Fundamentals of Signal Decompositions

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1. Vector spaces and Inner Products

Linear Algebra:

- Vectors over R or C that are of a finite dimension n: R^n or C^n
- Given $\{v_k\}$ a set of vectors in these spaces:
  - Does the set span the spaces, i.e. can every vector be represented as a linear combination of vectors from $\{v_k\}$?
  - Are the vectors linearly independent?
  - How can we find bases for the spaces to be spanned?
  - Given a subspace in R^n or C^n and a general vector x, can we find an approximation of x in the least-square sense that lies in the subspace?

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- A *vector space* over R or C is a set of vectors V, together with *addition* and scalar *multiplication*.
- Properties:
  - ✍ Commutativity: $x + y = y + x$
  - ✍ Associativity: $(x + y) + z = x + (y + z)$
  - ✍ Distributivity: $\alpha(x + y) = \alpha x + \alpha y$
  - ✍ Additive Identity: there exists 0 in V, such that: $x + 0 = x, \ \forall x \in V$
  - ✍ Multiplicative Identity: $1 \cdot x = x, \ \forall x \in V$
  - ✍ Additive inverse: for all x in V, there exists a (-x) in V, such that: $x + (-x) = 0$
- A subset M of V is a *subspace* of V if:
  - ✍ For all x and y in M, $x+y$ is in M
  - ✍ For all x in M, and α in R or C, αx is in M.
- Given $S \subset V$, the *span* of S is the subspace of V consisting of all linear combinations of vectors in S.
  - ✍ Example: $span(S) = \left\{ \sum_{i=1}^{n} \alpha_i x_i \mid \forall \alpha_i \in C, \forall x_i \in S \right\}$

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- The vectors $\{x_1, x_2, \ldots x_n\}$ are *linearly independent* if $\sum_i \alpha_i x_i = 0 \Rightarrow \alpha_i = 0, \forall i$
- A *basis* in *V* is a subset of linearly independent vectors $\{x_1, x_2, \ldots x_n\}$ for which: $E = span(x_1 \ldots x_n)$
- *V* is *infinite dimensional* if it contains an infinite linearly independent set of vectors;
  - eg.: the space of infinite sequences is spanned by the infinite set $\{\delta(n-k)\}_{k \in Z}$
- The *inner product* on *V* over *C* is a function defined on *V* x *V* with the properties:
  - $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$
  - $\langle x, \alpha y\rangle = \alpha\langle x, y\rangle$
  - $\langle x, y\rangle^* = \langle y, x\rangle$
  - $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ if and only if $x \equiv 0$
- Examples:
  - The inner product for complex-valued functions over *R*: $\langle f, g\rangle = \int_{-\infty}^{\infty} f(t) g^*(t) dt$
  - The inner product for complex-valued sequences over *Z*: $\langle x, y\rangle = \sum_{n=-\infty}^{\infty} x[n] y^*[n]$
- A vector space equipped with an inner product is called an *inner product space*

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- The *norm* of a vector is defined as: $\|x\| = \sqrt{\langle x, x \rangle}$
- Examples of *inner-product spaces*:
  - ✍ The real-numbers $\mathbb{R}$: $\langle x, y \rangle = xy$
  - ✍ The Euclidian space $\mathbb{R}^n$ for which the inner product is the *dot product* between two vectors: $\langle (x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n) \rangle = \sum_{i=1}^{n} x_i y_i$
- One can introduce the notion of angle between two vectors: $\langle x, y \rangle = \|x\| \cdot \|y\| \cdot \cos\left(\widehat{xy}\right)$

  => Cauchy-Buniakowski-Schwartz inequality:

$$\langle x, y \rangle \leq \|x\| \cdot \|y\| \Leftrightarrow \left(\sum_{i=1}^{n} x_i y_i \right)^2 \leq \left(\sum_{i=1}^{n} x_i^2 \right) \left(\sum_{i=1}^{n} y_i^2 \right)$$

