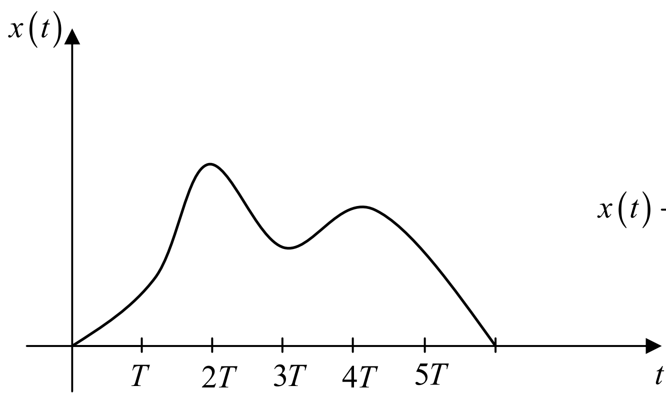
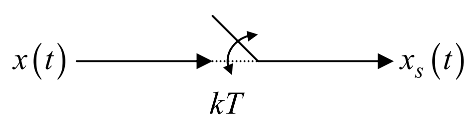
Sampling Theory

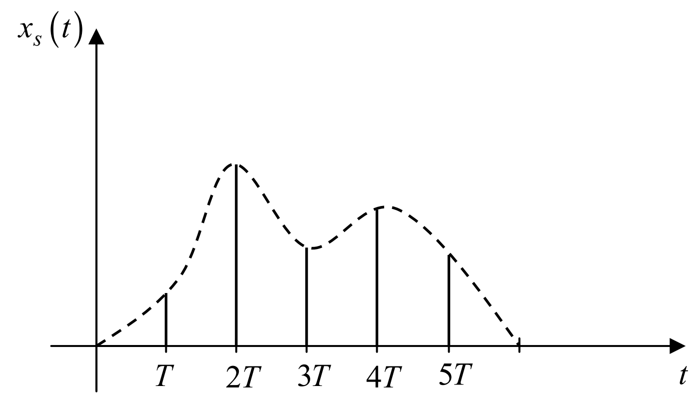
4. Sampling Theory

- In practice, most of information processing nowadays is done in the digital world
- Consequently, analog signals which are part of our real-world need to be converted to a digital format prior to processing
- This conversion process is called *analog-to-digital* conversion.
- The inverse process, which starts from a digital signal to produce an analog version is called *digital-to-analog* conversion
- A key element in analog-to-digital conversion is given by *sampling*
- By sampling, an analog signal is approximated by a series of samples at discrete moments in time.



Analog signal

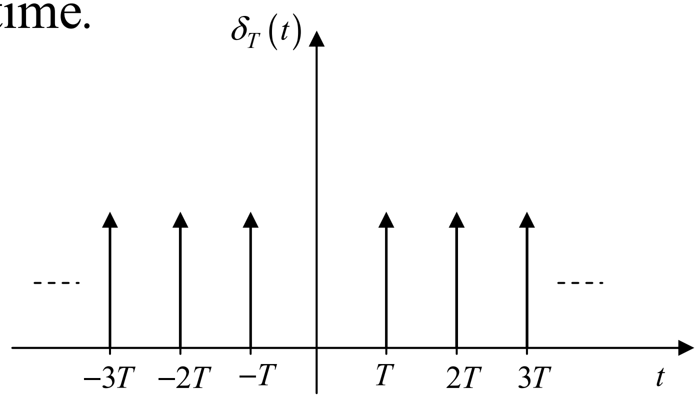


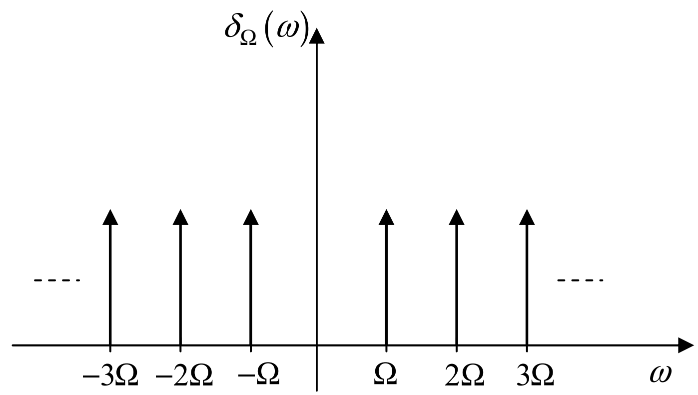


Sampled signal

4. Sampling Theory

- The sampled signal can be expressed as the product between the original signal, $x(t)$ and a periodic Dirac function $\delta_T(t)$ having a period T equal to the sampling period in time.





