

Assignment 2

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Absque sudore et labore nullum opus perfectum est

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Problem 1. Convert the following natural-language text into an equivalent set of propositional premises, together with an associated propositional conclusion. Your answers should identify the text that corresponds to each proposition variable you use and the proposition formulæ that correspond to the premises and the conclusion. Determine if the argument is valid.

1. If the Big Bang Theory is correct then either there was a time before anything existed or the world will come to an end. The world will not come to an end. Therefore, if there was no time before anything existed, the Big Bang Theory is incorrect.
2. To win a gold medal, an athlete must be very fit. If s/he does not win a gold medal, then either s/he arrived late for the competition or his/her training was interrupted. If s/he is not very fit, s/he will blame his/her coach. If s/he blames his/her coach, or his/her training is interrupted, then s/he will not get into the competition. Therefore, if s/he gets into the competition, s/he will not have arrived late.
3. If Hector wins the battle, he will plunder the city. If he does not win the battle, he will either be killed or go into exile. If he plunders the city, then Priam will lose his kingdom. If Priam loses his kingdom or Hector goes into exile, then the war will end. Therefore, if the war does not end, Hector will be killed.
4. If the colonel was out of the room when the murder was committed then he couldn't have been right about the weapon used. Either the butler is lying or he knows who the murderer was. If Lady Barntree was not the murderer then either the colonel was in the room at the time or the butler is lying. Either the butler knows who the murderer was or the colonel was out of the room at the time of the murder. Therefore, if the colonel was right about the weapon then Lady Barntree was the murderer.

Solution

1. P : Big Bang Theory is correct, Q : There was a time before anything existed, R : The world will come to an end

$$\frac{P \rightarrow \neg(Q \leftrightarrow R), \neg R,}{\neg Q \rightarrow \neg P}$$

1. *Show* $\neg Q \rightarrow \neg P$

2.	$\neg Q$	1 Ass CD
3.	<i>Show</i> $\neg P$	
4.	P	3 ID
5.	$P \rightarrow \neg(Q \leftrightarrow R)$	P1
6.	$\neg(Q \leftrightarrow R)$	4,5 MP
7.	$\neg R$	P1
8.	Q	5,6 MTP
9.	$\neg Q$	2

2. W : Win a gold medal, F : Athlete is fit, L : Athlete arrived late for the competition, I : Training was interrupted, B : Athlete blames the coach, C : Athlete gets into the competition

$$\frac{W \rightarrow F, \neg W \rightarrow \neg(L \leftrightarrow I), \neg(B \leftrightarrow I) \rightarrow \neg C}{C \rightarrow \neg L}$$

The given conclusion is not valid. The counter argument for this is: $W : T, F : T, L : T, I : F, B : F, C : T$

The conclusions is false because $C \rightarrow \neg L$ is false but all the premises are true i.e.

$W \rightarrow F : T$

$\neg W \rightarrow (L \leftrightarrow I) : T$

$\neg F \rightarrow B : T$

$\neg(B \leftrightarrow I) \rightarrow \neg C : T$

3. H : Hector wins the battle, P : City is plundered by Hector, K : Hector gets killed, E : Hector goes into the exile, L : Priam will lose his kingdom, W : War will end

$$\frac{H \rightarrow P, \neg H \rightarrow (E \oplus K), P \rightarrow L, (L \oplus E) \rightarrow W}{\neg W \rightarrow K}$$

1. *Show* $\neg W \rightarrow K$

2.	$\neg W$	1 Ass CD
3.	<i>Show</i> K	
4.	$\neg(L \leftrightarrow E) \rightarrow W$	4th Premise
5.	$(L \leftrightarrow E)$	2,4 Modus Tollens
6.	$P \rightarrow L$	3rd Premise
7.	$\neg L$	5 XOR
8.	$\neg P$	3,5 Modus Tollens
9.	$H \rightarrow P$	1st Premise
10.	$\neg H$	8,9 Modus Tollens
11.	$\neg E$	5 XOR
12.	$\neg H \rightarrow \neg(E \leftrightarrow K)$	2nd Premise
13.	$\neg(E \leftrightarrow K)$	10,12 Modus Ponens
14.	$\neg(K \rightarrow E)$	13 BC
15.	K	11,14 MP

Thus the given conclusion is true. This can also be established by backtracking the statements.

4. O : Colonel was out of the room when the murder was committed, R : Colonel is correct about the weapon used in the murder, B : Butler is lying, K : Butler knows who the murderer, L : Lady Barntree was the murderer

$$\frac{O \rightarrow \neg R, \neg(B \leftrightarrow K), \neg L \rightarrow \neg(B \leftrightarrow \neg O), \neg(K \leftrightarrow O)}{R \rightarrow \neg L}$$

The counter-model is: $O : F, R : F, B : F, K : T, L : T$. All the premises are true but the conclusion is false

The given conclusion is false. This can be established by backtracking the statements.

We know R , this implies that $\neg O$ from the first proposition. Using $\neg O$ in the second last proposition we can say that K is true. Using K in the second proposition we can say that B is false. Using all this we can say that $\neg(B \leftrightarrow \neg O)$ is *True*. But since $\neg L \rightarrow \neg(B \leftrightarrow \neg O)$ and $\neg(B \leftrightarrow \neg O)$ are both true we can not necessarily say that $\neg L$ is *True*

Problem 2. Which of the following are tautologies? Which are contradictions? Which are neither? Justify your answer with truth tables, or sufficient models as is necessary.