- Triangle inequality: $\|x + y\| \leq \|x\| + \|y\|$
- Parallelogram law: $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$
- Pythagorean theorem: If $\langle x, y \rangle = 0$ then: $\|x + y\|^2 = \|x\|^2 + \|y\|^2$

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- A *vector x* is said to be *orthogonal* to a set of vectors $S = \{y_i\}$, *if* $\langle x, y_i \rangle = 0, \forall i$
  - ✍ Notation: $x \perp S$
- *Two spaces* S_1 and S_2 are called to be *orthogonal* if for $\forall x_i \in S_1,\ x_i \perp S_2$
  - ✍ Notation: $S_1 \perp S_2$
- A *set of vectors* $\{x_1, x_2 ...\}$ is called *orthogonal* if $\forall i, j, i \neq j,\ x_i \perp x_j$
- A *orthonormal* set of vectors is an orthogonal set with unit norm: $\langle x_i, x_j \rangle = \delta[i-j]$

- A sequence of vectors $\{x_n\}$ in *V* is said to converge to a vector *x* in *V* if:

$$\|x_n - x\| \to 0\big|_{n\to\infty}$$

- A sequence of vectors $\{x_n\}$ is called a *Cauchy* sequence if:

$$\|x_n - x_m\| \to 0, as\ \ n, m \to \infty$$

- If *every*(!) Cauchy sequence in *V* converges to a vector in *V*, then the space is *V* is called *complete*
- Note: Not all Cauchy sequences necessarily need to converge to a vector in V, i.e. not all vector spaces are necessarily complete

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- *A complete normed vector space is called a Banach space.*
- *Banach* spaces: Every Cauchy sequence (with respect to the metric $d(x, y)=\|x-y\|$) in V has a limit in V
- Example: the space $C[a,b]$ of all continuous real-valued of complex functions on the interval $[a,b]$, with the norm defined as: $\|f\| = \sup\{|f(x)| : x \in [a,b]\}$
  - ✍ Continuous functions on an interval are bounded, hence, the space is complete under this norm
- Another example: define the l^p norm:

$$\|x\|_p = \left(|x_1|^p + |x_2|^p + \ldots\right)^{1/p}$$

- l^p is the Banach space of all infinite sequences $(x_1, x_2, \ldots.)$ of real or complex-valued elements, for which $\|x\|_p$ is finite.
  - ✍ E.g. $\left(1, \frac{1}{2}, \frac{1}{3}, \ldots \frac{1}{n} \ldots.\right)$ is not in l^1 but it is in l^p

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- *An inner-product space of which the norm is complete is called a Hibert space.*
- Every inner-product gives rise to a norm, via the association: $\langle x, x\rangle = \|x\|^2$
- If the norm is complete, then the space is called a Hilbert space
- Thus every Hilbert space is a Banach space by definition
- A necessary and sufficient condition for a Banach space *V* to be a Hilbert space is if the parallelogram rule is satisfied for all vectors *x*,*y* in *V*

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

- Example: $\mathbb{R}^n$ is a Banach space for *any norm* defined on it, and it is a Hilbert space only with respect to the Euclidean norm

- Another example: define the L^p norm: $\|f\|_p := \left(\int_a^b |f(x)|^p \, dx\right)^{1/p}$
- $L^p([a,b])$ is the Banach space of all real or complex-valued functions on $[a,b]$, for which L^p is finite $L^p < \infty$
- This Banach space is a Hilbert space only for $p = 2$

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- Examples of Hilbert spaces
  - ✍ *Space of square-summable sequences* - $l^2(Z)$
    - ☯ inner product: $\langle x, y\rangle = \sum_{n=-\infty}^{\infty} x[n]y^*[n]$
    - ☯ norm: $\|x\| = \sqrt{\langle x, x\rangle} = \sqrt{\sum_{n\in Z} |x[n]|^2} < \infty$
  - ✍ *Space of square-integrable functions* - $L^2(R)$
    - ☯ inner product: $\langle f, g\rangle = \int_{t\in R} f(t)g^*(t)\,dt$
    - ☯ norm: $\|f\| = \sqrt{\langle f, f\rangle} = \sqrt{\int_{t\in R} |f(t)|^2 dt}$
- Given a Hibert space V and a subspace S, the *orthogonal complement* of S in V, denoted by $S^\perp$ is the set $\{x\in V,\ x\perp S\}$
- Given a vector x in V, there exists a unique y in S and a unique z in $S^\perp$ such that $x = y + z$.
  - ✍ Result: $V = S \oplus S^\perp$