- Thus: $x_s(t) = x(t) \cdot \delta_T(t)$, where: $\delta_T(t) = \sum_k \delta(t - kT)$

- It can be shown that the Fourier Transform of $\delta_T(t)$ is proportional to a periodic Dirac function in frequency $\delta_\Omega(\omega) = \sum_k \delta(\omega - k\Omega)$ having the period:

$$\Omega = \frac{2\pi}{T}$$

- Using Fourier series, it will be later on shown that:

$$\mathcal{F}\{\delta_T(t)\} = \Omega \delta_\Omega(\omega) = \Omega \sum_k \delta(\omega - k\Omega)$$

4. Sampling Theory

- The Fourier transform of the sampled-signal is then given by:

$$\mathcal{F}\{x_s(t)\} = \mathcal{F}\{x(t)\cdot\delta_T(t)\} = \frac{1}{2\pi}\mathcal{F}\{x(t)\} * \mathcal{F}\{\delta_T(t)\} = \frac{1}{2\pi}X(\omega) * \left(\Omega\delta_\Omega(\omega)\right)$$

$$\mathcal{F}\{x_s(t)\} = \frac{\Omega}{2\pi}\left(X(\omega) * \sum_k \delta_\Omega(\omega - k\Omega)\right) = \frac{\Omega}{2\pi}\sum_k \left(X(\omega) * \delta_\Omega(\omega - k\Omega)\right)$$

Now: $X(\omega) * \delta_\Omega(\omega - k\Omega) = \int_{-\infty}^{\infty} X(u)\delta_\Omega(\omega - k\Omega - u)\,du = X(\omega - k\Omega)$

Also, $\frac{\Omega}{2\pi} = \frac{1}{T}$. Then:

$$\mathcal{F}\{x_s(t)\} = X_s(\omega) = \frac{1}{T}\sum_k X(\omega - k\Omega)$$

4. Sampling Theory

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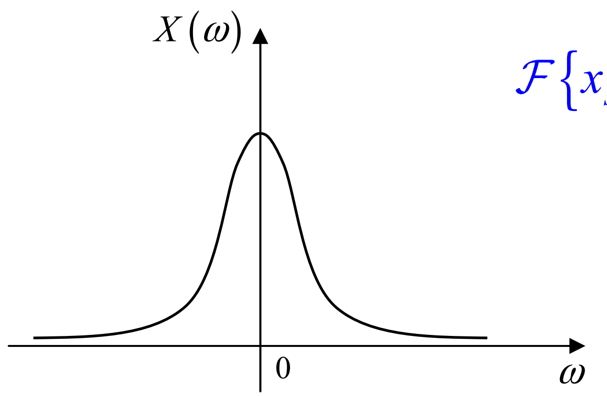
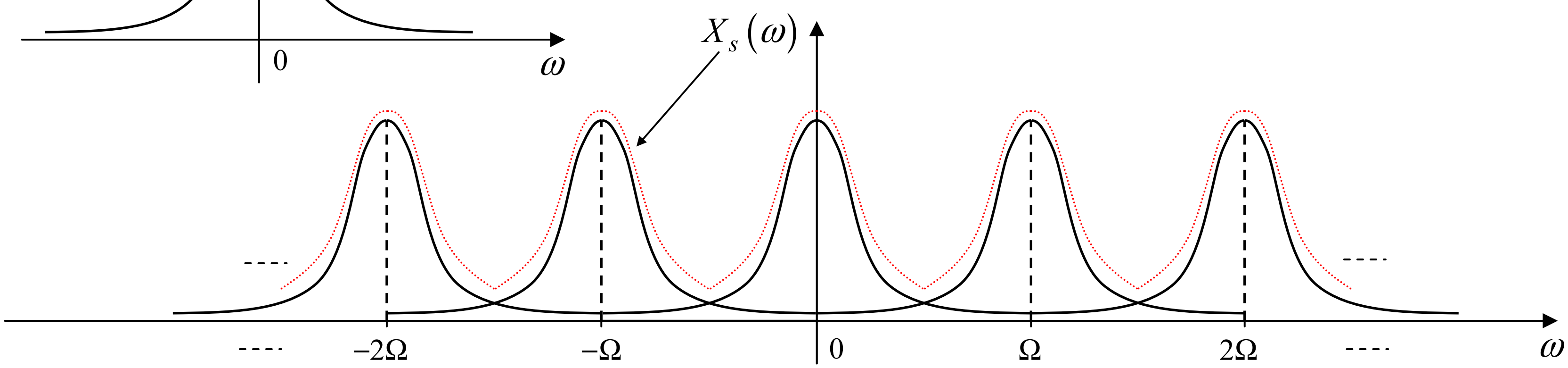


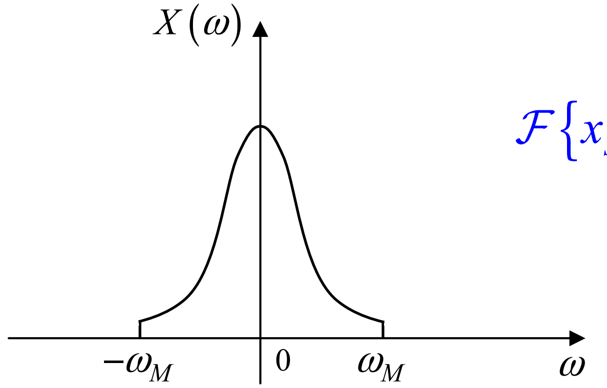
$$\mathcal{F}\{x_s(t)\} = X_s(\omega) = \frac{1}{T}\sum_k X(\omega - k\Omega)$$