1. $(P \wedge Q) \rightarrow (P \rightarrow Q)$
2. $(P \wedge Q) \leftrightarrow (P \rightarrow Q)$
3. $(\neg P \vee Q) \rightarrow (P \rightarrow \neg Q)$
4. $((P \rightarrow Q) \rightarrow P) \rightarrow Q$
5. $(P \rightarrow (Q \rightarrow (P \rightarrow Q)))$
6. $((P \wedge \neg Q) \rightarrow \neg R) \leftrightarrow ((P \wedge R) \rightarrow Q)$
7. $((P \vee Q) \vee R) \vee S \leftrightarrow (P \vee (Q \vee (R \vee S)))$
8. $((P \rightarrow Q) \rightarrow R) \rightarrow S \leftrightarrow (P \rightarrow (Q \rightarrow (R \rightarrow S)))$
9. $(P \rightarrow (\neg R \rightarrow \neg S)) \vee ((S \rightarrow (P \vee \neg T)) \vee (\neg Q \rightarrow R))$

Solution

Table 1: Part 1. Clearly a Tautology

P	Q	$P \rightarrow Q$	$P \wedge Q$	$(P \wedge Q) \rightarrow (P \rightarrow Q)$
F	F	T	F	T
F	T	T	F	T
T	F	F	F	T
T	T	T	T	T

Table 2: Part 2. Clearly a Satisfiable

P	Q	$P \rightarrow Q$	$P \wedge Q$	$(P \wedge Q) \leftrightarrow (P \rightarrow Q)$
F	F	T	F	F
F	T	T	F	F
T	F	F	F	T
T	T	T	T	T

Table 3: Part 3. Clearly satisfiable

P	Q	$(\neg P \vee Q)$	$(P \rightarrow \neg Q)$	$(\neg P \vee Q) \rightarrow (P \rightarrow \neg Q)$
F	F	T	T	T
F	T	T	T	T
T	F	F	T	T
T	T	F	F	F

Table 4: Part 4. Clearly satisfiable

P	Q	$(P \rightarrow Q)$	$((P \rightarrow Q) \rightarrow P)$	$((P \rightarrow Q) \rightarrow P) \rightarrow Q$
F	F	T	F	T
F	T	T	F	T
T	F	F	T	F
T	T	T	T	T

Table 5: Part 5. Clearly a Tautology

P	Q	$(P \rightarrow Q)$	$(Q \rightarrow (P \rightarrow Q))$	$(P \rightarrow (Q \rightarrow (P \rightarrow Q)))$
F	F	T	T	T
F	T	T	T	T
T	F	F	T	T
T	T	T	T	T

Table 6: Part 6. Clearly a Tautology

P	Q	R	$((P \wedge \neg Q) \rightarrow \neg R)$	$((P \wedge R) \rightarrow Q)$	$((P \wedge \neg Q) \rightarrow \neg R) \leftrightarrow ((P \wedge R) \rightarrow Q)$
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	F	F	T
T	F	F	T	T	T
F	T	T	T	T	T
F	T	F	T	T	T
F	F	T	T	T	T
F	F	F	T	T	T

Table 7: Part 7. Clearly a Tautology

P	Q	R	$((P \vee Q) \vee R) \vee S$	$(P \vee (Q \vee (R \vee S)))$	$((P \vee Q) \vee R) \vee S \leftrightarrow (P \vee (Q \vee (R \vee S)))$
T	T	T	T	T	T
T	T	F	F	F	T
T	F	T	F	F	T
T	F	F	F	F	T
F	T	T	F	F	T
F	T	F	F	F	T
F	F	T	F	F	T
F	F	F	F	F	T

Table 8: Part 8. Clearly satisfiable

P	Q	R	S	$((P \rightarrow Q) \rightarrow R) \rightarrow S$	$(P \rightarrow (Q \rightarrow (R \rightarrow S)))$	$((P \rightarrow Q) \rightarrow R) \rightarrow S \leftrightarrow (P \rightarrow (Q \rightarrow (R \rightarrow S)))$
T	T	T	T	T	T	T
T	F	T	F	F	T	F

Table 9: Part 9. Clearly a Tautology

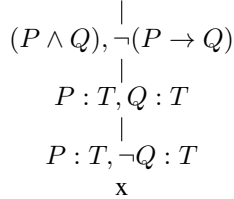
P	Q	R	S	T	$(P \rightarrow (\neg R \rightarrow \neg S))$	$(S \rightarrow (P \vee \neg T))$	$(P \rightarrow (\neg R \rightarrow \neg S)) \vee (S \rightarrow (P \vee \neg T)) \vee (\neg Q \rightarrow R)$
T	T	T	T	T	T	T	T
T	T	T	T	F	T	T	T
T	T	T	F	T	T	T	T
T	T	T	F	F	T	T	T
T	T	F	T	T	F	T	T
T	T	F	T	F	F	T	T
T	T	F	F	T	T	T	T
T	T	F	F	F	T	T	T
T	F	T	T	T	T	T	T
T	F	T	T	F	T	T	T
T	F	T	F	T	T	T	T
T	F	T	F	F	T	T	T
T	F	F	T	T	T	T	T
T	F	F	T	F	T	T	T
T	F	F	F	T	T	T	T
T	F	F	F	F	T	T	T
F	T	T	T	T	T	T	T
F	T	T	T	F	T	T	T
F	T	T	F	T	T	T	T
F	T	T	F	F	T	T	T
F	T	F	T	T	T	T	T
F	T	F	T	F	T	T	T
F	T	F	F	T	T	T	T
F	T	F	F	F	T	T	T
F	F	T	T	T	T	F	T
F	F	T	T	F	T	T	T
F	F	T	F	T	T	T	T
F	F	T	F	F	T	T	T
F	F	F	T	T	T	F	T
F	F	F	T	F	T	T	T
F	F	F	F	T	T	T	T
F	F	F	F	F	T	T	T

Problem 3. For the propositional formulae in Question 2, use Semantic Tableaux to show whether each is a tautology, contradiction, or neither.

