Wavelet Transform

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Topics

- Fundamentals of Signal Decompositions
- Why Wavelets?
- Time-Frequency Representations
- CWT, STFT and Frame Theory
- The Multiresolution Representation
- Applications
  - Wavelet Based Image Coding
  - Multiscale Edge Detection via CWT
  - Image Enhancement using Wavelets
  - Wavelet based Denoising
- Wavelet Bases & Filter Banks

2. Why Wavelets?

2.1. Review of the Fourier Theory

2.2. Drawbacks of the Fourier Analysis

2.2. Drawbacks of the Fourier Analysis

- The Fourier Transform and its variations are signal expansions in which the basis functions are complex exponentials
- *Analysis:*

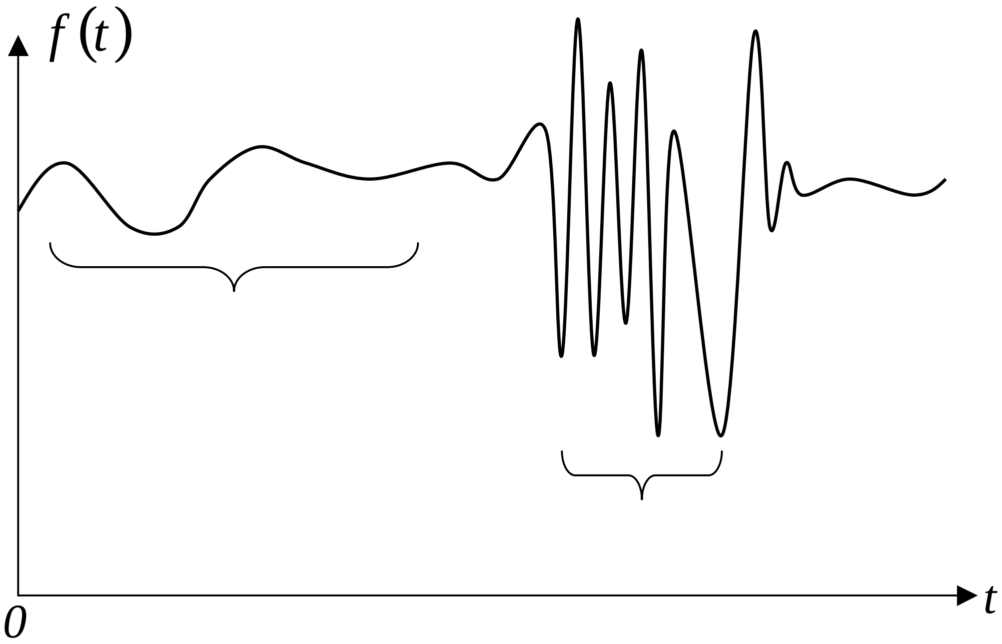
$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t} dt,$$

$$F(\omega) = \left\langle e^{j\omega t}, f(t) \right\rangle$$

- *Synthesis:*

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t} d\omega$$

$$f(t) = \left\langle e^{-j\omega t}, F(\omega) \right\rangle$$



- Localization property:

_ the “spread” of the basis function in time & frequency.

GOAL: localizing the analysis via localized basis functions in time <u>and</u> in frequency $\Rightarrow$ capturing of signal transients.

2.2. Drawbacks of the Fourier Analysis

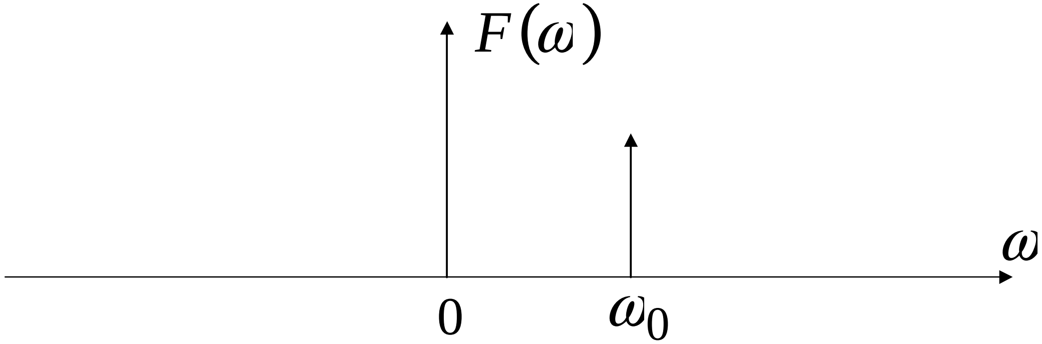
Localization properties for Fourier analysis

- No localization in time

$$\left.\begin{aligned} T &= \frac{2\pi}{\omega} \\ e^{j\omega t} &= e^{j\omega(t+kT)} \end{aligned}\right\} \Rightarrow \text{ infinite extent in time domain.}$$

- Perfect localization in frequency.

$$f(t) = e^{j\omega_0 t} \Rightarrow F(\omega) = \int_{-\infty}^{\infty} e^{-j(\omega-\omega_0)t}\,dt = \delta(\omega-\omega_0)$$



In order to capture large trends in the signal together with small variations (transients), basic modifications of the classical basis functions are required.

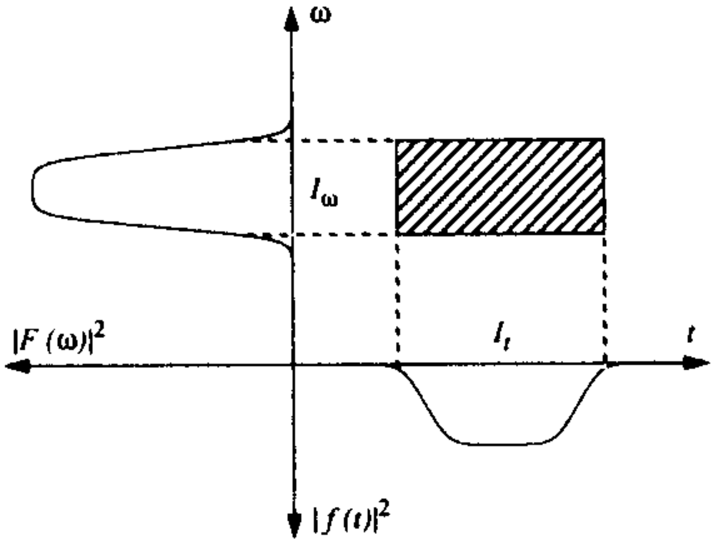
3. Time-Frequency Representations

Tiles in the time-frequency plane

- In signal expansions: primary concern is the localization of a given basis function in time and frequency
- Various ways to define localization
- All are related with the "spread" of the function in time and frequency

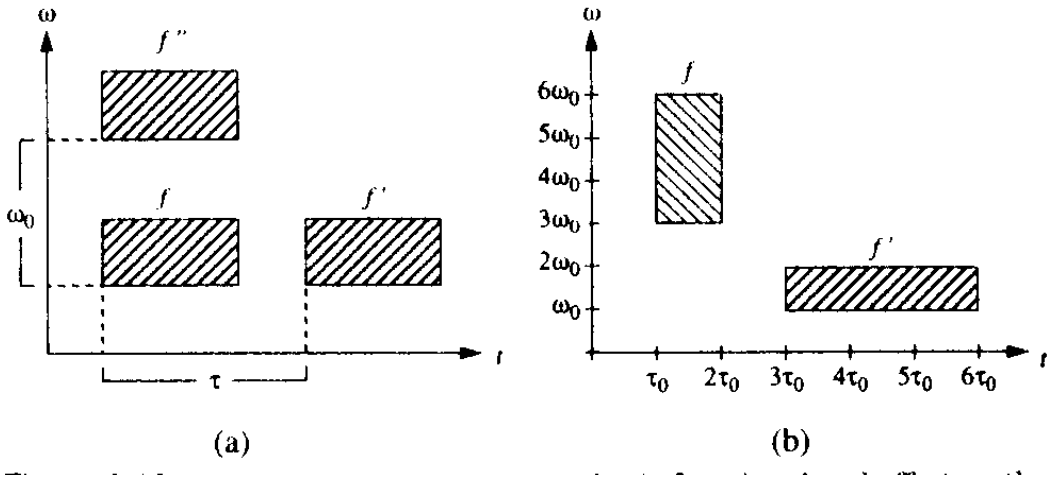
• $f(t)$ a basis function

• A ***tile*** contains 90% of the energy of the time and frequency-domain functions, and is centered around the centers of gravity of $|f(t)|^2$, respectively $|F(\omega)|^2$.



Properties of Tiles

- Time shift $\tau \Rightarrow$ shifting of the tile by τ
- Modulation by $e^{j\omega_0 t} \Rightarrow$ shift of the tile in frequency by ω_0
- Scaling by s, i.e. $f'(t) = f(st) \Rightarrow I_t' = I_t / s$ and $I_\omega' = sI_\omega$
  Resolution in frequency is traded for resolution in time.



3. Time-Frequency Representations

- **Time Support of** $f(t)$: $I_t = [-T, +T]$

Define: $f'(t) \overset{\Delta}{=} f(st), \quad s > 0$

$-T \le t \le T$

$-T \le t_1 = st \le T \Rightarrow \quad -\dfrac{T}{s} \le t \le \dfrac{T}{s}$

Time support of $f'(t)$ **is** $\boxed{I_t' = \dfrac{I_t}{s}}$

- **Frequency Support of** $F(\omega)$: $I_\omega = [\omega_{\min}, \omega_{\max}]$

$$F'(\omega) = \int_{-T/s}^{T/s} f'(t) e^{-j\omega t} dt = \int_{-T/s}^{T/s} f(st) e^{-j\omega t} dt;$$

$change \;\; st \to t_1$

$$F'(\omega) = \frac{1}{s} \int_{T}^{T} f(t_1) e^{-j(\omega/s)t_1} \, dt_1 = \frac{1}{s} \cdot F\left(\frac{\omega}{s}\right)$$

$$\omega_{\min} \le \omega \le \omega_{\max}$$

$$\omega_{\min} \le \omega_1 = \frac{\omega}{s} \le \omega_{\max} \Rightarrow$$

$$s\omega_{\min} \le \omega \le s\omega_{\max}$$

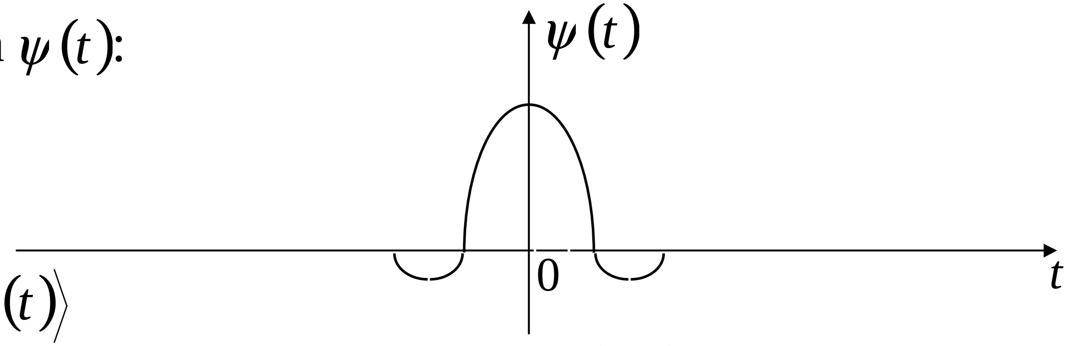
Frequency support of $f'(t)$ **is** $\boxed{I_\omega' = s \cdot I_\omega}$

3. Time-Frequency Representations

SCALING

- a fundamental operation used in the wavelet transform

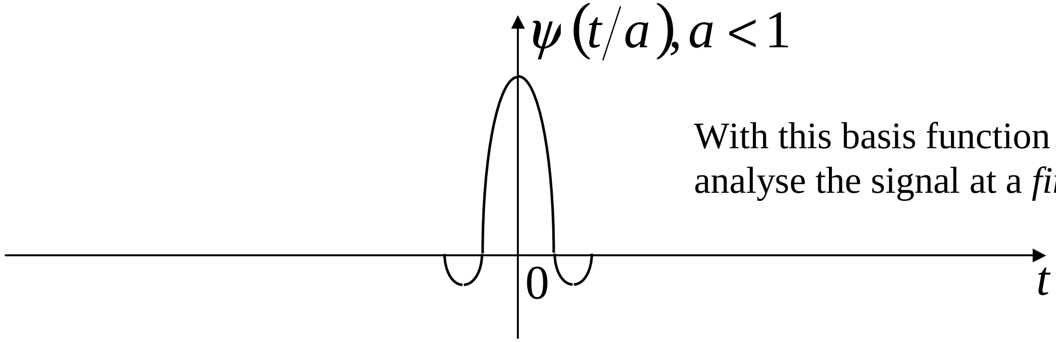
Scaling of a basis function $\psi(t)$:



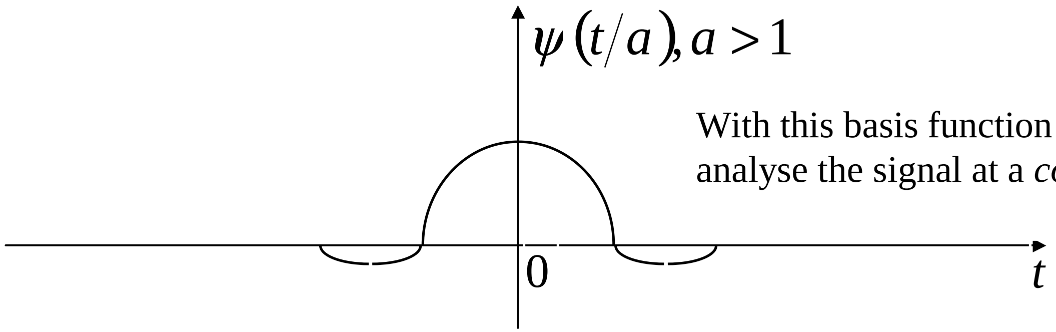
$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right)$$

$$Transform_{a,b} = \left\langle f(t), \psi_{a,b}(t) \right\rangle$$

1. Scaling by a, $a > 0$**.**



With this basis function we can analyse the signal at a *finer* scale



With this basis function we can analyse the signal at a *coarser* scale

3. Time-Frequency Representations

Scale $\Leftrightarrow a$:

$a > 1 \Rightarrow$ **large scale analysis**

$\psi_{a,b}(t)$ **identifies long term trends in** $f(t)$

$a < 1 \Rightarrow$ **short term (fine scale) analysis**

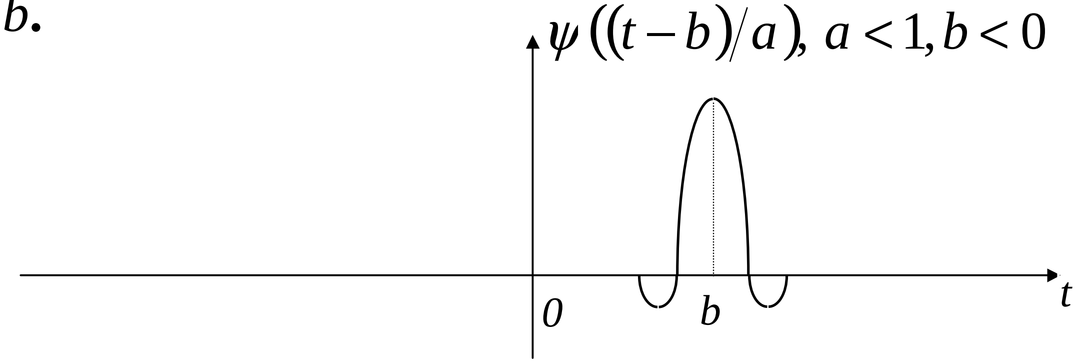
$\psi_{a,b}(t)$ **follows short-term behavior in** $f(t)$

- Assuming that the basis function is a band-pass filter (as in wavelet analysis):
  - high-frequency basis functions $\Rightarrow$ small scales
  - low-frequency basis functions $\Rightarrow$ large scales
  - Scale is loosely related to inverse frequency
- The notion of scale is similar to that used in geographical maps:
  - large scales $\Leftrightarrow$ coarse, global view
  - small scales $\Leftrightarrow$ fine, detailed view

3. Time-Frequency Representations

SCALING

2. Shifting by *b*.



3. Normalization factor.

Energy:

$$\int \left|\psi(t)\right|^2 dt = \int \psi(t)\psi^*(t)dt$$

$$\int \left|\psi_{a,0}(t)\right|^2 dt = \int \frac{1}{\sqrt{a}}\psi\left(\frac{t}{a}\right)\frac{1}{\sqrt{a}}\psi^*\left(\frac{t}{a}\right)dt = \int \psi(t')\,\psi^*(t')dt' = \int \left|\psi(t)\right|^2 dt$$

$\Rightarrow$ **Energy Conservation**

3. Time-Frequency Representations

Uncertainty Principle

- Sharpness of time analysis is traded off for sharpness in frequency and vice-versa.
- There is no way to get *arbitrarily* sharp analysis in both domains simultaneously.

- Consider a basis function $\psi(t)$ with Fourier transform $\Psi(\omega)$:
  - centered around origin in time and frequency
  - satisfying $\int t|\psi(t)|^2 dt = 0, \text{ and } \int \omega|\Psi(\omega)|^2 d\omega = 0$

- Sharpness in time/frequency quantified by:
  - *resolution in time*:

$$\Delta t^2 = \frac{\int t^2 |\psi(t)|^2 dt}{\int |\psi(t)|^2 dt}$$

  - *resolution in frequency*:

$$\Delta\omega^2 = \frac{\int \omega^2 |\Psi(\omega)|^2 d\omega}{\int |\Psi(\omega)|^2 d\omega}$$

- **If $\psi(t)$ vanishes faster than $1/\sqrt{t}$ as $t \to \infty$, then:**

$$\Delta t^2 \cdot \Delta\omega^2 \geq \frac{1}{4}$$

Equality holds for Gaussian functions, $\psi(t) = \sqrt[4]{\frac{2\alpha}{\pi}} e^{-\alpha t^2}$

3. Time-Frequency Representations

Uncertainty Principle - proof

$$\Delta t^2 \cdot \Delta\omega^2 \geq \frac{1}{4} \Leftrightarrow \left(\int_{-\infty}^{\infty} t^2 |\psi(t)|^2 dt\right)\left(\int_{-\infty}^{\infty} \omega^2 |\Psi(\omega)|^2 d\omega\right) \geq \frac{1}{4}\left(\int_{-\infty}^{\infty} |\psi(t)|^2 dt\right)\left(\int_{-\infty}^{\infty} |\Psi(\omega)|^2 d\omega\right)$$

Parseval's formula:

$$\int_{-\infty}^{\infty} |\psi(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\Psi(\omega)|^2 d\omega$$

$$\Delta t^2 \cdot \Delta\omega^2 \geq \frac{1}{4} \Leftrightarrow \left(\int_{-\infty}^{\infty} t^2 |\psi(t)|^2 dt\right)\left(\int_{-\infty}^{\infty} \omega^2 |\Psi(\omega)|^2 d\omega\right) \geq \frac{\pi}{2}\left(\int_{-\infty}^{\infty} |\psi(t)|^2 dt\right)^2$$

$$\left.\begin{array}{l} \psi'(t) \rightarrow j\omega\Psi(\omega) \\ \psi'(t)^* \rightarrow -j\omega\Psi^*(\omega) \end{array}\right\} \xrightarrow{Parseval} \int_{-\infty}^{\infty} |\psi'(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \omega^2 |\Psi(\omega)|^2 d\omega$$

What has to be proven:

$$\Delta t^2 \cdot \Delta\omega^2 \geq \frac{1}{4} \Leftrightarrow \left(\int_{-\infty}^{\infty} t^2 |\psi(t)|^2 dt\right)\left(\int_{-\infty}^{\infty} |\psi'(t)|^2 dt\right) \geq \frac{1}{4}\left(\int_{-\infty}^{\infty} |\psi(t)|^2 dt\right)^2$$

3. Time-Frequency Representations

Cauchy-Schwarz inequality

$$\left(\int_{-\infty}^{\infty}\left|f(t)\right|^2 dt\right)\left(\int_{-\infty}^{\infty}\left|g(t)\right|^2 dt\right) \geq \left|\int_{-\infty}^{\infty} f(t)g(t)dt\right|^2 \qquad f(t)=t\psi(t);\; g(t)=\psi'(t) \Rightarrow$$

$$\left(\int_{-\infty}^{\infty} t^2\left|\psi(t)\right|^2 dt\right)\left(\int_{-\infty}^{\infty}\left|\psi'(t)\right|^2 dt\right) \geq \left(\left|\int_{-\infty}^{\infty} t\psi(t)\psi'(t)dt\right|\right)^2 \overset{notation}{=} I^2$$

$$I = \int_{-\infty}^{\infty} t\psi(t)\psi'(t)dt = t\psi^2(t)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\left(\psi(t)+t\psi'(t)\right)\cdot\psi(t)dt = 0 - \int_{-\infty}^{\infty}\psi^2(t)dt - I \Rightarrow$$

$$I = \frac{1}{2}\int_{-\infty}^{\infty}\psi^2(t)dt \Rightarrow I^2 = \frac{1}{4}\left(\int_{-\infty}^{\infty}\psi^2(t)dt\right)^2$$

Equality in the Uncertainty Principle for:

$$\left.\begin{array}{l}\psi'(t) = kt\cdot\psi(t);\;\; k=-2\alpha \\ \int_{-\infty}^{\infty}\psi^2(t)dt = 1\end{array}\right\} \Rightarrow \psi(t) = \sqrt[4]{\frac{2\alpha}{\pi}}\, e^{-\alpha t^2}$$

4. STFT, CWT and Frames Theory

4.1. Continuous Short-Time Fourier Transform

4.2. Continuous Wavelet Transform

4.3. Frames of WT and STFT

- 4.3.1. Discretization of CWT
- 4.3.2. Discretization of STFT
- 4.3.3. Reconstruction in Frames
- 4.3.4. Frames of the CWT
- 4.3.5. Frames of the STFT

4.1. Continuous Short-Time Fourier Transform

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt,$$

$$\psi_\omega(t) = e^{j\omega t} \qquad F(\omega) = \left\langle e^{j\omega t}, f(t) \right\rangle$$

GOAL: Improve the localization properties of the Fourier transform.

- ***Forward transform***

$$STFT_f(\tau, \omega) = \int_{-\infty}^{\infty} f(t) \cdot w^*(t - \tau) \cdot e^{-j\omega t} dt$$

  - The inner product between the signal and the "shifts and modulates of an elementary window":

    $STFT_f(\tau,\omega) = \left\langle g_{\omega,\tau}(t), f(t) \right\rangle$, where $g_{\omega,\tau}(t) = w(t - \tau) e^{j\omega t}$

- ***Inverse transform***

$$f(t) = \frac{1}{2\pi \|w(t)\|^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} STFT_f(\omega, \tau) \cdot g_{\omega,\tau}(t) d\omega d\tau$$

4.1. Continuous Short-Time Fourier Transform

- Any classic window used for Fourier analysis is suitable for STFT transform; smooth windows are preferred; e.g. Hanning window:

$$w(t)=\begin{cases}[1+\cos(2\pi t/T)]/2, & t\in[-T/2,T/2]\\ 0, & otherwise\end{cases}$$

- It is convenient to choose the window such that $\|w(t)\|=\int |w(t)|^2 dt=1$
- The Gaussian window proposed by Gabor is given by:

  $w(t)=\beta e^{-\alpha t^2}, \alpha,\beta>0$, where α controls the width and β is a normalization factor.

