

Assignment 1

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Absque sudore et labore nullum opus perfectum est

Problem 1. Prove or Disprove:

- a) $R \subseteq S \Leftrightarrow R \subseteq ((S - T) \cup (R \cap T))$
- b) $(A \cap B) \subseteq (B \cap C) \longrightarrow (A \subseteq C)$
- c) $A \in B \wedge B \in C \longrightarrow A \in C$
- d) $A \in B \wedge B \in C \longrightarrow A \subseteq C$
- e) $A \in B \wedge B \subseteq C \longrightarrow A \in C$
- f) $A \in B \wedge B \subseteq C \longrightarrow A \subseteq C$

Solution

- a) This can be disproved by a counter example

$$R = \{1, 2, 3, 4\}, S = \{1, 2, 3\}, T = \{4\}$$

The right hand side is: $R \subseteq \{1, 2, 3\} \cup \{4\}$ which is $R \subseteq \{1, 2, 3, 4\}$ which is true, but on the other hand the left hand side $R \subseteq S$ is false and $R \not\subseteq S$. Thus, the given statement is false

- b) This can be disproved by a counter example

$$A = \{1, 2, 3, 4\}, C = \{1, 2, 3\}, B = \{1, 2, 3\}$$

The left hand side is: $\{1, 2, 3\} \subseteq \{1, 2, 3\}$ which is true, but on the other hand the right hand side $A \subseteq B$ is false and $A \not\subseteq B$. Thus, the given statement is false

- c) This can be disproved by a counter example

$$A = \{1, 2, 3, 4\}, B = \{\{1, 2, 3, 4\}, \{0\}\}, C = \{\{\{1, 2, 3, 4\}, \{0\}\}\}$$

The left hand side is: $A \in B \wedge B \in C$ which is true, but on the other hand the right hand side $A \in C$ is false and $A \notin C$. Thus, the given statement is false

d) This can be disproved by a counter example

$$A = \{1, 2, 3, 4\}, B = \{\{1, 2, 3, 4\}, \{0\}\}, C = \{\{\{1, 2, 3, 4\}, \{0\}\}\}$$

The left hand side is: $A \in B \wedge B \in C$ which is true, but on the other hand the right hand side $A \subseteq C$ is false and $A \not\subseteq C$

Thus, the given statement is false

e) if $A \in B$ and $B \subseteq C$ then we can say that \forall_x in B , x also exists in C and since A is an element of B we can safely say that A is an element of C and hence $A \in C$

f) This can be disproved by a counter example

$$A = 1, B = \{1, 2, 3, 4\}, C = \{1, 2, 3, 4\}$$

The left hand side is: $A \in B \wedge B \subseteq C$ which is true, but on the other hand the right hand side $A \subseteq C$ is false and $A \not\subseteq C$

Thus, the given statement is false

Problem 2. Given sets A and B , under what conditions does $A - B = B - A$?. Prove that this is the only conditions under which this is true

Solution

If $A - B = B - A$ then

for any $x \in A - B = B - A$ we know, $x \in A$ and $x \in B$ and $x \notin A$ and $x \notin B$.

That's a contradiction so $A - B = B - A$ is empty.

Thus there are no elements in A that are not in B . In other words A is a subset of B .

Likewise there are no elements of B that are in A . So B is a subset of A .

So $A = B$

Problem 3.

- a) Given $f: A \longrightarrow B$, what is the relation between f 's co-domain and its image if f is
- (i) injective
 - (ii) surjective
 - (iii) bijective
- b) Given $f: A \longrightarrow B$, what is the relation between the image of f^{-1} and the domain of f if f is
- (i) injective
 - (ii) surjective
 - (iii) bijective

Solution

- a) (i) injective : The image is a subset of the co-domain
(ii) surjective: The image is equal to the co-domain
(iii) bijective: The image is equal to the co-domain
- b) (i) injective : The image is equal to the domain
(ii) surjective: The inverse of f is not defined
(iii) bijective: The image is equal to the domain

Problem 4. Given sets X, Y such that $X \subseteq Y$. Prove or disprove:

- a) There exists an injection $f: X \longrightarrow Y$
b) There exists an surjection $f: Y \longrightarrow X$

Solution

We are given that $X \subseteq Y$ thus, we can say $\text{cardinality}(X) \leq \text{cardinality}(Y)$

- a) From above we can clearly say $\forall x \in X \exists y \in Y$ such that each x is mapped to a unique y and there might be elements in Y that have no pre-image in X .
Thus an injective function exists
- b) The given statement can be disproved by taking a counter example.
If the subset X is empty then there does not exist a well defined surjection from Y to X . Therefore the given statement is not true

Problem 5. Given the following functions, are they: injective, surjective, bijective, or none of these?