1. $(P \wedge Q) \rightarrow (P \rightarrow Q)$
2. $(P \wedge Q) \leftrightarrow (P \rightarrow Q)$
3. $(\neg P \vee Q) \rightarrow (P \rightarrow \neg Q)$
4. $((P \rightarrow Q) \rightarrow P) \rightarrow Q$
5. $(P \rightarrow (Q \rightarrow (P \rightarrow Q)))$
6. $((P \wedge \neg Q) \rightarrow \neg R) \leftrightarrow ((P \wedge R) \rightarrow Q)$
7. $((P \vee Q) \vee R) \vee S \leftrightarrow (P \vee (Q \vee (R \vee S)))$
8. $((P \rightarrow Q) \rightarrow R) \rightarrow S \leftrightarrow (P \rightarrow (Q \rightarrow (R \rightarrow S)))$
9. $(P \rightarrow (\neg R \rightarrow \neg S)) \vee ((S \rightarrow (P \vee \neg T)) \vee (\neg Q \rightarrow R))$

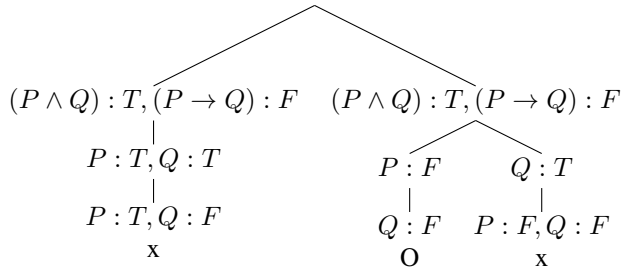
Solution

1. $((P \wedge Q) \rightarrow (P \rightarrow Q)) : F$



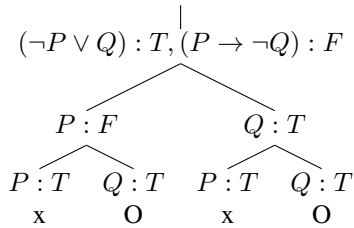
Therefore this statement is a Tautology

2. $(P \wedge Q) \leftrightarrow (P \rightarrow Q) : F$



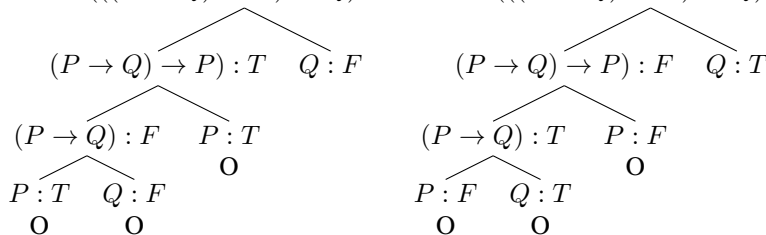
Therefore this statement is satisfiable

3. $(\neg P \vee Q) \rightarrow (P \rightarrow \neg Q) : F$



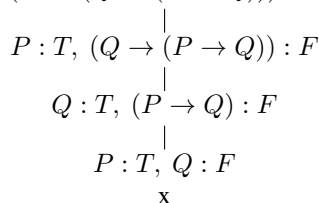
Therefore this statement is satisfiable

4. $((P \rightarrow Q) \rightarrow P) \rightarrow Q : F$ $((P \rightarrow Q) \rightarrow P) \rightarrow Q : T$

