Fourier Transform Flavors

5. Fourier Theory

Fourier Case

(a) *Continuous-time Fourier Transform* **(CTFT)** - *Fourier Transform*

(b) *Continuous-time Fourier Series* **(CTFS)** - *Fourier Series*

(c) *Discrete-time Fourier Transform* **(DTFT)**

(d) *Discrete-time Fourier Series* **(DTFS)**

5.1. Fourier Series

The **Fourier series** of a periodic signal, $f(t) = f(t+kT), k \in Z$ is given by:

$$f(t) = \sum_{k=-\infty}^{\infty} F[k] e^{jk\omega_0 t}, \omega_0 = \frac{2\pi}{T}; \qquad (3)$$

which is of the form:

$$x(t) = \sum_k X_k \psi_k(t) \Rightarrow \psi_k(t) = e^{jk\omega_0 t}$$

✍ linear combination of complex exponentials with frequencies $k\omega_0$

Fourier Coefficients:

$$F[k] = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-jk\omega_0 t} dt \qquad (4)$$

- Note 1: If $f(t)$ is continuous, then the series converges uniformly to $f(t)$
- Note 2: If $f(t)$ is square-integrable over a period, but not necessarily continuous, then the series converges to $f(t)$ in the L^2 sense.

$$k\Big|_{-N}^{N} \Rightarrow \hat{f}_N(t); \qquad \left\| f(t) - \hat{f}_N(t) \right\| \to 0 \Big|_{N\to\infty}$$

5.1. Fourier Series

Properties

- The set of functions used in (3) is a complete orthonormal system for the interval $[-T/2, T/2]$:

$$\varphi_k(t) = \left(1/\sqrt{T}\right)e^{jk\omega_0 t}, \; t \in [-T/2, T/2], k \in Z \Rightarrow \langle \varphi_k(t), \varphi_l(t) \rangle_{[-T/2, T/2]} = \delta[k-l]$$

- Parseval relation:

$$\langle g(t), f(t) \rangle = T \cdot \langle F[k], G[k] \rangle \Rightarrow \int_{-T/2}^{T/2} |f(t)|^2 \, dt = T \cdot \sum_{k=-\infty}^{\infty} |F[k]|^2$$

- Best Approximation Property

$$\left\| f(t) - \sum_{k=-N}^{N} \langle \varphi_k, f \rangle \varphi_k(t) \right\| \le \left\| f(t) - \sum_{k=-N}^{N} \alpha_k \varphi_k(t) \right\|, \; \alpha_k \text{ an arbitrary set of coefficients.}$$

✍ The Fourier series coefficients are the best ones for an approximation in the span of $\{\varphi_k(t)\}, k \in [-N, N]$

- Fourier series can be used for non-periodic signals via periodization. Problem: discontinuities at the boundaries.

5.2. Discrete-Time Fourier Transform

Given a sequence $\{f[n]\}_{n\in Z}$ in $l^1(Z)$, its **discrete-time Fourier transform** is given by

$$F\left(e^{j\omega}\right)=\sum_{n=-\infty}^{\infty} f[n]e^{-j\omega n}\ ,\ \text{which is } 2\pi \text{ periodic.} \qquad (5)$$

Inverse discrete-time Fourier transform:

$$f[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi} F\left(e^{j\omega}\right)e^{j\omega n}d\omega \qquad (6)$$

$$x[n]=\int X_{\omega}\psi_{\omega}[n]d\omega \Rightarrow \psi_{\omega}[n]=e^{j\omega n}$$

- Note 1: If $f[n]$ is in $l^2(Z)$, we have convergence in l^2 sense
- Note 2: $f[n]$ is obtained by sampling a continuous time signal $f(t)$ at instants nT.

Result: DTFT is related to the Fourier transform $F_c(\omega)$ of $f(t)$:

$$F\left(e^{j\omega}\right)=\frac{1}{T}\sum_{k=-\infty}^{\infty} F_c\left(w-k\frac{2\pi}{T}\right)$$

5.2. Discrete-Time Fourier Transform

Properties

- The properties of the FT are carried over by DTFT
- Convolution:

$$f[n]*g[n]=\sum_{l=-\infty}^{\infty} f[n-l]g[l]=\sum_{l=-\infty}^{\infty} f[l]g[n-l]\leftrightarrow F\left(e^{j\omega}\right)\cdot G\left(e^{j\omega}\right)$$

- Parseval's relation:

$$\sum_{n=-\infty}^{\infty} f^*[n]g[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi} F^*\left(e^{j\omega}\right)G\left(e^{j\omega}\right)d\omega$$

- Energy conservation:

$$\sum_{n=-\infty}^{\infty} \left|f[n]\right|^2=\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|F\left(e^{j\omega}\right)\right|^2 d\omega$$

5.3. Discrete-Time Fourier Series

- A periodic discrete signal, $f[n] = f[n + lN], l \in Z$ has its **discrete-time Fourier series** (DTFS) representation given by:

$$F[k] = \sum_{n=0}^{N-1} f[n] e^{-jn \cdot k 2\pi / N}, \qquad k \in Z \tag{7}$$