- a) $f: \mathbb{N} \longrightarrow \mathbb{N}$ where $f: x \mapsto x$
- b) $g: \mathbb{N} \longrightarrow \mathbb{N}$ where $g: x \mapsto x^2$
- c) $h: \mathbb{Q}^+ \longrightarrow \mathbb{Q}^+$ where $h: x \mapsto 1/x$
- d) If it is possible to compose f and g , f and h and g and h what is the result of the composition. If not explain why not?

Solution

- a) INJECTIVE: If $f(x_1) = f(x_2)$ then it implies that $x_1 = x_2$
 SURJECTIVE: Range of $f(x)$ is clearly \mathbb{N} which is same as it's co-domain.
 Since $f(x)$ is both injective and surjective, we can say that it is bijective
- b) INJECTIVE: If $g(x_1) = g(x_2)$ then it implies that $x_1 = x_2$ since we are only restricted to the positive square root in the set of natural numbers
 SURJECTIVE: We cannot say it is surjective since the numbers in \mathbb{N} which are not perfect squares do not lie in the range of $g(x)$
 Thus $g(x)$ is only injective
- c) INJECTIVE: If $h(x_1) = h(x_2)$ then it implies that $1/x_1 = 1/x_2 \implies x_1 = x_2$
 SURJECTIVE: Range of $h(x)$ is clearly \mathbb{Q}^+ since all numbers in \mathbb{Q}^+ also have their inverses in \mathbb{Q}^+ . This implies that $h(x)$ is surjective
 Since $h(x)$ is both injective and surjective, we can say that it is bijective
- d) (i) $f \circ g: \mathbb{N} \longrightarrow \mathbb{N}$ where $f \circ g: x \longrightarrow x^2$
 (ii) $f \circ h: \mathbb{N} \longrightarrow \mathbb{Q}^+$ where $f \circ h: x \longrightarrow 1/x$
 (iii) $g \circ h: \mathbb{N} \longrightarrow \mathbb{Q}^+$ where $g \circ h: x \longrightarrow 1/x^2$

Problem 6. Prove or disprove: Given a strict partial order $(X, <)$, the pair $(X, <^{ref})$ is a partial order

Solution Since $(X, <)$ is a strict partial order, thus we can say that we will need to add minimum number of terms for the reflexive closure.

The terms that we will add will be of the form $(x, x) \forall x \in X$. Adding these reflexive terms will make $(X, <^{ref})$ reflexive.

But adding these terms won't affect transitivity since $xRx \wedge xRy \Rightarrow xRy$ but xRy is already in the relation so the relation. Thus, $(X, <^{ref})$ also remains transitive

Also all the new added reflexive terms will inherently exhibit anti-symmetry.

Therefore $(X, <^{ref})$ is reflexive, anti-symmetric, transitive.

Thus according to the definition of a partial order, $(X, <^{ref})$ is a partial order

Problem 7. Consider the set of letter grades $G = \{ A, B, C, D, F, \text{INC}, \text{WD}, \text{AEG} \}$. A, B, C, D, and F are what you would intuitively think they are, with A better than B, which is better than C, etc.