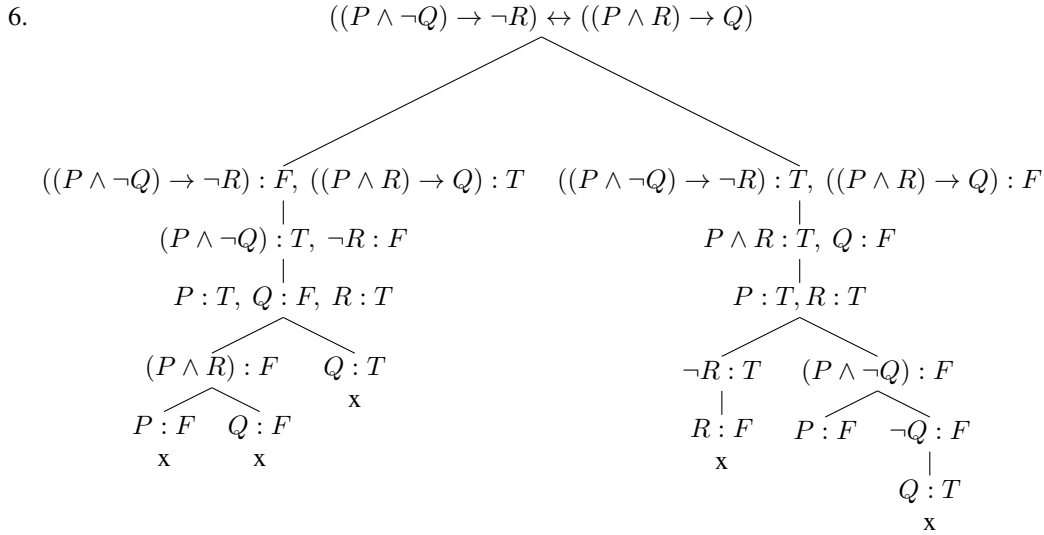


Since the proposition can be both True and False it is Satisfiable

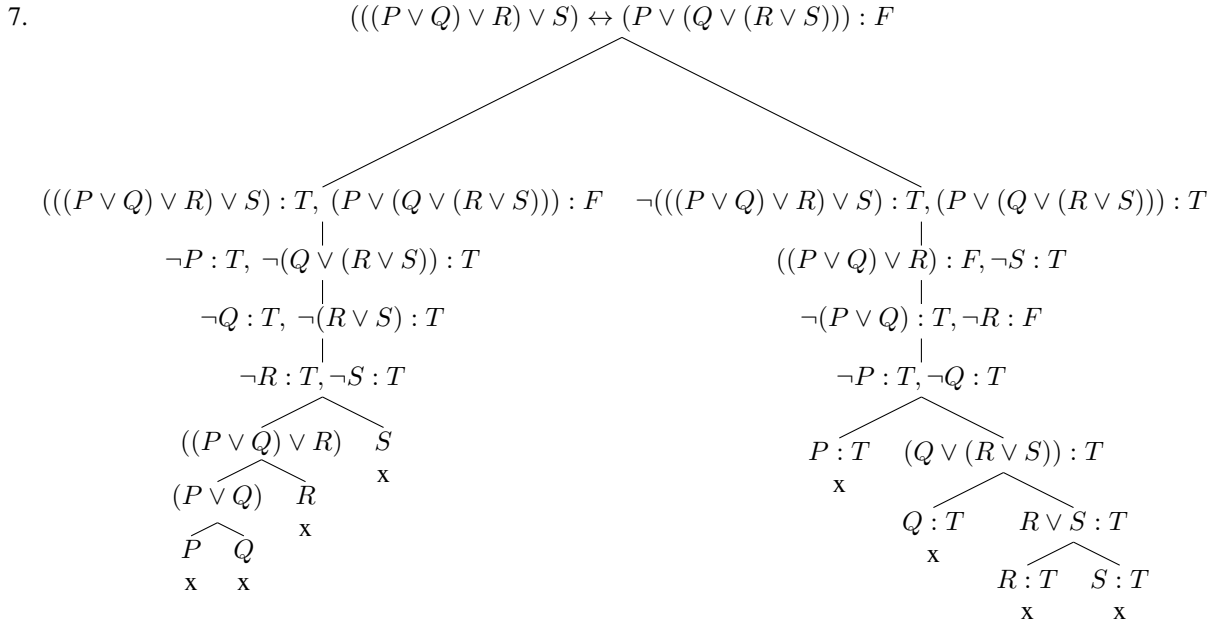
5. $(P \rightarrow (Q \rightarrow (P \rightarrow Q))) : F$



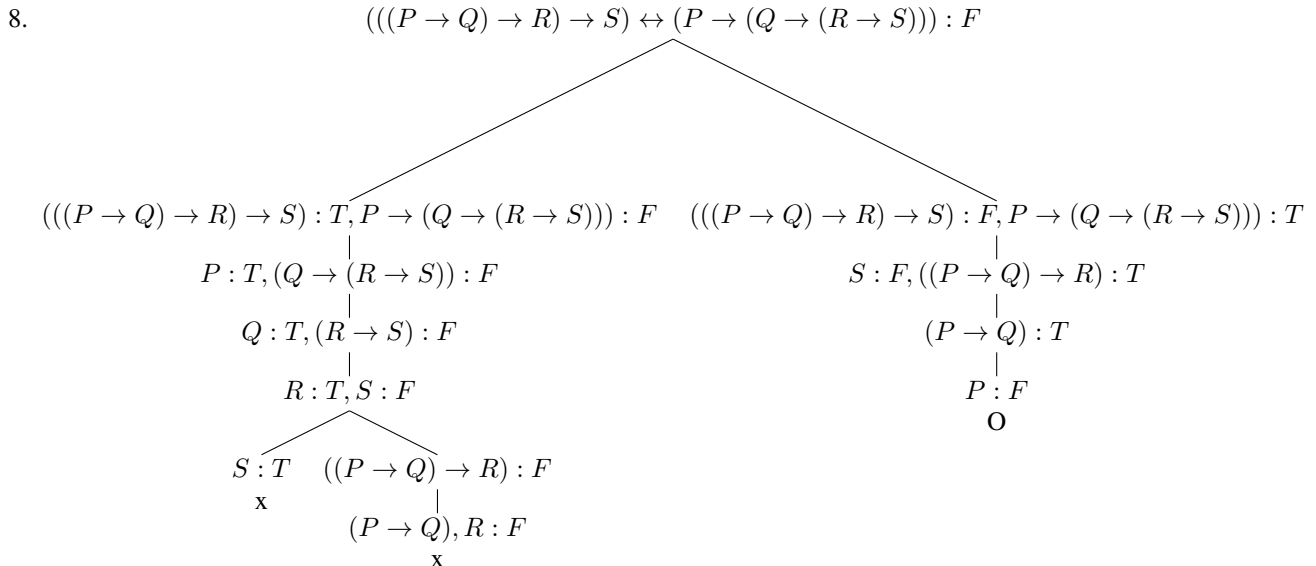
Therefore this statement is a Tautology



Therefore this statement is a Tautology



Therefore this statement is a Tautology



Thus the statement is Satisfiable

$$\begin{array}{c}
9. \quad ((P \rightarrow (\neg R \rightarrow \neg S)) \vee ((S \rightarrow (P \vee \neg T)) \vee (\neg Q \rightarrow R))) : False \\
\quad \neg(P \rightarrow (\neg R \rightarrow \neg S)) : True, \neg((S \rightarrow (P \vee \neg T)) \vee (\neg Q \rightarrow R)) : True \\
\quad \quad P : True, \neg(\neg R \rightarrow \neg S) : True \\
\quad \quad \quad \neg(S \rightarrow (P \vee \neg T)) : True, \neg(\neg Q \rightarrow R) : True \\
\quad \quad \quad \quad S, \neg(P \vee \neg T) : True \\
\quad \quad \quad \quad \quad \neg P : True, \neg\neg T : True \\
\quad \quad \quad \quad \quad \quad x
\end{array}$$

Therefore this statement is a Tautology

Problem 4. For the propositional formulae in Question 2, for any case where the formula was a tautology, prove that it is a tautology with a Kalish-Montague derivations.