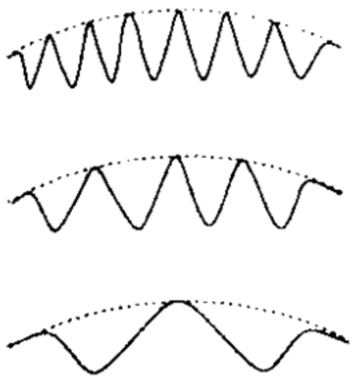
- Parseval's formula:

$$\int_{-\infty}^{\infty} |f(t)|^2 dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|STFT_f(\omega,\tau)|^2 d\omega d\tau$$

- The time-frequency resolution of each elementary function is constant, therefore it is natural to discretize the STFT on a rectangular grid $(m\omega_0,n\tau_0)$

4.1. Continuous Short-Time Fourier Transform

- Key question for STFT: the window size. Once the window size is chosen, all frequencies are analyzed with the same time and frequency resolutions, unlike what happens in the Wavelet Transform.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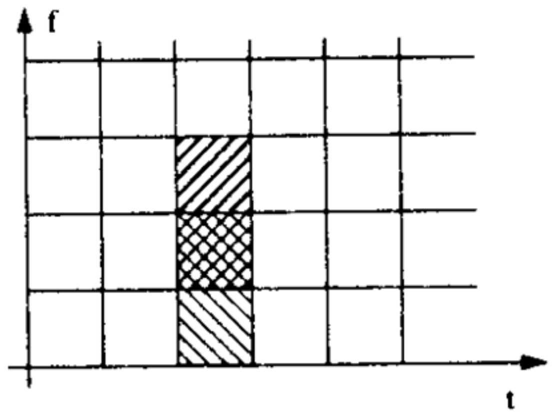


(b)

- It is not possible to distinguish different behaviors within a window spread.
- Alternative: generalized STFT with multiple window sizes $\Rightarrow$ overcomplete representations

4.2. Continuous Wavelet Transform

- Drawback introduced by the STFT is the constant resolution in time and frequency.
- Suppose equality in the Heisenberg principle:

$$\Delta t \cdot \Delta\omega = \frac{1}{2}$$

- New idea of CWT: *constant-relative bandwidth analysis*:

$$\frac{\Delta f}{f} = c \Rightarrow \Delta f = c \cdot f$$

$$\Delta t \cdot 2\pi\Delta f = \frac{1}{2} \Rightarrow \Delta t = \frac{1}{4\pi c \cdot f}$$

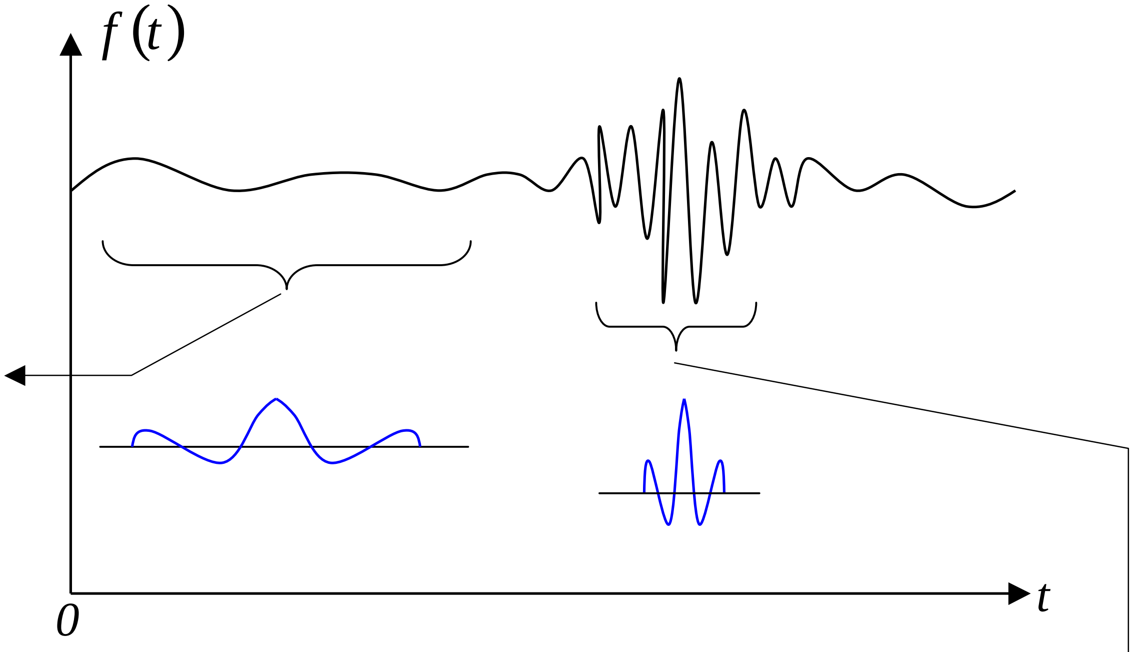
- High frequencies $\Rightarrow$

$f \nearrow$ | $\Delta f \nearrow$ Bad localization in frequency | $\Delta t \searrow$ Good localization in time

- Low frequencies $\Rightarrow$

$f \searrow$ | $\Delta f \searrow$ Good localization in frequency | $\Delta t \nearrow$ Bad localization in time

4.2. Continuous Wavelet Transform



Good localization in frequency:
discern between closed low-frequencies

Good localization in time:
discern between closed spikes

$f \nearrow$	$\Delta f \nearrow$ Bad localization in frequency	$\Delta t \searrow$ Good localization in time
$f \searrow$	$\Delta f \searrow$ Good localization in frequency	$\Delta t \nearrow$ Bad localization in time

4.2. Continuous Wavelet Transform

DEFINITION:

- Real band-pass filter with impulse response $\psi(t)$, zero mean, $\int_{-\infty}^{\infty} \psi(t) dt = 0$ and unit energy
- ***Forward transform***

$$CWT_f(a,\tau) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} f(t) \cdot \psi^* \left(\frac{t-\tau}{a} \right) dt, \quad for \quad f(t) \in L^2(R)$$

$\psi_{a,\tau}(t) = \frac{1}{\sqrt{a}} \cdot \psi\left(\frac{t-\tau}{a} \right)$ $\Rightarrow$ CWT measures the similarity between the signal and the scaled and shifted version of the elementary basis function $\psi(t)$:

$$CWT_f(a,\tau) = \left\langle f(t), \psi_{a,\tau}(t) \right\rangle$$

- ***Inverse transform***

If the wavelet $\psi(t)$ satisfies the *admissibility condition*: $C_\psi = \int_{-\infty}^{\infty} \frac{|\Psi(\omega)|^2}{|\omega|} d\omega < \infty$, with $\Psi(\omega)$ the Fourier transform of $\psi(t)$, then:

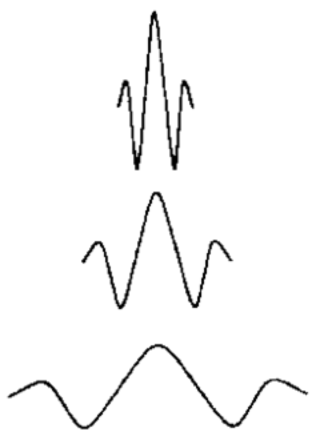
$$f(t) = \frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} CWT_f(a,\tau) \cdot \psi_{a,\tau}(t) \frac{da\, d\tau}{a^2}$$

4.2. Continuous Wavelet Transform

- Any $f(t) \in L^2(R)$ can be written as a superposition of shifted and dilated wavelets
- The reconstruction is in the L^2 sense (the L^2 norm of the reconstruction error is 0)
- $\left.\begin{array}{l} a > 1 \Rightarrow \psi_{a,\tau}(t) \text{ is "long" and of low frequency} \\ a < 1 \Rightarrow \psi_{a,\tau}(t) \text{ is "short" and of high frequency} \end{array}\right\} \Rightarrow$

$\Rightarrow$The discretization of the time-frequency space uses large time steps for large a $(a > 1)$, and fine time steps for small $a \Rightarrow$ tiling of the time-frequency plane.

$\Rightarrow$The discretization of (a, τ) is of the form $\left(a_0{}^n, a_0{}^n \cdot \tau_0\right)$



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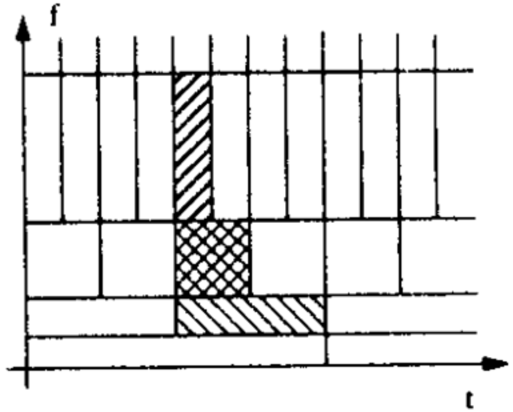


(d)

4.2. Continuous Wavelet Transform

PROPERTIES

1. Linearity - follows from the linearity of the inner product.

2. Shift Property

If $g(t) = f(t - \tau') \Rightarrow CWT_g(a, \tau) = CWT_f(a, \tau - \tau')$

3. Scaling Property

If $g(t) = \frac{1}{\sqrt{s}} f\left(\frac{t}{s}\right) \Rightarrow CWT_g(a, \tau) = CWT_f\left(\frac{a}{s}, \frac{\tau}{s}\right)$

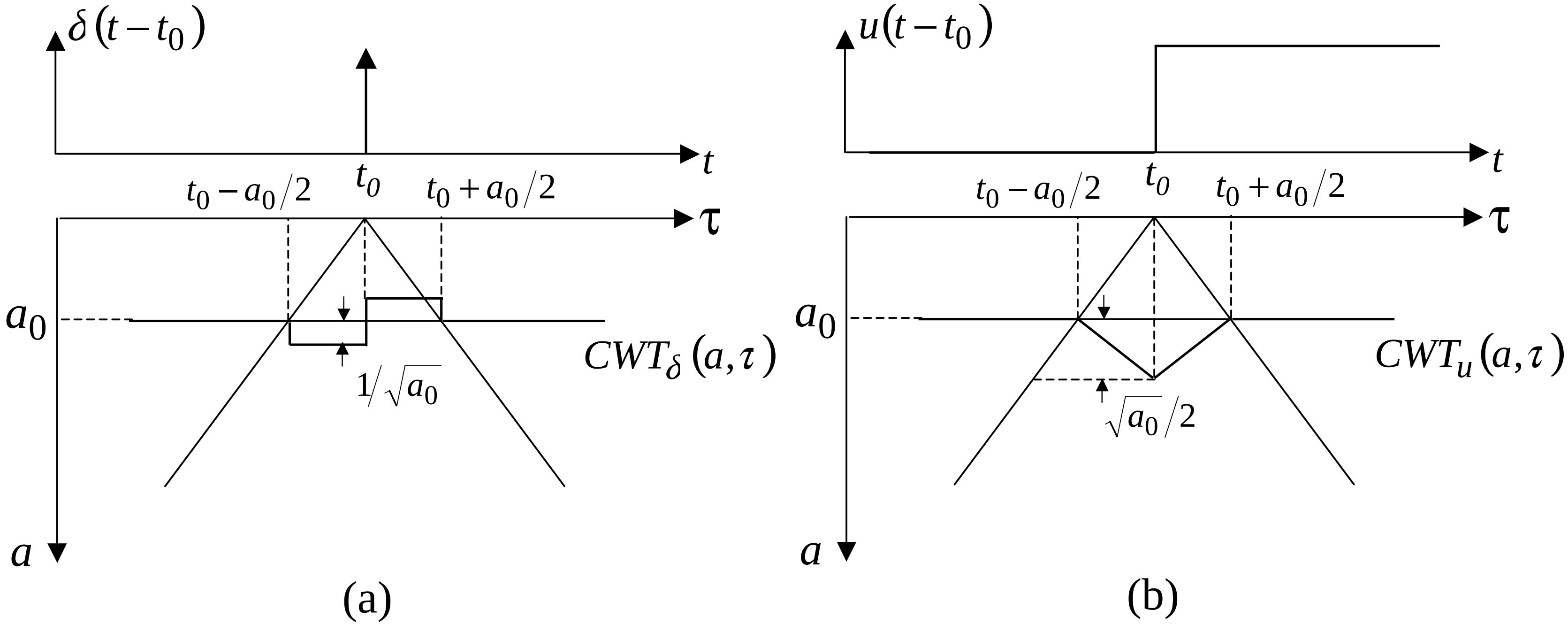
4. Time Localization

Example: The CWT of a Dirac pulse at time t_0 is:

$$CWT_\delta(a,\tau) = \frac{1}{\sqrt{a}} \psi^*\left(\frac{t_0 - \tau}{a}\right)$$

- The wavelet transform is equal to the scaled wavelet reversed in time and centered at the location of the Dirac.

4. Time Localization

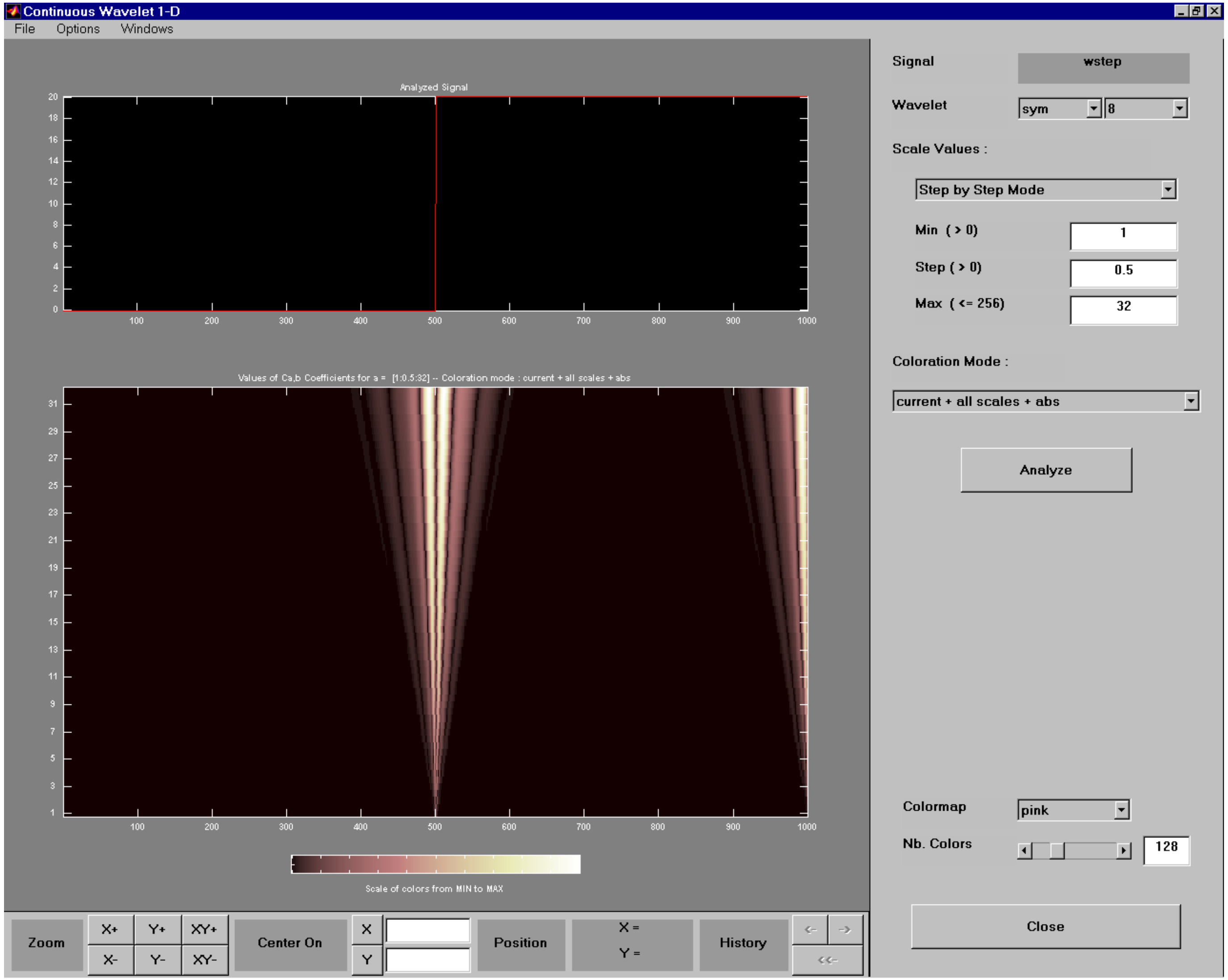


Time localization property, shown for the case of a zero-phase Haar wavelet.
(a) Behavior for $f(t)=\delta(t-t_0)$*. (b) Behavior for* $f(t)=u(t-t_0)$*.*

- The transform "zooms-in" to the Dirac or into the step function with a very good localization for very small scales *a*.

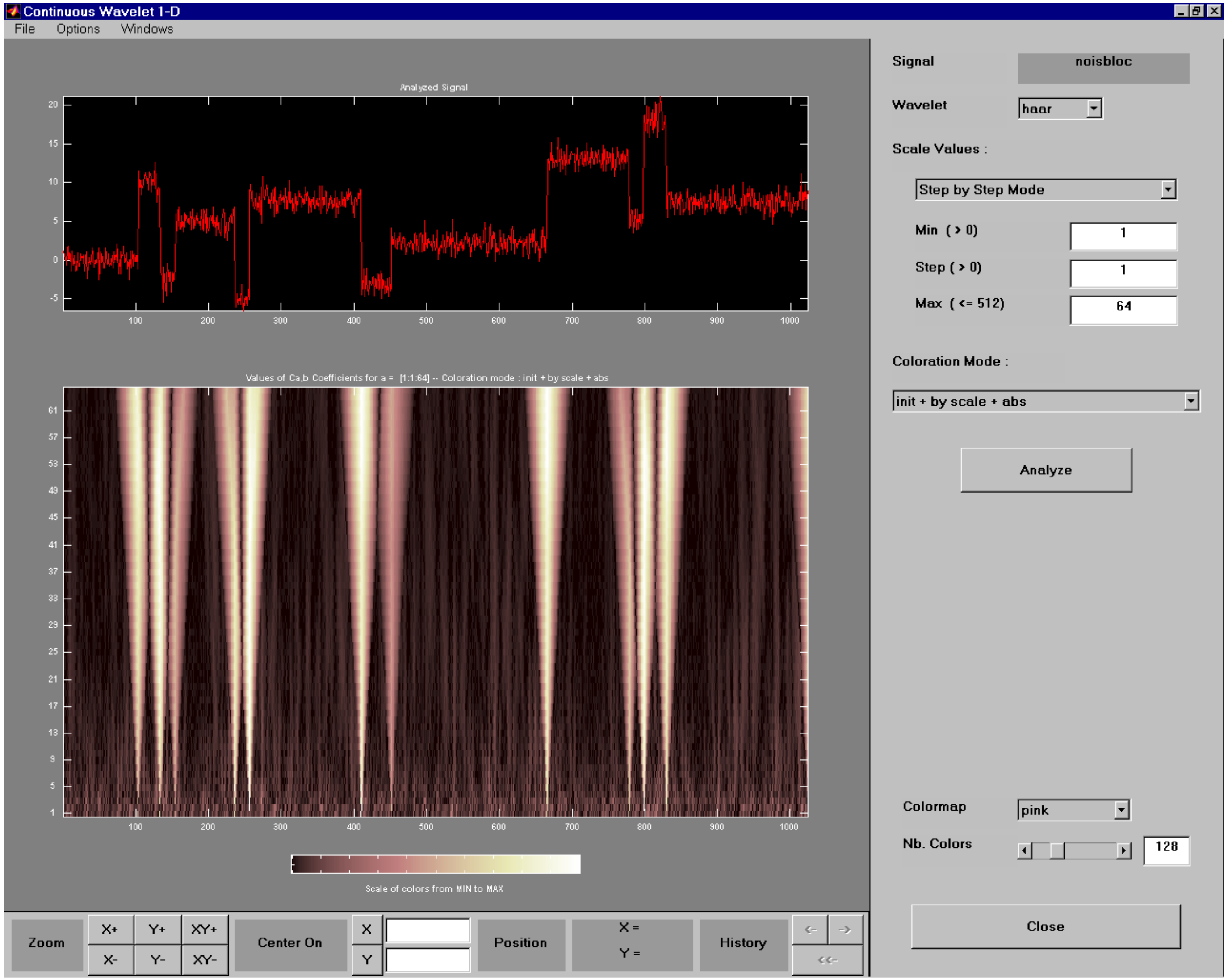
4.2. Continuous Wavelet Transform

Time Localization Property - Example



4.2. Continuous Wavelet Transform

Time Localization Property - Example

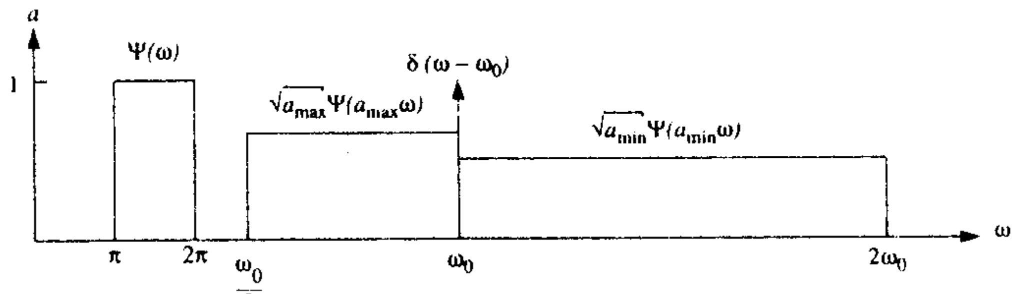


4.2. Continuous Wavelet Transform

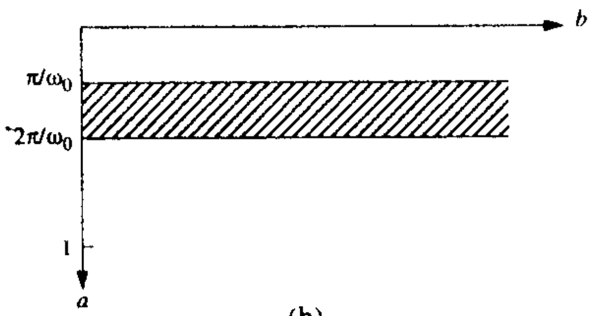
5. Frequency Localization

Example

- A complex sinusoide of unit magnitude at frequency ω_0.
- The “sinc wavelet” (a perfect bandpass filter) with the magnitude 1, for $|\omega| \in [\pi, 2\pi]$



(a)



(b)

1. The highest frequency wavelet that can analyze the signal has the scale $a_{\min} = \pi/\omega_0$

2. The lowest frequency wavelet that can analyze the signal has the scale $a_{\max} = 2\pi/\omega_0$

4.2. Continuous Wavelet Transform

5. Frequency Localization

$$\psi(t) \xrightarrow{FT} \Psi(\omega)$$

$$\psi\left(\frac{t}{a}\right) \xrightarrow{FT} a \cdot \Psi(a\omega); \; \textit{scaling}$$

$$\frac{1}{\sqrt{a}}\psi\left(\frac{t}{a}\right) \xrightarrow{FT} \sqrt{a} \cdot \Psi(a\omega); \; \textit{linearity}$$

1. $a\omega_0 = \omega, \quad \textit{and} \quad \omega = 2\pi \Rightarrow \quad a = \dfrac{2\pi}{\omega_0}$

Lowest frequency wavelet which can still analyze the signal $\Rightarrow a = a_{\max}$