INC means Incomplete. This means the final grade has not yet been assigned because some outstanding work is missing. After the outstanding work is done, a final grade will be assigned. If the outstanding work is never done, then an INC changes to an F after two terms. Is an INC better than an F? Well, it is at least better than or equal to an F, since it will never get worse than F, and let's be honest: it's really better than F because (a) it is easier to explain away an INC during an interview than an F (I was sick and so did not complete ...; my average prior to that was -state something plausible and consistent with other grades.) (b) the student will probably do better than an F when the outstanding work is submitted (otherwise why bother do the work, since it will automatically change to an F in two terms); in the meantime, reason (a) says an F now is worse than an F later. Is INC better than a D? That is harder to argue. It could become an F. It means there is outstanding work and it will become an F if that work is not completed. We will therefore claim that INC is better than or equal to a D but cannot say more than that with respect to D. How about a B or C? Realistically, we cannot claim any relationship between INC and B or C, since we do not know what the result of completing the outstanding work will be, so there can be no better than relation between those grades and INC. Finally, there is A? Is A better than INC? Clearly it is, because there is no grade better than A. Even when the outstanding work is done, the best the INC can become is an A, and an A now means there is nothing to even explain in an interview. Ergo, A is better than INC.

WD means Withdrawn and so the class was attended long enough that it will not be removed from a transcript, but not so long that a final-grade assessment will be made. It is somewhat awkward to have on a transcript (better to have dropped prior to the drop deadline), but far better than a failing grade on the transcript. Therefore WD is better than F. Is it better than D? Almost certainly it is. D is not a great grade. It is a pass, but barely. In a traditional setting, it means: pass, but this course cannot be used as a prerequisite for some subsequent course. If it is for an elective, unrelated to a major, it may be OK, but WD is arguably better since it says that the student recognized that it was better to drop the course than not. Ergo, WD is better than D. As with INC, it is not really comparable to C, since C is an OK, but not good, grade. How about B? B is a good grade. Is it better to have withdrawn from a course of got a B? This is unclear. It is probably reasonable to claim that B is better than or equal to a WD.

Finally, AEG is a curious grade that hopefully you will never experience, but is useful in some circumstances. AEG is short for Aegrotat which is Latin for s/he is ill. It is a grade that is awarded when a student has demonstrated enough ability to show that s/he has passed the course, but no better granularity is possible on the grade because there is insufficient information to make that determination. It is almost like a pass in a pass/fail system, except that what it is really saying is the grade is somewhere between A and D but we do not have enough information to say where (because the student was ill). Given that

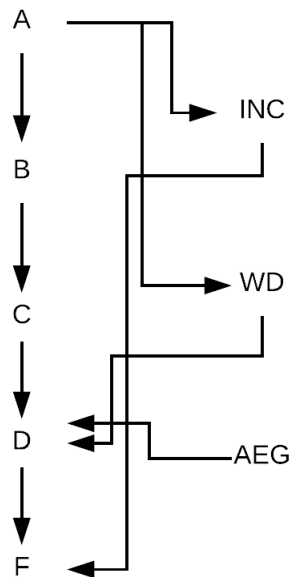
it is often viewed as a pass in a pass/fail system, it is reasonable to argue that an AEG is at least as good as a C, but not much more can be stated about it. Given the information above:

- Draw a graph of the better than relation for the set G.
- Draw a graph of the better than or equal to relation for the set G.
- Over the set G, for each of the relations (better than and better than or equal to), is there a least upper bound and a greatest lower bound, and if so what are they?
- Over the set $G - \{A, F\}$, for each of the relations, what are the minimal and maximal elements?
- Are the relations partial orders? Is there any missing information and/or information you are assuming that might affect that answer?

Solution

a)

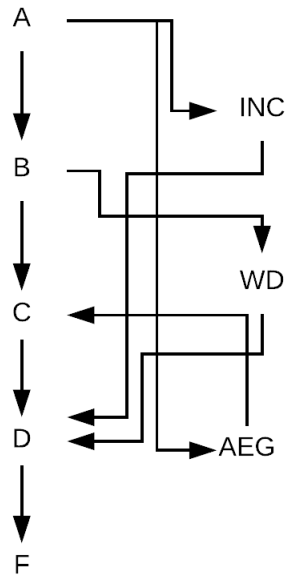
Better Than Relation over G



Note: Transitive relations excluded to preserve clarity

b)

Better Than or Equal to Relation over G



Note: Transitive relations and self relations excluded to preserve clarity

- c) The LUB in the better than relationship is A . As for the GLB in the better than relation's GLB is F . In the better than or equal to relationship, LUB is A and GLB does not exist since we cannot compare AEG and F
- d) i) For the "better than" relation:
 Minimal elements: $\{B, INC, AEG, WD\}$;
 Maximal elements: $\{D\}$
- ii) For the "better than or equal to" relation:
 Minimal elements: $\{B, INC, AEG\}$;
 Maximal elements: $\{D\}$
- e) "Better Than" relation is a partial order but due to lack of information the "Better Than or Equal To" is not a partial order

Problem 8. Given $R, S \subseteq A^2$ are relations on A . Define $T \subseteq A^2$ such that $xTy \Leftrightarrow (xRy \wedge xSy)$.