1. $(P \wedge Q) \rightarrow (P \rightarrow Q)$
2. $(P \wedge Q) \leftrightarrow (P \rightarrow Q)$
3. $(\neg P \vee Q) \rightarrow (P \rightarrow \neg Q)$
4. $((P \rightarrow Q) \rightarrow P) \rightarrow Q$
5. $(P \rightarrow (Q \rightarrow (P \rightarrow Q)))$
6. $((P \wedge \neg Q) \rightarrow \neg R) \leftrightarrow ((P \wedge R) \rightarrow Q)$
7. $((P \vee Q) \vee R) \vee S \leftrightarrow (P \vee (Q \vee (R \vee S)))$
8. $((P \rightarrow Q) \rightarrow R) \rightarrow S \leftrightarrow (P \rightarrow (Q \rightarrow (R \rightarrow S)))$
9. $(P \rightarrow (\neg R \rightarrow \neg S)) \vee ((S \rightarrow (P \vee \neg T)) \vee (\neg Q \rightarrow R))$

Solution

- | | | |
|----------|--|------------------------|
| 1. | <i>Show</i> $(P \wedge Q) \rightarrow (P \rightarrow Q)$ | 2–7 Conditionalization |
| 2. | <div style="border: 1px solid black; padding: 5px; display: inline-block;"> $(P \wedge Q)$ </div> | 1 Ass CD |
| 3. | <div style="border: 1px solid black; padding: 5px; display: inline-block;"> <i>Show</i> $(P \rightarrow Q)$ </div> | 4–5 Conditionalization |
| 1.
4. | <div style="border: 1px solid black; padding: 5px; display: inline-block;"> <div style="border: 1px solid black; padding: 5px; display: inline-block;"> P </div> </div> | 2 SIMP |
| 5. | <div style="border: 1px solid black; padding: 5px; display: inline-block;"> <div style="border: 1px solid black; padding: 5px; display: inline-block;"> Q </div> </div> | 2 SIMP |
-
- | | | |
|----------|---|------------------------|
| 1. | <i>Show</i> $(P \rightarrow (Q \rightarrow (P \rightarrow Q)))$ | 2–7 Conditionalization |
| 2. | <div style="border: 1px solid black; padding: 5px; display: inline-block;"> P </div> | 1 Ass CD |
| 3. | <div style="border: 1px solid black; padding: 5px; display: inline-block;"> <i>Show</i> $(Q \rightarrow (P \rightarrow Q))$ </div> | 3–7 Sub Derivation |
| 4. | <div style="border: 1px solid black; padding: 5px; display: inline-block;"> <div style="border: 1px solid black; padding: 5px; display: inline-block;"> Q </div> </div> | 3 Ass CD |
| 5.
5. | <div style="border: 1px solid black; padding: 5px; display: inline-block;"> <div style="border: 1px solid black; padding: 5px; display: inline-block;"> <i>Show</i> $(P \rightarrow Q)$ </div> </div> | 7–8 Sub Derivation |
| 6. | <div style="border: 1px solid black; padding: 5px; display: inline-block;"> <div style="border: 1px solid black; padding: 5px; display: inline-block;"> <div style="border: 1px solid black; padding: 5px; display: inline-block;"> P </div> </div> </div> | 5 CD |
| 7. | <div style="border: 1px solid black; padding: 5px; display: inline-block;"> <div style="border: 1px solid black; padding: 5px; display: inline-block;"> <div style="border: 1px solid black; padding: 5px; display: inline-block;"> Q </div> </div> </div> | 4 Repeat |

6. Citing De-Morgan's Law proof from question 5

1. *Show* $((P \wedge \neg Q) \rightarrow \neg R) \leftrightarrow ((P \wedge R) \rightarrow Q)$

2–21 Conditionalization

2. *Show* $((P \wedge \neg Q) \rightarrow \neg R) \rightarrow ((P \wedge R) \rightarrow Q)$

1 Subderivation

3. $((P \wedge \neg Q) \rightarrow \neg R)$

2 Ass CD

4. *Show* $((P \wedge R) \rightarrow Q)$

7–8 Sub-Derivation

5. $P \wedge R$

4 Conditional Derivation

6. P

5 Simplification

7. R

5 Simplification

8. $\neg \neg R$

7 Double Negation

9. $\neg(P \wedge \neg Q)$

3,7 Modus Tollens

10. $\neg P \vee Q$

9 de-Morgan's law

11. Q

6,10 Modus Tollendo Ponens

12. *Show* $((P \wedge R) \rightarrow Q) \rightarrow ((P \wedge \neg Q) \rightarrow \neg R)$

1 Subderivation

13. $((P \wedge R) \rightarrow Q)$

12 Ass CD

14. *Show* $((P \wedge \neg Q) \rightarrow \neg R)$

7–8 Sub-Derivation

15. $P \wedge \neg Q$

12 Conditional Derivation

16. P

5 Simplification

17. $\neg Q$

5 Simplification

18. $\neg(P \wedge R)$

13,5 Modus Tollens

19. $\neg P \vee \neg R$

18 de-Morgan's law

20. $\neg R$

16,19 Modus Tollendo Ponens

21. $((P \wedge \neg Q) \rightarrow \neg R) \leftrightarrow ((P \wedge R) \rightarrow Q)$

2,12 Conditional to Biconditional, DD

7.

Property 1: $\frac{\neg(P \vee Q)}{\neg P}$

$\neg(P \vee Q) : T, \neg P : F$

$P : T$

$P \vee Q : F$

$P : F, Q : F$

x

1. *Show* $((P \vee Q) \vee R) \vee S \leftrightarrow (P \vee (Q \vee (R \vee S)))$

2–13 Conditionalization

2. *Show* $((P \vee Q) \vee R) \vee S \rightarrow (P \vee (Q \vee (R \vee S)))$

1 Subderivation

3. $((P \vee Q) \vee R) \vee S$

2 Ass CD

4. *Show* $(P \vee (Q \vee (R \vee S)))$

Sub derivation

5. $\neg(P \vee (Q \vee (R \vee S)))$

4 ID

6. $\neg P$

5 Property 1

7. $\neg(Q \vee (R \vee S))$

5 Property 1

8. $\neg Q$

7 Property 1

9. $\neg(R \vee S)$

5 Property 1

10. $\neg R$

9 Property 1

11. $\neg S$

9 Property 1

12. $(P \vee Q) \vee R$

3, 11 MTP

13. $P \vee Q$

10,13 MTP

14. P

8,13 MTP

15. *Show* $(P \vee (Q \vee (R \vee S))) \rightarrow ((P \vee Q) \vee R) \vee S$

1 Subderivation

16. $(P \vee (Q \vee (R \vee S)))$

15 Ass CD

17. *Show* $((P \vee Q) \vee R) \vee S$

18–8 Sub-Derivation

18. $\neg(((P \vee Q) \vee R) \vee S)$

17 ID

19. $\neg S$

18 Property 1

20. $\neg((P \vee Q) \vee R)$

18 Property 1

21. $\neg R$

20 Property 1

22. $\neg(P \vee Q)$

20 Property 1

23. $\neg P$

22 Property 1

24. $\neg Q$

22 Property 1

25. $Q \vee (R \vee S)$

23,24 MTP

26. $(R \vee S)$

24,25 MTP

27. R

19,26 MTP

28. $((P \vee Q) \vee R) \vee S \leftrightarrow (P \vee (Q \vee (R \vee S)))$

2,15 CB,DD

9.