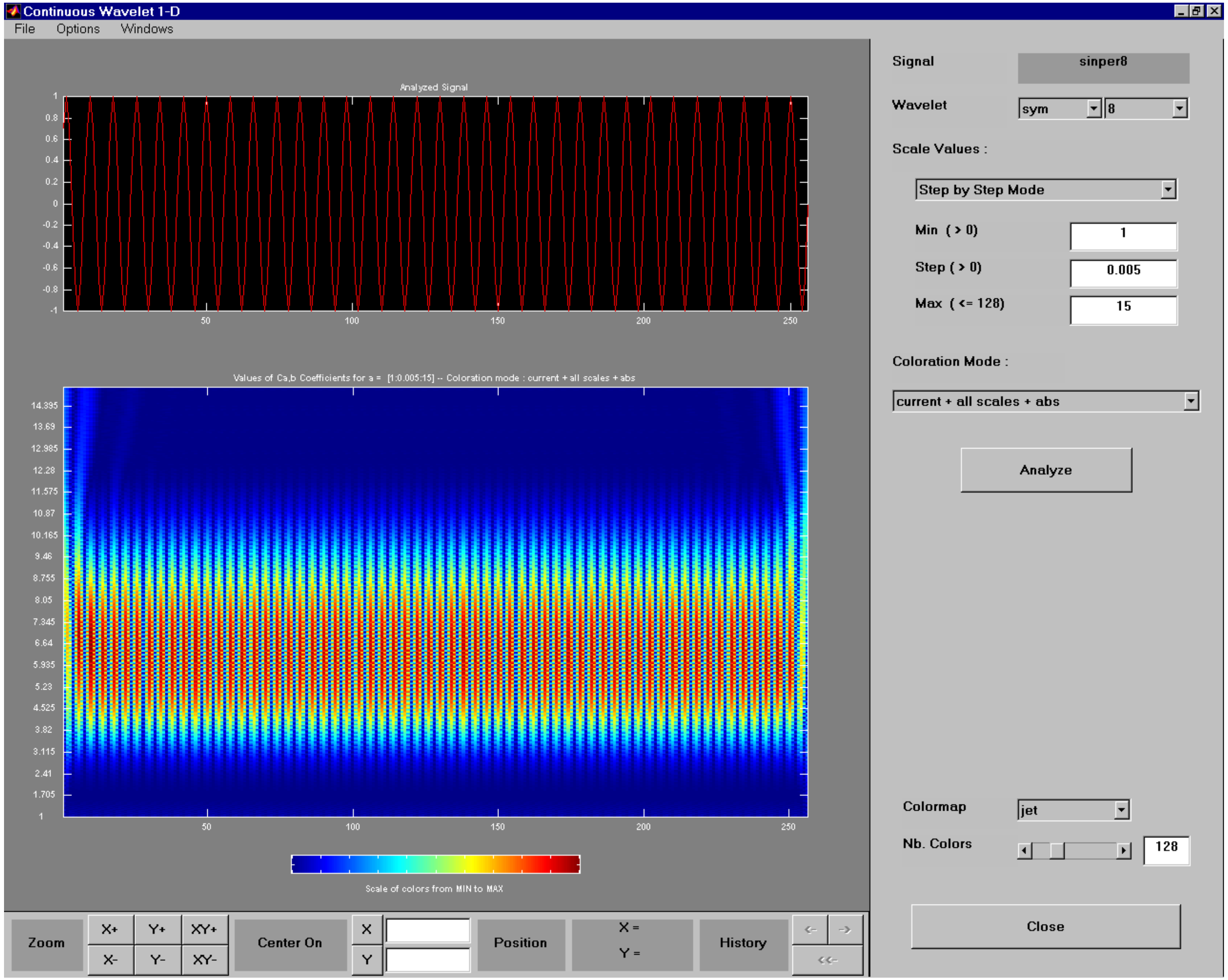
2. $a\omega_0 = \omega, \quad \textit{and} \quad \omega = \pi \Rightarrow \quad a = \dfrac{\pi}{\omega_0}$

Highest frequency wavelet which can still analyze the signal $\Rightarrow a = a_{\min}$

- Frequency resolution limited with this octave-band filter; solution: narrower band-pass filters - third of an octave for example.

4.2. Continuous Wavelet Transform

Frequency Localization Property - Example



4.3. Frames of WT and STFT

1. $$STFT_f(\tau,\omega)=\int_{-\infty}^{\infty} f(t)\cdot w^*(t-\tau)\cdot e^{-j\omega t}dt$$

$$STFT_f(\tau,\omega)=\langle f(t), g_{\omega,\tau}(t)\rangle, \quad where \quad g_{\omega,\tau}(t)=w(t-\tau)e^{j\omega t}$$

2. $$CWT_f(a,\tau)=\frac{1}{\sqrt{a}}\int_{-\infty}^{\infty} f(t)\cdot\psi^*\left(\frac{t-\tau}{a}\right)dt, \quad for \quad f(t)\in L^2(R)$$

$$CWT_f(a,\tau)=\langle f(t), \psi_{a,\tau}(t)\rangle, \quad where \quad \psi_{a,\tau}(t)=\psi\left(\frac{t-\tau}{a}\right)$$

- *Problem*: discretize these continuous-time transforms, represent the signal via a set of transform coefficients and reconstruct the signal in a stable way.

$$g_{\omega,\tau}\xrightarrow{discretize\ \omega,\tau} g_{m,n}, \quad such\ that \quad f=\sum_m\sum_n\langle g_{m,n}, f\rangle\tilde{g}_{m,n}$$

$$\psi_{a,\tau}\xrightarrow{discretize\ a,\tau} \psi_{m,n}, \quad such\ that \quad f=\sum_m\sum_n\langle \psi_{m,n}, f\rangle\tilde{\psi}_{m,n}$$

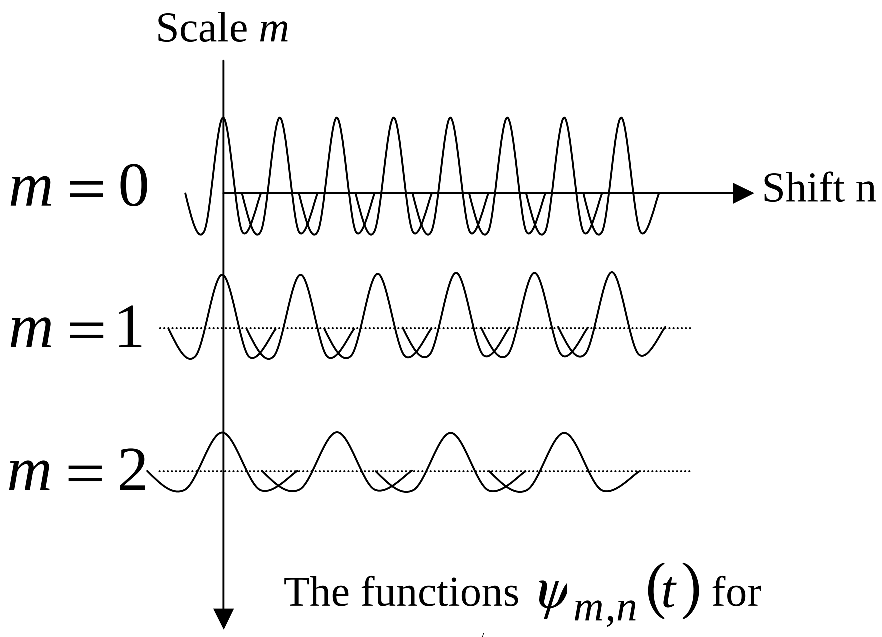
4.3.1. Discretization of the CWT

- Discretization of the scale parameter
  $a = a_0^{\,m}, m \in Z, a_0 \neq 1$
- Discretization of the time shift
  First idea : $\quad$ *For* $m = 0$, *take* $b = nb_0$
  - Choose b_0 such that $\psi_{1,b} = \psi(t - nb_0)$ covers the whole time axis
  - Moreover: choose b_0 such that for any m, $\psi_{a_0^{\,m},b}(t)$ covers the whole time axis.

Basis function:

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right)$$

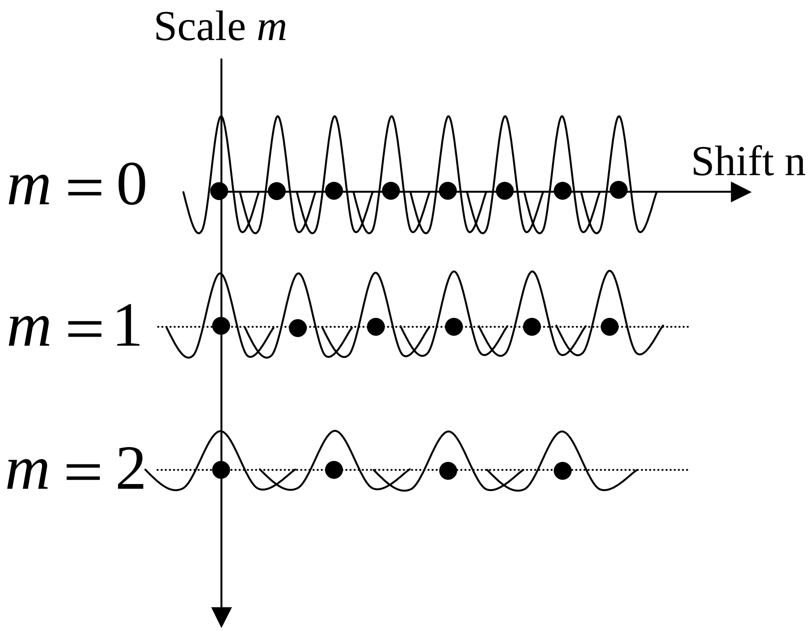
- Note:
  $$\Delta_t\left(\psi_{a_0^{\,m},0}(t)\right) = a_0^{\,m}\Delta_t\left(\psi_{1,0}(t)\right)$$
- Result:
  $$\boxed{a = a_0^{\,m} \ \textit{and}\ b = nb_0 a_0^{\,m}, \quad m,n \in Z, \quad a_0 > 1, b_0 > 0}$$
- Discretized family of wavelets:
  $$\psi_{m,n}(t) = a_0^{-m/2}\psi\left(a_0^{-m}t - nb_0\right)$$



The functions $\psi_{m,n}(t)$ for $a_0 = 2^{-1/2}$ and $b_0 = 1$

4.3.1. Discretization of CWT

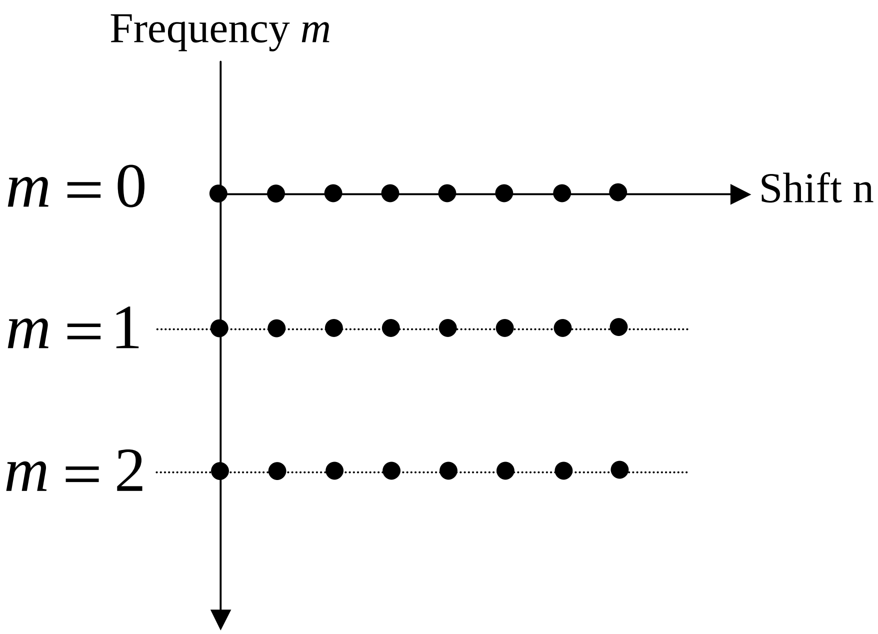
Sampling grid for the CWT



The sampling grid for CWT

$a_0 = 2^{-1/2}$ and $b_0 = 1$

Sampling grid for the STFT



The sampling grid for STFT

$t_0 = 1$

4.3.2. Discretization of the STFT

- Discretization of the frequency parameter

  $\omega = m\omega_0,\ \omega_0 > 0, m \in Z,\ \omega_0\ \ fixed$

Basis function:

$$g_{\omega,\tau}(t) = e^{j\omega t} w(t-\tau)$$

- Discretization of the time shift

  $\tau = nt_0,\ t_0 > 0,\ n \in Z, t_0\ \ fixed$

- Result:

  $g_{\omega,\tau}(t) \to g_{m,n}(t) = e^{jm\omega_0 t} w(t - nt_0)$

- Note:

  $\Delta_t\left(g_{m,0}(t)\right) = \Delta_t\left(g_{1,0}(t)\right)$

  $\Delta_\omega\left(g_{m,0}(t)\right) = \Delta_\omega\left(g_{1,0}(t)\right)$

- Different sampling grid for the discretization of the STFT if compared with the CWT

- Let's come back to the problem:

  $Given\ \left\langle g_{m,n}, f \right\rangle, in\ which\ conditions \quad f = \sum_m \sum_n \left\langle g_{m,n}, f \right\rangle \tilde{g}_{m,n}$

  $Given\ \left\langle \psi_{m,n}, f \right\rangle, in\ which\ conditions \quad f = \sum_m \sum_n \left\langle \psi_{m,n}, f \right\rangle \tilde{\psi}_{m,n}$

4.3.5. Frames of the STFT

Theorem (Balian-Low)

If $g_{m,n}(t) = e^{j2\pi mt} w(t-n),\ m,n \in Z$ constitutes a frame for $L^2(R)$ then either

$\int t^2 |w(t)|^2 dt = \infty$ either $\int \omega^2 |W(\omega)|^2 d\omega = \infty$

- Remark that the settings are $\omega_0 = 2\pi$ and $t_0 = 1 \Rightarrow \omega_0 t_0 = 2\pi$.
- Result of this theorem: *there is no way* to construct STFT orthonormal basis with good time-frequency localization.

Example

$$g_{m,n}(t) = e^{jm\omega_0 t} w(t - nt_0),$$

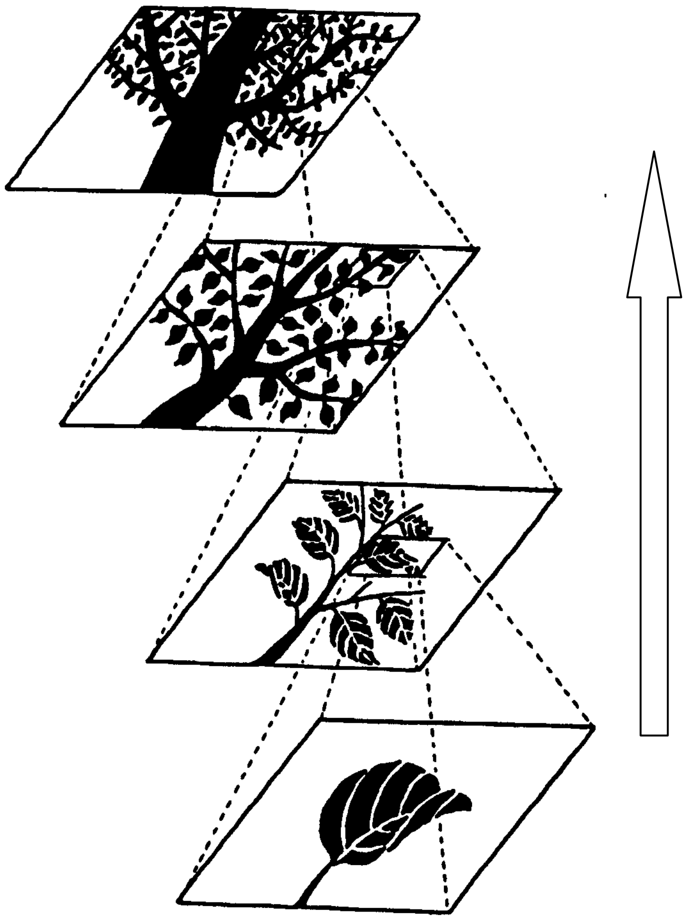
Gaussian window $w(t) = \pi^{-1/4} e^{-t^2/2}$

Dual frame:

$$\tilde{g}_{m,n}(t) = e^{jm\omega_0 t} \tilde{w}(t - nt_0)$$

The window $\tilde{w}(t)$ can be calculated with an iterative procedure.

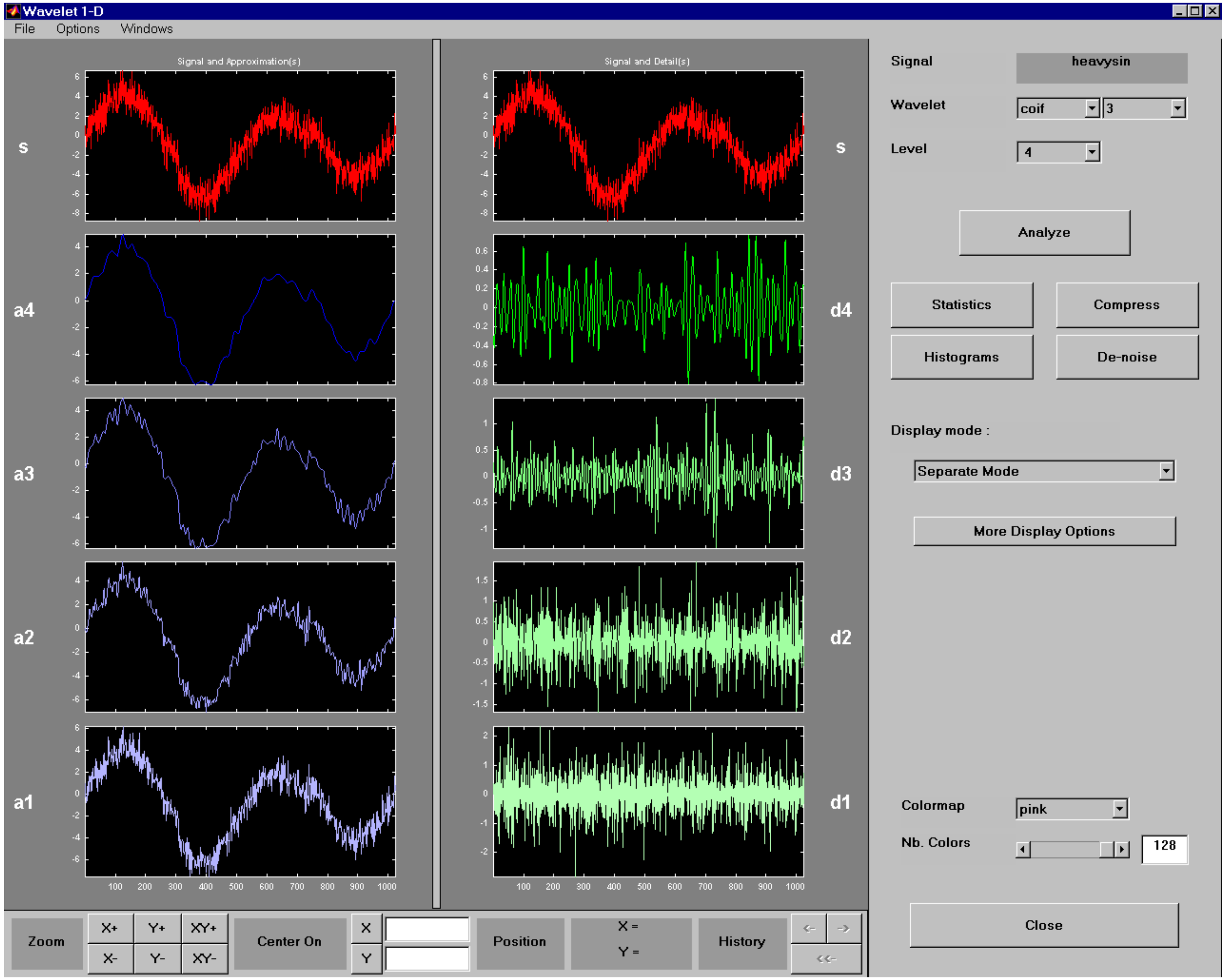
5. The Multiresolution Representation



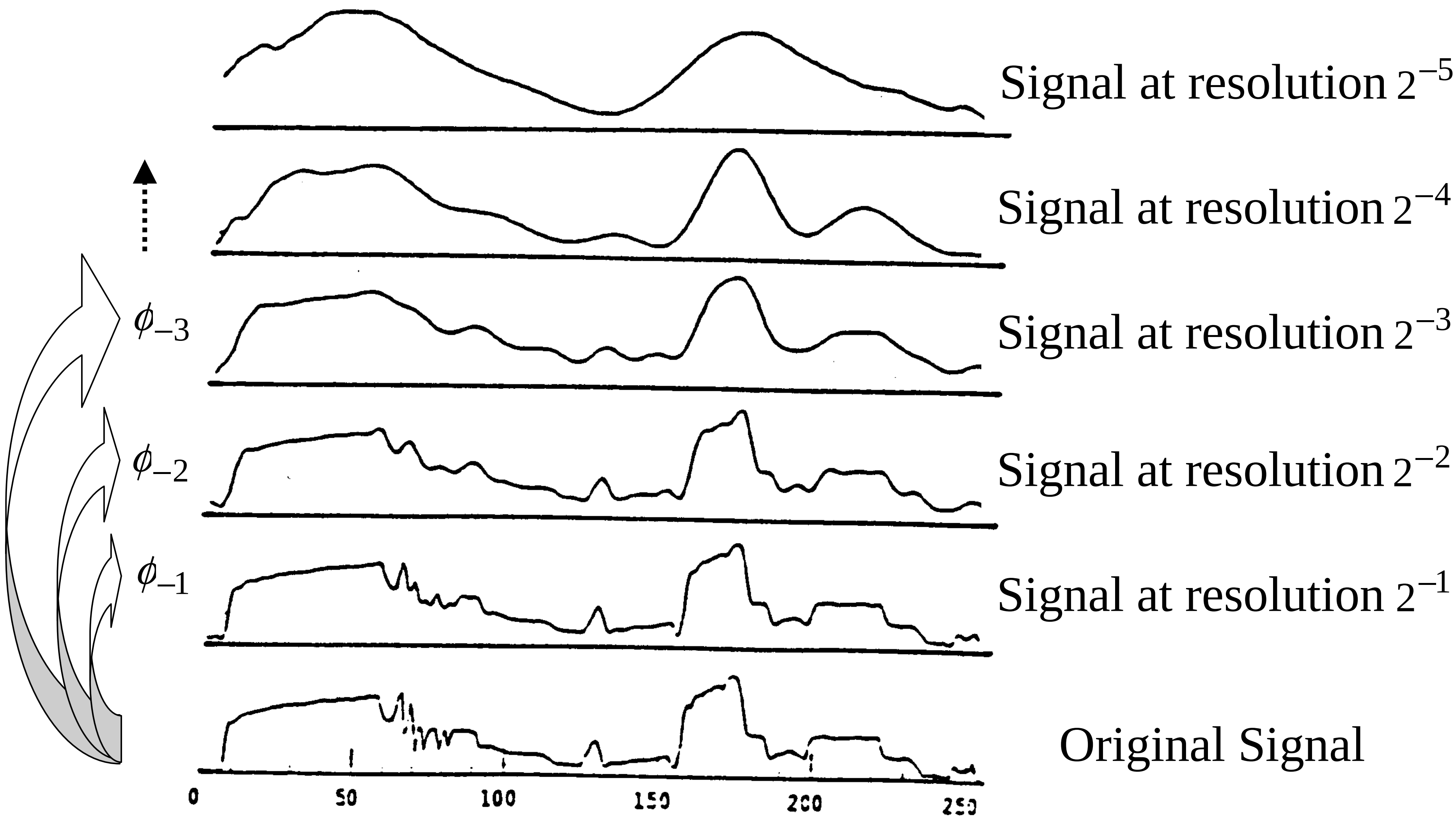
- **GOAL: to represent any arbitrary image as a set of successive approximations.**