- **Inverse DTFS** representation:

$$f[n] = \frac{1}{N} \sum_{k=0}^{N-1} F[k] e^{jn \cdot k 2\pi / N}, \qquad n \in Z \tag{8}$$

$$x[n] = \sum_{k} X_k \psi_k[n], \Rightarrow \psi_k[n] = e^{jn \cdot k 2\pi / N}$$

- All the properties of the FT hold.
- Convolution $\Rightarrow$ Circular Convolution

$\rightarrow f_0[n], g_0[n]$, one period of $f[n], g[n]$

$$\rightarrow f_0[n] = \begin{cases} f[n], 0 \le n \le N-1 \\ 0, \textit{otherwise} \end{cases}$$

5.3. Discrete-Time Fourier Series

- Circular Convolution

$$f[n]*g[n]=\sum_{l=0}^{N-1} f[n-l]g[l]=\sum_{l=0}^{N-1} f_0[(n-l)\bmod N]g_0[l]=$$
$$=f_0[n]\circ g_0[n]\leftrightarrow F[k]\cdot G[k]$$

- Parseval's relation:

$$\sum_{n=0}^{N-1} f^*[n]g[n]=\frac{1}{N}\sum_{k=0}^{N-1} F^*[k]G[k]$$

Discrete Fourier Transform (DFT)

The same formulas as (7) and (8), except that $f[n]$ and $F[k]$ are defined only for:
$n,k\in\{0,\ldots,N-1\}$

Convolution matrix C:

$$C=\begin{pmatrix} f[0] & f[1] & . & . & f[N-1] \\ f[N-1] & f[0] & . & . & f[N-2] \\ . & . & . & . & . \\ . & . & . & . & . \\ f[1] & f[2] & . & . & f[0] \end{pmatrix}$$

5.3. Discrete-Time Fourier Series

- Circular Convolution

$$f \circ g = \mathbf{C}g = \mathbf{F}^{-1}\boldsymbol{\Lambda}\mathbf{F}g, \quad \mathbf{F}[n][k] = e^{-jn\cdot k2\pi/N}, \qquad \mathbf{F}^{-1}[n][k] = (1/N)e^{jn\cdot k2\pi/N}$$

- Λ is a diagonal matrix with $F[k]$ on its diagonal.
- Another view:
  - ✍ **C** is diagonalized by **F**
  - ✍ The complex exponential sequences $\left\{e^{j(2\pi/N)\cdot nk}\right\}$ are eigenvectors for the convolution matrix **C**, with eigenvalues $F[k]$.
- Parseval's relation:

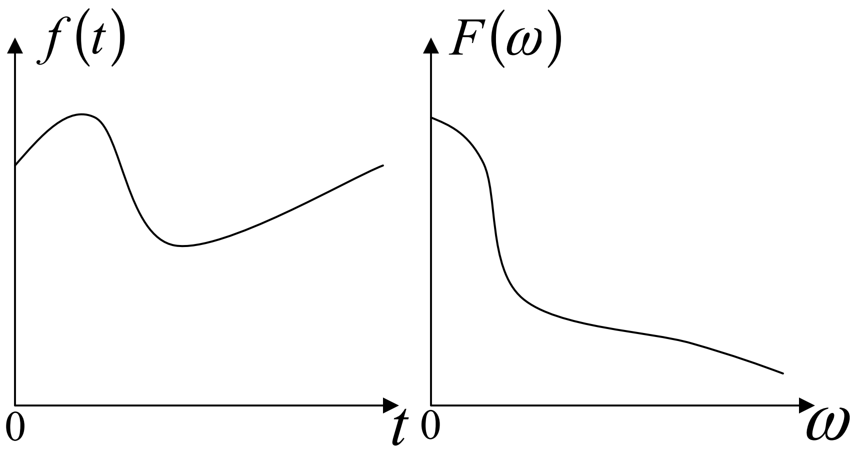
$$\hat{f} = \mathbf{F}f, \quad \hat{g} = \mathbf{F}g$$

$$\hat{f}^*\hat{g} = (\mathbf{F}f)^*(\mathbf{F}g) = f^*\mathbf{F}^*\mathbf{F}g = Nf^*g; \qquad \mathbf{F}^{-1} = \frac{1}{N}\mathbf{F}^*$$