Property 1: $\frac{\neg(P \vee Q)}{\neg P}$

$$\begin{array}{c} \neg(P \vee Q) : T, \neg P : F \\ | \\ P : T \\ | \\ P \vee Q : F \\ | \\ P : F, Q : F \\ \times \end{array}$$

Table 10: Proof for $P \rightarrow Q = \neg P \vee Q$

P	Q	$P \rightarrow Q$	$\neg P \vee Q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

Property 2: $P \rightarrow Q = \neg P \vee Q$

Referencing De-Morgan's law from Question 5

1. *Show* $\neg(P \rightarrow (\neg R \rightarrow \neg S)) \vee (S \rightarrow (P \vee \neg T)) \vee (\neg Q \rightarrow R)$

2–9 Conditionalization

2. $\neg(P \rightarrow (\neg R \rightarrow \neg S)) \vee (S \rightarrow (P \vee \neg T)) \vee (Q \vee R)$

1 ID

3. $\neg(P \rightarrow (\neg R \rightarrow \neg S))$

2 Property1

4. $\neg(\neg P \vee (\neg R \rightarrow \neg S))$

2 Property2

5. P

2 Property1

6. $\neg((S \rightarrow (P \vee \neg T)) \vee (Q \vee R))$

2 Property1

7. $\neg(S \rightarrow (P \vee \neg T))$

6 Property1

8. $\neg(\neg S \vee (P \vee \neg T))$

7 Property2

9. $\neg(P \vee \neg T)$

8 Property1

10. $\neg P$

10 Property1

11. P

5

Problem 5. If a set of premises is inconsistent, then attempting to prove things with these premises is necessarily useless. For example, given the (clearly inconsistent) premises:

1. P
2. $\neg P$

The proof for *any* statement Q is then:

1. Show Q
2.

P	Premise 1, ID
$\neg P$	Premise 2
- 3.

One technique for determining if a set of premises is inconsistent is to determine if their conjunction is a contradiction (*i.e.*, if there are N premises, identified as P_i for $i = 1$ to N , then the premises are inconsistent if $\forall_{Tr}^N E_P(P_1 \wedge P_2 \wedge \dots \wedge P_N) \rightarrow F$).

1. Considering all possible models to show that a set of premises is inconsistent will take $O(2^N)$ time, where N is the number of propositional variables. Proving inconsistency instead might be preferable. However, as noted above, proving things with a set of inconsistent premises is necessarily useless. How (precisely) can you prove a set of premises, P_i for $i = 1$ to N , is inconsistent using (a) Semantic Tableaux (b) Kalish-Montague derivations?
2. It is possible that a set of premises is collectively a tautology (*i.e.*, if there are N premises, identified as P_i for $i = 1$ to N , then $\models (P_1 \wedge P_2 \wedge \dots \wedge P_N)$). Does this cause a problem for proving things? Are there any other implications of having a set of premises whose conjunction is a tautology?
3. Consider the following set of premises: "Sales of houses fall off if interest rates rise. Auctioneers are not happy if sales of houses fall off. Interest rates are rising. Auctioneers are happy."
 - (a) Formalize these premises into a set of propositional formulae.
 - (b) Demonstrate that they are inconsistent using truth tables
 - (c) Prove that they are inconsistent using a Kalish-Montague derivation

Solution

1. If our premises form a contradiction, then we can say that

$$\neg(P_1 \wedge P_2 \dots \wedge P_N)T$$

We first convert it into a tautology then proceed normally with our Semantic Tableau
We would then use a semantic tableau to show contradiction occur if we start with:

$$\neg(P_1 \wedge P_2 \dots \wedge P_N) : F$$

which equals

$$== (P_1 \wedge P_2 \dots \wedge P_N) : T$$

We will then as in any other ST, split the terms inside the dis-junctions showing that we do have a tautology.
This would show the contradiction

We would also be able to use KM derivation and we will again start with

$$\text{Show } \neg(P_1 \wedge P_2 \dots \wedge P_N)$$

which will evidently turn to

$$(P_1 \wedge P_2 \dots \wedge P_N) \quad 1, ID$$

If this turns out to be True we would have proved that this is a tautology. But since we took the negation of our premises, we would have then proven that our premises form a contradiction

2. Having the dis-junction of premises as a tautology shouldn't cause a problem but just that the logical implication of that premise will also turn out to be a tautology. This just means that the logical implication is also basically part of the premises themselves and that there is no need of an explicit proof
3. (a) S : Sales of houses fall off, I : Interest rates rise, A : Auctioneers are happy
Propositional formulae: $I \rightarrow S, S \rightarrow \neg A, I, A$
(They are in the same order as the given statements)
- (b) According to the problem, the premises are inconsistent if $\forall_{Tr}^N E_P(P_1 \wedge P_2 \wedge \dots \wedge P_N) \rightarrow F$ for N premises identified as P_i for $i = 1$ to N
Therefore using the same property in our premises:
Conjunction of Premises = $(I \rightarrow S) \wedge (S \rightarrow \neg A) \wedge I \wedge A$

Table 11: The conjunction is clearly always false

I	S	A	$(I \rightarrow S)$	$S \rightarrow \neg A$	$(I \rightarrow S) \wedge (S \rightarrow \neg A) \wedge I \wedge A$
T	T	T	T	F	F
T	T	F	T	T	F
T	F	T	F	T	F
T	F	F	F	T	F
F	T	T	T	F	F
F	T	F	T	T	F
F	F	T	T	T	F
F	F	F	T	T	F

1. *Show* $\neg((I \rightarrow S) \wedge (S \rightarrow \neg A) \wedge I \wedge A)$ 2–8 Conditionalization
2.