5. The Multiresolution Representation

Example



5.1. Multiresolution Analysis



5.1. Multiresolution Analysis

SCALING FUNCTION

- $\varphi_j(x),\ j<0$ **- set of basis functions, all derived from one function** $\varphi(x)$:

$$\varphi_j(x) = 2^j \cdot \varphi\left(2^j x\right),\ j<0$$

$\varphi(x)$ - ***Scaling Function***

5.1. Multiresolution Analysis

Definitions

$$\left\langle f(u), g(u) \right\rangle = \int_{-\infty}^{\infty} f(u) \cdot g(u) du$$ **- Inner product**

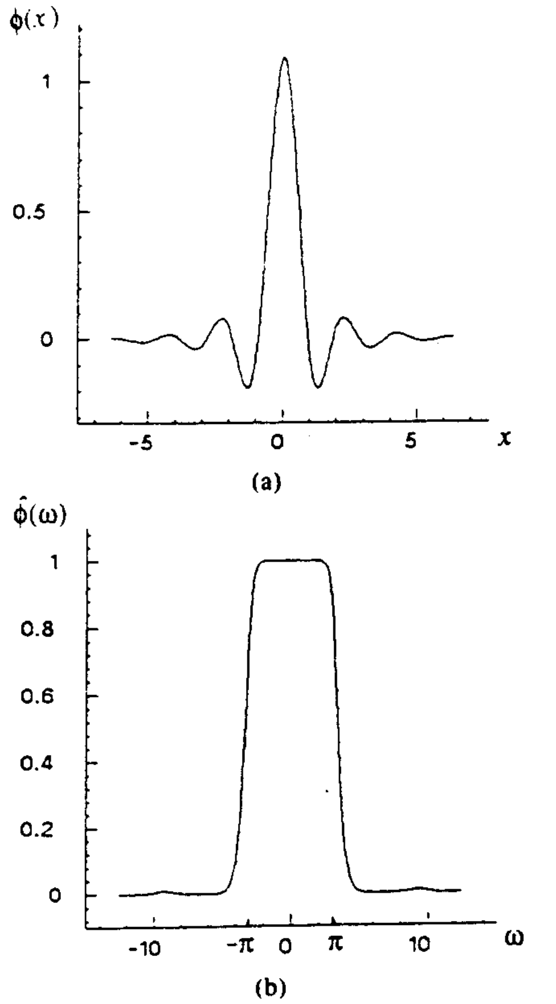
$A_j f(x)$**: Continuous approximation of** $f(x)$ **at the resolution** $2^j,\ j<0$

$$A_j f(x) = 2^{-j} \sum_{n=-\infty}^{\infty} \left\langle f(u), \phi_j\left(u - 2^{-j} n\right) \right\rangle \cdot \phi_j\left(x - 2^{-j} n\right)$$

$$\underline{s} = \sum_{n=-\infty}^{\infty} coeff(n) \cdot b(n) \Leftrightarrow \underline{s} = \underline{coeff} \cdot \underline{b}$$

5.1. Multiresolution Analysis

DISCRETE APPROXIMATIONS



- **A scaling function is a low pass filter.**
- **The set of inner products:**

$$A_j^d f(n) = \left\langle f(u), \phi_j\left(u - 2^{-j} n\right)\right\rangle_{n \in Z}$$

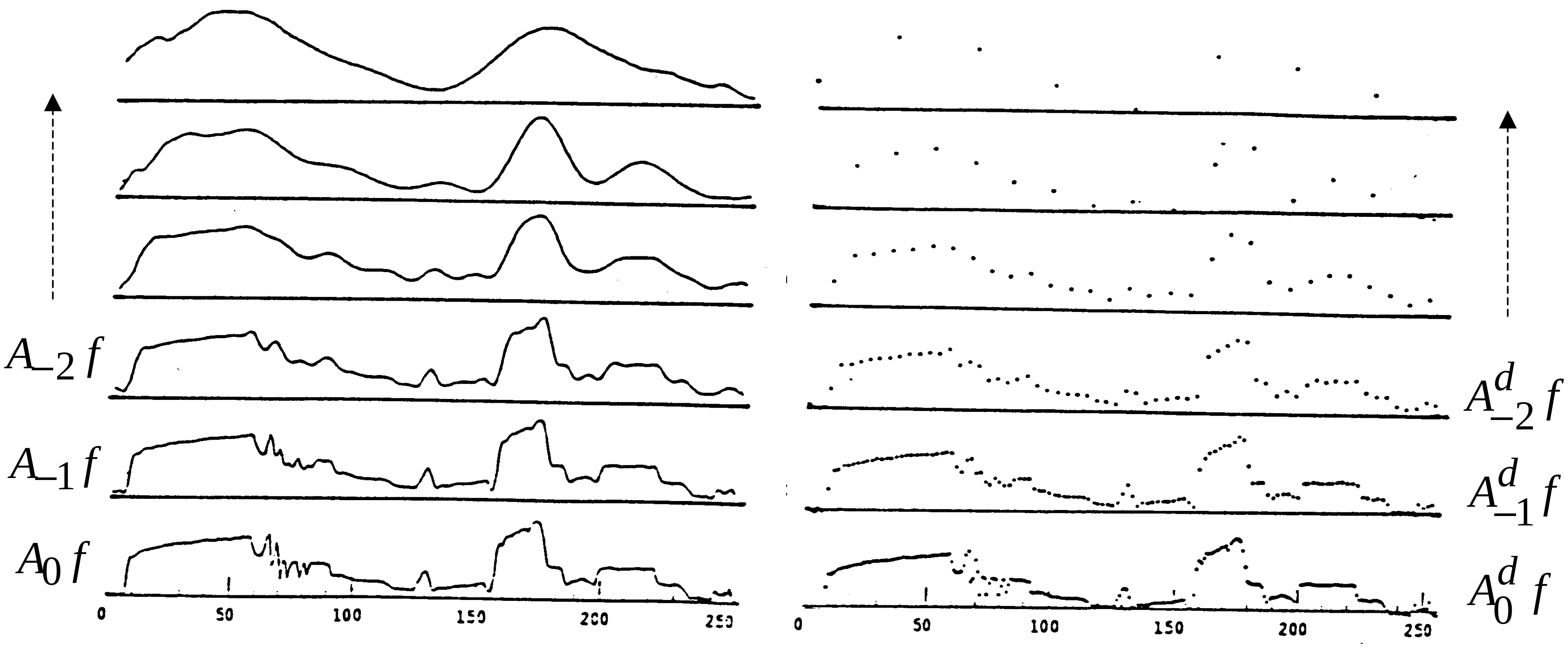
is called *discrete approximation* of $f(x)$ **at the resolution** 2^j**.**

$$A_j^d f(n) = \left(f(u) * \phi_j(-u)\right)\left(2^{-j} n\right)_{n \in Z}$$

- **Low pass filtering**
- **Uniform sampling at the rate** 2^j**.**

5.1. Multiresolution Analysis

Continuous and the discrete approximations at several resolutions



- **In practice we use discrete signals.** $A_0^d f = (\alpha_n)_{1 \le n \le N}$
- **Imaging device gives a finite number of samples**

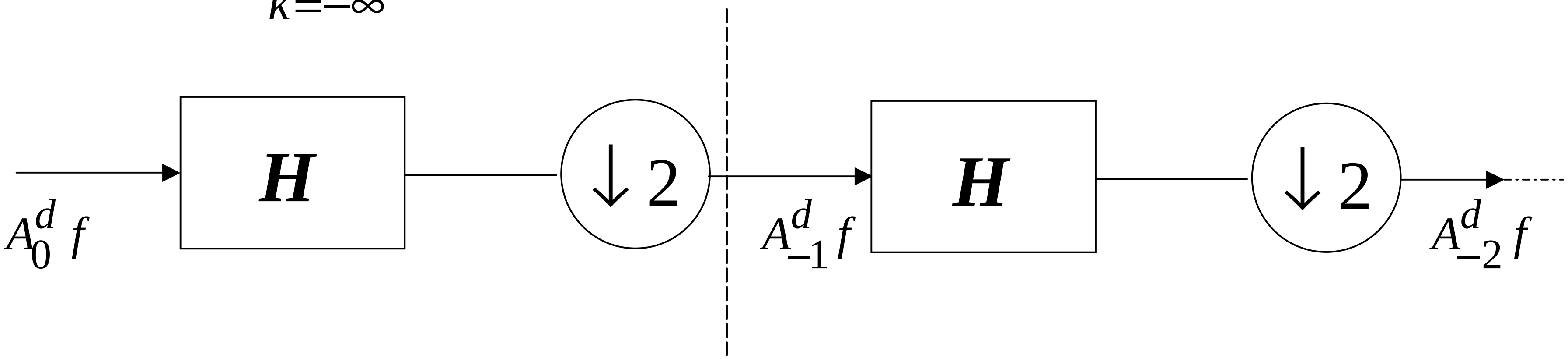
5.1. Multiresolution Analysis

FAST ALGORITHM FOR CALCULATING $A_j^d f$.

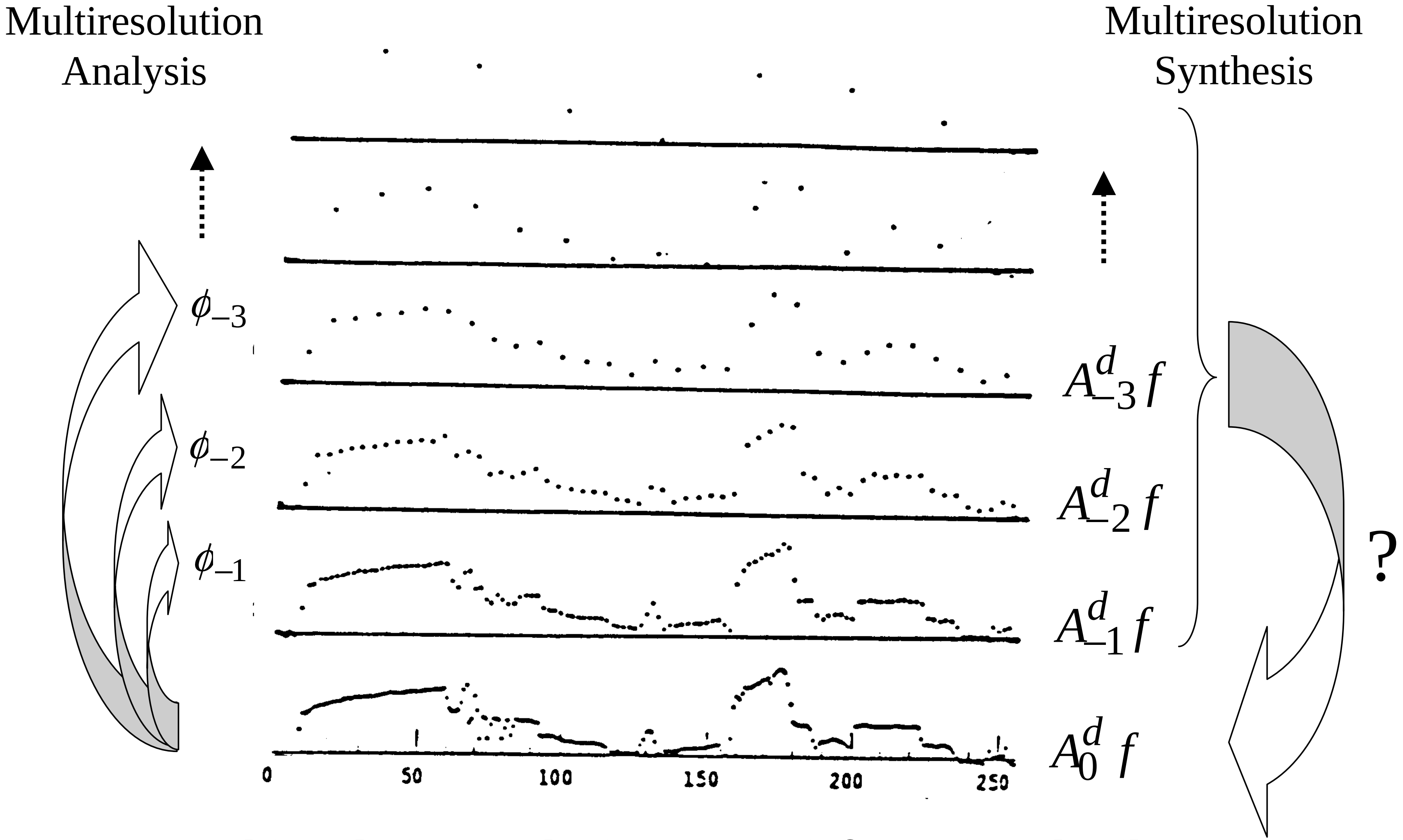
Pyramidal algorithm.

- $A_j^d f$ is computed by convolving $A_{j+1}^d f$ with H and downsampling by 2:

$$A_j^d f(n) = \sum_{k=-\infty}^{\infty} h(2n-k) \cdot A_{j+1}^d f(k), \quad j<0$$

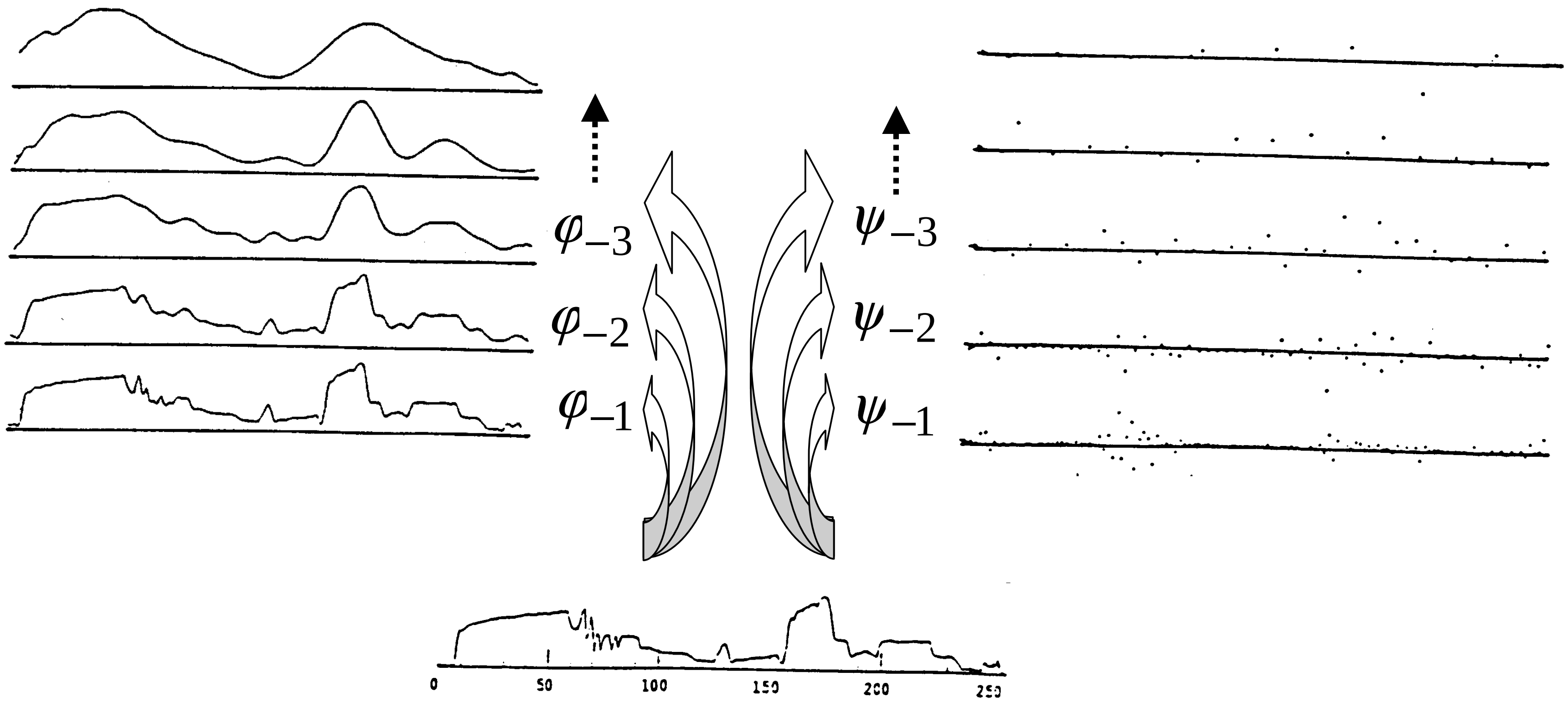


5.1. Multiresolution Analysis



Multiresolution synthesis $\Rightarrow$ extra information (details).

5.1. Multiresolution Analysis



WAVELET FUNCTION

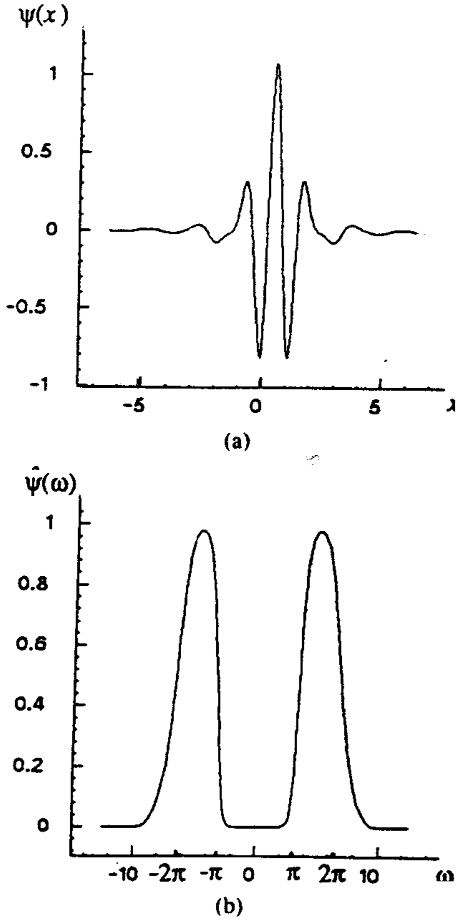
- $\psi_j(x), \quad j < 0$ **- set of basis functions, all derived from one function** $\psi(x)$:

$$\psi_j(x) = 2^j \cdot \psi\left(2^j x\right), \quad j < 0$$

$\psi(x)$ - ***Wavelet***

5.1. Multiresolution Analysis

DISCRETE DETAILS



- **The wavelet is a band pass filter.**
- **The set of inner products:**

$$D_j f(n) = \left\langle f(u), \psi_j\left(u - 2^{-j} n\right)\right\rangle_{n \in Z}$$

is called <u>*discrete detail*</u> of $f(x)$ at the resolution 2^j.

$$D_j f(n) = \left(f(u) * \psi_j(-u)\right)\left(2^{-j} n\right)_{n \in Z}$$

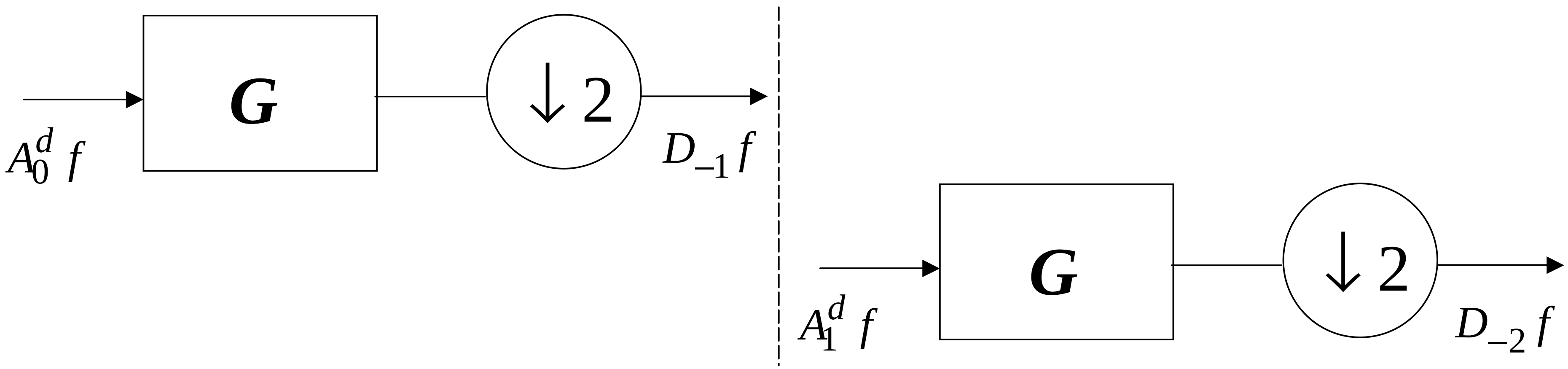
- **Band pass filtering**
- **Uniform sampling at the rate 2^j.**

FAST ALGORITHM FOR CALCULATING $D_j f$.

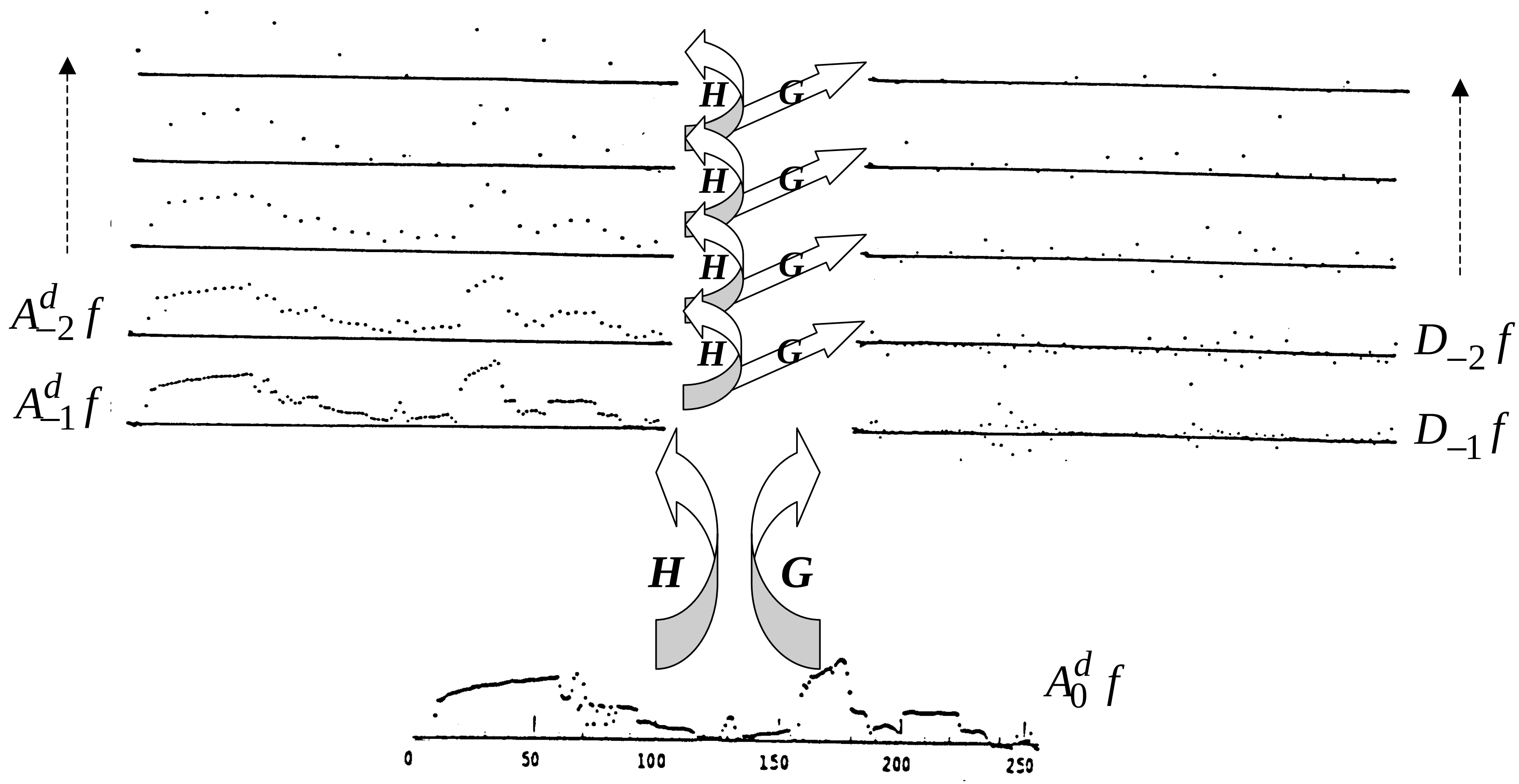
Pyramidal algorithm.