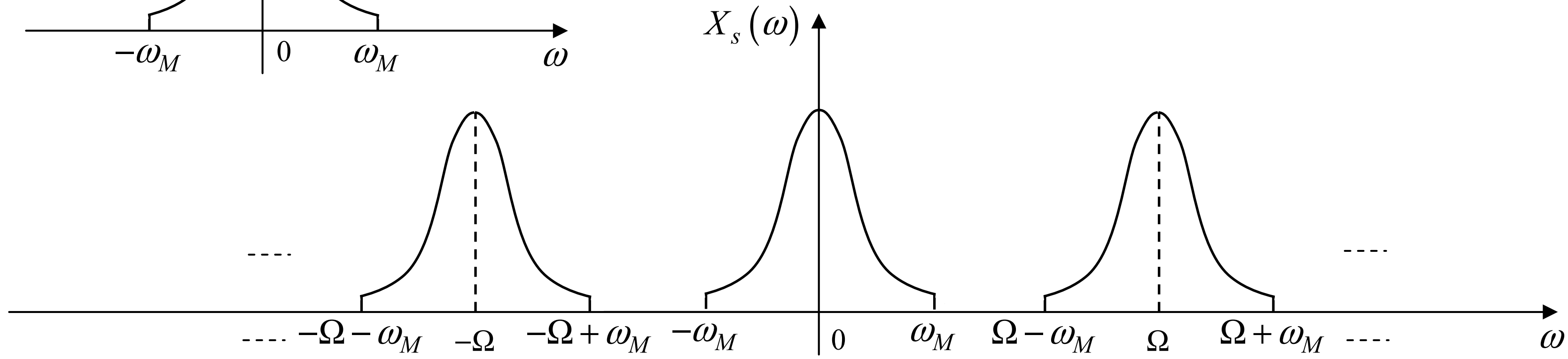
- If the spectra $X(\omega - k\Omega)$ overlap, then $X(\omega)$ cannot be reconstructed from $X_s(\omega)$
- If $X(\omega)$ has a finite support, that is, $X(\omega) = 0$, for $\omega > \omega_M$ then $X(\omega)$ can be reconstructed from $X_s(\omega)$

4. Sampling Theory

- Suppose that $X(\omega)$ has a finite support: $X(\omega)=0$, for $\omega > \omega_M$



$$\mathcal{F}\{x_s(t)\} = X_s(\omega) = \frac{1}{T}\sum_k X(\omega - k\Omega)$$



- **Nyquist condition**:

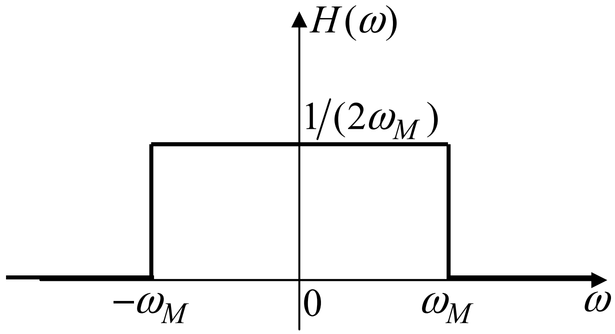
$$\Omega \geq 2\Omega_M \Leftrightarrow \frac{2\pi}{T} \geq 2(2\pi f_M) \Leftrightarrow T \leq \frac{1}{2f_M} = \frac{\pi}{\Omega_M}$$

- **Shanon's theorem**: Any signal $x(t)$ of which the spectral density has a finite support that is, $X(\omega)=0$, for $\omega > \omega_M$ is completely defined by its samples $x(nT)$ if :

$$T = \frac{\pi}{\Omega_M}$$

4. Sampling Theory

- How to reconstruct $x(t)$ from its samples $x(nT)$?



The finite-pulse function in frequency

$$\hat{X}(\omega) = X_s(\omega) \cdot H(\omega) \Leftrightarrow \hat{x}(t) = x_s(t) * h(t)$$

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) e^{j\omega t} d\omega = \frac{T}{2\pi} \int_{-\omega_M}^{\omega_M} e^{j\omega t} d\omega$$

$$h(t) = \mathrm{sinc}(\omega_M t)$$

$$\hat{x}(t) = x_s(t) * h(t)$$

$$\hat{x}(t) = \sum_n x(nT)\delta(t - nT) * h(t)$$

$$\hat{x}(t) = \sum_n x(nT)\,\mathrm{sinc}\left(\omega_M (t - nT)\right)$$

- Thus, the reconstructed signal is a linear superposition of sinc functions located in time at positions nT