I

2 SIMP
3.

$(I \rightarrow S)$

2 SIMP
4.

$(S \rightarrow \neg A)$

2 SIMP
- (c) 5.

A

2 SIMP
6.

$\neg(\neg A)$

5 Double Negation
7.

$\neg S$

3,6 Modus Tollens
8.

$\neg I$

2,7 Modus Tollens

Thus $((I \rightarrow S) \wedge (S \rightarrow \neg A) \wedge I \wedge A)$ is a contradiction

Problem 6. Convert the following propositions to Conjunctive Normal Form (CNF) with at most 3 literals per clause

1. $(P \wedge Q) \rightarrow (P \rightarrow Q)$
2. $(P \wedge Q) \leftrightarrow (P \rightarrow Q)$
3. $(\neg P \vee Q) \rightarrow (P \rightarrow \neg Q)$
4. $((P \rightarrow Q) \rightarrow P) \rightarrow Q$
5. $(P \rightarrow (Q \rightarrow (P \rightarrow Q)))$
6. $((P \wedge \neg Q) \rightarrow \neg R) \leftrightarrow ((P \wedge R) \rightarrow Q)$
7. $((P \vee Q) \vee R) \vee S \leftrightarrow (P \vee (Q \vee (R \vee S)))$
8. $((P \rightarrow Q) \rightarrow R) \rightarrow S \leftrightarrow (P \rightarrow (Q \rightarrow (R \rightarrow S)))$
9. $(P \rightarrow (\neg R \rightarrow \neg S)) \vee ((S \rightarrow (P \vee \neg T)) \vee (\neg Q \rightarrow R))$

Solution

Using De-Morgan's law in all the proofs. Citing Proof here from question 7 for clarity. Also using the property shown below:

Table 12: Proof for $P \rightarrow Q = \neg P \vee Q$

P	Q	$P \rightarrow Q$	$\neg P \vee Q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

$$P \rightarrow Q = \neg P \vee Q \text{ (Property)}$$

1. True
2. $(P \wedge Q) \leftrightarrow (P \rightarrow Q)$
 $= (P \wedge Q) \leftrightarrow (\neg P \vee Q)$
 $= ((P \wedge Q) \rightarrow (\neg P \vee Q)) \wedge ((\neg P \vee Q) \rightarrow (P \wedge Q))$
 $= (\neg(P \wedge Q) \vee (\neg P \vee Q)) \wedge (\neg(\neg P \vee Q) \vee (P \wedge Q))$
 $= (\neg P \vee \neg Q) \vee (\neg P \vee Q) \wedge ((P \wedge \neg Q) \vee (P \wedge Q))$
 $= (\neg P \vee \neg Q \vee \neg P \vee Q) \wedge ((P \wedge \neg Q) \vee (P \wedge Q))(P \wedge \neg Q) \vee (P \wedge Q)$
 $= ((P \wedge \neg Q) \vee P) \wedge ((P \wedge \neg Q) \vee Q)$
 $= ((P \vee P) \wedge (\neg Q \vee P) \wedge ((P \vee Q) \wedge (\neg Q \vee Q))$
 $= P \wedge (\neg Q \vee P) \wedge ((P \vee Q))$
 $= P$
3. $(\neg P \vee Q) \rightarrow (P \rightarrow \neg Q)$
 $= \neg(\neg P \vee Q) \vee (P \rightarrow \neg Q)$
 $= \neg(\neg P \vee Q) \vee (\neg P \vee \neg Q)$
 $= (P \wedge \neg Q) \vee (\neg P \vee \neg Q)$
 $= (P \vee (\neg P \vee \neg Q)) \wedge ((\neg Q) \vee (\neg P \vee \neg Q))$
 $= \neg P \vee \neg Q$
4. $((P \rightarrow Q) \rightarrow P) \rightarrow Q$
 $= (((\neg P \vee Q) \rightarrow P) \rightarrow Q)$
 $= ((\neg(\neg P \vee Q) \vee P) \rightarrow Q)$
 $= (\neg(\neg(\neg P \vee Q) \vee P) \vee Q)$
 $= (\neg((P \wedge \neg Q) \vee P) \vee Q)$
 $= (\neg((P \vee P) \wedge (\neg Q \vee P)) \vee Q)$
 $= ((\neg(P \vee P) \vee \neg(\neg Q \vee P)) \vee Q)$
 $= ((\neg P \vee (Q \wedge \neg P)) \vee Q)$
 $= (((\neg P \vee Q) \wedge (\neg P \vee \neg P)) \vee Q)$
 $= (((\neg P \vee Q) \wedge \neg P) \vee Q)$
 $= (\neg P \vee Q) \vee Q \wedge (\neg P \vee Q)$
 $= (\neg P \vee Q) \wedge (\neg P \vee Q)$
 $= \neg P \vee Q$
5. True
6. True
7. True

8. $(\neg P \vee Q \vee S) \wedge (P \vee \neg R \vee S)$

Table 13: Part 8. Clearly satisfiable

P	Q	R	S	$((P \rightarrow Q) \rightarrow R) \rightarrow S \leftrightarrow (P \rightarrow (Q \rightarrow (R \rightarrow S)))$
T	T	T	T	T
T	T	T	F	T
T	T	F	T	T
T	T	F	F	T
T	F	T	T	T
T	F	T	F	F
T	F	F	T	T
T	F	F	F	F
F	T	T	T	T
F	T	T	F	F
F	T	F	T	T
F	T	F	F	T
F	F	T	T	T
F	F	T	F	F
F	F	F	T	T
F	F	F	F	T