- $D_j f$ is computed by convolving $A_{j+1}^d f$ with G followed by a downsampling by 2:

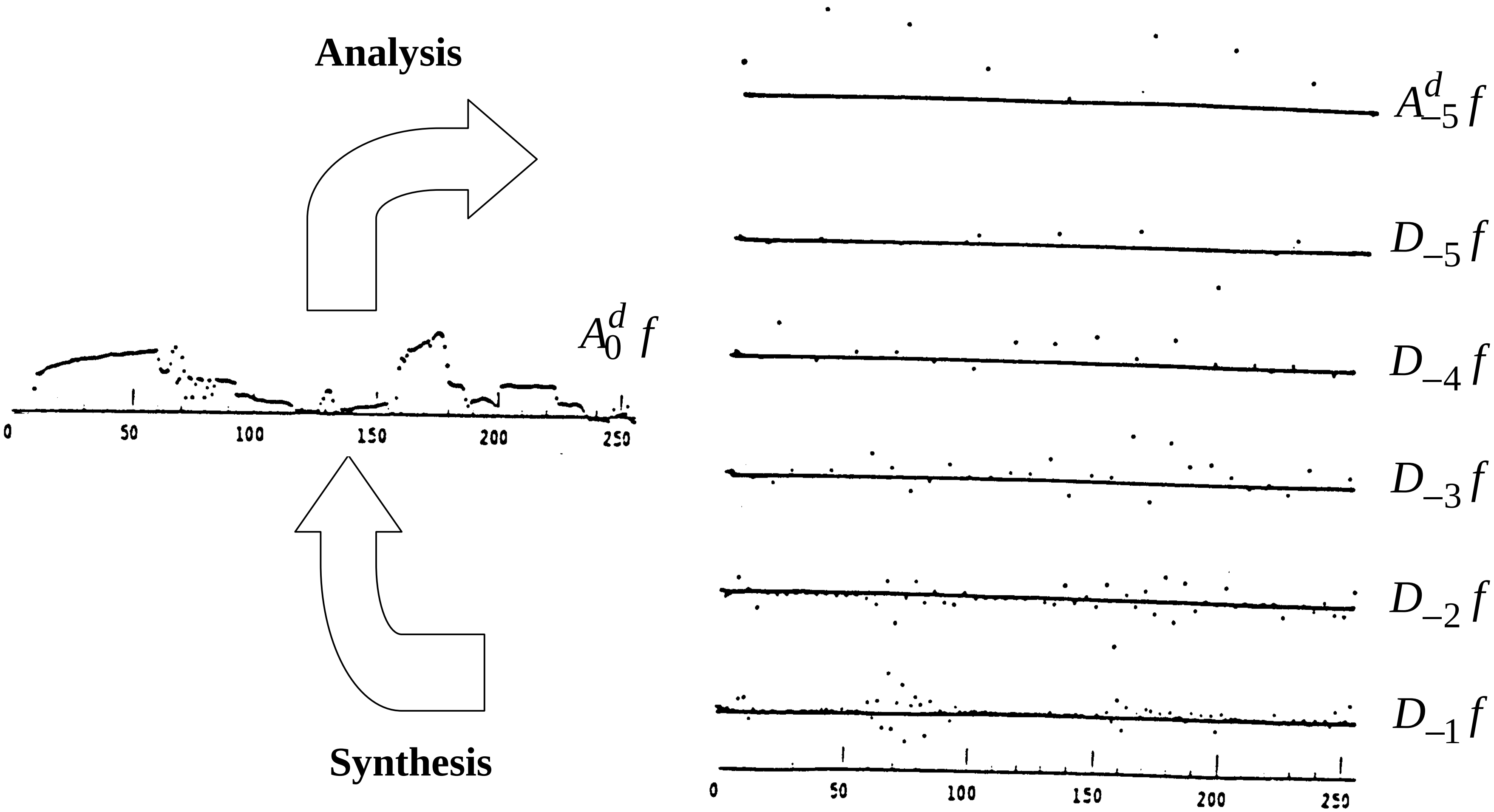
$$D_j f(n) = \sum_{k=-\infty}^{\infty} g(2n-k) \cdot A_{j+1}^d f(k), \quad j \le 0$$



5.2. Dyadic Wavelet Representation



5.2. Dyadic Wavelet Representation



5.2. Dyadic Wavelet Representation

$\boldsymbol{H}$ – analysis low pass filter
$\boldsymbol{G}$ – analysis high pass filter
$\tilde{\boldsymbol{H}}$ – synthesis low pass filter
$\tilde{\boldsymbol{G}}$ – synthesis high pass filter

$$\tilde{H}(n) = H(-n)$$

$$\tilde{G}(n) = G(-n)$$

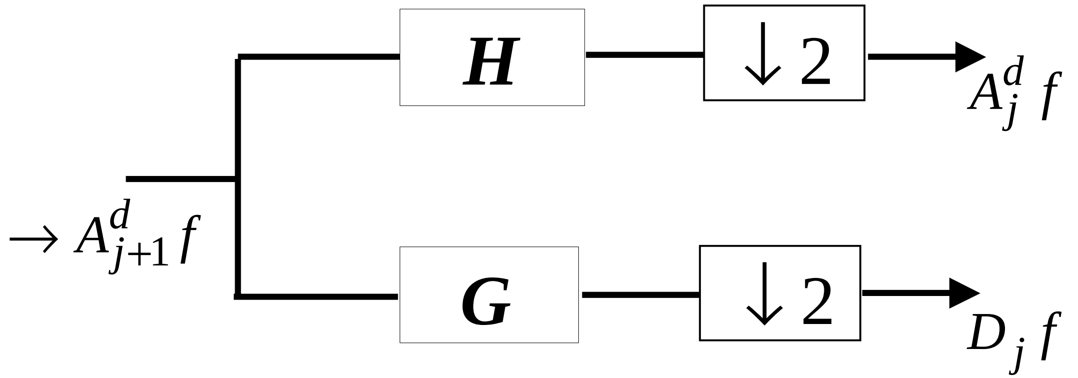
$$\Phi(2\omega) = H(\omega)\cdot\Phi(\omega)$$

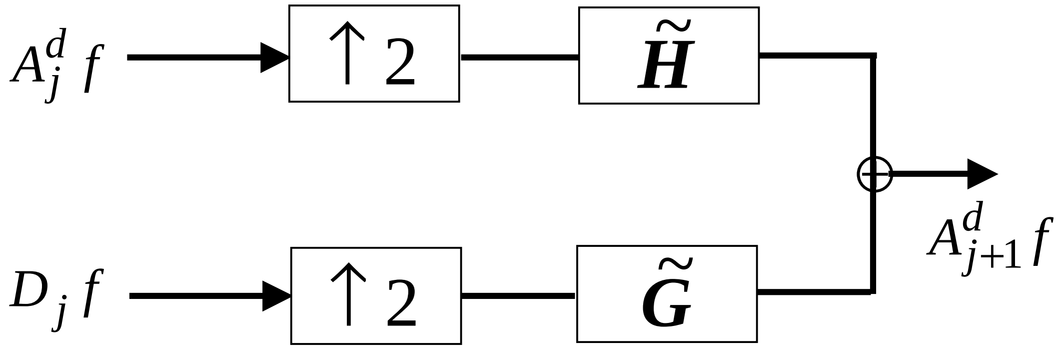
$$\Psi(2\omega) = G(\omega)\cdot\Phi(\omega)$$

$\boxed{\downarrow 2}$ - downsampling by 2

$\boxed{\uparrow 2}$ - upsampling by 2

Wavelet Analysis

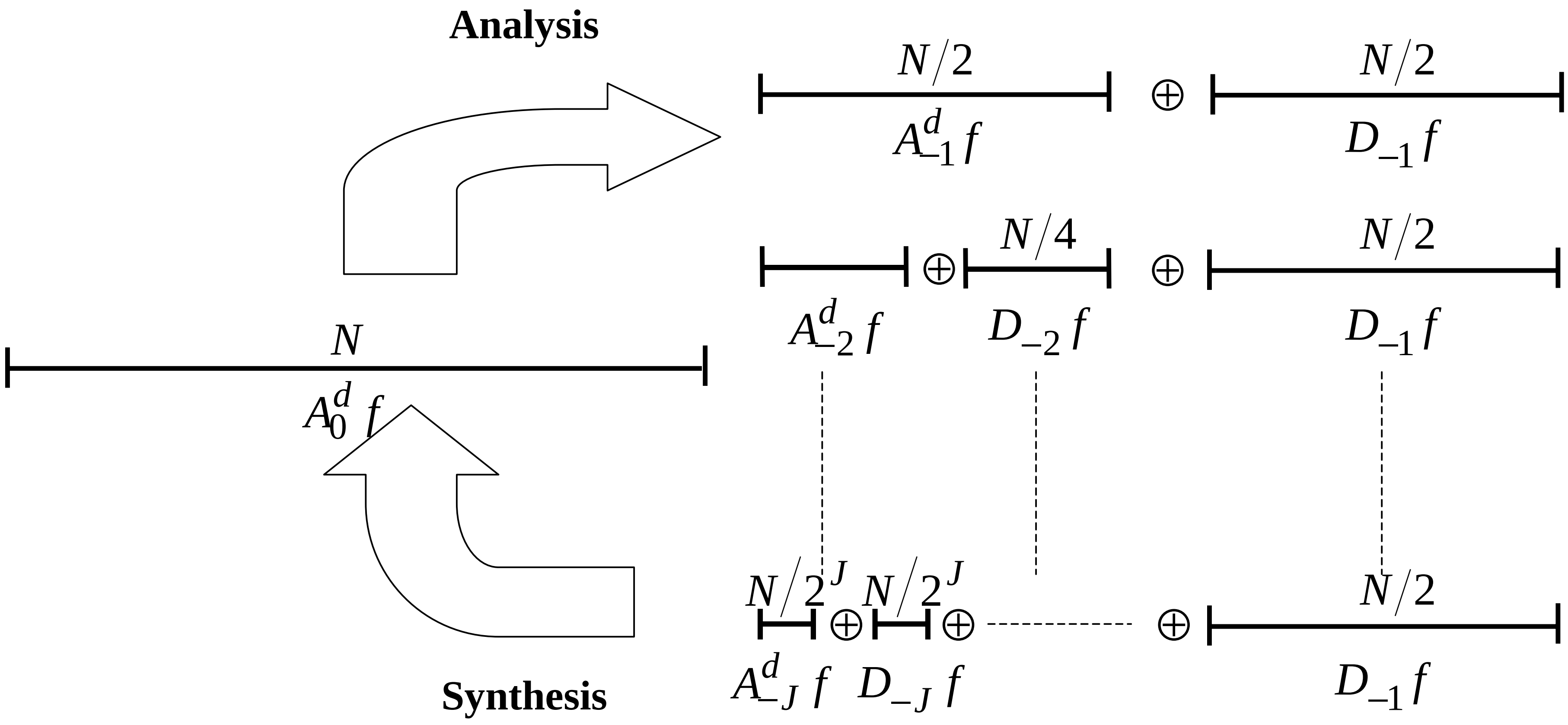




Wavelet Synthesis

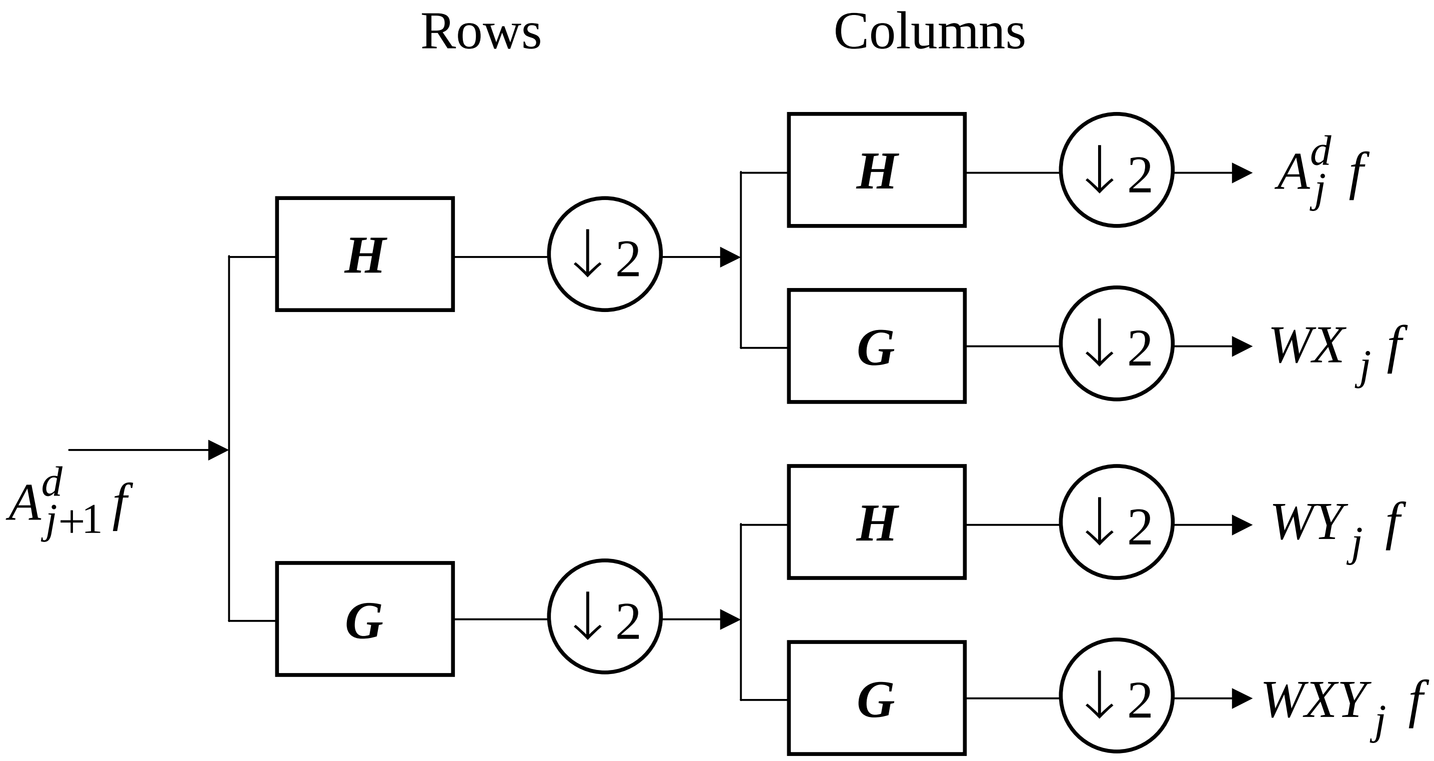
5.2. Dyadic Wavelet Representation

Orthonormal transform (critical sampling) $\Rightarrow$ the same number of samples in the transformed domain as in the spatial domain



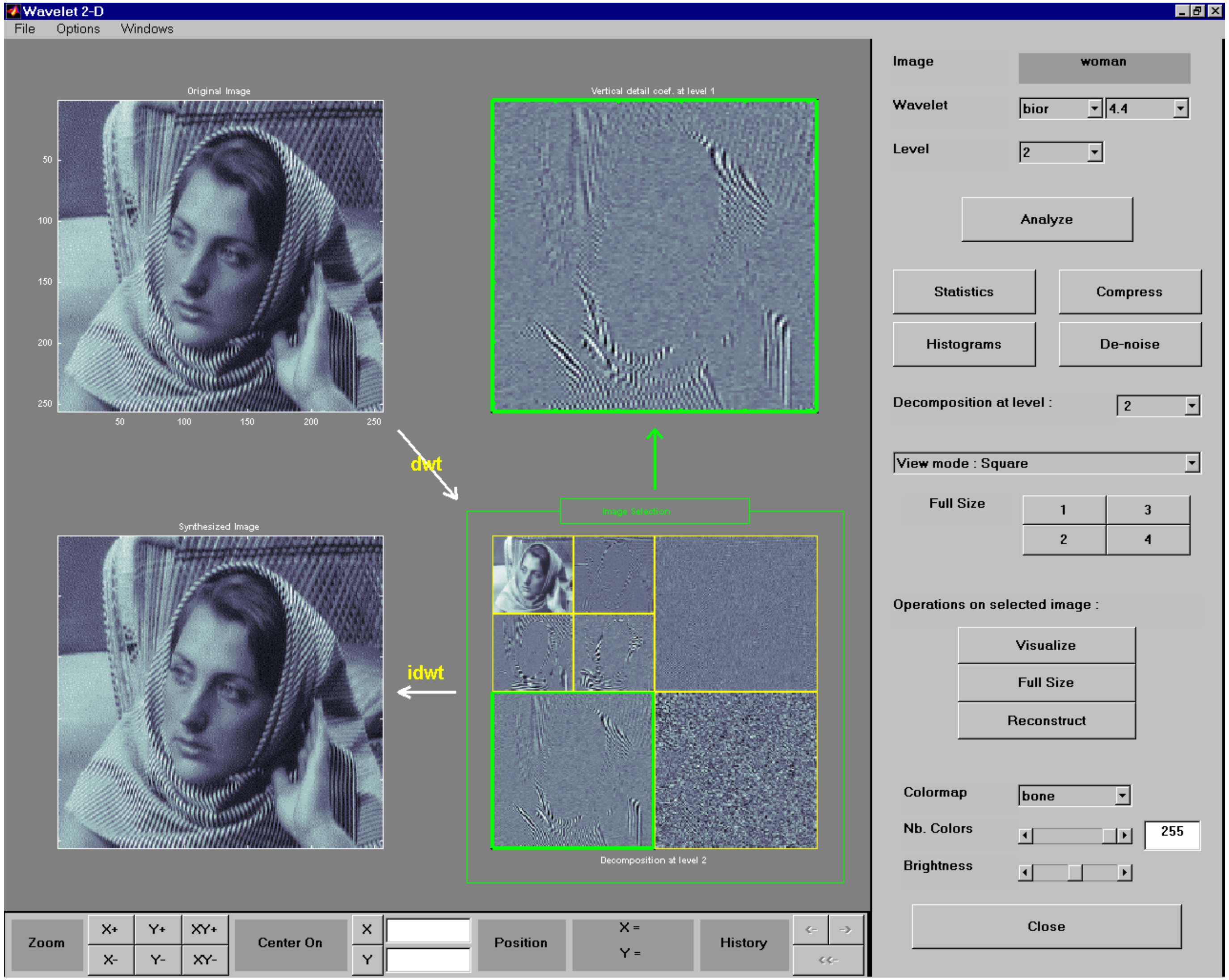
5.3. The 2D Wavelet Representation

Dyadic Wavelet Analysis of Images



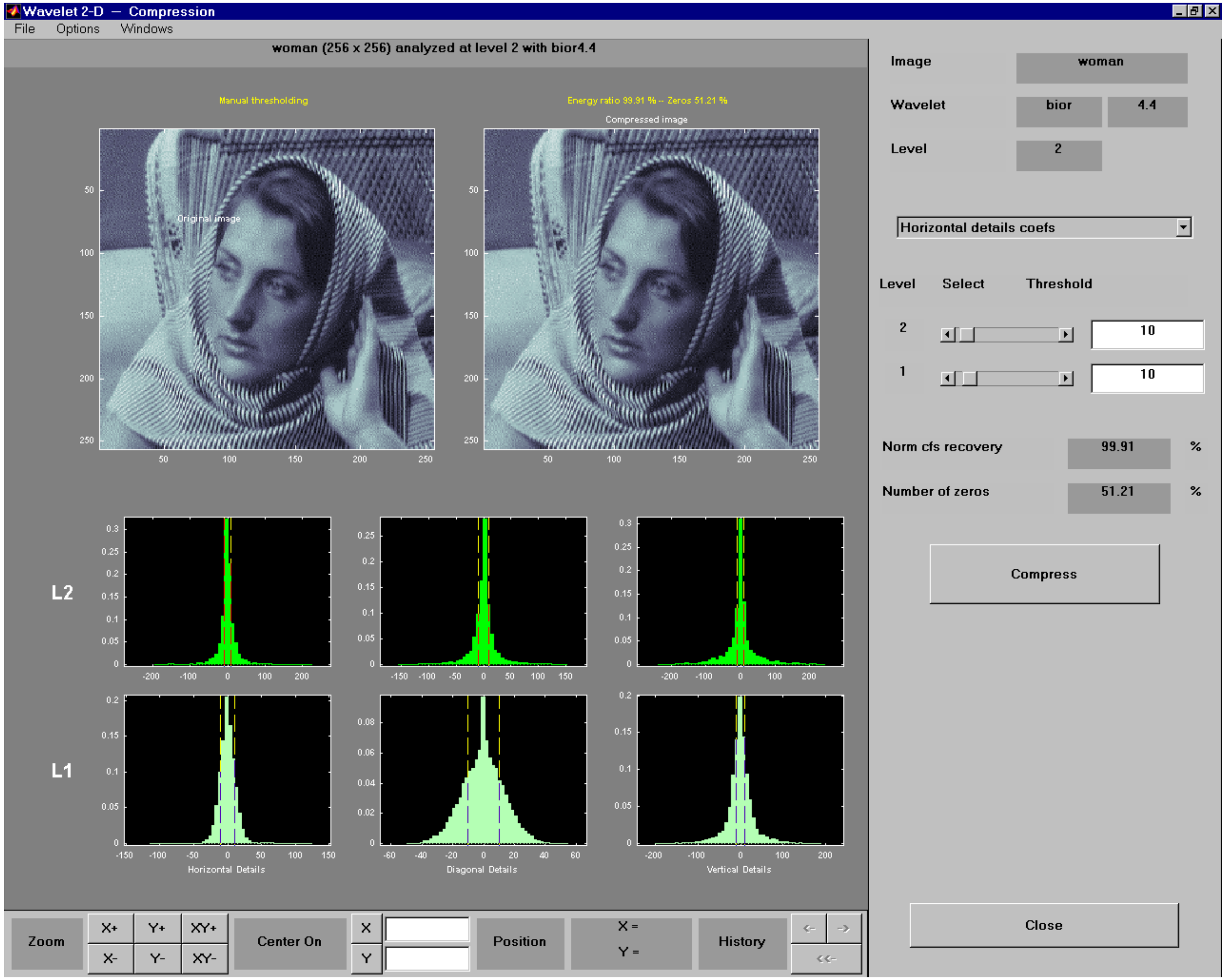
5.3. The 2D Wavelet Representation

Example 2-D Wavelet Analysis

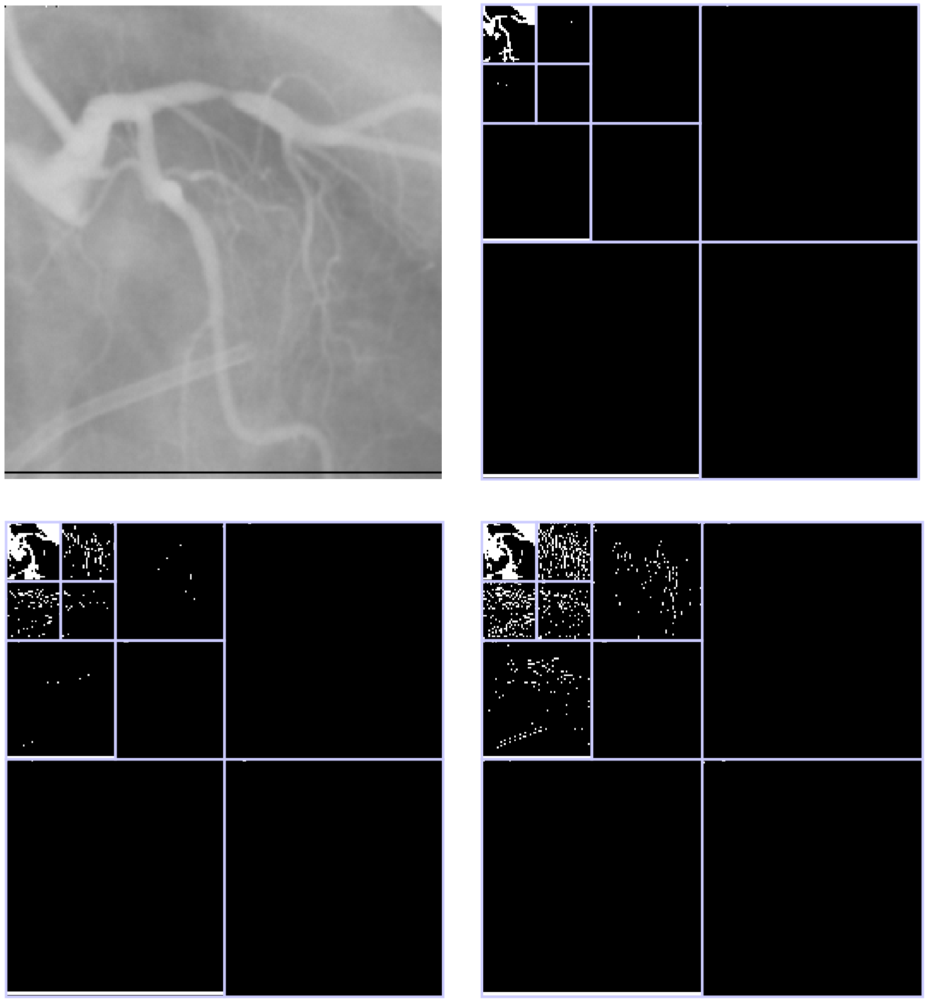


Brief Overview of Wavelet Based Image Coding

Basic Principles – Prioritization and Thresholding



Embedded Zerotree Coding of Wavelet Coefficients (EZW)