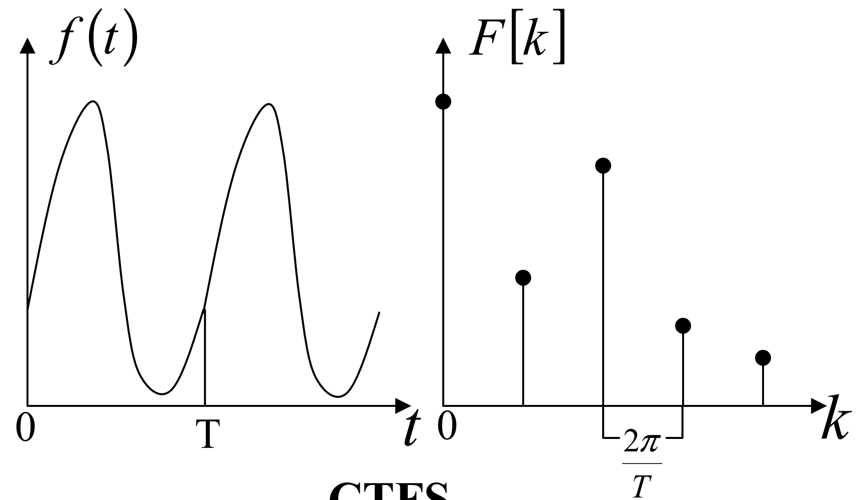
5.4. Various Flavors of Fourier Transforms - summary

Transform	Time	Frequency	Analysis Synthesis
Fourier Transform **CTFT**	Continuous	Continuous	$F(\omega) = \int_t f(t) e^{-j\omega t} dt$ $f(t) = 1/2\pi \int_\omega F(\omega) e^{j\omega t} d\omega$
Fourier Series **CTFS**	Continuous Periodic	Discrete	$F[k] = 1/T \int_{-T/2}^{T/2} f(t)\, e^{-j2\pi kt/T} dt$ $f(t) = \sum_k F[k]\, e^{j2\pi kt/T}$
Discrete-Time **Fourier Transform** **DTFT**	Discrete	Continuous Periodic	$F\left(e^{j\omega}\right) = \sum_n f[n] e^{-j\omega n}$ $f[n] = 1/2\pi \int_{-\pi}^{\pi} F\left(e^{j\omega}\right) e^{j\omega n} d\omega$
Discrete-Time **Fourier Series** **DTFS**	Discrete Periodic	Discrete Periodic	$F[k] = \sum_{n=0}^{N-1} f[n] e^{-jn\cdot k2\pi/N}$ $f[n] = 1/N \sum_{k=0}^{N-1} F[k] e^{jn\cdot k2\pi/N}$

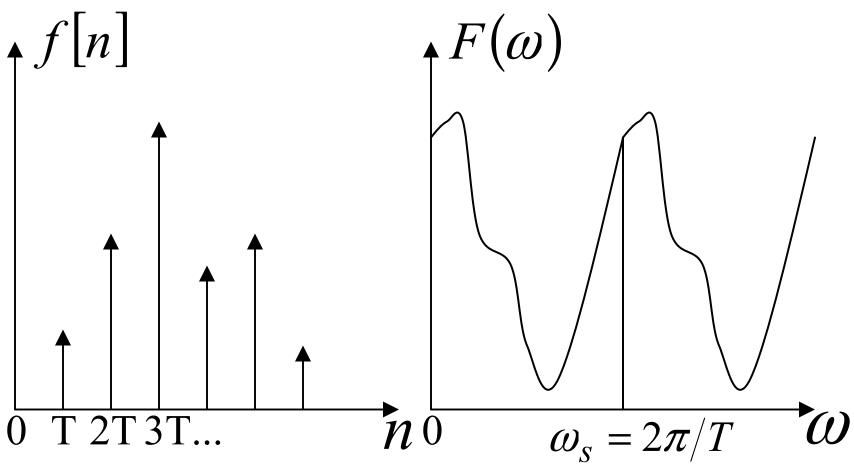
5.4. Various Flavors of Fourier Transforms - summary



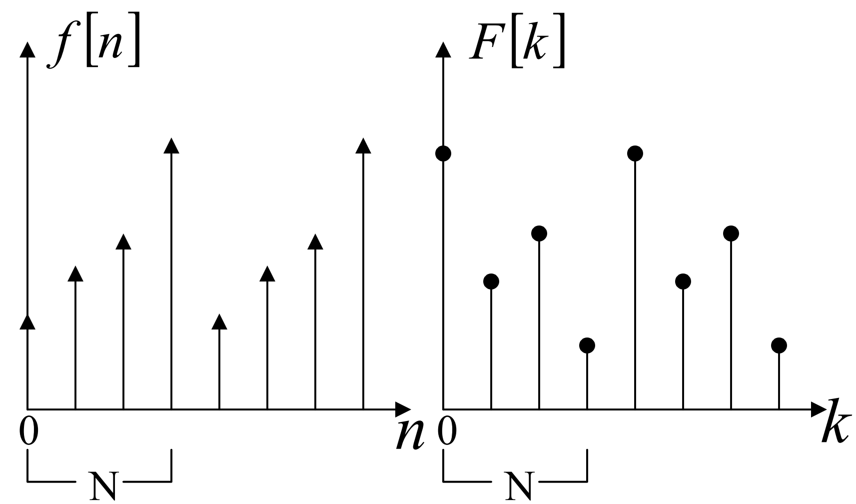
CTFT



CTFS



DTFT



DTFS