Prove or disprove: If R and S are equivalence relations, then T is also an equivalence relation.

Solution Suppose R and S are both equivalence relations on a set A . We will show that R and S both being equivalence relations on the set A implies that T is also an equivalence relation.

It is immediately apparent that since R and S are equivalence relations then:

$$x \in A \Rightarrow (xRx) \wedge (xSx) \Rightarrow ((x, x) \in R) \wedge ((x, x) \in S) \Rightarrow (x, x) \in T$$

Thus $x \in A \Rightarrow (x, x) \in T$, therefore T is reflexive.

$$\text{Now suppose } (x, y) \in T \Rightarrow (xRy) \wedge (xSy) \Rightarrow (yRx) \wedge (ySx)$$

$$[\text{Since both } R \text{ and } S \text{ are symmetric}] \Rightarrow ((y, x) \in R) \wedge ((y, x) \in S) \Rightarrow (y, x) \in T$$

Since both R and S are symmetric, T will also be symmetric as shown above

$$\text{Finally suppose } ((x, y) \in T \wedge (y, z) \in T) \Rightarrow ((xRy) \wedge (xSy)) \wedge ((yRz) \wedge (ySz)) \Rightarrow (xRz) \wedge (xSz)$$

[Since both R and S are transitive] $\Rightarrow ((x, z) \in R) \wedge ((x, z) \in S) \Rightarrow (x, z) \in T$ Thus we have shown T is transitive

It follows that T is an equivalence relation since it is reflexive, symmetric and transitive.

□

Problem 9. Given poset (X, \leq) , prove or disprove:

- a) $x \geq y \Leftrightarrow y \leq^{-1} x$
- b) $x \geq y \Leftrightarrow y \leq' x$
- c) $x < y \Leftrightarrow y \leq' x$
- d) $x > y \Leftrightarrow y(\leq^{-1})'x$
- e) $x > y \Leftrightarrow y(\leq')^{-1}x$

Solution Assuming this stands True for all posets:

- a) $x \geq y \Leftrightarrow x(\leq)^{-1}y$, therefore it cannot imply $y \leq^{-1} x$
 $7 \geq 5 \Leftrightarrow 7(\leq)^{-1}5$, but $5 \not\leq^{-1} 7$
- b) $x \geq y \Leftrightarrow x(<)'y$, therefore it cannot imply $y \leq' x$
 $7 \geq 5 \Leftrightarrow 7(<)'5$, but $5 \not\leq' 7$
- c) $x < y \Leftrightarrow y > x \Rightarrow (y \geq x \wedge y \neq x)$, therefore by anti-symmetry it implies that $y \leq^{-1} x$
- d) $x > y \Leftrightarrow x \leq^{-1} y \Leftrightarrow y(\leq^{-1})'x$, this statement is inherently true as shown from earlier
- e) $x > y \Leftrightarrow x \leq' y \Leftrightarrow y(\leq')^{-1}x$, this statement is inherently true as shown from earlier

Problem 10. Given $X \subseteq \mathbb{N}, \forall_{x,y \in \mathbb{N}} x \leq y \Leftrightarrow \exists_{z \in X} x + z = y$. Prove that, if (X, \leq) is a partial order:

- a) $0 \in X$
- b) $\forall_{x,y}(x \in X \wedge y \in X) \rightarrow x + y \in X$

Solution

- a) Given (X, \leq) is a partial order we can say that (X, \leq) is reflexive over X , thus

$$(x \leq x \Leftrightarrow \exists_{z \in X} x + z = x) \Rightarrow (z = 0 \wedge z \in X) \Rightarrow 0 \in X$$