Using the truth table to derive the product of sums format:

$$\begin{aligned}
& (\neg P \vee Q \vee \neg R \vee S) \wedge (\neg P \vee Q \vee R \vee S) \wedge (P \vee \neg Q \vee \neg R \vee S) \wedge (P \vee Q \vee \neg R \vee S) \\
&= (\neg P \vee Q \vee S \vee \neg R) \wedge (\neg P \vee Q \vee S \vee R) \wedge (P \vee \neg R \vee S \vee \neg Q) \wedge (P \vee \neg R \vee S \vee Q) \quad [\vee \text{ is commutative/associative}] \\
&= ((\neg P \vee Q \vee S) \vee \neg R) \wedge ((\neg P \vee Q \vee S) \vee R) \wedge ((P \vee \neg R \vee S) \vee \neg Q) \wedge ((P \vee \neg R \vee S) \vee Q) \\
&= ((\neg P \vee Q \vee S) \vee \neg R) \wedge ((\neg P \vee Q \vee S) \vee R) \wedge ((P \vee \neg R \vee S) \vee \neg Q) \wedge ((P \vee \neg R \vee S) \vee Q) \\
&= (((\neg P \vee Q \vee S) \vee \neg R) \wedge ((\neg P \vee Q \vee S) \vee R)) \wedge (((P \vee \neg R \vee S) \vee \neg Q) \wedge ((P \vee \neg R \vee S) \vee Q)) \\
&= (((\neg P \vee Q \vee S)) \wedge (\neg R \vee R)) \wedge (((P \vee \neg R \vee S)) \wedge (\neg Q \vee Q)) \quad [\text{De-Morgan's Law}] \\
&= (\neg P \vee Q \vee S) \wedge (P \vee \neg R \vee S) \quad [(\neg Q \vee Q) : T]
\end{aligned}$$

9. True

Problem 7. Define:

$$\bigvee_{i=1}^N P_i = P_1 \vee P_2 \dots \vee P_N$$

and

$$\bigwedge_{i=1}^N P_i = P_1 \wedge P_2 \dots \wedge P_N$$

Show that

$$\bigvee_{i=1}^N P_i \leftrightarrow \neg \bigwedge_{i=1}^N (\neg P_i)$$

and

$$\bigwedge_{i=1}^N P_i \leftrightarrow \neg \bigvee_{i=1}^N (\neg P_i)$$

are true for $N \in \mathbb{N}$

Hint: in addition to basic Kalish-Montegue derivation, you will need to add induction.

Solution

Table 14: Proof using Truth Table of base case $P_1 \wedge P_2 = \neg(\neg P_1 \vee \neg P_2)$

P_1	P_2	$P_1 \wedge P_2$	$(\neg P_1 \vee \neg P_2)$	$\neg(\neg P_1 \vee \neg P_2)$
T	T	T	F	T
T	F	F	T	F
F	T	F	T	F
F	F	F	T	F

Now let us assume

$$P(y) = \bigwedge_{i=1}^y P_i = P_1 \wedge P_2 \dots \wedge P_y$$

Assuming

$$P(n) = \neg \bigvee_{i=1}^n (\neg P_i)$$

as True we will try to show that $P(n+1)$ is True

We know $P(n) \wedge P_{n+1} = P(n+1)$

$$\Rightarrow P(n+1) = \left(\bigwedge_{i=1}^n P_i \right) \wedge P_{n+1} = \left(\neg \bigvee_{i=1}^n (\neg P_i) \right) \wedge P_{n+1}$$

$$\Rightarrow P(n+1) = \neg(\neg(\neg \bigvee_{i=1}^n (\neg P_i)) \vee (\neg P_{n+1})) \quad [Using the property proved from base case]$$

$$\Rightarrow P(n+1) = \neg(\bigvee_{i=1}^n (\neg P_i) \vee (\neg P_{n+1}))$$

$$\Rightarrow P(n+1) = \neg(\bigvee_{i=1}^{n+1} (\neg P_i))$$

$$\Rightarrow P(n+1) = \neg \bigvee_{i=1}^{n+1} (\neg P_i)$$

Thus we have proved $P(n+1)$ from $P(n)$, and thus using the Principle of Mathematical Induction we have proved

$$\bigwedge_{i=1}^N P_i \leftrightarrow \neg \bigvee_{i=1}^N (\neg P_i)$$

Table 15: Proof using Truth Table of base case $P_1 \vee P_2 = \neg(\neg P_1 \wedge \neg P_2)$

P_1	P_2	$P_1 \vee P_2$	$(\neg P_1 \wedge \neg P_2)$	$\neg(\neg P_1 \wedge \neg P_2)$
T	T	T	F	T
T	F	T	F	T
F	T	T	F	T
F	F	F	T	F

Now let us assume

$$P(y) = \bigvee_{i=1}^y P_i = P_1 \vee P_2 \dots \vee P_y$$

Assuming

$$P(n) = \neg \bigwedge_{i=1}^n (\neg P_i)$$

as True we will try to show that $P(n+1)$ is True

We know $P(n) \vee P_{n+1} = P(n+1)$

$$\Rightarrow P(n+1) = \left(\bigvee_{i=1}^n P_i \right) \vee P_{n+1} = \left(\neg \bigwedge_{i=1}^n (\neg P_i) \right) \vee P_{n+1}$$

$$\Rightarrow P(n+1) = \neg(\neg(\neg \bigwedge_{i=1}^n (\neg P_i)) \wedge (\neg P_{n+1})) \quad [Using the property proved from base case]$$

$$\Rightarrow P(n+1) = \neg(\bigwedge_{i=1}^n (\neg P_i) \wedge (\neg P_{n+1}))$$

$$\Rightarrow P(n+1) = \neg(\bigwedge_{i=1}^{n+1} (\neg P_i))$$

$$\Rightarrow P(n+1) = \neg \bigwedge_{i=1}^{n+1} (\neg P_i)$$

Thus we have proved $P(n+1)$ from $P(n)$, and thus using the Principle of Mathematical Induction we have proved

$$\bigvee_{i=1}^N P_i \leftrightarrow \neg \bigwedge_{i=1}^N (\neg P_i)$$