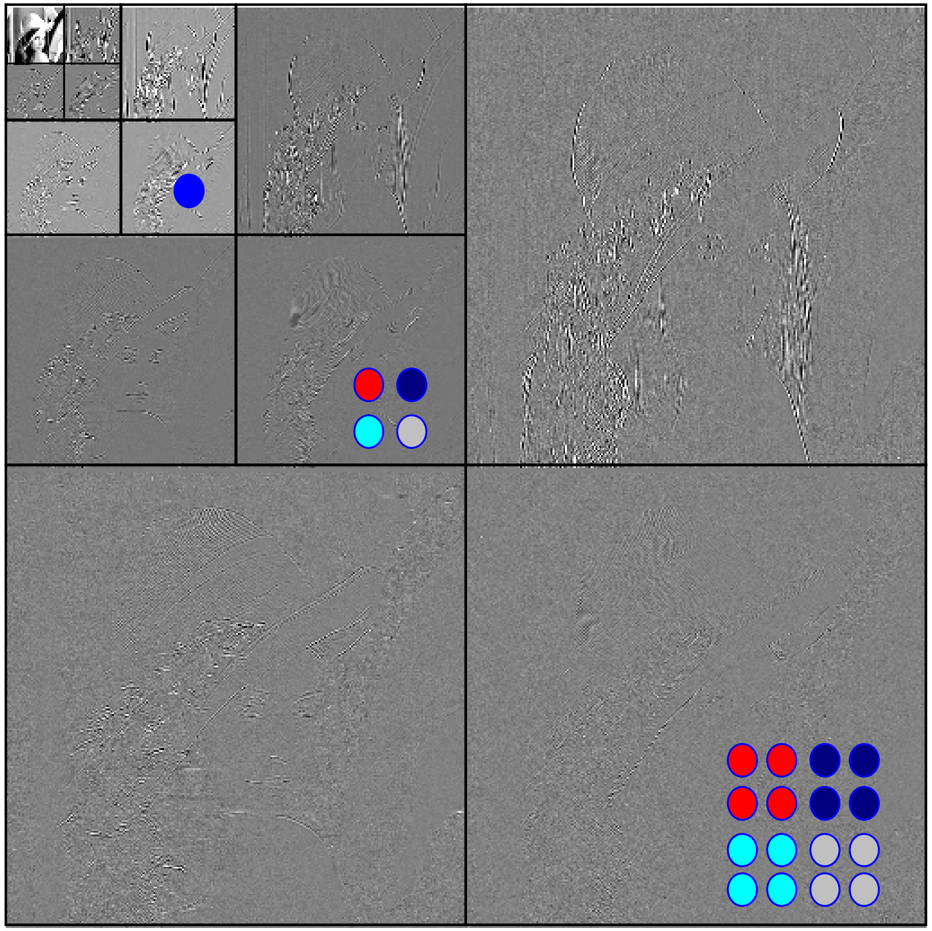


- Fixed rate coding: given the rate, find the optimal quantizers that minimize the distortion (MSE).
- Embedded wavelet coding: prioritize the significance information and encode the image in an embedded fashion.
- Successive approximation quantization (SAQ) applies $2^T, T_{max} \geq T \geq 0$
- Result: a set of *binary* maps indicating the positions of significant coefficients (*significance maps*)
- Embedded wavelet image coding deals with the *efficient coding* of the significance maps.

Embedded Zerotree Coding of Wavelet Coefficients (EZW)

- Organize the coefficients with similar locations in corresponding subbands in trees growing exponentially across the scales
- Apply SAQ.
- *Model: Zerotree hypothesis.*

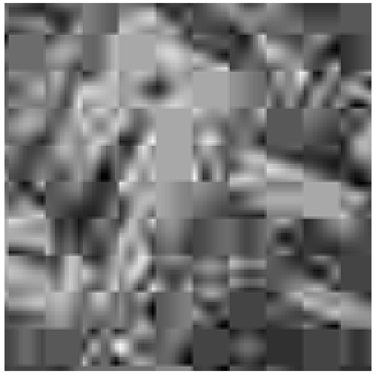
  If a coefficient is not significant with respect to a given threshold *T*, then all of its descendants are not significant either with respect to *T*.
- For every *T*, encode the corresponding significance map.



Comparison with JPEG

Original image

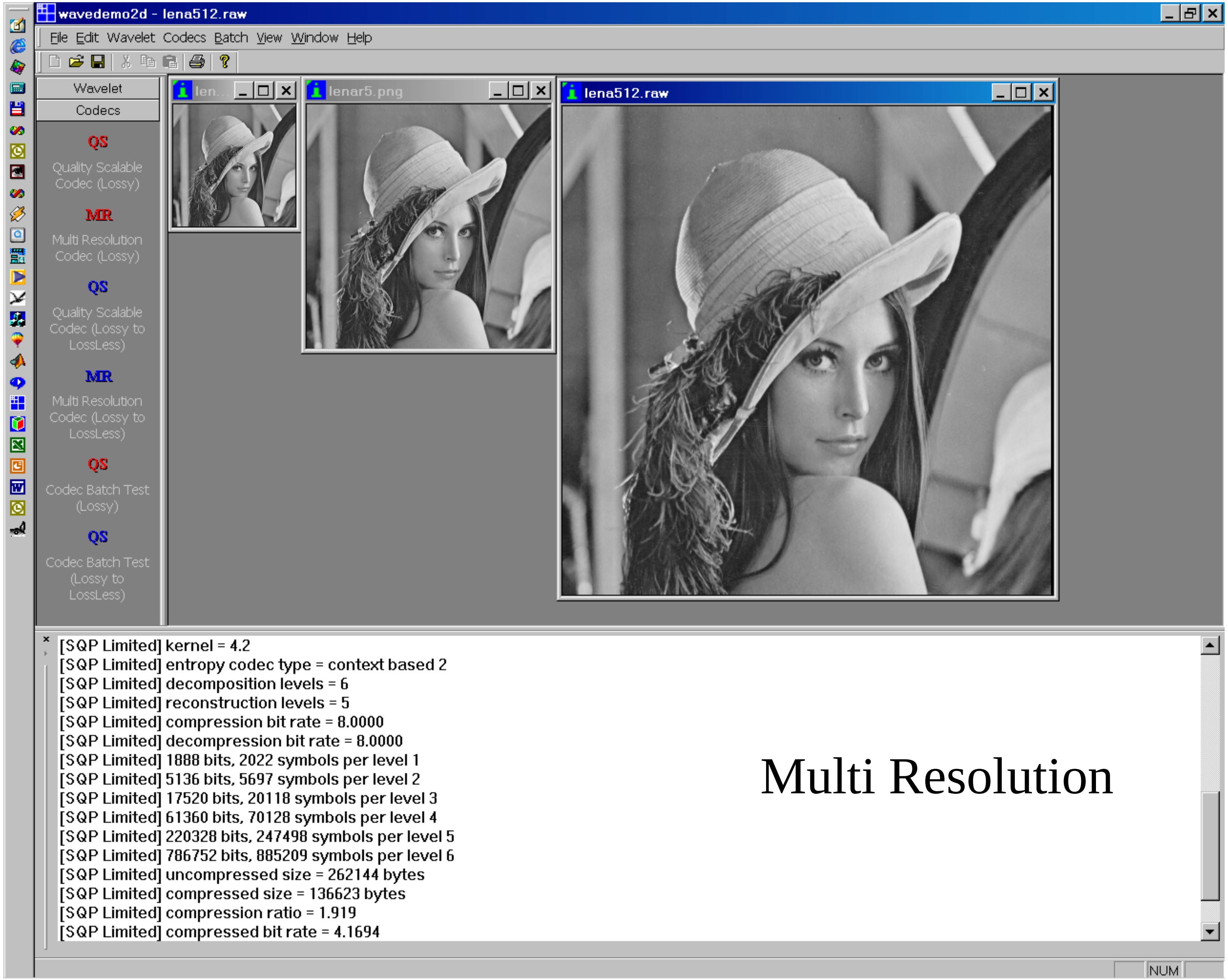
Zoomed area



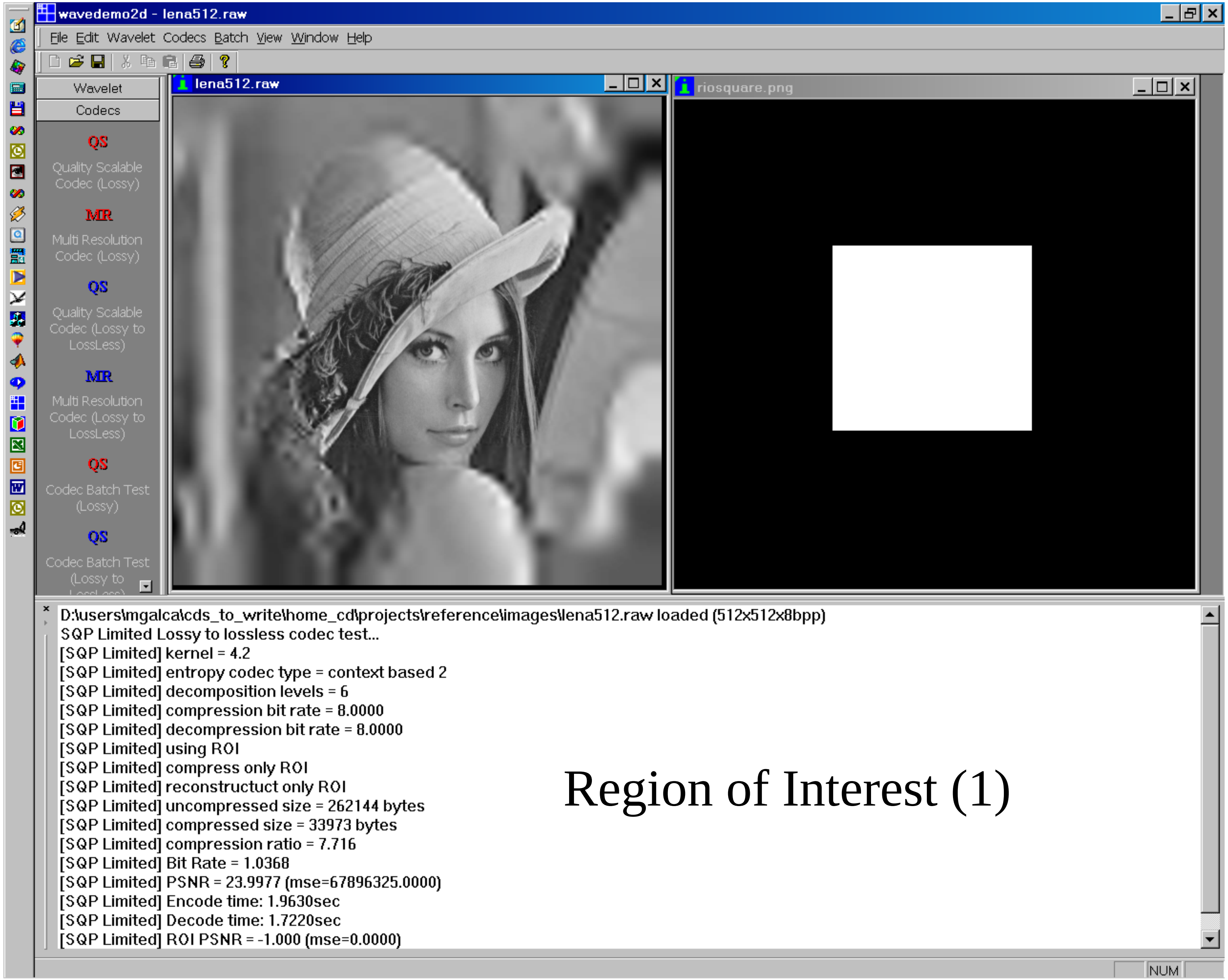
Zoomed area in the image compressed with DCT (JPEG)

Zoomed area in the image compressed with wavelets

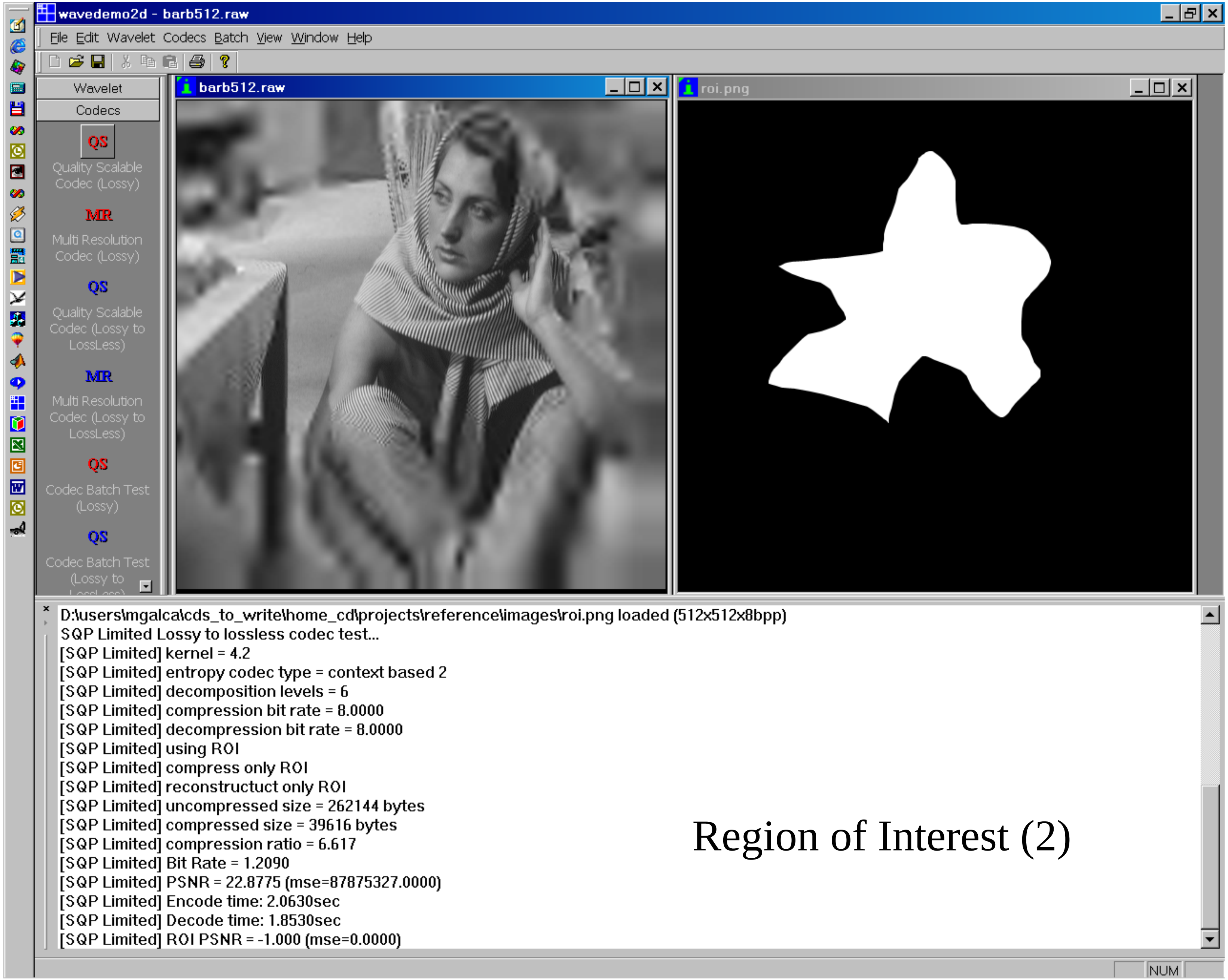
Multi Resolution



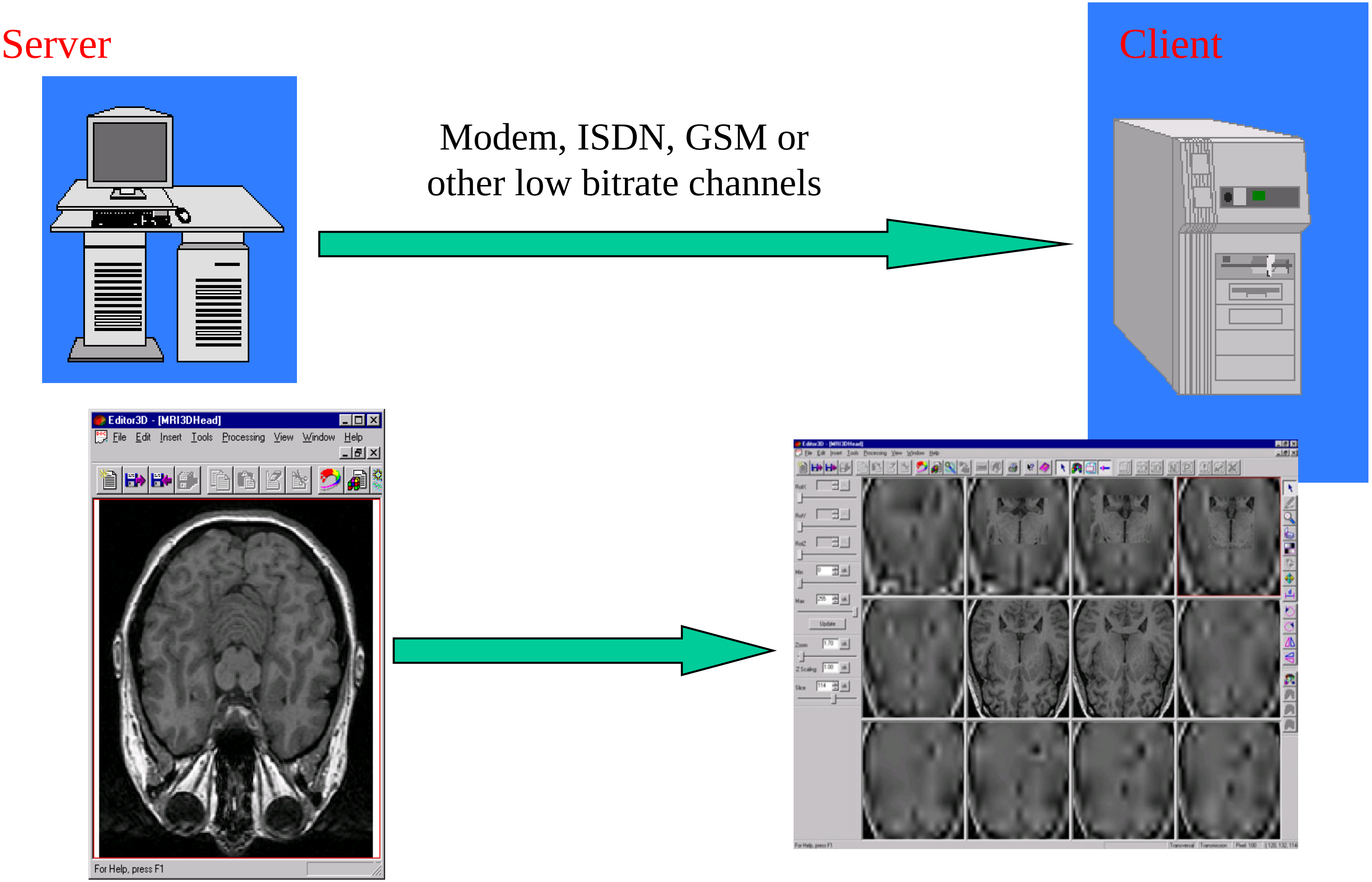
Region of Interest (1)



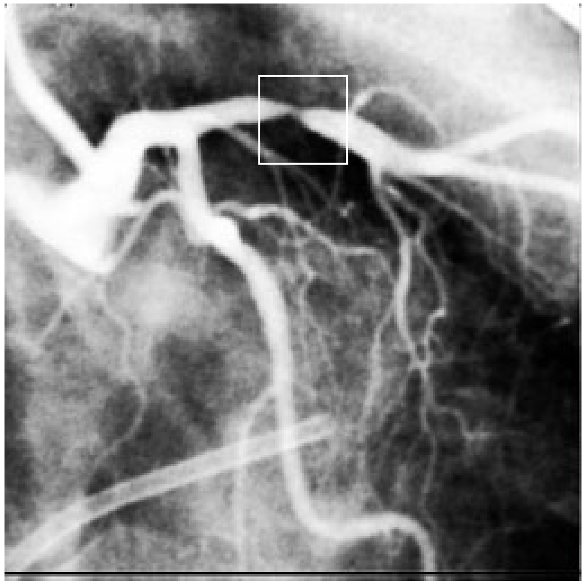
Region of Interest (2)