- b) Since $x, y \in X \wedge x \subseteq \mathbb{N} \Rightarrow x, y \in \mathbb{N} \Rightarrow x + y \in \mathbb{N}$

Also $\exists_{z \in X}$ such that $x + z = y$

Thus going from right to left in the given bi-directional implication and using that $z = 0$

$$(x + y) + 0 = (x + y) \Leftrightarrow (x + y) \leq (x + y)$$

And since (X, \leq) is a partial order, \leq is reflexive,

Therefore $\forall_{a \in X}, a \in X$

Therefore, $(x + y) \leq (x + y) \Rightarrow (x + y) \in X$

Problem 11. Given partially ordered set (X, \leq) , and $Y \subseteq X$, define a minimum element in Y as:

$$x \text{ is a minimum element in } Y \Leftrightarrow x \in Y \wedge \forall_{y \in Y} x \leq y$$

Note the difference between this definition and that of a minimal element ($x \in Y$ is a minimal element in Y if $\nexists_{y \in Y} y < x$). Prove or disprove:

- a) If x is a minimum element in Y then x is unique.
- b) If x is a minimum element in Y then x is a minimal element in Y .
- c) If x is a minimal element in Y then x is a minimum element in Y .
- d) If every nonempty subset of X has a minimum element, then X is totally ordered.

Solution

- a) Let us assume that x is not unique, which means there exists z such that $z \in Y \wedge \forall_{y \in Y} z \leq y$ and since x is also a minimum element $x \in Y \wedge \forall_{y \in Y} x \leq y$ which implies that [after combining those 2 statements]: $x \leq z \wedge z \leq x \Rightarrow x = z$. Therefore the minimum element is unique
- b) Since x is a minimum element in Y we can say that $x \in Y \wedge \forall_{y \in Y} x \leq y$ this clearly implies that there does not exist an element in Y that is smaller than x $\nexists_{y \in Y} y < x$. Therefore, x is also the minimal element
- c) Since x is a minimal element in Y we can say that $\nexists_{y \in Y} y < x$ this clearly implies that there does not exist an element in Y that is smaller than $x \Rightarrow \forall_{y \in Y} x \leq y$. Therefore, x is also the minimum element
- d) Given the fact that every subset has a minimum element, we can say that $\forall_{x_1, x_2} \in X$ for subset $A = \{x_1, x_2\}$ either: $x_1 \leq x_2$ or $x_2 \leq x_1$ and since this holds $\forall_{x_1, x_2} \in X$ we can imply that X is totally ordered

Problem 12. Given finite set A with cardinality N :

- a) How many distinct functions, $f: A \rightarrow A$, exist that map A to A ?
- b) How many of these functions are injective? surjective? bijective?
- c) How many distinct binary relations, $R \subseteq A \times A$, exist on set A ?
- d) How many of these relations are both symmetric and anti-symmetric?
- e) How many of these relations are both symmetric, anti-symmetric, and reflexive?
- f) How many of these relations are equivalence relations?

Solution

- a) By elementary combinatorics we can say that every element in A has N choices for it's output, therefore total number of function = N^N
- b) (i) Injective Only: The first input has N choices, the next one has N choices as well and so on. Thus the total number of functions are: N^N
(ii) Surjective Only: Since the size of domain and co-domain is same, all Surjective functions will be bijective by default. Therefore as explained below the total number of functions are: $N!$
(iii) Bijective: The first input has N choices, the next one has $N-1$ choice and so on. Thus the total number of functions are: $N!$
- c) cardinality($A \times A$) = N^2 , each element in that set may or may not be in the relation R therefore total number of relations is 2^{N^2}
- d) Since the relation is both symmetric and anti-symmetric this means $\forall_{x,y \in A} (xRy \Leftrightarrow yRx \wedge x = y)$ therefore the relation is $xRy \Leftrightarrow x = y$. Therefore the number of relations is 2^N since we can either choose or not choose any of the $(xRx \forall x \in A)$
- e) Extending from the previous part, there is only one relation that is symmetric, anti-symmetric as well as reflexive
- f) Extending from above there is only one relation that is reflexive, symmetric, anti-symmetric as well as transitive i.e. an Equivalence class