - \delta(\omega + \omega_0)\right)$$

3. Dirac and Heaviside distributions

- **Property 7 – Integration in time**

$$\mathcal{F}\left\{\int_{-\infty}^{t} x(\tau)\,d\tau\right\} = \frac{X(\omega)}{j\omega} \quad \textit{for signals with} \quad X(0)=0$$

In general: $\mathcal{F}\left\{\int_{-\infty}^{t} x(\tau)\,d\tau\right\} = \frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega)$

Proof:

$\int_{-\infty}^{t} x(\tau)\,d\tau = \int_{-\infty}^{\infty} x(\tau)u(t-\tau)\,d\tau = x(t)*u(t)$. Hence:

$$\mathcal{F}\left\{\int_{-\infty}^{t} x(\tau)\,d\tau\right\} = X(\omega)\cdot\left(\frac{1}{j\omega} + \pi\delta(\omega)\right) \Leftrightarrow$$

$$\mathcal{F}\left\{\int_{-\infty}^{t} x(\tau)\,d\tau\right\} = \frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega)$$