Prototype System for Archiving And Progressive Transmission of Medical Images



Progressive Transmission





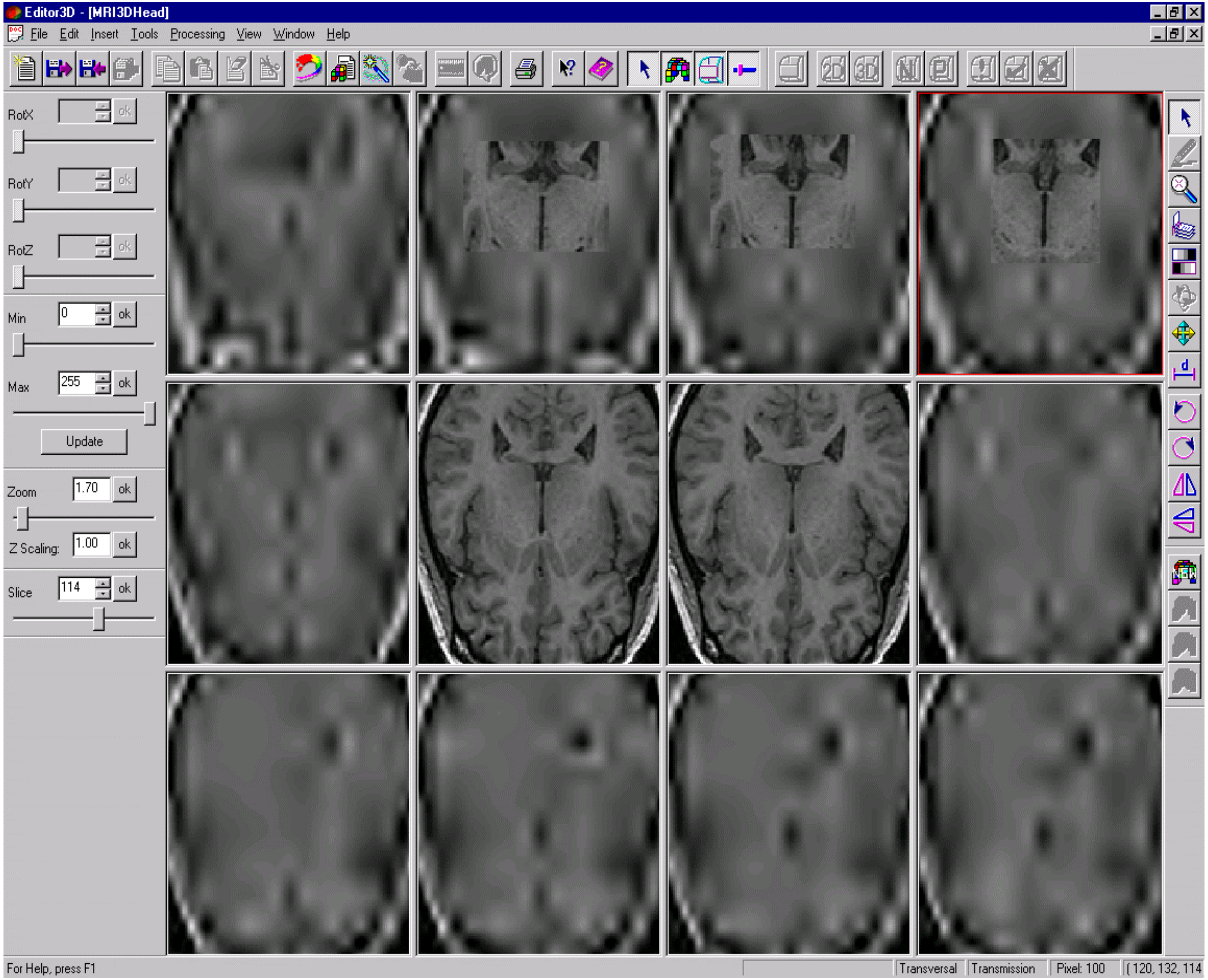
JPEG, CR=50 JPEG, CR=25 JPEG, CR=10

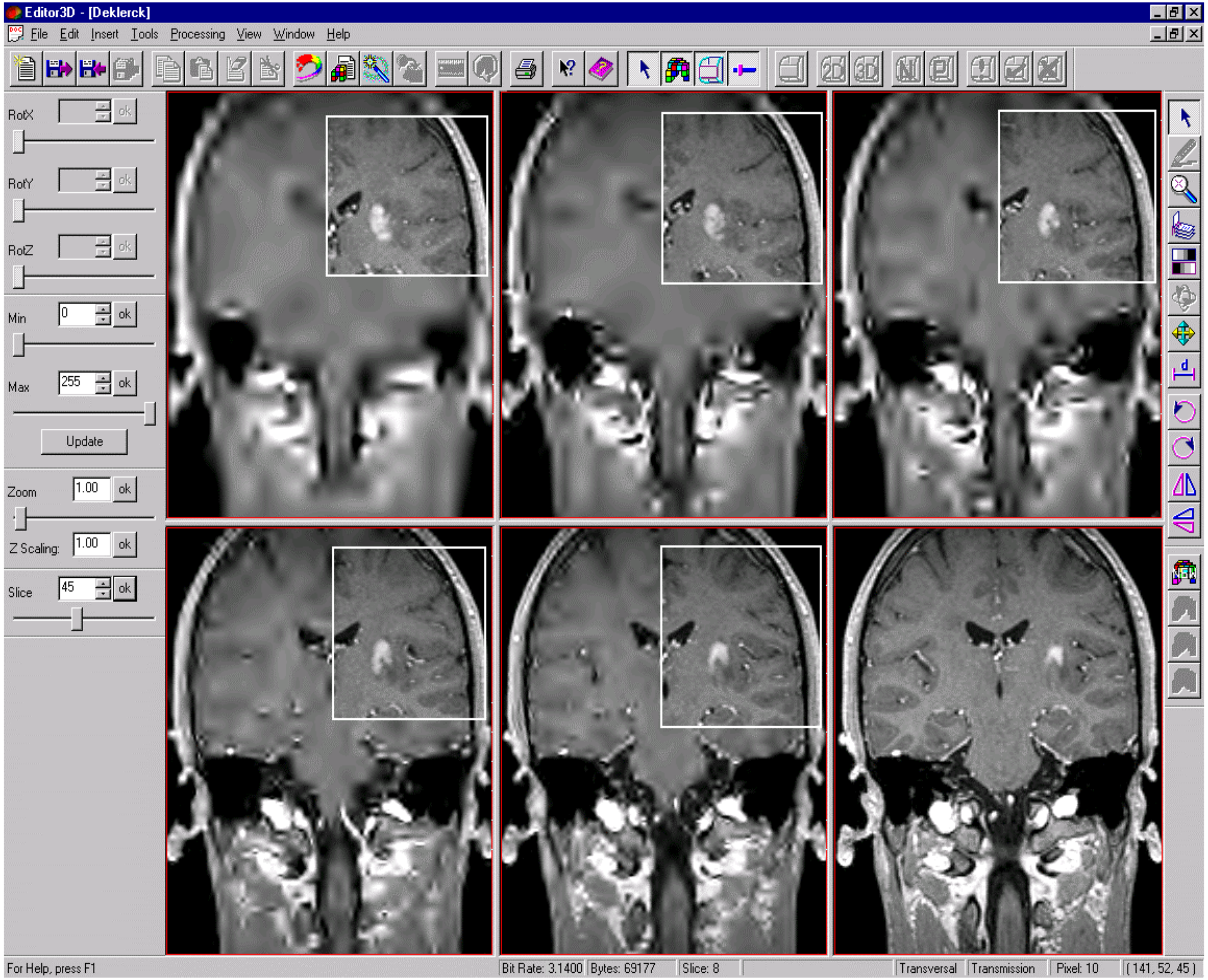


SQP, CR=50 SQP, CR=25 SQP, CR=10

A zoomed area of the coronary stenosis

The image on the left is compressed at 50:1, 25:1, and 10:1 with JPEG and SQP. The zoomed area of the coronary stenosis indicated by the white rectangle is progressively refined up to the lossless version depicted on the right.


Editor3D - [MRI3DHead]
File Edit Insert Tools Processing View Window Help
RotX
ok
RotY
ok
RotZ
ok
Min
0
ok
Max
255
ok
Update
Zoom
1.70
ok
Z Scaling:
1.00
ok
Slice
114
ok
For Help, press F1
Transversal
Transmission
Pixel: 100
(120, 132, 114



Progressive transmission: interruption during refinement of the images, only the ROI has been refined up to the lossless stage.

5.4. Applications

Multi-scale edge detection via CWT

- Smoothing function $\theta(x)$

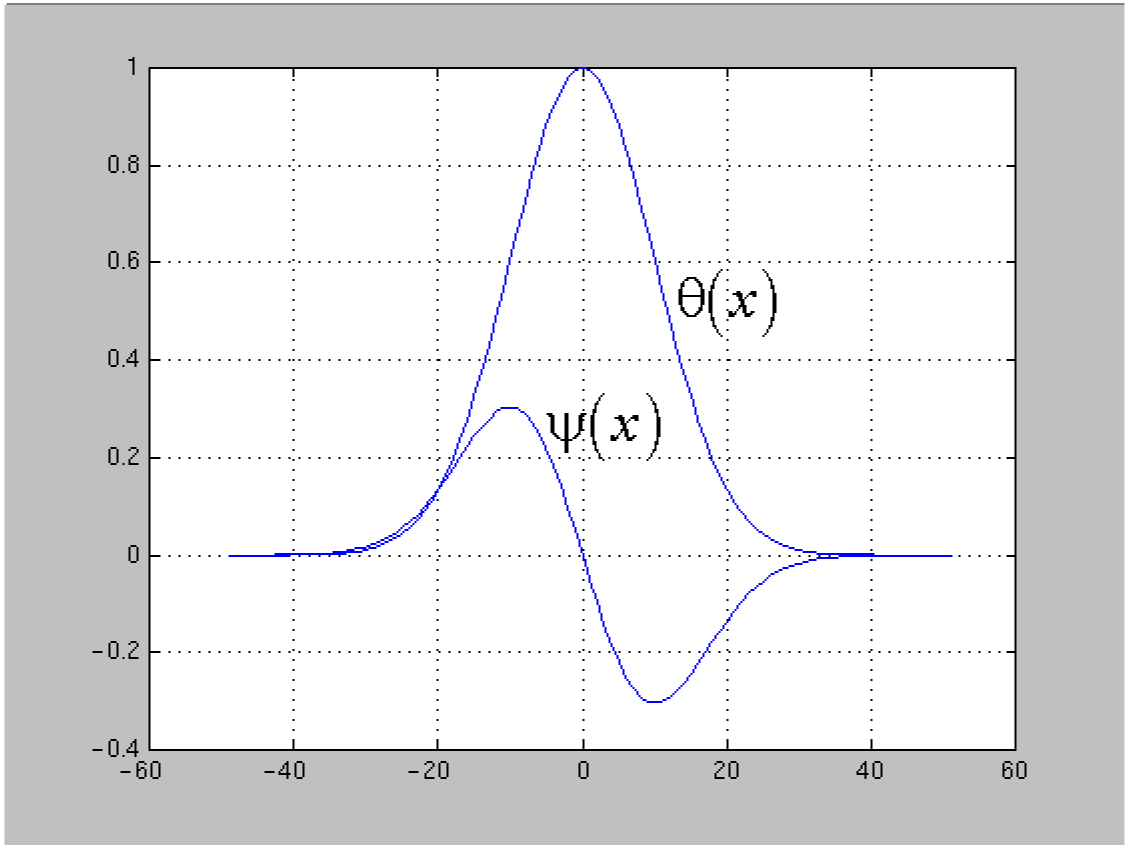
  for which $\int_{-\infty}^{\infty} \theta(x) dx = 1$

- Wavelet: $\psi(x) = \frac{d}{dx}\theta(x)$

- Continuous scale *s*:

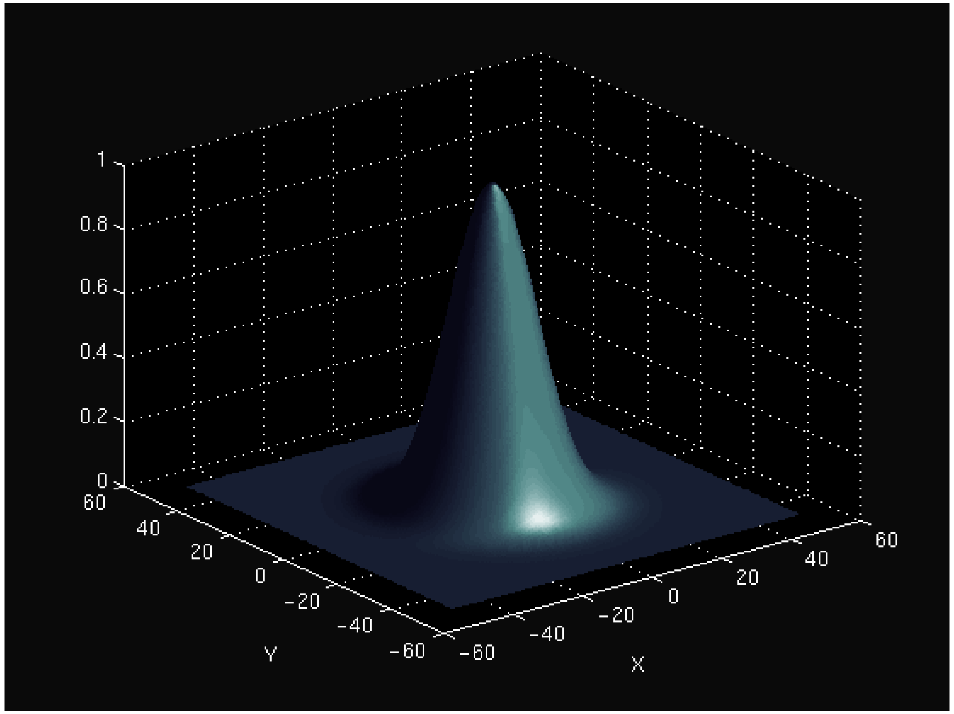
$$\theta_s(x) = \frac{1}{s}\theta\left(\frac{x}{s}\right)$$

$$\psi_s(x) = \frac{1}{s}\psi\left(\frac{x}{s}\right)$$



5.4. Applications

Multi-scale edge detection via CWT



- 2D smoothing function:

$$\theta(x,y)=\theta(x)\cdot\theta(y)$$

- Define two wavelet functions:

$$\psi^1(x,y)=\frac{d}{dx}\theta(x,y)$$

$$\psi^2(x,y)=\frac{d}{dy}\theta(x,y)$$

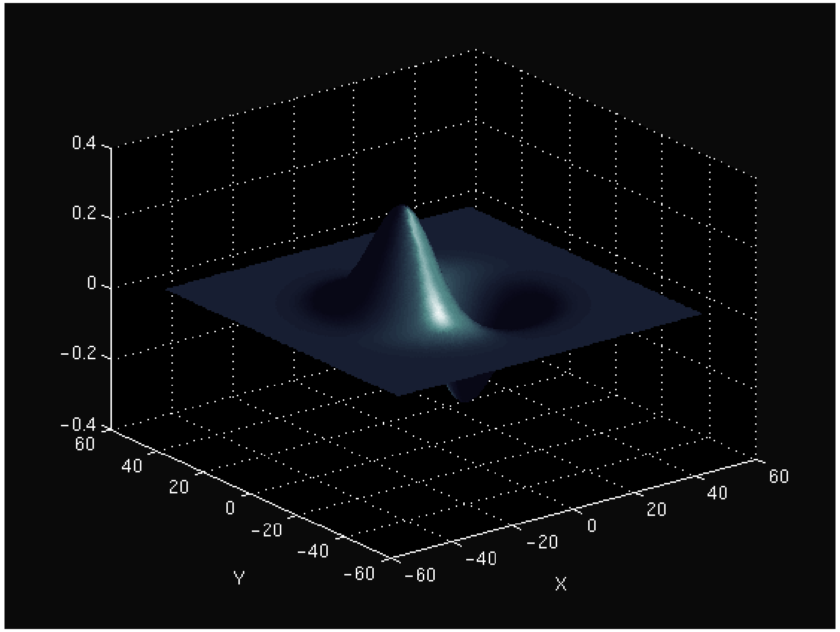
- 2D wavelet transform:

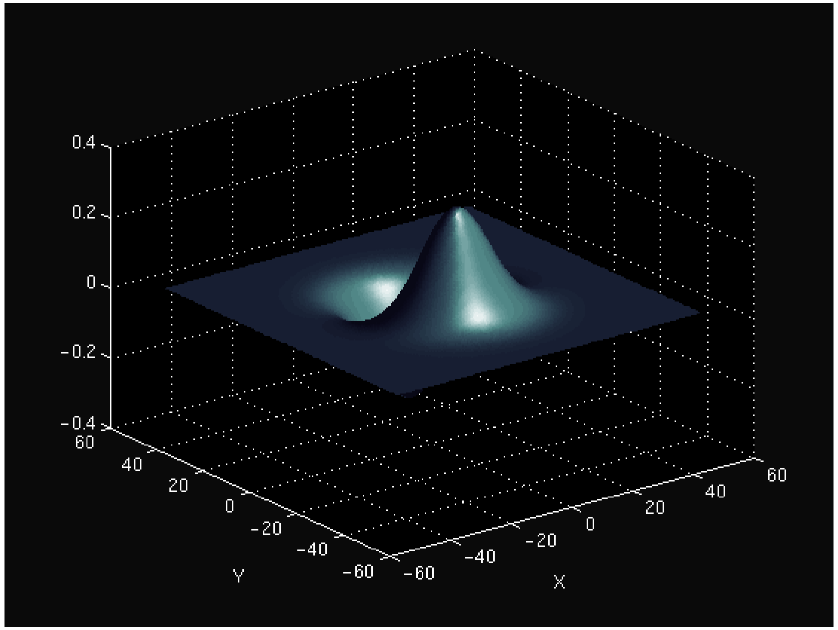
$$W_s^1 f(x,y)=\left(f*\psi_s^1\right)(x,y)=s\cdot\frac{d}{dx}(f*\theta_s)(x,y)$$

$$W_s^2 f(x,y)=\left(f*\psi_s^2\right)(x,y)=s\cdot\frac{d}{dy}(f*\theta_s)(x,y)$$

5.4. Applications

Multi-scale edge detection via CWT





$$\psi^1(x,y)=\frac{d}{dx}\theta(x,y) \qquad \psi^2(x,y)=\frac{d}{dy}\theta(x,y)$$

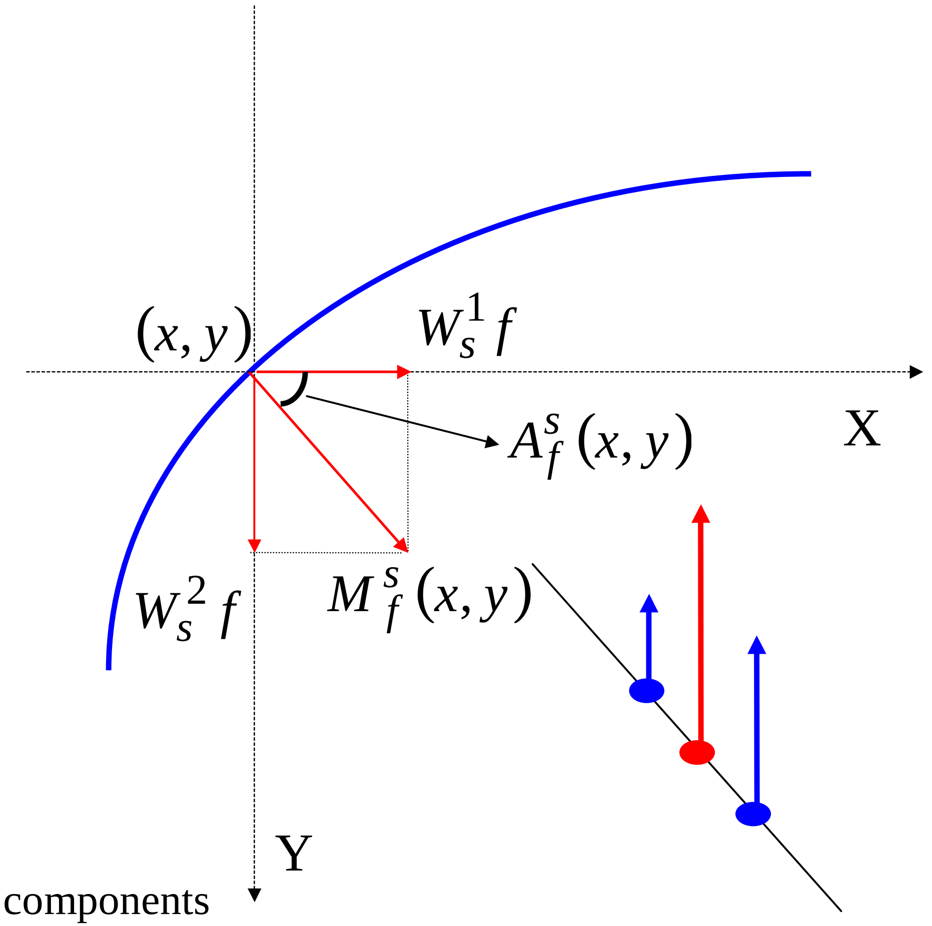
The modulus and the angle of $\vec{\nabla}(f*\theta_s)$ in any point (x,y)

$$M_f^s(x,y)=\left(W_s^1 f(x,y)\right)^2+\left(W_s^2 f(x,y)\right)^2$$

$$A_f^s(x,y)=\arg\left(W_s^1 f(x,y)+j\cdot W_s^2 f(x,y)\right)$$

5.4. Applications

Wavelet based edge detection: principles



- Compute the 2 gradient components
- Compute the gradient magnitude and direction
- Determine the edge points as the local maxima points of the gradient magnitude along the gradient direction