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Orthonormal Bases in Hilbert spaces

- $S = \{x_i\}$ form an orthonormal basis in V if $\langle x_i, x_j \rangle = \delta[i-j]$ and $\forall y \in V,\ \exists \alpha_k,$

  $y = \sum_k \alpha_k x_k$
- $\alpha_k = \langle x_k, y \rangle$

Orthogonal Projection and Least-square Approximation

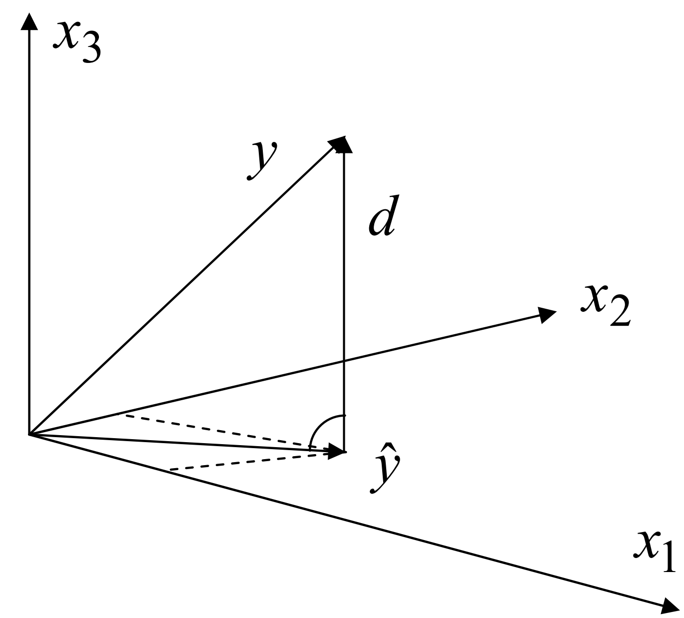
- Given V a Hilbert space, S a subspace of V, $\{x_1, x_2, ...\}$ an orthonormal basis in S, and a vector $y \in V$, find the best approximation $\hat{y}$ of y in S.

  ✍ $d = y - \hat{y}$

  ✍ $\|d\| = \|y - \hat{y}\|$ is minimal for $\hat{y} = \sum_i \langle x_i, y \rangle x_i$

  ✍ Note that: $d \perp S$.

  ✍ Note that: $\|y\|^2 = \|\hat{y}\|^2 + \|d\|^2$



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Biorthogonal Bases

A system $\{x_k, \tilde{x}_k\}$ constitutes *biorthogonal base* in a Hilbert space V if and only if:

- For all i, j in Z: $\langle x_i, \tilde{x}_j \rangle = \delta[i-j]$
- The sets $\{x_k\}$ and $\{\tilde{x}_k\}$ constitute each a *frame* in V, that is, for all y in V, there exist strictly positive constants A, B, $\tilde{A}$, $\tilde{B}$ (called *frame bounds*) such that:

  $$A\|y\|^2 \le \sum_k |\langle x_k, y\rangle|^2 \le B\|y\|^2$$

  $$\tilde{A}\|y\|^2 \le \sum_k |\langle \tilde{x}_k, y\rangle|^2 \le \tilde{B}\|y\|^2$$

✍ Bases that satisfy these constraints are called *Riesz bases*.

✍ If $A = B$ the frame $\{x_k\}$ is called a *tight frame*.

- Expansion formula:

$$y = \sum_k \langle x_k, y\rangle \tilde{x}_k = \sum_k \langle \tilde{x}_k, y\rangle x_k$$