5.4. Applications

Intermediary Results

Different scales,

Different gradient orientations

Scale 0, 0^o

Scale 0, - 45^o

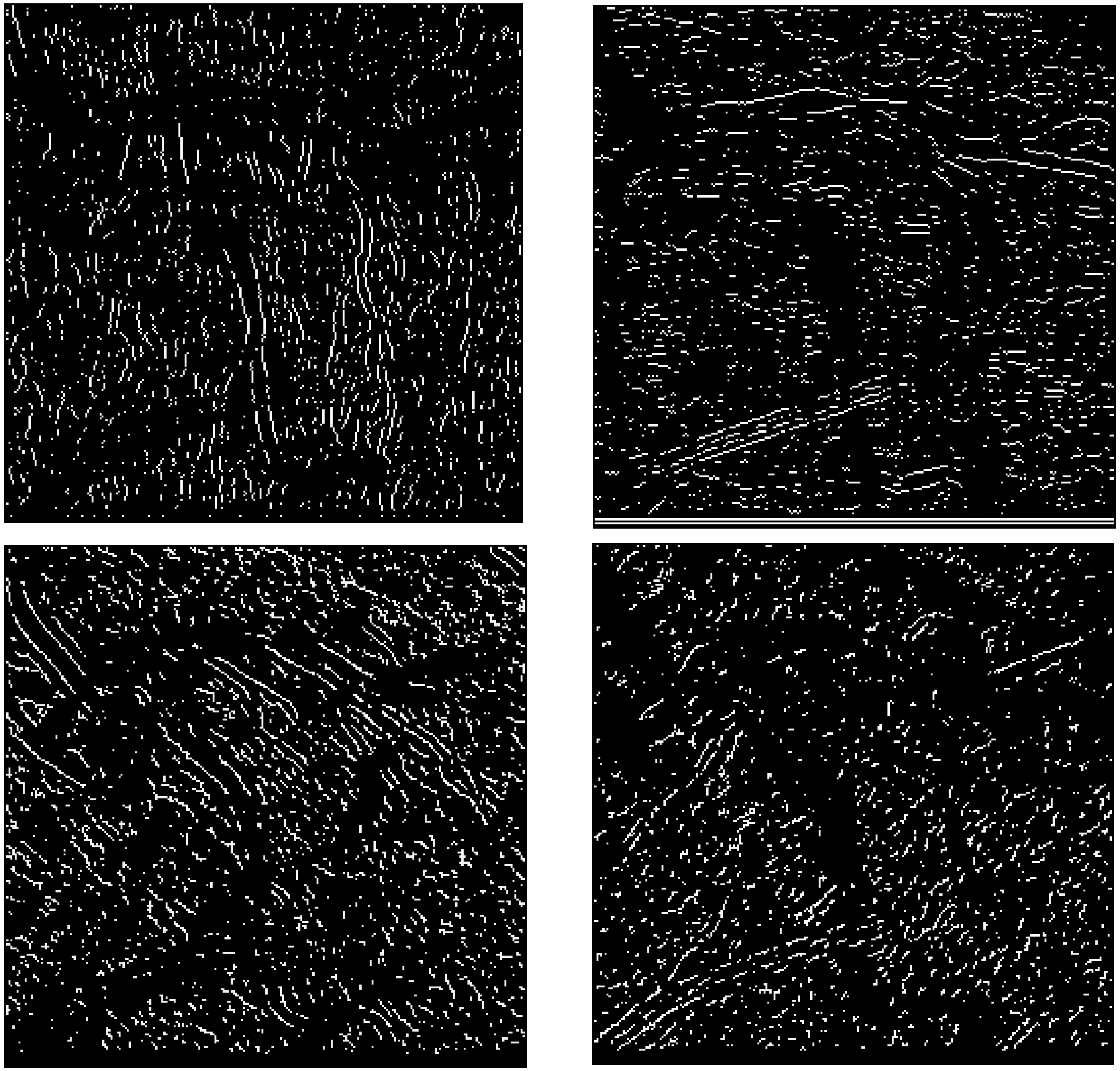
Scale 0, + 45^o

Scale 0, 90^o

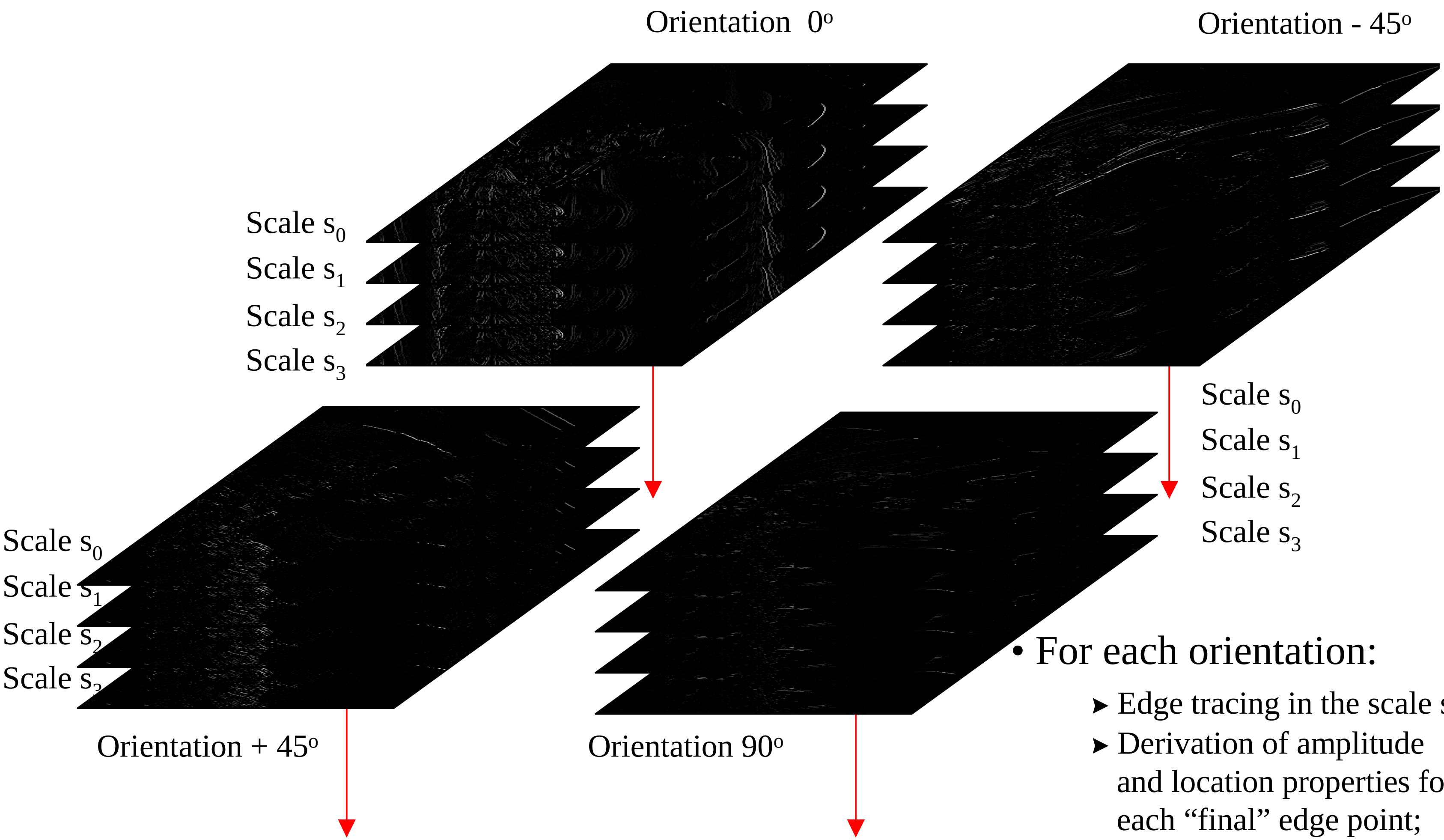
5.4. Applications

Intermediary Results

Coronary Angiogram



Multiscale-Edge Combining Algorithm



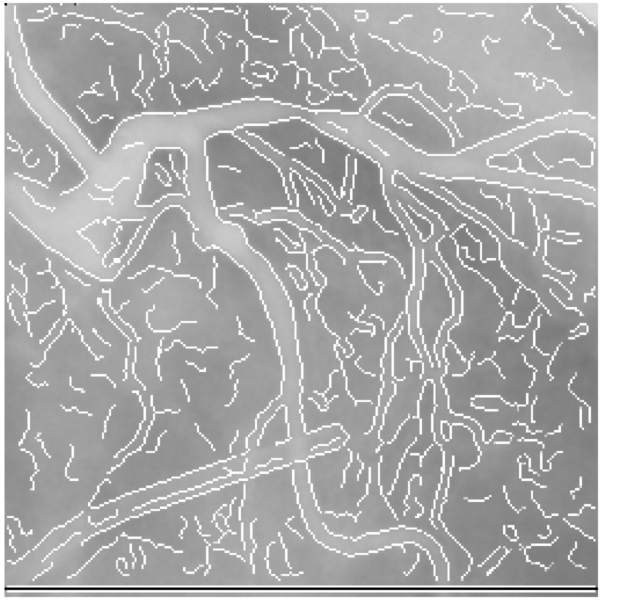
- For each orientation:
  - Edge tracing in the scale space;
  - Derivation of amplitude and location properties for each “final” edge point;
  - Result: 4 edge images corresponding to the 4 orientations

5.4. Applications

Wavelet edge detection applied to coronary angiograms

Original Angiographic Image

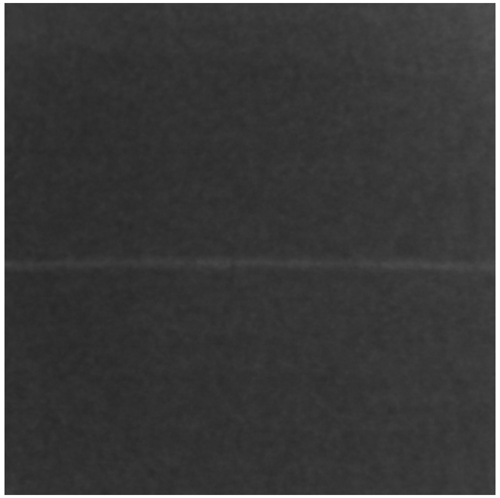
Canny Edge Detection
$\sigma = 1$



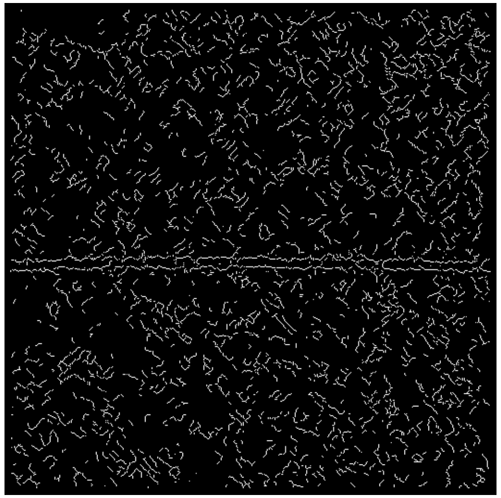
CWT based Edge Detection

5.4. Applications

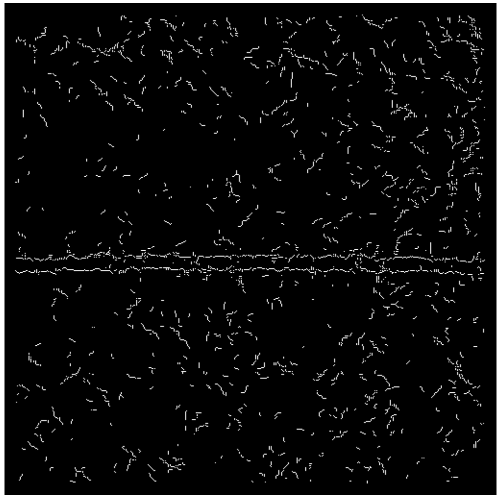
Wavelet based edge detection in low contrast images



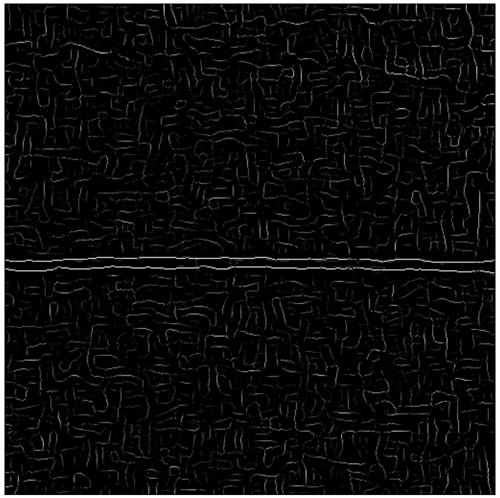
Phantom Vessel – 0.46mm



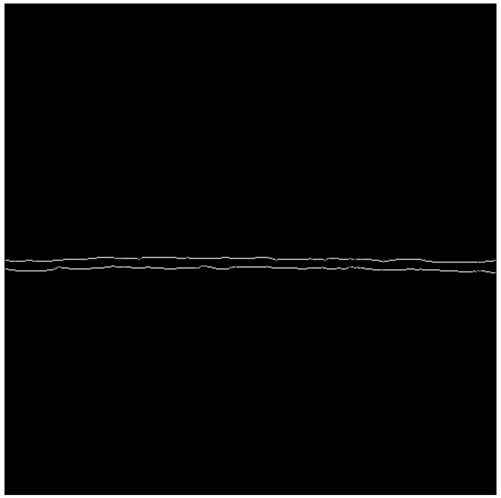
Canny Detection – $\sigma = 2$



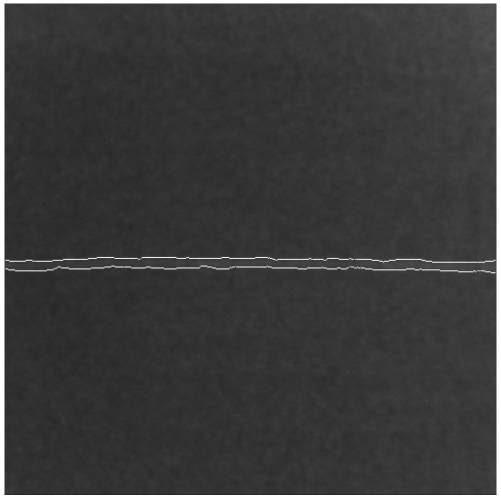
Canny Detection – $\sigma = 4$



CWT based Detection



CWT Detection – Vessel Retained

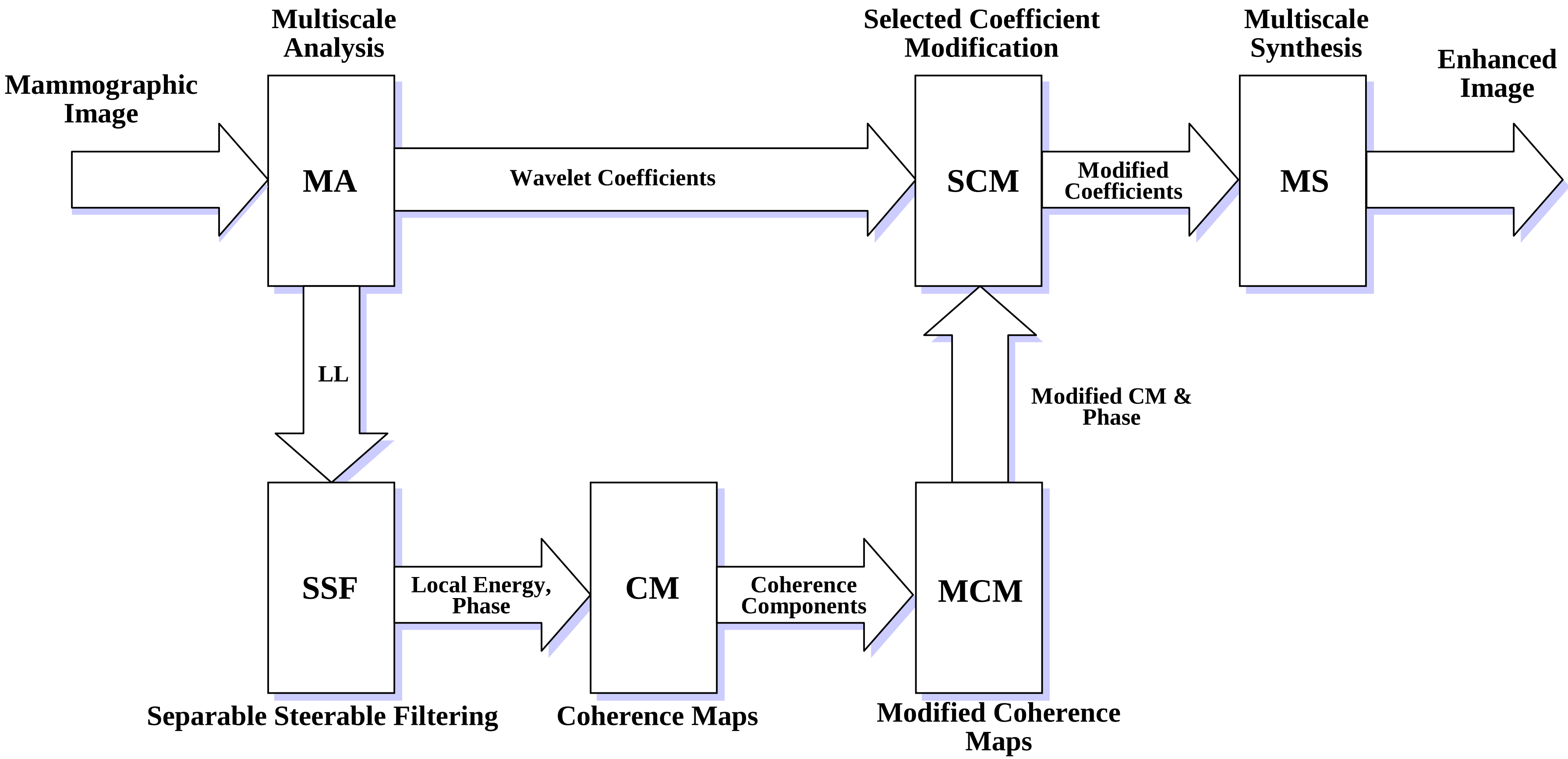


CWT Detection – Overlapped Edges

5.4. Applications

Wavelet-based enhancement technique for processing of mammograms

A. F. Laine – Biomedical Engineering Center, Columbia University



5.4. Applications

Wavelet-based enhancement technique for processing of mammograms

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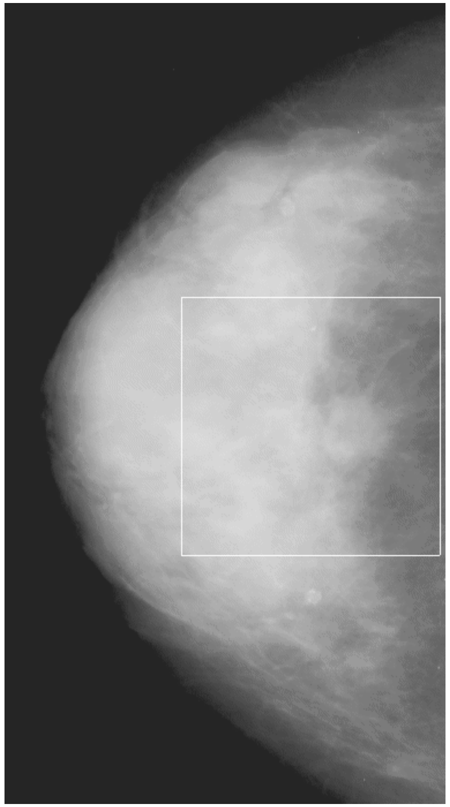
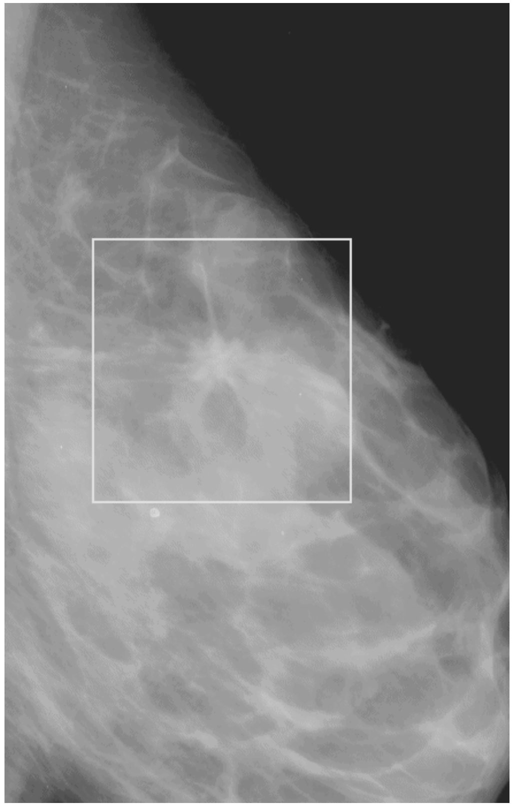


The craniocaudal view of the left breast shows an **irregular spiculated mass** in retroglandular fat.



This mammogram indicates a **stellate lesion** with a partially obscured irregular mass.

This mammogram indicates **microcalcification clusters**.

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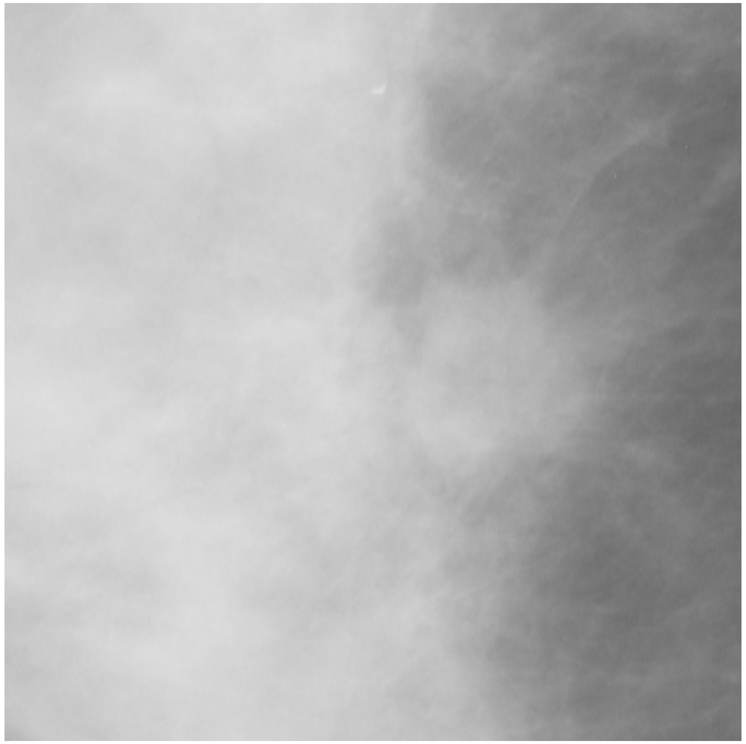

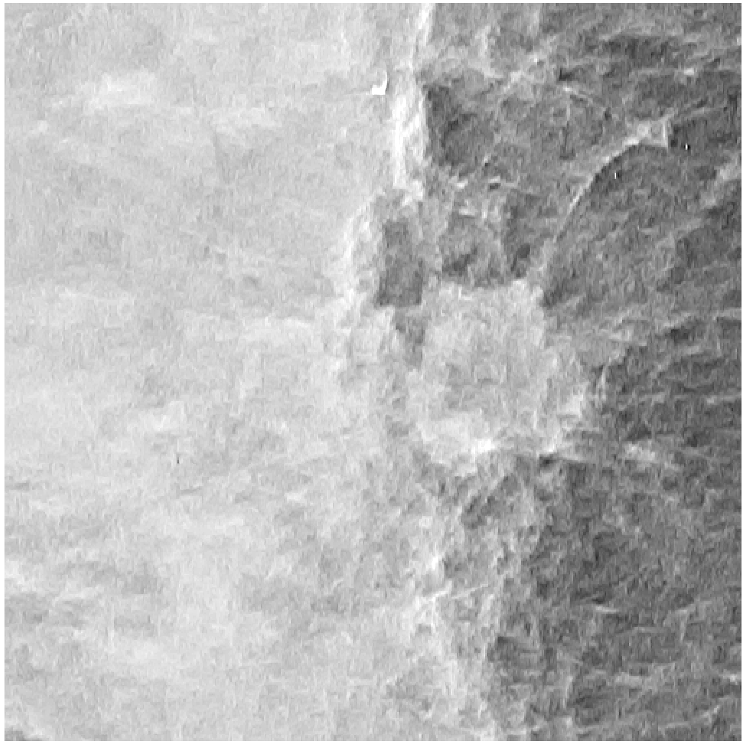

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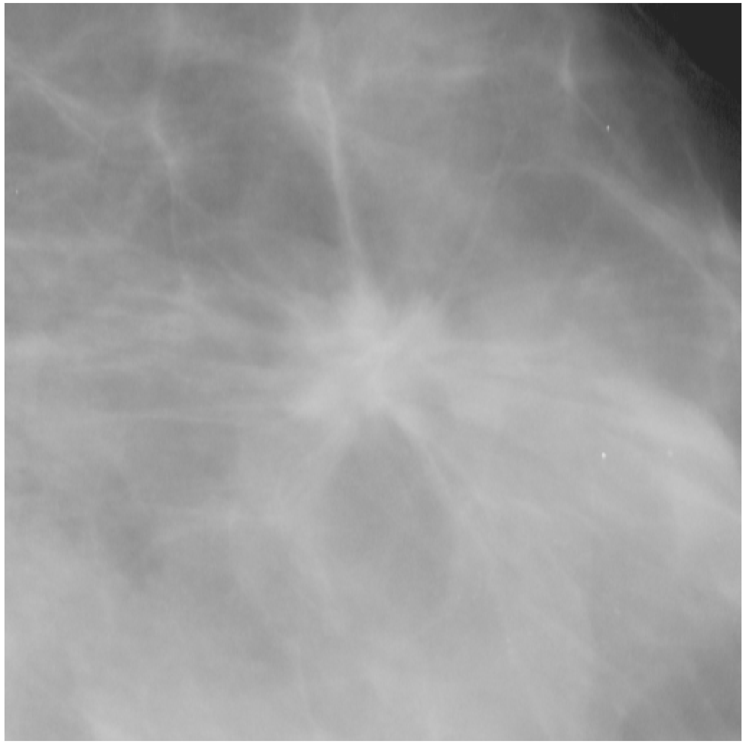

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